On Galvin's Theorem and Stable Matchings

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
August 6, 2016

Keywords: List Coloring, Stable Matchings, Edge Coloring

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Abstract

In this thesis, we will discuss list edge coloring and its relation to stable matchings. In particular, we will present three proofs of Galvin's famous theorem that bipartite graphs satisfy the list edge coloring conjecture. Galvin presented two of these proofs in his original paper, one by induction and another using stable matchings and the Gale-Shapley Theorem about stable matchings in bipartite graphs. The third proof we present is a corollary of a more general result proven by Borodin, Kostochka, and Woodall. We will study the techniques of these proofs to find a characterization of graphs that have stable matchings with respect to their preference lists, and two alternative proofs of the Gale-Shapley Theorem. We will also consider some new generalizations of stable matchings.

Acknowledgments

I would like to thank my advisor, Dr. Jessica McDonald for all of her guidance and assistance throughout my time as a graduate student at Auburn University. I would also like to thank all of my committee, Dr. Dean Hoffman and Dr. Pete Johnson, as well as Dr. Chris Rodger for their invaluable advice and suggestions for this thesis.

Table of Contents

List of Figures

On Galvin's Theorem and Stable Matchings

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Chapter 1

Introduction

In this introduction we will discuss several preliminary topics. We begin with list edge-coloring, proceeding to line graphs, and then stable matchings. We conclude the introduction with a short description of the content that will be discussed in this thesis.

1.1 List Edge Coloring

Let G be a graph which may contain parallel edges, but contains no loops. A function $c : V(G) \to \mathbb{N}$ is called a *coloring* of G if for any $v_1, v_2 \in V(G)$ such that $(v_1, v_2) \in E(G)$, $c(v_1) \neq c(v_2)$. The *chromatic number* of a graph G, denoted $\chi(G)$, which is the least amount of different natural numbers necessary to properly color G. A function f, $f : E(G) \to \mathbb{N}$ is an *edge coloring* if for any $e_1, e_2 \in E(G)$ such that e_1 meets e_2 at a vertex, $f(e_1) \neq f(e_2)$. The edge analogue of coloring can be viewed as vertex coloring the *line graph* $L(G)$; the vertex set of the line graph, $V(L(G))$ is $E(G)$ and two vertices are joined by an edge for each time they are incident in G. The *chromatic index*, denoted $\chi'(G)$, is the least amount of different natural numbers necessary to properly color $L(G)$.

A generalization of the coloring problem of graphs is called list coloring. Let L_v be a set of lists of natural numbers for each $v \in V(G)$. A *list coloring* is a function $g : V(G) \to L_v$ such that for any $v_1, v_2 \in V(G)$ with $(v_1, v_2) \in E(G)$, $g(v_1) \neq g(v_2)$. The *list chromatic number*, denoted $\chi_l(G)$, is the least size for which G can be properly colored with any lists of that size. To demonstrate that this is indeed a generalization, we first note that if all vertices are given the same list, this is the same as standard coloring. Below we give a theorem and example of a graph in which list coloring is indeed different from just coloring the graph.

Theorem 1.1. *There exists a graph G such that $\chi(G) \neq \chi_l(G)$*

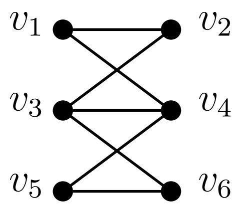


Figure 1.1: G_1 above is a graph which has $\chi(G) \neq \chi_l(G)$

Proof. Let G be the graph in Figure 1.1. By coloring v_1, v_3, v_5 with color 1 and v_2, v_4, v_6 with color 2 we see that $\chi(G) = 2$. Consider lists $S_{v_1} = \{1, 3\}$, $S_{v_2} = \{2, 3\}$, $S_{v_3} = \{1, 2\}$, $S_{v_4} = \{1, 2\}$, $S_{v_5} = \{2, 3\}$, $S_{v_6} = \{1, 3\}$. If G is 2-list colorable, then G can be properly colored using this list. In particular, v_3 must be colored either 1 or 2.

Suppose v_3 is colored 1. Then v_4 must be colored 2 and v_6 must be colored 3. From this, we see that v_5 cannot be colored.

Suppose v_3 is colored 2. Then v_2 is colored 3 and v_4 is colored 1. But with these colorings, v_1 is uncolorable. Therefore G is not 2-list colorable. In particular $\chi(G) < \chi_l(G)$. □

Now that we have an example of a graph in which the list chromatic number is different from the chromatic number, we are left to determine what classes of graphs $\mathcal{G}$ have the property $\chi(G) = \chi_l(G)$ for all $G \in \mathcal{G}$. The list edge coloring conjecture posits that equality actually holds for all line graphs. That is, $\chi'(G) = \chi'_l(G)$ where $\chi'_l(G)$ is the list chromatic index of G, which is the least size of lists such that the graph G can be properly edge colored for any lists of that size.

Conjecture 1.1.1 (List Edge Coloring Conjecture)**.** *All graphs G have the property that $\chi'(G) = \chi'_l(G)$.*

This conjecture has been made independently by many, but appeared first in print in a paper by Bollobás and Harris [12]. There has been significant study into this property, see Graph Edge Coloring, by Stiebitz, Scheide, Toft, and Favrholdt [13]. A graph G is considered *bipartite* if there exists a partition of $V(G)$, $\{X, Y\}$ such that no edge has both of its ends in either X or Y. Notice the graph in Figure 1.1 is an example of a bipartite graph. It is well known result of Kőnig [9] that $\chi'(G)$ is the maximum degree, Δ, of G for a bipartite graph G. Galvin's theorem gives a class of graphs which have equal chromatic number and list chromatic number, line graphs of bipartite graphs.

Theorem 1.2 (Galvin's Theorem[1])**.** *For every bipartite graph G,*

$$\chi_l{}'(G) = \chi'(G).$$

While this theorem provides a class of graphs in which list coloring is the same as coloring, it is not a characterization of such graphs as odd cycles are not bipartite but do have $\chi_l{}'(G) = \chi'(G)$. There are more results on classes of graphs with equal chromatic number and list chromatic number. For example, an analogue of the five color theorem has been studied by Thomassen [10], finding that all planar graphs have list coloring number at most 5. We note that planar graphs have chromatic number at most four, which is very close to satisfying our desired equality, but does not quite achieve it.

The main goal of Chapter two is to present three proofs of Galvin's Theorem. To present these proofs, we will need some basic information on line graphs and stable matchings, which will be provided in the next two sections of this chapter.

1.2 Line Graphs

Line graphs are a well understood and studied class of graphs, with several characterizations known. When working with edge coloring as vertex coloring a line graph, the following results will be very useful. Below we state without proof two theorems due to Beineke[2] on line graphs and some related line graph results and conjectures. These will be necessary as we further discuss theorems throughout the thesis.

Theorem 1.3 (Beineke's Theorem[2])**.** *The following are equivalent*

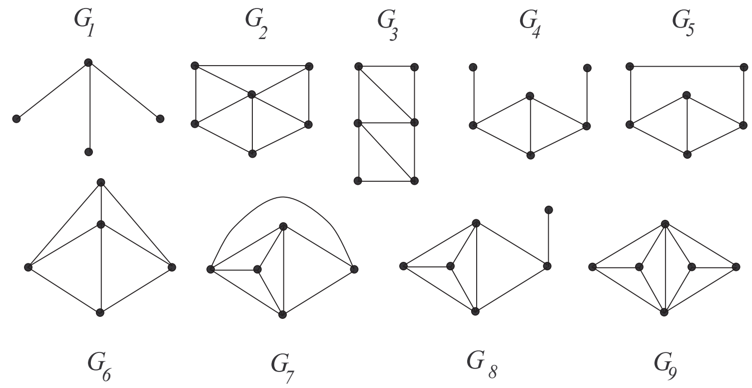


Figure 1.2: $G_1, G_2, \ldots, G_9$ are the forbidden induced subgraphs for line graphs of simple graphs.

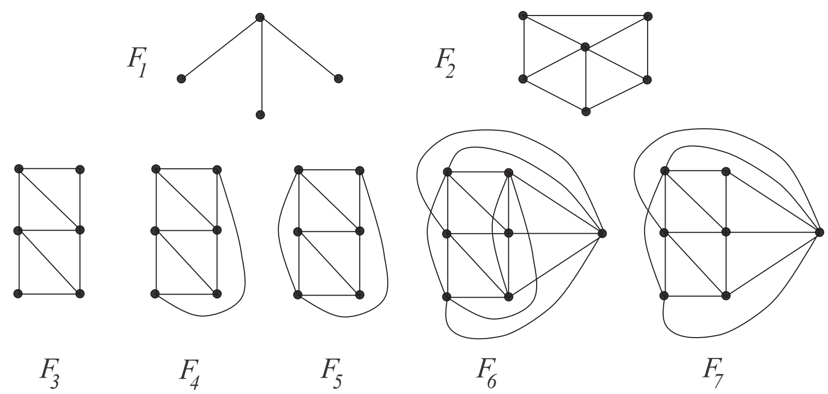


Figure 1.3: $F_1, F_2, \ldots, F_7$ are the forbidden induced subgraphs for line graphs of multigraphs.

1. *The graph G is the line graph of some simple graph.*

2. *The edges of G can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subgraphs.*

3. *The graph G does not contain any of the graphs in Figure 1.2 as an induced subgraph.*

Theorem 1.4. *Line graphs of multigraphs do not contain any of the graphs in Figure 1.3 as an induced subgraph.*

These characterizations, in particular the characterization about paritioning edges into complete graphs, will prove instrumental to our study.

At this point, one would hope to find an example of a graph that is not list χ-colorable for each forbidden subgraph to show that the list edge coloring conjecture may be the best possible class of graphs that can be χ-list colored. We have given above in Figure 1.1 an example which contains the claw, which is the complete bipartite graph with bipartition $\{X, Y\}$ where X contains one vertex and Y contains three vertices (i.e. $K_{1,3}$). Unfortunately examples containing the remaining forbidden subgraphs have proven difficult enough to inspire the following conjecture by Gravier and Maffray[3].

Conjecture 1.4.1 (Gravier and Maffray [3])**.** *Every claw-free graph G has $\chi_l(G) = \chi(G)$.*

The conjecture Gravier and Maffray is a more general than the original list edge coloring conjecture. It was proven by Maffray and Gravier [5] in 2004 that every claw-free perfect graph with $\chi(G) = 3$ has $\chi_l(G) = 3$, where a perfect graph is a graph in which the chromatic number of every induced subgraph is the size of the largest clique in that subgraph. Further study into finding larger classes of graphs which have this equality have been studied by Maffray, Gyarfas, and Esperet [6] [3] but no other class has been reported.

1.3 Stable Matchings

One of the proofs of Galvin's Theorem that will be provided in Chapter 2 uses the notion of *stable matchings*. A *matching* is a set of edges of a graph G such that no two edges are adjacent. Let L_v be a set of edge preference lists which is a complete ordering of the edges adjacent to any vertex $v \in V(G)$. We define a stable matching as follows: a matching M of G is said to be stable with respect to a set of preference lists L_v if, for any edge $e \notin M$ there exists some edge e' incident to e at $w \in V(G)$ such that $e' \in M$ and w prefers e' to e. The Gale-Shapley Theorem gives a characterization of bipartite graphs that a graph is bipartite if and only if for any preference lists, there exists a stable matching [8] [11].

Theorem 1.5 (The Gale-Shapley Theorem [8])**.** *Let G be a bipartite graph with edge preference lists L_v. Then there exists a stable matching of G with respect to L_v.*

There are several variations on an algorithmic proof that have been given to establish Theorem 1.5 in polynomial time, giving nice constructions for stable matchings. All have roughly the same steps. Create the set of all highest preferences of one part of the graph, and then remove from the graph some edges if this set is not independent. Repeat this process until the set is independent, at which point, one will find that the remaining set is indeed a stable matching. Below is roughly the proof given by D. West in his textbook, Introduction to Graph Theory [7].

Proof. **Algorithm:** Input preference rankings by n men and m women with $n \leq m$.

Iterate by having each man propose to the highest woman on his preference list who has not rejected him. If each woman receives at most one proposal, stop. Otherwise, every woman receiving more than one proposal rejects all men except the one that is the highest on her preference list. Every woman receiving a proposal says "maybe" to the man that she does not reject each round.

Claim 1: The algorithm provides a matching if it terminates.

If the stopping condition is reached, we have a matching since each man proposes to at most one woman, and each woman has at most one man to which she has said "maybe." So we need only observe that no man can be rejected by every woman. Suppose that a man is rejected by every woman, then at each iteration, the woman that he proposed to had also be proposed to by another man. But this means that each woman that the man has proposed to has some "maybe." after he proposes, and since men cannot propose to the same woman after being rejected, the man can be rejected at most $n - 1$ times. Further, at the end of the algorithm, every man must be married and no woman has two proposals.

Claim 2: The algorithm terminates.

Observe that the list of each male is non-increasing in size as the algorithm iterates and that if no male's list changes, the algorithm stops with a matching. Hence we have a strictly decreasing list (of all male preferences) which may terminate before the list is empty if no individual male's list decreases.

Claim 3: The algorithm provides a stable matching.

Suppose that the algorithm terminates, and for contradiction that the matching is not stable. Hence there exists some edge such that it is preferred to both the man and woman but is not a part of the matching. But, the man would have proposed to the woman before his current match, and the woman must have rejected him, meaning that she had a higher preference as a "maybe" and women will only reject for higher preferences so whoever the woman is married to must have a higher preference than the man, a contradiction with the edge being unstable.

□

We have now covered some preliminary results which are necessary as we progress through the main goals of this thesis. In chapter two, we will provide three proofs of Galvin's Theorem. Two of which were provided originally by Galvin [1], one proof by induction and one using stable matchings and the Gale-Shapley Theorem. As a part of this, we characterize when a graph has a stable matching with respect to its preference lists. Then we will provide a proof which uses a more general result, proven by Borodin, Kostochka, and Woodall [4]. In chapter three we will observe the relationships between the techniques of the proofs of Galvin's Theorem and stable matchings and two alternative proofs of the Gale-Shapley Theorem. We also introduce two notions which generalize stable matchings.

Chapter 2

Galvin's Theorem

In this chapter, we present three proofs of Galvin's theorem. We will first see a key lemma due to Bondy, Boppana, and Siegel, first reported by Galvin [1] and other common elements of each proof in section 2.1. We give the proofs due to Galvin in section 2.2. As part of this, we characterize when any graph has a stable matching. In section 2.3 we will discuss and prove a result due to Borodin, Kostochka, and Woodall and demonstrate how it implies Galvin's Theorem..

2.1 Shared Elements of the Proofs of Galvin's Theorem

A graph D is *directed* if each edge $(v_1, v_2) \in E(D)$ has an *orientation*, that is to say that it is an ordered pair. We say that $(v_1, v_2) \in E(D)$ points from v_1 to v_2. Bondy, Boppana and Siegel's result on directed graphs is based on the use of kernels. A *kernel* of a directed graph D is a set $K \subset V(D)$ such that K is *independent* (which is to say that for any $v_1, v_2 \in V(D)$ there is no edge (v_1, v_2) such that both $v_1 \in K$ and $v_2 \in K$) and such that for any $v \notin K$, there exists at least one edge $(v, k) \in E(D)$ such that $k \in K$. We will also use the term *kernel-perfect* to describe an directed graph D in which every subgraph of D has a kernel.

Theorem 2.1 (Bondy, Boppana, Siegel (see [1]). *Let H be a graph and $(S_v)_{v\in V(H)}$ a family of lists. If H has an orientation D such that $d^+(v) < |S_v|$ for every v and such that D is kernel-perfect, then H can be colored from the lists S_v.*

Proof. Proceed by induction on $|V(H)|$. If $|V(H)| = 1$ then by assumption, $|S_v| \geq 1$ hence the graph can be colored.

Let $|V(H)| > 1$ and suppose for all graphs of with vertex set of size $n < |V(H)|$ the lemma holds. Let α be a color appearing in at least one list. Consider the subgraph induced by all vertices containing α in their list. By supposition this subgraph has a kernel U. Color each vertex in U with the color α in H. Let H' be the graph $H \setminus U$ with lists $S'_v = (S_v \setminus \{\alpha\})_{v \in H \setminus U}$. Note that for a vertex $v \in V(H)$ if $\alpha \notin S_v$ in H, $d^+_{H'}(v) < |S_v| = |S'_v|$. Otherwise $v \notin U$ so $d^+_{H'}(v)$ is strictly less than $d^+_H(v)$ since U was a kernel of a graph containing v, so v had at least one edge to a vertex in U. $|S'_v|$ is exactly one less than $|S_v|$, hence $d^+_{H'}(v) < |S_v|$. Therefore for all $v \in H'$, $d^+(v) < |S_v|$ and by supposition since H' is an induced subgraph of H, every induced subgraph of H' has a kernel and $|H'| < |H|$. By induction hypothesis H' is colorable, with no vertices using α as their color, so H is colorable. □

Each of the proofs that we provide of Galvin's Theorem 1.2 will conclude by applying the result of Bondy, Boppana, Siegel. To this end, we will must prove that we can provide an orientation of the line graph of any bipartite graph such that the out degree of each vertex of the line graph is less than the size of each list (in this case, $\chi(G)$). We give three different proofs of the fact that the given oriented graph is indeed kernel-perfect.

Let G be a bipartite graph and let c be a proper edge coloring of G. Define $D(G, c)$ to be an orientation of $L(G)$ such that for any $e_1, e_2 \in E(G)$ with e_1 incident to e_2 and $c(e_1) < c(e_2)$, the edge $(e_1, e_2) \in E(L(G))$ is oriented as follows:

- If e_1 and e_2 meet only in X, orient from e_1 to e_2.
- If e_1 and e_2 meet only in Y, orient from e_2 to e_1.
- If e_1 and e_2 meet both in X and Y then by our construction of the line graph, there exist two parallel edges (e_1, e_2) so orient one from (e_1, e_2) and the other from (e_2, e_1).

Lemma 2.2 (Galvin [1]). *Let G be a bipartite graph with bipartition X, Y. Let c be a k-edge coloring of G where $k = \chi'(G)$. For each $v \in V(D(G, c))$, $d^+(v) < k$.*

Proof. Consider $e \in V(D(G, c))$. Let $c(e) = i$. Let $e' \in V(D(G, c))$ be adjacent to e such that $e'e$ is oriented from e' to e. If e' is not a parallel edge, then e' meets e only in X, so

$c(e') \in \{1, 2, 3, \ldots, i-1\}$ or if e' meets e only in Y, so $c(e') \in \{i+1, i+2, i+3, \ldots, k\}$ by construction. Then each color appearing adjacent to e not from a parallel edge is from the set $\{1, 2, \ldots, i-1, \hat{i}, i+1, i+2, \ldots, k\}$ so at most $k-1$ colors are adjacent from not parallel edges. Notice that if e' is a parallel edge, $e' \in \{1, 2, \ldots, i-1, \hat{i}, i+1, i+2, \ldots, k\}$ but e' is adjacent to all edges e is adjacent to, so $c(e')$ cannot be duplicated in either previous set since c is a proper edge coloring. Similarly, any two edges meeting e both in X or Y cannot have the same color. Hence, $d^+(e) \leq |\{1, 2, \ldots, i-1, \hat{i}, i+1, i+2, \ldots, k\} = k-1 < k$. □

With Lemma 2.2 proven, it remains only to show the following lemma so that we can apply the result of Bondy, Boppana, and Siegel to complete our proofs of Galvin's Theorem.

Lemma 2.3 (Galvin [1])**.** *Let G be a bipartite graph with bipartition $\{X, Y\}$ and c be a k-edge coloring of G where $k = \chi'(G)$. Then $D(G, c)$ is kernel-perfect.*

2.2 Galvin's Proofs

We now move to Galvin's two proofs showing $D(G, c)$ is kernel-perfect. We first go through a proof using the Gale-Shapley Theorem, using the fact that a stable matching in G will provide a kernel of $L(G)$.

We first define $D(G, L_v)$. Let G be a graph. Let L_v be a set of edge preference lists for each vertex $v \in G$. $D(G, L_v)$ is the orientation of $L(G)$ oriented as follows: for $e_1, e_2 \in V(L(G))$ with $(e_1, e_2) \in E(L(G))$ where e_1 meets e_2 at a vertex $v \in V(G)$ orient (e_1, e_2) from e_1 to e_2 if and only if $e_1 <_v e_2$.

Theorem 2.4. *A graph G has a stable matching with respect to edge preference lists L_v if and only if $D(G, L_v)$ has a kernel.*

Proof. We will first show that a kernel of $D(G, L_v)$ provides a stable matching of G. Let K be a kernel of $D(G, L_v)$. Let $k_1, k_2 \in K$. Then there are no edges between k_1 and k_2 since K is independent, therefore the k_1 and k_2 do not meet at any vertex. Hence, $K \in G$ is a matching. Let $k' \in V(D(G, L_v) \setminus K$, then since K is a kernel, there exists some edge (k, k')

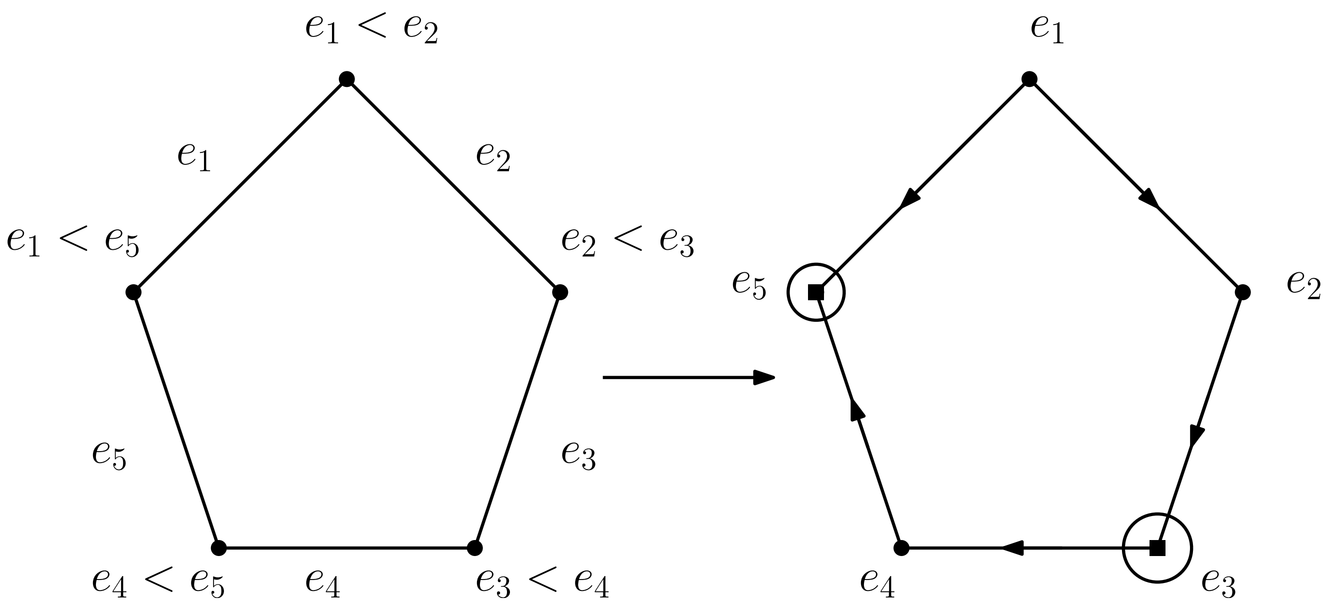


Figure 2.1: Here we see a graph that is not bipartite, with which we can apply the technique of checking $D(G, L_v)$ to determine that there is a stable matching. The kernel of $D(G, L_v)$ is circled on the right, which is indeed a stable matching of the graph on the left.

oriented from k' to k so by our definition of $D(G, L_v)$, k meets k' at a vertex v, and $k' <_v k$. Therefore, K is stable in G. Hence we have a stable matching of G as desired.

We now show that a stable matching of G with respect to preference lists $L(v)$ is a kernel. Let M be a stable matching of G and consider $M \in D(G, L_v)$. Let $m_1, m_2 \in G$. Since M is a matching, m_1 and m_2 do not meet at any vertex in G, therefore in the line graph, there is no edge (m_1, m_2). Thus, M is independent in $D(G, L_v)$. Let $m' \in E(G) \setminus M$. Since M is a stable matching, there exists some vertex v and some edge m such that m' and m meet at v and $m' <_v m$, then the edge (m', m) exists in $D(G, L_v)$ and further by our definition of $D(G, L_v)$, it is directed from m' to m. In particular, M absorbs all vertices not in M. Therefore M is a kernel of $D(G, L_v)$.

□

We will now use Lemma 2.4 to prove Lemma 2.3 completing our first proof of Galvin's Theorem.

Proof. (Proof 1 of Lemma 2.3) Let H be an induced subgraph of $D(G, c)$. To use the Gale-Shapley Theorem on G define preference lists for all $v \in V(G)$ by $e_1 <_v e_2$ if the edge $e_1 e_2 \in L(G)$ is directed from e_1 to e_2. We want to show H has a kernel. To see this, we show that a stable matching in G induces a kernel in $D(G, c)$, and as such, any subgraph of G created by removing edges will induce a kernel in an induced subgraph of $D(G, c)$. Let W be a stable matching in G which exists since G is bipartite.

We proceed with contradiction to show that W is a kernel in $D(G, c)$. Suppose that W is not a kernel in $D(G, c)$. Then there exists $e \in V(D(G, c))$ such that e does not have an edge oriented into the vertices $W \in V(L(G))$. If $e \in W$, then W can still be a kernel, so $e \in E(G) \setminus W$. If e is not adjacent to any edges in W, then e must be in W since W is a stable matching. So e is adjacent to some edge $e_1 \in W$, oriented from e_1 to e in $D(G, c)$. But then the vertex v_1 at which they meet has $e_1 <_{v_1} e$, so if e is not adjacent to another edge in W, this is not a stable matching. So e is adjacent to another edge $e_2 \in W$. Similarly, the edge in $D(G, c)$ must be oriented from e_2 to e. So by our preferences, the vertex v_2 at which e_2 and e meet has $e_2 <_{v_2} e$. Since $e_1 <_{v_1} e$ and $e_2 <_{v_1} e$, e is preferred at both vertices to the current pairing, in contradiction with W being a stable matching. Therefore, W must be a kernel.

□

Figure 2.2 provides a small example with which to check the above method of proof, with colors solid<dashed and the kernel of the line graph found with this method in squares.

This proof is perhaps the most simple of any that will be done to prove Lemma 2.4, but that is merely because it relies on the already known result of Gale-Shapley, meaning some of the work is concealed. That being said, this demonstrates a significant relationship between Gale-Shapley's Theorem and Galvin's Theorem, which is the guide for the rest of this thesis. The next proof was also provided by Galvin [1], but relies on no outside results. This is an inductive, self-contained proof.

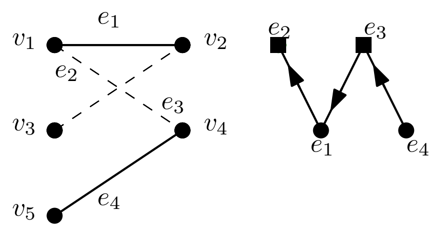


Figure 2.2: A small example with which to check proof 1 of lemma 2.3.

Proof. (Proof 2 of Lemma 2.3) We proceed by induction on $|V(H)|$ where H is an induced subgraph of $L(G)$. Suppose that $|V(H)| = 1$ then the vertex in H is a kernel of H. For each $x \in X$ induced by the set $V(H)$ let $e_x \in V(H)$ be the edge such that $c(e_x)$ is less than all the colors of each other edge incident to x that is in $V(H)$. Let $U = \{e_x\}$ for all $x \in X$. Let $e' \in V(H) \setminus U$. Clearly e' is adjacent to some $e_x \in U$ since G is bipartite. Furthermore, the edge $(e, e') \in H$ is directed from e' to e. So every vertex not in U has an edge into U.

It remains to show that U is independent. Suppose there exist some $e, e' \in U$ such that e and e' are adjacent. Let $c(e) < c(e')$. Since U is defined to take only one edge incident to each vertex in X, e and e' must meet in Y. So by our orientation, (e, e') is oriented from e to e'. By our induction hypothesis, the graph $V(H) \setminus \{e\}$ has a kernel U'. If $e' \in U'$ then U' is a kernel in H since (e, e') is directed from e to e'. So suppose $e' \notin U'$. Then there exists some edge $e'' \in U'$ such that $e'e''$ exists and is directed from e' to e''. Notice that again e' and e'' cannot meet in X since e' was chosen to have the least color for its corresponding vertex in X so $c(e') < c(e'')$, a contradiction with our orientation. Hence, e' and e'' meet in Y and $c(e') < c(e'')$ but also e and e' meet in Y, therefore e and e'' meet in Y. Also $c(e) < c(e') < c(e'')$, so since $c(e) < c(e'')$ and they meet in Y, (e, e'') is directed from e to e'', so e has an edge into U', therefore U' is a kernel in H. □

2.3 A Theorem of Borodin, Kostochka, and Woodall

Borodin, Kostochka, and Woodall provide an alternative proof of Galvin's Theorem through the following general theorem.

Theorem 2.5 (Borodin, Kostochka, and Woodall[4]). *An orientation D of a line-graph G is kernel-perfect if and only if every oriented odd cycle in D has a chord and every clique has a kernel.*

The theorem is self-contained, in that it does not rely on any major results like the Gale-Shapley theorem inherently, and can be applied to $D(G, c)$ for any bipartite graph G. The proof is both inductive and algorithmic, which gives some insight as to the depth of the theorem. Again, this theorem merely serves to provide the kernel-perfect result desired in Lemma 2.3.

Before going through this extensive proof, we will assume it and show how, if indeed it is true, this theorem implies Galvin's Theorem. For completeness, after we have applied it to Galvin's theorem, we will go back to give Borodin, Kostochka, and Woodall's proof [4].

Proof. (Proof 3 of Lemma 2.3) We want to use Borodin, Kostchka, and Woodall's Theorem to prove this. Namely, we must show that the $D(G, c)$ has the following properties:

1. Every clique has a kernel.

2. Every odd directed cycle has a chord.

We will consider the graph as described by Beineke's characterization by separating the vertices into cliques (representing each vertex). For each $v \in V(G)$ let K_v be the clique in $D(G, c)$ created by all edges in $E(G)$ meeting at v. Notice that each clique has a kernel, since for any $v \in X$, the highest colored edge is a sink of K_v. Respectively, for any $v \in Y$ the lowest colored edge is a sink of K_v.

Let C be a cycle in $D(G, c)$.Either the cycle contains at most one edge in K_v, or it contains more than one edge of at least K_v.

Suppose C contains at most one edge of each clique, then in the original bipartite graph, an induced cycle can be created in G which will, indeed, be a cycle since each clique of $L(G)$ is visited at most once, so each vertex will be visited at most once in G. Thus C corresponds to a cycle in G, hence C is not odd.

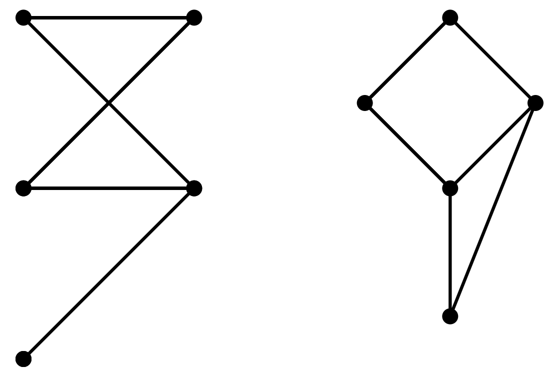


Figure 2.3: On the left is a bipartite graph, and on the right we have the line graph without orientation so that the reader can check any number of orientations.

Suppose that C contains at least two edges of K_v for some $v \in G$. If it uses all edges, it must be K_3, else C is not a cycle. If indeed it is K_3, since every K_v has a kernel $C = K_3$ has a sink and is not a directed cycle. So at least one edge e in K_v is not in C, in particular, that edge is a chord. □

See Figure 2.3 for an example on how the argument above works. This completes the third and final proof of Galvin's Theorem. We will now continue to discuss in greater detail the relationship between Galvin's Theorem and the Gale-Shapley Theorem in chapter 3.

Prior to working through the details of this general theorem, we will discuss the ideas behind each part of the proof. This is an inductive proof with an algorithm to find a kernel in a smallest counterexample, a contradiction. We first define a "preference list" based on each clique, and will use these preferences to create a kernel. We then create a set of "proposals" being the highest preferred each for each vertex like in the traditional proof of the Gale-Shapley theorem. In case 1, you will find a striking similarity to the induction proof provided by Galvin to find a kernel using the removal of one edge. In case 2 we find first of all that our set of highest preferences create only even cycles. We then label these even cycles in increasing order (paying special attention to the parity of each label) to either remove some set of the even cycles to create a smaller graph with a kernel similar to the inductive proof by Galvin, or we find an odd cycle with no chords if there is no set that can be removed. By ensuring in the algorithm that we never relabel a cycle we ensure that the algorithm terminates since our graph is finite.

Proof. (Proof of Theorem 2.5) Let G be the line graph of some multigraph H and proceed by induction on the number of edges in H. For $|E(H)| = 1$ the theorem is clearly true.

Suppose H is the multigraph with the least number of edges such that the theorem is not true, and let D be an orientation of $L(H)$ such that every odd cycle in D has a chord and every clique has a kernel. Since D is the smallest such counter example, D has no kernel and all subgraphs of D do contain a kernel.

Since each clique in D has a kernel we have a tournament, therefore we can create a topological sort which allows us to create the following labels. For each $v \in V(H)$, each edge e incident with v can be labelled by a number $l_v(e)$ so that the different edges get different labels and $l_v(e') < l_v(e'')$ implies that (e', e'') is an arc in D. We will treat this labelling $l_v(e)$ as a preference list, saying that v prefers e'' to e' if $l_v(e') < l_v(e'')$. For every vertex $v \in V(H)$, let $e(v)$ denote the edge incident to v with the maximum label $l_v(e)$. Then v prefers $e(v)$ to any other edge it is incident to.

Suppose $e(v) = e(w)$ for some $v, w \in V(H)$ such that $v \neq w$. Consider the line graph $L(H - v, w)$ which has a kernel Q by minimality of H. Therefore $Q \cup \{e(v)\}$ is a kernel in D, a contradiction. Notice that we must remove both v and w from H to ensure that the set $Q \cup \{e(v)\}$ is independent. Therefore all $e(v)$ are distinct.

Let $M = \{e(v) | v \in V(H)\}$, the set of all edges with highest preference for each vertex. Since all $e(v)$ are distinct, the ends of any $e \in M$ can be marked as $x(e)$ and $y(e)$ such that $x(e)$ is the vertex which has e as its highest preference, and $y(e)$ is the other end. Note that $y(e)$ does not have e as its highest preference for any e.

<u>Case 1.</u> For some $e_1 \in M$, there exists $e_2 \in E(H)$ incident with $y(e_1)$ such that $l_{y(e_1)}(e_2) < l_{y(e_1)}(e_1)$, in other words, $y(e_1)$ prefers e_1 to e_2.

Choose such an e_2 with lowest preference with respect to $y(e_1)$. We proceed similarly to proof 2 of lemma 2.3 by deleting vertex e_2 to get a kernel W in $D - e_2$. Notice that if e_1 is in the kernel, we are done since there is an edge from e_2 to e_1 and adding e_2 will not affect independence of the kernel. Otherwise, e_1 is the highest preference of $x(e_1)$ so it cannot point to anything incident to $x(e_1)$. Therefore e_1 points to some edge e_3 incident to $y(e_1)$ in the kernel. Therefore e_3 is preferred to e_1 at $y(e_1)$, but e_1 is preferred to e_2 hence e_3 is preferred to e_2. Therefore there exists an edge pointing from e_2 to e_3. So again, we have a kernel in D.

Case 2. For each $e_1 \in M$ and every $e_2 \in E(H)$ incident with $y(e_1)$ and distinct from e_1, $l_{y(e_1)}(e_1) < l_{y(e_1)}(e_2)$. Meaning that each edge $e \in M$ has the lowest preference amongst all edges incident to $y(e)$.

Notice that all $y(e)$ for all $e \in M$ should be distinct, otherwise two edges are incident to one vertex and $y(e)$ must have a preference between the two, so one of the is not the least preference. Note also that $|M| = |V(G)|$ since each $e(v)$ is distinct. Therefore each vertex must be $y(e)$ for exactly one $e \in M$, so M forms a 2-factor in H. Suppose one of those cycles is odd, then that creates an odd directed cycle in D because of the assumption of our case. But this cycle is not chorded, because any chord would imply that an edge was incident to three vertices, a contradiction with the assumption that all directed odd cycles were chorded. Therefore all cycles formed by M must be even.

We now describe an algorithm with which to find either a kernel in D or a directed odd cycle without chords. In either case, we will have arrived at a contradiction, so D will be kernel-perfect.

Step 0. Among the cycles created by the edges in M, choose an arbitrary cycle C_1. Label the vertices of C_1 $v_{1,1}, v_{1,2}, \ldots, v_{1,2r_1}$ where C_1 contains $2r_1$ vertices such that $v_{1,i}$ is adjacent to $v_{1,i-1}$ and $v_{1,i+1}$ and $v_{1,1}$ is adjacent to $v_{1,2r_1}$. Set $W_1 = \{v_{1,2j} | 1 \leq j \leq r_1\}$ and $B_1 = V(C_1) \setminus W_1$ so W_1 is all even labelled vertices and B_1 is all odd labelled vertices. Proceed to step 1.

Step k. Stop if either

(i) W_k is not independent in H; or

(ii) no vertex in $V(H) \setminus (B_k \cup W_k)$ is adjacent to W_k.

Otherwise, choose a vertex $v \in V(H) \setminus (B_k \cup W_k)$ adjacent to W_k which must exist or we would have stopped in (ii). Let $C_k + 1$ be the cycle in M containing v. Let $v = v_{k+1,1}$ and label the vertices in the cycle with increasing second index clockwise similarly to how C_1 was labelled. Set $W_{k+1} = W_k \cup \{v_{k+1,2j}$ and $B_{k+1} = B_k \cup \{V(C_{k+1}) \setminus W_{k+1}\}$, so W_{k+1} is all vertices that have appeared in a chosen cycle of M given an even label, and B_{k+1} is all those which have an odd label. Go to step k+1.

Notice that this algorithm must terminate since there are only a finite number of cycles in M able to be chosen, and each step uses a new cycle in M. Suppose we terminate in step m. Assume that W_m is independent in H, so we are in stopping condition (ii). Then each edge in H incident with W_m must also be incident to B_m, otherwise we have a vertex adjacent to W_m that is in $V(H) \setminus (B_k \cup W_k)$, a contradiction with the stopping condition. Since H is chosen to be the smallest graph such that its orientation D has no kernel, the subgraph of D induced by $H - W_m - B_m$ has a kernel, call it Q, or is empty. Let $M' = \{e(v) | v \in B_m\}$ is the set of all of the highest preferences of B_m is a matching, and absorbs all vertices in D corresponding to edges incident with B_m. Notice now that all edges in H but not in $H - W_m - B_m$ are incident to B_m, hence $Q \cup M'$ is a kernel in D, a contradiction.

Now assume we are in stopping condition (i) so W_m is not independent in H. Let $e = (a, b)$ be an edge such that $a, b \in W_m$. Let $a \in C_p$ and $b \in C_q$. Suppose that $p = q$. Since e goes between two even labelled vertices, e creates two odd cycles with C_p. Notice that one of these cycles will induce a directed odd cycle, since we can take the cycle created by following e to the highest preference of b around to e, where the other edge on the cycle adjacent to e is the lowest preference of a. So we have an odd directed cycle, and similarly to before, we cannot have a chord similar to before, a contradiction with our assumption that every directed odd cycle had a chord. Suppose instead $p \neq q$. There exist some paths along edges only contained in the cycles of M or the edge from $v_{k,1}$ to the cycle C_k for $2 \leq k \leq m$ chosen in the algorithm following the edges of highest preference from $v \in C_1$ adjacent to $v_2, 1$ to

a which goes through cycles of strictly increasing index, call it P_a, and a similar such path to vertex b following along lowest preferences, call it P_b. Let t be the index of the cycle of highest index which contains edges used in both paths P_a and P_b. We begin our odd cycle by first following P_a from C_t to a, then to b. From b, follow P_2 back to C_t, and either the cycle is complete, or we must follow part of C_t to close the cycle, in which case follow along lowest preferences in C_t. Note that this is an odd cycle because by construction we always go from an even labeled vertex to an odd, or odd to even in cycles, and to move from one cycle to another we must move from an odd labelled vertex to an even labelled except for (a, b), which goes from even to even. This cycle creates an odd directed cycle due to the preference designations of the paths, and is not chorded.

□

This concludes our discussion of the three proofs of Galvin's Theorem. In chapter 3 we will use the techniques of these three proofs on stable matchings to find two alternative proofs of the Gale-Shapley Theorem and discuss two new generalizations of stable matchings.

Chapter 3

The Gale-Shapley Theorem

We will begin with two alternative proofs of the Gale-Shapley Theorem in Section 3.1. The second proof is by induction (similar to the induction proof of Galvin's Theorem) which leads to two generalizations of stable matchings which we will explore in Section 3.2. Our first alternative proof of the Gale-Shapley Theorem will rely on the result of Borodin, Kostochka, and Woodall, and will be very similar to the application of the theorem to Galvin's Theorem.

3.1 Two Alternative Proofs of the Gale-Shapley Theorem

Below we use Lemma 2.4 and Theorem 2.5 to give our first alternative proof of the Gale-Shapley Theorem.

Proof. (Alternative Proof 1 of Theorem 1.5, The Gale-Shapley Theorem)

Let G be a bipartite graph with preference lists L_v. To use Lemma 2.4 we will consider the graph $D(G, L_v)$ to find a kernel which induces as stable matching of G.

We will apply the theorem of Borodin, Kostochka, and Woodall to find a kernel of $D(G, L_v)$. To do so, we need to show two things about the graph $D(G, L_v)$

- Every clique in $D(G, L_v)$ has a kernel
- Every oriented odd cycle of $D(G, L_v)$ has a chord

To show that every clique in $D(G, L_v)$ has a kernel. Since G is bipartite, each kernel is induced by some vertex $v \in V(G)$ via Beineke's Theorem, label each kernel K_v. Since K_v is induced by v and v has some highest preference of edges, that edge induces a sink in K_v. In particular, each clique in $D(G, L_v)$ has a kernel.

We now want to show that every oriented odd cycle of $D(G, L_v)$ has a chord. This argument is identical to that of Proof 3 of Lemma 2.3. We note that any any cycle contains edges of some largest K_v in terms of vertices. If it uses all of the edges of K_v either that K_v was a clique on two vertices, or three vertices, since any larger clique is not a cycle. If K_v was a clique on two vertices, C induces a cycle in G, hence it is even. If K_v is a clique on three vertices, it is all of C, in particular C is a clique, hence has a kernel so it is not a directed odd cycle. If there exists some K_v which has edges in C for which some edge of K_v is not in C, that edge is a chord. So every oriented odd cycle has a chord. By Borodin, Kostochka, and Woodall's Theorem $D(G, L_v)$ is kernel-perfect, in particular it has a kernel. Therefore G has a stable matching by Lemma 2.4.

□

It is important to note here that we found that $D(G, L_v)$ was kernel-perfect, a much stronger condition than was necessary for finding a stable matching. We can also use the first proof of Lemma 2.3 to produce another alternative, non-algorithmic proof of the Gale-Shapley Theorem which states that for a bipartite graph and any preference lists, there exists a stable matching.

Proof. (Alternative Proof 2 of Theorem 1.5, the Gale-Shapley Theorem)

Let G have bipartition $\{X, Y\}$.

We first note that a kernel of this oriented line graph corresponds to a stable matching of G. Let K be a kernel of $D(G, L_v)$. K is independent by definition, hence the corresponding edges of G are not incident at any vertex, so each vertex has degree at most 1. In particular, we have a matching. Now let e be an edge not in the matching induced by K. Then in $D(G, L_v)$, e is absorbed by some vertex $e' \in K$. Therefore at a vertex $w \in G$ where e and e' meet, $e <_w e'$. In particular, for any edge $e \notin K$, at least one vertex incident to e prefers the edge in K to which it is incident to e, so the matching is stable.

Now we must show that a kernel always exists in $D(G, L_v)$ which will be shown by induction. Suppose G has one edge, e. Clearly e is a kernel of $D(G, L_v)$.

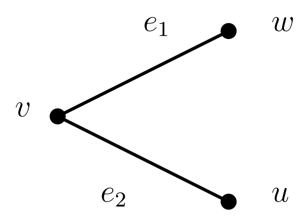
Suppose for all G with $|E(G)| \leq n$ there exists a kernel of $D(G, L_v)$. Let G be a bipartite graph with preference lists L_v, total orderings of edges adjacent to each vertex $v \in G$, with $n+1$ edges. Let e_v be the highest preference of v with respect to L_v for each $v \in V$. Let $e_X = \{e_v | v \in X\}$. Claim: If e_X is independent, e_X is a kernel of $D(G, L_v)$. Let $e \notin e_X$. Since G is bipartite, e has one end in X, so there exists $x \in X$ such that $e <_x e_x$ so in $D(G, L_v)$ (e, e_x) is directed from e to e_x. So e_X absorbs all vertices not in e_X.

Suppose e_X is not independent. Then for some $v, w \in V(G)$ e_v is adjacent to e_w, but they cannot meet in X since we chose only one edge incident to each vertex in X. Therefore they meet in Y, call the vertex at which they meet y. Since L_y is a total ordering, one edge is preferred by y to the other. Without loss of generality, say $e_w <_y e_v$. We consider the graph $G^- = G \setminus e_w$ with preference lists $L_v \setminus e_w$. Then G^- has n edges, and by assumption has a kernel K of $D(G^-, L_v)$. We consider K in $D(G, L_v)$. K is independent and absorbs all edges of $G \setminus K$ except potentially e_w. Notice that if $e_v \in K$, since e_v and e_w meet at y and $e_w <_y e_v$ the edge $(e_w, e_v) \in D(G, L_v)$ is oriented from e_w to e_v so e_w is absorbed by K. If $e_v \notin K$ then e_v is absorbed by K. Therefore there exists some edge $e' \in K$ such that $e_v <_z e'$ but e_v is the highest preference of v, so e' must meet e_v at y. In particular $e_v <_y e'$. But also $e_w <_y e_v$ so $e_w <_y e_v <_y e'$, hence $e_w <_y e'$ so the edge (e_w, e') is an edge of $L(G)$ and is oriented from e_w to e', therefore e_w is absorbed by K. We now have that all edges of $D(G, L_v)$ are absorbed by K and K is independent, so K is a kernel of $D(G, L_v)$. So $D(G, L_v)$ has a kernel, as desired.

□

3.2 Generalizations of Stable Matchings

The second method of proof in Section 3.1 provides a slight generalization of Galvin's theorem. To see this, first we must generalize the concept of stable matching. Let G be a graph and $M \subset E(G)$ be a matching. We will say that a matching M is *loosely stable* if for any edge $e \notin M$ there exists some edge e' in M such that e is incident to e' and at



$$e_1 =_v e_2$$

Figure 3.1: An example of a bipartite graph with no strictly stable matchings, and a family of graphs with no strictly stable matchings.

a vertex at which they are incident, e' is preferred to e or e' is tied with e. We define a matching as *strictly stable* if for any edge $e \notin M$ there exists some edge e' in M such that e is incident to e' and at a vertex at which they are incident, e' is preferred to e. We now want to determine if the notions of strictly stable and loosely stable matchings are different from that of a stable matching.

Theorem 3.1. *There exist bipartite graphs G that do not have a strictly stable matching.*

As a proof, we notice that any graph containing Figure 3.1 provides on the left, a simple example of such a graph. Either $\{e_1\}$ or $\{e_2\}$. If the matching is $\{e_1\}$, it is unstable since $\{e_2\}$ is not incident to any edge in the matching to which it is less preferred. Similarly the matching $\{e_2\}$ is unstable, so there are no strictly stable matchings. Further, any graph which contains Figure 3.1 as a subgraph such that e_1 is the highest preference of w, e_2 is the highest preference of u, and v has no higher preference than e_1 and e_2 would also be unstable. We make a special note of this result since it demonstrates that we do not have a result analogous to the Gale-Shapley Theorem for strictly stable matchings, a notable difference in our definition.

To find a difference between loosely stable matchings and stable matchings we will have to redefine a trait analogous to kernels for graphs with ties. Here we will orient the same way as in $D(G, L_v)$ (pointing to higher preferences) but leave equal edges unoriented. We will define a kernel of a mixed graph G as an independent set of vertices $K \subset V(G)$ such that for any vertex $v \notin K$ there exists some vertex $w \in K$ such that v is incident to w and either v points to w or there is an undirected edge between v and w.

It is important to notice here that trying to find a larger class of graphs in which for any set of preference lists, there exists a stable matching is impossible. It has been proven that if a graph G has a stable matching for any preference list, then the graph is bipartite [11]. This forces us to relax some other conditions on stable matchings.

Theorem 3.2. *Let G be a bipartite graph with bipartition $\{X, Y\}$ and preference lists L_v for each $v \in V(G)$. For $x \in X$ the L_x has no ties, but for each $y \in Y$ L_y may contain ties. Then G has a stable matching.*

The proof of the theorem is identical to that of the ungeneralized proof above Theorem 1.5, only we break a tie arbitrarily if there are ties where the set e_X is dependent and replace all instances of $<$ with $\leq$ in the proof. We notice that we cannot allow ties in X. Otherwise, let $e_v \in E(G)$ be the highest preference of a vertex $v \in X$. The removal of e_v may take us out of our inductive hypothesis. That is to say that our highest preference of $v \in E(G) \setminus e_v$ may be a tie and we have not defined how to create the set e_X if there is a tie for highest preference in X. We notice now that this generalization is actually inferior to a quick observation that can be made once the idea of arbitrarily breaking ties has occurred. Indeed, if one breaks all ties arbitrarily we can simply apply the original theorem of Gale-Shapley to find a kernel, which will provide a loosely stable matching. A question that may be asked is if loosely stable matchings have any properties that differ from stable matchings. Indeed, we find that the following theorem about stable matchings is untrue of loosely stable matchings. Below is a theorem I learned from D. Hoffman in a course, but have included my own proof of the theorem for completeness, as his proof relies on a slightly different algorithm than the one provided in the introduction.

Theorem 3.3 (Hoffman)**.** *Let G be a bipartite graph and L_v be a set of edge preference lists for each vertex $v \in V(G)$. Let M and N of G be stable matchings of G with respect to L_v. Then $|M| = |N|$.*

Proof. We consider instead the graph H induced by the edges of $M \cup N$. Suppose that $|N| < |M|$. Let C be a component of H. Since G is bipartite, either C is an even cycle, or it is a path. If the component is an even cycle, there are the same numbers of edges in C from $|N|$ as there are from $|M|$. So there exists at least one path P such that $|P \cap N| < |P \cap M|$. Let $e \in E(H)$ be an end of P, which must be in $|M|$. Now let $e' \in E(H)$ be the edge adjacent to e meeting at a vertex $v \in V(P)$. Since e is an end of P then $e' <_v e$ else e' is unstable. But again since e is an end of P, and N is stable, $e <_v e'$ else e is unstable. But it cannot be the cease that both $e' <_v e$ and $e <_v e'$ a contradiction. Hence $|M|$ cannot be greater than $|N|$.

□

We now want to observe that this theorem about bipartite graphs with respect to stable matchings provides a difference between stable matchings and loosely stable matchings. The theorem below gives us such a difference and will be proven with Figure 3.2.

Theorem 3.4. *Let G be a graph and L_v be a set of edge preference lists for each vertex $v \in V(G)$. Let M and N of G be loosely stable matchings of G with respect to L_v. Then it is not neccessarily the case that $|M| = |N|$.*

Proof. See Figure 3.2. At the top of the figure, we have a graph G with preference lists given. Since we know that arbitrarily breaking ties will provide a loosely stable matching, on the bottom left we set $e_1 < e_2$. Here both of the middle vertices prefer e_2 to any other edge, hence $\{e_2\}$ provides a loosely stable matching of size 1. If we instead set $e_2 < e_1$ to break our tie (on the bottom right), we find the set $\{e_1, e_3\}$ is a loosely stable matching of size 2. We have now provided a bipartite graph with preference lists that have loosely stable matchings of varying sizes. □

As this example demonstrates, our definition of loosely stable matching is different from stable matchings thanks to this finding.

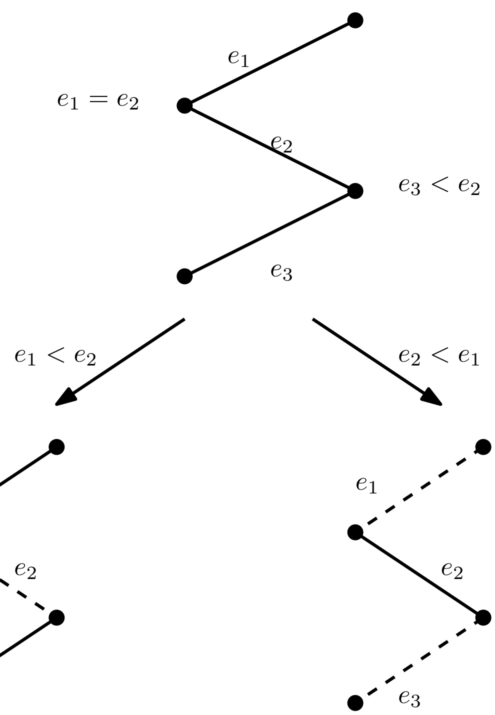


Figure 3.2: A graph with stable matchings of two different sizes.

Chapter 4

Conclusion

In this thesis, we studied the relationship between Galvin's Theorem and stable matchings. We showed two proofs of Galvin's Theorem which were originally provided by Galvin: a proof by induction which was nicely self-contained, and a proof which relied on stable matchings and the Gale-Shapley Theorem. The proof by induction led us to a non-algorithmic proof of the Gale-Shapley Theorem. We also provided a proof of Galvin's Theorem which used the powerful theorem of Borodin, Kostochka, and Woodall, which allowed us to give a second alternative proof of the Gale-Shapley Theorem, though this theorem was perhaps stronger than necessary to accomplish this goal. We also used the idea of creating directed line graphs (used in all of the proofs of Galvin's Theorem) to find a characterization of graphs with stable matchings. We recall that the first alternative proof of the Gale-Shapley Theorem in Section 3.1 provided a much stronger result than was necessary, that $D(G, L_v)$ was kernel-perfect for any bipartite graph G and any lists L_v. One may ask if there exists some weakened version of this proof technique which finds only a kernel, instead of kernel-perfect which may be applied back to Galvin's Theorem?

Through the second of the alternative proofs of the Gale-Shapley Theorem, we were motivated to give two new generalizations of stable matchings, loosely stable matchings and strictly stable matchings, both of which we have proven are indeed different from stable matchings. In finding this, several interesting questions arise. We found a forbidden subgraph with special preferences for a graph to have a strictly stable matching. One might ask if there are other forbidden subgraphs, or if there is a way to characterize graphs with strictly stable matchings based on forbidden subgraphs, analogous to Beineke's Theorem. Additionally, is there a polynomial time algorithm with which to find a strictly stable matching?

For loosely stable matchings, we noted that the traditional algorithm of the Gale-Shapley Theorem can be used to find a loosely stable matching. To demonstrate a difference between loosely stable matchings and stable matchings, we provided a bipartite graph with preferences for which the size of a loosely stable matching can vary. We then may ask if there exist connected bipartite graphs for which the difference in size of loosely stable matchings is unbounded? Without the constraint of connectedness, the answer is obvious as copies of Figure 3.2 provide any desired size difference. Additionally one might ask what the proportion of the size of loosely stable matchings might be. Our example provides a graph with proportion $\frac{1}{2}$, but can it approach 0? We note now that we can achieve a proportion of 1 simply by taking a bipartite graph with no ties, in which all loosely stable matchings will be stable matchings of the same size.

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