Atomic Characterization of L_1 And The Lorentz-Bochner Space $L^X(p,1)$ for $1 \leq p < \infty$ With Some Applications

by

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Abstract

In the 1950s, G. G Lorentz introduced the spaces $\Lambda(\alpha)$ and $M(\alpha)$, for $0 < \alpha < 1$ and showed that the dual of $\Lambda(\alpha)$ is equivalent to $M(\alpha)$ in his paper titled 'Some New Functional Spaces' (see [10]). Indeed, Lorentz mentioned that for the excluded value $\alpha = 1$, the space $\Lambda(1)$ is L_1 and $M(1)$ is L_∞. In 2010, De Souza [4] motivated by a theorem by Guido Weiss and Elias Stein on operators acting on $\Lambda(\alpha)$, showed that there is a simple characterization for the space $\Lambda(\alpha)$ for $0 < \alpha < 1$. The theorem by Stein and Weiss is an immediate consequence of the new characterization by De Souza. In this work, we seek to investigate the decomposition of L_1 which is the case $\alpha = 1$, and also extend the result to the well-known Lorentz-Bochner space $L^X(p, 1)$ for $p \geq 1$, and X is a Banach space, that is, the Lorentz space of vector-valued functions. As a by product, we will use these new characterizations to study some operators defined on these spaces into some well-known Banach spaces.

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Chapter 1

Introduction

1.1 Background

The Lorentz spaces, denoted $L_{p,q}$, were introduced by G. G Lorentz in the 1950s and they are generalizations of the traditional Lebesgue L_p spaces. These Lorentz spaces has been studied and generalized in many aspects by some authors. Examples of such generalizations include the (weighted) Lorentz-Orlicz Spaces (see [18, 19, 20]), Lorentz-Bochner Spaces (see [23]), Lorentz-Karamata Spaces (see [25]) and Lorentz Spaces with variable exponents (see [26]). For the purpose of this work, we will restrict ourselves to a special case of the Lorentz and Lorentz-Bochner Spaces.

We recall a few basic definitions, properties and notations. Throughout this work, $(T, \mathcal{M}, \mu)$ denotes a finite, complete, nonatomic measure space.

Definition 1.1. *Let f be a real-valued measurable function defined on T. The distribution function of f is the function $\mu_f : [0,\infty) \to [0,\infty)$ defined by*

$$\mu_f(\alpha) = \mu(\{x \in T : |f(x)| > \alpha\}), \quad for \quad \alpha \in [0,\infty).$$

Example 1.1. *For any given measurable set $A \in \mathcal{M}$, the distribution function of $f = \chi_A$ is given by $\mu_f(\alpha) = \mu(A)\chi_{[0,1)}(\alpha)$. Indeed, we can easily verify that if f is nonnegative simple function, that is,*

$$f(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x)$$

where the sets A_j are pairwise disjoint and $a_1 > a_2 > \cdots > a_n > 0$ then we have that

$$\mu_f(\alpha) = \sum_{j=0}^{n} b_j \chi_{[a_{j+1}, a_j)}(\alpha)$$

where $b_j = \sum_{k=1}^{j} \mu(A_k)$, for $j = 1, \cdots, n$, $b_0 = 0$ and $a_0 = \infty$.

For more information about the distribution function, we refer the reader to [9].

Definition 1.2. *Let f be a real-valued measurable function defined on T. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by;*

$$f^*(t) = \inf\{\alpha \geq 0 : \mu_f(\alpha) \leq t\}$$

Remark 1.1. *f^* is a decreasing function supported in $[0, \mu(T)]$.*

Example 1.2. *The decreasing rearrangement of $f = \chi_A$ is $f^*(t) = \chi_{[0,\mu(A))}(t)$*

The following are some useful properties of the decreasing rearrangement for which the details and other properties can be found in [9].

Proposition 1.1. *For any measurable function f, we have*

(1) $f^*(\mu_f(\alpha)) \leq \alpha$ whenever $\alpha > 0$.

(2) $\mu_f(f^*(t)) \leq t$, for all $t \geq 0$.

(3) f^* is right continuous on $[0, \infty)$. That is, $\lim_{t \to a^+} f^*(t) = f^*(a)$, for any $a \in [0, \infty)$.

(4) $t \leq \mu(\{x \in T : |f(x)| \geq f^*(t)\})$, since $\mu(T) < \infty$.

(5) $\|f\|_{L_\infty} = f^*(0)$.

Proposition 1.2. *For any $A \in \mathcal{M}$ and any measurable function f, we have*

$$(f\chi_A)^*(t) \leq f^*(t)\chi_{[0,\mu(A))}(t), \quad for \quad all \quad t \in [0, \infty).$$

Proof. Given $\lambda > 0$, we have that $\mu_f(\lambda) = \mu(\{x \in T : |f(x)| > \lambda\})$. Thus, for $A \in \mathcal{M}$,

$$\begin{aligned}
\mu_{f\chi_A}(\lambda) &= \mu(\{x \in T : |(f\chi_A)(x)| > \lambda\}) \\
&= \mu(\{x \in A : |f(x)| > \lambda\}) \\
&\leq \mu(\{x \in T : |f(x)| > \lambda\}) = \mu_f(\lambda)
\end{aligned}$$

That is, $\mu_{f\chi_A}(\lambda) \leq \mu_f(\lambda)$. Hence, for $t \geq 0$ we have that

$$C := \{\lambda > 0 : \mu_f(\lambda) \leq t\} \subseteq \{\lambda > 0 : \mu_{f\chi_A}(\lambda) \leq t\} =: D.$$

So, $(f\chi_A)^*(t) = \inf D \leq \inf C = f^*(t)$. That is, $(f\chi_A)^*(t) \leq f^*(t)$, for $t \geq 0$. Now for $t \geq \mu(A)$, we have that given $\lambda > 0$, $\mu_{f\chi_A}(\lambda) \leq \mu(A) \leq t$. That is, $\mu_{f\chi_A}(\lambda) \leq t$, for all $\lambda > 0$. Hence, $(f\chi_A)^*(t) = 0$, for $t \geq \mu(A)$ and thus,

$$(f\chi_A)^*(t) \leq f^*(t)\chi_{[0,\mu(A))}(t), \quad for \quad all \quad t \in [0, \infty).$$

□

Proposition 1.3. *For any $a \in (0, \infty)$, and measurable function f, there exist a measurable set $\tilde{A} \in \mathcal{M}$ such that $\mu(\tilde{A}) = a$ and*

$$(f\chi_{\tilde{A}})^*(t) = f^*(t)\chi_{[0,a)}(t), \quad for \quad all \quad t \in [0, \infty).$$

Proof. Let $A_1 = \{x \in T : |f(x)| > f^*(a)\}$ and $A_2 = \{x \in T : |f(x)| \geq f^*(a)\}$. We have, $A_1 \subseteq A_2$ and by (2) and (4) of Proposition 1.1, it follows that

$$\mu(A_1) = \mu_f(f^*(a)) \leq a \leq \mu(A_2).$$

That is, $\mu(A_1) \leq a \leq \mu(A_2)$. By the property of nonatomic measures, choose $\tilde{A} \in \mathcal{M}$ such that $A_1 \subseteq \tilde{A} \subseteq A_2$ and $\mu(\tilde{A}) = a$. We claim that $(f\chi_{\tilde{A}})^*(t) = f^*(t)\chi_{[0,a)}, \ \ for \ \ t \geq 0$. To see this, first we observe that $(f\chi_{\tilde{A}})^*(t) \leq f^*(t)\chi_{[0,a)}$ by Proposition 1.2. To get the reverse inequality, let $t < a$ and note that $f^*(a) \leq f^*(t)$. Thus, if $f^*(t) \leq \lambda$ then $f^*(a) \leq \lambda$. Moreover, for $\lambda > 0$ with $f^*(a) \leq \lambda$, we have

$$\begin{aligned}
\mu_f(\lambda) &= \mu(\{x \in T : |f(x)| > \lambda\}) \\
&= \mu(\{x \in \tilde{A} : |f(x)| > \lambda\}) + \mu(\{x \in \tilde{A}^c : |f(x)| > \lambda\}) \\
&= \mu(\{x \in \tilde{A} : |f(x)| > \lambda\}), \ \ \text{since} \ \ \tilde{A}^c \subseteq A_1^c \\
&= \mu_{f\chi_{\tilde{A}}}(\lambda)
\end{aligned}$$

That is, $\mu_f(\lambda) = \mu_{f\chi_{\tilde{A}}}(\lambda)$, for $\lambda > 0$ with $f^*(a) \leq \lambda$. Now, since $f^*(a) < |f\chi_{A_1}| \leq |f\chi_{\tilde{A}}|$, we have that $f^*(a) \leq (f\chi_{\tilde{A}})^*(t)$. Take $\lambda = (f\chi_{\tilde{A}})^*(t)$. By the above, we have that $\mu_f((f\chi_{\tilde{A}})^*(t)) = \mu_{f\chi_{\tilde{A}}}((f\chi_{\tilde{A}})^*(t)) \leq t$, by (2) of Proposition 1.1. That is, $\mu_f((f\chi_{\tilde{A}})^*(t)) \leq t$ and so $f^*(t) \leq f^*(\mu_f((f\chi_{\tilde{A}})^*(t))) \leq (f\chi_{\tilde{A}})^*(t)$, by (4) of Proposition 1.1. Thus, $f^*(t) \leq (f\chi_{\tilde{A}})^*(t)$, for $t < a$. Hence,

$$(f\chi_{\tilde{A}})^*(t) = f^*(t)\chi_{[0,a)}(t), \quad for \ \ all \ \ t \in [0, \infty).$$

□

The following is the definition of the Lorentz spaces.

Definition 1.3. *Given a measurable function f defined on T and $0 < p, q \leq \infty$, define*

$$\|f\|_{L(p,q)} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty \left(f^*(t) t^{\frac{1}{p}} \right)^q \dfrac{dt}{t} \right)^{1/q} & q \in (0, \infty) \\ \sup\limits_{t \geq 0} t^{1/p} f^*(t) & q = \infty \end{cases}$$

The Lorentz space with indices p and q, denoted by $L(p,q)$ or $L_{p,q}$, is the set of all measurable functions f for which $\|f\|_{L(p,q)} < \infty$.

Remark 1.2. *For $0 < p, q \leq \infty$, the Lorentz space $L(p,q)$ is a quasi-Banach space. The special case with $p = q$, $L(p,p) = L_p$, the Lebesgue space. The case $q = \infty$, the Lorentz space $L(p, \infty)$ is the same as the weak L_p spaces.*

Of particular importance for this work is the case $q = 1$ and $1 \leq p < \infty$. That is;

$$L(p,1) = \left\{ f : T \to \mathbb{R} : \quad \|f\|_{L(p,1)} = \frac{1}{p} \int_0^\infty f^*(t) t^{\frac{1}{p}-1} dt < \infty \right\} \quad \text{for} \quad 1 \leq p < \infty$$

Remark 1.3. *The space $L(p,1)$ is the space introduced in [10] by G. G Lorentz and denoted $\Lambda(\alpha)$ with α replaced by $1/p$. The case $\alpha = 1$ gives the space of integrable functions. That is, $L(1,1) = L_1$. G. G Lorentz also showed in his paper that the dual space (i.e, the space of all bounded linear functionals) of $L(p,1)$ is equivalent to the space $M(1/p)$ defined as follows;*

$$M(1/p) = \left\{ g : T \to \mathbb{R} : \quad \|g\|_{M(1/p)} = \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)^{1/p}} \int_A |g(t)| d\mu(t) < \infty \right\}$$

for $1 < p < \infty$. It can also be easily verified that the space $M(1/p)$ is equivalent to the Lorentz space $L(p', \infty)$ where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, that is, $p' = \dfrac{p}{p-1}$.

The following is a recall of the definition of bounded (continuous) linear operators.

Definition 1.4. *Given two normed linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the map $T : X \to Y$ is said to be a bounded linear operator if and only if*

(1) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$, and

(2) $\|T(x)\|_Y \leq M\|x\|_X$, for all $x \in X$ and some $M \geq 0$.

We define the space $\mathcal{L}(X, Y)$ by $\mathcal{L}(X, Y) = \{T : X \to Y : T \text{ is a bounded linear operator}\}$ and endow it with the norm $\|T\| = \sup\{\|T(x)\|_Y : \|x\|_X \leq 1\}$. If Y is Banach space then $\mathcal{L}(X, Y)$ is a Banach space. The particular case where $Y = \mathbb{R}$, the space $\mathcal{L}(X, \mathbb{R})$ is called the dual space of X and denoted by $X^\star$.

Definition 1.5. *Two normed linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are said to be equivalent if and only if $\alpha\|x\|_Y \leq \|x\|_X \leq \beta\|x\|_Y$, for all $x \in X$ and some absolute positive constants α and β. We write $X \cong Y$ to mean that X is equivalent to Y.*

1.2 Preliminary Results and Motivation

Definition 1.6. *The special atom space $B(\mu, 1/p)$ for $1 \leq p < \infty$ is defined as;*

$$B(\mu, 1/p) = \left\{ f : T \to \mathbb{R} : f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)^{1/p}} \chi_{A_n}(t) \quad \text{and} \quad \sum_{n\geq 1} |c_n| < \infty \right\},$$

where the c_n's are real numbers, and $A_n \in \mathcal{M}$ for each $n \geq 1$. We endow $B(\mu, 1/p)$ with the "norm" $\|f\|_{B(\mu,1/p)} = \inf \sum_{n\geq 1} |c_n|$, where the infimum is taken over all possible representations of f.

Theorem 1.1. *The space $B(\mu, 1/p)$ is a Banach space with respect to the norm $\|\cdot\|_{B(\mu,1/p)}$ for $1 \leq p < \infty$.*

The proof of Theorem 1.1 for the case $p = 1$ is provided in the next chapter and very similar to the general case. The following is the result obtained by De Souza in 2010 which gives the atomic characterization of $L(p, 1)$ for $p > 1$.

Theorem 1.2 (De Souza,[4]). *For $1 < p < \infty$, the special atom space $B(\mu, 1/p)$ is equivalent to the Lorentz space $L(p, 1)$. That is, there exist absolute postive constants α and β such that $\alpha \|f\|_{B(\mu,1/p)} \leq \|f\|_{L(p,1)} \leq \beta \|f\|_{B(\mu,1/p)}$.*

This characterization obtained by De Souza provides a simple proof of the theorem by Guido Weiss and Elias Stein concerning linear operators acting on the Lorentz space $L(p, 1)$ which is stated below;

Theorem 1.3 (Stein and Weiss,[11]). *If T is a linear operator on the space of measurable functions and $\|T\chi_A\|_Y \leq M\mu(A)^{1/p}, 1 < p < \infty, A \in \mathcal{M}$ where Y is a Banach space, then T can be extended to all $L(p, 1)$; that is $T : L(p, 1) \to Y$ and $\|Tf\|_Y \leq M \|f\|_{L(p,1)}$.*

Motivated by the result obtained by De Souza and Theorem 1.3, we seek to investigate the situation for the case $p = 1$, and obtain similar results for the Lorentz-Bochner space, that is, the case of vector-valued functions.

1.3 Outline of the Dissertation

In chapter 2, we will discuss the atomic characterization of L_1. This will be done in two parts. We will consider the case when $\mu$is the Lebesgue measure and when μ is any arbitrary measure. Chapter 3 extends the result by De Souza for the Lorent-Bochner space. In chapter 4 we study some applications of these characterizations. Particularly, we study the boundedness of some well-known operators acting on L_1 and the Lorentz-Bochner space.

Chapter 2

The L_1 Space

Many authors have studied the atomic decomposition of Banach spaces of functions. That is, they seek to determine if every element in the Banach space is of the form $\sum_{n=1}^{\infty} \lambda_n a_n$, where the λ_n's are scalars with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ and the a_n (called atoms) satisfy some simple properties and belong to some given subset of the Banach space under consideration. This decomposition provides a better and easy understanding of the classical results of these spaces, such as the dual representation, interpolation and some fundamental inequalities in harmonic analysis like boundedness of operators. For instance, R. R. Coifman and many other authors have studied the atomic decomposition of the Hardy H^p spaces and its variants [27, 28, 29, 30, 31]. On the other hand, Coifman, Weiss and Rochberg [32, 33] obtained decomposition theorems for the Bergman spaces. Recently, De Souza [4] obtained a decomposition for the Lorentz-space $L(p, 1)$ for $p > 1$.

In this chapter, we study an atomic characterization of the Lebesgue L_1 space. This would be done in two parts; namely, the particular case with respect to the Lebesgue measure and the more general case for any arbitrary finite measure space.

2.1 The Lebesgue measure case

For simplicity, we take $T = [0, 2\pi]$; our results will hold for any finite interval $[0, a]$ as well. The first observation towards the atomic characterization of L_1 is the relationship between between the derivatives of Lipschitz functions and L_∞. Indeed, as we will show later, a function belongs to L_∞ if and only it is the derivative of a Lipschitz function. It became necessary to consider the derivatives of Lipschitz functions since they appear natural as the dual of a special atom space as we will see later. We give a little introduction and properties of Lipschitz functions in the following;

A Brief Note on Lipschitz Functions

The Lipschitz space often denoted by Lip^1 is the space of real-valued functions f defined on the interval $[0, 2\pi]$, for which $|f(x+h) - f(x)| \leq Mh$ for some positive constant M. This space has been studied and generalized in several different ways. The first generalization is to replace h with h^α where $0 < \alpha \leq 1$ to obtain the so called Lipschitz spaces Lip^α of order α. Another generalization obtained by replacing h with a positive function $\rho(h)$ playing the role of a weight (See [3],[5]). Recently, De Souza in [4] gave a generalization related to the space Lip^α for general measures on subsets of the interval $[0, 2\pi]$ for $0 < \alpha < 1$. In this note, we are concerned with a similar extension we denote by $Lip(\mu, 1)$ related to the case $\alpha = 1$, where μ is a general measure on $[0, 2\pi]$ with certain properties. In particular, we will show that $Lip(\mu, 1)$ is L_∞. However, its representation give us an easy way to obtain the dual space of the generalized special atom space $B(\mu, 1)$ of the special atom space B to general measures μ on $[0, 2\pi]$. This representation of L_∞ provides a clear connection between the derivatives of Lipschitz functions and L_∞ functions as we will show later. We start with the definition of the Lipschitz condition for some functions, which the reader can find in any undergraduate or graduate text in Analysis, including [6].

Definition 2.1. *A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is said to satisfy the Lipschitz condition with respect to x if there exists $K > 0$ such that*

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|, \quad \forall (t, x_1), (t, x_2) \in D. \tag{2.1}$$

In ordinary differential equations, the Lipschitz condition is used in the existence and uniqueness theorem: that is, if $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ satisfies the Lipschitz condition in D with respect to x, then the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t,x) \quad \text{for } (t,x) \in D \\ \\ x(t_0) = x_0 \end{cases}$$

has a unique solution in D. In analysis, any function that satisfies the Lipschitz condition is said to be Lipschitz continuous or simply Lipschitz. It is known that functions with bounded derivative are Lipschitz functions. Lipschitz functions are absolutely continuous. A function f defined on the interval $[a,b]$ is said to be absolutely continuous if and only if there exist a Lebesgue integrable function g on $[a,b]$ such that

$$f(x) = f(a) + \int_a^x g(t) \;\; dt \;\; \text{for all} \;\; x \in [a,b]$$

and where $g = f'$ almost everywhere. Thus, Lipschitz functions are almost everywhere differentiable (e.g. $f(x) = |x|, \quad f(x) = \sin x$ and so on). The next definition is a generalization of Lipschitz functions, see [12], [6].

Definition 2.2. *A function $f : [0,2\pi] \to \mathbb{R}$ is said to be a Lipschitz function of order α if $\forall x \in [0,2\pi]$, and $h > 0$,*

$$\frac{|f(x+h) - f(x)|}{h^\alpha} \leq M, \quad \textit{for some } M \geq 0 \textit{ and } 0 < \alpha \leq 1.$$

Definition 2.3. *We denote by Lip^{α}, $\ 0 < \alpha \leq 1$ the space of all Lipschitz functions of order α and endow it with the norm*

$$\|f\|_{Lip^{\alpha}} = \sup_{\substack{x\in[0,2\pi] \\ h>0}} \frac{|f(x+h) - f(x)|}{h^{\alpha}}.$$

Here the constants are identified as the zero vector.

Remark 2.1. *It is worth noting that if $\alpha \geq 1$ then $Lip^{\alpha} = \{constant\ functions\}$*

In this work, we are concerned with the case where $\alpha = 1$. That is, the space of Lipschitz functions of order 1. The next definition is the space of derivatives of Lipschitz functions of order 1.

Definition 2.4. *We define the space $(Lip^1)'$ as follows;*

$$(Lip^1)' = \left\{ g' : S \subseteq [0, 2\pi] \to \mathbb{R} : g \in Lip^1 \right\}$$

where the prime denotes the derivative. We endow $(Lip^1)'$ with the "norm"

$$\|g'\|_{(Lip^1)'} := \|g\|_{Lip^1}, \, where \quad g \in Lip^1.$$

Theorem 2.1. *$((Lip^1)', \|\cdot\|_{(Lip^1)'})$ is a Banach space.*

Proof. The proof follows directly from the fact that Lip^1 is a Banach space. □

Proposition 2.1. *$(Lip^1)' \cong L_{\infty}$ with $\|f\|_{\infty} = \|f\|_{(Lip^1)'}$ for every $f \in L_{\infty}$.*

Proof. Let $g' \in (Lip^1)'$. We have that $g \in Lip^1$ with $\|g\|_{Lip^1} \leq C < \infty$. Thus, we have

$$\frac{|g(t+h) - g(t)|}{h} \leq C, \text{ for all } \quad t \in [0, 2\pi] \text{ and } \quad h > 0.$$

Hence, $|g'(t)| \leq C$, for almost all $t \in [0, 2\pi]$. Thus, $g' \in L_\infty(X)$ and $\|g'\|_\infty \leq \|g'\|_{(Lip^1)'}$. To see the converse, let $g \in L_\infty \subseteq L_1$, i.e $g \in L_1$ and define $G : [0, 2\pi] \to \mathbb{R}$ by $G(t) = \displaystyle\int_0^t g(s)ds$, for $t \in [0, 2\pi]$. G is well-defined and Lipschitz with $G'(t) = g(t)$ a.e. Thus, $g \in (Lip^1)'$ and $\|g\|_{(Lip^1)'} \leq \|g\|_\infty$.

$\square$

Remark 2.2. *Though we have seen in Proposition 2.1 that $(Lip^1)'$ is the same as L_∞, it became necessary to consider the derivatives of Lipschitz functions as they appear natural as the dual of the special atom space B (defined below) as we have shown in the following results.*

The next definition is the definition of the special atom space which is a slight modification of the space introduced by De Souza in [4].

Definition 2.5. *The special atom space is the space B of functions defined by*

$$B = \left\{ f : [0, 2\pi] \to \mathbb{R} : f(t) = \sum_{n \geq 1} c_n \frac{1}{|I_n|} \chi_{I_n}(t), \quad \textit{and} \quad \sum_{n \geq 1} |c_n| < \infty \right\}$$

where the c_n's are real numbers, χ_{I_n} is the characteristic function of the interval I_n in $[0, 2\pi]$ and $|I_n|$ denotes the length of the interval.

We endow B with the "norm" $\|f\|_B = \inf \displaystyle\sum_{n=1}^{\infty} |c_n|$ where the infimum is taken over all the representations of f.

Theorem 2.2. $(B, \|\cdot\|_B)$ *is Banach space.*

Note that Theorem 2.2 is a particular case of Theorem 2.8 in Section 2.2 and the proof is similar up to minor modifications. The next results are special cases of much general results obtained in Section 2.2, showing that the $(Lip^1)'$ is the dual space of the special atom space B. First, we give a Hölder's type inequality between the space of derivatives of Lipschitz functions and the special atom space B.

Theorem 2.3 (Hölder's type inequality)**.**
If $f \in B$ and $g' \in (Lip^1)'$, then

$$\left| \int_0^{2\pi} f(t) g'(t) dt \right| \leq \|f\|_B \|g'\|_{\left(Lip^1\right)'}.$$

Proof. Let $g \in \mathrm{Lip}^1$, we observe that $\dfrac{|g(t+h) - g(t)|}{h} \leq \|g\|_{\mathrm{Lip}^1}$, for all $t \in [0, 2\pi]$ and $h > 0$. Thus $|g'(t)| \leq \|g\|_{\mathrm{Lip}^1}$, for almost all $t \in [0, 2\pi]$. Now let $f \in B$ with

$$f(t) = \sum_{n \geq 1} c_n \frac{1}{|I_n|} \chi_{I_n}(t), \quad \text{and} \sum_{n \geq 1} |c_n| < \infty,$$

we have that

$$\int_0^{2\pi} f(t) g'(t) dt = \int_0^{2\pi} \sum_{n \geq 1} \left(c_n \frac{1}{|I_n|} g'(t) \chi_{I_n}(t) \right) dt.$$

Thus

$$\begin{aligned} \left| \int_0^{2\pi} f(t) g'(t) dt \right| &\leq \sum_{n \geq 1} |c_n| \frac{1}{|I_n|} \int_{I_n} |g'(t)| \, dt \\ &\leq \sum_{n \geq 1} |c_n| \frac{1}{|I_n|} |I_n| \|g\|_{\mathrm{Lip}^1}, \qquad \text{since} \quad |g'(t)| \leq \|g\|_{\mathrm{Lip}^1} \\ &\leq \left(\sum_{n \geq 1} |c_n| \right) \|g\|_{\mathrm{Lip}^1}. \end{aligned}$$

Taking the infimum on the R.H.S. of the latter over all representations of f, we have

$$\left|\int_0^{2\pi} f(t)g'(t)dt\right| \le \|f\|_B \|g'\|_{\left(\mathrm{Lip}^1\right)'}.$$

$\square$

Remark 2.3. *Theorem 2.3 implies that, given any $g' \in (Lip^1)'$ the map $\phi : B \to \mathbb{R}$ defined by*

$$\phi(f) = \int_0^{2\pi} f(t)g'(t)dt, \quad \textit{for all} \;\; f \in B,$$

is a bounded linear functional. That is, $\phi \in B^\star$.

The next result gives a characterization of all bounded linear functionals defined on B.

Theorem 2.4 (Duality)**.**
The dual space $B^\star$ of B, is equivalent to $(Lip^1)'$. That is, $\phi \in B^\star$ if and only if there exists $g' \in (Lip^1)'$ so that $\phi(f) = \int_0^{2\pi} f(t)g'(t)dt$, $\forall f \in B$ and $\|\phi\|_{B^\star} = \|g'\|_{\left(Lip^1\right)'}$.

Proof. $\Longleftarrow$. Fix $g' \in (\mathrm{Lip}^1)'$ and define $\phi_g(f) = \int_0^{2\pi} f(t)g'(t)dt$ for all $f \in B$. ϕ_g is a linear map on B, and $|\phi_g(f)| \le \|f\|_B \|g'\|_{(\mathrm{Lip}^1)'}$, by Theorem 2.3. Hence $\phi_g \in B^\star$

$\Longrightarrow$. Consider the map $\psi : (\mathrm{Lip}^1)' \to B^\star$ defined by $\psi(g') = \phi_g$, ϕ_g defined as above. We want to show that ψ is onto, i.e. given $\phi \in B^\star$, there exists $g' \in (\mathrm{Lip}^1)'$ such that $\phi = \phi_g$. Let $\phi \in B^\star$, and define $g(t) = \phi\left(\chi_{(0,t]}\right)$, $t \in [0, 2\pi]$.

<u>Claim:</u> $g \in \mathrm{Lip}^1$ and hence $g' \in (\mathrm{Lip}^1)'$.

In fact, observe that

$$g(t+h) - g(t) = \phi(\chi_{(0,t+h]} - \chi_{(0,t]}) = \phi(\chi_{[t,t+h]}).$$

Thus

$$|g(t+h) - g(t)| = |\phi(\chi_{[t,t+h]})| \leq \|\phi\|_{B^\star} \|\chi_{[t,t+h]}\|_B \leq \|\phi\|_{B^\star} h.$$

It follows that

$$\frac{|g(t+h) - g(t)|}{h} \leq \|\phi\|_{B^\star} < \infty, \quad \forall h > 0.$$

Hence the claim is proved. Thus, we have that $g'(t)$ exists almost everywhere.

This implies that

$$\phi\left(\chi_{(0,t]}\right) = g(t) = \int_0^t g'(s)ds = \int_0^{2\pi} g'(s)\chi_{[0,t]}(s)ds.$$

Now since

$$\chi_{[a,b]}(t) = \chi_{[0,b]}(t) - \chi_{[0,a]}(t) \quad \text{for } a < b,$$

we have that

$$\begin{aligned}
\phi(\chi_{[a,b]}) &= \phi(\chi_{[0,b]}) - \phi(\chi_{[0,a]}) \ \text{ since } \phi \text{ is linear} \\
&= \int_0^{2\pi} g'(t)\chi_{[0,b]}(t)dt - \int_0^{2\pi} g'(t)\chi_{[0,a]}(t)dt \\
&= \int_0^{2\pi} g'(t)\left(\chi_{[0,b]}(t) - \chi_{[0,a]}(t)\right)dt \\
&= \int_0^{2\pi} g'(t)\chi_{[a,b]}(t)dt.
\end{aligned}$$

Therefore

$$\phi\left(\frac{1}{b-a}\chi_{[a,b]}\right) = \int_0^{2\pi} \frac{1}{b-a} g'(t)\chi_{[a,b]}(t)dt.$$

For $f(t)=\sum_{n\geq 1} c_n \frac{1}{|I_n|}\chi_{I_n}(t)$ with $\sum_{n\geq 1}|c_n|<\infty$, we have

$$f(t)=\lim_{k\to\infty} f_k(t) \quad \text{where} \quad f_k(t)=\sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}(t), k\in\mathbb{N}.$$

For each $k\in\mathbb{N}$,

$$\begin{aligned}
\phi(f_k) &= \phi\left(\sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}\right)\\
&= \sum_{n=1}^{k} c_n \frac{1}{|I_n|}\phi\left(\chi_{I_n}\right)\\
&= \sum_{n=1}^{k} c_n \frac{1}{|I_n|}\int_0^{2\pi} \chi_{I_n}(t) g'(t)dt\\
&= \int_0^{2\pi}\left(\sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}(t)\right) g'(t)dt\\
&= \int_0^{2\pi} f_k(t)g'(t)dt.
\end{aligned}$$

That is,

$$\phi(f_k)=\int_0^{2\pi} f_k(t)g'(t)dt.$$

Now since $\phi\in B^\star$, it follows that

$$\lim_{k\to\infty}\phi(f_k)=\phi(f).$$

On the other hand, we have that

$$\int_0^{2\pi} f_k(t)g'(t)dt \to \int_0^{2\pi} f(t)g'(t)dt.$$

To see this, let $h_k(t) = f_k(t)g'(t)$ and $p_k(t) = \displaystyle\sum_{n=1}^{k} |c_n|\frac{1}{|I_n|}|g'(t)|\chi_{I_n}(t)$. We observe that

$$|h_k(t)| \le p_k(t) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad t \in [0, 2\pi].$$

In addition,

$$0 \le p_k(t) \le p_{k+1}(t), \quad \text{for} \quad t \in [0, 2\pi] \quad \text{and} \quad p_k(t) \to p(t) := \sum_{n\ge 1} |c_n|\frac{1}{|I_n|}|g'(t)|\chi_{I_n}(t).$$

So by the Monotone convergence theorem (see [8], page 83), we have that

$$\int_0^{2\pi} p_k(t)dt \to \int_0^{2\pi} p(t)dt = \sum_{n\ge 1} |c_n|\frac{1}{|I_n|}\int_{I_n} |g'(t)|dt \le \|g\|_{Lip^1} \sum_{n\ge 1} |c_n| < \infty.$$

That is

$$\int_0^{2\pi} p_k(t)dt \to \int_0^{2\pi} p(t)dt < \infty.$$

Hence by the Dominated convergence theorem (see [8], page 89), we have that

$$\lim_{k\to\infty} \int_0^{2\pi} h_k(t)dt = \int_0^{2\pi} \lim_{k\to\infty} h_k(t)dt.$$

Thus

$$\int_0^{2\pi} f_k(t)g'(t)dt \to \int_0^{2\pi} f(t)g'(t)dt.$$

Hence

$$\phi(f) = \int_0^{2\pi} f(t)g'(t)dt.$$

That is, $\phi = \phi_g$. Therefore, ψ is onto. In addition we have,

$$\|\phi\|_{B^\star} = \sup_{\|f\|_B \leq 1} |\phi(f)| \leq \|g'\|_{(Lip^1)'} \text{ by the Hölder's inequality.}$$

That is

$$\|\phi\|_{B^\star} \leq \|g'\|_{(Lip^1)'}.$$

On the other hand, for $f_h(t) = \frac{1}{h}\chi_{[x,x+h]}(t), \quad h > 0$, we have $f_h \in B$ with

$$\|f_h\|_B \leq 1 \quad \text{and } \phi(f_h) = \frac{1}{h}\int_0^{2\pi} \chi_{[x,x+h]}(t)g'(t)dt = \frac{1}{h}\int_x^{x+h} g'(t)dt = \frac{g(x+h)-g(x)}{h}.$$

This implies that

$$|\phi(f_h)| = \frac{|g(x+h)-g(x)|}{h} \leq \|\phi\|_{B^\star}.$$

Taking the supremum over $x \in [0, 2\pi]$ and $h > 0$, we obtain $\|g\|_{Lip^1} \leq \|\phi\|_{B^\star}$. So that

$$\|\phi\|_{B^\star} = \|g'\|_{(Lip^1)'}$$

$\square$

Remark 2.4. *By Proposition 2.1 and Theorem 2.4, we deduce that* $B^\star \cong L_\infty$.

Theorem 2.5. *The special atom space* B *is continuously contained in* L_1 *and*

$$\|f\|_1 \leq C\,\|f\|_B\,, \quad \textit{for} \;\; f \in B$$

.

Proof. Let $f \in B$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{|I_n|}\chi_{I_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$ and consider

$$\int_0^{2\pi} |f(t)|dt \leq \sum_{n\geq 1} |c_n| \frac{1}{|I_n|} \int_{I_n} 1 \;\; dt = \sum_{n\geq 1} |c_n| < \infty.$$

So $f \in L_1$ and $\|f\|_1 \leq \|f\|_B$, for $f \in B$.

□

The following theorem is a classical result in Functional Analysis, which can be found in [7] (see page 160).

Theorem 2.6. *Let X and Y be two normed linear spaces, and let $T \in \mathcal{L}(X, Y)$. Let $T^\star$ be the adjoint operator of T defined by $T^\star f = f \circ T$ for all $f \in Y^\star$. Then*

(1) $T^\star \in \mathcal{L}(Y^\star, X^\star)$ and $\|T^\star\| = \|T\|$

(2) $T^\star$ is injective if and only if the range of T is dense in Y. In addition, if X and Y are Banach spaces then $T^\star$ is invertible if and only if T is invertible.

Now, we have the following situations;

(1) $B \subseteq L_1$ with $\|f\|_1 \leq \|f\|_B$, for $f \in B$ by Theorem 2.5.

(2) $B^\star \cong L_1^\star$ by Remark 2.4.

(3) B is dense in L_1. This can be verified with standard techniques and a corollary of the Hahn-Banach Theorem.

As a consequence of these facts and Theorem 2.6, the embedding operator $I : B \to L_1$ defined by $I(f) = f$ is a Banach space isomorphism. So, we have the following result;

Theorem 2.7. *$B \cong L_1$ with equivalent norms, i.e, there exist $f \in B \iff f \in L_1$ and $\alpha \|f\|_B \leq \|f\|_1 \leq \beta \|f\|_B$ form some absolute positive constants α and β.*

In the following section, we extend this result for arbitrary measures which is the case $p = 1$ in the result obtained by De Souza [4].

2.2 Extension to Arbitrary measures

Here we consider a general nonatomic, finite measure space $(T, \mathcal{M}, \mu)$ where $T \subset \mathbb{R}$. The next definition is a natural extension of the special atom space B to general measures which was first proposed by De Souza in [4].

Definition 2.6. *We define the space $B(\mu, 1)$ as*

$$B(\mu, 1) = \left\{ f : T \to \mathbb{R} : f(t) = \sum_{n \geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t) \quad and \quad \sum_{n \geq 1} |c_n| < \infty \right\},$$

where the c_n's are real numbers, and $A_n \in \mathcal{M}$ for each $n \geq 1$. We endow $B(\mu, 1)$ with the "norm" $\|f\|_{B(\mu,1)} = \inf \sum_{n \geq 1} |c_n|$, where the infimum is taken over all possible representations of f.

Remark 2.5. *The space $B(\mu, 1)$ is the case $p = 1$ in the space $B(\mu, 1/p)$ by De Souza.*

Theorem 2.8.

(a) $\| \cdot \|_{B(\mu,1)}$ is a norm of $B(\mu, 1)$.

(b) $\big(B(\mu, 1), \| \cdot \|_{B(\mu,1)}\big)$ is a Banach space.

The proof of Theorem 2.8 is similar to the one obtained by De Souza but we produce it here for completion.

Proof. (a) To show that $\|\cdot\|_{B(\mu,1)}$ is a norm, observe that $\|f\|_{B(\mu,1)} \geq 0, \forall f \in B(\mu,1)$ and $f = 0$ implies that $\|f\|_{B(\mu,1)} = 0$. On the other hand, suppose $\|f\|_{B(\mu,1)} = 0$. We want to show that $f = 0,\ \ \mu\ a.e.$ Let $(c_{nk})_{n,k\in\mathbb{N}}$ be a sequence of real numbers and $(A_{nk})_{n,k\in\mathbb{N}}$ be a sequence of measurable subsets of X such that $f(t) = \sum_{n\geq 1} c_{nk}\frac{1}{\mu(A_{nk})}\chi_{A_{nk}}(t)$ with $\sum_{n\geq 1}|c_{nk}| < \infty$ for each $k \in \mathbb{N}$ and $\sum_{n\geq 1}|c_{nk}| \to 0$ as $k \to \infty$. So we have for each $n \in \mathbb{N}$, $|c_{nk}| \to 0$ as $k \to \infty$. Thus, the coefficients of the representations of f converges to zero and hence $f = 0,\ \ \mu\ a.e.$

For $\alpha \in \mathbb{R}$ and $f \in B(\mu,1)$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $\sum_{n\geq 1}|c_n| < \infty$, we have $(\alpha f)(t) = \sum_{n\geq 1}\alpha c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and this implies that

$$\begin{aligned}
\|\alpha f\|_{B(\mu,1)} &= \inf\sum_{n\geq 1}|\alpha c_n| \\
&= |\alpha|\inf\sum_{n\geq 1}|c_n| \\
&= |\alpha|\,\|f\|_{B(\mu,1)}.
\end{aligned}$$

Finally, for $f, g \in B(\mu,1)$, to show that $\|f+g\|_{B(\mu,1)} \leq \|f\|_{B(\mu,1)} + \|g\|_{B(\mu,1)}$, let $\epsilon > 0$ be given, and let $(c_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\mathbb{N}}$ be sequences of real numbers such that $f(t) = \sum_{n\geq 1} c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $g(t) = \sum_{n\geq 1} b_n\frac{1}{\mu(B_n)}\chi_{B_n}(t)$, for some sequence $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, and such that $\sum_{n\geq 1}|c_n| < \|f\|_{B(\mu,1)} + \epsilon/2,\quad \sum_{n\geq 1}|b_n| < \|g\|_{B(\mu,1)} + \epsilon/2.$ Note that we can write

$$(f+g)(t) = \sum_{n\geq 1} d_n\frac{1}{\mu(D_n)}\chi_{D_n}(t),$$

with $\sum_{n\geq 1}|d_n| = \sum_{n\geq 1}|c_n| + |b_n|$ where

$$d_n = \begin{cases} c_{\frac{n}{2}} & \text{if } n \text{ is even} \\ b_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

and

$$D_n = \begin{cases} A_{\frac{n}{2}} & \text{if } n \text{ is even} \\ B_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

It follows that

$$\begin{aligned} \|f+g\|_{B(\mu,1)} &\leq \sum_{n\geq 1}|d_n| \\ &= \sum_{n\geq 1}|c_n| + \sum_{n\geq 1}|b_n| \\ &< \|f\|_{B(\mu,1)} + \|g\|_{B(\mu,1)} + \epsilon \end{aligned}$$

Thus since ϵ is arbitrary, we have

$$\|f+g\|_{B(\mu,1)} \leq \|f\|_{B(\mu,1)} + \|g\|_{B(\mu,1)}.$$

(b) To prove completeness, it suffices to show that for any sequence $(f_m)_{m\geq 1} \subseteq B(\mu,1)$, we have

$$\|\sum_{m\geq 1} f_m\|_{B(\mu,1)} \leq \sum_{m\geq 1}\|f_m\|_{B(\mu,1)}.$$

Note that given $\epsilon > 0$ and for each $m \geq 1$, there are sequence real numbers (c_{m_n}) and sequence of sets $A_{m_n} \in \mathcal{M}$ such that $f_m(t) = \sum_{n\geq 1}\frac{c_{m_n}}{\mu(A_{m_n})}\chi_{A_{m_n}}(t)$ with $\sum_{n\geq 1}|c_{m_n}| <$

$\|f_m\|_{B(\mu,1)} + \frac{\epsilon}{2^m}$. It follows that

$$\sum_{m\geq 1}\sum_{n\geq 1}|c_{m_n}| < \sum_{m\geq 1}\|f_m\|_{B(\mu,1)} + \epsilon\sum_{m\geq 1}\frac{1}{2^m} = \sum_{m\geq 1}\|f_m\|_{B(\mu,1)} + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\|\sum_{m\geq 1} f_m\|_{B(\mu,1)} \leq \sum_{m\geq 1}\|f_m\|_{B(\mu,1)}.$$

$\square$

The next definition, is the candidate for the dual space of $B(\mu, 1)$.

Definition 2.7. *Define the space $Lip(\mu, 1)$ as*

$$Lip(\mu,1) = \left\{ g : [0, 2\pi] \to \mathbb{R} : \frac{1}{\mu(A)}\left|\int_A g(t)d\mu(t)\right| \leq C < \infty, \quad \forall A \in \mathcal{M}, \quad \mu(A) \neq 0 \right\}.$$

Endow $Lip(\mu, 1)$ with the "norm"

$$\|g\|_{Lip(\mu,1)} = \sup_{\mu(A)\neq 0}\frac{1}{\mu(A)}\left|\int_A g(t)d\mu(t)\right|.$$

Remark 2.6. *This space $Lip(\mu, 1)$ as we shall show later is L_∞. However, the representation of the norm provides an easy way to see the connection between the derivatives of Lipschitz functions and L_∞ functions. Indeed, we observe that if we take $\mu =$Lebesgue measure and the measurable sets to be intervals then we have a more general representation of the norm on $(Lip^1)'$. That is if $g \in Lip^1$ and $A = [x, x+h]$ then $\frac{1}{\mu(A)}\left|\int_A g'(t)d\mu(t)\right| = \frac{|g(x+h) - g(x)|}{h}$.*

Lemma 2.1. *If $g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ then*

$$\int_A |g(t)|d\mu(t) \leq \mu(A)\|g\|_{Lip(\mu,1)}.$$

Proof. Let $g \in \mathrm{Lip}(\mu, 1)$ and $A \in \mathcal{M}$. Now let $A_+ = \{t \in A : g(t) \geq 0\}$ and $A_- = \{t \in A : g(t) < 0\}$. We have that $A_-, A_+ \in \mathcal{M}, \quad A = A_+ \cup A_-$ and $A_+ \cap A_- = \emptyset$. Now consider

$$\begin{aligned} \int_A |g(t)|d\mu(t) &= \int_{A_+} g(t)d\mu(t) - \int_{A_-} g(t)d\mu(t) \\ &\leq \left|\int_{A_+} g(t)d\mu(t)\right| + \left|\int_{A_-} g(t)d\mu(t)\right| \\ &\leq \mu(A_+)\|g\|_{\mathrm{Lip}(\mu,1)} + \mu(A_-)\|g\|_{\mathrm{Lip}(\mu,1)} \\ &= \mu(A)\|g\|_{\mathrm{Lip}(\mu,1)} \end{aligned}$$

That is,

$$\int_A |g(t)|d\mu(t) \leq \mu(A)\|g\|_{\mathrm{Lip}(\mu,1)}.$$

Hence the Lemma is proved. □

Theorem 2.9.

(a) $\|\cdot\|_{Lip(\mu,1)}$ *is a norm on* $Lip(\mu, 1)$.

(b) $\big(Lip(\mu, 1), \|\cdot\|_{Lip(\mu,1)}\big)$ *is Banach space.*

Proof.

(a) To show that $\|\cdot\|_{Lip(\mu,1)}$ is a norm, first observe that $\|g\|_{Lip(\mu,1)} \geq 0, \quad \forall g \in Lip(\mu, 1)$. Now suppose $\|g\|_{Lip(\mu,1)} = 0$. Then $\left|\int_A g(t)d\mu(t)\right| = 0, \quad \forall A \in \mathcal{M}$ with $\mu(A) \neq 0$. Thus $\int_A g(t)d\mu(t) = 0, \quad \forall A \in \mathcal{M}$ with $\mu(A) \neq 0$. This implies that $g = 0, \quad \mu - a.e.$

For $\alpha \in \mathbb{R}$ and $g \in Lip(\mu, 1)$, we have

$$\begin{aligned}
\|\alpha g\|_{Lip(\mu,1)} &= \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)} \left| \int_A \alpha g(t) d\mu(t) \right| \\
&= \sup_{\mu(A)\neq 0} |\alpha| \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&= |\alpha| \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&= |\alpha| \, \|g\|_{Lip(\mu,1)}.
\end{aligned}$$

Finally, for $f, g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\begin{aligned}
\frac{1}{\mu(A)} \left| \int_A (f(t) + g(t)) \, d\mu(t) \right| &\leq \frac{1}{\mu(A)} \left| \int_A f(t) d\mu(t) \right| + \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&\leq \|f\|_{Lip(\mu,1)} + \|g\|_{Lip(\mu,1)}
\end{aligned}$$

Taking the supremum on the L.H.S of the above inequality, we get

$$\|f + g\|_{Lip(\mu,1)} \leq \|f\|_{Lip(\mu,1)} + \|g\|_{Lip(\mu,1)}.$$

Thus $\| \cdot \|_{Lip(\mu,1)}$ is a norm on $Lip(\mu, 1)$. To complete the proof, we need to proved that $Lip(\mu, 1)$ is complete. In order to do so, it is sufficient to prove that for any sequence $(g_n)_{n\in\mathbb{N}} \subseteq Lip(\mu, 1)$ such that $\sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)} \leq C < \infty$, we have

$$\sum_{n\geq 1} g_n \in Lip(\mu, 1) \quad \text{and } \| \sum_{n\geq 1} g_n\|_{Lip(\mu,1)} \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)}.$$

Let $A \in \mathcal{M}$ with $\mu(A) \neq 0$. We have that

$$\begin{aligned}
\left|\int_A \sum_{n\geq 1} g_n(t) d\mu(t)\right| &\leq \int_A \left|\sum_{n\geq 1} g_n(t)\right| d\mu(t) \\
&\leq \int_A \sum_{n\geq 1} |g_n(t)|\, d\mu(t) \\
&= \sum_{n\geq 1} \int_A |g_n(t)| d\mu(t) \\
&\leq \sum_{n\geq 1} \mu(A) \|g_n\|_{Lip(\mu,1)} \quad \text{by Lemma 2.1}
\end{aligned}$$

That is, $\dfrac{1}{\mu(A)}\left|\int_A \sum_{n\geq 1} g_n d\mu\right| \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)} \leq C < \infty$. Hence $\sum_{n\geq 1} g_n \in Lip(\mu, 1)$. Taking the supremum on the L.H.S of the latter over all $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\|\sum_{n\geq 1} g_n\|_{Lip(\mu,1)} \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)}.$$

□

Theorem 2.10. *$Lip(\mu, 1) \cong L_\infty$ with $\|g\|_{L(\mu,1)} = \|g\|_\infty$.*

Proof. To see this, we first observe that if $g \in L_\infty$ then, we have for any $A \in \mathcal{M}$ with $\mu(A) \neq 0$,

$$\frac{1}{\mu(A)}\left|\int_A g(t) d\mu(t)\right| \leq \frac{1}{\mu(A)} \int_A |g(t)| d\mu(t) \leq \|g\|_\infty \frac{1}{\mu(A)} \int_A 1 d\mu(t) \leq \|g\|_\infty.$$

So, $g \in Lip(\mu, 1)$ and $\|g\|_{Lip(\mu,1)} \leq \|g\|_\infty$. On the other hand, given $g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we obtain from Lemma 2.1 that

$$\int_A |g(t)| d\mu(t) \leq \mu(A) \|g\|_{\mathrm{Lip}(\mu,1)}.$$

By Proposition 1.3, there exist a measurable subset $\tilde{A}$ of T with $\mu(\tilde{A}) = \mu(A)$ such that $\int_0^{\mu(A)} g^*(t)dt = \int_{\tilde{A}} |g(t)|d\mu(t)$, where g^* denotes the decreasing rearrangement of $|g|$. Thus, $\int_0^{\mu(A)} g^*(t)dt \leq \mu(A)\|g\|_{\mathrm{Lip}(\mu,1)}$ by Lemma 2.1. Since g^* is a decreasing function on $[0,\infty)$, we have that $g^*(\mu(A)) \leq g^*(t)$ for all $t \in [0,\mu(A)]$. Hence, $g^*(\mu(A))\mu(A) \leq \int_0^{\mu(A)} g^*(t)dt \leq \mu(A)\|g\|_{\mathrm{Lip}(\mu,1)}$. So, $g^*(\mu(A)) \leq \|g\|_{\mathrm{Lip}(\mu,1)}$. Let $\mu(A) \to 0$ to obtain $g^*(0) \leq \|g\|_{\mathrm{Lip}(\mu,1)}$. But $\|g\|_\infty = g^*(0)$. Thus, $g \in L_\infty$, and hence $\|g\|_\infty = \|g\|_{\mathrm{Lip}(\mu,1)}$.

□

In the following results, we show that the dual space of $B(\mu,1)$ is equivalent to $Lip(\mu,1)$. To do this, we recall the following result in Analysis which can be found in [7], page 55.

Theorem 2.11. *Suppose that $\{f_n\}$ is a sequence in $L_1(\mu)$ such that $\sum_{n\geq 1}\int_T |f_n|d\mu < \infty$. Then $\sum_{n\geq 1} f_n$ converges a.e to a function in $L_1(\mu)$, and $\int_T \sum_{n\geq 1} f_n d\mu = \sum_{n\geq 1}\int_T f_n d\mu$.*

As a consequence of Theorem 2.11, we have the following result.

Lemma 2.2. *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such $\sum_{n\geq 1}|c_n| < \infty$, $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of T and $g \in Lip(\mu,1)$. For each $n \in \mathbb{N}$, define $h_n : T \to \mathbb{R}$ by $h_n(t) := c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)$. Then $\sum_{n\geq 1} h_n$ converges a.e to a function in $L_1(\mu)$, and $\int_T \sum_{n\geq 1} h_n(t)d\mu(t) = \sum_{n\geq 1}\int_T h_n(t)d\mu(t)$.*

Proof. Let $n \in \mathbb{N}$ and consider

$$\begin{aligned}
\int_T |h_n(t)|d\mu(t) &= \int_T |c_n|\frac{1}{\mu(A_n)}\chi_{A_n}(t)|g(t)|\mu(t) \\
&= |c_n|\frac{1}{\mu(A_n)}\int_{A_n}|g(t)|d\mu(t) \\
&\leq |c_n|\|g\|_{\mathrm{Lip}(\mu,1)} < \infty, \quad \text{by Lemma 2.1}
\end{aligned}$$

Thus, $h_n \in L_1(\mu)$ and

$$\sum_{n\geq 1} \int_T |h_n(t)| d\mu(t) \leq \|g_n\|_{\mathrm{Lip}(\mu,1)} \left(\sum_{n\geq 1} |c_n| \right) < \infty.$$

The conclusion follows from Theorem 2.11 $\square$

Similar to the case of the Lebesgue measure, we have the following Hölder's Type Inequality for the special atom space $B(\mu,1)$ and $Lip(\mu,1)$.

Theorem 2.12 (Hölder's Type Inequality)**.**
If $f \in B(\mu,1)$ and $g \in Lip(\mu,1)$, then

$$\left| \int_T f(t)g(t)d\mu(t) \right| \leq \|f\|_{B(\mu,1)} \|g\|_{Lip(\mu,1)}.$$

Proof. Let $g \in \mathrm{Lip}(\mu,1)$ and $f \in B(\mu,1)$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$, we have

$$\int_T f(t)g(t)d\mu(t) = \int_T \sum_{n\geq 1} \left(c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t) g(t) \right) d\mu(t).$$

It follows that

$$\begin{aligned} \left| \int_T f(t)g(t)d\mu(t) \right| &\leq \sum_{n\geq 1} |c_n| \frac{1}{\mu(A_n)} \int_T |g(t)| \, d\mu(t) \\ &\leq \sum_{n\geq 1} |c_n| \frac{1}{\mu(A_n)} \mu(A_n) \|g\|_{\mathrm{Lip}(\mu,1)}, \quad \text{by Lemma 2.1} \\ &= \|g\|_{\mathrm{Lip}(\mu,1)} \left(\sum_{n\geq 1} |c_n| \right). \end{aligned}$$

Taking the infimum over all possible representations of f, we have

$$\left|\int_T f(t)g(t)d\mu(t)\right| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}.$$

$\square$

Theorem 2.13 (Duality)**.**

$B^\star(\mu,1) \cong Lip(\mu,1)$ with equivalent norms, that is, $\varphi \in B^\star(\mu,1)$ if and only if there exists $g \in Lip(\mu,1)$ such that $\varphi(f) = \int_T f(t)g(t)d\mu(t), \quad \forall f \in B(\mu,1)$*. Moreover,*

$$\|\varphi\| = \|g\|_{Lip(\mu,1)}.$$

Proof.

"$\Longleftarrow$:" Fix $g \in Lip(\mu,1)$ and define $\varphi_g : B(\mu,1) \to \mathbb{R}$ by

$$\varphi_g(f) = \int_T f(t)g(t)d\mu(t), \quad \forall f \in B(\mu,1). \tag{2.2}$$

Then clearly φ_g is a linear map and by Theorem 2.12, we have

$$|\varphi_g(f)| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}. \text{ Thus } \varphi_g \in B^\star(\mu,1).$$

"$\Longrightarrow$:" Consider the map $\psi : Lip(\mu,1) \to B^\star(\mu,1)$ define by $\psi(g) = \varphi_g$ where φ_g is defined as in (2.2). We want to show that ψ is onto. Let $\varphi \in B^\star(\mu,1)$. Define $\lambda : \mathcal{M} \to \mathbb{R}$ by $\lambda(A) = \varphi(\chi_A), \forall A \in \mathcal{M}$. We observe that

$$|\lambda(A)| = |\varphi(\chi_A)| \leq \|\varphi\|\|\chi_A\|_{B(\mu,1)} \leq \|\varphi\|\mu(A).$$

Thus, if $\mu(A) = 0$ then $\lambda(A) = 0$ and thus $\lambda << \mu$. Hence by the Radon-Nikodym Theorem, we have that $\lambda(A) = \int_A g d\mu$ for some $g \in L_1(\mu)$. In particular, $g \in Lip(\mu, 1)$ since $\int_A g d\mu = \varphi(\chi_A)$ implies $\left| \int_A g d\mu \right| \leq \|\varphi\| \mu(A)$. Thus

$$\frac{1}{\mu(A)} \left| \int_A g d\mu \right| \leq \|\varphi\| < \infty, \quad \forall A \in \mathcal{M} \text{ with } \mu(A) \neq 0.$$

So we have

$$\varphi(\chi_A) = \int_A g d\mu = \int_T g \chi_A d\mu.$$

That is,

$$\varphi(\chi_A) = \int_T \chi_A(t) g(t) d\mu(t).$$

Now, given $f \in B(\mu, 1)$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$, we have that $\varphi(f) = \varphi\left(\sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n} \right) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \varphi(\chi_{A_n})$, since $\varphi \in B^\star(\mu, 1)$. So we get,

$$\varphi(f) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \int_T \chi_{A_n}(t) g(t) d\mu(t) = \sum_{n\geq 1} \int_T \left(c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t) g(t) \right) d\mu(t)$$

That is,

$$
\begin{aligned}
\varphi(f) &= \sum_{n\geq 1}\int_T \left(c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)\right) d\mu(t) \\
&= \int_T \sum_{n\geq 1}\left(c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)\right) d\mu(t), \quad \text{by Lemma 2.2} \\
&= \int_T \left(\sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)\right) g(t)d\mu(t) \\
&= \int_T f(t)g(t)d\mu(t).
\end{aligned}
$$

Thus

$$\varphi(f) = \int_T f(t)g(t)d\mu(t).$$

This shows that $\varphi_g(f) = \varphi(f), \quad \forall f \in B(\mu,1)$ and for some $g \in Lip(\mu,1)$. That is, $\psi(g) = \varphi_g = \varphi$. It follows that the inclusion map $i : Lip(\mu,1) \to B^\star(\mu,1)$ is a bijection. Thus, $B^\star(\mu,1) \cong Lip(\mu,1)$. Moreover, it follows from the inequality $|\varphi(f)| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}$ that

$$\|\varphi\| = \sup_{\|f\|_{B(\mu,1)}\leq 1} |\varphi(f)| \leq \|g\|_{Lip(\mu,1)}.$$

Let $A \in \mathcal{M}$ with $\mu(A) \neq 0$, and let $f = \dfrac{1}{\mu(A)}\chi_A$. We have that $f \in B(\mu,1)$ with

$$\|f\|_{B(\mu,1)} \leq 1 \quad \text{and} \quad \varphi(f) = \frac{1}{\mu(A)}\int_T \chi_A(t)g(t)d\mu(t) = \frac{1}{\mu(A)}\int_A g(t)d\mu(t).$$

Thus

$$|\varphi(f)| = \frac{1}{\mu(A)}\left|\int_A g(t)d\mu(t)\right| \leq \|\varphi\|.$$

Taking the supremum on the L.H.S over all $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\|g\|_{Lip(\mu,1)} \leq \|\varphi\|, \quad \text{and hence } \|\varphi\| = \|g\|_{Lip(\mu,1)}.$$

$\square$

Remark 2.7. *Theorem 2.10 and Theorem 2.13 implies that* $B^{\star}(\mu, 1) \cong L_\infty$.

With similar arguments as the Lebesgue case, we have the following result;

Theorem 2.14. $B(\mu, 1) \cong L_1(\mu)$ *with* $m \|f\|_{B(\mu,1)} \leq \|f\|_1 \leq M \|f\|_{B(\mu,1)}$ *for some absolute positive constants* m *and* M.

An application of this characterization of $L_1(\mu)$ is considered in Chapter 4

Chapter 3

Lorentz-Bochner Space

3.1 Background, Preliminary Definitions and Notations

The Bochner spaces, denoted by $L^p(X)$, (where $(X, \|\cdot\|_X)$ is a Banach space) are a generalization of the classical L^p spaces to functions whose values are in X. That is, $L^p(X)$ consist of all functions f with values in X and whose norm $\|f\|_X$ belongs to the classical L^p space. It is this generalization that led to a similar extension for the Lorentz spaces known as the Lorentz-Bochner spaces which we will denote in this note by $L^X(p, q)$.

We let $(T, \mathcal{M}, \mu)$ denote a complete, finite and nonatomic measure space, $(X, \| \ \|_X)$ a Banach space and $X^\star$ the (real) dual space of X with the dual norm $\| \ \|_\star$ given by $\|\psi\|_\star = \sup\{|\psi(x)| : \|x\|_X \leq 1\} = \sup\{\psi(x) : \|x\|_X \leq 1\}$, for every $\psi \in X^\star$. In the following, we recall some basic definitions.

Definition 3.1. *The canonical duality paring $\langle \cdot, \cdot \rangle : X^\star \times X \to \mathbb{R}$ is given by $\langle \psi, x \rangle = \psi(x)$ for all $\psi \in X^\star$ and all $x \in X$. We have the following Cauchy-Schwartz type inequality; $|\langle \psi, x \rangle| \leq \|\psi\|_\star \|x\|_X$ for $\psi \in X^\star$ and $x \in X$.*

Definition 3.2. *We say that a function $\phi : T \to X^\star$ is $w^\star-$measurable if the real-valued function $t \mapsto \langle \phi(t), x \rangle$ is $\mu-$measurable for all $x \in X$.*

Definition 3.3. *Let $m : \mathcal{M} \to X^\star$ be a vector measure (an $X^\star$-valued measure). The total variation of m, denoted by $\|m\|$, is define as*

$$\|m\|(A) = \sup \sum_{i=1}^{n} \|m(A_i)\|_\star,$$

where the supremum is taken over all the partitions $A = \cup_{i=1}^{n} A_i$ of A into a finite number of disjoints measurable sets, for all $A \in \mathcal{M}$.

We now recall the formal definition of the Lorentz-Bochner spaces.

Definition 3.4. *For $1 \le p, q \le \infty$, we define*

$$\|f\|_{L^X(p,q)} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty \left(\|f\|_X^* (t) t^{\frac{1}{p}} \right)^q \dfrac{dt}{t} \right)^{1/q} & 1 \le p, q < \infty \\ \sup\limits_{t>0} t^{1/p} \|f\|_X^* (t) & 1 \le p \le \infty, \quad q = \infty \end{cases}$$

where the function $\|f\|_X : T \to \mathbb{R}$ is defined by $\|f\|_X(w) = \|f(w)\|_X$ and g^* denotes the decreasing rearrangement of $|g|$. The Lorentz-Bochner space with indices p and q, denoted $L^X(p,q)$, consist of all X-valued functions f such that $\|f\|_{L^X(p,q)} < \infty$.

The Lorentz-Bochner space, like the Bochner spaces, has most of the properties of the classical Lorentz spaces. For example, they are Banach spaces and the (real) dual of $L^X(p,q)$ is the Lorentz-Bochner space $L^{X^\star}(p',q')$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Of particular interest to this work is the case with $1 \le p < \infty$ and $q = 1$. That is the space

$$L^X(p,1) = \left\{ f : T \to X : \|f\|_{L^X(p,1)} = \frac{1}{p} \int_0^\infty \|f\|_X^*(t) t^{\frac{1}{p}-1} dt < \infty \right\}$$

whose dual is given by the space

$$L^{X^\star}(p',\infty) := \left\{ \phi : T \to X^\star : \|\phi\|_{L^{X^\star}(p',\infty)} = \sup_{t>0} t^{1/p'} \|\phi\|_\star^*(t) < \infty \right\},$$

where the ϕ's are $w^\star-$measurable. In the next section, we discuss the atomic characterization of the space $L^X(p,1)$ for $1 \le p < \infty$.

3.2 The Atomic Characterization of the Lorentz-Bochner Space $L^X(p,1)$

To begin with, we give the definition of the special atom space for vector valued functions.

Definition 3.5. *For $p \ge 1$, we define the space $A^X(p,\mu)$ as follows;*

$$A^X(p,\mu) := \left\{ f : T \to X : f(t) = \sum_{n\ge1} x_n \chi_{A_n}(t) \quad \text{and} \quad \sum_{n\ge1} \|x_n\|_X \mu(A_n)^{1/p} < \infty \right\},$$

where the x_n's are in X, and $A_n \in \mathcal{M}$ for each $n \ge 1$. We endow $A^X(p,\mu)$ with the "norm" $\|f\|_{A^X(p,\mu)} = \inf \sum_{n\ge1} \|x_n\|_X \mu(A_n)^{1/p}$, where the infimum is taken over all possible representations of f.

Remark 3.1. *This space is an extension of the space $B(\mu, 1/p)$ introduced in [4] by De Souza for vector-valued functions.*

Theorem 3.1. *$\left(A^X(p,\mu), \|\cdot\|_{A^X(p,\mu)}\right)$ is a Banach space.*

The proof of this theorem is similar to the real-valued case studied in [4] by De Souza up to slight modification and therefore omitted. The next definition is the candidate for the dual space of $A^X(p,\mu)$.

Definition 3.6. *We define the space $M(p, X^\star)$ as follows;*

$$M(p, X^\star) = \left\{ \phi : T \to X^\star; \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)^{1/p}} \int_A \|\phi(t)\|_\star d\mu(t) < \infty \right\},$$

where the ϕ's are $w^\star-$measurable and we endow it with the "norm"

$$\|\phi\|_{M(p,X^\star)} = \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)^{1/p}} \int_A \|\phi(t)\|_\star d\mu(t)$$

Remark 3.2. *The space $M(p, X^\star)$ is an extension of the space $M(\alpha)$ with $\alpha = 1/p$ introduced in [10] by Lorentz for vector valued functions.*

Theorem 3.2. *$(M(p, X^\star), \|\cdot\|_{M(p,X^\star)})$ is a Banach space.*

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $M(p, X^\star)$ such that $\sum_{n=1}^{\infty} \|\phi_n\|_{M(p,X^\star)} < \infty$. It suffices to show that $\sum_{n=1}^{\infty} \phi_n$ converges in $M(p, X^\star)$. To do this, first we observe that $\int_T \sum_{n=1}^{\infty} \|\phi_n(t)\|_\star d\mu(t) = \sum_{n=1}^{\infty} \int_T \|\phi_n(t)\|_\star d\mu(t) \leq \sum_{n=1}^{\infty} \|\phi_n\|_{M(p,X^\star)} \mu(T)^{1/p} < \infty$. Thus, we have that $\sum_{n=1}^{\infty} \|\phi_n(t)\|_\star < \infty$ $\mu-$a.e $t \in T$ and since $X^\star$ is a Banach space it follows that $\sum_{n=1}^{\infty} \phi_n(t)$ converges in $X^\star$ and $\|\sum_{n=1}^{\infty} \phi_n(t)\|_\star \leq \sum_{n=1}^{\infty} \|\phi_n(t)\|_\star$ $\mu-$a.e $t \in T$. Now, let $A \in \mathcal{M}$ with $\mu(A) \neq 0$ and consider

$$\int_A \|\sum_{n=1}^{\infty} \phi_n(t)\|_\star d\mu(t) \leq \sum_{n=1}^{\infty} \int_A \|\phi_n(t)\|_\star \leq \sum_{n=1}^{\infty} \|\phi_n\|_{M(p,X^\star)} \mu(A)^{1/p}$$

Thus

$$\frac{1}{\mu(A)^{1/p}} \int_A \|\sum_{n=1}^{\infty} \phi_n(t)\|_\star d\mu(t) \leq \sum_{n=1}^{\infty} \|\phi_n\|_{M(p,X^\star)} \leq C < \infty.$$

Hence $\sum_{n=1}^{\infty} \phi_n$ converges in $M(p, X^\star)$ and $\| \sum_{n=1}^{\infty} \phi \|_{M(p,X^\star)} \leq \sum_{n=1}^{\infty} \|\phi\|_{M(p,X^\star)}$. $\square$

Remark 3.3. *Indeed, we observe that for any Banach space X, the space $M(p, X)$, $p \in [1, \infty)$ defined by*

$$M(p, X) = \left\{ f : T \to X; \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)^{1/p}} \int_A \|f(t)\|_X d\mu(t) < \infty \right\},$$

endowed with the norm

$$\|f\|_{M(p,X)} = \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)^{1/p}} \int_A \|f(t)\|_X d\mu(t)$$

is a Banach space. Similar to the real-valued case, we also have that

$$M(p, X) \cong L^X(p', \infty)$$

with equivalent norms where $\frac{1}{p} + \frac{1}{p'} = 1$.

The following result is a Hölder's Type inequality involving the spaces $A^X(p, \mu)$ and $M(p, X^\star)$.

Theorem 3.3 (Hölder's Type Inequality)**.** *If $\phi \in M(p, X^\star)$ and $f \in A^X(p, \mu)$, then*

$$\left| \int_T \langle \phi(t), f(t) \rangle d\mu(t) \right| \leq \|\phi\|_{M(p,X^\star)} \|f\|_{A^X(p,\mu)}.$$

Proof. Let $\phi \in M(p, X^\star)$ and $f \in A^X(p, \mu)$ with $f(t) = \sum_{n \geq 1} x_n \chi_{A_n}(t)$, and $\sum_{n \geq 1} \|x_n\|_X \mu(A)^{1/p} < \infty$. So $\int_T \langle \phi(t), f(t) \rangle d\mu(t) = \int_T \sum_{n=1}^{\infty} \langle \phi(t), x_n \rangle \chi_{A_n}(t) d\mu(t)$. It follows that

$$
\begin{aligned}
\left|\int_T \langle\phi(t), f(t)\rangle d\mu(t)\right| &\leq \int_T \sum_{n=1}^{\infty} |\langle\phi(t), x_n\rangle|\, \chi_{A_n}(t) d\mu(t) \\
&\leq \sum_{n=1}^{\infty} \|x_n\|_X \int_{A_n} \|\phi(t)\|_\star d\mu(t) \\
&\leq \left(\sum_{n=1}^{\infty} \|x_n\|_X \mu(A_n)^{1/p}\right) \|\phi\|_{M(p,X^\star)}.
\end{aligned}
$$

Taking the infimum on the R.H.S over all possible representations of f, we have

$$
\left|\int_T \langle\phi(t), f(t)\rangle d\mu(t)\right| \leq \|\phi\|_{M(p,X^\star)} \|f\|_{A^X(p,\mu)}.
$$

□

The above result implies that for any given $\phi \in M(p, X^\star)$ the map $\Gamma : A^X(p,\mu) \to \mathbb{R}$ with $\Gamma(f) = \int_T \langle\phi(t), f(t)\rangle d\mu(t)$ is a well-defined bounded linear functional. To characterize all bounded linear functionals defined on $A^X(p,\mu)$, consider the following theorems;

Theorem 3.4 (Radon-Nikodým type II). *If $m : \mathcal{M} \to X^\star$ is a vector measure and m is absolutely continuous with respect to μ, then there exists $\phi : T \to X^\star$, $w^\star$-measurable satisfying the following three conditions;*

(1) The function $t \mapsto \|\phi(t)\|_\star$ is $\mu-$measurable and belongs to $L_1(\mu)$

(2) For all $x \in X$ and all $A \in \mathcal{M}$

$$
m(A)(x) = \int_A \langle\phi(t), x\rangle d\mu(t)
$$

(3) For all $A \in \mathcal{M}$, $\|m\|(A) = \int_A \|\phi(t)\|_\star d\mu(t)$

The proof of Theorem 3.4 is outside the scope of this work and can be found in [13], page 22.

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ with $\sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p} < \infty$ and $\phi \in M(p, X^\star)$. For each $n \in \mathbb{N}$, define $h_n : T \to \mathbb{R}$ by $h_n(t) := \langle \phi(t), x_n\rangle \chi_{A_n}(t)$. Then $h_n \in L_1(\mu), n \in \mathbb{N}$ and $\sum_{n\geq 1} \|h_n\|_1 < \infty$.*

Proof. Let $n \in \mathbb{N}$. Observe that h_n is $\mu-$measurable (since ϕ is $w^\star-$measurable) and consider $|h_n(t)| = |\langle\phi(t), x_n\rangle| \chi_{A_n}(t) \leq \|x_n\|_X \|\phi(t)\|_\star \chi_{A_n}(t)$, $t \in T$. Thus $\int_T |h_n(t)|\, d\mu(t) \leq \|x_n\|_X \int_{A_n} \|\phi(t)\|_\star \, d\mu(t) \leq \|x_n\|_X \mu(A_n)^{1/p} \|\phi\|_{M(p,X^\star)} < \infty$. That is, $h_n \in L_1(\mu)$ and $\sum_{n\geq 1} \|h_n\|_1 \leq \left(\sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p}\right) \|\phi\|_{M(p,X^\star)} < \infty$. □

As a consequence of Theorem 3.1 and Theorem 2.11 we have that

$$\int_T \sum_{n\geq 1} h_n(t) d\mu(t) = \sum_{n\geq 1} \int_T h_n(t) d\mu(t).$$

Theorem 3.5 (Duality)**.** *$(A^X(p,\mu))^\star \cong M(p, X^\star)$; that is, $\Gamma \in (A^X(p,\mu))^\star$ if and only if there exists a unique $\phi \in M(p, X^\star)$ such that*

$$\Gamma(f) = \int_T \langle\phi(t), f(t)\rangle d\mu(t),$$

for all $f \in A^X(p,\mu)$. In addition, $\|\Gamma\|_\star = \|\phi\|_{M(p,X^\star)}$.

Proof.

"$\Longleftarrow$": Follows from Theorem 3.3.

"$\Longrightarrow$": Let $\Gamma \in (A^X(p,\mu))$ and consider the vector measure $m : \mathcal{M} \to X^\star$ associated with Γ defined by $m(A)(x) = \Gamma(x\chi_A)$, for all $A \in \mathcal{M}$ and all $x \in X$. We observed that

$\Gamma(x\chi_A) \leq |\Gamma(x\chi_A)| \leq \|\Gamma\|_\star \|x\chi_A\|_{A^X(p,\mu)}$. But $\|x\chi_A\|_{A^X(p,\mu)} \leq \|x\|_X \mu(A)^{1/p}$. So we get that $\Gamma(x\chi_A) \leq \|x\|_X \mu(A)^{1/p} \|\Gamma\|_\star$, for $A \in \mathcal{M}$ and $x \in X$. In particular, we have that for each $A \in \mathcal{M}$, the set $\{\Gamma(x\chi_A) : \|x\|_X \leq 1\}$ is bounded above by $\mu(A)^{1/p} \|\Gamma\|_\star$.

Claim: Given $A \in \mathcal{M}$, we have that $\|m\|(A) \leq \mu(A)^{1/p} \|\Gamma\|_\star$.

Justification: It suffices to show that $\sum_{i=1}^{n} \|m(A_i)\|_\star \leq \mu(A)^{1/p} \|\Gamma\|_\star$, for any partition $A = \cup_{i=1}^{n} A_i$, $A_i \in \mathcal{M}$ and $A_i \cap A_j = \emptyset, i \neq j$. To see this, consider

$$\begin{aligned}
\sum_{i=1}^{n} \|m(A_i)\|_\star &= \sum_{i=1}^{n} \sup\{|m(A_i)(x)| : \|x\|_X \leq 1\} \\
&= \sum_{i=1}^{n} \sup\{|\Gamma(x\chi_{A_i})| : \|x\|_X \leq 1\} \\
&= \sum_{i=1}^{n} \sup\{\Gamma(x\chi_{A_i}) : \|x\|_X \leq 1\}, \quad \text{since} \quad \Gamma \in (A^X(p,\mu))^\star \\
&= \sup\left\{\sum_{i=1}^{n} \Gamma(x\chi_{A_i}) : \|x\|_X \leq 1\right\} \\
&= \sup\{\Gamma(x\chi_A) : \|x\|_X \leq 1\} \\
&\leq \mu(A)^{1/p} \|\Gamma\|_\star .
\end{aligned}$$

Thus, $\sum_{i=1}^{n} \|m(A_i)\|_\star \leq \mu(A)^{1/p} \|\Gamma\|_\star$ and hence $\|m\|(A) \leq \mu(A)^{1/p} \|\Gamma\|_\star$. Therefore, m is absolutely continuous with respect to μ. So let $\phi : T \to X^\star$ as in Theorem 3.4. By condition (3) of Theorem 3.4, we have that for each $A \in \mathcal{M}$ with $\mu(A) \neq 0$,

$$\int_A \|\phi(t)\|_\star \, d\mu(t) = \|m\|(A) \leq \mu(A)^{1/p} \|\Gamma\|_\star .$$

So,

$$\frac{1}{\mu(A)^{1/p}} \int_A \|\phi(t)\|_\star \, d\mu(t) \leq \|\Gamma\|_\star < \infty.$$

Hence $\phi \in M(p, X^\star)$ and $\|\phi\|_{M(p,X^\star)} \leq \|\Gamma\|_\star$. Now, by condition (2) of Theorem 3.4, we have that $\Gamma(x\chi_A) = m(A)(x) = \int_A \langle \phi(t), x \rangle d\mu(t) = \int_T \langle \phi(t), x\chi_A(t) \rangle d\mu(t)$. That is,

$$\Gamma(x\chi_A) = \int_T \langle \phi(t), x\chi_A(t) \rangle d\mu(t).$$

Let $f \in A^X(p, \mu)$ with $f(t) = \sum_{n\geq 1} x_n \chi_{A_n}(t)$ and $\sum_{n\geq 1} \|x_n\|_X \, \mu(A_n)^{1/p}$. We have that

$$\begin{aligned}
\Gamma(f) &= \Gamma\left(\sum_{n\geq 1} x_n \chi_{A_n}\right) = \sum_{n\geq 1} \Gamma\left(x_n \chi_{A_n}\right) = \sum_{n\geq 1} \int_T \langle \phi(t), x_n \chi_{A_n}(t) \rangle d\mu(t) \\
&= \sum_{n\geq 1} \int_T h_n(t) d\mu(t), \quad \text{with} \quad h_n \quad \text{as in Lemma 3.1} \\
&= \int_T \sum_{n\geq 1} h_n(t) d\mu(t) = \int_T \sum_{n\geq 1} \langle \phi(t), x_n \chi_{A_n}(t) \rangle d\mu(t) \\
&= \int_T \langle \phi(t), \sum_{n\geq 1} x_n \chi_{A_n}(t) \rangle d\mu(t) \\
&= \int_T \langle \phi(t), f(t) \rangle d\mu(t).
\end{aligned}$$

Thus,

$$\Gamma(f) = \int_T \langle \phi(t), f(t) \rangle d\mu(t).$$

Now, $|\Gamma(f)| = \left| \int_T \langle \phi(t), f(t) \rangle d\mu(t) \right| \leq \|\phi\|_{M(p,X^\star)} \|f\|_{A^X(p,\mu)}$, by Theorem 3.3 and thus $\|\Gamma\|_\star \leq \|\phi\|_{M(p,X^\star)}$. So we have $\|\Gamma\|_\star = \|\phi\|_{M(p,X^\star)}$. This completes the proof. $\square$

In the following we will show that $A^X(p, \mu) \cong L^X(p, 1)$ for $p \geq 1$, with equivalent norms. That is, there exist absolute positive constants α and β such that $\alpha \|f\|_{A^X(p,\mu)} \leq \|f\|_{L^X(p,1)} \leq \beta \|f\|_{A^X(p,\mu)}$. The first observation is that $M(p, X^\star) \cong L^{X^\star}(p', \infty)$, where

$1/p + 1/p' = 1$ and thus $(L^X(p,1))^\star \equiv (A^X(p,\mu))^\star$. In the next theorem, we will show that $A^X(p,\mu)$ is continuously contained in $L^X(p,1)$.

Theorem 3.6. *The space $A^X(p,\mu)$ is continuously contained in $L^X(p,1)$ with*

$$\|f\|_{L^X(p,1)} \leq \|f\|_{A^X(p,\mu)}$$

for $1 \leq p < \infty$.

Proof. First, we observe that given $x \in X$ and $A \in \mathcal{M}$, we have $(\|x\chi_A\|_X)^*(t) = \|x\|_X \chi_A^*(t) = \|x\|_X \chi_{[0,\mu(A))}(t)$. Thus $\|x\chi_A\|_{L^X(p,1)} = \dfrac{1}{p}\displaystyle\int_0^\infty \|x\|_X \chi_{[0,\mu(A))}(t) t^{\frac{1}{p}-1} dt = \|x\|_X \mu(A)^{1/p}$. That is, $\|x\chi_A\|_{L^X(p,1)} = \|x\|_X \mu(A)^{1/p}$. Now, let $f \in A^X(p,\mu)$ with $f(t) = \displaystyle\sum_{n\geq 1} x_n \chi_{A_n}(t)$ and $\displaystyle\sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p} < \infty$. That is, $\displaystyle\sum_{n\geq 1} \|x_n\chi_{A_n}\|_{L^X(p,1)} < \infty$. Thus, $f(t) = \displaystyle\sum_{n\geq 1} x_n\chi_{A_n}(t)$ converges in $L^X(p,1)$ and $\|f\|_{L^X(p,1)} \leq \displaystyle\sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p}$. Taking the infimum over all the representations of f, we get $\|f\|_{L^X(p,1)} \leq \|f\|_{A^X(p,\mu)}$ for $1 \leq p < \infty$.

□

We therefore have the following situations:

(1) $A^X(p,\mu) \subseteq L^X(p,1)$ and $\|f\|_{L^X(p,1)} \leq \|f\|_{A^X(p,\mu)}$ for $1 \leq p < \infty$.

(2) $(A^X(p,\mu))^\star \cong (L^X(p,1))^\star$, since $M(p, X^\star) \cong L^{X^\star}(p',\infty)$, $1/p + 1/p' = 1$.

(3) $A^X(p,\mu)$ is dense in $L^X(p,1)$. Easily verified with standard technique.

As a consequence of these facts, the embedding operator $I : A^X(p,\mu) \to L^X(p,1)$ defined by $I(f) = f$ is a Banach space isomorphism. Thus, we have the following result.

Theorem 3.7. *$A^X(p,\mu) \cong L^X(p,1)$ with equivalent norms, for $1 \le p < \infty$. That is, $f \in A^X(p,\mu) \iff f \in L^X(p,1)$ and $\alpha \|f\|_{A^X(p,\mu)} \le \|f\|_{L^X(p,1)} \le \beta \|f\|_{A^X(p,\mu)}$ where α and β are absolute positive constants.*

Remark 3.4. *Theorem 3.7 implies that given $f \in L^X(p,1)$, $p \ge 1$, we have that $f(t) = \sum_{n\ge1} x_n \chi_{A_n}(t)$ with $\sum_{n\ge1} \|x_n\|_X \mu(A_n)^{1/p}$ and $\|f\|_{L^X(p,1)} \cong \inf \sum_{n\ge1} \|x_n\|_X \mu(A_n)^{1/p}$. As we shall see in the next chapter this characterization gives a way to extend the result by Stein and Weiss for operators defined on measurable vector-valued functions. This provides a simple way to study the boundedness of operators on $L^X(p,1)$.*

Chapter 4

Applications

4.1 Application of the Atomic Characterization of L_1

In this section, we use the characterization of L_1 to show that the study of the boundedness of linear operators defined on L_1 reduces to the study of the action on characteristic functions. More specifically, we will study the boundedness of the Multiplication and Composition operators defined into other Banach spaces such as the Lebesgue L_p and Lorentz $L(p,1)$ spaces for $p \geq 1$. The following result is the result by Stein and Weiss given in Theorem 1.3 for $p = 1$.

Theorem 4.1. *If T is a linear operator on the space of measurable functions and $\|T(\chi_A)\|_Y \leq M\mu(A), A \in \mathcal{M}$ where Y is a Banach space, then T can be extended to all L_1; that is $T : L_1 \to Y$ and $\|T(f)\|_Y \leq M \|f\|_1$.*

Proof. Define $T : L_1 \to Y$ by $T(f) = \sum_{n \geq 1} c_n T(\chi_{A_n})$ for every $f \in L_1$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$. We have that T is well-defined on L_1 and it is linear. Now consider, $\|T(f)\|_Y \leq \sum_{n\geq 1} |c_n| \frac{1}{\mu(A_n)} \|T(\chi_{A_n})\|_Y$. So, $\|T(f)\|_Y \leq M \sum_{n\geq 1} |c_n|$. Taking the infimum on the R.H.S over all representations of f, we get $\|T(f)\|_Y \leq M\|f\|_1$. □

Remark 4.1. *As mentioned earlier, Theorem 4.1 shows that to study the boundedness of an operator defined on L_1 it is enough to study the action on the characteristic function of a measurable set as we shall see in the following examples.*

4.1.1 Multiplication Operator on L_1

The Multiplication operator, defined as the point-wise multiplication by a measurable function, is one of the commonly studied operators. It has been studied by many authors in the past decades, especially on the Lebesgue and Lorentz spaces, and their variants. For more information, we refer to [15, 16, 17, 21, 22, 23, 24]. In this section, we study the multiplication operator defined on L_1 into the Lebesgue space L_p and the Lorentz spaces $L(p, 1)$ for $1 \leq p < \infty$.

Let $g : T \to \mathbb{R}$ be a measurable function and Y be a Banach space of functions defined on T. The Multiplication operator $M_g : L_1 \to Y$ is defined as follows;

$$M_g(f) = g \cdot f, \quad \text{for} \quad f \in L_1$$

where $g \cdot f$ is the point-wise product of g and f, that is, $(g \cdot f)(t) = g(t) \cdot f(t)$ for $t \in T$.

We recall the following Banach spaces, which is a generalization of $M(\alpha)$ introduced by G. G Lorentz in [10] and denoted $M(\alpha, p)$ for $0 < \alpha \leq 1$ and $1 \leq p < \infty$:

$$M(\alpha, p) = \left\{ g : T \to \mathbb{R}; \quad \|g\|_{M(\alpha,p)} = \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)^\alpha} \left(\int_A |g(t)|^p d\mu(t) \right)^{1/p} < \infty \right\}$$

Remark 4.2. *$M(\alpha, 1) = M(\alpha)$ and thus $M(1, 1) = L_\infty$.*

Theorem 4.2. *The multiplication operator $M_g : L_1 \to L_p$, for $p \geq 1$ is bounded if and only if $g \in M(1, p)$ and $\|M_g\| = \|g\|_{M(1,p)}$.*

Proof. Suppose $g \in M(1, p)$ and let $A \in \mathcal{M}$. Consider

$$\|M_g(\chi_A)\|_p = \left(\int_T |g(t)\chi_A(t)|^p d\mu(t) \right)^{1/p} = \left(\int_A |g(t)|^p d\mu(t) \right)^{1/p} \leq \|g\|_{M(1,p)} \mu(A).$$

That is,

$$\|M_g(\chi_A)\|_p \leq \|g\|_{M(1,p)} \mu(A).$$

So by Theorem 4.1, we have that $\|M_g(f)\|_p \leq \|g\|_{M(1,p)} \|f\|_1$ for all $f \in L_1$ and

$$\|M_g\| \leq \|g\|_{M(1,p)}.$$

Conversely, suppose M_g is bounded, that is $\|M_g(f)\|_p \leq \|M_g\| \|f\|_1$ for all $f \in L_1$. In particular, for any $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have $\|M_g(\chi_A)\|_p \leq \|M_g\| \|\chi_A\|_1$. That is,

$$\left(\int_A |g(t)|^p d\mu(t)\right)^{1/p} \leq \|M_g\| \mu(A).$$

So,

$$\sup_{\mu(A)\neq 0} \frac{1}{\mu(A)} \left(\int_A |g(t)|^p d\mu(t)\right)^{1/p} \leq \|M_g\| < \infty$$

Hence, $g \in M(1,p)$ and $\|g\|_{M(1,p)} \leq \|M_g\|$. Thus, $\|M_g\| = \|g\|_{M(1,p)}$.

□

Remark 4.3. *In particular, the multiplication operator $M_g : L_1 \to L_1$ is bounded if and only if $g \in L_\infty$ and $\|M_g\| = \|g\|_\infty$.*

To study the multiplication operator $M_g : L_1 \to L(p,1)$, we consider the following space;

$$\Delta(p) = \left\{ g : T \to \mathbb{R}; \quad \|g\|_{\Delta(p)} = \sup_{\mu(A)\neq 0} \frac{1}{p\mu(A)} \int_0^{\mu(A)} g^*(t) t^{\frac{1}{p}-1} dt < \infty \right\}$$

Remark 4.4. $\Delta(p) \subseteq L_\infty$ *and* $\Delta(1) = L_\infty$.

Theorem 4.3. *The multiplication operator $M_g : L_1 \to L(p,1)$, for $p \geq 1$ is bounded if and only if $g \in \Delta(p)$ and $\|M_g\| = \|g\|_{\Delta(p)}$.*

Proof. Suppose $g \in \Delta(p)$ and let $A \in \mathcal{M}$. By Proposition 1.2, we have $(g\chi_A)^*(t) \leq g^*(t)\chi_{[0,\mu(A))}(t), \quad for \;\; all \;\; t \in [0,\infty)$. Consider

$$\|M_g(\chi_A)\|_{L(p,1)} = \frac{1}{p}\int_0^\infty (g\chi_A)^*(t)t^{\frac{1}{p}-1}dt \leq \frac{1}{p}\int_0^{\mu(A)} g^*(t)t^{\frac{1}{p}-1}dt.$$

So,

$$\|M_g(\chi_A)\|_{L(p,1)} \leq \|g\|_{\Delta(p)}\,\mu(A).$$

Hence, by Theorem 4.1 we have $\|M_g(f)\|_{L(p,1)} \leq \|g\|_{\Delta(p)}\|f\|_1$ for all $f \in L_1$ and

$$\|M_g\| \leq \|g\|_{\Delta(p)}$$

Conversely, suppose M_g is bounded, that is $\|M_g(f)\|_{L(p,1)} \leq \|M_g\|\,\|f\|_1$ for all $f \in L_1$. Let $A \in \mathcal{M}$ with $\mu(A) \neq 0$. By Proposition 1.3, choose $\tilde{A} \in \mathcal{M}$ with $\mu(\tilde{A}) = \mu(A)$ and $(g\chi_{\tilde{A}})^*(t) = g^*(t)\chi_{[0,\mu(A))}(t), \quad for \;\; all \;\; t \in [0,\infty)$. Now consider,

$$\begin{aligned}
\frac{1}{p}\int_0^{\mu(A)} g^*(t)t^{\frac{1}{p}-1}dt &= \frac{1}{p}\int_0^\infty \left(g^*(t)\chi_{[0,\mu(A))}(t)\right)t^{\frac{1}{p}-1}dt \\
&= \frac{1}{p}\int_0^\infty \left(g\chi_{\tilde{A}}\right)^*(t)t^{\frac{1}{p}-1}dt \\
&= \|M_g(\chi_{\tilde{A}})\|_{L(p,1)} \\
&\leq \|M_g\|\,\|\chi_{\tilde{A}}\|_1 \\
&= \|M_g\|\,\mu(A)
\end{aligned}$$

So, we have

$$\sup_{\mu(A)\neq 0} \frac{1}{p\mu(A)} \int_0^{\mu(A)} g^*(t)t^{\frac{1}{p}-1}dt \leq \|M_g\| < \infty$$

Thus, $g \in \Delta(p)$ and $\|g\|_{\Delta(p)} \leq \|M_g\|$. Hence, $\|M_g\| = \|g\|_{\Delta(p)}$ $\square$

4.1.2 Composition Operator on L_1

Just as the multiplication operator, the composition operator has also received considerable attention over the years. In this section we study the boundedness of the composition operator define on L_1 in to L_p and $L(p, 1)$.

Given a measurable and non-singular function $g : T \to T$ and a Banach space Y, the composition operator $C_g : L_1 \to Y$ is defined by;

$$C_g(f) = f \circ g, \quad \text{for all} \quad f \in L_1$$

where $f \circ g$ is the composite of f and g.

Remark 4.5. *We observe that for any $A \subseteq T$ and $g : T \to T$, $\chi_A \circ g = \chi_{g^{-1}(A)}$, where g^{-1} denotes the pre-image of A under g,that is, $g^{-1}(A) = \{t \in T : g(t) \in A\}$.*

Theorem 4.4. *The composition operator $C_g : L_1 \to L_p$, $1 \leq p < \infty$ is bounded if and only if $\mu(g^{-1}(A))^{1/p} \leq C\mu(A)$ for some $C > 0$ and all $A \in \mathcal{M}$, and*

$$\|C_g\| = \sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)}.$$

Proof. Suppose g satisfies the condition $\mu(g^{-1}(A))^{1/p} \leq C\mu(A)$ for all $A \in \mathcal{M}$ and some $C > 0$. So, we have $\|C_g(\chi_A)\|_p = \left\|\chi_{g^{-1}(A)}\right\|_p = \mu(g^{-1}(A))^{1/p} \leq C\mu(A)$. Thus,

$$\|C_g(\chi_A)\|_p \leq C\mu(A), \quad \text{for all} \quad A \in \mathcal{M}$$

Hence, $\|C_g(f)\|_p \leq \left(\sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)}\right) \|f\|_1, \quad f \in L_1$ by Theorem 4.1 and

$$\|C_g\| \leq \sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)}.$$

Conversely, if $C_g : L_1 \to L_p$ is bounded, then for any $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\|C_g(\chi_A)\|_p \leq \|C_g\| \, \|\chi_A\|_1$$

That is,

$$\mu(g^{-1}(A))^{1/p} \leq \|C_g\| \, \mu(A)$$

So,

$$\sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)} \leq \|C_g\| < \infty, \quad \text{and thus} \quad \|C_g\| = \sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)}$$

$\square$

Theorem 4.5. *The composition operator $C_g : L_1 \to L(p,1), \quad 1 \leq p < \infty$ is bounded if and only if $\mu(g^{-1}(A))^{1/p} \leq C\mu(A)$ for some $C > 0$ and all $A \in \mathcal{M}$, and*

$$\|C_g\| = \sup_{\mu(A)\neq 0} \frac{\mu(g^{-1}(A))^{1/p}}{\mu(A)}.$$

The proof of Theorem 4.5 is similar to that of Theorem 4.4 and therefore omitted.

In the next section we study the Multiplication and Composition operators on the Lorentz-Bochner spaces $L^X(p,1)$.

4.2 Application of the Atomic Characterization of the Lorentz-Bochner Space $L^X(p,1)$

As a consequence of the characterization given in Theorem 3.7, we have a simple way to study the boundedness of operators defined on the Lorentz-Bochner space $L^X(p,1)$ for $p \geq 1$. First, we have the following result which is an extension of Theorem 1.3 by E. Stein and G. Weiss for Lorentz-Bochner spaces.

Theorem 4.6. *Let T be a linear operator defined on the space of measurable vector-valued functions into a Banach space Y such that $\|T(x\chi_A)\|_Y \leq M \|x\|_X (\mu(A))^{1/p}$, $1 \leq p < \infty$, $x \in X$ and $A \in \mathcal{M}$. Then T can be extended to all $L^X(p,1)$; that is $T : L^X(p,1) \to Y$ and $\|T(f)\|_Y \leq M \|f\|_{L^X(p,1)}$ for all $f \in L^X(p,1)$.*

Proof. Indeed T can be extended to all of $L^X(p,1)$ as follows; $T : L^X(p,1) \to Y$ with $T(f) \doteq \sum_{n\geq 1} T(x_n\chi_{A_n})$, $f \in L^X(p,1)$ with $f(t) = \sum_{n\geq 1} x_n\chi_{A_n}$ and $\sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p} < \infty$. T is well-defined on $L^X(p,1)$ and $\|T(f)\|_Y \leq \sum_{n\geq 1} \|T(x_n\chi_{A_n})\|_Y \leq M \sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p}$. That is, $\|T(f)\|_Y \leq M \sum_{n\geq 1} \|x_n\|_X \mu(A_n)^{1/p}$ and so $\|T(f)\|_Y \leq M \|f\|_{L^X(p,1)}$.

□

Thus, to study the boundedness of a linear operator defined on $L^X(p,1)$ it is enough to study the situation for $f = x\chi_A$, for $x \in X$ and $A \in \mathcal{M}$ as we shall show with the multiplication and composition operators. These operators have been studied by many authors in particular by Arora et al in [23]. However, we believe that our technique and simplicity of our approach is worth noting to enrich the subject. To do this, we first consider the following definitions.

Let $\mathcal{B}(X) = \{L : X \to X : L$ is linear and bounded$\}$ with norm $\|L\| = \sup\{\|L(x)\|_X : \|x\|_X = 1\}$ for $L \in \mathcal{B}(X)$. For a strongly measurable function $u(\cdot) : T \to \mathcal{B}(X)$, define

$\|u(\cdot)\|:T\to\mathbb{R}$ by $\|u(\cdot)\|(w)=\|u(w)\|$ and $\|u(\cdot)(x)\|X:T\to\mathbb{R}$ by $\|u(\cdot)(x)\|_X(w)=\|u(w)(x)\|_X$ for $w\in T$ and $x\in X$. We define the spaces $M_1^p(\mathcal{B}(X))$ and V_p as follows;

$$M_1^p(\mathcal{B}(X))=\left\{u(\cdot):T\to\mathbb{B}(X):\|u\|_{M_1^p(\mathcal{B}(X))}=\sup_{\mu(A)\neq 0}\frac{1}{p\mu(A)^{1/p}}\int_0^{\mu(A)}\|u(\cdot)\|^*(t)t^{\frac{1}{p}-1}\ \ dt<\infty\right\}$$

and

$$V_p=\left\{g:T\to T:\|g\|_{V_p}=\sup_{\mu(A)\neq 0}\left\{\frac{\mu(g^{-1}(A))}{\mu(A)}\right\}^{1/p}<\infty\right\},$$

where g is non-singular and measurable.

Remark 4.6. *We note that $M_1^p(\mathcal{B}(X))\cong L_\infty(\mathcal{B}(X))$ with $\|u\|_{M_1^p(\mathcal{B}(X))}=\|u\|_{L_\infty(\mathcal{B}(X))}$. This is an extension of the result obtained by De Souza et al and we refer the reader to [24].*

For a strongly measurable function $u(\cdot):T\to\mathcal{B}(X)$, the Multiplication operator $M_u:L^X(p,1)\to L(T,X)$ is defined as

$$(M_uf)(w)=u(w)(f(w)),\quad\text{for all}\quad w\in T,$$

where $L(T,X)$ is the space of all measurable functions from T to X. For a non-singular measurable function $g:T\to T$, the Composition operator $C_g:L^X(p,1)\to L(T,X)$ is defined as

$$(C_gf)(w)=f(g(w)),\quad\text{for all}\quad w\in T.$$

Theorem 4.7. *The multiplication operator $M_u:L^X(p,1)\to L^X(p,1)$ is bounded if and only if $u\in M_1^p(\mathcal{B}(X))$. Furthermore,*

$$\|M_u\|=\|u\|_{M_1^p(\mathcal{B}(X))}$$

Proof. Suppose $u \in M_1^p(\mathcal{B}(X))$ and let $f = x\chi_A$ for $A \in \mathcal{M}$ and $x \in X$. It is straight forward to see that $\|M_u x\chi_A\|_X^*(t) \leq \|u(\cdot)\|^*(t) \|x\|_X \chi_{[0,\mu(A))}(t)$ and thus

$$\|M_u x\chi_A\|_{L^X(p,1)} \leq \frac{1}{p} \|x\|_X \int_0^{\mu(A)} \|u(\cdot)\|^*(t) t^{\frac{1}{p}-1} \; dt.$$

Hence, it follows that

$$\|M_u x\chi_A\|_{L^X(p,1)} \leq \|u\|_{M_1^p(\mathcal{B}(X))} \|x\|_X \mu(A)^{1/p}.$$

Thus, by Theorem 4.6, we have that

$$\|M_u f\|_{L^X(p,1)} \leq \|u\|_{M_1^p(\mathcal{B}(X))} \|f\|_{L^X(p,1)}, \text{ for all } f \in L^X(p,1) \text{ and } \|M_u\| \leq \|u\|_{M_1^p(\mathcal{B}(X))}.$$

Now suppose that $M_u : L^X(p,1) \to L^X(p,1)$ is bounded. Let $A \in \mathcal{M}$ with $\mu(A) \neq 0$. Given $\epsilon > 0$, choose $x_{(\cdot)} \in X$ with $\|x_{(\cdot)}\|_X = 1$ such that

$$\|u(\cdot)\| \leq \|u(\cdot)(x_{(\cdot)})\|_X + \epsilon.$$

It follows that,

$$\|u(\cdot)\|^*(t) \leq \|u(\cdot)(x_{(\cdot)})\|_X^*(t/2) + \epsilon, \quad \text{for} \quad t \geq 0.$$

So,

$$\begin{aligned}
\int_0^{\mu(A)} \|u(\cdot)\|^*(t) t^{\frac{1}{p}-1} \; dt &\leq \int_0^{\mu(A)} \|u(\cdot)(x_{(\cdot)})\|_X^*(t/2) t^{\frac{1}{p}-1} \; dt + p\epsilon\mu(A)^{1/p} \\
&= 2^{1/p} \int_0^{\mu(A)/2} \|u(\cdot)(x_{(\cdot)})\|_X^*(t) t^{\frac{1}{p}-1} \; dt + p\epsilon\mu(A)^{1/p} \\
&= 2^{1/p} \int_0^{\infty} \|u(\cdot)(x_{(\cdot)})\|_X^*(t) \chi_{[0,\mu(A)/2)}(t) t^{\frac{1}{p}-1} \; dt + p\epsilon\mu(A)^{1/p}.
\end{aligned}$$

Choose $\tilde{A} \in \mathcal{M}$ with $\mu(\tilde{A}) = \mu(A)/2$ such that

$$(\left\|u(\cdot)(x_{(\cdot)})\right\|_X \chi_{\tilde{A}})^*(t) = \left\|u(\cdot)(x_{(\cdot)})\right\|_X^* (t)\chi_{[0,\mu(\tilde{A}))}(t).$$

Thus, we have

$$\begin{aligned}
\int_0^{\mu(A)} \|u(\cdot)\|^* (t)t^{\frac{1}{p}-1} \ \ dt &\leq 2^{1/p} \int_0^{\infty} (\left\|u(\cdot)(x_{(\cdot)})\right\|_X \chi_{\tilde{A}})^*(t)t^{\frac{1}{p}-1} \ \ dt + p\epsilon\mu(A)^{1/p} \\
&= 2^{1/p} \int_0^{\infty} (\left\|u(\cdot)(x_{(\cdot)}\chi_{\tilde{A}})\right\|_X)^*(t)t^{\frac{1}{p}-1} \ \ dt + p\epsilon\mu(A)^{1/p} \\
&= p2^{1/p} \left\|M_u x_{(\cdot)}\chi_{\tilde{A}}\right\|_{L^X(p,1)} + p\epsilon\mu(A)^{1/p} \\
&\leq p2^{1/p} \|M_u\| \left\|x_{(\cdot)}\right\|_X \mu(\tilde{A})^{1/p} + p\epsilon\mu(A)^{1/p}, \ \ \text{since} \ \ M_u \ \ \text{is bounded} \\
&= p\mu(A)^{1/p}(\|M_u\| + \epsilon).
\end{aligned}$$

Hence,

$$\frac{1}{p\mu(A)^{1/p}} \int_0^{\mu(A)} \|u(\cdot)\|^* (t)t^{\frac{1}{p}-1} \ \ dt \leq \|M_u\| + \epsilon.$$

Letting $\epsilon \to 0$, we get that $u \in M_1^p(\mathcal{B}(X))$ and $\|u\|_{M_1^p(\mathcal{B}(X))} \leq \|M_u\|$. Thus,

$$\|M_u\| = \|u\|_{M_1^p(\mathcal{B}(X))} .$$

□

Theorem 4.8. *The composition operator $C_g : L^X(p,1) \to L^X(p,1)$ is bounded if and only if $g \in V_p$. Furthermore,*

$$\|C_g\| = \|g\|_{V_p}$$

Proof. Suppose $g \in V_p$. Let $A \in \mathcal{M}, x \in X$ and consider $(C_g x\chi_A)(w) = x\chi_{g^{-1}(A)}(w),\ \ w \in T$ and thus $\|C_g x\chi_A\|_X^*(t) = \|x\|_X \chi_{[0,\mu(g^{-1}(A)))}$. So,

$$\begin{aligned}
\|C_g x\chi_A\|_{L^X(p,1)} &= \frac{1}{p}\int_0^\infty \|C_g x\chi_A\|_X^*(t) t^{\frac{1}{p}-1}\ \ dt \\
&= \frac{1}{p}\|x\|_X \int_0^{\mu(A)} t^{\frac{1}{p}-1}\ \ dt \\
&= \|x\|_X (\mu(g^{-1}(A)))^{1/p}.
\end{aligned}$$

So, we have that

$$\|C_g x\chi_A\|_{L^X(p,1)} = \|x\|_X (\mu(g^{-1}(A)))^{1/p},\ \text{ and thus }\ \|C_g x\chi_A\|_{L^X(p,1)} \le \|g\|_{V_p} \|x\|_X \mu(A)^{1/p}.$$

Hence, by Theorem 4.6, we get

$$\|C_g f\|_{L^X(p,1)} \le \|g\|_{V_p} \|f\|_{L^X(p,1)},\ \ \text{for all}\ \ f \in L^X(p,1)\ \ \text{and}\ \ \ \|C_g\| \le \|g\|_{V_p}.$$

Conversely, suppose $C_g : L^X(p,1) \to L^X(p,1)$ is bounded, and let $A \in \mathcal{M}$ with $\mu(A) \neq 0$ and $x \in X$ with $\|x\|_X = 1$. We have

$$\|C_g x\chi_A\|_{L^X(p,1)} \le \|C_g\| \|x\|_X \mu(A)^{1/p},\ \ \text{since}\ \ C_g\ \ \text{is bounded.}$$

That is,

$$\|x\|_X (\mu(g^{-1}(A)))^{1/p} \le \|C_g\| \|x\|_X \mu(A)^{1/p}.$$

Hence, we have

$$\left\{\frac{\mu(g^{-1}(A))}{\mu(A)}\right\}^{1/p} \le \|C_g\|.$$

So, we conclude that $g \in V_p$ and $\|g\|_{V_p} \le \|C_g\|$. Thus, $\|C_g\| = \|g\|_{V_p}$. $\square$

4.3 Conclusion

A decomposition of the Lebesgue space L_1, and the Lorentz-Bochner space $L^X(p, 1)$ for $p \geq 1$ has been studied in this work. It is worth noting that these decompositions are in terms of characteristic functions of measurable sets. These decompositions are used to study the boundedness of some linear operators defined on the spaces into other Banach spaces. The computations provided are quiet simple and straightforward since we only have to consider the action on characteristic functions which are simple functions to deal with. Similar decompositions will be studied for other well known Banach spaces in future work.

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