Euclidean Szlam Numbers

by

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Abstract

Suppose $X \subseteq \mathbb{R}^n$, for some positive integer n, is closed under vector addition, ρ a translation invariant distance function on X, and $D \subseteq (0, \infty)$. The distance graph $G_\rho(\mathbb{R}^n, D)$ is the graph with vertex set $\mathbb{R}^n$ with $u, v \in \mathbb{R}^n$ adjacent if and only if $\rho(u, v) \in D$. A rather red coloring of G is a coloring of X with red and blue such that no two points adjacent in G are both blue. The Szlam number of G is the minimum cardinality, over all rather red colorings of G, of $F \subseteq X$ such that no translate of F is all red. We exploit results of Johnson, Szlam, and Kloeckner to show that for every positive integer n there exists ρ such that the Szlam number of $G_\rho(\mathbb{R}^2, \{1\})$ is n.

Let $K, X \subseteq \mathbb{R}^n$ for some positive integer n and suppose that $K = -K$ and $(0, \ldots, 0) \notin K$. We will call such a set a *K-set*. Define $G(X, K)$, a *K-graph* on X, to be the graph whose vertex set is X and $x, y \in X$ are adjacent if and only if $x - y \in K$. We show that for every K-set there exists a translation invariant distance function on $\mathbb{R}^n$ such that $G(\mathbb{R}^n, K) = G_\rho(\mathbb{R}^n, \{1\})$ and for every translation invariant distance function ρ on $\mathbb{R}^n$ and $D \subseteq (0, \infty)$ there exists a K-set such that $G(\mathbb{R}^n, K) = G_\rho(\mathbb{R}^n, D)$. For certain K-sets we find the Szlam number of $G_\rho(\mathbb{R}^n, D)$ where $G_\rho(\mathbb{R}, D) = G(\mathbb{R}^n, K)$.

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Chapter 1

Introduction

If we color every point in the real plane either red or blue such that no two points colored blue are unit distance apart in the Euclidean sense, then one can pose the following problem. Find a set of points in the plane that has no translation in which all points are red (an all red set). As a consequence to a result by Johnson and Szlam [10], the cardinality of such a set is an upper bound of the *chromatic number of the plane*, that is, the minimum number of colors needed to color every point in the real plane such that no two points of unit distance are monochromatic; again, distance is Euclidean.

In light of the above result a natural question arises, which is, does there exist a red and blue coloring of the plane as described above and a set in the plane with no translation that is all red whose cardinality is the same as the chromatic number of the plane? It is well known that the chromatic number of the plane is either 4, 5, 6, or 7. We define the *Szlam number of the plane* to be the minimum size of a set such that no translation of the set is all red for some red and blue coloring of the plane with no two blue points unit distance apart. If one could show that the Szlam number of the plane is less than 7, thus lowering the upper bound on the chromatic number of the plane, it would be an extraordinary result, giving light to the chromatic number of the plane problem which has eluded many for over 60 years. Of course, the value of the Szlam number of the plane is unknown and may be as elusive to find as the chromatic number of the plane. The following work is motivated by a desire to better understand this relationship between the Szlam and chromatic numbers of the plane. In particular we ask the above question for different notions of distance that are translation invariant.

In Chapter 1 we introduce basic definitions needed in understanding the following work as well as a proof of the result by Johnson and Szlam from which it follows that the Szlam number of the plane is an upper bound of the chromatic number of the plane. We conclude by abstracting the notion of distance and proving some elementary results when distance is defined in terms of the Euclidean metric, taxi-cab metric, and the max norm.

In Chapter 2 we characterize the possible values for the Szlam number of the plane problem when the notion of distance is required to be translation invariant and to induce the usual topology on the plane.

In Chapter 3 we introduce the definition of a K-graph on $\mathbb{R}^n$ and find that the class of all K-graphs is equivalent to the class of all distance graphs $G_\rho(\mathbb{R}^n, \{1\})$ on $\mathbb{R}^n$ where ρ is a translation invariant distance function on $\mathbb{R}^n$. We then look at a class of K-graphs on $\mathbb{R}$ for which the chromatic numbers and Szlam numbers are equal and find the Szlam numbers for graphs within this class.

In Chapter 4 we look at a class of K-graphs on $\mathbb{R}^2$ for which the chromatic numbers and Szlam numbers are equal and find Szlam numbers for graphs within this class. In other cases we estimate the Szlam numbers of K-graphs, when K is a convex closed curve symmetric around $(0, 0)$.

1.1 Preliminaries

The following definitions and examples are fundamental in our investigations of the Szlam number. For the reader familiar with graph theory and in particular distance graphs most of what follows will be known, but not all since some definitions and notation are our own. All of our work lies within $\mathbb{R}^n$ for n a positive integer.

1.1.1 Distance Functions

Let $X \subseteq \mathbb{R}^n$ be a set, and $\rho : X \times X \to [0, \infty)$ be a function that satisfies the following, for all $x, y \in X$:

$$i. \quad \rho(x,y) = 0 \text{ if and only if } x = y$$

$$ii. \quad \rho(x,y) = \rho(y,x).$$

Then we say ρ is a *distance function* on X. It is standard to require a third property, $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$ for all $x, y, z \in \mathbb{R}^n$, also know as the *triangle inequality*, when defining a notion of distance. A distance function that satisfies the triangle inequality is called a *metric*. The reason for defining the notion of distance without the triangle inequality is that there are many distance functions that are not metrics; for instance, $\rho(x,y) = (x-y)^2$ on $\mathbb{R}$. However, every distance function defined in the following work is a metric.

Now, a property of a distance function that is not a standard in the notion of distance, but is of great importance for reasons that will become clear when we state Szlam's lemma, is that of translation invariance. Let ρ be a distance function on $X \subseteq \mathbb{R}^n$ for some positive integer n, and suppose that X is closed under vector addition. Then ρ is *translation invariant* on X if $\rho(a+b, a+c) = \rho(b,c)$ for all $a, b, c \in X$, where $+$ denotes vector addition.

There are several classic translation invariant distance functions on $\mathbb{R}^n$. One such class of distance functions is the *p-norm* which is defined on $\mathbb{R}^n$ as follows. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and p a positive real number such that $p \geq 1$. Define $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$\rho_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

We write $|x-y|_p$ and for $\left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$ when expressing ρ_p. The 1-norm $|\cdot|_1$ is called the *taxi-cab metric*, the 2-norm $|\cdot|_2$ is called the *Euclidean metric* and, the infinity-norm $\lim_{p \to \infty} |\cdot|_p = |\cdot|_\infty$ is called the *max norm*: $|x-y|_\infty = \max(|x_1-y_1|, \ldots, |x_n-y_n|)$. Notationally, for the Euclidean metric it is common practice to drop the subscript and write $|\cdot|$.

1.1.2 Distance Graphs

Let ρ be a distance function on $\mathbb{R}^n$ for some positive integer n, $X \subseteq \mathbb{R}^n$ and, $D \subseteq (0, \infty)$. We define the *distance graph*, $G_\rho(X, D)$, on X to be the graph with vertex set X, with $x, y \in X$ adjacent if and only if $\rho(x, y) \in D$. The most well known distance graph on $\mathbb{R}^n$ is where ρ is $|\cdot|$ and $D = \{1\}$. This distance graph on $\mathbb{R}^n$ we denote by $G(\mathbb{R}^n)$. For convenience, if $\rho = |\cdot|$ and $D = \{1\}$ we write $G_\rho(X, D)$ by simply $G(X)$. We note that these notational liberties do not have to be used in conjunction.

1.1.3 Coloring Problems

There are two properties of a distance graph of general interest, both of which involve the notion of coloring a graph. Let G be a graph with vertex set X, and let C be a set. A *coloring* of the graph G is a function $\phi : X \to C$. The elements of C are called *colors* and the set $X_i = \{x \in X : \phi(x) = i\}$ is called the *ith color class* of ϕ. A coloring ϕ of G is a *proper coloring* of G if for every $x, y \in X$ that are adjacent in G it follows that $\phi(x) \neq \phi(y)$. A classic coloring problem and the first property of concern is finding the minimum cardinality of the set C such that there is a proper coloring $\phi : X \to C$ of G, called the *chromatic number* of G and denoted by $\chi(G)$. Notationally, we write the chromatic number of a distance graph $G_\rho(X, D)$ as $\chi_\rho(X, D)$ and, again, we drop the ρ when ρ is $|\cdot|$ and do not write the D when $D = \{1\}$. The problem of finding the *chromatic number of the plane* [13] is then the same as finding the chromatic number $\chi(\mathbb{R}^2)$ of the distance graph $G(\mathbb{R}^2)$.

Let $\phi : X \to \{r, b\}$ be a coloring of G, a graph with vertex set X, with colors r (red) and b (blue), such that no two vertices adjacent in G are both blue. We denote the red and blue color classes as $R = \{x \in X : \phi(x) = r\}$ and $B = \{x \in X : \phi(x) = b\}$ respectively and say G has a *rather red coloring* $\{R, B\}$. Let $G_\rho(X, D)$ be a distance graph with a rather red coloring $\{R, B\}$ where $X \subseteq \mathbb{R}^n$ is closed under vector addition. Then a nonempty set $F \subseteq X$ is said to be *forbidden* by the rather red coloring $\{R, B\}$ of $G_\rho(X, D)$ if no translate of F is all red, that is, $v + F \not\subseteq R$ for all $v \in X$. Equivalently, $(v + F) \cap B \neq \emptyset$ for all $v \in X$.

We can now state the second property of interest. Let $X \subseteq \mathbb{R}^n$ be closed under vector addition and $G_\rho(X, D)$ a distance graph. We define the *Szlam number* of $G_\rho(X, D)$, denoted $Szl_\rho(X, D)$, by $Szl_\rho(X, D) = \min\{|F| : F$ is forbidden by a rather red coloring of $G_\rho(X, D)\}$. Again, we drop the subscript ρ when ρ is $|\cdot|$ and do not write the D when $D = \{1\}$. Hence the *Szlam number of the plane* is written as $Szl(\mathbb{R}^2)$.

1.2 History

The history of the Szlam number of a given distance graph $G_\rho(X, D)$ where $X \subseteq \mathbb{R}^n$, $D \subseteq (0, \infty)$ and ρ is a translation invariant distance function is two fold. The flavor of the problem may have its origins in Ramsey Theory but, because of the importance Szlam's Lemma plays in this work in relating the Szlam number of a distance graph $G_\rho(X, D)$ to its chromatic number, the history of the chromatic number of a distance graph $G_\rho(X, D)$ must also be given its due.

In 2009 Soifer [13] published *The Mathematical Coloring Book* in which he gives a detailed account of the history of the chromatic number of the plane problem. The following is a summary of some of his findings. In 1950 Edward Nelson, at the time an undergraduate of the University of Chicago, asked John Isbell, a fellow student, the following question:

"*What is the minimum number of colors needed to color the real plane such that no two points of distance one apart are the same color?*"

Shortly after Nelson proved that you needed at least 4 colors and Isbell, using techniques of Hugo Hadwiger [8], showed that it could be done in 7. These bounds have not been improved. The problem first appeared in a publication by Martin Gardner [7] in 1960 and the proofs of both bounds appeared in a publication by Hugo Hadwiger [9] in 1961.

Throughout the years that followed the problem gained popularity and the problem began to be seen as finding the chromatic number of the distance graph $G(\mathbb{R}^2)$. Thus the above bounds of Nelson and Isbell are stated as $4 \leq \chi(\mathbb{R}^2) \leq 7$. Of course there are many variations of the problem of finding $\chi_\rho(X, D)$ where $X \subseteq \mathbb{R}^n$, for some positive integer n, $D \subseteq (0, \infty)$ and, ρ a distance function. A great many of these problems involve letting ρ be the Euclidean metric and looking at different sets X and D. For more information we refer the reader to [13]. In this work we are motivated to look at the opposite, which is to fix X and D, and investigate the problem of finding $\chi_\rho(X, D)$ for different distance functions ρ. Moreover, since Szlam's Lemma , as stated in Lemma 1.1 below, requires ρ to be a translation invariant distance function we only concern our selfs with distance functions of this type. In 1991 Chilakamarri [2] looked at the chromatic numbers of a distance graphs $G_\rho(\mathbb{R}^2)$ on $\mathbb{R}^2$ where ρ is a Minkowski metric (the details of which we explore in Chapter 4), and Kloeckner in 2015 explored the chromatic numbers of distance graphs $G_\rho(\mathbb{R}^2)$ on $\mathbb{R}^2$ where ρ is a translation invariant distance function that induces the usual topology (the details of which we look at in Chapter 2).

For the origins of the Szlam number of a distance graph we need to look into Ramsey Theory. The history and prehistory of Ramsey Theory is given in great detail in two works by Soifer [14, 13]. In the following we extract from these two works the problems and results that we find relevant to the history of developing the problem which is the Szlam number of a distance graph. In 1973 Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [5] published a paper in which they were interested in problems of the following type which we give in their own words.

> *Is it true that for any partition of the Euclidean plane into two classes (we say that the plane is two-colored), there exists a set of three points all in the same class forming the vertices of an equilateral triangle of side length* 1 *?(We call such a set monochromatic.)*

The answer to the above was no and the authors provide a construction of a two-coloring of the plane for which no triangle of side length 1 is monochromatic. When the above question is asked about $\mathbb{R}^4$ instead of $\mathbb{R}^2$ the answer is interestingly yes. In a follow up paper [6] in 1975 by the same authors the following question was posed:

> Is it possible to color the Euclidean plane with two colors, say red and blue, so that no two blue points are Euclidean distance 1 apart and no four red points are the vertices of a unit square, a square of side length 1?

Anysuch coloring of the plane is clearly a rather red coloring of $G(\mathbb{R}^2)$, though the authors did not express their results in terms of distance graphs. The origins of this problem are speculative since Johnson [1] recounts hearing the problem in 1973 from Don Greenwell, who may well have gotten the problem from Lazio Lovose, who was his office mate for a period when Greenwell was a graduate student at Vanderbilt. Not much later in 1979 Juhász [11] not only showed the answer to be no but proved that given any coloring of the Euclidean plane with two colors (red and blue) so that no two blue points are Euclidean distance 1, then for any 4 point set $F \subseteq \mathbb{R}^2$ there exists a set F' congruent (F is *congruent* to F' if and only if F' is a composition of translations and rotations of F) to F such that F' is all red.

If we express Juhász's result in the terms we have defined in the previous section by strengthening our definition of a nonempty set $F \subseteq \mathbb{R}^2$ being forbidden by a rather red coloring $\{R, B\}$ of $G(\mathbb{R}^2)$ by defining F to be *forbidden** by a rather red coloring $\{R, B\}$ of $G(\mathbb{R}^2)$ if there does not exist an $F' \subseteq \mathbb{R}^2$ congruent to F that is all red, and defining $Szl^*(\mathbb{R}^2) = \min\{|F| : F \subseteq \mathbb{R}^2$ is a forbidden* by a rather red coloring of $G(\mathbb{R}^2)\}$. The above Theorem by Juhász is equivalent to saying $Szl^*(\mathbb{R}^2) > 4$. Juhász [11], in the same paper, went on to show that $Szl^*(\mathbb{R}^2) \leq 12$ and in 1994 Csizmadia and Tóth [3] improved the upper bound by proving $Szl^*(\mathbb{R}^2) \leq 8$. These bounds have not been improved.

In 1999 Arthur Szlam, at the time participating in a summer research experience at Auburn University for undergraduates, restricted the notion of a set being forbidden to the definition we use in Section 1.1.3 and showed that if $F \subseteq \mathbb{R}^n$ is forbidden by a rather red coloring of $G(\mathbb{R}^n)$, for n a positive integer, then $\chi(\mathbb{R}^n) \leq |F|$. The immediate consequence noted by both Johnson and Szlam [10, 15] is that since $\chi(\mathbb{R}^2) \geq 4$, for any set F forbidden by a rather red coloring of $\mathbb{R}^2$ it follows that $|F| \geq 4$, or in our notation $Szl(\mathbb{R}^2) \geq 4$. Szlam [15] showed there exists a 7 point set forbidden by a rather red coloring of $G(\mathbb{R}^2)$ which is equivalent in our notation to saying $Szl(\mathbb{R}^2) \leq 7$. Though the idea of the Szlam number was clearly evident in both Szlam's work [15] and Johnson and Szlam's work [10], a definition of the Szlam number did not appear in publication until 2011 in a work by Berkert ad Johnson [1]. To date, excluding the work that follows, there has been no other results on finding the Szlam number of a distance graph $G_\rho(X, D)$ where $X \subseteq \mathbb{R}^n$, $D \subseteq (0, \infty)$ and ρ is a translation invariant distance function.

1.3 Szlam's Lemma and some Elementary Results

In 2001 Johnson and Szlam published a paper [10] in *Geombinatorics* titled, "A New Connection Between Two Kinds of Euclidean Coloring Problems". The following is a corollary of their main result.

Theorem 1.1. *Let $X \subseteq \mathbb{R}^n$ be closed under vector addition for some positive integer n, $D \subseteq (0, \infty)$, ρ a translation invariant distance function on X and $G_\rho(X, D)$ the distance graph associated with X, D, and ρ. Suppose the set $F \subseteq X$ is forbidden by a rather red coloring of $G_\rho(X, D)$. Then*

$$\chi_\rho(X, D) \leq |F|.$$

Proof. Let $F \subseteq X$ be forbidden by the rather red coloring $\{R, B\}$ of $G_\rho(X, D)$. For each $x \in X$ choose an $f \in F$ such that $f + x \in B$ and define $f_x = f$ (we note that if F is denumerable we do not need the axiom of choice here). We define a coloring $\phi : X \to F$ of $G_\rho(X, D)$ by $\phi(x) = f_x$. Suppose $\phi(x) = \phi(y)$. Then $f_x = f_y$. Thus $f_x + y \in B$ and $f_x + x \in B$. It follows that $f_x + y$ and $f_x + x$ are not adjacent in $G_\rho(X, D)$. Then $\rho(x, y) = \rho(f_x + x, f_x + y) \notin D$. Hence x and y are not adjacent in $G_\rho(X, D)$. $\square$

An immediate consequence of Johnson and Szlam's result, which we call *Szlam's lemma*, is the following.

Lemma 1.1 (Szlam's Lemma). *Let X, D, and ρ be as in Theorem 1.1. Then*

$$\chi_\rho(X, D) \leq Szl_\rho(X, D).$$

We can now see that not only is the Szlam number of the plane an upper bound to chromatic number of the plane but also a corresponding inequality holds for any translation invariant distance function ρ on $X \subseteq \mathbb{R}^n$ and any set $D \subseteq (0, \infty)$. We also notice that if we have a set F forbidden by a rather red coloring of a distance graph $G_\rho(X, D)$ such that $Szl_\rho(X, D) = |F|$, then by the proof of Johnson and Szlam's result above we can "construct" a proper coloring using $|F|$ colors. A natural question arises, for which distance graphs $G_\rho(X, D)$ does $\chi_\rho(X, D) = Szl_\rho(X, D)$? The breadth of this question is large to say the least, so for now we refine our inquiry by letting $X = \mathbb{R}^2$, $D = \{1\}$ and ask the above question for the Euclidean metric, the taxi-cab metric and the max-norm.

Theorem 1.2. $4 \leq Szl(\mathbb{R}^2) \leq 7$

Proof. Since $|\cdot|$ is translation invariant the lower bound follows directly from Szlam's lemma, since $4 \leq \chi(\mathbb{R}^2) \leq 7$ [13]. To prove the upper bound we construct a rather red coloring and find a forbidden set whose cardinality is seven. We start with a Hadwiger tile which consists of seven regular hexagons each with diameter slightly less than 1 as shown in Figure 1.1. We color the center hexagon along with its boundary blue and other six hexagons along with the sections of their boundaries that do not intersect with the center hexagon red. We can then tile the plane with this colored Hadwiger tile as shown in Figure 1.2. Let B be the set of points in $\mathbb{R}^2$ colored blue and R be the set of points in $\mathbb{R}^2$ colored red. $\{R, B\}$ is a rather red coloring. Let F be the set of seven hexagon centers as shown in Figure 1.2. F is a forbidden set of $\{R, B\}$. To see this, think of moving F by translation, in any direction. Clearly, when the point that was originally the center of the blue hexagon reaches the boundary of that hexagon, a former center of a neighboring red hexagon will have reached the opposite blue boundary segment, and will advance into the blue interior if the former blue center continues into red territory, and will reach the blue boundary when the former blue center reaches the red boundary of the original Hadwiger tile. Thus it is impossible to translate F so that the blue center is in the original Hadwiger tile and all 7 elements of F are red. Thus, it is impossible to translate F in any way so that F is entirely red. Hence $Szl(\mathbb{R}^2) \leq 7$. $\square$

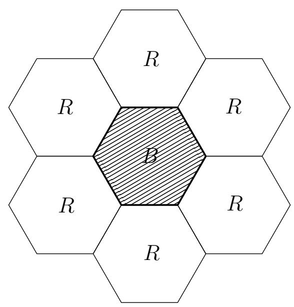

Figure 1.1: Hadwiger Tile

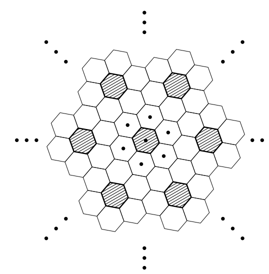

Figure 1.2: Tiling of Plane

Theorem 1.3. $\chi_{|\cdot|_1}(\mathbb{R}^2) = Szl_{|\cdot|_1}(\mathbb{R}^2) = 4$

Proof. Let $K = \{(0,0),(1/2,1/2),(1/2,-1/2),(1,0)\}$. Then $G_{|\cdot|_1}(K)$ is a subgraph of $G_{|\cdot|_1}(\mathbb{R}^n)$ and $G_{|\cdot|_1}(K)$ is isomorphic to K_4. Hence $\chi_{|\cdot|_1}(\mathbb{R}^2) \geq 4$. Clearly, $|\cdot|_1$ is translation invariant. Thus by Szlam's lemma $Szl_{|\cdot|_1}(\mathbb{R}^2) \geq 4$. Define $B = \{(x,y) \in \mathbb{R}^2 : y \geq -x+n-1/2,\ y < -x+n+1/2,\ y \geq x+m-1/2,\ y < x+m+1/2 \text{ for some } n,m \in \mathbb{Z}\}$, $R = \mathbb{R}^2 \backslash B$, and $F = \{(0,0),(1/2,1/2),(1/2,-1/2),(1,0)\}$ as shown in the Figure 1.3 below. Let $v \in \mathbb{R}^2$. It is straightforward to see that $(v+F) \cap B \neq \emptyset$. (See the proof of Theorem 1.4) Therefore F is forbidden by $\{R,B\}$. Hence, $4 \leq \chi_{|\cdot|_1}(\mathbb{R}^2) \leq Szl_{|\cdot|_1}(\mathbb{R}^2) \leq |F| = 4$. $\square$

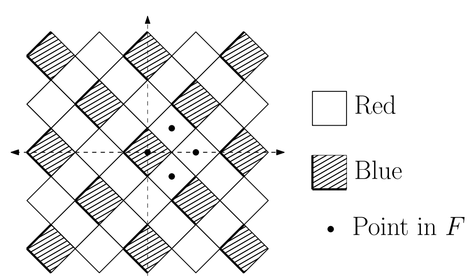

Figure 1.3: Rather red coloring and forbidden set of $G_{|\cdot|_1}(\mathbb{R}^2)$

Theorem 1.4. $\chi_{|\cdot|_\infty}(\mathbb{R}^2) = Szl_{|\cdot|_\infty}(\mathbb{R}^2) = 4$

Proof. Let $K = \{(0,0),(0,1),(1,0),(1,1)\}$. Then $G_{|\cdot|_\infty}(K)$ is a subgraph of $G_{|\cdot|_\infty}(\mathbb{R}^n)$ and $G_{|\cdot|_\infty}(K)$ is isomorphic to a K_4. Hence $\chi_{|\cdot|_\infty}(\mathbb{R}^2) \geq 4$. Clearly $|\cdot|_\infty$ is translation invariant. Thus by Szlam's lemma $Szl_{|\cdot|_\infty}(\mathbb{R}^2) \geq 4$. Define $B = \{x, y \in \mathbb{R}^2 : 2n \leq x < 2n+1, 2m \leq y < 2m+1 \text{ for } n, m \in \mathbb{Z}\}$, $R = \mathbb{R}^2 \backslash B$ and $F = \{(1/2,1/2),(1/2,3/2),(3/2,3/2),(3/2,1/2)\}$ as shown in the Figure 1.4 below. Let $v \in \mathbb{R}^2$. Then for some integers s, t and $\theta_1, \theta_2 \in [0,1)$,

$$\begin{aligned}
v + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) &= (s + \theta_1, t + \theta_2), \\
v + \left(\tfrac{3}{2}, \tfrac{1}{2}\right) &= (s + 1 + \theta_1, t + \theta_2), \\
v + \left(\tfrac{1}{2}, \tfrac{3}{2}\right) &= (s + \theta_1, t + 1 + \theta_2), \text{ and} \\
v + \left(\tfrac{3}{2}, \tfrac{3}{2}\right) &= (s + 1 + \theta_1, t + 1 + \theta_2).
\end{aligned}$$

Since one of $s, s+1$ is even, and one of $t, t+1$ is even, it follows that one of $v + u$, $u \in F$, is in B. Thus $\{v + F\} \cap B \neq \emptyset$. Therefore F is forbidden by $\{R, B\}$. Hence, $4 \leq \chi_{|\cdot|_\infty}(\mathbb{R}^2) \leq Szl_{|\cdot|_\infty}(\mathbb{R}^2) \leq |F| = 4$. $\square$

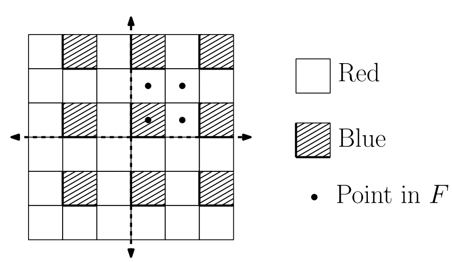

Figure 1.4: Rather red coloring and forbidden set of $G_{|\cdot|_\infty}(\mathbb{R}^2)$

Let G be a graph and define the clique number $\omega(G)$ of G to be the largest integer n such that the complete graph K_n is a subgraph of G. We denote the clique number of a distance graph $G_\rho(X, D)$ where $X \subseteq \mathbb{R}^n$ for some integer n, $D \subseteq (0, \infty)$ and ρ a distance function on X by $\omega_\rho(X, D)$. From Theorem 1.2 and 1.3 one might observe the following.

Theorem 1.5. *Let $G_\rho(X, D)$ be a distance graph on X where $X \subseteq \mathbb{R}^n$ is closed under vector addition for some integer n, $D \subseteq (0, \infty)$ and ρ is a translation invariant distance function on X. If $\chi_\rho(X, D) = \omega_\rho(X, D)$, then $Szl_\rho(X, D) = \chi_\rho(X, D)$.*

Proof. Let $\phi : X \to C$ be a proper coloring of $G_\rho(X, D)$ such that $|C| = \chi_\rho(X, D)$. Let $i \in C$ and C_i be ith color class of ϕ. Let $B = C_i$ and $R = X \backslash B$. Since ϕ is a proper coloring $\{R, B\}$ is a rather red coloring of $G_\rho(X, D)$. Let $F \subseteq X$ such that $G_\rho(F, D) \cong K_m$ where $m = \omega_\rho(X, D)$. Let $v \in X$. Since ρ is translation invariant, $G_\rho(v+F, D) \cong K_m$. Furthermore, since $\chi_\rho(X, D) = \omega_\rho(X, D)$, there exists an $f \in F + v$ such that $\phi(f) = C_i = B$. Hence $(F + v) \cap B \neq \emptyset$ for all $v \in X$. Therefore F is forbidden by $\{R, B\}$ of $G_\rho(X, D)$. Since $|F| = \omega_\rho(X, D)$, by Szlam's Lemma $\omega_\rho(X, D) = \chi_\rho(X, D) \leq Szl_\rho(X, D) \leq |F| = \omega_\rho(X, D)$. $\square$

The above results, although disappointing with regards to shedding any light on finding $\chi(\mathbb{R}^2)$, do provoke some natural questions. First, does there exist a translation invariant distance function ρ such that $\chi_\rho(\mathbb{R}^2) < Szl_\rho(\mathbb{R}^2)$? This question in particular becomes of greater interest throughout the following work since for every translation invariant distance ρ we discuss where $\chi_\rho(\mathbb{R}^2)$ is known it is the case that $\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2)$. Another question, on which we can shed some light, is the following. What can be said about $Szl_\rho(\mathbb{R}^2)$ where ρ is a p-norm and $p > 2$? In Chapter 4 we prove that for all $p \geq 1$, $4 \leq Szl_\rho(\mathbb{R}^2) \leq 7$ where ρ a p-norm.

Chapter 2

Spectrum of Planar Szlam Numbers

In 2015 Kloeckner [12] investigated the possible values for $\chi_\rho(\mathbb{R}^2)$ where ρ ranges over a class of translation invariant distance functions. Let $\mathcal{U}(\mathbb{R}^2) = \{\rho : \rho$ is a translation invariant metric on $\mathbb{R}^2$ which induces the usual topology on $\mathbb{R}^2\}$. Kloeckner begins by looking at $\rho \in \mathcal{U}(\mathbb{R}^2)$ and defines $PLANAR = \{\chi_\rho(\mathbb{R}^2) : \rho \in \mathcal{U}(\mathbb{R}^2)\}$. He proves $PLANAR = \mathbb{N} \cup \aleph_0$ where $\mathbb{N} = \{1, 2, \ldots\}$. To our delight, for every value in $PLANAR$ Kloeckner provides a corresponding translation invariant metric. It is only natural that for each of these metrics ρ we ask what is the value of $Szl_\rho(\mathbb{R}^2)$. Define $SZLAM = \{Szl_\rho(\mathbb{R}^2) : \rho \in \mathcal{U}(\mathbb{R}^2)\}$. In the following we take the methods used by Kloeckner in characterizing $PLANAR$ to prove similar results for $SZLAM$. For each $\rho \in \mathcal{U}(\mathbb{R}^2)$ introduced by Kloeckner, we show that $\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2)$.

2.1 Planar Szlam Numbers

Theorem 2.1. $1 \in SZLAM$. *Further, there is a such that*

$$\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2) = 1.$$

Proof. Let $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ where $|\cdot|$ is the Euclidean metric. Then ρ is a translation invariant metric that induces the usual topology on $\mathbb{R}^2$ and thus $\rho \in \mathcal{U}(\mathbb{R}^2)$. Since there does not exist an $x, y \in \mathbb{R}^2$ such that $\rho(x, y) = 1$, it follows that any partition of $\mathbb{R}^2$ into sets R and B is a rather red coloring of $G_\rho(\mathbb{R}^2)$. Let $B = \mathbb{R}^2$ and $R = \emptyset$. Then $\{R, B\}$ is a rather red coloring of $G_\rho(\mathbb{R}^2)$. Let $F = \{(0, 0)\}$. Clearly F is forbidden by the rather red coloring $\{R, B\}$ and since $|F| = 1$, $Szl_\rho(\mathbb{R}^2) \leq 1$. Then by Szlam's lemma, $1 \leq \chi_\rho(\mathbb{R}^2) \leq Szl_\rho(\mathbb{R}^2) \leq 1$. The claims of the Theorem follow. $\square$

To show $\mathbb{N}\backslash\{1\} \subseteq PLANAR$ Kloeckner uses the metric

$$\rho_d\left((x_1, x_2), (y_1, y_2)\right) =$$

$$\max\left(\min(|x_1 - y_1|, 1), \frac{1}{d}|x_1 - y_1|, \frac{|x-y|}{1+|x-y|}\right)$$

for $d \in \mathbb{N}$ and proves $\chi_{\rho_d}(\mathbb{R}^2) = d + 1$. We note that ρ_d is translation invariant and induces the usual topology. The crux of his argument relies on the fact $\chi(\mathbb{R}, [1, d]) = d + 1$ for $d \in \mathbb{N}$. In 1985 Eggleton, Erdős, and Skilton [4] showed that for $d \in \mathbb{R}$ such that $d \geq 1$, $\chi(\mathbb{R}, [1, d]) = \lceil d \rceil + 1$. We provide our own proof of this result in Chapter 3. For the purposes of finding the elements of $PLANAR$ Kloeckner need only let $d \in \mathbb{N}$. Similarly, we need only let $d \in \mathbb{N}$ to find the elements of $SZLAM$. Yet this is not our only concern; as discussed in chapter 1 we wish to determine for which translation invariant distance functions do the Szlam number and the chromatic number differ. For these reasons we let d be any real number ≥ 1 even though it is an unnecessary generalization in determining $SZLAM$.

To show $\mathbb{N}\backslash\{1\} \subseteq SZLAM$ first we need to prove some facts about ρ_d.

Lemma 2.1. *$\rho_d(x, y) = 1$ if and only if $|x_1 - y_1| \in [1, d]$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.*

Proof. First note that $\frac{|x-y|}{1+|x-y|} < 1$ for all $x, y \in \mathbb{R}^2$. Suppose $\rho_d(x, y) = 1$. Then either $\min(|x_1 - y_1|, 1) = 1$ and $\frac{1}{d}|x_1 - y_1| \leq 1$, or $\min(|x_1 - y_1|, 1) \leq 1$ and $\frac{1}{d}|x_1 - y_1| = 1$. Let $\min(|x_1 - y_1|, 1) = 1$ and $\frac{1}{d}|x_1 - y_1| \leq 1$. It follows that $|x_1 - y_1| \in [1, d]$. Next let $\min(|x_1 - y_1|, 1) \leq 1$ and $\frac{1}{d}|x_1 - y_1| = 1$. It follows that $|x_1 - y_1| = d$. Therefore, if $\rho_d(x, y) = 1$ then $|x_1 - y_1| \in [1, d]$.

Suppose $|x_1 - y_1| \in [1, d]$. Then $\min(|x_1 - y_1|, 1) = 1$ and $\frac{1}{d}|x_1 - y_1| \leq 1$. Thus $\rho_d(x, y) = 1$. $\square$

Lemma 2.2. *Let $d \geq 1$. Then $\chi_{\rho_d}(\mathbb{R}^2) = \chi(\mathbb{R}, [1, d]) = \lceil d \rceil + 1$*

Proof. Let C be a set of colors and $\phi : \mathbb{R} \to C$ be a proper coloring of $G(\mathbb{R}, [1, d])$. We define $\pi : \mathbb{R}^2 \to C$ by $\pi((x_1, x_2)) = \phi(x_1)$. Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then,

$$\begin{aligned} \pi(x) = \pi(y) &\Rightarrow \phi(x_1) = \phi(y_1) \\ &\Rightarrow |x_1 - y_1| \notin [1, d] \\ &\Rightarrow \rho_d(x, y) \neq 1 \\ &\Rightarrow x \not\sim y \text{ in } G_{\rho_d}(\mathbb{R}^2, \{1\}). \end{aligned}$$

Hence π is a proper coloring of $G_{\rho_d}(\mathbb{R}^2)$ with $|C|$ colors; it follows that $\chi_{\rho_d}(\mathbb{R}^2) \leq \chi(\mathbb{R}, [1, d])$. Next, let C be a set and $\Pi : \mathbb{R}^2 \to C$ be a proper coloring of $G_{\rho_d}(\mathbb{R}^2)$. We define $\Phi : \mathbb{R} \to C$ by $\Phi(x) = \Pi((x, 0))$. Let $x, y \in \mathbb{R}$:

$$\begin{aligned} \Phi(x) = \Phi(y) &\Rightarrow \Pi((x, 0)) = \Pi((y, 0)) \\ &\Rightarrow \rho_d((x, 0), (y, 0)) \neq 1 \\ &\Rightarrow |x - y| \notin [1, d] \\ &\Rightarrow x \not\sim y \text{ in } G(\mathbb{R}, [1, d]). \end{aligned}$$

Hence Φ is a proper coloring on $G(\mathbb{R}, [1, d])$ with $|D|$ colors; it follows that $\chi_{\rho_d}(\mathbb{R}^2) \geq \chi(\mathbb{R}, [1, d])$. Therefore $\chi_{\rho_d}(\mathbb{R}^2) = \chi(\mathbb{R}, [1, d]) = \lceil d \rceil + 1$. $\square$

Theorem 2.2. *$\mathbb{N} \backslash \{1\} \in SZLAM$. Further, for each $d \geq 1$, $\chi_{\rho_d}(\mathbb{R}^2) = Szl_{\rho_d}(\mathbb{R}^2) = \lceil d \rceil + 1$.*

Proof. Let $d \geq 1$, $B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [k(\lceil d \rceil + 1), k(\lceil d \rceil + 1) + 1) \text{ for some } k \in \mathbb{Z}\}$ and $R = \mathbb{R}^2 \backslash B$. Suppose $(x_1, x_2), (y_1, y_2) \in B$. Then $|x_1 - y_1| \in [0, 1)$ or $|x_1 - y_1| > d$. Therefore $\rho_d(x, y) \neq 1$ by Lemma 2.1. Hence $\{R, B\}$ is a rather red coloring of $G_{\rho_d}(\mathbb{R}^2)$. Next, let $F = \{(0, 0), (1, 0), \ldots, (\lceil d \rceil, 0)\}$ and $v \in \mathbb{R}^2$. Then $(v + F) \cap B \neq \emptyset$. It follows F is forbidden by the rather red coloring $\{R, B\}$ of $G_{\rho_d}(\mathbb{R}^2)$. Therefore $Szl_{\rho_d}(\mathbb{R}^2) \leq \lceil d \rceil + 1$. Thus, by Szlam's lemma and lemma 2.2, $\lceil d \rceil + 1 = \chi_{\rho_d}(\mathbb{R}^2) \leq Szl_{\rho_d}(\mathbb{R}^2) \leq \lceil d \rceil + 1$. The claims of the Theorem follows. $\square$

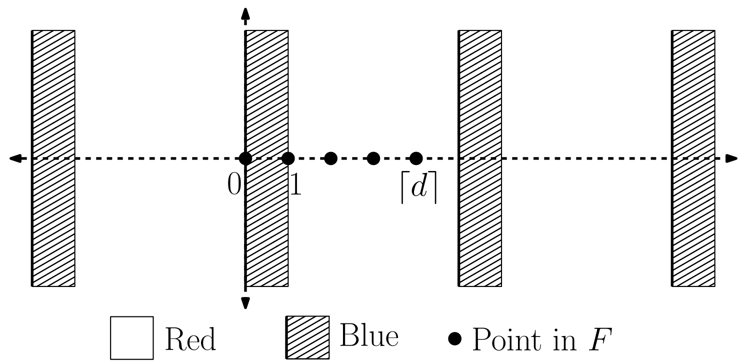

Figure 2.1: Rather red coloring and forbidden set of $G_{\rho_d}(\mathbb{R}^2)$ where $3 < d \leq 4$.

Theorem 2.3. $\aleph_0 \in SZLAM$. *Further, there is a* $\rho \in \mathcal{U}(\mathbb{R}^2)$ *such that*

$$\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2) = \aleph_0.$$

Proof. Let $\rho(x, y) = \min(|x-y|, 1)$ for all $x, y \in \mathbb{R}^2$, where $|\cdot|$ is the Euclidean metric. Clearly $\rho \in \mathcal{U}(\mathbb{R}^2)$. Let $B = \{(x, y) \in \mathbb{R}^2 \mid -1/2 \leq x < 1/2, -1/2 \leq y < 1/2\}$ and $R = \mathbb{R}^2 \backslash B$ as seen in Figure 2.2 below. Let $F = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$. Then $\{R, B\}$ is a rather red coloring and F is a forbidden by $\{R, B\}$ with respect to ρ. Thus $\chi_\rho(\mathbb{R}^2) \leq Szl_\rho(\mathbb{R}^2) \leq \aleph_0$. On the other hand $\chi_\rho(\mathbb{R}^2) \geq \aleph_0$, since for any partition of $\mathbb{R}^2$ into finitely many sets, at least one set must contain points further apart than 1 in the Euclidean distance and therefore at ρ-distance 1. Hence $\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2) = \aleph_0$ and $\aleph_0 \in SZLAM$. $\square$

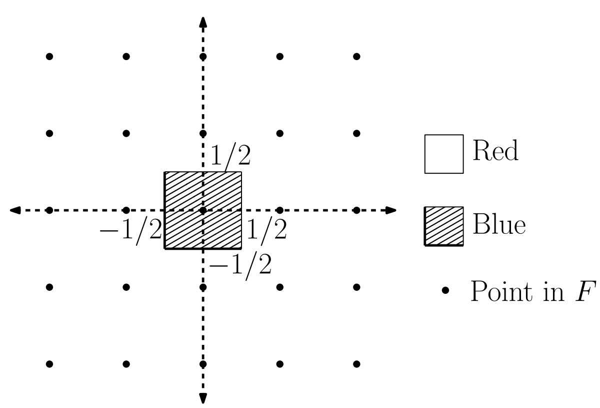

Figure 2.2: Rather red coloring and forbidden set of $G_\rho(\mathbb{R}^2)$ where $\rho(x, y) = \min(|x - y|, 1)$.

Theorem 2.4. *$Szl_\rho(\mathbb{R}^2) \leq \aleph_0$ for any $\rho \in \mathcal{U}(\mathbb{R}^2)$.*

Proof. Let $B_\rho(0,1) = \{y : \rho(0,y) < 1\}$ and $B(0,\epsilon) = \{y : |y| < \epsilon\}$. Since ρ induces the usual topology, there exists an $r \in \mathbb{R}$ such that $B(0,r) \subseteq B_\rho(0,1)$. Let C be a closed square centered at $(0,0)$ with Euclidean diameter d where $d < r$. Suppose $x, y \in C$. It follows that $|x - y| \leq d < r$. Hence $x - y \in B(0,r)$ which implies $x - y \in B_\rho(0,1)$. Thus $\rho(0, x-y) < 1$. Furthermore, since ρ is translation invariant $\rho(y,x) = \rho(0,x-y) < 1$. Let $C = B$ and $R = \mathbb{R}^2 \backslash C$. Then $\{R, B\}$ is a rather red coloring of $G_\rho(\mathbb{R}^2)$. Next, let $F = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = \frac{nd}{\sqrt{2}}, x_2 = \frac{md}{\sqrt{2}}$ where $m, n \in \mathbb{Z}\}$. Then $(F + v) \cap B \neq \emptyset$ for all $v \in \mathbb{R}^2$. Hence F is forbidden by the rather red coloring $\{R, B\}$ of $G_\rho(\mathbb{R}^2)$. Also, $|F| = \aleph_0$. Therefore $Szl_\rho(\mathbb{R}) \leq \aleph_0$ □

The following is trivial given the above theorems.

Theorem 2.5. *$SZLAM = \mathbb{N} \cup \aleph_0$.*

2.2 Proper Planar Szlam Numbers

Now, in view of the above results one may be inclined to remark that a stronger restriction is needed on the class of translation invariant distance functions in question. Kloeckner asks the same question and proposes to look at *proper planar metrics*; ρ is proper planar if and only if $\rho \in \mathcal{U}(\mathbb{R}^2)$ and every closed ball of ρ is compact. Kloeckner proves several results about the realizable values of $\chi_\rho(\mathbb{R}^2)$ where ρ is a proper planar metric. Let $\mathcal{U}^*(\mathbb{R}^2) = \{\rho :$ ρ is a proper planar metric on $\mathbb{R}^2\}$ and define $SZLAM^* = \{Szl_\rho(\mathbb{R}^2) : \rho \in \mathcal{U}^*(\mathbb{R}^2)\}$. We continue to use the methods of Kloeckner to prove the following results about $SZLAM^*$ which are similar to those of Kloeckner.

Theorem 2.6. *If $n \in SZLAM^*$, then $n > 2$.*

Proof. Kloeckner shows if $\rho \in \mathcal{U}^*(\mathbb{R}^2)$, then $\chi_\rho(\mathbb{R}^2) > 2$. By Szlam's lemma the theorem is trivial. □

Theorem 2.7. $SZLAM^* \subset \mathbb{N}$

Proof. Let $\rho \in \mathcal{U}^*(\mathbb{R}^2)$. By the argument in Theorem 2.4, there exists a square C centered at $(0,0)$ whose side is of Euclidean length c, such that $\rho(x,y) < 1$ for all $x, y \in C$. Moreover, since the closed balls of ρ are compact, they are bounded in the Euclidean sense. Then there exists a square D whose side is of Euclidean distance d such that $B_\rho(0,1) \subset D$. Let $B = \{(x,y) \in \mathbb{R}^2 : (d+c)n - \frac{c}{2} \leq x < (d+c)n + \frac{c}{2}, (d+c)m - \frac{c}{2} \leq y < (d+c)m + \frac{c}{2}, \text{ for } m, n \in \mathbb{Z}\}$ and $R = \mathbb{R}^2 \backslash B$. If $x, y \in B$, then either x and y both belong in some translate of C, hence $\rho(x,y) < 1$, or x and y are not together in any translate of D, hence $\rho(x,y) > 1$. Thus $\{R, B\}$ is a rather red coloring of $G_\rho(\mathbb{R}^2)$. Let $F = \{(x,y) \in \mathbb{R}^2 : x = nc, y = mc$ where $-(c+d) \leq nc, mc \leq (c+d)$ for $n, m \in \mathbb{Z}\}$. Then F is forbidden by the rather red coloring $\{R, B\}$. Since F is finite, $Szl_\rho(\mathbb{R}^2)$ is finite; thus $SZLAM^* \subset \mathbb{N}$. $\square$

Let ρ_1 and ρ_2 be metrics on $X_1, X_2 \subseteq \mathbb{R}^n$ respectively. Define $\rho_1 \overset{\infty}{\times} \rho_2 : X_1 \times X_2 \to [0, \infty)$ by

$$\rho_1 \overset{\infty}{\times} \rho_2 \left((x_1, x_2), (y_1, y_2)\right) = \max\left(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\right).$$

It is easy to see that $\rho_1 \overset{\infty}{\times} \rho_2$ is a metric on $X_1 \times X_2$, and is translation invariant if ρ_1 and ρ_2 are.

Lemma 2.3. *Let ρ_1 and ρ_2 be translation invariant metrics on $X_1, X_2 \subseteq \mathbb{R}^n$, respectively, where X_1 and X_2 are closed under vector addition. Then*

$$Szl_{\rho_1 \overset{\infty}{\times} \rho_2}(X_1 \times X_2) \leq Szl_{\rho_1}(X_1) \cdot Szl_{\rho_2}(X_2).$$

Proof. Since $\rho_1 \overset{\infty}{\times} \rho_2$ is a translation invariant metric, $Szl_{\rho_1 \overset{\infty}{\times} \rho_2}(X_1 \times X_2,)$ is well defined. Let $\{R_1, B_1\}$, $\{R_2, B_2\}$ be rather red colorings of $G_{\rho_1}(X_1)$,$G_{\rho_2}(X_2)$ respectively and F_1, F_2 be forbidden by $\{R_1, B_1\}$, $\{R_2, B_2\}$ respectively. Let $(x_1, x_2), (y_1, y_2) \in B_1 \times B_2$. Then $\rho_1(x_1, y_1) \neq 1$ and $\rho_2(x_2, y_2) \neq 1$. It follows that $\rho_1 \overset{\infty}{\times} \rho_2 \left((x_1, x_2), (y_1, y_2)\right) \neq 1$. Let $R = X_1 \times X_2 \backslash B_1 \times B_2$ and $B = B_1 \times B_2$. Then $\{R, B\}$ is a rather red coloring of $G_{\rho_1 \overset{\infty}{\times} \rho_2}(X_1 \times X_2)$.

Let $F = F_1 \times F_2$ and $v = (v_1, v_2) \in X_1 \times X_2$. Then $F + v = (F_1 + v_1) \times (F_2 + v_2)$. There exist a $f_1 \in F_1$ and $f_2 \in F_2$ such that $f_1 + v_1 \in B_1$ and $f_2 + v_2 \in B_2$. Therefore $(f_1 + v_1, f_2 + v_2) \in B$, from which it follows that $(F + v) \cap B \neq \emptyset$. Hence F is forbidden by $\{R, B\}$. Since $|F| = |F_1| \cdot |F_2|$, the theorem follows. $\square$

Theorem 2.8. *$SZLAM^*$ contains all non-prime integers greater than 1. Further, for each such integer n there is a $\rho \in \mathcal{U}^*(\mathbb{R}^2)$ such that*

$$\chi_\rho(\mathbb{R}^2) = Szl_\rho(\mathbb{R}^2) = n.$$

Proof. Let $n > 1$ be a non-prime integer and $\rho_i : \mathbb{R} \to [0, \infty)$ defined by $\rho_i(x, y) = \max\left(\min(|x - y|, 1), \frac{1}{d_i}|x - y|\right)$ where $i = 1, 2$ and $d_i \in \mathbb{N}$ such that $(d_1 + 1)(d_2 + 1) = n$. Then $G_{\rho_i}(\mathbb{R}) = G(\mathbb{R}, [1, d_i])$. Let $\rho = \rho_1 \overset{\infty}{\times} \rho_2$. Then $\rho \in \mathcal{U}^*(\mathbb{R}^2)$. It follows that

$$\begin{aligned} Szl_\rho(\mathbb{R}^2) &\leq Szl_{\rho_1}(\mathbb{R}) \cdot Szl_{\rho_2}(\mathbb{R}) \\ &= Szl(\mathbb{R}, [1, d_1]) \cdot Szl(\mathbb{R}, [1, d_2]) \\ &= (1 + d_1)(1 + d_2) = n. \end{aligned}$$

Kloeckner shows that $\chi_\rho(\mathbb{R}^2) = n$. Thus, by Szlam's lemma, the theorem holds. $\square$

With regards to finding $Szl(\mathbb{R}^2)$, the above results, though only reducing the spectrum of Szlam numbers of $G_\rho(\mathbb{R}^2)$ at the extremes when ρ is required to be proper, give rise to the thought that if one could characterize $SZLAM^*$ one then might reduce the possible values of $Szl(\mathbb{R}^2)$, since the Euclidean metric is indeed proper. The opposite direction may be needed in finding the elusive distance function ρ, if it exists, such that $\chi_\rho(\mathbb{R}^2) < Szl_\rho(\mathbb{R}^2)$. It might be the case that such a distance function ρ is neither in $\mathcal{U}(\mathbb{R}^2)$ or $\mathcal{U}^*(\mathbb{R}^2)$. If we lessen the restriction on the distance function ρ so that ρ does not necessarily induce the usual topology we do get an increase in the spectrum of Szlam numbers. One can easily show that when ρ is the discrete metric $\chi_p(\mathbb{R}^2) = Szl_p(\mathbb{R}^2) = \aleph_1$.

We conclude by generalizing a result due to Szlam [15] which concerns itself with what a proper coloring must look like if indeed the Szlam number and chromatic number differ. Let $\phi : \mathbb{R}^n \to C$ be a proper coloring of $G_\rho(\mathbb{R}^n)$ where ρ is a translation invariant distance function and let $C_i = \{x \in \mathbb{R}^n : \phi(x) = i\}$, $i \in C$, be the *color classes* of ϕ. Fix $j \in C$. Then ϕ is called a *regular proper coloring* if for each $i, j \in C$ there exists an $f_{ij} \in \mathbb{R}^n$ such that $C_j = C_i + f_{ij}$. Szlam proves that if $\phi : \mathbb{R}^n \to C$ is a regular proper coloring of $G(\mathbb{R}^n)$ where $|C| = k \in \mathbb{N}$, then there exist a rather red coloring $\{R, B\}$ and a set $F \in \mathbb{R}^n$ forbidden by $\{R, B\}$ such that $|F| = k$. We generalize his proof here to encompass all translation invariant distance functions, let $D \subseteq (0, \infty)$ and not require the number of color classes to be finite. We must give credit to Szlam since the proof here required little modification of his original proof.

Theorem 2.9. *Let ρ be a translation invariant distance function on $\mathbb{R}^n$, $D \subseteq (0, \infty)$ and $\phi : \mathbb{R}^n \to C$ be a regular proper coloring of $G_\rho(\mathbb{R}^n, D)$. Then there exist a rather red coloring $\{R, B\}$ of $G_\rho(\mathbb{R}^n, D)$ and a set $F \in \mathbb{R}^n$ forbidden by $\{R, B\}$ such that $|F| = |C|$.*

Proof. Let $\{C_i : i \in C\}$ be the collection of color classes of ϕ. Fix $j \in C$. Then for each $i \in C$ choose an $f_{ij} \in \mathbb{R}^n$ such that $C_j = C_i + f_{ij}$. Let $B = C_j$ and $R = \mathbb{R}^n \backslash B$. Because ϕ is a proper coloring, $\{R, B\}$ is a rather red coloring of $G_\rho(\mathbb{R}^n, D)$. Let $F = \{f_{ij} : i \in C\}$. Then $|F| = |C|$. Let $v \in \mathbb{R}^n$. For some $k \in C$, $v \in C_k$. It follows that $v + f_{kj} \in C_j = B$ and thus $(v + F) \cap B \neq \emptyset$. Hence F is forbidden by $\{R, B\}$. $\square$

The above can be restated as: if $\chi_\rho(\mathbb{R}^n, D) < Szl_\rho(\mathbb{R}^n, D)$ then there does not exist a regular proper coloring $\phi : \mathbb{R}^n \to C$ of $G_\rho(\mathbb{R}^n, D)$ such that $|C| = \chi_\rho(\mathbb{R}^n)$. We note that for every translation invariant distance function ρ given above in this chapter there exists a regular proper coloring of $G_\rho(\mathbb{R}^n, D)$ that achieves the chromatic number. Clearly, if we wish to find a translation invariant distance function ρ and $D \subseteq (0, \infty)$ such that $\chi_\rho(\mathbb{R}^n, D) < Szl_\rho(\mathbb{R}^n, D)$ its corresponding distance graph cannot have a regular proper coloring that achieves the chromatic number.

Chapter 3

K-Graphs and the Szlam Numbers of the Real Line

3.1 K-Graphs

Kloeckner [12] proposes to define a graph in the following way. Let $K, X \subseteq \mathbb{R}^n$, for some positive integer n, where $K = -K$ and $(0, \ldots, 0) \notin K$. We will call such a set a *K-set*. Define $G(X, K)$, a *K-graph* on X, to be the graph whose vertex set is X and $x, y \in X$ are adjacent if and only if $x - y \in K$.

Theorem 3.1. *Let ρ be a translation invariant distance function on $\mathbb{R}^n$, for some positive integer n, $X \subseteq \mathbb{R}^n$, and $D \subseteq (0, \infty)$. Let $K = \{x \in \mathbb{R}^n : \rho(0, x) \in D\}$, then $G(X, K) = G_\rho(X, D)$.*

Proof. Let $k \in K$. Since ρ is translation invariant, $\rho(0, k) = \rho(0 - k, k - k) = \rho(-k, 0) = \rho(0, -k)$. Thus $-k \in K$. Since $0 \notin D$, $(0, \ldots, 0) \notin K$. Hence K is a K-set and thus $G(X, K)$ is well defined. Let $x, y \in X$ be adjacent in $G_\rho(X, D)$. Since $\rho(0, x - y) = \rho(y, x) \in D$, $x - y \in K$. Therefore x and y are adjacent in $G(X, K)$. Let $x, y \in X$ be adjacent in $G(X, K)$. Since $x - y \in K$, $\rho(y, x) = \rho(0, x - y) \in D$. Therefore x and y are adjacent in $G_\rho(X, D)$. Hence $G_\rho(X, D) = G(X, K)$. $\square$

For X, D, and ρ, as defined in Theorem 3.1, the graph $G_\rho(X, D)$ is a K-graph. Conversely we get that for $X, K \subseteq \mathbb{R}^n$, for some positive integer n, and K a K-set, the K-graph $G(X, K)$ can be defined as a distance graph $G_\rho(X, D)$ where where $D = \{1\}$.

Theorem 3.2. *Let $K, X \subseteq \mathbb{R}^n$, for some positive integer n, and let K be a K-set. Then there exists a translation invariant distance function ρ on $\mathbb{R}^n$ such that $G(X, K) = G_\rho(X)$.*

Proof. Define $\rho : \mathbb{R}^n \to [0, \infty)$ by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x - y \in K \\ 1.1 & \text{otherwise} \end{cases}$$

Clearly $\rho(x,y) = 0$ if and only if $x = y$. Further, since $K = -K$, for $x, y \in X$, $x - y \in K$ if and only if $y - x \in K$. Thus $\rho(x,y) = \rho(y,x)$. Let $v, x, y \in X$. Since $x - y \in K$ if and only if $x + v - (y + v) \in K$ and, $x - y = 0$ if and only if $x + v - (y + v) = 0$, then $\rho(x + v, y + v) = \rho(x, y)$. Hence ρ is a translation invariant distance function (we note that ρ is also a metric).

Let x and y be adjacent in $G_\rho(X)$. Then $\rho(x,y) = 1$ which implies $x - y \in K$. Whence x and y are adjacent in $G(X,K)$. Let x and y be adjacent in $G(X,K)$. Then $x - y \in K$ which implies $\rho(x,y) = 1$. Therefore x and y are adjacent in $G_\rho(X)$ and hence $G(X,K) = G_\rho(X)$. $\square$

We can now see that for ρ a translation invariant distance function on $\mathbb{R}^n$, for some positive integer n, $X \subseteq \mathbb{R}^n$, and $D \subseteq (0, \infty)$ the class of distance graphs $G_\rho(X, D)$ is equivalent to the class of K-graphs $G(X,K)$ for K a K-set. Furthermore, we observe if X, D, and ρ are defined as above, by the above two Theorems there exist a K-graph $G(X,K)$ and a distance graph $G_\tau(X)$ where τ is a translation invariant distance function on $\mathbb{R}^n$ such that $G_\rho(X,D) = G(X,K) = G_\tau(X)$. Thus, when looking at distance graphs $G_\rho(X,D)$ we may assume $D = \{1\}$ when ranging over all translation invariant distance functions ρ on $\mathbb{R}^n$ for $X \subseteq \mathbb{R}^n$. Of course, in practice this observation may not be useful; as we saw in Chapter 2 it can behoove us to look at distance graphs where $D \neq \{1\}$. Furthermore, as also seen in chapter 2, defining translation invariant distance functions ρ can a be a tedious affair. We are motivated to look at K-graphs for certain classes of K-sets and their corresponding Szlam and chromatic numbers without resorting to their status as distance graphs.

Let $X, K \subseteq \mathbb{R}^n$, for some positive integer n, and let K be a K-set. We denote the chromatic number of the K-graph $G(X,K)$ by $\chi(X,K)$. Let $G(X,K)$ be a K-graph with a rather red coloring $\{R, B\}$, and suppose that X is closed under vector addition. Then a set $F \subseteq X$ is said to be *forbidden* by the rather red coloring $\{R, B\}$ of $G(X, K)$ if no translate of F is all red; that is, $v + F \nsubseteq R$ for all $v \in X$. Further, define the *Szlam number* of $G(X, K)$, denoted $Szl(X, K)$, by $Szl(X, K) = \min\{|F| : F$ is a forbidden by a rather red coloring of $G(X, K)\}$. Clearly, if $G(X, K) = G_\rho(X, D)$, then $Szl(X, K) = Szl_\rho(X, D)$ and $\chi(X, K) = \chi_\rho(X, D)$.

For $X, K \in \mathbb{R}^n$, for some positive integer n, where X is closed under vector addition, if K is a K-set, $G(X, K)$ is a K-graph and $G_\rho(X)$ is its equivalent distance graph as defined in the proof of Theorem 3.2, it follows that $\chi_\rho(X) = \chi(X, K)$ and $Szl_\rho(X) = Szl(X, K)$. By Szlam's Lemma $\chi(X, K) = \chi_\rho(X) \leq Szl_\rho(X) = Szl(X, K)$. Therefore $\chi(X, K) \leq Szl(X, K)$. For reasons of self satisfaction we provide the following proof of the K-graph version of Szlam's Lemma without invoking Szlam's Lemma itself.

Theorem 3.3 (Szlam's Lemma for K-graphs)**.** *Let $X, K \subseteq \mathbb{R}^n$, for some positive integer n, X closed under vector addition, K a K-set, and $G(X, K)$ a K-graph. Then $\chi(X, K) \leq Szl(X, K)$.*

Proof. Let $F \in X$ be forbidden by the rather red coloring $\{R, B\}$ on $G(X, K)$. For each $x \in X$ choose an $f \in F$ such that $f + x \in B$ and define $f_x = f$ (we note that if F is denumerable we do not need the axiom of choice here). We define a coloring $\phi : X \to F$ of $G(X, K)$ by $\phi(x) = f_x$. Suppose $\phi(x) = \phi(y)$. Then $f_x = f_y$. It follows that $f_x + x \in B$ and $f_x + y \in B$. Thus $f_x + y$ and $f_x + x$ are not adjacent in $G(X, K)$. Hence $x - y = f_x + x - (f_x + y) \notin K$. Therefore x and y are not adjacent in $G(X, K)$. Thus ϕ is a proper coloring of $G(X, K)$ from which the claim of the Theorem follows. □

Now, Kloeckner [12] does not prove anything about K-graphs, but notes that it may be useful in finding the chromatic number of a given distance graph or in our case the Szlam number. We continue our investigation of the relationship between the Szlam and chromatic numbers by looking at K-graphs for certain classes of a K-set K. The first class of interest is when K is a subset of a line l in $\mathbb{R}^n$, for some positive integer n. For K to be a K-set the line l must pass through the origin. By this observation we get the following.

Theorem 3.4. *Let $l \subseteq \mathbb{R}^n$, for some positive integer n, be a line that passes through the origin and let $K \subseteq l$ be a K-set. Then there exists a K-set $M \subseteq \mathbb{R}$ such that $\chi(\mathbb{R}^n, K) = \chi(\mathbb{R}, M)$ and $Szl(\mathbb{R}^n, K) \leq Szl(\mathbb{R}, M)$.*

Proof. Without loss of generality assume l to be the x-axis. Then for each $k \in K$ there exists $k_1 \in \mathbb{R}$ such that $k = (k_1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $M = \{m \in \mathbb{R} : k = (m, 0, \ldots, 0) \in K\}$. For each $m \in M$ there exists $k \in K$ where $k = (m, 0 \ldots, 0)$. Since $K = -K$, $-k = (-m, 0, \ldots, 0) \in K$ and hence $-m \in M$. Thus $M = -M$. Further, since $(0, \ldots, 0) \notin K$, $0 \notin M$. Hence M is a K-set and thus the K-graph $G(\mathbb{R}, M)$ is well defined. Let $x = (x_1 \ldots, x_n)$ and $y = (y_1 \ldots, y_n)$, then $x - y \in K$ if and only if $x_1 - y_1 \in M$.

Let C be a set and $\phi : \mathbb{R} \to C$ be a proper coloring of $G(\mathbb{R}, M)$. Define $\Phi : \mathbb{R}^n \to C$ by $\Phi(x) = \phi(x_1)$ where $x = (x_1 \ldots, x_n)$. Let x and y be adjacent in $G(\mathbb{R}^n, K)$ where $x = (x_1 \ldots, x_n)$ and $y = (y_1 \ldots, y_n)$. Then $x - y \in K$ implies $x_1 - y_1 \in M$, and thus x_1 and y_1 are adjacent in $G(\mathbb{R}, M)$. Hence $\Phi(x) = \phi(x_1) \neq \phi(y_1) = \Phi(y)$. Therefore Φ is a proper coloring of $G(\mathbb{R}^n, K)$. Then $\chi(\mathbb{R}^n, K) \leq \chi(\mathbb{R}, M)$. It is easy to see that $G(\mathbb{R}, M) \cong G(l, K)$. Therefore $\chi(\mathbb{R}, M) = \chi(l, K)$. Since $G(l, K)$ is a proper subgraph of $G(\mathbb{R}^n, K)$, $\chi(\mathbb{R}^n, K) \geq \chi(l, K) = \chi(\mathbb{R}, M)$. Hence $\chi(\mathbb{R}^n, K) = \chi(\mathbb{R}, M)$.

Let $\pi : \mathbb{R} \to \{r, b\}$ be a rather red coloring of $G(\mathbb{R}, M)$ and let $B_1 = \{x \in \mathbb{R} : \pi(x) = b\}$ and $R_1 = \{x \in \mathbb{R} : \pi(x) = r\}$. Define $\Pi : \mathbb{R}^n \to \{r, b\}$ by $\Pi(x) = \pi(x_1)$. Let $B = \{x \in \mathbb{R}^n : \Pi(x) = b\}$ and $R = \{x \in \mathbb{R}^n : \Pi(x) = r\}$ where $x = (x_1 \ldots, x_n)$. Suppose $x, y \in B$ where $x = (x_1 \ldots, x_n)$ and $y = (y_1 \ldots, y_n)$. Then $x_1, y_1 \in B_1$. Therefore $x_1 - y_1 \notin M$ from which it follows that $x - y \notin K$. Hence x and y are not adjacent in $G(\mathbb{R}^n, K)$. Thus $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}^n, K)$. Let $F_1 \subseteq \mathbb{R}$ be forbidden by the rather red coloring $\{R_1, B_1\}$ of $G(\mathbb{R}, M)$. Then for all $i \in \mathbb{R}$, there exists an $f \in F_1$ such that $i + f \in B_1$. Let $F = \{(f, 0, \ldots, 0) \in \mathbb{R}^n : f \in F_1\}$. Let $v \in \mathbb{R}^n$ where $v = (v_1, \ldots, v_n)$. Since there exists an $f \in F_1$ such that $f + v_1 \in B_1$, there exists $(f, 0, \ldots, 0) = s \in F$ such that

$v + s \in B$. Thus F is forbidden by $\{R, B\}$ in $G(\mathbb{R}^n, K)$. Since $|F| = |F_1|$, it follows that $Szl(\mathbb{R}^n, K) \le Szl(\mathbb{R}, M)$. $\square$

The above Theorem is somewhat curious since $G(\mathbb{R}, M)$ is isomorphic to a subgraph of $G(\mathbb{R}^n, K)$, for M and K as defined above, from which one might think that $Szl(\mathbb{R}^n, K) \ge Szl(\mathbb{R}, M)$ and thus $Szl(\mathbb{R}^n, K) = Szl(\mathbb{R}, M)$. The proof, if the Szlam numbers are indeed equal, becomes troublesome since there does not seem to be an obvious method to define a rather red coloring and forbidden set of $G(\mathbb{R}, M)$ from a given rather red coloring and forbidden set of $G(\mathbb{R}^n, K)$. We do note that if $Szl(\mathbb{R}, M) = \chi(\mathbb{R}, M)$, then

$$Szl(\mathbb{R}, M) = \chi(\mathbb{R}, M) = \chi(\mathbb{R}^n, K) \le Szl(\mathbb{R}^n, K) \le Szl(\mathbb{R}, M)$$

and hence $Szl(\mathbb{R}^n, K) = Szl(\mathbb{R}, M)$. This motivates us to find K-graphs $G(\mathbb{R}, K)$ on $\mathbb{R}$ where $\chi(\mathbb{R}, K) = Szl(\mathbb{R}, K)$. Since for any K-graph $G(\mathbb{R}, K)$ on $\mathbb{R}$, there exists a distance graph $G(\mathbb{R}, D)$ on $\mathbb{R}$, where $D = \{|d| \in \mathbb{R} : d \in K\}$, such that $G(\mathbb{R}, K) = G(\mathbb{R}, D)$, it behooves us to return to distance graphs and exploit the known results about chromatic numbers of some distance graphs on $\mathbb{R}$.

3.2 Szlam Numbers on the Real Line

In 1984 Eggleton, Erdős, and Skilton [4] published a paper titled "Coloring the Real line" in the *Journal of Combinatorial Theory* in which they determined the chromatic numbers for distance graphs $G(\mathbb{R}, D)$ on $\mathbb{R}$ for certain sets $D \subseteq (0, \infty)$. We use their methods to prove similar results about the Szlam number of distance graphs $G(\mathbb{R}, D)$ on $\mathbb{R}$ for certain sets $D \subseteq (0, \infty)$ and show that for every set D introduced by Eggleton, Erdős, and Skilton [4] $\chi(\mathbb{R}, D) = Szl(\mathbb{R}, D)$; thus determining both the chromatic numbers and Szlam numbers of the K-graphs $G(\mathbb{R}^n, K)$ on $\mathbb{R}^n$, for some positive integers n, for certain K-sets where K is a subset of a line in $\mathbb{R}^n$.

Eggleton, Erdős, and Skilton [4] prove that $\chi(\mathbb{R},[1,d]) = \lceil d\rceil + 1$ for $d \in \mathbb{R}$ such that $d \geq 1$. We provide our own proof here which we discovered independently.

Lemma 3.1. *Let $d \in \mathbb{R}$ such that $d \geq 1$. Then $\chi(\mathbb{R},[1,d]) = \lceil d\rceil + 1$.*

Proof. First we show that $\chi(\mathbb{R},[1,d]) \geq \lceil d\rceil + 1$. Let $d \notin \mathbb{N}$. By way of contradiction assume that $\chi(\mathbb{R},[1,d]) \leq \lceil d\rceil$. Then there exists a proper coloring $\phi : \mathbb{R} \to \{0,\ldots,\lfloor d\rfloor\}$ of $G(\mathbb{R},[1,d])$.

Let $i \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$ such that $0 \leq \epsilon \leq d - \lfloor d\rfloor$. Define $A_{i,\epsilon} = \{k(1+\frac{\epsilon}{\lfloor d\rfloor}) : k \in \mathbb{Z}, i \leq k \leq i + \lfloor d\rfloor\}$. Then $G(A_{i,\epsilon},[1,d]) \cong K_{\lceil d\rceil}$ and $G(A_{i,\epsilon} \cap A_{i+1,\epsilon},[1,d]) \cong K_{\lfloor d\rfloor}$. It follows for ϕ to be a proper coloring, $G(A_{i,\epsilon} \cap A_{i+1,\epsilon},[1,d])$ must use $\lfloor d\rfloor$ colors which leaves only one possible choice to color the vertices $i(1+\frac{\epsilon}{\lfloor d\rfloor})$ and $(i+\lceil d\rceil)(1+\frac{\epsilon}{\lfloor d\rfloor})$. Hence $\phi\left(i(1+\frac{\epsilon}{\lfloor d\rfloor})\right) = \phi\left((i+\lceil d\rceil)(1+\frac{\epsilon}{\lfloor d\rfloor})\right)$. It follows that if $i \equiv j \mod \lceil d\rceil$, then $\phi\left(i(1+\frac{\epsilon}{\lfloor d\rfloor})\right) = \phi\left(j(1+\frac{\epsilon}{\lfloor d\rfloor})\right)$.

Without loss of generality suppose $\phi(0) = 0$ and $\phi(1) = 1$. Let $\epsilon = 0$. Then for all $x \equiv 1 \mod \lceil d\rceil$, it follows that $\phi(x) = \phi\left(x(1+\frac{\epsilon}{\lfloor d\rfloor})\right) = 1$. Next, let $\epsilon = \frac{\lfloor d\rfloor(\lceil d\rceil+1)}{n}$ where $n \geq \frac{\lfloor d\rfloor(\lceil d\rceil+1)}{d-\lfloor d\rfloor}$ and $n \equiv 0 \mod \lceil d\rceil$. Then ϵ satisfies the condition $0 \leq \epsilon \leq d - \lfloor d\rfloor$ and thus $\phi\left(n(1+\frac{\epsilon}{\lfloor d\rfloor})\right) = 0$. Furthermore, $n(1+\frac{\epsilon}{\lfloor d\rfloor}) = n + 1 + \lceil d\rceil \equiv 1 \mod \lceil d\rceil$. Hence $\phi\left(n(1+\frac{\epsilon}{\lfloor d\rfloor})\right) = 1$ which is a contradiction to ϕ being a proper coloring. Therefore $\chi(\mathbb{R},[1,d]) \geq \lceil d\rceil + 1$ for $d \notin \mathbb{N}$.

Let $d \in \mathbb{N}$, $X = \{0,1,\ldots d\}$. Then $G(X,[1,d]) \cong K_{d+1}$. Hence $\chi(\mathbb{R},[1,d]) \geq d+1$ for $d \in \mathbb{N}$. Therefore $\chi(\mathbb{R},[1,d]) \geq \lceil d\rceil + 1$ for $d \geq 1$.

Next we show that $\chi(\mathbb{R},[1,d]) \leq \lceil d\rceil + 1$. Define $\phi : \mathbb{R} \to \{0,\ldots,\lceil d\rceil\}$ by $\phi(x) = \lfloor x\rfloor \mod (\lceil d\rceil + 1)$. Suppose $\phi(x) = \phi(y)$ and without loss of generality let $y \leq x$. Then for some $b, c \in \mathbb{R}$ such $0 \leq b, c < 1$ and some $k \in \mathbb{N} \cup \{0\}$

$$\phi(x) = \phi(y) \Rightarrow \lfloor x\rfloor = \lfloor y\rfloor + k(\lceil d\rceil+1) \Rightarrow x+c = y+b+k(\lceil d\rceil+1) \Rightarrow x-y = b-c+k(\lceil d\rceil+1).$$

If $k = 0$, then $y - x = b - c \in (-1, 1)$. If $k \geq 1$, then $k + b - c > 0$ and hence $x - y > k\lceil d \rceil$. Therefore $|x - y| \notin [1, d]$, from which it follows x is not adjacent to y in $G(\mathbb{R}, [1.d])$. Hence ϕ is a proper coloring of $G(\mathbb{R}, [1.d])$. □

Theorem 3.5. *Let $d \geq 1$. Then $\chi(\mathbb{R}, [1, d]) = Szl(\mathbb{R}, [1, d]) = 1 + \lceil d \rceil$*

Proof. Let $d \geq 1$, $B = \bigcup_{k \in \mathbb{Z}} [k(\lceil d \rceil + 1), k(\lceil d \rceil + 1) + 1)$ and $R = \mathbb{R} \backslash B$. Suppose $x, y \in B$. Then $|x - y| \in [0, 1)$ or $|x - y| > d$. Hence $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}, [1, d])$. Next, let $F = \{0, 1, \ldots, \lceil d \rceil\}$ and $v \in \mathbb{R}$. Then $(v + F) \cap B \neq \emptyset$. It follows that F is forbidden by the rather red coloring $\{R, B\}$ of $G(\mathbb{R}, [1, d])$. Therefore $Szl(\mathbb{R}, [1, d]) \leq \lceil d \rceil + 1$. Thus, by Szlam's lemma, $\lceil d \rceil + 1 = \chi(\mathbb{R}, [1, d]) \leq Szl(\mathbb{R}, [1, d]) = \lceil d \rceil + 1$. Whence the theorem follows. □

The K-graph equivalent to the distance graph $G(\mathbb{R}, [1, d])$ where $d \geq 1$ is $G(\mathbb{R}, [-d, -1] \cup [1, d])$. Thus $\chi(\mathbb{R}, [-d, -1] \cup [1, d]) = Szl(\mathbb{R}, [-d, -1] \cup [1, d]) = 1 + \lceil d \rceil$. Thus by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, ([-d, -1] \cup [1, d]) \times \{0\}^{n-1}) = \lceil d \rceil + 1$. We continue by looking at the Szlam number of distance graphs $G(\mathbb{R}, D)$ where D is an open interval.

Lemma 3.2. *Let $D, D' \subseteq (0, \infty)$ such that $D \subseteq D'$. Then $Szl(\mathbb{R}, D) \leq Szl(\mathbb{R}, D')$.*

Proof. Let $\{R, B\}$ be a rather red coloring of $G(\mathbb{R}, D')$ and $F \subseteq \mathbb{R}$ be forbidden by $\{R, B\}$. Clearly $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}, D)$. Hence $Szl(\mathbb{R}, D) \leq Szl(\mathbb{R}, D')$. □

Theorem 3.6. *Let $d \geq 1$. Then $\chi(\mathbb{R}, (1, d)) = Szl(\mathbb{R}, (1, d)) = 1 + \lceil d \rceil$*

Proof. Since $(1, d) \subseteq [1, d]$, by Lemma 3.2 $Szl(\mathbb{R}, (1, d)) \leq Szl(\mathbb{R}, [1, d])$. Eggleton, Erdős, and Skilton [4] show that $\chi(\mathbb{R}, (1, d)) = 1 + \lceil d \rceil$. Hence, by Szlam's Lemma, $1 + \lceil d \rceil = \chi(\mathbb{R}, (1, d)) \leq Szl(\mathbb{R}, (1, d)) \leq Szl(\mathbb{R}[1, d]) = 1 + \lceil d \rceil$. □

The K-graph equivalent to the distance graph $G(\mathbb{R}, (1, d))$ where $d \geq 1$ is $G(\mathbb{R}, (-d, -1) \cup (1, d))$. Hence $\chi(\mathbb{R}, (-d, -1) \cup (1, d)) = Szl(\mathbb{R}, (-d, -1) \cup (1, d)) = 1 + \lceil d \rceil$. Thus by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, (-d, -1) \cup (1, d) \times \{0\}^{n-1}) = \lceil d \rceil + 1$.

Theorem 3.7. *For $d \geq 1$, let $D = \bigcup_{k=0}^{\infty}[k(\lceil d \rceil + 1) + 1, k(\lceil d \rceil + 1) + d]$. Then $\chi(\mathbb{R}, D) = Szl(\mathbb{R}, D) = \lceil d \rceil + 1$.*

Proof. Since $[1, d] \subseteq D$, by Theorem 3.5 and Lemma 3.2, $1 + \lceil d \rceil = Szl(\mathbb{R}, [1, d]) = \chi(\mathbb{R}, [1, d]) \leq \chi(\mathbb{R}, D) \leq Szl(\mathbb{R}, D)$. Let $B = \bigcup_{k \in \mathbb{Z}}[k(\lceil d \rceil + 1), k(\lceil d \rceil + 1) + 1)$ and $R = \mathbb{R} \backslash B$. Suppose $x, y \in B$ and $x < y$. Then there exist integers r, s such that $r \leq s$, $r(\lceil d \rceil + 1) \leq x < r(\lceil d \rceil + 1) + 1$, and $s(\lceil d \rceil + 1) \leq y < s(\lceil d \rceil + 1) + 1$. If $s = r$, then $0 < y - x < 1$ and hence y is not adjacent to x in $G(\mathbb{R}, D)$. Suppose $r < s$; then $(s - r)(\lceil d \rceil + 1) - 1 < y - x < (s - r)(\lceil d \rceil + 1) + 1$. If x and y are adjacent in $G(\mathbb{R}, [1, d])$, then there exists an integer $k \geq 0$ such that $[k(\lceil d \rceil + 1) + 1, k(\lceil d \rceil + 1) + d] \cap ((s - r)(\lceil d \rceil + 1) - 1, (s - r)(\lceil d \rceil + 1) + 1) \neq \emptyset$. The previous holds when $k(\lceil d \rceil + 1) + 1 < (s - r)(\lceil d \rceil + 1) + 1$ and $k(\lceil d \rceil + 1) + d > (s - r)(\lceil d \rceil + 1) - 1$. This implies $k < (s - r)$ and $k + 1 > (s - r)$ which is a contradiction. Hence x is not adjacent to y in $G(\mathbb{R}, D)$. Therefore $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}, D)$. Next, let $F = \{0, 1, \ldots, \lceil d \rceil\}$ and $v \in \mathbb{R}$. Then $(v + F) \cap B \neq \emptyset$. It follows that F is forbidden by the rather red coloring $\{R, B\}$ of $G(\mathbb{R}, D)$. Therefore, by Szlam's Lemma, $1 + \lceil d \rceil = Szl(\mathbb{R}, [1, d]) = \chi(\mathbb{R}, [1, d]) \leq \chi(\mathbb{R}, D) \leq Szl(\mathbb{R}, D) \leq \lceil d \rceil + 1$. The claim of the Theorem follows. □

The K-graph equivalent to the distance graph $G(\mathbb{R}, D)$ where $D = \bigcup_{k=0}^{\infty}[k(\lceil d \rceil + 1) + 1, k(\lceil d \rceil + 1) + d]$ is $G(\mathbb{R}, -D \cup D)$. Thus $\chi(\mathbb{R}, -D \cup D) = Szl(\mathbb{R}, -D \cup D) = 1 + \lceil d \rceil$. Thus by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, -D \cup D \times \{0\}^{n-1}) = \lceil d \rceil + 1$.

Theorem 3.8. *Let $D = \{1, 2, \ldots, m\}$ for m a positive integer. Then $\chi(\mathbb{R}, D) = Szl(\mathbb{R}, D) = m + 1$.*

Proof. Since $D \subseteq [1, m]$, by Theorem 3.5 and Lemma 3.2, $\chi(\mathbb{R}, D) \leq Szl(\mathbb{R}, D) \leq Szl(\mathbb{R}, [1, m]) = m + 1$. Moreover, the vertices $0, 1, \ldots, m$ induce a subgraph K_{m+1} of $G(\mathbb{R}, D)$. Hence, by Szlam's Lemma, $m + 1 \leq \chi(\mathbb{R}, D) \leq Szl(\mathbb{R}, D) \leq Szl(\mathbb{R}, [1, m]) = m + 1$ □

The K-graph equivalent to the distance graph $G(\mathbb{R}, D)$ where $D = \{1, 2, \ldots, m\}$ for m a positive integer is $G(\mathbb{R}, -D \cup D)$. Thus $\chi(\mathbb{R}, -D \cup D) = Szl(\mathbb{R}, -D \cup D) = m + 1$. Thus by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, -D \cup D \times \{0\}^{n-1}) = m + 1$.

Theorem 3.9. *Let C be a nonempty subset of positive odd integers. Then $\chi(\mathbb{R}, C) = Szl(\mathbb{R}, C) = 2$.*

Proof. Let D be the set as defined in Theorem 3.7 where $d = 1$. Then $D = \bigcup_{k=0}^{\infty}[2k+1, 2k+1]$ is precisely the set of positive odd integers and by Theorem 3.7 $\chi(\mathbb{R}, D) = Szl(\mathbb{R}, D) = 2$. Since $C \subseteq D$, by Lemma 3.2 $Szl(\mathbb{R}, C) \leq 2$. Moreover the edge set of $G(\mathbb{R}, C)$ is nonempty and thus $G(\mathbb{R}, C)$ has K_2 as a subgraph. Hence, by Szlam's Lemma, $2 \leq \chi(\mathbb{R}, C) \leq Szl(\mathbb{R}, C) \leq 2$, from which the theorem follows. $\square$

The K-graph equivalent to the distance graph $G(\mathbb{R}, C)$, were C is a nonempty subset of positive odd integers, is $G(\mathbb{R}, -C \cup C)$. Thus $\chi(\mathbb{R}, -C \cup C) = Szl(\mathbb{R}, -C \cup C) = 2$. Hence by the observation after Theorem 3.4 $Szl(\mathbb{R}^n, -C \cup C \times \{0\}^{n-1}) = 2$.

Theorem 3.10. *Let r, s be relatively prime positive integers of opposite parity such that $r < s$. Then $\chi(\mathbb{R}, \{r, s\}) = Szl(\mathbb{R}, \{r, s\}) = 3$.*

Proof. Eggleton, Erdős, and Skilton [4] show that $\chi(\mathbb{R}, \{r, s\}) = 3$. Then by Szlam's Lemma $Szl(\mathbb{R}, \{r, s\}) \geq 3$. Let D be the set as defined in Theorem 3.7 where $d = 2$. Then $D = \bigcup_{k=0}^{\infty}[3k + 1, 3k + 2]$. Let $c \in \mathbb{R}$ such that $c > 0$ and define $cD = \{cd : d \in D\}$. Clearly $G(\mathbb{R}, D) \cong G(\mathbb{R}, cD)$.

In the proof of $\chi(\mathbb{R}, \{r, s\}) = 3$ Eggleton, Erdős, and Skilton [4] show that there exists $c \in \mathbb{R}$ such that $\{r, s\} \subseteq cD$. We repeat their argument here. $\{r, s\} \subseteq cD$ precisely when there exists an integer k such that $c \leq r \leq 2c$ and $(3k + 1)c \leq s \leq (3k + 2)c$. These inequalities are equivalent to $r/2 \leq c \leq r$ and $s/(3k + 2) \leq c \leq s/(3k + 1)$. There exist a c and a k that satisfy these conditions when $s/(3k + 2) \leq r$ and $s/(3k + 1) \geq r/2$. Again these conditions are equivalent to

$$\frac{1}{3}\left(\frac{s}{r}-2\right) \leq k \leq \frac{1}{3}\left(\frac{2s}{r}-1\right).$$

Let $s/r = 3a+2$ for $a \in \mathbb{R}$. It follows that $a \leq k \leq 2a+1$. Without loss of generality suppose $s/r > 1$. Then $a > -1/3$. If $a > 0$ the interval $[a, 2a+1]$ contains at least 1 positive integer n and we let $k = n$. If $-1/3 < a \leq 0$, then $2a+1 > 0$ and we let $k = 0$. Hence there exists an integer $k \geq 0$ and a real number $c > 0$ that satisfy the above conditions. Therefore $\{r, s\} \subseteq cD$ for some $c > 0$. Thus, by Szlam's Lemma, $3 = \chi(\mathbb{R}, \{r, s\}) \leq Szl(\mathbb{R}, \{r, s\}) \leq Szl(\mathbb{R}, cD) = Szl(\mathbb{R}, D) = 3$. □

The K-graph equivalent to the distance graph $G(\mathbb{R}, \{r, s\})$, where r, s are relatively prime positive integers of opposite parity, is $G(\mathbb{R}, \{-s, -r, r, s\})$. Thus $\chi(\mathbb{R}, \{-s, -r, r, s\}) = Szl(\mathbb{R}, \{-s, -r, r, s\}) = 3$. Hence by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, \{-s, -r, r, s\} \times \{0\}^{n-1}) = 3$.

Theorem 3.11. *Let P denote the set of all prime numbers. Then $\chi(\mathbb{R}, P) = Szl(\mathbb{R}, P) = 4$.*

Proof. Eggleton, Erdős, and Skilton [4] prove that $\chi(\mathbb{R}, P) = 4$. Hence, by Szlam's Lemma $Szl(\mathbb{R}, P) \geq 4$. Let $B = \bigcup_{k\in\mathbb{Z}}[4k, 4k+1)$ and $R = B\backslash\mathbb{R}$. Let $x, y \in B$ and $x < y$. Then there exist integers r, s such that $r \leq s$, $4r \leq x < 4r+1$, and $4s \leq y < 4s+1$. If $s = r$, then $0 < y - x < 1$ and hence y is not adjacent to x in $G(\mathbb{R}, P)$. Suppose $r < s$. Then $4(s-r) - 1 < y - x < 4(s-r) + 1$. Suppose $y - x \notin \mathbb{Z}$. Hence $y - x \notin P$. Therefore x and y are not adjacent in $G(\mathbb{R}, P)$. Hence $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}, P)$. Suppose $y - x \in \mathbb{Z}$. Since both $4(s-r) - 1$ and $4(s-r) + 1$ are both odd and differ by 2 it follows that $y - x$ must be even. Furthermore, since $r < s$, $3 < y - x$. Hence $y - x \notin P$. Therefore x and y are not adjacent in $G(\mathbb{R}, P)$ and hence $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}, P)$. Let $F = \{0, 1, 2, 3\}$ and $v \in \mathbb{R}$. Then $\{v + F\} \cap B \neq \emptyset$. Since $|F| = 4$, $\chi(\mathbb{R}, P) = Szl(\mathbb{R}, P) = 4$. □

The K-graph equivalent to the distance graph $G(\mathbb{R}, P)$, were P denotes the set of all prime numbers, is $G(\mathbb{R}, -P \cup P)$. Thus $\chi(\mathbb{R}, -P \cup P) = Szl(\mathbb{R}, -P \cup P) = 4$. Hence by the observation after Theorem 3.4, $Szl(\mathbb{R}^n, (-P \cup P) \times \{0\}^{n-1}) = 4$.

Chapter 4

Szlam Numbers of K-graphs for K a convex closed curve in $\mathbb{R}^2$

In this chapter we look at the chromatic and Szlam numbers of the K-graph for K-sets that are convex closed curves. The motivation to look at K-sets of this type comes from a paper by Chilakamarri [2] in which he looks at the chromatic numbers of certain distance graphs $G_\rho(\mathbb{R}^2)$ where ρ is a *Minkowski metric*. Let C be a convex closed centrally symmetric ($-C = C$) curve in $\mathbb{R}^2$. Define the *Minkowski norm* $||\cdot||_C$ on $\mathbb{R}^2$ by $||(0,0)||_C = 0$ and for $u \in \mathbb{R}^2 \backslash \{(0,0)\}$, $||u||_C = \frac{|u|}{|P_u|}$, where P_u is the point at the intersection of C and the half-line with one end at $(0,0)$ which goes through u. We then define the *Minkowski metric* $\rho_C : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ by $\rho_C(x,y) = ||x-y||_C$.

4.1 Szlam Numbers of K-graphs for K a convex closed curve in $\mathbb{R}^2$

Lemma 4.1. *Let $C \subseteq \mathbb{R}^2$ be a centrally symmetric convex closed curve. Then $G_{\rho_C}(\mathbb{R}^2) = G(\mathbb{R}^2, C)$*

Proof. Since C is a centrally symmetric convex closed curve, $C = -C$ and $(0,0) \notin C$. Thus C is a K-set. Hence $G(\mathbb{R}^2, C)$ is well defined. Let $x, y \in \mathbb{R}^2$ be adjacent in $G_{\rho_C}(\mathbb{R}^2)$. Then $\rho_C(x,y) = 1$, which implies that $x - y = P_{x-y}$. Hence $x - y \in C$, and thus x and y are adjacent in $G(\mathbb{R}^2, C)$. Let $x, y \in \mathbb{R}^2$ be adjacent in $G(\mathbb{R}^2, C)$. Clearly $\rho_C(x,y) = 1$. Hence x and y are adjacent in $G_{\rho_C}(\mathbb{R}^2)$. Therefore, $G_{\rho_C}(\mathbb{R}^2) = G(\mathbb{R}^2, C)$. □

It is known that if there exists an isomorphism between graphs G and H then $\chi(G) = \chi(H)$. One can easily convince oneself that for two isomorphic distance graphs $G_\rho(X,D)$ and $H_\tau(X',D')$ where $X, X' \subseteq \mathbb{R}^n$, for some positive integer n, X and X' are closed under vector addition, $D, D' \subseteq (0,\infty)$, and ρ, τ are distance functions it follows that $Szl_\rho(X,D) =$

$Szl_\tau(X', D')$, a fact we used without note in Theorem 3.10. We provide the following Lemma so that the proof of the Theorem that follows is easier to see.

Lemma 4.2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^n$, for some positive integer n, be a linear isomorphism and K be a K-set. Then $G(\mathbb{R}^n, K) \cong G(\mathbb{R}^n, \pi(K))$ and $Szl(\mathbb{R}^n, K) = Szl(\mathbb{R}^n, \pi(K))$.*

Proof. Let $k \in K$. Then $-k \in K$. Let $y \in \pi(K)$. Then there exists $x \in K$ such that $\pi(x) = y$. Thus $-y = -\pi(x) = \pi(-x) \in \pi(K)$. Hence $\pi(K) = -\pi(K)$. Further, since $(0, \ldots, 0) \notin K$, $\pi((0, \ldots, 0)) = (0, \ldots, 0)$ and π is one-to-one, $(0, \ldots, 0) \notin \pi(K)$. Hence $\pi(K)$ is a K-set and $G(\mathbb{R}^n, \pi(K))$ is well defined. Let $x, y \in \mathbb{R}^n$ be adjacent in $G(\mathbb{R}^n, K)$. Then $x - y \in K$. It follows that $\pi(x - y) = \pi(x) - \pi(y) \in \pi(K)$. Hence $\pi(x)$ and $\pi(y)$ are adjacent in $G(\mathbb{R}^n, \pi(K))$. Let $n, m \in \mathbb{R}^n$ be adjacent in $G(\mathbb{R}^n, \pi(K))$. Then $n - m \in \pi(K)$. Then there exist $x, y \in \mathbb{R}^n$ such that $\pi(x) = n$ and $\pi(y) = m$. Hence $\pi^{-1}(n - m) = \pi^{-1}(m) - \pi^{-1}(m) = x - y \in \pi^{-1}(K) = K$. Hence x and y are adjacent in $G(\mathbb{R}^n, K)$. Therefore $G(\mathbb{R}^n, K) \cong G(\mathbb{R}^n, \pi(K))$.

Let $\{R, B\}$ be a rather red coloring of $G(\mathbb{R}^n, K)$ and $F \subseteq \mathbb{R}^n$ be forbidden by $\{R, B\}$. Let $x, y \in \pi(B)$. Then x and y are adjacent in $G(\mathbb{R}^n, \pi(B))$ if and only if $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are adjacent in B. Since $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}^n, K)$, $\{\pi(R), \pi(B)\}$ is a rather red coloring of $G(\mathbb{R}, \pi(K))$. Let $v \in \mathbb{R}^n$. Then $(v + F) \cap B \neq \emptyset$ implies that $(\pi(v) + \pi(F)) \cap \pi(B) \neq \emptyset$. Since for every $u \in \mathbb{R}^n$ there exists a $v \in \mathbb{R}^n$ such that $\pi(v) = u$, $\pi(F)$ is a forbidden by $\{\pi(B), \pi(R)\}$ of $G(\mathbb{R}^n, \pi(K))$. Hence $Szl(\mathbb{R}^n, K) \geq Szl(\mathbb{R}^2, \pi(K))$. Replacing π with π^{-1} and K by $\pi(K)$ we obtain the reverse of this inequality. Therefore $Szl(\mathbb{R}^n, K) = Szl(\mathbb{R}^n, \pi(K))$.

□

Chilakamarri shows that for any parallelogram P centered at the origin, $\chi_{\rho_P}(\mathbb{R}^2) = 4$. We get the following similar result.

Theorem 4.1. *Let $P \subseteq \mathbb{R}^2$ be a parallelogram centered at the origin. Then $\chi(\mathbb{R}^2, P) = Szl(\mathbb{R}^2, P) = 4$.*

Proof. Clearly P is a K-set. Let $S = \{s \in \mathbb{R}^2 : |0 - s|_\infty = 1\}$. Then S is a square centered at the origin and S is a K-set. Therefore $G_{|\cdot|_\infty}(\mathbb{R}^2) = G(\mathbb{R}^2, S)$. By Theorem 1.4, $4 = \chi(\mathbb{R}^2, S) = Szl(\mathbb{R}^2, S)$. Without loss of generality we can assume that two sides of P are parallel to the x-axis. Let $(x_1, y_1), (x_2, -y_1), (-x_1, -y_1)$ and $(-x_2, y_1)$ be the 4 vertices of P as shown in the figure 4.1 below. Define $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\pi(x, y) = \left(\frac{(x_1+x_2)x}{2} + \frac{(x_1-x_2)y}{2}, yy_1\right)$. Then π is a linear isomorphism and $\pi(S) = P$. Moreover π^{-1} exists and $\pi^{-1}(P) = S$. Hence, by Lemma 4.2, $G(\mathbb{R}^n, P) \cong G(\mathbb{R}^n, S)$ and $Szl(\mathbb{R}^n, P) = Szl(\mathbb{R}^n, S)$. Thus $\chi(\mathbb{R}^n, P) = \chi(\mathbb{R}^n, S)$. Therefore $\chi(\mathbb{R}^2, P) = Szl(\mathbb{R}^2, P) = 4$. $\square$

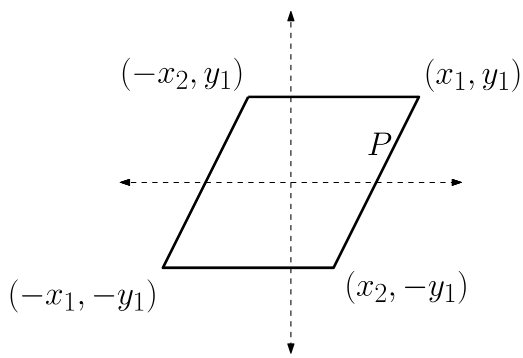

Figure 4.1: Parallelogram P

Lemma 4.3. *Let $K \subseteq \mathbb{R}^2$ be a K-set such that K is a convex closed curve. Then $Szl(\mathbb{R}^2, K) \geq 4$.*

Proof. Since K is a K-set, then K is a centrally symmetric convex closed curve. Chilakamarri [2] shows that for any centrally symmetric convex closed curve $C \in \mathbb{R}^2$, $\chi_{\rho_C}(\mathbb{R}^2) \geq 4$ where ρ_C is the Minkowski metric for C on $\mathbb{R}^2$. Since $G_{\rho_K}(\mathbb{R}^2) = G(\mathbb{R}^2, K)$ by Lemma 4.1, it follows that $\chi(\mathbb{R}^2, K) \geq 4$. Hence, by Szlam's Lemma, $Szl(\mathbb{R}^2, K) \geq 4$ $\square$

Theorem 4.2. *Let $P \subseteq \mathbb{R}^2$ be centrally symmetric convex 6-gon. Then $\chi(\mathbb{R}^2, P) = Szl(\mathbb{R}^2, P) = 4$.*

Proof. Since P is a centrally symmetric convex closed curve, P is a K-set. Hence $G(\mathbb{R}^2, P)$ is well defined. We color $\mathbb{R}^2$ by first constructing a tile which consist of four copies of

$\frac{1}{2}P = \left\{\frac{p}{2} : p \in P\right\}$ as seen in Figure 4.1. We color the interior of the bottom left $\frac{1}{2}P$ blue. We color the boundary of this 6-gon by coloring 3 non-adjacent sides blue represented by the bold lines and we color the end points of one of the blue sides blue as shown in Figure 4.1. We color the rest of the tile red. We can then tile $\mathbb{R}^2$ as seen in Figure 4.2. Let B be the set of all points in $\mathbb{R}^2$ colored blue and R be the set of all points in $\mathbb{R}^2$ colored red. It is easy to see that $\{R, B\}$ is a rather red coloring of $G(\mathbb{R}^2, P)$. Let F be the set of 4 centers of the original tile as seen in Figure 4.2. It is easy to see that for $v \in \mathbb{R}^2$, $(v + F) \cap B \neq \emptyset$. Hence F is forbidden by $\{R, B\}$ of $G(\mathbb{R}^2, P)$. Therefore $Szl(\mathbb{R}^2, P) \leq 4$. Since P is a centrally symmetric convex closed curve, by Lemma 4.3 $Szl(\mathbb{R}^2, P) \geq 4$. The claim follows.

□

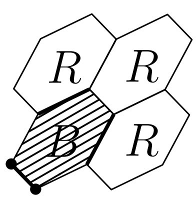

Figure 4.2: 4-Hexagon Tile

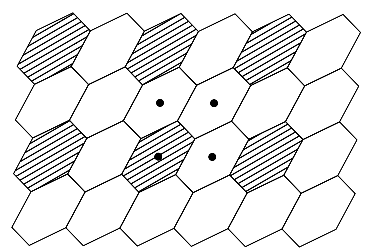

Figure 4.3: Tiling of Plane

Theorem 4.3. *Let C be a centrally symmetric convex closed curve. Then*

$$4 \leq Szl(\mathbb{R}^2, C) \leq 7.$$

Proof. The lower bound follows by Lemma 4.3. To prove the upper bound we look at the coloring Chilakamarri [2] provides to show that $\chi_{\rho C}(\mathbb{R}^2) \leq 7$. Chilakamarri inscribes a convex centrally symmetric hexagon in $\frac{1}{2}C = \{\frac{c}{2} : c \in C\}$ by the following.

Let O denote the origin, and let A and A^* be the points of intersection of $\frac{1}{2}C$ with the positive and negative x-axis, respectively. If we translate the line segment OA in the upper half plane in such a way that the point A is always on the curve $\frac{1}{2}C$ then there exists a translate of OA such that both the end-points F and B are on the curve C. We note that the uniqueness of both B and F depend on $\frac{1}{2}C$. If C is strictly convex then both B and F are both unique. We let $B^* = -B$ and $F^* = -F$ as shown in the Figure 4.3 below.

Chilakamarri shows that $H = ABFA^*B^*F^*$ is a centrally symmetric convex hexagon. He then constructs a Hadwiger tile of 7 copies of H and assigns colors 1 through 7 to the interiors of the hexagons as shown in the bold Hadwiger tile in Figure 4.4 below. If a hexagon is colored i we also give the same color to three non-adjacent sides and also give the same color to the endpoints of one of the sides. We then tile $\mathbb{R}^2$ with these 7 tiles as shown in Figure 4.4. If the same choice of colored sides and colored endpoints is made for every color then as Chilakamarri shows, this is a proper coloring of $G(\mathbb{R}^2, C)$. Moreover, it is easy to see this coloring is a regular proper coloring and thus, by Theorem 2.9, $Szl(\mathbb{R}^2, C) \leq 7$. $\square$

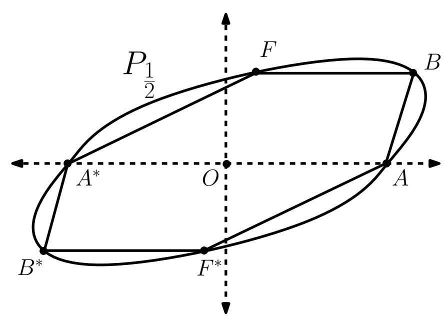

Figure 4.4: Hexagon inscribed in $\frac{1}{2}C$

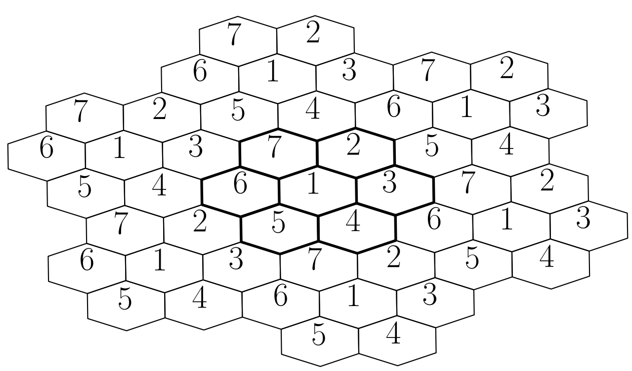

Figure 4.5: Proper coloring of $G(\mathbb{R}^2, C)$

Corollary 1. *Let ρ_p be the p-norm for $p \geq 1$. Then $4 \leq Szl_{\rho_p}(\mathbb{R}^2) \leq 7$*

Proof. Let $C = \{c \in \mathbb{R}^n : |0 - c|_p = 1\}$. Then C is a centrally symmetric convex closed curve and $G_{\rho_p}(\mathbb{R}^2) = G(\mathbb{R}^2, C)$. It follows from Theorem 4.3. that $4 \leq Szl(\mathbb{R}^2, C) \leq 7$. Hence $4 \leq Szl_{\rho_p}(\mathbb{R}^2) \leq 7$. $\square$

4.2 Open Problems

At the start of our investigation of the Szlam number there were two main questions of interest. The first was to find $Szl(\mathbb{R}^2)$ and the second question was: does there exist $X \subseteq \mathbb{R}^n$ closed under vector addition for some n a positive integer, $D \subsetneq (0, \infty)$, and ρ a translation invariant distance function such that $\chi_\rho(X, D) < Szl_\rho(X, D)$? Both of these questions remain unanswered. As to the first, as shown in Theorem 1.2, we know the answer is either 4,5,6, or 7. As to the second, as seen as the end of Chapter 2, we know that for the question to be in the affirmative there cannot exist a regular proper coloring that achieves the chromatic number of the distance graph. We note that for every distance graph or K-graph discussed where the chromatic number is known there exists a regular proper coloring of said distance graph that achieves its chromatic number. In our research of we have not come across a distance graph $G_\rho(X, D)$ where X, D, and ρ are as above, that does not have a regular proper coloring that achieves the chromatic number.

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