The Fractional Chromatic Number and the Hall Ratio

by

Johnathan B. Barnett

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 6, 2016

Keywords: Fractional chromatic number, Hall ratio

Approved by

Peter D. Johnson, Chair, Professor of Discrete Mathematics
Chris Rodger, Professor of Discrete Mathematics
Dean Hoffman, Professor of Discrete Mathematics
Jessica McDonald, Professor of Discrete Mathematics

Abstract

All graphs in this paper are both finite and simple. $\alpha(G)$ is the vertex independence of a graph, G. The Hall ratio of G is defined as $\rho(G) = max[\frac{n(H)}{\alpha(H)} \mid n(H) = |V(H)| \text{ and } H \subseteq G]$ where $H \subseteq G$ means that H is a subgraph of G. The chromatic number of G, denoted $\chi(G)$, is the smallest number of colors needed to label the vertices of G such that no two adjacent vertices receive the same color.

A *b-fold coloring* of G is an assignment to each vertex of G a set of b colors so that adjacent vertices receive disjoint sets of colors. We then say that G is *a:b-colorable* if it has a b-fold coloring in which the b colors come from a palette of a colors. The least a for which G has a b-fold coloring from $\{1, ..., a\}$ is the *b-fold chromatic number* of G and is denoted $\chi_b(G)$. We can now define the *fractional chromatic number* of G to be $\chi_f(G) = \inf_b \frac{\chi_b}{b}$. It is known that $\chi(G) \geq \chi_f(G) \geq \rho(G)$ for all G.

It is known that, on the class of Kneser graphs, the ratio of the chromatic number to the Hall ratio is unbounded. However, if K is a Kneser graph, then it happens that $\chi_f(K) = \rho(K)$. This begs the question, "Is $\frac{\chi_f}{\rho}$ bounded on the domain of all finite simple graphs?" In Chapter 2, we define a function that should help to identify the Hall ratio of a given graph and we discuss general properties of said function. In Chapter 3, we give results toward answering the question above by considering the lexicographic and disjunctive powers of the graph, W_5. In Chapter 4, we give results toward answering the question above by considering the Mycielski graphs.

Acknowledgments

Donna, Rick, and Josh - the family that encouraged me to achieve all.

Lindsey - whose love and faith kept me alive.

Susie Q - pure optimism incarnate.

CJ and Kat - whose friendship and support were unwavering.

Dr. Johnson - whose guidance got me this far.

Thank you for all that you are and have been.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Preliminary Definitions

All graphs in this dissertation are finite and simple. Unless otherwise specified, G and H stand for graphs.

Definition 1.1.1. *The vertex independence number of G, denoted $\alpha(G)$, is the size of a maximum independent set of vertices in G.*

Definition 1.1.2. *A clique in a graph G is a complete subgraph of G.*

Definition 1.1.3. *The clique number of G, $\omega(G)$, is the maximum number of vertices in a clique of G.*

Definition 1.1.4. *The matching number of G, denoted $\alpha'(G)$, is the size of a maximum independent set of edges in G.*

Definition 1.1.5. *$\beta(G)$ denotes the vertex cover number of G and is the size of a minimum set of vertices of G such that each edge in G is incident to at least one vertex in the set.*

Definition 1.1.6. *Assuming G has no isolated vertices, $\beta'(G)$ is the edge cover number of G and is the size of a minimum set of edges of G such that each vertex of G is incident with at least one edge in the set.*

Definition 1.1.7. $n(G) = |V(G)|$.

Definition 1.1.8. $\rho(G) = max[\frac{n(H)}{\alpha(H)} \mid H \text{ is an induced subgraph of } G]$ *is the Hall ratio of G. Deletion of the word "induced" gives an equivalent definition of $\rho(G)$.*

Definition 1.1.9. *The chromatic number of G, denoted $\chi(G)$, is the smallest number of colors needed to label the vertices of G such that adjacent vertices receive distinct colors. Alternatively, $\chi(G)$ is the smallest number of independent sets of vertices into which it is possible to partition $V(G)$.*

Definition 1.1.10. *A b-fold coloring of G is an assignment to each vertex of G a set of b colors so that adjacent vertices receive disjoint sets of colors.*

Definition 1.1.11. *G is a:b-colorable if it has a b-fold coloring in which the b colors come from a palette of a colors. Such a coloring is an a:b coloring of G.*

Definition 1.1.12. *The least a for which G has an a:b coloring is the b-fold chromatic number of G and is denoted $\chi_b(G)$.*

Definition 1.1.13. *The fractional chromatic number of G is $\chi_f(G) = \inf_b \frac{\chi_b(G)}{b}$.*

Definition 1.1.14. *The lexicographic product of G and H, denoted here as $GLexH$, is a graph such that $V(GLexH) = V(G) \times V(H)$ and $(x_1, y_1), (x_2, y_2) \in V(GLexH)$ are adjacent if and only if either x_1 is adjacent to x_2 in G or $x_1 = x_2$ and y_1 is adjacent to y_2 in H.*

Definition 1.1.15. *The disjunctive product of G and H, denoted $GDisjH$, is a graph such that $V(GDisjH) = V(G) \times V(H)$ and $(x_1, y_1), (x_2, y_2) \in V(GDisjH)$ are adjacent if and only if either x_1 is adjacent to x_2 in G or y_1 is adjacent to y_2 in H.*

1.2 History

Several years ago, Dr. Peter Johnson proffered the following conjecture: "the ratio of the chromatic number of a graph to its Hall ratio is bounded on the domain of finite simple graphs." However, this conjecture was shown to be false; this ratio is unbounded on the class of Kneser graphs [4]. It is interesting to note, though, that if K is a Kneser graph, then the fractional chromatic number of K is equal to the Hall ratio of K. With this fact in mind, Johnson went on to pose the question, "Is $\frac{\chi_f}{\rho}$ bounded?" This question has remained open for more than a decade. The following results relate the Hall ratio and chromatic numbers of a graph.

Proposition 1.2.1. [3] $\rho(G) \leq \chi_f(G) \leq \chi(G)$.

Theorem 1.2.2. *(Johnson 2009) $\chi_f - \rho$ is unbounded.*

Although the previous theorem tells us the difference between χ_f and ρ can be arbitrarily large, an interesting fact is that, prior to this research, the largest known value of $\frac{\chi_f}{\rho}$ known to us was $\frac{6}{5}$. This record was achieved by a graph of A. Daneshgar's appearing in [1]. One of the results of this paper will be to provide a graph in Chapter 3 for which this ratio is equal to $\frac{343}{282} > \frac{6}{5}$.

Chapter 2

Two New Functions

In this chapter we define two new functions whose jobs are to help determine the Hall ratio of a given graph. We then give some properties of these functions as well as provide examples of these functions in action.

2.1 The w-Function

$H \subseteq G$ means that H is a subgraph of G.

Definition 2.1.1. $w(a, G) = max[n(H) \mid H \subseteq G \text{ and } \alpha(H) = a]$, $1 \leq a \leq \alpha(G)$.

We can now express the Hall ratio of G as $\rho(G) = max\{\frac{w(a,G)}{a}\}$ where the maximum is taken over all $a \in \{1, 2, ..., \alpha(G)\}$. We almost immediately get the following results concerning this new function:

Proposition 2.1.2. *$w(1, G) = \omega(G)$ where $\omega(G)$ denotes the clique number of G.*

Proof. This fact follows from the simple observation that a graph has a vertex independence number of 1 if and only if the graph is complete. The size of a largest complete subgraph is, by definition, the clique number of a graph.

Proposition 2.1.3. *$w(\alpha(G), G) = n(G)$.*

Proof. This result follows from the fact that no subgraph of G can have more vertices than G.

Proposition 2.1.4. *For $a < \alpha(G), w(a,G) < w(a+1,G)$.*

Proof. To prove this, let $H \subseteq G$ be such that $\alpha(H) = a$ and $n(H) = w(a,G)$. Let $v \in V(G)\backslash V(H)$ and now consider $H \cup v$. It follows from the definition that $\alpha(H)+1 \geq \alpha(H \cup v) > \alpha(H) = a$ since H contains the maximum number of vertices of any subgraph of G with independence number a. Thus, $w(a+1,G) \geq n(H \cup v) = n(H)+1 = w(a,G)+1$.

Corollary 2.1.5. *For all $a \in \{1,...,\alpha(G)\}, w(a,G) = n(H)$ for some induced subgraph H of G such that $\alpha(H) = a$.*

Proof. The claim holds for $a \in \{1, \alpha(G)\}$ by Propositions 2.1.2 and 2.1.3, so suppose $1 < a < \alpha(G)$. Suppose that H' is a subgraph of G such that $\alpha(H') = a$ and $n(H') = w(a,G)$. Let H be the subgraph of G induced by $V(H')$. Then $n(H) = n(H')$ and $\alpha(H) \leq \alpha(H') = a$. If $\alpha(H) < a$, then $n(H) = n(H') = w(a,G) \leq w(\alpha(H),G) < w(a,G)$, by Proposition 2.1.4. This contradiction implies that $\alpha(H) = a$. ■

Theorem 2.1.6. *If G is a bipartite graph with no isolated vertices, then $w(a,G) = 2a$ for $1 \leq a \leq \alpha'(G)$ and $w(a,G) = a + \alpha'(G)$ for $\alpha'(G) < a \leq \alpha(G)$.*

Proof. The proof of this theorem relies on the following well known results [5] which are true for bipartite graphs containing no isolated vertices:

(i) $\alpha(G) + \beta(G) = n(G)$

(ii) $\alpha'(G) + \beta'(G) = n(G)$ *(Gallai)*

(iii) $\alpha'(G) = \beta(G) \Longrightarrow \alpha(G) = \beta'(G)$ *(König)*

(iv) $\alpha(G) \geq \alpha'(G)$

Suppose $1 \leq a \leq \alpha(G)$ and let H be an induced subgraph of G such that $n(H) = w(a,G)$ and $\alpha(H) = a$. This implies that H has no isolated vertices. To see this, assume that v is an isolated vertex in H. Then we can find $u \in V(G)\backslash V(H)$ to which v is adjacent in G since, by supposition, G has no isolated vertices. Then, letting H' be the graph induced by $V(H) \cup \{u\}$, $\alpha(H') = a$ and $n(H') = n(H) + 1$ thus contradicting $\alpha(H) = a$ and $n(H) = w(a, G)$.

So, using (i)-(iv) with H replacing G, we have

$w(a, G) = n(H) = \alpha(H) + \beta(H) = a + \alpha'(H) \leq a + \alpha(H) = 2a$. If $a \leq \alpha'(G)$, then we can find an H with $\alpha'(H) = \alpha(H) = a$ and $n(H) = 2a$. To do this, consider a matching in G with a edges as well as the subgraph H of G induced by those $2a$ vertices. Then $a \leq \alpha'(H) \leq \frac{n(H)}{2} = a$ and $\alpha(H) \geq \alpha'(H) = a$, while any subset of $V(H)$ with more than a elements must contain two vertices on the same edge from the original matching. So $\alpha(H) = a$. Thus $w(a, G) = 2a$ if $a \leq \alpha'(G)$.

If $a > \alpha'(G)$, then, as above, $w(a, G) = a + \alpha'(H) \leq a + \alpha'(G)$, for some H. We can find an H with $\alpha(H) = a$ and $\alpha'(H) = \alpha'(G)$, which will show that $w(a, G) = a + \alpha'(G)$ in this case, by taking a maximum matching M of G. Let $\tilde{H}$ be the graph induced by the vertices of our matching, M. By the argument above, $\alpha(\tilde{H}) = \alpha'(\tilde{H}) = \alpha'(G) < a \leq \alpha(G)$. Now, add vertices to $\tilde{H}$ to obtain a subgraph H of G with $\alpha(H) = a$ and $\alpha'(\tilde{H}) = \alpha'(G) \leq \alpha'(H) \leq \alpha'(G) \Longrightarrow \alpha'(H) = \alpha'(G)$. ∎

Theorem 2.1.7. *Let $G = K_{p_1, p_2, \ldots, p_k}, 1 \leq p_1 \leq p_2 \leq \ldots \leq p_k$, be a complete multipartite graph with k parts. Then*

(i) if $1 \leq a \leq p_1$, then $w(a, G) = ka$.

(ii) if $p_i \leq a < p_{i+1}$ for some $i \in \{1, 2, \ldots, k-1\}$, then $w(a, G) = \sum_{j=1}^{i} p_j + (k - i)a$.

(iii) if $a = p_k$, then $w(a, G) = n(G)$.

Proof. Every induced subgraph H of G is a complete multipartite graph $K_{q_1, q_2, \ldots, q_r}, 1 \leq r \leq k$ with $\alpha(H) = a = max[q_i : 1 \leq i \leq r]$. To maximize $n(H)$ with $\alpha(H) = a$, we can take $r = k$. If $1 \leq a \leq p_1$ then let $H = K_{a, a, \ldots, a} \Longrightarrow$ (i). If $p_i \leq a \leq p_{i+1}$ for some $i \in \{1, 2, \ldots, k-1\}$, let $H = K_{p_1, p_2, \ldots p_i, a, a, \ldots, a} \Longrightarrow$ (ii). If $a = p_k = \alpha(G)$, then let $H = G \Longrightarrow$ (iii). ∎

Definition 2.1.8. *The join of graphs G and H, denoted $G \vee H$, is a graph obtained from disjoint copies of G and H by joining each vertex of G to each vertex of H.*

Theorem 2.1.9. *Suppose that $\alpha(G) \leq \alpha(H)$. Then for $1 \leq a \leq \alpha(G \vee H) = \alpha(H)$,*

$$w(a, G \vee H) = \begin{cases} w(a, G) + w(a, H), & 1 \leq a \leq \alpha(G) \\ n(G) + w(a, H), & \alpha(G) \leq a \leq \alpha(H) \end{cases}$$

Proof. The proof of Theorem 2.1.9 follows from some simple observations. We know that we can find an $X_1 \subseteq G$ and $X_2 \subseteq H$ such that $n(X_1) = w(a, G)$, $\alpha(X_1) = a$, $n(X_2) = w(a, H)$, and $\alpha(X_2) = a$ for $1 \leq a \leq \alpha(G) \leq \alpha(H)$. It is elementary that $\alpha(X_1 \vee X_2) = max[\alpha(X_1), \alpha(X_2)] = a$. Since $X_1 \vee X_2 \subseteq G \vee H$, we have $n(X_1 \vee X_2) = n(X_1) + n(X_2) = w(a, G) + w(a, H) \leq w(a, G \vee H)$.

On the other hand, suppose that X is an induced subgraph of $G \vee H$, $\alpha(X) = a \leq \alpha(G)$, and $n(X) = w(a, G \vee H)$. Let $X_1 = X \cap G$ and $X_2 = X \cap H$. Since X is induced, $X = X_1 \cup X_2$. Since $a = \alpha(X) = max[\alpha(X_1, X_2]$, by Proposition 2.1.3, we have $n(X) = w(a, G \vee H) = n(X_1) + n(X_2) \leq w(a, G) + w(a, H)$. Thus we have $w(a, G \vee H) = w(a, G) + w(a, H)$.

Now assume that $\alpha(G) \leq a \leq \alpha(H)$. Then let $X_1 = G$ and we have $n(X_1 \vee X_2) = n(X_1) + n(X_2) = n(G) + w(a, H) \leq w(a, G \vee H)$. The reverse inequality follows easily by an argument similar to that in the preceding paragraph. ■

At this point, we can also point out that Theorem 2.1.7 can be derived from Theorem 2.1.9 since $\overline{K_{p_1}} \vee ... \vee \overline{K_{p_k}}$

Below are two examples of the w-function in action.

Example 2.1.10.

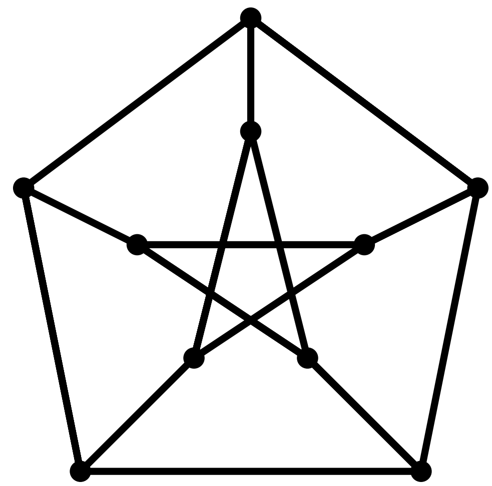

Figure 2.1: Petersen Graph

Figure 2.1 above is a typical representation of the Petersen graph. Let it be denoted by P. Note that it is triangle free and contains at least one edge. Thus, by Proposition 2.1.2, $w(1, P) = \omega(P) = 2$. Additionally, by Proposition 2.1.3., we have $w(4, P) = w(\alpha(P), P) = n(P) = 10$. Although less trivial, it is not hard to prove to oneself that $w(2, P) = 5$ and $w(3, P) = 7$. Relating this back to our goal of finding the Hall ratio of a given graph, we find that $\rho(P) = max\{\frac{2}{1}, \frac{5}{2}, \frac{7}{3}, \frac{10}{4}\} = \frac{5}{2}$. Furthermore, this ratio is achieved by P itself and any of the subgraphs of P that are 5-cycles.

Example 2.1.11.

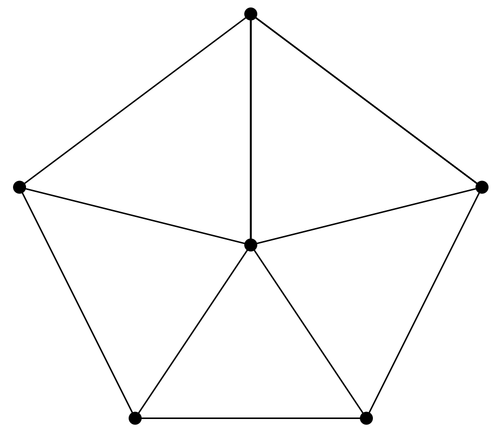

Figure 2.2: W_5

Figure 2.2 above is W_5 - the wheel with five spokes. With a quick check, it can be confirmed that $\omega(W_5) = w(1, W_5) = 3$ and $w(2, W_5) = n(W_5) = 6$. $\rho(W_5)$ is thus equal to 3 and is achieved by W_5 itself and any of its triangles.

2.2 The b-function

Definition 2.2.1. *Let* $b(t, G) = min[\alpha(H) \mid H \subseteq G \text{ and } n(H) = t], 1 \leq t \leq n.$

Proposition 2.2.2. $b(w(a, G), G) = a$ *for* $1 \leq a \leq \alpha(G)$

Proof. Note that $b(w(a, G), G) \leq a$ by the definitions of w and b. Now assume $b(w(a, G), G) = c < a$ and let $H \subseteq G$ be a subgraph of order $t = w(a, G)$ with $\alpha(H) = c$. Then $w(c, G) \geq n(H) = w(a, G)$ while $c < a$, which contradicts the result of Proposition 2.1.4. ■

Proposition 2.2.3. *For* $1 \leq t \leq n(G)$*,* $w(b(t,G),G) \geq t$*.*

Proof. Let $H \subseteq G$ be a subgraph of order t such that $\alpha(H) = b(t,G)$. Then $t \leq w(\alpha(H),H) \leq w(\alpha(H),G)$. ■

Example 2.2.4. *Let* K_n *denote the complete graph on* $n > 1$ *vertices. By definition,* $\alpha(K_n) = 1$ *and thus* $b(t,K_n) = 1$ *for all* $t \in \{1,...,n\}$*.*

The above example may seem a bit trivial, but it stands to highlight the fact that $b(t,G)$ is not necessarily increasing with t if G has even one edge.

Chapter 3

The Lexicographic and Disjunctive Products of W_5

It is our belief that the ratio of the fractional chromatic number to the Hall ratio is unbounded on some class of graphs. In this chapter, we discuss two potential candidates for which this might occur: the lexicographic and disjunctive powers of W_5.

3.1 Lexicographic Products

Lemma 3.1.1. [3] *Suppose that X is a subgraph of $GLexH$. For each $u \in V(G)$, let $H(u, X)$ be the subgraph of H induced by $\{v \in V(H) \mid (u, v) \in V(X)\}$. Then*
$\alpha(X) = max[\sum_{u \in I} \alpha(H(u, X)) \mid I \subseteq V(G) \text{ is independent in } G]$

The following results by Johnson, Sheinerman, and Ullman provide useful information involving the independence number, Hall ratio, and fractional chromatic number of graphs formed by lexicographic products.

Corollary 3.1.2. [3] $\alpha(GLexH) = \alpha(G)\alpha(H)$.

Theorem 3.1.3. [3] $\rho(GLexH) \geq \rho(G)\rho(H)$.

Theorem 3.1.4. [4] $\chi_f(GLexH) = \chi_f(G)\chi_f(H)$.

Theorem 3.1.5. [3] $\rho(GLexH) \leq \chi_f(G)\rho(H)$.

We now expand on the topic by looking at how the w-function behaves on $GLexH$.

Proposition 3.1.6. $w(1, GLexH) = \omega(GLexH) = \omega(G)\omega(H)$.

Proof. Assume K_G and K_H are maximum cliques in G and H respectively. Then it should be obvious that the graph induced in $GLexH$ by $V(K_G) \times V(K_H)$ is a clique in $GLexH$. Thus we have $\omega(GLexH) \geq \omega(G)\omega(H)$. Now consider a maximum clique, K, in $GLexH$. Considering the vertices of this clique as ordered pairs, it must be the case that the first coordinates induce a clique in G while for each first coordinate $x \in V(G)$, the set $\{y \in V(H) \mid (x, y) \in V(K)\}$ induces a clique in H. Therefore, we have $\omega(GLexH) \leq \omega(G)\omega(H)$. Combining the two inequalities gives the desired result. ■

Proposition 3.1.7. $w(\alpha(GLexH), GLexH) = n(GLexH) = n(G)n(H)$.

Proof. This follows directly from the definition of the lexicographic product of two graphs as well as Proposition 2.1.3.

Theorem 3.1.8. *Suppose that a, b, and c are positive integers. Then $w(a, GLexH) \geq w(b, G)w(c, H)$ wherever $b \leq \alpha(G)$, $c \leq \alpha(H)$, and $a = bc$.*

Proof. Let G' be a subgraph of G such that $\alpha(G') = b$ and $n(G') = w(b, G)$. Let H' be a subgraph of H such that $\alpha(H') = c$ and $n(H') = w(c, H)$. We have $G'LexH' \subseteq GLexH$. By [1], $\alpha(G'LexH') = \alpha(G')\alpha(H') = bc$. Thus, if $bc = a$, then we have the above theorem. ■

W_5 is the wheel with 5 spokes, pictured in Figure 2.2. Let W_5^n denote the graph formed by $W_5LexW_5Lex...LexW_5$ where Lex appears $n - 1$ times. By use of the above theorem and propositions, we have the following corollaries:

Corollary 3.1.9. $\chi_f(W_5^n) = \chi_f(W_5)^n = (\frac{7}{2})^n$.

Corollary 3.1.10. *(i)* $w(1, W_5^n) = \omega(W_5)^n = 3^n$*;*

(ii) $w(2^n, W_5^n) = n(W_5^n) = n(W_5)^n = 6^n$*;*

(iii) $w(a, W_5^n) \geq w(b, W_5^x)w(c, W_5^y)$*, whenever* $n, a, b, c, x,$ *and* y *are positive integers satisfying* $1 \leq b \leq 2^x, 1 \leq c \leq 2^y$*,* $bc = a$*, and* $x + y = n$*.*

Proposition 3.1.11. $w(a, W_5^n) \leq 6^n - 5^n + \lfloor a(\frac{5}{2})^n \rfloor$.

Proof. Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, with 0 being the vertex of degree five, and $1, ..., 5$ being the vertices, in natural order, on the 5-cycle. Notice that there are 5 distinct independent sets of vertices on the 5-cycle of cardinality 2: {1,3}, {1,4}, {2,4}, {2,5}, {3,5}. It should then be obvious that there are 5^n distinct maximum independent sets of vertices in W_5^n and, by Corollary 3.1.2, each set has cardinality 2^n. If $V = \{0, ..., 5\}^n$ is the vertex set of W_5^n, then observe that each $v \in V$ is in exactly 2^n of the 5^n maximum independent sets in W_5^n.

Thus, to obtain a graph $X \subseteq W_5^n$ with $n(X) = w(a.W_5^n)$ and $\alpha(X) = a$, we must remove at least $(\alpha(W_5^n) - a) = (2^n - a)$ vertices from each of the 5^n sets. And since each of these vertices are in 2^n of those sets, we get the following result:
$w(a, W_5^n) \leq 6^n - \frac{(2^n - a)5^n}{2^n} = 6^n - 5^n + a(\frac{5}{2})^n$. ■

Lemma 3.1.12. $w(2, W_5^n) \geq \frac{1}{2}(5 * 3^n - 3)$.

Proof. Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, with 0 being the vertex of degree five, and $1, ..., 5$ being the vertices, in natural order, on the 5-cycle. Consider the case when $n = 1$. Because $2 = \alpha(W_5)$, we know that $w(2, W_5) = n(W_5) = 6$. Now, let us assume that $n > 1$ and $X \subseteq W_5^n$ such that $\alpha(X) = 2$ and $n(X) = w(2, W_5^n)$. Then $S = (V(X) \times \{0, 1, 2\}) \cup \{(0, ..., 0, 3), (0, ..., 0, 4), (0, ..., 0, 5)\}$ is a set of vertices in W_5^{n+1} which induces a subgraph with vertex independence number 2. As such, we know that $n(S) = 3n(X) + 3 = 3w(2, W_5^n) + 3 \leq$

$w(2, W_5^{n+1})$.

Let $b_n = 3b_{n-1} + 3$ for $n > 1$ and $b_1 = 6$. By standard difference equation techniques, we obtain $b_n = \frac{5}{2} * 3^n - \frac{3}{2}$. Therefore, $w(2, W_5^n) \geq \frac{5}{2} * 3^n - \frac{3}{2} = \frac{1}{2}(5 * 3^n - 3)$. ∎

Theorem 3.1.13. $w(2, W_5^n) = \frac{1}{2}(5 * 3^n - 3)$

Proof. Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, with 0 being the vertex of degree five, and $1, ..., 5$ being the vertices, in natural order, on the 5-cycle. Let $\{0, ..., 5\}^n$ be the vertex set of W_5^n. We proceed by induction on n. The theorem claim holds for $n = 1$. Suppose that $n > 1$ and let X be an induced subgraph of W_5^n such that $\alpha(X) = 2$ and $n(X) = w(2, W_5^n)$. We have $n(X) = \sum_{i=0}^{5} |V_i|$ where $V_i = \{v \in \{0, ..., 5\}^{n-1} \mid iv \in V(X)\}$. Let X_i be the subgraph of W_5^{n-1} induced by V_i, $i = 0, ..., 5$. By Lemma 3.1.1, we have $2 = \alpha(X) = max[\alpha(X_0),$ $\alpha(X_1) + \alpha(X_3)$, $\alpha(X_1) + \alpha(X_4)$, $\alpha(X_2) + \alpha(X_4)$, $\alpha(X_2) + \alpha(X_5)$, $\alpha(X_3) + \alpha(X_5)]$.

We may as well assume $\alpha(X_0) = 2$ and $n(V_0) = w(2, W_5^{n-1})$. (If $\alpha(X_0) < 2$, then we can replace X_0 by a larger subgraph of W_5^{n-1} with independence number 2 and thus enlarge X without increasing $\alpha(X) = 2$.) If, say, $\alpha(X_1) = 2$, then $\alpha(X_3) = \alpha(X_4) = 0$. i.e., $V_3, V_4 = \emptyset$. So there are essentially only the following possibilities for $X_1, ..., X_5$:

(i) $\alpha(X_i) = 1$, $i = 1, ..., 5$

Then we may as well have $n(V_i) = w(1, W_5^{n-1}) = \omega(W_5^{n-1}) = 3^{n-1}, i = 1, ..., 5$. With $n(V_0)$ $= w(2, W_5^{n-1}) = \frac{1}{2}(5 * 3^{n-1} - 3)$ by the induction hypothesis, this gives
$n(X)$=5*$3^{n-1} + w(2, W_5^{n-1}) = \frac{1}{2}(5 * 3^n - 3)$.

(ii) $\alpha(X_1) = 2$, $\alpha(X_3) = \alpha(X_4) = 0$, $\alpha(X_2) = \alpha(X_5) = 1$.

This gives $n(X) = 2 * 3^{n-1} + 2w(2, W_5^{n-1}) = 7 * 3^{n-1} - 3$

(iii) $\alpha(X_1) = \alpha(X_2) = 2$, $\alpha(X_i) = 0$, $i = 3, 4, 5$

This gives $n(X) = 3w(2, W_5^{n-1}) = \frac{3}{2}(5 * 3^{n-1} - 3)$.

From here, it is easy to see that the maximum of the possible values of $n(X)$ in these three cases occurs in case (i), and by induction on n we find that $w(2, W_5^n) = \frac{1}{2}(5 * 3^n - 3)$. ■

Notice that the proof of Theorem 3.1.13 does not use Lemma 3.1.12. The value of Lemma 3.1.12 is that its proof can be used to construct a particular subgraph X of W_5^n with $\alpha(X) = 2$ and $n(X) = \frac{1}{2}(5 * 3^n - 3)$.

Using the result of Theorem 3.1.11 and arguments parallel to those above, we produce the following results; the beginnings of a recursive approach to finding $w(a, W_5^n)$ for any given a and n.

Theorem 3.1.14. *$w(3, W_5^n) = 4 * 3^n - 3(n+1)$ for $n \geq 2$.*

Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, with 0 being the vertex of degree five, and $1, ..., 5$ being the vertices, in natural order, on the 5-cycle. Let $\{0, ..., 5\}^n$ be the vertex set of W_5^n. Suppose that $n > 2$ and let X be an induced subgraph of W_5^n such that $\alpha(X) = 3$ and $n(X) = w(3, W_5^n)$. Now, $n(X) = \sum_{i=0}^{5} |V_i|$ where $V_i = \{v \in \{0, ..., 5\}^{n-1} \mid iv \in V(X)\}$. Let X_i be the subgraph of W_5^{n-1} induced by V_i, $i = 0, ..., 5$. By Lemma 3.1.1, we have $3 = \alpha(X) = max[\alpha(X_0),\ \alpha(X_1) + \alpha(X_3),\ \alpha(X_1) + \alpha(X_4),\ \alpha(X_2) + \alpha(X_4),\ \alpha(X_2) + \alpha(X_5),\ \alpha(X_3) + \alpha(X_5)]$. We will proceed by induction on $n \geq 2$. When $n = 2$, we may as well suppose that $\alpha(X_0) = 2 = \alpha(X_1) = \alpha(X_2)$, $\alpha(X_3) = \alpha(X_4) = \alpha(X_5) = 1$, $n(X_i) = 6$ for $i = 0, 1, 2$, and $n(X_i) = 3$ for $i = 3, 4, 5$. Then $w(3, W_5^2) = 3*6+3*3 = 27 = 4*3^2-3(2+1)$. Now suppose that $n \geq 3$.

We may as well assume $\alpha(X_0) = 3$ and $n(V_0) = w(3, W_5^{n-1})$. (If $\alpha(X_0) < 3$, then we can replace X_0 by a larger subgraph of W_5^{n-1} with independence number 3 and thus enlarge X without increasing $\alpha(X) = 3$.) If, say, $\alpha(X_1) = 3$, then $\alpha(X_3) = \alpha(X_4) = 0$. i.e., $V_3, V_4 = \emptyset$. So there are essentially only the following possibilities for $X_1, ..., X_5$:

(i) $\alpha(X_i) = 1$, $i = 1, 2, 3$ and $\alpha(X_i) = 2$, $i = 4, 5$.

We may as well have $n(V_i) = w(1, W_5^{n-1}) = \omega(W_5^{n-1}) = 3^{n-1}$, $i = 1, 2, 3$ and $n(V_i) = w(2, W_5^{n-1}) = \frac{1}{2}(5 * 3^{n-1} - 3)$, $i = 4, 5$. With $n(V_0) = w(3, W_5^{n-1}) = 4 * 3^{n-1} - 3n$, this gives $n(X) = w(3, W_5^{n-1}) + 3^n + (5*3^{n-1} - 3) = 4*3^{n-1} - 3n + 3^n + 5*3^{n-1} - 3 = 4*3^n - 3(n+1)$.

(ii) $\alpha(X_1) = \alpha(X_2) = 3$, $\alpha(X_3) = \alpha(X_4) = \alpha(X_5) = 0$.

This gives $n(X) = 3w(3, W_5^{n-1}) = 12 * 3^{n-1} - 9n = 4 * 3^n - 9n$

(iii) $\alpha(X_1) = 3$, $\alpha(X_2) = 2$, $\alpha(X_3) = \alpha(X_4) = 0$, $\alpha(X_5) = 1$

This gives $n(X) = 8 * 3^{n-1} - 6n + \frac{1}{2}(5 * 3^{n-1} - 3) + 3^{n-1} = \frac{23}{6}3^n - 3(2n + \frac{1}{2})$.

From here, it is easy to see that the maximum of the possible values of $n(X)$ in these three cases occurs in case (i), and by induction on n we find that $w(3, W_5^n) = 4*3^n - 3(n+1)$.

■

Theorem 3.1.15. $w(4, W_5^n) = \frac{1}{4}(25 * 3^n - 30n - 21)$ *for* $n \geq 2$.

Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, with 0 being the vertex of degree five, and $1, ..., 5$ being the vertices, in natural order, on the 5-cycle. Let $\{0, ..., 5\}^n$ be the vertex set of W_5^n.

Suppose that $n \geq 2$ and let X be an induced subgraph of W_5^n such that $\alpha(X) = 4$ and $n(X) = w(4, W_5^n)$. Now, $n(X) = \sum_{i=0}^{5} |V_i|$ where $V_i = \{v \in \{0, ..., 5\}^{n-1} \mid iv \in V(X)\}$. Let X_i be the subgraph of W_5^{n-1} induced by V_i, $i = 0, ..., 5$. By Lemma 3.1.1, we have $4 = \alpha(X) = max[\alpha(X_0), \alpha(X_1) + \alpha(X_3), \alpha(X_1) + \alpha(X_4), \alpha(X_2) + \alpha(X_4), \alpha(X_2) + \alpha(X_5), \alpha(X_3) + \alpha(X_5)]$.

We proceed by induction on $n \geq 2$. Since $4 = \alpha(W_5^2)$,
$w(4, W_5^2) = 36 = \frac{1}{4}(25 * 3^2 - 30 * 2 - 21)$.

Now suppose that $n > 2$. We may as well assume $\alpha(X_0) = 4$ and $n(V_0) = w(4, W_5^{n-1})$. (If $\alpha(X_0) < 4$, then we can replace X_0 by a larger subgraph of W_5^{n-1} with independence number 4 and thus enlarge X without increasing $\alpha(X) = 4$.) If, say, $\alpha(X_1) = 4$, then $\alpha(X_3) = \alpha(X_4) = 0$. i.e., $V_3, V_4 = \emptyset$. So there are essentially only the following possibilities for $X_1, ..., X_5$:

(i) $\alpha(X_i) = 2$, $i = 1, 2, 3, 4, 5$.

We may as well have $n(V_i) = w(2, W_5^{n-1}) = \frac{1}{2}(5 * 3^{n-1} - 3)$, $i = 1, 2, 3, 4, 5$. With $n(V_0) = w(4, W_5^{n-1}) = \frac{1}{4}(25 * 3^{n-1} - 30(n-1) - 21)$, this gives $n(X) = w(4, W_5^{n-1}) + 5w(2, W_5^{n-1}) = \frac{1}{4}(25 * 3^{n-1} - 30(n-1) - 21) + \frac{5}{2}(5 * 3^{n-1} - 3) = \frac{1}{4}(25 * 3^n - 30n - 21)$.

(ii) $\alpha(X_1) = \alpha(X_2) = 4$, $\alpha(X_3) = \alpha(X_4) = \alpha(X_5) = 0$.

This gives $n(X) = 3w(4, W_5^{n-1}) = \frac{3}{4}(25 * 3^{n-1} - 30n + 9)$.

(iii) $\alpha(X_1) = \alpha(X_2) = 3$, $\alpha(X_3) = \alpha(X_4) = \alpha(X_5) = 1$.

This gives $n(X) = \frac{1}{4}(25 * 3^{n-1} - 30n + 9) + 8 * 3^{n-1} - 6n + 3^n = \frac{1}{4}(23 * 3^n - 54n + 9)$.

(iv) $\alpha(X_1) = 3$, $\alpha(X_2) = 2$, $\alpha(X_3) = \alpha(X_4) = 0$, $\alpha(X_5) = 1$.

This gives $n(X) = \frac{1}{4}(25 * 3^{n-1} - 30n + 9) + 4 * 3^{n-1} - 3n + \frac{1}{2}(5 * 3^{n-1} - 3) + 3^{n-1} = \frac{1}{4}(55 * 3^{n-1} - 42n + 3)$.

(v) $\alpha(X_1) = 4$, $\alpha(X_2) = 3$, $\alpha(W_3) = \alpha(X_4) = 0$, $\alpha(X_5) = 1$.

This gives $n(X) = \frac{2}{4}(25*3^{n-1} - 30(n-1) - 21) + 4*3^{n-1} - 3n + 3^{n-1} = \frac{1}{2}(35*3^{n-1} - 36n + 9)$.

(vi) $\alpha(X_1) = 4$, $\alpha(X_2) = \alpha(X_5) = 2$, $\alpha(X_3) = \alpha(X_4) = 0$.

This gives $\frac{2}{4}(25 * 3^{n-1} - 30(n-1) - 21) + \frac{2}{2}(5 * 3^{n-1} - 3) = \frac{1}{2}(35 * 3^{n-1} - 30n + 3)$.

From here, it is easy to see that the maximum of the possible values of $n(X)$ in these six cases occurs in case (i), and by induction on n we find that $w(4, W_5^n) = \frac{1}{4}(25*3^n - 30n - 21)$. ■

The proofs for the above three theorems are very similar, but we include them to highlight two observations. The first is that we have essentially developed a recursive approach to find $w(a, W_5^n)$ for any values of a and n, $1 \leq a \leq 2^n$. This means we have a method of finding $\rho(W_5^n)$ for a specified n. The second observation is that in each of the three theorems above, $n(X) = w(a, W_5^n) = w(a, W_5^{n-1}) + 3w(\lfloor \frac{a}{2} \rfloor, W_5^{n-1}) + 2w(\lceil \frac{a}{2} \rceil, W_5^{n-1})$. Finding out whether this is true for all a would be an excellent task for future research.

Proposition 3.1.16. $\rho(W_5^3) = \frac{141}{4}$

Proof. Recall that $\rho(W_5^n) = max[\frac{w(a,W_5^n)}{a}]$, $1 \leq a \leq \alpha(W_5^n) = 2^n$. By Corollary 3.1.10 and Theorems 3.1.13-3.1.15, we know $w(1, W_5^3) = 3^3 = 27$, $w(2, W_5^3) = \frac{1}{2}(5 * 3^3 - 3) = 66$, $w(3, W_5^3) = 4 * 3^3 - 3(3+1) = 96$, and $w(4, W_5^3) = \frac{1}{4}(25 * 3^3 - 30(3) - 21) = 141$. Additionally, by Proposition 3.1.11, we have the following: $w(5, W_5^3) \leq 169$, $w(6, W_5^3) \leq 184$, $w(7, W_5^3) \leq 200$, and $w(8, W_5^3) = 216$.

We thus have $\rho(W_5^3) \leq max[\frac{27}{1}, \frac{66}{2}, \frac{96}{3}, \frac{141}{4}, \frac{169}{5}, \frac{184}{6}, \frac{200}{7}, \frac{216}{8}] = \frac{141}{4}$. Since this maximum is achieved for $a = 4$, we actually have equality by Theorem 3.1.15. ■

The significance of Proposition 3.1.16 is that we now know $\frac{\chi_f(W_5^3)}{\rho(W_5^3)} = \frac{(\frac{7}{2})^3}{\frac{141}{4}} = \frac{343}{282} > \frac{6}{5}$ which was the previously known record high for this ratio. A subgraph X of W_5^3 such that $\frac{n(X)}{\alpha(X)} = \frac{141}{4} = \rho(W_5^3)$ is the graph induced by the vertices $(\{1,2,3,4,5\} \times \{1,2,3,4,5\} \times \{2,3\}) \cup \{v = (v_1, v_2, v_3) \in \{0, ..., 5\}^3 \mid v_i = 0, \text{ for some } i \in \{1,2,3\}\}$.

3.2 Disjunctive Products

We now shift our focus to the disjunctive product of graphs. Although it is entirely possible for the ratio, $\frac{\chi_f}{\rho}$, to be unbounded on a class of graphs defined by the disjunctive product, do note that any graph obtained by the disjunctive product will potentially contain more edges than the corresponding graph formed by the lexicographic product. This means that it is likely the case that $\frac{\chi_f}{\rho}$ will be smaller for a graph obtained by the disjunctive product of two graphs than it will be for the graph obtained by the lexicographic product of the same two graphs. We thus only consider this product because it may be the case that finding the value of certain graph parameters may prove easier under the disjunctive product than the lexicographic product.

Theorem 3.2.1. [4] $\chi_f(GDisjH) = \chi_f(G)\chi_f(H)$.

Let $(W_5^n)_D = (W_5 Disj W_5 Disj ... Disj W_5)$ where the product is taken $(n-1)$ times.

Corollary 3.2.2. $\chi_f((W_5^n)_D) = (\frac{7}{2})^n$.

Proof. We know that $\chi_f(W_5) = \frac{7}{2}$. Using Theorem 3.2.1, we find that $\chi_f((W_5^n)_D) = \chi_f(W_5)\chi_f((W_5^{n-1})_D)$. Repeated application of the theorem gives $\chi_f((W_5^n)_D) = (\chi_f(W_5))^n = (\frac{7}{2})^n$. ■

Theorem 3.2.3. $w(a, GLexH) \leq w(a, GDisjH)$.

Proof. Let X be an induced subgraph of $(GLexH)$ such that $\alpha_{GLexH}(X) = a$, $w(a, GLexH) = n(X)$. Let Y be the subgraph of $GDisjH$ induced by $V(X)$. Then $\alpha_{GDisjH}(Y) = b \leq a$. So $w(a, GLexH) = n(X) = n(Y) \leq w(b, GDisjH) \leq w(a, GDisjH)$. ■

Proposition 3.2.4. $\omega((W_5^2)_D) \geq 10 > \omega(W_5)^2$.

Proof. Let $V(W_5) = \{0, 1, 2, 3, 4, 5\}$, as before. Consider the subgraph of $W_5 Disj W_5$ induced by the following set of vertices:

$S = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0), (2,4), (3,2), (4,5), (5,3)\}$. Note that $|S| = 10$ and that the subgraph is a complete graph. Thus $\omega(W_5)^2 = 3^2 = 9 < 10 \leq \omega((W_5^2)_D)$. ■

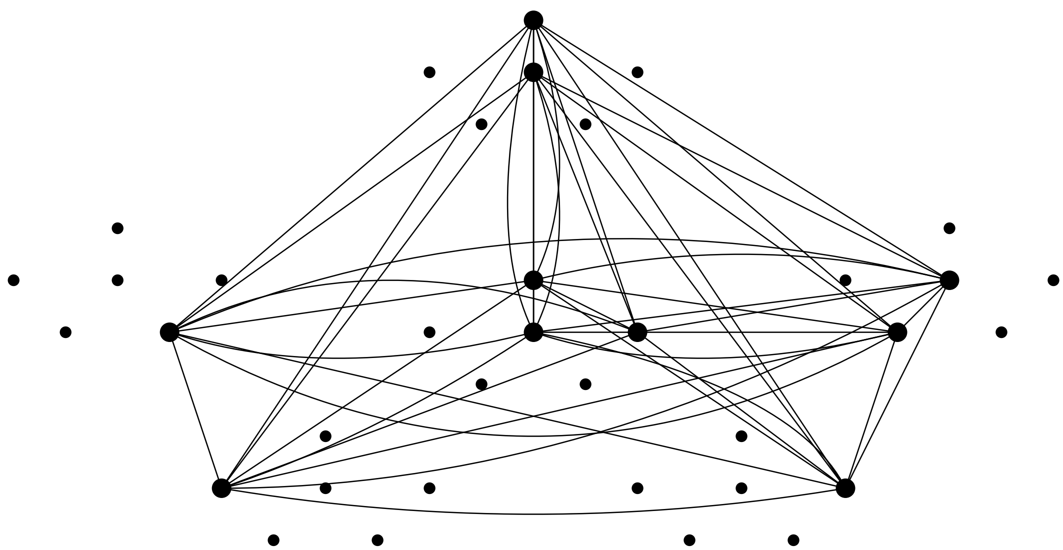

Figure 3.1: A clique of order 10 in $W_5^{2'}$

Theorem 3.2.5. *For all* $a \in \{1, ..., 2^n\}$, $w(a, W_5^n) \leq w(a, (W_5^n)_D)$

$$\leq \sum_{k=0}^{\lceil log_2(a) \rceil - 1} \binom{n}{k} 5^k + a \sum_{k=\lceil log_2(a) \rceil}^{n} \binom{n}{k} (\frac{5}{2})^k.$$

Proof. Let $V = \{0, ..., 5\}^n$ be the vertex set for both W_5^n and $(W_5^n)_D$. For $S \subseteq \{1, ..., n\}$, let $V_0(S) = \{w \in V \mid w_t = 0 \iff t \in S\}$. Note that S_1, $S_2 \subseteq \{1, ..., n\}$, $S_1 \neq S_2 \implies V_0(S_1) \cap V_0(S_2) = \emptyset$. Also, $|V_0(S)| = 5^{n-|S|}$.

Suppose $k \in \{0, ..., n\}$ and let $S \subseteq \{1, ..., n\}$, $|S| = n - k$. The subgraph $G(S)$ induced in $(W_5^n)_D$ by $V_0(S)$ is isomorphic to $(C_5^k)_D$ and has vertex independence number 2^k.

By arguments given in the proof of Proposition 3.1.11, there are 5^k different independent sets $I \subseteq V_0(S)$ such that $|I| = 2^k$, and every $w \in V_0(S)$ is in 2^k of those I. Let $\mathcal{I}(S) = \{I \subseteq V_0(S) \mid I$ is an independent set of vertices in $G(S)$ and $|I| = 2^k\}$.

Suppose $1 \leq a \leq 2^k$, which holds if and only if $k \geq \lceil log_2(a) \rceil$. Let Q be a smallest set of words in $V_0(S)$ such that $|(V_0(S)\backslash Q) \cap I| = |I \backslash Q| = |I \backslash (I \cap Q)| = |I| - |I \cap Q| = 2^k - |I \cap Q| \leq a$ for all $I \in \mathcal{I}(S)$. Let M be the number of ordered pairs (u, I), $u \in I \cap Q$, $I \in \mathcal{I}(S)$. Let $q_k = |Q|$. Because each $u \in Q$ is in 2^k different $I \in \mathcal{I}$, it follows that $M = q_k * 2^k$. On the other hand, since $|I \cap Q| \geq 2^k - a$ for each $I \in \mathcal{I}(S)$, $q_k * 2^k = M \geq (2^k - a)|\mathcal{I}(S)| = (2^k - a)5^k \Longrightarrow q_k \geq (1 - \frac{a}{2^k})5^k$.

Suppose X is a subgraph of $(W_5^n)_D$ such that $\alpha(X) = a$ and $n(X) = w(a, (W_5^n)_D)$. For each $k \in \{\lceil log_2(a) \rceil, ..., n\}$ and each $S \subseteq \{1, ..., n\}$, $|S| = n - k$, $V(X)$ can contain no more than a elements of I for each $I \in \mathcal{I}(S)$. Therefore, $|V_0(S)\backslash(V(X) \cap V_0(S))| = |V_0(S)| - |V(X) \cap V_0(S)| = 5^k - |V(X) \cap V_0(S)| \geq q_k \Longrightarrow |V(X) \cap V_0(S)| \leq 5^k - q_k \leq 5^k - (1 - \frac{a}{2^k})5^k = \frac{a}{2^k}5^k$.

As k ranges from 0 to n and, for each k, S ranges over the $\binom{n}{k}$ $(n-k)$-subsets of $\{1, .., n\}$, the sets $V(X) \cap V_0(S)$ range over pairwise disjoint subsets of $V(X)$ which cover all of $V(X)$. Therefore, $|V(X)| = w(a, (W_5^n)_D) = \sum_{k=0}^{n} \sum_{\substack{S \subseteq \{1,...,n\} \\ |S| = n-k}} |V(X) \cap V_0(S)| \leq$

$$\sum_{k=0}^{\lceil log_2(a) \rceil - 1} \binom{n}{k} 5^k + a \sum_{k=\lceil log_2(a) \rceil}^{n} \binom{n}{k} (\frac{5}{2})^k. \quad \blacksquare$$

Chapter 4

The Mycielski Graphs

In this chapter, we discuss another class of graphs for which we believe the ratio, $\frac{\chi_f}{\rho}$, may be unbounded. We bgein, however, with a well-known construction and some results pertaining to the construction.

4.1 The Mycielskian Construction

Definition 4.1.1. *Let* $V(G) = \{x_1, x_2, ..., x_n\}$*. We define the Mycielskian of* G*, denoted* $M(G)$*, to be the graph whose vertex set is* $V(M(G)) = \{x_1, ..., x_n, y_1, ..., y_n, z\}$ *and whose adjacency is as follows:*

(i) x_i *is adjacent to* x_j *if and only if* x_i *is adjacent to* x_j *in* G

(ii) x_i *is adjacent to* y_j *if and only if* x_i *is adjacent to* x_j *in* G

(iii) y_i *is adjacent to* z *for all* $i \in \{1, 2, ..., n\}$.

At this point, we typically denote $V(M(G)) = X \cup Y \cup \{z\}$ where $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$.

Example 4.1.2.

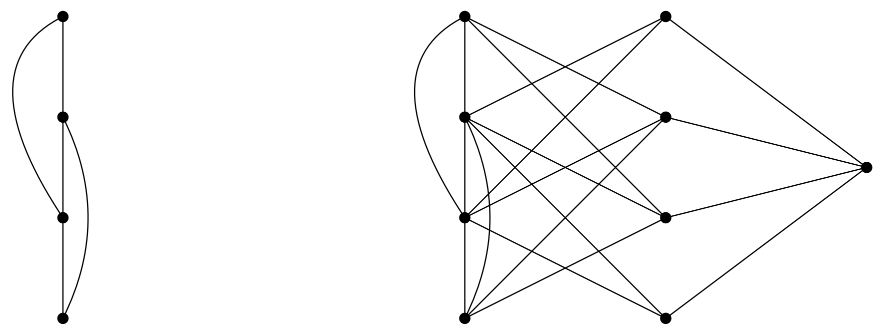

Figure 4.1: A graph and its Mycielskian

Notice that if G has n vertices, then $M(G)$ has $2n+1$ vertices. Additionally, the set of vertices in Y is an independent set. This means that $\alpha(M(G)) \geq n$. Because the independence number is at least roughly half the total number of vertices in any given $M(G)$, it seems a possibility that a class of graphs built around this Mycielskian construction could result in a scenario in which $\frac{\chi_f}{\rho}$ tends to infinity. The following are some important and well-known results concerning this construction.

Proposition 4.1.3. *If G has at least one edge, then $\omega(M(G)) = \omega(G)$.*

Proof. First, we note that G is always isomorphic to a subgraph of $M(G)$. Thus we know immediately that any clique of G will be isomorphic to a clique in $M(G)$. So we have $\omega(M(G)) \geq \omega(G)$. To prove $\omega(M(G)) \leq \omega(G)$, we consider cliques in $M(G)$ in the following cases:

(i) Assume the given clique contains the vertex z.

Because z is only adjacent to y_i, $i = \{1, ..., n\}$ and because none of the vertices in Y are adjacent to each other, we get that the size of such a clique is at most 2.

(ii) Assume given clique contains the vertex y_i and not z.

We already know that such a clique cannot contain 2 vertices from Y. Thus every other vertex in the clique must belong to X. However, those vertices in X that are adjacent to y_i are exactly the same vertices that are adjacent to x_i. Thus the order of a clique containing y_i and not z is equal to the order of a clique entirely contained in X and containing x_i. Since the graph induced by the vertices of X is isomorphic to G, we then have the order of any clique containing y_i is at most the clique number of G.

Together, these two cases give us $\omega(M(G)) \leq \omega(G)$ since $\omega(G)$ is at least 2 due to G containing an edge. Combining the inequalities above gives us the result of the proposition.

■

One of the more significant results concerning this construction is given in the following well-known theorem:

Theorem 4.1.4. $\chi(M(G)) = \chi(G) + 1$.

Proof. Suppose $\chi(G) = k$ and define $f : V(G) \to \{1, ..., k\}$ to be a proper k coloring of G. We can define a proper $k+1$ coloring of $M(G)$, $g : V(M(G)) \to \{1, ..., k, k+1\}$, as follows: set $g(x_i) = g(y_i) = f(x_i)$ for each $i \in \{1, ..., n\}$ and set $g(z) = k + 1$. By construction, we see that this is a proper $k + 1$ coloring of $M(G)$. Thus we have proven $\chi(M(G)) \leq k + 1$.

Now consider a proper coloring, g, of $M(G)$. We define a coloring, f, on G as follows: set $f(x_i) = g(x_i)$ if $g(x_i) \neq g(z)$ and $f(x_i) = g(y_i)$ if $g(x_i) = g(z)$. f thus colors G properly without the use of the color $g(z)$. We have then proven that $\chi(M(G)) - 1 \geq \chi(G) = k$ or equivalently $\chi(M(G)) \geq k + 1$.

Combining these two results gives $\chi(M(G)) = \chi(G) + 1$ as desired. ■

Theorem 4.1.5. [4] $\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$.

4.2 The Mycielski Graphs

Applying the Mycielskian construction repeatedly, starting with the graph of a single edge, gives rise to a sequence of graphs known as the Mycielski graphs. By convention, we typically denote the n^{th} graph in this sequence as M_{n+1}. The reason for this potentially confusing labeling is due the result of Theorem 4.1.4 above. The first graph in the sequence, M_2, is a single edge with 2 vertices and $\chi(M_2) = 2$. By Theorem 4.1.4, we then have $\chi(M_n) = n$. Thus the subscript simply refers to the Mycielski graph with a specified chromatic number.

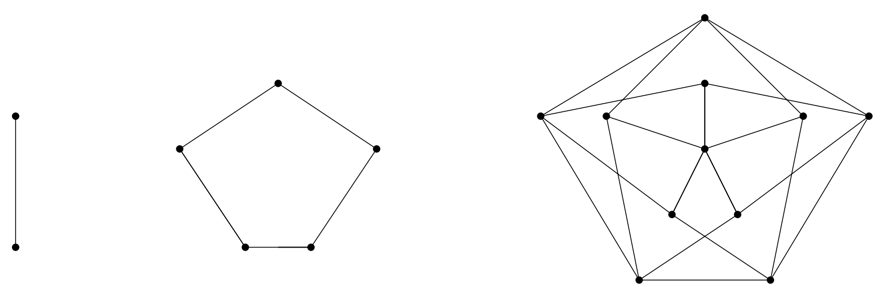

Figure 4.2: M_2, M_3, and M_4

As with the lexicographic and disjunctive powers of W_5, another benefit of working with the Mycielski graphs is that we already know how the fractional chromatic number behaves for this class of graphs. In addition to Theorem 4.1.5, we have the following result:

Theorem 4.2.1. *$\chi_f(M_n)$ is asymptotically equivalent to $\sqrt{2n}$.*

Proof. For $n \geq 3$, let $X_n = X_{n-1} + \frac{1}{X_{n-1}}$ with $X_2 = 2$. Let $U_n = X_n^2 = (X_{n-1} + \frac{1}{X_{n-1}})^2 = 2 + U_{n-1} + \frac{1}{U_{n-1}}$. Then we claim that, for all $n \geq 2$, $U_n \geq 2 + U_{n-1} \geq 2 + 2(n-1)$. The first inequality follows from before. To show the second, first note that $U_2 = (X_2)^2 = 2^2 = 4 \geq 2 + 2(2-1) = 4$. Thus the inequality holds for $n = 2$. We now proceed by induction on n. Assume $U_k \geq 2k$, $2 \leq k \leq n-1$. We then have that $U_{k+1} = 2+U_k+\frac{1}{U_k} \geq 2+2k+\frac{1}{U_k} \geq 2(k+1)$. So $n \geq 2 \implies U_n \geq 2n$ (i).

So, for $n > 2$,

$$
\begin{aligned}
U_n &= 2 + U_{n-1} + \tfrac{1}{U_{n-1}} \\
&\leq 2 + U_{n-1} + \tfrac{1}{2n-2} \\
&\leq 4 + U_{n-2} + \tfrac{1}{2(n-2)} + \tfrac{1}{2(n-1)} \\
&. \\
&. \\
&. \\
&\leq 2(n-2) + U_2 + \tfrac{1}{2}[\tfrac{1}{2} + ... + \tfrac{1}{n-1}] \\
&= 2n + \tfrac{1}{2}[\tfrac{1}{2} + ... + \tfrac{1}{n-1}] \\
&\leq 2n + c * ln(n) \qquad (ii)
\end{aligned}
$$

for some constant c. Combining (i) and (ii) gives

$$
\begin{aligned}
2n \leq X_n^2 \leq 2n + c * ln(n) \Longrightarrow \\
\sqrt{2n} \leq X_n \leq \sqrt{2n + c * ln(n)} \Longrightarrow \\
1 \leq \tfrac{X_n}{\sqrt{2n}} \leq \sqrt{1 + \tfrac{ln(n)}{2n}} \longrightarrow 1 \text{ as } n \longrightarrow \infty \quad \blacksquare
\end{aligned}
$$

Proposition 4.2.2. *For $k \geq 2$, $n(M_k) = 3 * 2^{k-2} - 1$.*

Proof. By the construction, we know that $|V(M_k)| = 2|V(M_{k-1})| + 1$ for $k > 2$. Let $a_k = 2a_{k-1} + 1$, $k > 2$ and $a_2 = 2$. By standard difference equation techniques, we obtain $a_k = \frac{3}{4} * 2^k - 1 = 3 * 2^{k-2} - 1$. Therefore, $n(M_k) = \frac{3}{4} * 2^k - 1 = 3 * 2^{k-2} - 1$. $\blacksquare$

For $S \subseteq V(M_n)$, let $N_k(S)$ be the open neighbor set of S in M_k and $N_k[S]$ be the closed neighbor set of S in M_k. For $k > 2$, define the function, $Y_k : V(M_{k-1}) \longrightarrow V(M_k)$, and let z_k be the added vertex such that $N_k(z_k) = Y_k(V(M_{k-1}))$.

Lemma 4.2.3. *For all $k \geq 2$ and all $S \subseteq V(M_k)$, $|S| \leq |N_k(S)|$.*

Proof. The proof is by induction on k. The case in which $k = 2$ is straightforward. Now suppose that $k > 2$. Suppose that $S \subseteq V(M_k)$.

If $S \cap V(M_{k-1}) = \emptyset$, then the conclusion is easily obtained. First, if $S = \{z_k\}$, then $|S| = 1 < 2 \le |Y_k(V(M_{k-1}))| = |N_k(z)| = n(M_{k-1})$. Otherwise, if $S \cap V(M_{k-1}) = \emptyset$ but $S \cap Y_k(V(M_{k-1})) = Y_k(U) \neq \emptyset$ for some $U \subseteq V(M_{k-1})$, then $N_k(S) = \{z_k\} \cup N_{k-1}(U)$. So $|N_k(S)| = 1 + |N_{k-1}(U)| \ge 1 + |U| \ge |S|$.

Now suppose that $T = S \cap V(M_{k-1}) \neq \emptyset$. Let $T_1 = \{v \in T \mid Y_k(v) \in S\}$, $T_2 = \{v \in T \mid Y_k(v) \notin S\}$, and $U = \{v \in V(M_{k-1}) \mid v \notin S \text{ and } Y_k(v) \in S\}$. Note that T_1, T_2, and U are pairwise disjoint subsets of $V(M_{k-1})$. We have $S = T_1 \cup Y_k(T_1) \cup T_2 \cup Y_k(U) \cup R$ where

$$R = \begin{cases} \{z_k\} \text{ if } z_k \in S \\ \emptyset \qquad \text{otherwise} \end{cases}.$$

Therefore, $|S| = 2|T_1| + |T_2| + |U| + |R|$. Also, $N_k(S) = N_{k-1}(T_1) \cup Y_k(N_{k-1}(T_1)) \cup N_{k-1}(T_2) \cup Y_k(N_{k-1}(T_2)) \cup N_{k-1}(U) \cup N_k(R) \cup Q$, where $Q = \begin{cases} \{\mathrm{z}_k\} \text{ if } T_1 \cup U \neq \emptyset \\ \emptyset \qquad \text{otherwise} \end{cases}.$

Rewriting the previous, we have $N_k(S) = N_{k-1}(T_1 \cup T_2 \cup U) \cup Y_k(N_{k-1}(T_1 \cup T_2)) \cup N_k(R) \cup Q$. By the induction hypothesis,

$$\begin{aligned}
|N_k(S)| &\ge |N_{k-1}(T_1 \cup T_2 \cup U) \cup Y_k(N_{k-1}(T_1 \cup T_2))| + |N_k(R)| \\
&\ge |N_{k-1}(T_1 \cup T_2 \cup U)| + |Y_k(N_{k-1}(T_1 \cup T_2))| \\
&= |N_{k-1}(T_1 \cup T_2 \cup U)| + |N_{k-1}(T_1 \cup T_2)| \\
&\ge |T_1 \cup T_2 \cup U| + |T_1 \cup T_2| \\
&\ge 2(|T_1| + |T_2|) + |U| \\
&\ge 2|T_1| + |T_2| + |U| + |R| \\
&= |S|
\end{aligned}$$

unless $T_2 = \emptyset$ and $R = \{z_k\}$. But if $T_2 = \emptyset$, then $T_1 \neq \emptyset$ (because $S \cap V(M_{k-1}) \neq \emptyset$). And if $R = \{z_k\}$, then $N_k(R) = N_k(z_k) = Y_k(V(M_{k-1}))$. So

$$\begin{aligned}
|N_k(S)| &= |N_{k-1}(T_1 \cup U)| \cup |Y_k(V(M_{k-1}))| + |Q| \\
&\ge |T_1| + |U| + |V(M_{k-1})| + 1 \\
&\ge 2|T_1| + 2|U| + 1
\end{aligned}$$

$$\geq 2|T_1| + |U| + 1$$
$$= |S|. \qquad \blacksquare$$

Theorem 4.2.4. *$\alpha(M_k) = \frac{n(M_k)-1}{2} = n(M_{k-1})$ for $k \geq 3$.*

Proof. Obviously, $\alpha(M_k) \geq |Y_k(V(M_{k-1}))| = n(M_{k-1})$. Now suppose I is a maximum independent set of vertices in M_k. We want to show that $|I| \leq n(M_{k-1})$. If $I \cap V(M_{k-1}) = \emptyset$, we are done. So suppose that $\emptyset \neq I \cap V(M_{k-1}) = S$. Let $U = V(M_{k-1}) \backslash N_{k-1}[S]$. Then $V(M_{k-1})$ is the disjoint union (since S is independent) of S, $N_{k-1}(S)$, and U. Clearly $I \cap Y_k(V(M_{k-1})) \subseteq Y_k(S) \cup Y_k(U)$. And if $z_k \notin I$, then

$$I = S \cup Y_k(S) \cup Y_k(U)$$
$$\implies |I| = 2|S| + |U|$$
$$= |S| + (n(M_{k-1}) - |N_{k-1}(S)|)$$
$$\leq n(M_{k-1})$$

by Lemma 4.2.3. If $z_k \in I$, then $|I| = 1 + |S| \leq 1 + \alpha(M_{k-1})$ and the last is clearly no greater than $n(M_{k-1})$. $\blacksquare$

Although the Mycielski graphs are technically the first class of graphs we considered, not much attention has been given to the graphs in recent months due to the difficulty in determining the Hall ratio of M_n. Future research will include a refocusing on these graphs. Specifically, we need to try and observe how the w-function behaves on M_n since this function was not yet defined when we considered this class of graphs. However, by theorem 4.2.4, we know $w(a, M_n)$ to be defined for $1 \leq a \leq n(M_{n-1}) = 3 * 2^{n-3} - 1$. We also get the following quick results from Proposition 2.1.2 and Proposition 2.1.3:

Corollary 4.2.5. *$w(1, M_n) = 2$ for all n.*

Corollary 4.2.6. *$w(\alpha(M_n), M_n) = w(3 * 2^{n-3} - 1, M_n) = 3 * 2^{n-2} - 1$.*

Bibliography

[1] A. Daneshgar, A.J.W. Hilton, and P.D. Johnson Jr., Relations among the fractional chromatic, choice, Hall, and Hall-condition numbers of simple graphs, *Discrete Math.* 241 (2001), 189-199.

[2] P.D. Johnson Jr., The Hall-condition number of a graph, *Ars Combinatoria* 3 (1994), 183-190.

[3] P.D. Johnson Jr., The fractional chromatic number, the Hall ratio, and the lexicographic product, *Discrete Math* 309 (2009), 4746-4749.

[4] E.R. Scheinerman and D.H. Ullman, *Fractional Graph Theory*, Wiley Interscience, John Wiley and Sons, 1997.

[5] D.B. West, *Introduction to Graph Theory*, 2^{nd} edition, Prentice-Hall, 2001.