Progress on the Marcus-de Oliveira Determinantal Conjecture

by

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Abstract

We discuss the well known conjecture of Marcus and de Oliveira of the determinant of the sum of normal matrices. We present the motivation of this conjecture, along with detailed history of its origin. Then we compile several situations in which the conjecture holds. That is the matrices satisfying the equation:

$$\Delta(A,B) \subseteq \mathrm{co}\{\prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}) : \sigma \in S_n\} \tag{1}$$

We also display further relations and links between Marcus and de Oliveira conjecture and other topics and conjectures, and indicate the cases that will be implied from them. Moreover, we provide some consequential results in various aspects regarding this conjecture.

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Chapter 1

Introduction

The Marcus-de Oliveira determinantal conjecture (MOC) presumes that for two normal matrices A and B with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively, the determinant of the sum of A and B satisfies

$$\Delta(A, B) \subseteq \text{co}\{\prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}) : \sigma \in S_n\} \tag{1.1}$$

where

$$\Delta(A, B) = \{\det(A + UBU^*) : U \in U_n\} \tag{1.2}$$

and co$\{\}$ is the convex hull of the indicated complex numbers. Due to the fact that the quantity $\det(A + UBU^*)$ is invariant under simultaneous unitary similarity of A and B, we may assume that $A = \text{diag}(\alpha_1, \ldots, \alpha_n)$ and $B = \text{diag}(\beta_1, \ldots, \beta_n)$. In addition, $\Delta(A, B)$ is clearly a compact connected subset of the complex plane.

In this thesis, we present a detailed development of the progress on MOC. In chapter one, we begin with motivations and origin of this conjecture. Then, in chapter two, we construct a classification of several basic cases in which MOC holds. In chapter 3, we discuss extended results and connections of MOC to some other topics and conjectures, and show the results that approach MOC in certain cases.

1.1 History of The Problem

In 1971, Fiedler published his result concerning the best possible lower and upper bounds for the determinant of the sum of two hermitian matrices in terms of their eigenvalues [18]. He proved that for two $n \times n$ hermitian matrices A and B with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$

and $\beta_1 \geq \cdots \geq \beta_n$ respectively,

$$\min_\sigma \prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}) \leq \det(A+B) \leq \max_\sigma \prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}), \quad \sigma \in S_n \tag{1.3}$$

where S_n denote the symmetric group of degree n. In particular, if A and B are positive semidefinite, then

$$\prod_{i=1}^{n}(\alpha_i + \beta_i) \leq \det(A+B) \leq \prod_{i=1}^{n}(\alpha_i + \beta_{n+1-i}). \tag{1.4}$$

These estimates are best possible bounds in terms of eigenvalues of A and B.

Besides Fiedler's result, the ***numerical range*** of a normal matrix A with eigenvalues $\alpha_1, \ldots, \alpha_n$ which is given by

$$W(A) = \{x^* Ax : ||x|| = 1\} \tag{1.5}$$

is the convex hull of the points in the complex plane corresponding to the eigenvalues of A [20, Theorem 3], that is:

$$W(A) = \text{co}\{\alpha_1, \ldots, \alpha_n\}. \tag{1.6}$$

Motivated by the above significant results, Marcus in 1973 provided a relative result to the classical Toepolitz-Hausdroff theorem which states that the numerical range (1.5) is a convex set in the complex plane for any given bounded linear operator A on an n-dimensional (complex or real) Hilbert space V [27]. For $1 \leq r \leq m \leq n$ and $(x_1, \ldots, x_m)$ is an orthonormal set in V, define the complex numbers

$$f_{r,m}(x_1, x_2, \ldots, x_m) = \sum_{\omega \in Q_{r,m}} \det[(Ax_{\omega(i)}, x_{\omega(j)})]_{r \times r} \tag{1.7}$$

where $Q_{r,m}$ is the set of all $\binom{m}{r}$ sequences ω of length r chosen from $1, \ldots, m$, which satisfy $\omega(1) < \cdots < \omega(r)$. The set of all complex numbers $f_{r,m}(x_1, x_2, \ldots, x_m)$ is denoted by $W^n_{r,m}(A)$. Marcus described the convexity of the set $W^n_{r,m}(A)$, and he proved that

Theorem 1.1. *[27, Theorem 1] If $1 \leq r \leq n-1$ and $A \in Hom(V,V)$, then $W^n_{r,n-1}(A)$ is convex. If $2 \leq r < n-1$, then there exists a normal $A \in Hom(V,V)$ such that $W^n_{r,r}(A)$ is not convex.*

The above theorem is reformulated as follow,

Theorem 1.2. *[27, Theorem 3] If $1 \leq r \leq n-1$ and $A \in Hom(V,V)$, then the set of all complex numbers*

$$trC_r(PAP), \tag{1.8}$$

where P runs over all orthogonal projections of rank $n-1$, and C_r is the r^{th} compound of (PAP), is convex. If $2 \leq r < n-1$, then there exists a normal $A \in Hom(V,V)$, such that the set of numbers

$$\det(PAP), \tag{1.9}$$

where P runs over all orthogonal projections of rank r, is not convex.

The transformation (PAP) in the preceding theorem is regarded as being restricted to the range of P [27].

However, the technique that used to prove the preceding result is intricate to be applied on $W^n_{r,m}(A)$ where $2 \leq r \leq m \leq n-2$. If $1 \leq m \leq n$ and $x_1, \dots, x_m$ is an orthonormal set in n-dimensional complex or real space V, then corresponding to (1.7) let $f_\delta(x_1, \dots, x_m)$ refer to some fixed complex valued function δ of the $m \times m$ matrix $X = [(Ax_i, x_j)]$, that is

$$f_\delta(x_1, \dots, x_m) = \delta(X). \tag{1.10}$$

Marcus considered a general formulation of the above result when $W(A,\delta)$ is the collection of complex numbers (1.10), then $W(A,\delta)$ will be convex under some conditions. For instance, if

$$\delta(X) = \sum_{k=1}^{m} c_k x_{kk}, \quad c_k \neq 0, \quad k = 1, \dots, m \tag{1.11}$$

then $W(A,\delta)$ is the set of complex numbers

$$\delta(X) = \sum_{k=1}^{m} c_k (Ax_k, x_m), \quad (x_i, x_j) = \delta_{ij}, \quad i, j = 1, \ldots, m. \tag{1.12}$$

If A is normal with eigenvalues $\alpha_1, \ldots, \alpha_n$ and δ is defined as in (1.12), then

$$W(A,\delta) \subset \mathrm{co}\{\sum_{k=1}^{n} c_k \alpha_{\omega(k)}, \quad \omega \in D_{m,n}\} \tag{1.13}$$

where co{} denote the convex hull of the indicated numbers. $\Gamma_{m,n}$ is the totality of sequences ω of length m chosen from $1, \ldots, n$, and $D_{m,n}$ is the subset of $\Gamma_{m,n}$ consisting of those ω which satisfy $\omega(i) \neq \omega(j)$ whenever $i \neq j$ [27, Theorem4].

Marcus gives another interesting possibility for δ if A is normal which is

$$\delta(X) = \det(B + X) \tag{1.14}$$

where B is fixed $m \times m$ normal matrix. For $m = n$, the problem of proving the convexity of $W(A,\delta)$ is equivalent to determining the structure of the set $W(A,\delta)$ of all complex numbers

$$\det(B + UAU^*). \tag{1.15}$$

In case A and B are hermitian, $W(A,\delta)$ is a closed interval on the real line according to Fiedler's result. Therefore, as a generalization of Fiedler's result, Marcus conjectured that:

$$\det(B + UAU^*) \in \mathrm{co}\{\prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)}) : \sigma \in S_n\} \tag{1.16}$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are the eigenvalues of the normal matrices A and B respectively [27] .

Independently, in 1982, de Oliveira was also motivated by Fiedler's result [9]. He conjectured that for two $n \times n$ normal matrices A and B with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$

respectively, we have

$$\det(A+B) \in \mathrm{co}\{\prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}) : \sigma \in S_n\}. \tag{1.17}$$

In fact, Fiedler's result immediately implies de Oliveira's conjecture in the case that all the numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are real, that is A and B are hermitian. In addition, de Oliveira observed the fact that there are some true relations such that when we replace sums with products and products with sums, we obtain other true relations. Consequently, Oliveira applied this idea on his conjecture replacing determinant with trace and sum with product, and he briefly proved that for $n \times n$ normal matrices A and B,

$$W(A,B) \subseteq \mathrm{co}\{\prod_{i=1}^{n}(\alpha_i \beta_{\sigma(i)}) : \sigma \in S_n\} \tag{1.18}$$

where $W(A,B)$ is the B**- *numerical range*** of A given by:

$$W(A,B) = \{\mathrm{tr}(AUBU^*) : U \in U_n\}. \tag{1.19}$$

Based on the independent work in [27] and [9], the speculation (1.1) is now known as Marcus-de Oliveira determinantal conjecture (MOC).

1.2 Preliminaries

We utilize certain standard notations throughout this thesis. We let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ complex matrices, and U_n denote the set of all $n \times n$ unitary matrices. P_σ refers to the permutation matrix corresponding to $\sigma \in S_n$. $P_{IJ}(\sigma)$ is an extension of the usual permutation matrix where $I, J \subseteq \{1, 2, \ldots, n\}$ and $|I| = |J|$, and it is defined as

$$P_{IJ}(\sigma) = \begin{cases} 1, & \text{if } \sigma(I) = J, \\ 0, & \text{otherwise.} \end{cases} \tag{1.20}$$

The $n \times n$ identity matrix is denoted by I_n, or I if the dimension is clear from context. Given matrix A, we let A^T denote the transpose of A and A^* denote the conjugate transpose of A. $\sigma(A)$ refers to the spectrum of A, that is, the set of eigenvalues of A. The notation $\mathrm{diag}(a_1, \ldots, a_n)$ represents the diagonal matrix with $a_1, \ldots, a_n$ as diagonal entries. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be **hermitian** if it equals its own conjugate transpose ($A = A^*$); this implies that all eigenvalues of a hermitian matrix A are real. However, if $A \in \mathbb{C}^{n\times n}$ satisfies $A^* = -A$, then A is called **skew-hermitian** matrix which possess purely imaginary or zero eigenvalues. An $n \times n$ hermitian matrix A is called **positive definite** (respectively **positive semidefinite**, **negative definite**, **negative semidefinite**) if all its eigenvalues are positive (respectively nonnegative, negative, nonpositive). The **singular values** of a matrix A are the square roots of the eigenvalues of A^*A. $e_m(x_1, x_2, \ldots, x_n)$ is the m'th elementary symmetric function of $x_1, x_2, \ldots, x_n \in \mathbb{C}$, while $E_m(A)$ is the m'th elementary symmetric function of the eigenvalues of $A \in \mathbb{C}^{n\times n}$.

Additionally, we define principal concepts and notations associated with Marcus-de Oliveira conjecture. Given $A, B \in \mathbb{C}^{n\times n}$ with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, we define the following region:

$$\Delta(A, B) = \{\det(A + UBU^*) : U \in U_n\}, \tag{1.21}$$

and let $\partial\Delta(A, B)$ refer to the boundary of this region. The B-numerical range of A is given by:

$$W(A, B) = \{\mathrm{tr}(AUBU^*) : U \in U_n\}. \tag{1.22}$$

We also define z_σ-points which belong to $\Delta(A, B)$ by

$$z_\sigma = \prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)}), \quad \sigma \in S_n. \tag{1.23}$$

Therefore,

$$\mathrm{co}\{z_\sigma : \sigma \in S_n\} \tag{1.24}$$

presents the convex hull of complex numbers $\{z_\sigma : \sigma \in S_n\}$. Let $\Re z_\sigma$ denote the real part of z_σ, while $\Im z_\sigma$ denote the imaginary part.

Chapter 2

Progress on MOC

In this chapter we show basic progress of MOC. We classify essential results in which MOC is true, and arrange them in categories according to common features such as size, eigenvalues, and determinant. Heretofore, MOC has been proved in several cases like $n \leq 3$ (section 2.1), A and B being hermitian or one hermitian and the other skew-hermitian, or A and B being scalar multiples of unitaries (section 2.2). In addition to these result, MOC is true in several other cases which we will see in the following sections.

2.1 Size

Considering the size n of two normal matrices in MOC, the conjecture is known to be true for $n = 2$. In this case, it is proved that for two arbitrary complex matrices $A, B \in \mathbb{C}^{2\times 2}$, $\Delta(A, B)$ is an elliptical disc [19, Theorem 2.1]. The result is stated as follow.

Theorem 2.1. *Let A and B be 2×2 complex matrices with eigenvalues α_1, α_2 and β_1, β_2 respectively. $\Delta(A, B)$ is an elliptical disc with foci*

$$(\alpha_1 + \beta_1)(\alpha_2 + \beta_2), \quad (\alpha_1 + \beta_2)(\alpha_2 + \beta_1) \tag{2.1}$$

and length of minor semi-axis equal to $(ac - bd)$, where $a \geq b$ and $c \geq d$ are the singular values of $A - \frac{1}{2}Tr(A)I_2$ and $B - \frac{1}{2}Tr(B)I_2$ respectively. Specifically,

1. *$\Delta(A, B)$ is a singleton if and only if A or B is a scalar matrix.*

2. *$\Delta(A, B)$ is a non-degenerated line segment if and only if A and B are non-scalar normal matrices.*

It has been already proved that $W(A, B)$ is an elliptical disc for $A, B \in \mathbb{C}^{2\times 2}$ [24]. This beneficial technique concerning the numerical range can be generalized to verify the convexity in this case.

In addition, applying the famous theorem of Birkhoff on doubly stochastic matrices, MOC is established for $n = 3$ [6]. For 3×3 normal matrices A and B, the result is obtained in the following theorem.

Theorem 2.2. *Let* $A = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$ *and* $B = \operatorname{diag}(\beta_1, \beta_2, \beta_3)$*. We have*

$$\Delta(A, B) \subset \operatorname{co}\{z_\sigma : \sigma \in S_3\}. \tag{2.2}$$

Proof. Let $U = (u_{jk})$ be 3×3 unitary matrix. We have

$$\det(A + UBU^*) = \det(AU + UB) \det U^*. \tag{2.3}$$

Let $1 \leq j, k \leq 3$. The cofactor U_{jk} of U corresponding to j, k satisfies $U_{jk} = \bar{u}_{jk} \det U$. Using Cauchy- Binet formula we have

$$\det(AU + UB) \det U^* = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 + \alpha^T V \bar{\beta} + \bar{\alpha}^T V \beta \tag{2.4}$$

where $\alpha^T = (\alpha_1, \alpha_2, \alpha_3)$, $\bar{\alpha}^T = (\alpha_2\alpha_3, \alpha_1\alpha_3, \alpha_1\alpha_2)$, $\beta^T = (\beta_1, \beta_2, \beta_3)$, $\bar{\beta}^T = (\beta_2\beta_3, \beta_1\beta_3, \beta_1\beta_2)$, and $V = (|u_{jk}|^2)$, a doubly stochastic matrix, is a convex combination of permutation matrices. Thus, there exist $t_1, \ldots, t_6$ where $0 \leq t_1, \ldots, t_6 \leq 1$ and $\sum_{j=1}^{6} t_j = 1$ such that:

$$V = \sum_{j=1}^{6} t_j P_{\sigma(j)} = \begin{bmatrix} t_1 + t_4 & t_2 + t_6 & t_3 + t_5 \\ t_3 + t_6 & t_1 + t_5 & t_2 + t_4 \\ t_2 + t_5 & t_3 + t_4 & t_1 + t_6 \end{bmatrix}, \quad \sigma \in S_3. \tag{2.5}$$

Substituting in (2.4) we get:

$$\det(AU + UB)\det U^* = \sum_{j=1}^{6} t_j \prod_{k=1}^{3}(\alpha_k + \beta_{\sigma_j(k)}) = \sum_{j=1}^{6} t_j z_{\sigma(j)}. \tag{2.6}$$

$\square$

On the other hand, MOC has not been proved in general for $n = 4$, even though there are several attempts and appreciable efforts trying to demonstrate the conjecture in this case. A stronger conjecture on compounds of unitary matrices, called the Merikoski-Virtanen conjecture (MVC), asks if there exist real numbers t_σ summing to unity such that:

$$|u_{IJ}|^2 = \sum_{\sigma \in S_n} t_\sigma P_{IJ}(\sigma) \tag{2.7}$$

where u_{IJ} is a minor of U corresponding to the subsets I and J of $\{1, 2, \ldots, n\}$ [28]. Unfortunately, a counter example (example 3.1) disproved this conjecture for the case $n = 4$.

However, MVC has been investigated for Householder reflections, and it is proved that MVC is true for Householder reflection matrices U when $n \leq 4$ but fails for $n \geq 5$ [13]. As a result, MOC is demonstrated to be true for Householder reflections when $\operatorname{rank}(\alpha_i+\beta_j)_{n\times n} \leq 2$ or $n \leq 4$ [13].

A ***Householder reflection*** is a unitary matrix of the form $U = I - 2\zeta\zeta^*$ where ζ is a unit vector. Since $\det(A + UBU^*)$ is invariant under simultaneous unitary similarity of A and B, we have

$$\begin{aligned} \det(A + UBU^*) &= \det[\operatorname{diag}(\alpha_1, \ldots, \alpha_n) + U \operatorname{diag}(\beta_1, \ldots, \beta_n)U^*] \\ &= \det[\operatorname{diag}(\alpha_1, \ldots, \alpha_n)U + U \operatorname{diag}(\beta_1, \ldots, \beta_n)] \det(U^*) \\ &= \det(((\alpha_j + \beta_k)u_{jk})) \det(U^*) \\ &= \frac{\det(Z \circ U)}{\det(U)} \end{aligned} \tag{2.8}$$

where Z is the matrix $(\alpha_j+\beta_k)_{jk}$, and the notation $\circ$ denotes the ***Hadamard (entry wise) product*** of matrices. MOC for Householder reflections can be expressed as follow:

$$\frac{\det(Z\circ U)}{\det(U)} \in \mathrm{co}\{v(\sigma) : \sigma \in S_n\} \tag{2.9}$$

where $Z = (\alpha_j + \beta_k)_{jk}$, U is unitary, and $v(\sigma) = \prod_{j=1}^{n} z_{j,\sigma(j)}$.

Let S_n^0 be the subset of S_n containing the identity permutation ϵ, the transpositions, and the three-cycles.

Theorem 2.3. *[13, Theorem 7] Let $n \le 4$, and let U be $n\times n$ Householder reflection matrix. There exist nonnegative numbers t_σ indexed over S_n summing to unity, vanishing off S_n^0 and such that* (2.7) *holds for all subsets J and K of $\{1,\dots,n\}$ with $|J| = |K|$.*

Corollary 2.4. *MOC is true for Householder reflection matrix U when $n \le 4$.*

A stronger version of MOC is achieved for Householder reflections [13, Corollary 3].

Theorem 2.5. *For $U = I - 2\zeta\zeta^*$ a Householder reflection and Z complex $n\times n$ matrix of rank 2, we have*

$$\frac{\det(Z\circ U)}{\det(U)} \in \mathrm{co}\{v(\sigma) : \sigma \in S_n^0\} \tag{2.10}$$

The preceding theorem is proved by the following technical result [13, Theorem 2].

Theorem 2.6. *Let Z be a complex $n\times n$ matrix of rank 2, and let λ be a complex number. We have*

$$\det(Z\circ(I+\lambda\zeta\otimes\zeta^*)) \in \mathrm{co}\{(1+\frac{1}{2}\lambda)^2 v(\epsilon) + \frac{1}{4}\lambda^2 v(\sigma) : \sigma \in S_n^0\}. \tag{2.11}$$

In fact, this result is slightly stronger because the left hand side of (2.11) lies in the convex hull of $(1+\lambda)v(\epsilon)$, $(1+\frac{1}{2}\lambda)^2 v(\epsilon) + \frac{1}{4}\lambda^2 v(\tau)$ as τ runs over the transpositions and $(1+\frac{1}{2}\lambda)^2 v(\epsilon) + \frac{1}{8}\lambda^2 v(\rho^2)$ as ρ runs over the three cycles.

A technical conjecture, the external vertices conjecture (EVC), considers z_σ points as vertices [14]. The EVC claims that if these vertices satisfy that $\Re z_\sigma \ge 1$ for all $\sigma \in S_n$, there exist nonnegative numbers t_{IJ} such that

$$\Re z_\sigma = 1 + \sum_{IJ} t_{IJ} P_{IJ}(\sigma), \quad \forall \sigma \in S_n. \tag{2.12}$$

It has been proved in [14] that EVC holds for $n = 4$ (see theorem 3.3). As a consequence of this result, the following weak form of MOC holds for $n = 4$ [14, Corollary 1.2].

Theorem 2.7. *Let A and B be normal 4×4 matrices with prescribed complex eigenvalues $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ respectively. Suppose that $0 \notin \mathrm{co}\{z_\sigma : \sigma \in S_4\}$. We have*

$$\det(A + B) \in \mathrm{co}\{r z_\sigma : \sigma \in S_4, r \geq 1\}. \tag{2.13}$$

Proof. In a closed half-space H not containing zero but containing z_σ for all $\sigma \in S_4$, we may assume without loss of generality that $H = \{z : \Re z \geq 1\}$. Depending on the validity of EVC for $n = 4$, there exist real numbers t_{IJ} and a 4×4 unitary matrix U with determinant 1 satisfying that

$$\Re \det(A + B) = 1 + \sum_{IJ} t_{IJ} |u_{IJ}|^2 \tag{2.14}$$

implying that $\det(A + B) \in H$. However, if we suppose that $\det(A + B) \notin H$ then $s \det(A + B) \notin \Delta(A, B)$ for $0 \leq s \leq 1$,where

$$\Delta(A, B) = \mathrm{co}\{z_\sigma : \sigma \in S_4\}. \tag{2.15}$$

Thus, $\Delta(A, B)$ and the line segment joining zero to $\det(A+B)$ do not meet. By the separation theorem of convex sets, we can construct H containing $\Delta(A, B)$, but does not contain zero and $\det(A + B)$ which is a contradiction. $\square$

2.2 Eigenvalues

Eigenvalues of the matrices A and B generate the largest number of valid cases of MOC. The first general result related to eigenvalues of two normal matrices is an implication of Fiedler's original result [18]. Obviously, the conjecture holds when both A and B are hermitian matrices; that is all eigenvalues α_i and β_j are real.

Theorem 2.8. *MOC holds in the case that all α_i and β_j are real.*

Another remarkable result in which MOC holds is obtained when A is positive definite hermitian matrix and B is a skew-hermitian matrix [8, Theorem 3.1].

Theorem 2.9. *If A and B are $n \times n$ hermitian and skew-hermitian matrices, having eigenvalues $\alpha_1, \ldots, \alpha_n$ and $i\beta_1, \ldots, i\beta_n$ respectively, and the α_j's, $j = 1, 2, \ldots, n$ are positive and distinct real numbers, then MOC holds.*

We need the following definition:

Definition 2.10. *A supporting line of $\Delta(A, B)$ is a line such that,*

1. *$\Delta(A, B)$ belongs to one of a half planes it defines.*

2. *It intersects $\Delta(A, B)$ at least at one point.*

Proof. If $\det(A+B)$ belongs to every closed disk that contains the z_σ points, then it belongs to the convex hull of z_σ points because this convex hull is the intersections of all closed disks.

For a fixed complex number w, $\det(A + B)$ belongs to $\Delta(A, B)$ if $|\det(A + B) - w|$ is maximum. Obviously, $\det(A, B)$ lies on a supporting line. Therefore, $\det(A, B)$ is either a z_σ point or there exists a curve in $\Delta(A, B)$ having a zero curvature at $\det(A, B)$. The existence of such a curve is impossible because the boundary of the disk centered at w and of radius $|\det(A + B) - w|$ has nonzero curvature at $\det(A, B)$. $\square$

The validity of MOC has also been confirmed for a certain class of normal matrices. Specifically, for the ones that satisfy the condition that all eigenvalues of A and B have the same absolute value [4, Theorem 3].

Theorem 2.11. *If $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ and $B = \operatorname{diag}(\beta_1, \ldots, \beta_n)$, and $|\alpha_1| = |\alpha_2| = \cdots = |\alpha_n| = |\beta_1| = \cdots = |\beta_n| = \rho$, then $\Delta(A, B)$ is a line segment and MOC holds with equality.*

Proof. We have two cases to consider:

1. If $\rho \notin \sigma(A) \cup \sigma(-B)$, in this case we consider the skew-hermitian matrices:

$$C = \frac{1}{2}\frac{A + \rho I}{A - \rho I} \quad \text{and} \quad D = -\frac{1}{2}\frac{B - \rho I}{B + \rho I} \tag{2.16}$$

then, it is clear that

$$A = \rho(I + \frac{I}{C - \frac{1}{2}}) \quad \text{and} \quad B = \rho(-I + \frac{I}{D + \frac{1}{2}}). \tag{2.17}$$

From Fiedler's theorem we have $\Delta(C, D)$ is a line segment with z_σ points as end points. Consequently, (by remark 2.36) $\Delta(A, B)$ is also a line segments with z_σ points as end points.

2. If $\rho \in \sigma(A) \cup \sigma(-B)$, we choose a real φ such that $\rho \notin \sigma(Ae^{i\varphi}) \cup \sigma(-Be^{i\varphi})$. Since $\Delta(A, B) = \Delta(Ae^{i\varphi}, Be^{i\varphi})e^{in\varphi}$, $\Delta(Ae^{i\varphi}, Be^{i\varphi})$ is a line segment according to the previous case.

□

A major result of MOC is demonstrated for a wide class of pairs of essentially hermitian matrices [5].

Definition 2.12. *A normal matrix A is said to be essentially hermitian matrix if it possesses collinear eigenvalues.*

A combination of the following theorems which extend the idea of theorem 2.9 prove MOC for a large class of essentially hermitian matrices[5, Theorem 1,1′].

Theorem 2.13. *Assume $\alpha_1, \ldots, \alpha_n$ to be real pairwise distinct positive numbers, and assume the complex numbers $\beta_1, \ldots, \beta_n$ lie along a line through the origin, then MOC holds.*

Theorem 2.14. *Assume that $\alpha_1, \ldots, \alpha_n$ to be pairwise distinct complex numbers lying on a line l , and $\beta_1, \ldots, \beta_n$ to lie on a parallel line to l, then MOC holds.*

We obtain this result from purely geometrical point of view, and utilizing the following theorem and proposition [5, Theorem 2, Proposition 3].

Definition 2.15. *A support point of a compact region is a simultaneous boundary point of the region and of some closed half space containing that region.*

Theorem 2.16. *With the assumption of 2.13 and 2.14. Let $z \in \partial\Delta(A, B)$ be a regular boundary point such that $z \neq 0$ and z is not a z_σ point. There passes a regular curve through z contained in $\Delta(A, B)$ and has zero curvature at z.*

Proposition 2.17. *Given a compact region $\Delta \subseteq \mathbb{C}$, assume that each support point $z \in \Delta$ is either belong to a certain finite set $S \subseteq \Delta$ or has the property that there passes a curve through z contained in Δ and of curvature zero in z. Then $\Delta \subseteq \operatorname{co} S$.*

If we choose the set S to be $S = \{z_\sigma : \sigma \in S_n\}$, and consider any support point z of the region $\Delta(A, B)$. If $z \neq 0$ and $z \notin S$, then z is a regular boundary point. If $\partial\Delta(A, B)$ does not contain zero, then $\Delta(A, B) \subseteq S$. However, if it does contain zero, then by 2.29 we obtain similar result.

Furthermore, the conjecture is proved in the case that $n - 2$ of the eigenvalues β_i of normal matrix B are equal [21].

Theorem 2.18. *Let A and B be $n \times n$ normal matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively. MOC is true for the case that $n - 2$ of β_i are equal.*

In this case, the spectrum of B is of the form $\beta_1, \beta_2, \beta, \ldots, \beta$. Clearly, the validity of this case will hold, if it is true for $\beta = 0$. In fact, this case of MOC will hold if the equivalent case of MVC (section 3.1) holds as follow [21, Theorem 1]:

Theorem 2.19. *Given $n \times n$ unitary matrix U, there exist nonnegative numbers t_σ , $\sigma \in S_n$, such that for all pairs (I, K) having the same cardinality in which $I \subseteq \{1, 2\}$,*

$$|\det U[I|K]|^2 = \sum_{\sigma:\sigma(I)=K} t_\sigma. \tag{2.18}$$

The problem is reduced to the question whether or not there exists $n \times n$ matrix S of nonnegative entries S_{ij} and with zero diagonal, whose k^{th} row sum is $|u_k|^2$, k^{th} column sum is $|v_k|^2$, and the ij entry of whose real part $S + S^T$ is $|u_i v_j - u_j v_i|^2$, that is, to show that the following system of linear equations has nonnegative solution.

$$\sum_j S_{kj} = |u_k|^2, \qquad k = 1, \ldots, n \tag{2.19}$$

$$\sum_j S_{jk} = |v_k|^2, \qquad k = 1, \ldots, n \tag{2.20}$$

$$S_{ij} + S_{ji} = |u_i v_j - u_j v_i|^2, \qquad i, j \in [n] \tag{2.21}$$

where $u = (u_k), v = (v_k) \in \mathbb{C}^n$ are mutually orthogonal unit vectors, and $S_{ij} \geq 0$. A real nonnegative solution is obtained implying that (2.18) holds, and hence proves MOC in this case.

A desired result is found when A and B are $n \times n$ normal matrices with pairwise distinct eigenvalues. In this case, $\Delta(A, B)$ under a sufficient condition is equal to the convex hull of z_σ points which is a line segment [3, Theorem 3.1].

Theorem 2.20. *Let A and B be $n \times n$ normal complex matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, such that $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are pairwise distinct. The set $\Delta(A, B)$ is a line segment of a line passing through the origin if and only if all the α_i and β_j lie on a common circle or straight line.*

Using the following lemma we can easily confirm this result,

Lemma 2.21. *Let $n \geq 2$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be complex row vectors such that $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are pairwise distinct. The convex hull* $\mathrm{co}\{z_\sigma : \sigma \in S_n\}$ *is a line segment of a line passing through the origin if and only if all the α_i and β_j lie on a common circle or straight line.*

Proof. Assume that $\Delta(A, B)$ be a line segment of a line through the origin, then its end points are corners. If z is a corner then $z = z_\sigma$ for some $\sigma \in S_n$. Thus, $\Delta(A, B) = \mathrm{co}\{z_\sigma; \sigma \in S_n\}$ is a line segment of a line through the origin, and all α_i and β_j lie on a common circle or straight line. Conversely, if allα_i and β_j lie on a common circle or straight line, then by the preceding lemma $\mathrm{co}\{z_\sigma; \sigma \in S_n\}$ is a line segment of a line passing through the origin. □

Definition 2.22. *A point z is a corner if $z \in \partial\Delta(A, B)$ and Δ is contained in an angle with vertex at z and measuring less than π in the neighborhood of z.*

Additionally, MOC is true in the case that $\sigma(A)$ lies on a line or circle and $\sigma(B)$ lies on another line or circle. A key observation in proving this case is that MOC is unchanged under fractional linear (Möbius) transformation of the eigenvalues $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ (see definition 2.33). As a special case of this result, the validity of MOC is obtained in the case A and B are scalar multiples of unitaries [16, Theorem 2].

Theorem 2.23. *Let A and B be $n \times n$ unitary matrices with eigenvalues $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ respectively. For any scalars t and s we have,*

$$\det(tA + sB) \in \mathrm{co}\{\prod_{j=1}^{n}(t\alpha_j + s\beta_{\sigma(j)} : \sigma \in S_n\}. \tag{2.22}$$

In other words, MOC holds if $\sigma(A)$ and $\sigma(B)$ belong to non intersecting circles in the complex plane C_A and C_B respectively. However, non intersecting circles in $\mathbb{C}$ can be mapped by some Möbius transformation μ to a pair of concentric circles centered at the origin, so we can replace A and B by $\mu(A)$ and $\mu(B)$ respectively.

Furthermore, a general case of this result is established for an arbitrary circles [15]. In this case, $B = \kappa C$ with eigenvalues $\beta_1, \dots, \beta_n$ where C is a hermitian matrix with eigenvalues

$\gamma_1, \ldots, \gamma_n$, κ is a complex number, and $\beta_i = \kappa\gamma_i$. This case is similar to theorem 2.13, but the purpose here is to weaken the hypothesis that A is positive definite to A is hermitian [15, Theorem 1.2].

Theorem 2.24. *MOC holds in case that A is hermitian and B a non-real scalar multiple of a hermitian matrix.*

Corollary 2.25. *Let C_A and C_B be circles in the complex plane. If $\alpha_1, \ldots, \alpha_n \in C_A$ and $\beta_1, \ldots, \beta_n \in C_B$, then MOC holds.*

The case $C_A = C_B$ is reduced to the case where both A and B are hermitian. For non-intersecting circles case, the result is established in theorem 2.23. When C_A touch C_B, the case is reduced to considering that C_B is the real axis and $C_A = \{z; z \in \mathbb{C}, \Im z = 1\}$. Finally, the case when the circles intersect at two points can be obtained using Möbius transformation.

Proof. (Theorem 2.24) Proving by induction on n. The results are easy to be verified by direct calculation for $n = 1$ and $n = 2$. For $n \geq 3$, we supposed that $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ are real and that $\beta_i = \kappa\gamma_i$ for some fixed $\kappa \in \mathbb{C}\backslash\mathbb{R}$ with $|\kappa| = 1$.

It is clear that MOC is steady under perturbations.This means if MOC holds for $\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \beta_n^{(r)}$ for every $r = 1, 2, \ldots$ and if $\lim_{r\to\infty} \alpha_j^{(r)} = \alpha_j$ and $\lim_{r\to\infty} \beta_j^{(r)} = \beta_j$ for $j = 1, 2, \ldots, n$ then it also holds for $\alpha_1, \alpha_2, \ldots, \beta_n$. Therefore, it suffices to consider generic sets of the eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ where we suppose that they are nonzero, distinct, distinct from their negatives and that the n^2 numbers $\beta_i\alpha_j^{-1}$ ($1 \leq i, j \leq n$ are distinct. Now suppose that $\Delta \nsubseteq \Delta(A, B)$, where

$$\Delta = \{\det(U^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n)U + V^* \operatorname{diag}(\beta_1, \ldots, \beta_n)V) : U, V \text{unitary}\} \tag{2.23}$$

then it follows that there is an extreme point z of $\operatorname{co}(\Delta)$ which is not almost flat and such that $z \notin \Delta(A, B)$. We exclude that $z \neq 0$ since we know (by theorem 2.29) that

$0 \in \Delta(A, B)$. Let A and B be the corresponding matrices. We can conclude that A and B possess a common nontrivial invariant linear subspace. The orthogonal complement is also simultaneously invariant. Due to the fact that eigenvalues sets contains distinct elements, allows the matrices to be decomposed simultaneously on space of lower dimension. This is a contradiction, so $\Delta \subseteq \Delta(A, B)$. □

Remark 2.26. *MOC holds for all pairs of normal matrices $A, B \in \mathbb{C}^{n\times n}$ for which at most one matrix has at most one non-real eigenvalue [22, Corollary 4].*

A particular case concerning the eigenvalues of A and B holds when these eigenvalues are necessarily real [11]. That is A and B are hermitian and hence $T = A + B$ is also hermitian with real eigenvalues $t_1, \dots, t_n$, we have

$$\prod_{j=1}^{n}(\lambda + t_j) \in \text{co}\{\prod_{j=1}^{n}(\lambda + \alpha_j + \beta_{\sigma(j)}) : \sigma \in S_n\}. \tag{2.24}$$

This result is established from a new aspect which is equivalent to considering the symmetric function f of the eigenvalues $t_1, \dots, t_n$, so we have

$$f(t_1, \dots, t_n) \in \text{co}\{f(\alpha_1 + \beta_{\sigma(1)}, \dots, \alpha_n + \beta_{\sigma(n)}) : \sigma \in S_n\} \tag{2.25}$$

where co denotes the convex hull in the space in which f takes values. Therefore, the result is restated as follows [11, Theorem 2,3].

Theorem 2.27. (2.25) *holds for $f(t_1, \dots, t_n) = \sum_{j=1}^{n} t_j^m$, where m is an integer ≥ 2.*

Theorem 2.28. (2.25) *holds for $f(t_1, \dots, t_n) = \prod_{j=1}^{n}(\lambda + t_j)$.*

The convex hull in theorem 2.28 is taken in the space of degree n polynomials with real coefficients. These theorems holds using differential calculus and derivatives of second and higher orders while they both fail in the complex case.

2.3 Determinant

In respect of the determinant of the sum of two normal matrices, MOC is proved in the case that $A+B$ is singular, that is, $\det(A+B)=0$ belongs to the convex hull of z_σ-points [17].

Theorem 2.29. *Let A and B be $n\times n$ normal matrices. In the special case $A+B$ is singular, we have*

$$0 \in \mathrm{co}\{z_\sigma : \sigma \in S_n\}. \tag{2.26}$$

It is convenient to replace B by its negative; therefore, we may assume that $A-B$ is singular and prove (2.26) . The following theorem (Wielandt's theorem) is applied in proving this result.

Theorem 2.30. *Let $\alpha_1,\dots,\alpha_n$, $\beta_1,\dots,\beta_n$, and v be given complex numbers. The following are equivalent:*

1. *It is impossible to separate the subsets $\{\alpha_1,\dots,\alpha_n\}$ and $\{v+\beta_1,\dots,v+\beta_n\}$ of $\mathbb{C}$ with a straight line or a circle.*

2. *There exist normal matrices A and B with eigenvalues $(\alpha_1,\dots,\alpha_n)$, $(\beta_1,\dots,\beta_n)$ respectively such that v is an eigenvalue of $A-B$.*

Proof. (of theorem 2.29) 2 of theorem 2.30 holds with $v=0$ by the hypothesis that $A-B$ is singular $v=0$. It follows that (by Wielandt's theorem) it is impossible to separate $\{\alpha_1,\dots,\alpha_n\}$ from $\{\beta_1,\dots,\beta_n\}$ with circle or straight lines. In view of speration theorem [23, p. 61-63], in this case there is a choice of five points from $\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\}$ which violate speration. The number of selected α's produces six various cases which are excluded except when the number of selected α's is 2 or 3. By reindexing $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$, we may assume that $\{\alpha_1,\alpha_2,\alpha_3\}$, and $\{\beta_1,\beta_2,\beta_3\}$ cannot be seperated by a straight lines or circles. Perceiving the geometrical argument, the following is established

$$0 \in \text{co}\{\prod_{j=1}^{3}(\alpha_j - \beta_{\rho(j)}) : \rho \in S_3\} \tag{2.27}$$

which evidently holds since MOC is true for $n = 3$ [6]. Now, we show(2.26) follows from (2.27). Let $z = \prod_{j=4}^{n}(\alpha_j - \beta_j)$, we have

$$0 \in \text{co}\{z\prod_{j=1}^{3}(\alpha_j - \beta_{\rho(j)}) : \rho \in S_3\} \tag{2.28}$$

□

2.4 z_σ-points

In view of z_σ points, MOC is confirmed in the case that all z_σ are collinear [28, Theorem 3].

Theorem 2.31. *Let $A = \text{diag}(\alpha_1, \ldots, \alpha_n)$, and $B = \text{diag}(\beta_1, \ldots, \beta_n)$ be in $\mathbb{C}^{n\times n}$. If all z_σ are collinear, then there exist real numbers $t_\sigma, \sigma \in S_n$, such that $0 \le t_\sigma \le 1$ and $\sum_{\sigma\in S_n} t_\sigma = 1$ satisfy*

$$\det(A + UBU^*) = \sum_{\sigma\in S_n} t_\sigma z_\sigma. \tag{2.29}$$

Proof. Assuming that all z_σ are collinear, then $\Delta(A, B)$ lies on a closed line due to its compactness and connectedness. We have two cases to consider:

1. If the end points of $\Delta(A, B)$ are nonzero, they are z_σ points, and since any point of this line segment is a convex combination of the end points, the theorem follows.

2. If $\det(A + UBU^*) = 0$ is an end point of $\Delta(A, B)$, then $A + UBU^*$ is singular. By theorem 2.29 we have $0 \in \text{co}\{z_\sigma : \sigma \in S_n\}$. Since 0 is a z_σ point for A, B, we can continue as above.

□

2.5 Equality of MOC

Examining the extremal set of pairs of complex matrices for which equality of (1.1) holds yields a characterization of this extremal set and sufficient conditions for pairs of matrices to be in it [19]. In theorem 2.20, the equality of MOC holds for two normal matrices under the condition that all α_i and β_j lie on a common circle or straight line. Another case that holds with equality exists if the boundary of $\Delta(A, B)$ is convex [19, Theorem 3.4].

Theorem 2.32. *Let A and B be $n \times n$ complex matrices. The equality of* (1.1) *holds if and only if the boundary of $\Delta(A, B)$ is a convex polygon in $\mathbb{C}$.*

Proof. If the equality holds of (1.1), then it follows immediately from the definition of the convex envelope $\mathrm{co}\{z_\sigma : \sigma \in S_n\}$ that the boundary of $\Delta(A, B)$ is a convex polygon. The other inclusion $\mathrm{co}\{z_\sigma : \sigma \in S_n\} \subseteq \Delta(A, B)$ is direct consequence of the hypothesis. In fact, since $\partial\Delta(A, B)$ is a convex polygon, it has well defined vertices which are corners of $\partial\Delta(A, B)$ and hence z_σ points. Therefore, every vertex of $\Delta(A, B)$ is in $\mathrm{co}\{z_\sigma : \sigma \in S_n\}$. Hence, the result follow. □

It worth noting that the characterization of the extremal set of complex matrices is established in the following statements. For arbitrary $n \times n$complex matrices A and B, we have:

1. The equality of (1.1) holds for all $B \in \mathbb{C}^{n\times n}$ if and only if A is a scalar matrix [19, Proposition 3.1].

2. Assume that B has pairwise distinct eigenvalues and the equality of MOC holds. Then A is a normal matrix [19, Theorem 3.5]

3. If B is unitarily similar to $B_1 \oplus \cdots \oplus B_k$, where $B_i \in T_{n_i}$ (an upper triangular), $i = 1, 2, \ldots, k$, $n_1 + \cdots + n_k = n$ and $\sigma(B_i) \cap \sigma(B_j) = \phi$ for $i \neq j$. Assume that the equality of MOC holds. Then A is unitarily similar to $A_1 \oplus \cdots \oplus A_k$, where $A_i \in T_{n_i}$ and $i = 1, 2, \ldots, k$ [19, Corollary 3.8].

2.6 Mӧbius transformation

Analyzing MOC from geometrical perspective such as Mӧbius transformation yields advantageous results in this area as in theorems 2.24 and 2.23.

Definition 2.33. *Let $a, b, c, d \in \mathbb{C}$, with $ad-bc \neq 0$. Let $A \in \mathbb{C}^{n\times n}$, and suppose that $cA+dI_n$ is invertible. The following is called a Mӧbius transformation of A on $\mathbb{C}\backslash\frac{-d}{c}$ ($\mathbb{C}$, if $c = 0$):*

$$\mu(A) = (aA + bI_n)(cA + dI_n)^{-1}. \tag{2.30}$$

As a consequence of applying this concept on MOC, we obtain the following interested fact [26, Lemma 3.4].

Theorem 2.34. *Let $A, B \in \mathbb{C}^{n\times n}$, and suppouse $\mu(A)$ and $\mu(B)$ are well defined. If MOC is true for $(A, -B)$, then it will be also true for $(\mu(A), -\mu(B))$. In this case we have*

$$\Delta(\mu(A), -\mu(B)) = \frac{|cA + dI|}{|cB + dI|}(ad - bc)^n\Delta(A, -B). \tag{2.31}$$

As a special case of the above result, if MOC is true for (A, B), then it will be true for $((A + tI)^{-1}, (B - tI)^{-1})$ for any appropriate $t \in \mathbb{C}$ [4, Theorem 2].

Corollary 2.35. *Let $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ and $B = \mathrm{diag}(\beta_1, \dots, \beta_n)$. Let $t \in \mathbb{C}$ such that $t \notin \sigma(-A) \cup \sigma(B)$, and $A' = (A + tI)^{-1}$ and $B' = (B - tI)^{-1}$. If*

$$\Delta(A, B) \subseteq \mathrm{co}\{\prod_{i=1}^{n}(\alpha_i + \beta_{\sigma(i)}) : \sigma \in S_n\} \tag{2.32}$$

then

$$\Delta(A', B') \subseteq \mathrm{co}\{\prod_{i=1}^{n}(\alpha'_i + \beta'_{\sigma(i)}) : \sigma \in S_n\} \tag{2.33}$$

where $\alpha'_j = (\alpha_j + t)^{-1}$ and $\beta'_j = (\beta_j - t)^{-1}$ are the eigenvalues of A' and B' respectively.

Remark 2.36. *If $\Delta(A,B)$ is a line segment, then $\Delta(A',B')$ is a line segment and* (1.1) *holds with equality.*

Proof. As a consequence of the equality

$$(A+tI)^{-1}+(UBU^*-tI)^{-1}=(A+tI)^{-1}(A+UBU^*)(UBU^*-tI)^{-1} \tag{2.34}$$

we have

$$\det[(A+tI)^{-1}+(UBU^*-tI)^{-1}]=\det(A+UBU^*)\prod_{j=1}^{n}\alpha'_j\prod_{j=1}^{n}\beta'_j \tag{2.35}$$

where $\alpha'_j=(\alpha_j+t)^{-1}$ and $\beta'_j=(\beta_j-t)^{-1}$, and since

$$\prod_{k=1}^{n}(\alpha'_k+\beta'_{\sigma(k)})=\prod_{k=1}^{n}\alpha'_k\prod_{k=1}^{n}\beta'_k\prod_{k=1}^{n}(\alpha_k+\beta_{\sigma(k)}) \tag{2.36}$$

and by hypothesis that $\det(A+UBU^*)$ is in the convex hull of the points $\prod_{k=1}^{n}(\alpha_k+\beta_{\sigma(k)})$, the theorem follows. $\square$

Chapter 3

Connections of MOC

MOC depends on the determinant which is a branched fundamental concept in linear algebra associated with several concepts such as eigenvalues, trace, upper and lower bounds, and derivatives. Due to this fact, MOC has various connections and consequential results. In this chapter, we present the relationship between MOC and several important results, and we display how they technically approach MOC in various ways.

3.1 Merikoski-Virtanen conjecture(MVC)

A stronger conjecture (MVC) on compounds of unitary matrices was established by Merikoski and Virtanen [28]. MVC conjectures the following problem which would imply MOC if it is answered affirmatively:

Problem: For $A, B \in \mathbb{C}^{n\times n}$ and U unitary. $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ and $B = \operatorname{diag}(\beta_1, \ldots, \beta_n)$. Do there exist real numbers t_σ , $\sigma \in S_n$ satisfying

$$0 \le t_\sigma \le 1, \tag{3.1}$$

and

$$\sum_{\sigma\in S_n} t_\sigma = 1, \tag{3.2}$$

such that

$$|u_{IJ}|^2 = \sum_{\sigma\in S_n} t_\sigma P_{IJ}(\sigma) \tag{3.3}$$

for any $I, J \subseteq \{1, 2, \ldots, n\}$ and $|I| = |J| = m$, $1 \le m \le n$, and u_{IJ} is a minor of U.

We perceive that the connection between this conjecture and MOC follows from the identity [12, (1.1)],

$$\det(A+UBU^*) = \sum \alpha_I \beta_{J^c} |u_{IJ}|^2 \tag{3.4}$$

where $\alpha_I = \prod_{i\in I} \alpha_i$ and $\beta_J = \prod_{j\in J} \beta_j$. In addition, I, J are subsets of $\{1, 2, \ldots, n\}$ having the same cardinality. J^c is the complement of J, and u_{IJ} is the corresponding minor of U. If we assume that MVC is true, we would obtain

$$\begin{aligned}
\det(A+UBU^*) &= \sum \alpha_I \beta_{J^c} |u_{IJ}|^2 \\
&= \sum \alpha_I \beta_{J^c} (\sum_{\sigma\in S_n} t_\sigma P_{IJ}(\sigma)) \\
&= \sum_{\sigma\in S_n} t_\sigma (\sum \alpha_I \beta_{J^c} P_{IJ}(\sigma)) \\
&= \sum_{\sigma\in S_n} t_\sigma z_\sigma.
\end{aligned} \tag{3.5}$$

The last equality is due to the fact that $z_\sigma = \sum \alpha_I \beta_{J^c} P_{IJ}(\sigma)$. Thus, the validity of MVC would immediately imply MOC's veracity.

Excluding condition (3.1) Merikoski and Virtanen proved that MOC is true if all z_σ's are real (or pure imaginary), and if all z_σ's are collinear (theorem 2.31). In addition, without condition (3.1), MVC is also solved and proved to be true.

Unfortunately, further investigations show that MVC is true for $n \le 3$, but false for $n = 4$ [10]. The following counterexample is presented to illustrate that MVC is not true in general (fails for $n = 4$ and hence for $n \ge 4$) [10, Section 4].

Example 3.1. *Let $n = 4$, and $\mathcal{T} = \{\sum t_\sigma |P_\sigma^{(2)}| : t_\sigma \ge 0, \sum t_\sigma = 1\}$ be the polytope with vertices at 2-ply permutation matrices. It is important to understand the nature of the facets of this polytope. A facet of a polytope of dimension n is a face that has dimension $n-1$. $\mathcal{T}$ possess three basic types of facets:*

1. *Facets containing* 20 *of the* $|P_\sigma^{(2)}|$. *These facets consist of* 6×6 *matrices in* $\mathcal{T}$ *vanishing at a given entry. In addition, these facets generate the polytope of all doubly stochastic* 6×6 *matrices with block symmetry.*

2. *Facets containing* 18 *of the* $|P_\sigma^{(2)}|$. *These facets correspond to the vanishing of a given entry of* $|P_\sigma^{(1)}|$ *which is a linear function of* $|P_\sigma^{(2)}|$. *The facets of types* 1 *and* 2 *generate a polytope containing all* $|U^{(2)}|^2$ *for unitary matrix* U.

3. *Facets containing* 13 *of the* $|P_\sigma^{(2)}|$. *These facets are* 12*-dimensional simplexes. The equation of a facet of this type is* $tr(TD) = 1$, *where* $T \in \mathcal{T}$ *and:*

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

For every matrix T *we have* $tr(TD) \leq 1$. *For our counterexample,* U *is constructed such that* $|U^{(2)}|^2 D > 1.06$. *Specifically, let:*

$$U = \begin{bmatrix} \alpha & \beta & \beta & \beta \\ -\beta & \alpha & \beta & \beta \\ -\beta & -\beta & \alpha & \beta \\ -\beta & \beta & -\beta & \alpha \end{bmatrix} \tag{3.7}$$

where α and β are real and related by $(\alpha^2 + 3\beta^2) = 1$. The matrix $U^{(2)}$ is given by,

$$U^{(2)} = \begin{bmatrix} \gamma & \delta & \zeta & \eta & \delta & \zeta \\ \zeta & \gamma & \delta & \zeta & \eta & \delta \\ \delta & \zeta & \gamma & \delta & \zeta & \eta \\ \eta & \delta & \zeta & \gamma & \delta & \zeta \\ \zeta & \eta & \delta & \zeta & \gamma & \delta \\ \delta & \zeta & \eta & \delta & \zeta & \gamma \end{bmatrix} \tag{3.8}$$

where $(\gamma = \alpha^2 + \beta^2)$, $(\delta = \beta(\beta + \alpha))$, $(\zeta = \beta(\beta - \alpha))$, and $(\eta = -2\beta^2)$. We find that

$$tr(|U^{(2)}|^2 D) = 2\delta^2 + 2\eta^2 - \gamma^2 \tag{3.9}$$

which obtain its maximum value for $\beta = \sqrt{k}$, where k is the real root of the equation $16k^3 - 8k^2 + 4k - 1 = 0$. Computer calculation provides approximate values for the above variables. $k \approx 0.323899$, $\beta \approx 0.569122$, $\alpha \approx 0.168231$, $\delta \approx 0.419643$, $\gamma \approx 0.352201$, $\zeta \approx 0.228155$, and $\eta \approx -0.647799$.

$$tr(|U^{(2)}|^2 D) = 2\delta^2 + 2\eta^2 - \gamma^2 \approx 1.06744 \tag{3.10}$$

implying that $|U^{(2)}|^2 \notin \mathcal{T}$.

On the other hand, MVC has been investigated for Householder reflection matrices U and proved to be true when $n \leq 4$ (see theorem 2.3)[13]. Nevertheless, the following counterexample disproves MVC for Householder reflections in the case $n = 5$ [13, Section 3].

Example 3.2. *Let $U = I - 2\zeta\zeta^*$ be a Householder reflection with $n = 5$, such that $t_j = |\zeta_j|^2 = \frac{1}{5}$ and $j = 1, \ldots, 5$. Calculation reveals that for $|J| = |K| = 2$, we have*

$$|u_{JK}|^2 = \begin{cases} \frac{1}{25}, & \text{if } J = K, \\ \frac{4}{25}, & \text{if } |J\Delta K| = 1 \\ 0, & \text{if } |J\Delta K| = 2. \end{cases} \tag{3.11}$$

We construct a matrix (m_{JK}) by

$$m_{JK} = \begin{cases} \frac{1}{2}, & \text{if } J = K, \\ \frac{-1}{6}, & \text{if } |J\Delta K| = 1 \\ \frac{1}{6}, & \text{if } |J\Delta K| = 2. \end{cases} \tag{3.12}$$

Calculation shows that,

$$\sum_{J,K} m_{JK}|u_{JK}|^2 = 10 \cdot \frac{1}{2} \cdot 125 + 60 \cdot (\frac{-1}{6}) \cdot \frac{4}{25} = \frac{-7}{5} \tag{3.13}$$

while $\sum_{J,K} m_{JK} P_{JK}(\sigma) = v(\sigma)$, and $v(\sigma) = \prod_{j=1}^{n} z_{j,\sigma(j)}$. v is given on the conjugacy classes of S_5 by $v(1^5) = 5$, $v(2 \cdot 1^3) = 1$, $v(2^2 \cdot 1) = 1$, $v(3 \cdot 1^2) = -1$, $v(3 \cdot 2) = 1$, $v(4 \cdot 1) = -1$, and $v(5) = 0$. The minimum value of v is -1 which is strictly greater than $\frac{-7}{5}$, so (3.3) fails in the case $n = 5$.

In contrast, there is no evidence of MOC's failure in any special or general case. Hence, MOC remains open.

3.2 Elementary symmetric functions

Another conjecture attempting to imply MOC from the perspective of elementary symmetric functions was published by Merikoski and Virtanen in 1992 [29]. This conjecture involves elementary symmetric functions of the eigenvalues of the sum and product of normal matrices. For $A, B \in \mathbb{C}^{n\times n}$ normal matrices with eigenvalues $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$, the following are conjectured:

$$(E_m) \qquad E_m(A+B) \in \mathrm{co}\{e_m(\alpha_1+\beta_{\sigma(1)}, \dots, \alpha_n+\beta_{\sigma(n)}), \quad \sigma \in S_n\}, \tag{3.14}$$

$$(F_m) \qquad E_m(AB) \in \mathrm{co}\{e_m(\alpha_1\beta_{\sigma(1)}, \dots, \alpha_n\beta_{\sigma(n)}), \quad \sigma \in S_n\}. \tag{3.15}$$

Obviously, MOC is identical to the case

$$(E_n) \quad \det(A+B) \in \{z_\sigma : \sigma \in S_n\} \tag{3.16}$$

which is known to be true in several cases as we studied in chapter two. In this conjecture, we have (E_1) and (F_n) are trivially true because of the properties $\mathrm{tr}(A+B) = \mathrm{tr}A + \mathrm{tr}B$ and $\det(AB) = \det A \det B$. In addition, (E_2) and (E_n) are true in the hermitian case according to Fiedler's result [18]. Moreover, de Oliveira's observation [9] implies that (F_1) which is used to prove the validity of (E_2), (E_3) and (F_{n-1}). The proof of (E_3) relies on Newton's formula. However, the same technique fails to work on (E_m) for $m \geq 4$.

3.3 External vertices conjecture (EVC)

A reformulation of the two conjectures MOC and MVC concerns vertices projections, which are nonnegative functions on S_n [12]. The z_σ points in MOC are defined to be MO (Marcus-de Oliveira) vertices. If MO vertices satisfy $\Re z_\sigma \geq 1$ for all $\sigma \in S_n$. Then there exist nonnegative numbers t_{IJ} such that

$$\Re z_\sigma = 1 + \sum_{IJ} t_{IJ} P_{IJ}(\sigma), \quad \forall \sigma \in S_n. \tag{3.17}$$

The sum is taken over over all pairs of subsets I and J of $\{1, \ldots, n\}$ with the same number of elements. This conjecture is known as the external vertices conjecture (EVC). The progress of EVC focus on the configuration of MO vertices and identifying the extreme vertex projections that would verify MOC. In fact, EVC is verified in the cases that $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are all real and in the case $n \leq 3$ [1]. As a consequence of examining the MO vertices, EVC has been proved for the case $n = 4$, which implies a weak form of MOC (theorem 2.7).

Theorem 3.3. *[14, Theorem 1.1] The EVC is true for $n = 4$.*

The proof of this result depends on the representation theory of the symmetric group S_n; especially, Saxl representations and Saxl functions on S_n. In addition, some calculations that were done by computer were involved in this proof.

3.4 The region $\Delta(A, B)$

The region $\Delta(A, B)$ is a compact connected subset of the complex plane. It has a great significance in MOC since it is described by the set of determinants of the sum of normal matrices. We observe several consequential result concerning this region throughout the progress of MOC. As one result, we note that for 2×2 normal matrices A and B, $\Delta(A, B)$ is an elliptical disk as we discussed in theorem 2.1. We also realize that in theorem 2.20 the region $\Delta(A, B)$ is a line segment of a line passing through the origin. In addition to these results, Bebiano and Queiró obtained a description of $\Delta(A, B)$ when A and B run over all normal matrices with prescribed eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively [7]. As we know that z_σ points trivially belong to $\Delta(A, B)$, the following results are verified [7, Theorem 1,2,3].

Theorem 3.4. *For $\sigma, \tau \in S_n$, if σ and τ differ by a transposition, then the segment $[z_\sigma, z_\tau]$ is contained in $\Delta(A, B)$.*

Theorem 3.5. *For σ, τ, and $\phi \in S_n$, if τ and ϕ both differ from σ by a transposition, then the region $[z_\sigma, z_\tau, z_\phi, z_{\phi\sigma^{-1}\tau}]$ is contained in $\Delta(A, B)$.*

Theorem 3.6. *Given z_σ for $\sigma \in S_n$, consider the points z_τ generated by τ which differ from σ by a transposition. If these points do not all lie in a half plane determined by a straight line through z_σ, then z_σ is an interior point of $\Delta(A, B)$.*

In addition, $\Delta(A, B)$ is shown to have an empty interior if and only if it is a line segment or a point [26, Theorem 3.2, 3.3].

Theorem 3.7. *Let $A, B \in \mathbb{C}^{n\times n}$ with $n \geq 3$. $\Delta(A, B) = \{\delta\}$ if and only if one of the following holds.*

1. *$\delta = 0$, and there is $\mu \in \mathbb{C}$ such that $\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n) < n$.*

2. *$\delta \neq 0$, one of the matrices A or B is a scalar matrix, and $\det(A + B) = \delta$.*

Theorem 3.8. *Suppose $A, B \in \mathbb{C}^{n\times n}$ are such that $\Delta(A, B)$ is not a singleton. The following conditions are equivalent.*

1. *$\Delta(A, B)$ has empty interior.*

2. *$\Delta(A, B)$ is non degenerate line segment.*

3. *$\{z_\sigma : \sigma \in S_n\}$ is not a singleton, that is, there are at least two distinct z_σ points, and one of the following conditions holds:*

   (a) A and B are normal matrices with eigenvalues lying on the same straight line or the same circle.

   (b) There is $\mu \in \mathbb{C}$ such that one of matrices $A - \mu I_n$ or $B + \mu I_n$ is rank one normal, and the other one is invertible normal such that the inverse matrix has collinear eigenvalues.

   (c) There is $\mu \in \mathbb{C}$ such that $A - \mu I_n$ is unitarily similar to $\tilde{A} \oplus 0_{n-k}$ and $B + \mu I_n$ is unitarily similar to $0_k \oplus \tilde{B}$ so that $\tilde{A} \in \mathbb{C}_{k\times k}$ and $\tilde{B} \in \mathbb{C}_{(n-k)\times(n-k)}$ are invertible.

Furthermore, since z_σ points shape a convex hull, it worth noting that some of these points are sharp points [26, Section 3.2].

Definition 3.9. *A boundary point μ of compact set S in $\mathbb{C}$ is a sharp point if there exists $d > 0$ and $0 \leq t_1 < t_2 < t_1 + \pi$ such that*

$$S \cap \{z \in \mathbb{C} : |z - \mu| \leq d\} \subseteq \{\mu + \rho e^{i\zeta} : \rho \in [0, d], \zeta \in [t_1, t_2]\}. \tag{3.18}$$

Theorem 3.10. *Let $A, B \in \mathbb{C}^{n\times n}$. Every sharp point of $\Delta(A, B)$ is a z_σ point.*

3.5 Determinantal inequality

One of the relative results of MOC is an inequality for the modulus of the determinant of the sum of two normal matrices [2]. This inequality involves the eigenvalues of the two matrices. Let A and B be $n \times n$ normal matrices with respective eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$. We have

$$|\det(A + B)| \leq \min\{\prod_{i=1}^{n} \max_j|\alpha_i + \beta_j|, \prod_{j=1}^{n} \max_i|\alpha_i + \beta_j|\}. \tag{3.19}$$

The equality case of (3.19) is established as follow:

Theorem 3.11. *[2, Theorem] Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be given . The equality of* (3.19) *is obtained for some normal matrices A and B with eigenvalues α_i and β_i respectively, if and only if there exists $\tau \in S_n$ such that:*

$$max_j|\alpha_i + \beta_j| = |\alpha_i + \beta_{\tau(i)}|, \quad i = 1, 2, \ldots, n. \tag{3.20}$$

or

$$max_i|\alpha_i + \beta_j| = |\alpha_{\tau(j)} + \beta_j|, \quad j = 1, 2, \ldots, n. \tag{3.21}$$

Moreover, in this situation,

$$|\det(A + B)| = max_{\sigma\in S_n} \prod_{i=1}^{n} |\alpha_i + \beta_{\sigma(i)}|. \tag{3.22}$$

The validity of MOC would directly imply that

$$|\det(A+B)| \leq \max_{\sigma \in S_n} \prod_{i=1}^{n} |\alpha_i + \beta_{\sigma(i)}|. \tag{3.23}$$

This result shows that the upper bound in (3.22) is maximum only if it satisfies MOC. On the other hand, a lower and upper bound of $|\det(A+B)|$ are given in terms of the singular values of A and B [25].

Theorem 3.12. *[25, Theorem 1] There exist $n \times n$ matrices A and B over real or complex field F with singular values $a_1 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_n \geq 0$, respectively, such that $\det(A+B) = z \in F$ if and only if;*

$$\prod_{j=1}^{n}(a_j + b_{n-j+1}) \geq |z| \geq \begin{cases} 0, & \text{if } [a_n, a_1] \cap [b_n, b_1] \neq \phi, \\ |\prod_{j=1}^{n}(a_j - b_{n-j+1})| & \text{otherwise.} \end{cases} \tag{3.24}$$

3.6 Linear operator

Linear preserver problem deals with linear maps on $\mathbb{C}^{n\times n}$ that fix several properties of a matrix. The interest here is linear maps $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ that leave the region $\Delta(A,B)$ of MOC invariant, that is, ones that satisfy the relation $\Delta(A,B) = \Delta(L(A),B)$. Matrices from both H_n, the real space of $n \times n$ hermitian matrices, and $\mathbb{C}^{n\times n}$ investigated and characterized respectively [3, Theorem 4.1, 4.2].

Theorem 3.13. *A linear operator $L : H_n \to H_n$ satisfies $\Delta(A,B) = \Delta(L(A),B)$, for all $A \in H_n$ and for all $B \in \mathbb{C}^{n\times n}$ if and only if there exists a unitary matrix U such that L is of the form $A \mapsto UAU^*$ or $A \mapsto UA^TU^*$*

Theorem 3.14. *A linear operator $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ satisfies $\Delta(A,B) = \Delta(L(A),B)$, for all $A,B \in \mathbb{C}^{n\times n}$ if and only if there exists a unitary matrix U such that L is of the form $A \mapsto UAU^*$ or $A \mapsto UA^TU^*$*

Proof. Let L be linear operator in $\mathbb{C}^{n\times n}$ such that $\Delta(A,B) = \Delta(L(A),B)$ for all $A, B \in \mathbb{C}^{n\times n}$. Suppose that A is Hermitian matrix, we prove that $L(A)$ is Hermitian. By hypothesis, there exists $B \in H_n$ such that eigenvalues of B and A are pairwise distinct and also the eigenvalues of B and$L(A)$ are pairwise distinct. Since A and B are Hermitian it follows that:

$$\Delta(A,B) = [\mathrm{min}_\sigma z_\sigma, \mathrm{max}_\sigma z_\sigma], \quad \sigma \in S_n. \tag{3.25}$$

As $\Delta(A,B) = \Delta(L(A),B)$, we can conclude that the eigenvalues of B and $L(A)$are real, and they belong to the same straight line or to the same circle. From (3.25) the endpoints of this line segment are corner, so $L(A)$ is normal and so Hermitian and $L(H_n) \subseteq H_n$. Consider $A \in \mathbb{C}^{n\times n}$ the cartesian decomposition, that is

$$A = \Re A + i\Im A, \quad \text{where} \quad \Re A = (A + A^*)/2, \quad \Im A = (A - A^*)/2i \tag{3.26}$$

are Hermitian matrices. If $L(A) = UAU^*$ hold, then

$$L(A) = L(\Re A) + iL(\Im A) = U(\Re A)U^* + iU(\Im A)U^* = UAU^*. \tag{3.27}$$

Similar argument for $L(A) = UA^TU^*$. □

3.7 Rank of the sum of matrices from unitary orbits

MOC which can be viewed as an analog of the generalized numerical rang of two normal matrices is a useful concept in pure and applied area. A further connection is studying basic property of matrices from unitary orbits. The unitary orbit $\mathcal{U}(A)$ of a matrix A is the set of matrices that are unitary similar to A. Given two $n \times n$ matrices A and B, and $X \in \mathcal{U}(A), Y \in \mathcal{U}(B)$. Studying matrices of the form $(X + Y)$ yields a best upper bound and a lower bound of the set

$$R(A, B) = \{\mathrm{rank}(X + Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\}. \tag{3.28}$$

Theorem 3.15. *[26, Theorem 2.1] Let $A, B \in \mathbb{C}^{n\times n}$ and*

$$\begin{aligned} m &= min\{rank(A - \mu I_n) + rank(B + \mu I_n) : \mu \in \mathbb{C}\} \\ &= min\{rank(A - \mu I_n) + rank(B + \mu I_n) : \mu \text{ is an eigenvalue of } A \oplus -B\} \end{aligned} \tag{3.29}$$

We have $max\{rank(UAU^* + VBV^*) : U, V\, unitary\} = min\{m, n\}$.

Focusing on $\mathrm{rank}(A-\mu I_n)+\mathrm{rank}(B+\mu I_n)$ for each eigenvalue μ of $A\oplus -B$ we can easily determine the value of m. In particular, if μ is an eigenvalue of A, then $\mathrm{rank}(A-\mu I_n) = n-k$, where k is the geometric multiplicity of μ; otherwise, $\mathrm{rank}(A - \mu I_n) = n$. Similarly, we can determine $\mathrm{rank}(B + \mu I_n)$.

For further researches in the direction of the sum of matrices from the unitary orbits one can also consider the determinant of these matrices. The study of the determinants of the sum of matrices from unitary orbits undergoes the action of Lie groups [30], [31]. As an application, the authors expand Fiedler's result (1.3) to the corresponding unitary orbits in some Lie algebras.

Chapter 4

Conclusion

We have examined and organized the progress of MOC and related areas of interest that encourage some mathematicians to investigate further situations aiming for solving this conjecture or confirming more valid cases. Despite the impressive studies and effort that have been accomplished in this conjecture, MOC is still open for $n \geq 4$. Although some attempts provide us with remarkable results, the two conjectures MVC (section 3.1) and EVC (section 3.3) fail to achieve a confirmation for $n \geq 4$. It seems difficult to prove or disprove MOC because we cannot characterize the set $\Delta(A, B)$.

As we discussed in chapter two and three, we perceive that many comprehensive topics related to matrices are involved in investigating MOC. Each one of them can be inspected deeply due to its extensiveness. For instance, the eigenvalues of A and B which have provided us with most valid cases of MOC is one of the essential topics in this conjecture. This topic is expandable, and it may study the set of eigenvalues itself, their algebraic or geometric multiplicity, or the linear transformation of them. It may also consider the singular values of A and B due to the fact that singular values of normal matrices is equal to the absolute value of their eigenvalues. Beside that, the geometry of $\Delta(A, B)$ is also a complicated topic in which the convexness of the set $\Delta(A, B)$ and the points of its boundary have been examined. This geometrical view shows some true cases of MOC as the other topics do. Further investigation in these topics individually or interdependently may play a key rule in solving MOC.

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