Mathematical and Numerical Analysis for Linear Peridynamic Boundary Value Problems

by

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Abstract

Peridynamics is motivated in aid of modeling the problems from continuum mechanics which involve the spontaneous discontinuity forms in the motion of a material system. By replacing differentiation with integration, peridynamic equations remain equally valid both on and off the points where a discontinuity in either displacement or its spatial derivatives is located.

A functional analytical framework was established in literature to study the linear bond-based peridynamic equations associated with a particular kind of nonlocal boundary condition. Investigated were the finite-dimensional approximations to the solutions of the equations obtained by spectral method and finite element method; as a result, two corresponding general formulas of error estimates were derived. However, according to these formulas, one can only conclude that the optimal convergence is algebraic.

Based on this theoretical framework, first we show that analytic data functions produce analytic solutions. Afterwards, we prove these finite-dimensional approximations will achieve exponential convergence under the analyticity assumption of data. At the end, we validate our results by conducting a few numerical experiments.

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Introduction

1.1 Origin of Peridynamics.

The purpose of mechanics is to study and describe the motion of material systems. As a branch of mechanics, continuum[1] mechanics deals with the analysis of the kinematics and the mechanical behavior of materials such as solids, liquids and gases (fluids). For instance, linear elastic behavior of solids is well described in classical solid mechanics[2] by the partial differential equaion[3]

$$\rho(\boldsymbol{x})\partial_t^2\boldsymbol{u}(\boldsymbol{x},t) = (L\boldsymbol{u})(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1.1}$$

where

$$(L\boldsymbol{u})(\boldsymbol{x},t) := (\lambda+\mu)\,\mathrm{grad}\ \mathrm{div}\ \boldsymbol{u}(\boldsymbol{x},t) + \mu\Delta\boldsymbol{u}(\boldsymbol{x},t),$$

which is derived from Newton's Second Law. In the equation, ρ describes the density of the body; variable $\boldsymbol{u} : \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ with $\Omega \subset \mathbb{R}^d$ and $d \in \{1,2,3\}$ is the displacement field; the right-hand side consists of the external force density $\boldsymbol{b}$ as well as inner tensions[4] and macroscopic forces with Lamé parameters λ and μ. This equation is based on the assumptions that all internal forces are contact forces (interactions between particles that are in direct contact with each other), and the deformation is twice continuously differentiable.

[1] A material modeled as a *continuum* is assumed the matter in the body is continuously distributed and fills the entire region of space it occupies.

[2] An area of continnum mechanics which studies the physics of continuous materials with a defined rest shape.

[3] The classical Navier equation of linear elasticity.

[4] In continuum mechanics, the internal forces are not, in general, determined by the current positions of the points alone, but also relates to the deformation of the body in the macroscopic sense. In contrast, Molecular dynamics requires only the current positions of atoms to determine the internal forces on the atoms. [35]

However, some materials may naturally form discontinuities such as cracks or fractures in the deforming structure. In such cases, the classical equations of continuum mechanics can not be applied directly because the displacement field $\boldsymbol{u}$ is discontinuous on these features. To overcome this difficulty, various remedies are formulated. For example, by redefining the body, one can shift the crack to boundary. Such a redefinition of the body has been an ingredient in essentially all of the work that has been done on the stress fields surrounding cracks; see Hellan [30] for a summary of this work. On other aspect, the techniques of fracture mechanics introduce relations[5] that are extraneous to the basic field equations of the classical theory. Specifically, linear elastic fracture mechanics (LEFM) considers a crack to evolve according to a separate constitutive model that predicts, on the basis of nearby conditions, how fast a crack grows, in what direction, whether it should arrest, branch, and so on.

Both these techniques, the redefinition of the body in the case of cracks, and the supplemental constitutive equations for determining the growth of defects, require us to know where the discontinuity is located. This limits the usefulness of these techniques in the problems[6] involving the *spontaneous* formation of discontinuities, in which we might not know their location in advance. Moreover, for certain methods like that provided by fracture mechanics, it is not clear to what extent they can meet the future needs of fracture modeling in complex media under general conditions, particularly at small length scales.

Materials modeled in continuum mechanics are conventionally treated in idealized cases by assuming they are continuous mass, meaning the substance of the object completely fills the space it occupies. Modeling objects in this way ignores the fact that matter is

[5] A constitutive relation, or constitutive equation, or constitutive model, is a relation between two physical quantities that is specific to a material. It is the mathematical description of how materials repond to external stimuli, usually as applied fields or forces. They are combined with other equations governing physical laws to solve physical problems. The first constitutive equation was developed by Robert Hooke and is known as Hooke's law which deals with the case of linear elastic materials.

[6] A good example is *concrete*, a material in which the standard assumptions of LEFM do not apply, at least on the macroscale, becasue it is heterogeneous and brittle unless large compressive confining stress is present. The process of cracking in concrete tends to occur through the accumulation of damage over a significant volume before localizing into a discontinuity, which itself usually follows a complex, three dimensional path.

made of atoms separated by "empty" space, so is not continuous; however, on length scales much greater than that of inter-atomic distance, such models are highly accurate. Nevertheless, technology increasingly involves the design and fabrication of devices at smaller and smaller length scales, even inter-atomic dimensions. Therefore, it is worthwhile to investigate whether the classical theory can be extended to permit relaxed assumptions of continuity, to include the modeling of discrete particles such as molecules and atoms.

Molecular dynamics (MD) provides an approach to understand the mechanics of materials at the smallest length scales, and has met with important successes in recent years. However even with the fastest computers, it is widely recognized that MD can not model systems of sufficient size to make it a viable replacement for continuum modeling.

Peridynamics, a new approach for continuum mechanics, proposed by Dr.Stewart Silling in 2000, attempts to unit the mathematical modeling of continuous media, cracks, and discrete particles within a single framework. It does this with two considerations:

I. Replacing the partial differential equations (embracing smooth displacement field and its partial derivatives with respect to the spatial coordinates) of the classical theory of solid mechanics with integral or integro-differential equations.

II. Assuming a model of internal forces within a body in which material points separated by a finite distance may exert forces on each other.

Part I enables minimal regularity assumptions on the deformation, with which the evolution of discontinuities are treated according to the same field equations as for continuous deformation; on the other side, part II allows discrete particles to employ the same field equations as for continuous media, which suggests that peridynamics is a multiscale material model for length scale ranging from MD (microscale) to those of classical elasticity (macroscale). In addition, part II manifests that the method falls into the category of *nonlocal* theories.[7] The maximum distance across which a pair of material points can exert forces

[7]A kind of theory that takes into account effects of long-range interaction. Some of their applications to problems of solid and fracture mechanics are discussed in [22, 28, 34].

is called the **horizon**[8] for the material which is treated as a constant material property. A given point can not "see" past its horizon. The term "peridynamics" is originated from the Greek roots *peri* and *dyna* which are for *near* and *force* respectively.

1.2 Peridynamic equation of motion.

Let $\mathcal{B}$ be the reference configuration of a closed, bounded body with reference mass density ρ. Let $\boldsymbol{y}(\cdot,\cdot)$ be a *deformation* of $\mathcal{B}$, so $\boldsymbol{y}(\boldsymbol{x},t)$ is the position at time $t \geq 0$ of a material point $\boldsymbol{x} \in \mathcal{B}$. Define the velocity field by

$$\boldsymbol{v}(\boldsymbol{x},t) = \partial_t \boldsymbol{y}(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \mathcal{B},\ t \geq 0.$$

Let $\boldsymbol{b}$ be the external body force density field. Let $\boldsymbol{L}(\boldsymbol{x},t)$ be the force per unit volume at time t on $\boldsymbol{x}$ due to interactions with other points in the body. The force vector on a subregion $\mathcal{P} \subset \mathcal{B}$ is given by

$$\int_{\mathcal{P}} (\boldsymbol{L} + \boldsymbol{b})\, dV.$$

Applying Newton's Second Law to this subregion,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\, \partial_t \boldsymbol{y}\, dV = \int_{\mathcal{P}} \rho\, \partial_t^2 \boldsymbol{y}\, dV = \int_{\mathcal{P}} (\boldsymbol{L} + \boldsymbol{b})\, dV, \tag{1.2}$$

hence, by localization, the equation of motion in terms of $\boldsymbol{L}$ is

$$\rho(\boldsymbol{x})\partial_t^2 \boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{L}(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t) \qquad \forall \boldsymbol{x} \in \mathcal{B},\ t \geq 0. \tag{1.3}$$

Newton's Second Law applied to $\mathcal{B}$ requires that

$$\frac{d}{dt}\int_{\mathcal{B}} \rho\, \partial_t \boldsymbol{y}\, dV = \int_{\mathcal{B}} \boldsymbol{b}\, dV. \tag{1.4}$$

[8] In Greek, "peri" has the meaning of "horizon".

Comparing (1.4) with (1.2) shows that $\boldsymbol{L}$ must be self-equilibrated:

$$\int_{\mathcal{B}} \boldsymbol{L}(\boldsymbol{x},t)dV_{\boldsymbol{x}} = 0 \qquad \forall t \geq 0.$$

Now let $\boldsymbol{u}(\boldsymbol{x},t)$ be the displacement of $\boldsymbol{x} \in \mathcal{B}$, and $\boldsymbol{f}$ be a vector-valued function such that

$$\boldsymbol{L}(\boldsymbol{x},t) = \int_{\mathcal{B}} \boldsymbol{f}(\boldsymbol{x'},\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x'},t),\boldsymbol{u}(\boldsymbol{x},t),t)dV_{\boldsymbol{x'}} \qquad \forall \boldsymbol{x} \in \mathcal{B},\ t \geq 0. \tag{1.5}$$

The function $\boldsymbol{f}$, which plays a fundamental role in the peridynamic theory, is called the *pairwise force density* whose value is the force vector (per unit volume squared) that the point $\boldsymbol{x'}$ exerts on the point $\boldsymbol{x}$.

In the following we will use the notation

$$\boldsymbol{\xi} = \boldsymbol{x'} - \boldsymbol{x}, \quad \boldsymbol{\eta} = \boldsymbol{u}(\boldsymbol{x'},t) - \boldsymbol{u}(\boldsymbol{x},t)$$

for relative position vectors and relative displacement vectors in the reference configuration, repectively. Note that

$$\boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t), \tag{1.6}$$

so $\boldsymbol{\xi}+\boldsymbol{\eta}$ $(= \boldsymbol{y}(\boldsymbol{x'},t) - \boldsymbol{y}(\boldsymbol{x},t))$ represents the current relative position vectors (in the deformed configuration).

The vector $\boldsymbol{\xi}$ is called a **bond**[9] (connected to $\boldsymbol{x}$).

Certain restrictions on $\boldsymbol{f}$ arise from basic mechanical considerations. For example, if the system is assumed to be invariant under rigid body motion and if the internal forces are independent of time, then

$$\boldsymbol{f}(\boldsymbol{x'},\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x'},t),\boldsymbol{u}(\boldsymbol{x},t),t) = \boldsymbol{f}(\boldsymbol{x'},\boldsymbol{x},\boldsymbol{\eta}).$$

[9]The concept of a bond that extends over a finite distance is a fundamental difference between peridynamics and classical continuum mechanics.

The body is *homogeneous* if

$$\boldsymbol{f}(\boldsymbol{x'}, \boldsymbol{x}, \boldsymbol{\eta}) = \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta})$$

is fulfilled for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

An important restriction on the form of $\boldsymbol{f}$ is provided by Newton's Third Law which gives

$$\boldsymbol{f}(-\boldsymbol{\xi}, -\boldsymbol{\eta}) = -\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) \qquad \forall \boldsymbol{\xi},\ \boldsymbol{\eta}.$$

Finally, for a given material, there is a positive number δ (horizon), such that

$$|\boldsymbol{\xi}| > \delta \quad \Longrightarrow \quad \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{0} \quad \forall \boldsymbol{\eta}.$$

In the following, $B_\delta(\boldsymbol{x})$ will denote the spherical neighborhood of $\boldsymbol{x}$ with radius δ.

Incorporating these discussions on $\boldsymbol{f}$ with (1.6) into (1.3), we obtain the *peridynamic equation of motion*

$$\rho(\boldsymbol{x})\partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t) = \int_{B_\delta(\boldsymbol{x})} \boldsymbol{f}\big(\boldsymbol{x'} - \boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x'}, t) - \boldsymbol{u}(\boldsymbol{x}, t)\big) dV_{\boldsymbol{x'}} + \boldsymbol{b}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \mathcal{B},\ t \geq 0. \quad (1.7)$$

By setting $\partial_t^2 \boldsymbol{u} = 0$, the *equilibrium equation* is found to be

$$\int_{B_\delta(\boldsymbol{x})} \boldsymbol{f}\big(\boldsymbol{x'} - \boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x'}) - \boldsymbol{u}(\boldsymbol{x})\big) dV_{\boldsymbol{x'}} + \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \mathcal{B}.$$

A body composed of *discrete particles* (e.g., atoms) can be represented as a peridynamic body and so applies the same equation of motion. For example [7], suppose a set of discrete particles is given with reference positions $\boldsymbol{x}_i$ and mass m_i, $i = 1, 2, \ldots, n$. Let the force exerted by particles j on particles i after deformation of the system be denoted by $\boldsymbol{F}_{j,i}(t)$. With Dirac delta function $\delta(\boldsymbol{x})$, we define a peridynamic body by

$$\rho(\boldsymbol{x}) = \sum_i m_i \delta(\boldsymbol{x} - \boldsymbol{x}_i)$$

and the corresponding pairwise force density function as

$$\boldsymbol{f}(\boldsymbol{x'}, \boldsymbol{x}, t) = \sum_i \sum_{j \neq i}^{n} \boldsymbol{F}_{j,i}(t)\, \delta(\boldsymbol{x'} - \boldsymbol{x}_j)\delta(\boldsymbol{x} - \boldsymbol{x}_i) \qquad \text{for all } \boldsymbol{x}, \boldsymbol{x'} \text{ in } \mathbb{R}^3.$$

Therefore, from (1.2), (1.5) and (1.6) with region $\mathcal{P} \subset \mathbb{R}^3$ enclosing only $\boldsymbol{x}_i$,

$$\begin{aligned}
\int_{\mathcal{P}} \rho(\boldsymbol{x})\, \partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} &= \int_{\mathcal{P}} \boldsymbol{L}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} \\
&= \int_{\mathcal{P}} \int_{\mathbb{R}^3} \boldsymbol{f}(\boldsymbol{x'}, \boldsymbol{x}, t)\, dV_{\boldsymbol{x'}}\, dV_{\boldsymbol{x}} \\
&= \int_{\mathcal{P}} \int_{\mathbb{R}^3 \setminus \mathcal{P}} \sum_i \sum_{j \neq i}^{n} \boldsymbol{F}_{j,i}(t)\, \delta(\boldsymbol{x'} - \boldsymbol{x}_j)\delta(\boldsymbol{x} - \boldsymbol{x}_i)\, dV_{\boldsymbol{x'}}\, dV_{\boldsymbol{x}},
\end{aligned}$$

by noticing the fact that

$$\int_{\mathcal{P}} \int_{\mathcal{P}} \boldsymbol{f}(\boldsymbol{x'}, \boldsymbol{x}, t)\, dV_{\boldsymbol{x'}}\, dV_{\boldsymbol{x}} = 0$$

due to Newton's Third Law $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x'}, t) = -\boldsymbol{f}(\boldsymbol{x'}, \boldsymbol{x}, t)$. Substituting the expression of $\rho(\boldsymbol{x})$ into left-hand side yields

$$\sum_i m_i \partial_t^2 \boldsymbol{u}(\boldsymbol{x}_i, t) = \sum_i \sum_{j \neq i}^{n} \boldsymbol{F}_{j,i}(t),$$

i.e.,

$$m_i \partial_t^2 \boldsymbol{u}(\boldsymbol{x}_i, t) = \sum_{j \neq i}^{n} \boldsymbol{F}_{j,i}(t) \qquad i = 1, 2, \ldots, n,\ t \geq 0,$$

which is the familiar statement of Newton's Second Law in the particle mechanics setting.

Notice that the integral in (1.7) expresses that the internal force at $\boldsymbol{x}$ is a summation of forces over all bonds connected to $\boldsymbol{x}$; moreover, the summands are independent from each other. However, this assumption is an oversimplification for most materials and leads to restrictions on the types of materials that can be modeled. In particular, it effectively limits

Poisson ratio[10] to a value of 1/4 for linear[11] isotropic solid materials, as demonstrated by Silling [44, §11]. In the same paper, a generalization for the linear theory is presented that augments the integral with the term $e(v(\boldsymbol{x}))$, where[12]

$$v(\boldsymbol{x}) = \int_{B_\delta(\boldsymbol{x})} j\big(|\boldsymbol{x'} - \boldsymbol{x}|\big)|\boldsymbol{y}(\boldsymbol{x'}, t) - \boldsymbol{y}(\boldsymbol{x}, t)|\, dV_{\boldsymbol{x'}}.$$

The quantity v is a weighted average of the deformation of all the bonds $\boldsymbol{x'} - \boldsymbol{x}$. It may be thought of as essentially giving the volume of a deformed sphere that is centered at $\boldsymbol{x}$ in the reference configuration; the quantity e then acts as a volume-dependent strain energy term that incorporates the collective motion of all the bonds $\boldsymbol{x'} - \boldsymbol{x}$ simultaneously. The modified formula is then shown to circumvent the restriction of Poisson ratio to a value of one-fouth to its allowable values.

Silling et al. [40] developes a new peridynamic theory, a subsequent generalization to the approach introduced above. In the new theory, the forces within each bond are not conceived as being determined independently of each other. Instead, each bond force depends on the collective deformation of all the bonds connected to its endpoints. This strategy is realized by introducing a mathematical object called *force state* that is in some ways similar to the traditional stress tensor of classical continuum mechanics as a replacement for pairwise force density function $\boldsymbol{f}$. The integral in (1.7) hence becomes

$$\int_{B_\delta(\boldsymbol{x})} \Big(\underline{\mathbf{T}}[\boldsymbol{x}, t]\langle \boldsymbol{x'} - \boldsymbol{x}\rangle - \underline{\mathbf{T}}[\boldsymbol{x'}, t]\langle \boldsymbol{x} - \boldsymbol{x'}\rangle\Big)\, dV_{\boldsymbol{x'}},$$

[10]Poisson ratio is a measure of the *Poisson effect*, the phenomenon in which a material tends to expand in directions perpendicular to the direction of compression, being the amount of transversal expansion divided by the amount of axial compression. If the material is stretched rather than compressed, it usually tends to contract in the directions transverse to the direction of stretching (imaging a rubber band), in which case the Poisson ratio will be the ratio of relative contraction to relative expansion. The Poisson ratio of a stable, isotropic, linear elastic material will be greater than -1 or less than 0.5. Most materials have Poisson ratio values ranging between 0 and 0.5.

[11]A peridynamic material, peridynamic model or a peridynamic theory is called *linear* if the pairwise force density $\boldsymbol{f}(\boldsymbol{\xi}, \cdot)$ is a linear function of $\boldsymbol{\eta}$ while $\boldsymbol{\xi}$ is fixed.

[12]Both e and j are scalar-valued functions.

where the *force state* $\underline{\mathbf{T}}[\boldsymbol{x}, t]$[13](or $\underline{\mathbf{T}}[\boldsymbol{x}', t]$) is a mapping from the bond $\boldsymbol{x}' - \boldsymbol{x}$ ($\boldsymbol{x} - \boldsymbol{x}'$) to a force density at $\boldsymbol{x}$ ($\boldsymbol{x}'$) and is assumed to be zero outside the horizon δ. Because of the analogy to stress tensors, it is possible to apply constitutive equations[14] in the classical theory more or less directly in the peridynamic theory. By using this new concept, it is shown [40] that the generalized peridynamic theory can include materials with any Poisson ratio.

Since the mathematical objects that convey information about the collective deformation of bonds are called *states*[15], the resulting modified theory is named **state-based**. As an opposite, the theory of original version is called **bond-based**.

1.2.1 States and equation of motion in terms of force states.

Consider a body $\mathcal{B}$. Let δ be the horizon. For a given $\boldsymbol{x} \in \mathcal{B}$, let $B_\delta(\boldsymbol{x})$ be the neighborhood of radius δ with center $\boldsymbol{x}$. Define the family of bonds connected to $\boldsymbol{x}$ by

$$\mathcal{H} = \{\boldsymbol{\xi} \in (\mathbb{R}^3 \backslash \mathbf{0}) \,\big|\, (\boldsymbol{\xi} + \boldsymbol{x}) \in B_\delta(\boldsymbol{x}) \cap \mathcal{B}\}.$$

A *state* $\underline{\mathbf{A}}\langle\cdot\rangle$ is a function on $\mathcal{H}$. The angle brackets $\langle\cdot\rangle$ enclose the bond vector. A state need not be a linear, differentiable or continuous function of the bonds in $\mathcal{H}$.

If the value $\underline{\mathbf{A}}\langle\cdot\rangle$ is a scalar, i.e., $\underline{\mathbf{A}}$ maps vectors (bonds) into scalars, then $\underline{\mathbf{A}}$ is called a *scalar state*. The set of all scalar states is denoted $\mathcal{S}$. Scalar states are usually written as lower case, non-bold font with an underscore, e.g., $\underline{a}$. Two special scalar states are the *zero state* and the *unity state* defined respectively by

$$\underline{0}\langle\boldsymbol{\xi}\rangle = 0, \qquad \underline{1}\langle\boldsymbol{\xi}\rangle = 1 \qquad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

[13]Square brackets indicate dependencies of the state on the position $\boldsymbol{x}$ and time t.

[14]They can be used to show how force states depend on deformed vectors $\boldsymbol{y}(\boldsymbol{x}', t) - \boldsymbol{y}(\boldsymbol{x}, t)$ and $\boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t)$, and thus tell how force state $\underline{\mathbf{T}}$ is determined for an equation of motion.

[15]The term "states" is chosen in analogy with the traditional usage of this term in thermodynamics: these objects contain descriptions of all the relevant variables that affect the conditions at a material point in the body. In the case of peridynamics, these variables are the nonlocal interactions between a point and its neighbors.

If the value of $\underline{\mathbf{A}}\langle\cdot\rangle$ is a vector, then $\underline{\mathbf{A}}$ is a *vector state*. The set of all vector states is denoted $\mathcal{V}$. Two special vector states are the *null vector state* and the *identity state* defined by

$$\underline{\mathbf{0}}\langle\boldsymbol{\xi}\rangle = \mathbf{0}, \qquad \underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathcal{H}$$

where $\mathbf{0}$ is the null vector.

An example of a scalar state is given by

$$\underline{a}\langle\boldsymbol{\xi}\rangle = 3\mathbf{c}\cdot\boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathcal{H},$$

where $\mathbf{c}$ is a constant vector. An example of a vector state is given by

$$\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} + \mathbf{c} \qquad \forall\boldsymbol{\xi} \in \mathcal{H}.$$

Some elementary operations on states can be defined. In the following, $\underline{a}$ and $\underline{b}$ are scalar states, $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are vector states, and $\mathbf{V}$ is a vector. Then for any $\boldsymbol{\xi} \in \mathcal{H}$:

$$\begin{aligned}
(\underline{a}+\underline{b})\langle\boldsymbol{\xi}\rangle &= \underline{a}\langle\boldsymbol{\xi}\rangle + \underline{b}\langle\boldsymbol{\xi}\rangle, \qquad & (\underline{a}\underline{b})\langle\boldsymbol{\xi}\rangle &= \underline{a}\langle\boldsymbol{\xi}\rangle\underline{b}\langle\boldsymbol{\xi}\rangle,\\
(\underline{\mathbf{A}}+\underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle &= \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle, \qquad & (\underline{\mathbf{A}}\cdot\underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle &= \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle,\\
(\underline{\mathbf{A}}\otimes\underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle &= \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle\otimes\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle, \qquad & (\underline{\mathbf{A}}\circ\underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle &= \underline{\mathbf{A}}\big\langle\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle\big\rangle,\\
(\underline{a}\underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle &= \underline{a}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle, \qquad & (\underline{\mathbf{A}}\cdot\mathbf{V})\langle\boldsymbol{\xi}\rangle &= \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle\cdot\mathbf{V},
\end{aligned}$$

where the symbol $\cdot$ indicates the usual scalar product of two vectors in $\mathbb{R}^3$ and $\otimes$ denotes the dyadic (tensor) product of two vectors. Also define a scalar state $|\underline{\mathbf{A}}|$, i.e., the *magnitude state* of $\underline{\mathbf{A}}$ by

$$|\underline{\mathbf{A}}|\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle| \tag{1.8}$$

and the *dot products*

$$\underline{a} \bullet \underline{b} = \int_{\mathcal{H}} \underline{a}\langle\boldsymbol{\xi}\rangle \underline{b}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}, \qquad \underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}$$

where once again, the symbol $\cdot$ denotes the scalar product of two vectors in $\mathbb{R}^3$. The *norm* of a scalar state or a vector state is defined by

$$\|\underline{a}\| = \sqrt{\underline{a} \bullet \underline{a}}, \qquad \|\underline{\mathbf{A}}\| = \sqrt{\underline{\mathbf{A}} \bullet \underline{\mathbf{A}}}.$$

It is readily verified that both $\mathcal{S}$ and $\mathcal{V}$ are infinite dimensional real Euclidean spaces [32] (assuming that $\mathcal{H}$ contains an infinite number of bonds).

The *direction* of a state $\underline{\mathbf{A}}$ can be defined to be the state $\mathbf{Dir}\,\underline{\mathbf{A}}$ given by

$$(\mathbf{Dir}\,\underline{\mathbf{A}})\langle\boldsymbol{\xi}\rangle = \begin{cases} \mathbf{0} & \text{if } |\underline{\mathbf{A}}|\langle\boldsymbol{\xi}\rangle = 0, \\ \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle / |\underline{\mathbf{A}}|\langle\boldsymbol{\xi}\rangle & \text{otherwise} \end{cases} \qquad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

A *state field* is defined by

$$\underline{\mathbf{A}}[\boldsymbol{x}, t],$$

a state valued function of position in $\mathcal{B}$ and time[16]. An example of a scalar state field is given by

$$\underline{a}[\boldsymbol{x}, t]\langle\boldsymbol{\xi}\rangle = |\boldsymbol{\xi} + \boldsymbol{x}| t \qquad \forall \boldsymbol{\xi} \in \mathcal{H},\ \boldsymbol{x} \in \mathcal{B},\ t \geq 0.$$

A vector state is analogous[17] to a second order tensor of the classical theory, because it maps vectors into vectors. It therefore provide the fundamental objects on which constitutive models act in peridynamics. In the classical theory, a constitutive model for a simple material specifies a tensor (stress) as a function of another tensor (deformation gradient). In the

[16] Note that square brackets are employed to enclose the dependencies. With this notation, a sequence of functions $f_n \;\; n = 1, 2, \ldots$ will be rewritten as $f[n] \;\; n = 1, 2, \ldots$

[17] In fact, vector states are more complex than second order tensors in that the mapping may be nonlinear and even discontinuous. It can be precisely shown [47, §3] that second order tensors are in some sense a special case of vector states.

peridynamic theory, a constitutive model instead provide a vector state (called the *force state*) as a function of another vector state (called the *deformation state*).

The state that maps bonds connected to $\boldsymbol{x}$ into their deformed images is called the *deformation state* and denoted $\underline{\mathbf{Y}}[\boldsymbol{x}, t]$. For a motion $\boldsymbol{y}$, at any $t \geq 0$,

$$\underline{\mathbf{Y}}[\boldsymbol{x}, t]\langle \boldsymbol{x'} - \boldsymbol{x} \rangle = \boldsymbol{y}(\boldsymbol{x'}, t) - \boldsymbol{y}(\boldsymbol{x}, t)^{18}$$

for any $\boldsymbol{x} \in \mathcal{B}$ and any $\boldsymbol{x'} \in \mathcal{B}$ such that $\boldsymbol{x'} - \boldsymbol{x} \in \mathcal{H}$. Angle brackets are used to indicate a bond that this state operates on. The *force state* $\underline{\mathbf{T}}[\boldsymbol{x}, t]$ is a state that maps the bond $\boldsymbol{x'} - \boldsymbol{x}$ to a *force density* (per unit volume) at $\boldsymbol{x}$ which is given by

$$\boldsymbol{t}(\boldsymbol{x'}, \boldsymbol{x}, t) = \underline{\mathbf{T}}[\boldsymbol{x}, t]\langle \boldsymbol{x'} - \boldsymbol{x} \rangle. \tag{1.9}$$

With this definition, the absorbed power density [42, §2.4 (42)] takes the form

$$p_{\text{abs}} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}}$$

where the dot product has just been defined. This absorbed power density is the peridynamic analogue of the stress power $\boldsymbol{\sigma} \cdot \dot{\mathbf{F}}$, where $\boldsymbol{\sigma}$ is the Piola stress tensor and $\mathbf{F} = \partial \boldsymbol{y}/\partial \boldsymbol{x}$ is the deformation gradient tensor.

In terms of the force state, the equation of motion has the form

$$\rho(\boldsymbol{x})\partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t) = \int_{B_\delta(\boldsymbol{x})} \Big(\underline{\mathbf{T}}[\boldsymbol{x}, t]\langle \boldsymbol{x'} - \boldsymbol{x} \rangle - \underline{\mathbf{T}}[\boldsymbol{x'}, t]\langle \boldsymbol{x} - \boldsymbol{x'} \rangle \Big)\, dV_{\boldsymbol{x'}} + \boldsymbol{b}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \mathcal{B},\ t \geq 0. \tag{1.10}$$

[18] It is assumed that at any $t \geq 0$, $\boldsymbol{y}(\cdot, t)$ is invertible, i.e., $\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \implies \boldsymbol{y}(\boldsymbol{x}_1, t) \neq \boldsymbol{y}(\boldsymbol{x}_2, t)$, which means that two distinct particles never occupy the same point as the deformation progresses. This assumption implies $\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle \neq \mathbf{0} \quad \forall \boldsymbol{\xi} \in \mathcal{H}$.

The equilibrium equation is then

$$\int_{B_\delta(\boldsymbol{x})} \Big(\underline{\mathbf{T}}[\boldsymbol{x}]\langle \boldsymbol{x'} - \boldsymbol{x} \rangle - \underline{\mathbf{T}}[\boldsymbol{x'}]\langle \boldsymbol{x} - \boldsymbol{x'} \rangle \Big)\, dV_{\boldsymbol{x'}} + \boldsymbol{b}(\boldsymbol{x}) = \mathbf{0} \qquad \forall \boldsymbol{x} \in \mathcal{B}.$$

The force state will be provided by constitutive model (also called *material model*). For a *simple* material and a homogeneous body, the force state depends only on the deformation state[19]:

$$\underline{\mathbf{T}}[\boldsymbol{x}, t] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\boldsymbol{x}, t])$$

where $\hat{\underline{\mathbf{T}}} : \mathcal{V} \to \mathcal{V}$ is a function whose value is a force state. Suppressing from the notation dependence on $\boldsymbol{x}$ and t,

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$$

which is analogous to the Piola stress in a simple material in the classical theory, $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$. If the body is heterogeneous, an explicit dependence on $\boldsymbol{x}$ is included:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \boldsymbol{x}).$$

If the material is rate dependent, the constitutive model would additionally depend on the time derivative of the deformation state:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \dot{\underline{\mathbf{Y}}}, \boldsymbol{x}).$$

An example of a simple[20] peridynamic material model is given by

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) = a(|\underline{\mathbf{Y}}| - |\underline{\mathbf{X}}|)\, \mathbf{Dir}\, \underline{\mathbf{Y}}, \qquad \mathbf{Dir}\, \underline{\mathbf{Y}} = \frac{\underline{\mathbf{Y}}}{|\underline{\mathbf{Y}}|} \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \tag{1.11}$$

[19] This is actually how a *simple* material is defined in peridynamics.

[20] An example of *non-simple* material is *plastic*, which involves the history of deformation as well as the current deformation, discussed in [47, §16].

where a is a constant. Writing this out in detail,

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = a\big(|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|\big)\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|} \qquad \forall\underline{\mathbf{Y}} \in \mathcal{V},$$

for any bond $\boldsymbol{\xi} \in \mathcal{H}$. In this material, the magnitude of the force density $\boldsymbol{t}$ defined in (1.9) is proportional to the bond extension (change in length of the bond), and its direction is parallel to the deformed bond. In this example, the bonds respond independently of each other: $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$ depends only on $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$. Material with such property is called *bond-based* which has been mentioned previously.

A much larger class of materials incorporates the *collective* response of bonds. This means that the force density in each bond depends not only on its own deformation, but also on the deformation of other bonds. A simple example is given by

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = a\big(|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\underline{\mathbf{Y}}\langle-\boldsymbol{\xi}\rangle|\big)\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}.$$

In this material, the force density for any bond $\boldsymbol{\xi}$ is proportional to the difference in deformed length between itself and the bond opposite to $\boldsymbol{\xi}$. (Note that in general $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \neq \underline{\mathbf{Y}}\langle-\boldsymbol{\xi}\rangle$, since the two bonds $\boldsymbol{\xi}$ and $-\boldsymbol{\xi}$ can deform independently of each other.) This is an example of the material called *bond-pair*. A general form of such material is discussed in [42, §4.12], which demonstrates that the bond-based materials are a special case of bond-pair materials.

From each example provided above, we can read that the state-based theory include the bond-based theory as a special case. In general, for a given bond-based material, the pairwise force density function $\boldsymbol{f}(\boldsymbol{x'} - \boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x'}, t) - \boldsymbol{u}(\boldsymbol{x}, t))$ can be recovered via the force states. In fact, we can let (the force density) $\underline{\mathbf{T}}[\boldsymbol{x}, t]\langle\boldsymbol{x'}-\boldsymbol{x}\rangle = 1/2\boldsymbol{f}(\boldsymbol{x'}-\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x'}, t)-\boldsymbol{u}(\boldsymbol{x}, t))$

in that it depends only on the bond $\boldsymbol{x'}-\boldsymbol{x}$ and relative displacement $\boldsymbol{u}(\boldsymbol{x'},t)-\boldsymbol{u}(\boldsymbol{x},t)$, then

$$\begin{aligned}
&\underline{\mathbf{T}}[\boldsymbol{x},t]\langle\boldsymbol{x'}-\boldsymbol{x}\rangle-\underline{\mathbf{T}}[\boldsymbol{x'},t]\langle\boldsymbol{x}-\boldsymbol{x'}\rangle\\
&=\frac{1}{2}\boldsymbol{f}(\boldsymbol{x'}-\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x'},t)-\boldsymbol{u}(\boldsymbol{x},t))-\frac{1}{2}\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{x'},\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{u}(\boldsymbol{x'},t))\\
&=\boldsymbol{f}(\boldsymbol{x'}-\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x'},t)-\boldsymbol{u}(\boldsymbol{x},t)).
\end{aligned}$$

This identification reveals another important distinction between the bond-based and state-based theories: that force interaction is carried by the bond in the former theory while the interaction is split between the force density at $\boldsymbol{x}$ and $\boldsymbol{x'}$ in the latter theory.

Example in (1.11) is one of the simplest *nonlinear* bond-based material models that has been suggested in the literature, modeling the *microelastic material*[21]. By substituting $\boldsymbol{\xi}+\boldsymbol{\eta}$ for $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$, we obtain

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle=a\left(|\boldsymbol{\xi}+\boldsymbol{\eta}|-|\boldsymbol{\xi}|\right)\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|}.$$

Let

$$s(\boldsymbol{\xi},\boldsymbol{\eta})=\frac{|\boldsymbol{\xi}+\boldsymbol{\eta}|-|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|},$$

then the pairwise force density function will be

$$\begin{aligned}
&\underline{\mathbf{T}}[\boldsymbol{x},t]\langle\boldsymbol{x'}-\boldsymbol{x}\rangle-\underline{\mathbf{T}}[\boldsymbol{x'},t]\langle\boldsymbol{x}-\boldsymbol{x'}\rangle\\
&=2\,\underline{\mathbf{T}}[\boldsymbol{x},t]\langle\boldsymbol{x'}-\boldsymbol{x}\rangle=2\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle\\
&=2a\,s(\boldsymbol{\xi},\boldsymbol{\eta})\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|}\,|\boldsymbol{\xi}|\qquad\forall\boldsymbol{\xi}\in\mathcal{H}.
\end{aligned}\tag{1.12}$$

$s(\boldsymbol{\xi},\boldsymbol{\eta})$ denotes the *bond stretch* that is the relative change of the length of a bond.

[21] If a material is microelastic, every pair of points $\boldsymbol{x}$ and $\boldsymbol{x'}$ is connected like by a spring in the sense that the force between them depends only on their distance in the deformed configuration, i.e., $|\boldsymbol{\xi}+\boldsymbol{\eta}|$; see [44, §4] for more details.

1.3 Linear bond-based model with initial data and its mathematical analysis.

1.3.1 Linearization.

Let $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta})$ be the pairwise force density function of a bond-based model. A Taylor expansion of $\boldsymbol{f}$ justifies for small $\boldsymbol{\eta}$ the *linear ansatz*

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{f}_0(\boldsymbol{\xi}) + \boldsymbol{C}(\boldsymbol{\xi})\eta$$

with a stiffness tensor (or *micromodulus function*) $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{\xi})$ and $\boldsymbol{f}_0$ denoting forces in the reference configuration. Without loss of generality, we may assume $\boldsymbol{f}_0 \equiv 0$ since otherwise $\boldsymbol{f}_0$ can be incorporated into the right-hand side $\boldsymbol{b}$.

In general the stiffness tensor $\boldsymbol{C}$ is not definite. However, $\boldsymbol{C}$ has to be symmetric with respect to Newton's Third Law as well as with respect to its tensor structure such that

$$\boldsymbol{C}(\boldsymbol{\xi}) = \boldsymbol{C}(-\boldsymbol{\xi}) \qquad \text{and} \qquad \boldsymbol{C}(\boldsymbol{\xi})^T = \boldsymbol{C}(\boldsymbol{\xi}).$$

In view of horizon, we shall require

$$\boldsymbol{C}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \text{if } |\boldsymbol{\xi}| \geq \delta.$$

In the following, we only consider a linear microelastic material, then the stiffness tensor can be shown to read as

$$\boldsymbol{C}(\boldsymbol{\xi}) = \lambda_{d,\delta}(|\boldsymbol{\xi}|)\boldsymbol{\xi} \otimes \boldsymbol{\xi}$$

where $\otimes$ denotes the dyadic product. The function $\lambda_{d,\delta} : \mathbb{R}_0^+ \to \mathbb{R}$ with $\lambda_{d,\delta}(r) = 0$ for $r \geq \delta$ determines the specific constitutive model and depends on the dimension d and the horizon

δ. The *linear* peridynamic equation of motion now read as

$$\rho(\boldsymbol{x})\partial_t^2 \boldsymbol{u}(\boldsymbol{x},t) = (L_{d,\delta}\boldsymbol{u})(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t) \qquad (\boldsymbol{x},t) \in \Omega \times (0,T)^{22}, \tag{1.13}$$

with

$$(L_{d,\delta}\boldsymbol{u})(\boldsymbol{x},t) := \int_{B_\delta(\boldsymbol{x})} \lambda_{d,\delta}\big(|\boldsymbol{x'}-\boldsymbol{x}|\big)(\boldsymbol{x'}-\boldsymbol{x}) \otimes (\boldsymbol{x'}-\boldsymbol{x}) \cdot (\boldsymbol{u}(\boldsymbol{x'},t) - \boldsymbol{u}(\boldsymbol{x},t))dV_{\boldsymbol{x'}}$$

Du et al. [19] also demonstrates how the one-dimensional equation above can be written as two first-order in time nonlocal advection equations. Note that $\lambda_{d,\delta}$ can have a singularity at $r = 0$. The standard example is the linearization of (1.12) with

$$\lambda_{d,\delta}(r) = \frac{2a}{r^3}, \qquad r \in (0,\delta).$$

Unfortunately, in this model, the interaction jumps to zero if $r = \delta$. This jump discontinuity can be avoided by taking

$$\lambda_{d,\delta}(r) = \frac{c}{r^3}\exp(-\delta^2/(\delta^2 - r^2)) \qquad r \in (0,\delta)$$

with a suitable constant of proportionality c, see also Emmrich & Weckner [26]. This is of advantage also to the numerical approximation relying on quadrature.

Since there are no spatial derivatives, boundary conditions are not needed in general for the partial integro-differential equation (1.13) (although this depends on the singularity behavior of the integral kernel and the functional analytic setting). Nevertheless, "boundary" conditions can be imposed by prescribing $\boldsymbol{u}$ in a strip along the boundary which constrains

[22] In order to introduce the related theoretical results, here we assume the body is defined in a set $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$, t is finite, and both $\boldsymbol{u}$ as well as $\boldsymbol{b}$ are the $\mathbb{R}^d$-valued functions, i.e., $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t) : \Omega \times [0,T] \to \mathbb{R}^d$ and $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x},t) : \Omega \times [0,T] \to \mathbb{R}^d$.

the solution along a nonzero volume. Hence, (1.13) is complemented with the initial data

$$\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0 \qquad \text{and} \qquad \partial_t\boldsymbol{u}(\cdot,0)=\dot{\boldsymbol{u}}_0. \tag{1.14}$$

1.3.2 Mathematical analysis of the linear bond-based model in L^2.

As usual, we denote by $\mathcal{C}$ the space of continuous functions, by $\mathcal{C}_b$ the space of bounded continuous functions, by $L^p\,(1\leq p<\infty)$ the space of Lebesgue-measurable functions u such that $|u|^p$ is Lebesgue-integrable, by L^∞ the space of essentially bounded Lebesgue-measurable functions and by $W^{k,p}$ $(1\leq p\leq\infty)$ the Sobolev space consisting of the functions whose derivatives (in the weak sense [39, p.343]) up to k order are belong to L^p [23], written $W^{k,p}=H^k$ as $p=2$. The canonical norm in a normed function space X is denoted by $||\cdot||_X$. Moreover, let $\mathcal{C}^m([0,T];X)$ with $m\in\mathbb{N}$ be the space of m-times continuously differentiable abstract functions $u:[0,T]\to X$ with norm

$$\|u\|_{C^m([0,T];X)}=\max_{t\in[0,T]}\sum_{j=0}^{m}\left\|\frac{d^ju(t)}{dt^j}\right\|_X.$$

We also write $C([0,T];X)$ if $m=0$. The function space $L^1(0,T;X)$ consists of Bochner-integrable abstract functions $u:[0,T]\to X$ such that $t\mapsto\|u(t)\|_X$ is Lebesgue-integrable and is equipped with the norm

$$\|u\|_{L^1(0,T;X)}=\int_0^T\|u(t)\|dt.$$

First results on existence, uniqueness and qualitative behavior of solutions in L^2 to the linear peridynamic equation of motion have been presented in Emmrich & Weckner [25] for the infinite bar. Besides well-posedness in L^∞ also nonlinear dispersion relations as well as jump relations for discontinuous solutions have been studied.

[23] $W^{k,p}(\Omega):=\{w\in L^p(\Omega)\ |D^jw\in L^p(\Omega)\text{ for }j\leq k\}$.

In [24], Emmrich and Weckner have proved results on existence, uniqueness, and continuous dependence of the solution for the linear model with data in an L^p-setting for $p > 2$ if $d = 2$ and $p > 3/2$ if $d = 3$. Moreover, a formal representation of the exact solution and a priori estimates are given. In [26], Emmrich and Weckner proved well-posedness of the linear model in $L^\infty(\Omega)$ and in $L^2(\Omega)$ under the condition

$$\int_0^\delta |\lambda_{d,\delta}(r)|\, r^{d+1} dr < \infty. \tag{1.15}$$

Theorem 1.1. *If (1.15) is fulfilled, then there exists for every $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0 \in L^2(\Omega)$, $\boldsymbol{b} \in L^1(0,T; L^2(\Omega))$ a unique solution $\boldsymbol{u} \in \mathcal{C}^1([0,T]; L^2(\Omega))$ to the initial value problem (1.13), (1.14) that satisfies the priori estimate*

$$\|\boldsymbol{u}\|_{\mathcal{C}^1([0,T];L^2(\Omega))} \le C_{d,\delta} \left(\|\boldsymbol{u}_0\|_{L^2(\Omega)} + \|\dot{\boldsymbol{u}}_0\|_{L^2(\Omega)} + \|\boldsymbol{b}\|_{L^1(0,T;L^2(\Omega))} \right).$$

If $\boldsymbol{b} \in \mathcal{C}([0,T]; L^2(\Omega))$, then $\boldsymbol{u} \in \mathcal{C}^2([0,T]; L^2(\Omega))$.

Moreover, other properties of the peridynamic integral operator defined through (1.13) such as dissipativity and self-adjointness are analyzed in Emmrich & Weckner [26].

In [21, 20], Du and Zhou consider the case $\Omega = \mathbb{R}^d$. Let $\mathcal{M}_\lambda(\mathbb{R}^d)$ be the space of functions $\boldsymbol{u} \in L^2(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} (\mathcal{F}\boldsymbol{u})(\boldsymbol{y}) \cdot (\boldsymbol{I} + \boldsymbol{M}_\delta(\boldsymbol{y}))(\mathcal{F}\boldsymbol{u}) d\boldsymbol{y} < \infty,$$

depending on $\lambda_{d,\delta}$ since

$$\boldsymbol{M}_\delta(\boldsymbol{y}) = \int_0^\delta \lambda_{d,\delta}(|\boldsymbol{x'}|)(1 - \cos(\boldsymbol{y} \cdot \boldsymbol{x'}))\boldsymbol{x'} \otimes \boldsymbol{x'} dV_{\boldsymbol{x'}}$$

which is a real-valued and symmetric positive semi-definite $d \times d$ matrix. Here $\mathcal{F}\boldsymbol{u}$ denotes the Fourier transform of $\boldsymbol{u}$.

A natural condition coming from the comparison of the deformation energy density which arises from peridynamics with the energy density that is known from the classical linear elasticity theory is

$$\int_0^\delta \lambda_{d,\delta}(r) r^{d+3} dr < \infty. \tag{1.16}$$

Theorem 1.2. *Assume* $\lambda_{d,\delta}(r) > 0$ *for* $0 < r < \delta$*, (1.16) and* $\boldsymbol{u}_0 \in \mathcal{M}_\lambda(\mathbb{R}^d)$*,* $\dot{\boldsymbol{u}}_0 \in L^2(\mathbb{R}^d)$ *and* $\boldsymbol{b} \in L^2(0,T; L^2(\mathbb{R}^d))$*. Then the initial value problem (1.13), (1.14) has a unique solution* $\boldsymbol{u} \in \mathcal{C}([0,T], \mathcal{M}_\lambda(\mathbb{R}^d))$ *with* $\boldsymbol{u}_t \in L^2(0,T; L^2(\mathbb{R}^d))$*.*

If in addition (1.15) is valid, then Du and Zhou show that the space $\mathcal{M}_\lambda(\mathbb{R}^d)$ is equivalent to the space $L^2(\mathbb{R}^d)$.

1.3.3 Linear bond-based model in H^σ ($\sigma \in (0,1)$).

The solution of Theorem 1.2 can take values in a fractional Sobolev space. Indeed, if

$$c_1 r^{-2-d-2\sigma} \le \lambda_{d,\delta}(r) \le c_2 r^{-2-d-2\sigma}, \qquad \forall\, 0 < r \le \delta$$

for some exponent $\sigma \in (0,1)$ and positive constant c_1 and c_2, then Theorem 1.2 remains true and the space $\mathcal{M}_\lambda(\mathbb{R}^d)$ is equivalent to the fractional[24] Sobolev space $H^\sigma(\mathbb{R}^d)$.

Additionally also the stationary problem is investigated in [21, 20].

1.3.4 A summary of the results on nonlinear bond-based model and state-based model.

A first result towards the nonlinear model is Erbay, Erkip & Muslu [27] analyzing the nonlinear elastic bar. They consider the one-dimensional initial value problem

$$u_{tt} = \int_{\mathbb{R}} \alpha(x'-x) g\big(u(x',t) - u(x,t)\big) dx', \qquad x \in \mathbb{R},\, t > 0, \tag{1.17}$$

$$u(x,0) = u_0, \qquad u_t(x,0) = \dot{u}_0, \qquad x \in \mathbb{R}.$$

[24] $\sigma > 0$ is an noninteger.

Applying Banach's fixed point theorem the following theorems ([27]) are proven.

Theorem 1.3. *Let $X = \mathcal{C}_b(\mathbb{R})$ or $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $1 \leq p \leq \infty$. Assume $\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$ with $g(0) = 0$. Then there exists $T > 0$ such that the Cauchy problem (1.17) is locally well-posed with solution in $\mathcal{C}^2([0,T], X)$ for initial data $u_0, \dot{u}_0 \in X$.*

Theorem 1.4. *Let $X = \mathcal{C}_b^1(\mathbb{R})$ or $W^{1,p}(\mathbb{R})$ with $1 \leq p \leq \infty$. Assume $\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R})$ with $g(0) = 0$. Then there exists $T > 0$ such that the Cauchy problem (1.17) is locally well-posed with solution in $\mathcal{C}^2([0,T], X)$ for initial data $u_0, \dot{u}_0 \in X$.*

The authors of [27] remark that the proofs of the above theorems can be easily adapted to the more general peridynamic equation with a nonlinear pairwise force function $f(\xi, \eta)$, where f is continuously differentiable in η for almost every ξ and fulfils additional assumptions. For a more specific type of nonlinearities, Erbay, Erkip & Muslu [27] proved well-posedness in fractional Sobolev spaces.

Theorem 1.5. *Let $\sigma > 0$ and $u_0, \dot{u}_0 \in H^\sigma(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume $\alpha \in L^1(\mathbb{R})$ and $g(\eta) = \eta^3$. Then there exists $T > 0$ such that the Cauchy problem (1.17) is locally well-posed with solution in $\mathcal{C}^2([0,T], H^\sigma(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.*

Furthermore, blow up conditions for these solutions are investigated, which we shall not present here.

As for the state-based model, Du et al. [16, 17] consider a *nonlocal vector calculus* building upon the ideas of Gunzburger & Lehoucq [29]. The nonlocal vector calculus is applied to establish the well-posedness of the *linear peridynamic state equilibrium equation*; see Du et al. [15] for the details.

1.3.5 Limit of vanishing nonlocality.

A fundamental question of the peridynamic theory was if it generalizes the conventional linear elastic theory. More precisely, if a deformation is classically smooth, does the nonlocal[25] linear peridynamic equation of motion *converge* towards the Navier equation of linear elasticity (as $\delta \to 0$)? Indeed in [26], Emmrich and Weckner proved convergence in an interior subdomain under smoothness assumptions of the solution. Therefore, let $\Lambda \subset \mathbb{R}^+$ be a null sequence bounded by some $\delta_0 > 0$, Ω_0 be the interior subdomain defined as all $\boldsymbol{x} \in \Omega$ such that $\operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \delta_0$, $L_{d,\delta}$ the linear operator defined through (1.13) and L the operator corresponding to the Navier equation of linear elasticity defined through (1.1) (with $\lambda = \mu$).

Theorem 1.6. *([26]) Let (1.15) be valid for all $\delta \in \Lambda$ and $\lambda_{d,\delta}$ be nonnegative. If $\boldsymbol{v} \in \mathcal{C}^2(\Omega)$ then*

$$\|L_{d,\delta}\boldsymbol{v} - L\boldsymbol{v}\|_{L^\infty(\Omega_0)} \to 0 \qquad \text{as } \delta \to 0 \quad (\delta \in \Lambda).$$

In addition, an expension of $L_{d,\delta}\boldsymbol{v}$ in terms of a series of differential operators of even order $2n$ ($n = 1, 2, \dots$) applied to $\boldsymbol{v}$ can be shown for smooth $\boldsymbol{v}$, where the second-order differential operator is the Navier operator and where the coefficients of the differential operators behave like $\delta^{2(n-1)}$.

Furthermore in [21], Du and Zhou have also investigated the limit of vanishing nonlocality in the case of the full space $\Omega = \mathbb{R}^d$ being then able to show convergence of the sequence of solutions.

Theorem 1.7. *Let (1.16) be valid and $\lambda_{d,\delta}(r) > 0$ for $0 < r < \delta$. If $\boldsymbol{u}_0 \in H^1(\mathbb{R}^d)$, $\dot{\boldsymbol{u}}_0 \in L^2(\mathbb{R}^d)$ and $\boldsymbol{b} \in L^2(0,T;L^2(\mathbb{R}^d))$, then the solution of the initial value problem (1.13), (1.14) converges to the solution of the initial value problem (1.1), (1.14) as $\delta \to 0$ in the conventional norms of $L^2(0,T;\mathcal{M}_\lambda(\mathbb{R}^d)) \cap H^1(0,T;L^2(\mathbb{R}^d))$ if*

$$\int_{B_\delta(\mathbf{0})} \lambda_{d,\delta}(|\boldsymbol{x}|)\, |\boldsymbol{x}|^4 \, d\boldsymbol{x} \to 2d(d+2)\mu \quad \text{as} \quad \delta \to 0,$$

where μ is the Lamé parameter appearing in (1.1).

[25] Given a function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, the operator L acted on it (such as $L_{d,\delta}$ in (1.13)) is deemed *nonlocal* if the value of $L\boldsymbol{u}$ at point $\boldsymbol{x}$ requires information about $\boldsymbol{u}$ at $\boldsymbol{x'} \neq \boldsymbol{x}$; this is contrasted with *local* operators, e.g., the value of $\Delta\boldsymbol{u}$ at a point $\boldsymbol{x}$ requires information about $\boldsymbol{u}$ only at $\boldsymbol{x}$.

Note that here no extra regularity of the solution is assumed.

The limit of vanishing nonlocality of the *state-based model* is investigated in Silling & Lehoucq [41].

1.4 Numerical analysis for peridynamic models.

Nearly all of the applications of the peridynamic model to date rely on numerical solutions. A numerical technique for approximating the peridynamic equations was proposed in [43] . This numerical method simply replaces the volume integral in (1.7) with a finite sum:

$$\frac{\rho_i}{h^2}\left(\boldsymbol{u}_i^{n+1} - 2\boldsymbol{u}_i^n + \boldsymbol{u}_i^{n-1}\right) = \sum_{j\in\mathcal{H}} \boldsymbol{f}(\boldsymbol{u}_j^n - \boldsymbol{u}_i^n, \boldsymbol{x}_j - \boldsymbol{x}_i)V_i + \boldsymbol{b}_i^n$$

where i is the node number, n is the time step number, h is the time step size, and V_i is the volume (in the reference configuration) of node i. This numerical method is meshless in the sense that there are no geometrical connections, such as elements, between the discretized nodes. Adaptive refinement and convergence of the discretized method in one dimension are discussed in [10]. Some examples and more details about the numerical method can be found in [46, 48].

Finite element (FE) discretization techniques for the peridynamic equations have been proposed by Zimmermann [51] and by Weckner et al. [49]. Macek [37] demonstrated that standard truss elements available in the Abaqus commercial FE code can be used to represent peridynamic bonds. These peridynamic elements can be applied in part of an FE mesh with standard elements in the remainder of the mesh. The resulting FE model of the peridynamic equations was applied in [37] to penetration problems. A FE formualtion was also developed by Chen and Gunzburger [13], who consider the one dimensional equations for a finite bar. Weckner and Emmrich investigated certain discretizations of the peridynamic equation of motion, including Gauss-Hermite quadrature, and applied these to initial value problems to demonstrate convergence [23, 50]. Du and Zhou [20] discussed finite-dimensional approximations to nonlocal boundary value problems and the corresponding error estimates which appeared to be the first of their kind in the literature. Convergence analysis and conditioning estimates for the discretized system have been given in [4, 6, 16, 20]. Du et al. [18] proposed a general abstract framework for a posteriori error analysis of finite element methods for solving linear nonlocal diffusion and bond-based peridynamic models.

Among applications of the peridynamic model to real systems, Bobaru [8, 9] demonstrated the application of a numerical model to small scale structures, including nanofibers and nanotubes. The meshless property of the numerical method, as well as the ability to treat long-range forces, is helpful in these applications because of the need to generate models of complex, random structures. Small scale numerical applications of the peridynamic equations are also demonstrated by Agwai, Guven, and Madenci [1, 2].

1.5 Linear bond-based model on a finite bar with boundary conditions and finite dimensional approximations to its solutions.

In (1.13), let

$$\lambda_{d,\delta}(|\boldsymbol{x'} - \boldsymbol{x}|) = \frac{c_\delta}{\sigma(|\boldsymbol{x'} - \boldsymbol{x}|)}$$

in which $c_\delta > 0$ is a normalization constant; $\sigma = \sigma(|\boldsymbol{x'} - \boldsymbol{x}|)$ is a function depending only on the scalar $|\boldsymbol{x'} - \boldsymbol{x}|$, called the **kernel function** (of the stiffness tensor). Then we have the peridynamic equation of the form

$$\partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{L}}_\delta \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{b}(\boldsymbol{x}, t)$$

with

$$\boldsymbol{\mathcal{L}}_\delta \boldsymbol{u}(\boldsymbol{x}, t) = c_\delta \int_{B_\delta(\boldsymbol{x})} \frac{(\boldsymbol{x'} - \boldsymbol{x}) \otimes (\boldsymbol{x'} - \boldsymbol{x})}{\sigma(|\boldsymbol{x'} - \boldsymbol{x}|)} \big(\boldsymbol{u}(\boldsymbol{x'}, t) - \boldsymbol{u}(\boldsymbol{x}, t)\big) dV_{\boldsymbol{x'}},$$

where we can see the kernel function uniquely determines the properties of solutions. Such equation (with different kernel functions) is studied in many existing works [5, 10, 13, 25, 26, 29, 40, 45].

In the following, we will mainly discuss the one-dimensional stationary (equilibrium) problem, so the equation can be further simplified as:

$$\mathcal{L}_\delta u(x) + b(x) = 0,$$

i.e.,

$$-\mathcal{L}_\delta u(x) = b(x)$$

where

$$-\mathcal{L}_\delta u(x) = -c_\delta \int_{x-\delta}^{x+\delta} \frac{|x'-x|^2}{\sigma(|x'-x|)} \big(u(x') - u(x)\big) dx'.$$

We call "$-\mathcal{L}_\delta$" the *peridynamic (PD) operator.*

Assuming that the equation is defined on the interval $I = (0, \pi)$ and is associated with the simple boundary conditions $u(0) = 0$ and $u(\pi) = 0$, we derive the following (nonlocal) boundary value problem (BVP):

$$\begin{cases} -\mathcal{L}_\delta u(x) = b(x) & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{1.18}$$

Du and Zhou [20] studied (1.18) in depth; several of their results are original and general. Here we briefly go over their work and leave the detailed investigation to the following chapters.

First, by using Fourier expansions, they obtained a Fourier series representation of the solution. Next, they defined an appropriate solution space explicitly relying on kernel function, in preparation for discussing the well-posedenss and regularity problems. Under suitable assumptions of kernel function, the relations between the solution space and Sobolev spaces (especially the fractional Sobolev spaces) were then built; meanwhile, the corresponding inequalities of space embedding were derived. By virtue of variatonal theory as well as the prescribed conditions for kernel function, they proved the BVP (1.18) is well-posed; further, an additional assumption introduced previously on kernel function enabled them to acquire an useful regularity result of the unique solution: an estimate of the solution in terms of the data function b with lower regularity. So far the theorectical foundation had been set up. Afterwards, the authors turned to seek approximations to the solution u by picking two kinds of finite-dimensional subspaces (of the solution space). Applying the results that had been established, first they showed, for the finite-dimensional subspace V_n spanned by the first n Fourier sine modes, i.e., $V_n = \{v\,|v(x) = \sum_{k=1}^{n} v_k \sin(kx)\}$,

Theorem 1.8. *Let kernel function σ satisfy some conditions with a constant $\beta \in [0,2)$. Then for $b \in H^m$, we have*

$$\|u - u_n\|_{H^\gamma} \leq C_\delta(\beta)^{-2}\,\frac{\|b\|_{H^m}}{n^{m+\beta-\gamma}} \quad \text{for any } \gamma \in [0,m], \tag{1.19}$$

where $u_n \in V_n$; $C_\delta(\beta)$ is a constant only depending on β and δ.

H^s represents the fractional Sobolev space on the interval I for $s \in [0,1)$.

Second they proved, for subspace V_n[26] formed by continuous piecewise polynomials that of degree $m(\geq 1)$ and are subject to the boundary conditions of (1.18), with a mesh (a partition of I into n subintervals) having meshwidth parameter h (the maximum length of those subintervals),

Theorem 1.9. *If the kernel function σ also satisfy some additional conditions, with a constant α such that $0 \leq \beta \leq \alpha \in (0,2)$. Then for $b \in H^{m'-\beta}$ with $\beta \leq m' \leq m+1$, we have*

$$\|u - u_n\|_{H^{\beta/2}} \leq cC_\delta(\alpha)C_\delta(\beta)^{-3}\,h^{m'-\alpha/2}\|b\|_{H^{m'-\beta}}, \tag{1.20}$$

where $u_n \in V_n$; the constant c is independent of h, δ and b; $h \to 0$ as $n \to \infty$.

[26]For convenience we keep using the same notation for the other finite-dimensional subspace.

The first way to find approximations of the solution is called **Fourier spectral method** which is quite natural and can be easily implemented. As for the second one, because the *finite element polynomials* are used to approximate the solution, the method demonstrated is the **Finite Element Method**.

1.6 Exponential approximations to solutions.

Observing (1.19) and (1.20), one will find that the rate of convergence can be improved by lifting the regularity of the data function b, i.e., by increasing m or m' (β is fixed); in other words, the smoother the function b, the more rapid the convergence. In this regard, one may be concerned with if the best convergence rate is achievable by resorting to a suitable data function.

The reply is affirmative! We are able to prove that the estimates in (1.19) and (1.20) fulfill *exponential convergence* with the assumption that function b is (real) analytic[27] (the function that is not only infinitely differentiable, but also identical to its Taylor expansion everywhere in the domain). Likewise, we merely roughly state our results, the elaboration of which will be presented at Chapter 3.

For Fourier spectral method, we have:

Theorem 1.10. *Assume kernel function σ, finite-dimensional space V_n and constant β are same as those stated in the theorem 1.8. Then for $b(x)$ analytic on $\bar{I}$ and any τ with $0 < \tau < \tau_0$, we have*

$$\|u - u_n\|_{H^\gamma} \leq c_\tau C_\delta(\beta)^{-2} \, \frac{e^{-\tau n}}{n^{\beta-\gamma}} \qquad \text{for any } \gamma \geq 0,$$

where $u_n \in V_n$; c_τ is a constant only depends on b and τ.

With finite element method, we possess:

Theorem 1.11. *Assume kernel function σ with constants β & α are maintained as those in the theorem 1.9, but V_n is made up by all the continuous piecewise polynomials that of degree*

[27] Smoothness brings about the algebraic convergence rate (i.e., $O(n^{-k})$) at best, which will be explained in Chapter 3.

through 0 *to* n *and satisfy that boundary conditions. Then for* $b(x)$ *analytic on* $\bar{I}$*, we have*

$$\|u - u_n\|_{H^{\beta/2}} \leq cC_\delta(\alpha)C_\delta(\beta)^{-1}e^{-\tau n},$$

where $u_n \in V_n$*;* c *and* $\tau > 0$ *are some constants independent of* n*,* δ *and* b*.*

This dissertation is organized as follows. The theoretical foundation (developed by [20]) is constructed in Chapter 2, where we precisely define the nonlocal BVP we shall discuss and build the associated solution space, based on which we investigate the well-posedness and regularity issues. In addition, our first finding about analyticity of solutions is presented at the end of this chapter. We devote Chapter 3 to the expositions of our main results on exponential convergence of finite-dimensional approximations. Some numerical experiments are demonstrated in the last chapter with the aim of validating the results.

Chapter 2

Theoretical foundation: nonlocal BVPs and solutions

We begin with defining the types of functions we will consider and the PD operator.

Definition 2.1. *Assume $u \in L^2$ defined on the interval $(-\delta, \pi+\delta)$ satisfies either*

$$\textit{odd in } (-\delta,\delta) \textit{ and } (\pi-\delta,\pi+\delta)^1, \tag{2.1}$$

or

$$\textit{even in } (-\delta,\delta) \textit{ and } (\pi-\delta,\pi+\delta)^2. \tag{2.2}$$

The PD operator $-\mathcal{L}_\delta$ is defined by

$$-\mathcal{L}_\delta u(x) = -c_\delta \int_{x-\delta}^{x+\delta} \frac{|x'-x|^2}{\sigma(|x'-x|)} \big(u(x') - u(x)\big) dx' \quad \forall x \in (0,\pi), \tag{2.3}$$

where $c_\delta > 0$, and for a nonnegative function $\rho = \rho(|x|)$ in $L^1(B_\delta(0))$, the kernel function $\sigma = \sigma(|y|)$ satisfies

$$\frac{|y|^2}{\sigma(|y|)} \geq \rho(|y|) \quad \forall y \in (-\delta,\delta) \quad \textit{and} \quad \tau_\delta := c_\delta \int_{-\delta}^{\delta} \frac{|y|^4}{\sigma(|y|)} dy < \infty. \tag{2.4}$$

Remark. The restrictions (2.1) and (2.2) will allow us to more easily form the natural nonlocal boundary conditions and formulate the spectrum of the corresponing PD operator, as the following shows.

[1] It means that the graph of $u(x)$ on this interval is symmetric with respect to the point $(\pi, 0)$.

[2] $x = \pi$ is the symmetric line of the graph within this interval.

For smooth enough functions, (2.1) implies $u(0) = u(\pi) = 0$, while (2.2) gives $u_x(0) = u_x(\pi) = 0$. Moreover, with Fourier sine and cosine series expansions, we have

$$u(x) = \sum_{k=1}^{\infty} u_k^o \sin(kx) \quad \text{and} \quad u(x) = \sum_{k=1}^{\infty} u_k^e \cos(kx) \tag{2.5}$$

with the coefficients $\{u_k^o\}$ or $\{u_k^e\}$ given by

$$u_k^o = \frac{2}{\pi}\int_0^{\pi} u(x)\sin(kx)dx \quad \forall k \geq 1, \qquad u_k^e = \frac{2}{\pi}\int_0^{\pi} u(x)\cos(kx)dx \quad \forall k \geq 1. \tag{2.6}$$

Denoting the PD operator by $-\mathcal{L}_\delta^o$ for functions satisfying (2.1) and by $-\mathcal{L}_\delta^e$ for those meeting (2.2), we have the following representations of operators:

Proposition.

$$-\mathcal{L}_\delta^o u(x) = \sum_{k=1}^{\infty} \eta_\delta(k) u_k^o \sin(kx), \tag{2.7}$$

$$-\mathcal{L}_\delta^e u(x) = \sum_{k=1}^{\infty} \eta_\delta(k) u_k^e \cos(kx), \tag{2.8}$$

where

$$\eta_\delta(k) = c_\delta \int_{-\delta}^{\delta} (1 - \cos(ky)) \frac{|y|^2}{\sigma(|y|)} dy \qquad \forall k \geq 1. \tag{2.9}$$

Proof. We only show (2.7). (2.8) can be derived by perfoming the similar process.

As to (2.6), the coefficients of $-\mathcal{L}_\delta^o u(x)$ in sine expansion are given by

$$\frac{2}{\pi}\int_0^{\pi} \big(- \mathcal{L}_\delta^o u(x)\big) \sin(kx)dx. \tag{2.10}$$

With (2.3), we get

$$
\begin{aligned}
(2.10) &= -\frac{2}{\pi}c_\delta \int_0^\pi \int_{x-\delta}^{x+\delta} \frac{|x'-x|^2}{\sigma(|x'-x|)}\big(u(x')-u(x)\big)\sin(kx)dx'dx \\
&= -\frac{2}{\pi}c_\delta \int_0^\pi \int_{x-\delta}^{x+\delta} \frac{|x'-x|^2}{\sigma(|x'-x|)}\Big(\sum_{k'} u^o_{k'}\big(\sin(k'x')-\sin(k'x)\big)\Big)\sin(kx)dx'dx \\
&= -\frac{2}{\pi}c_\delta \int_0^\pi \int_{x-\delta}^{x+\delta} \frac{|x'-x|^2}{\sigma(|x'-x|)}\Big(\sum_{k'} u^o_{k'}\sin(k'x')\sin(kx) - \sum_{k'} u^o_{k'}\sin(k'x)\sin(kx)\Big)dx'dx
\end{aligned}
$$

let $y = x' - x$,

$$
= -\frac{2}{\pi}c_\delta \int_{-\delta}^{\delta} \frac{|y|^2}{\sigma(|y|)}\Big(\sum_{k'} u^o_{k'}\int_0^\pi \sin(k'x+k'y)\sin(kx)dx - \sum_{k'} u^o_{k'}\int_0^\pi \sin(k'x)\sin(kx)dx\Big)dy.
$$

Notice that the first integral in the parenthese has value of $\frac{\pi}{2}\cos(ky)$ as $k' = k$; otherwise it is zero. The second integral takes the value of $\frac{\pi}{2}$ if $k' = k$, equal to zero for any $k' \neq k$. Thus, we finally gain

$$
\begin{aligned}
(2.10) &= -\frac{2}{\pi}c_\delta \int_{-\delta}^{\delta} \frac{|y|^2}{\sigma(|y|)}\Big(\frac{\pi}{2}u^o_k\cos(ky) - \frac{\pi}{2}u^o_k\Big)dy \\
&= u^o_k\Big(c_\delta \int_{-\delta}^{\delta} \frac{|y|^2}{\sigma(|y|)}\big(1-\cos(ky)\big)dy\Big) \\
&= u^o_k\ \eta_\delta(k),
\end{aligned}
$$

which gives (2.7). □

Similar to [20], we will mainly focus on the functions satisfying condition (2.1) with the Fourier sine expansion; nevertheless, all the paralell conclusions are derivable to the functions concerning condition (2.2).

Now we form our nonlocal BVP as follows:

$$
\begin{cases}
-\mathcal{L}^o_\delta u = f & \text{in } (0,\pi), \\
u(0) = u(\pi) = 0,
\end{cases}
\tag{2.11}
$$

where u satisfies (2.1).

2.1 Solution space M_σ^o

Prior to a further investigation into (2.11), it is necessary to assign a solution space:

Definition 2.2. *The space M_σ^o, which depends on the kernel function σ, consists of all functions $u \in L^2$ for which $(-\mathcal{L}_\delta^o u, u) < \infty$. The M_σ^o-norm is defined by*

$$\|u\|_{M_\sigma^o} = \Big[\frac{2}{\pi}(-\mathcal{L}_\delta^o u, u)\Big]^{1/2} = \Big(\sum_{k=1}^{\infty} \eta_\delta(k) u_k^{o2}\Big)^{1/2}.$$

The corresponding inner product in M_σ^o is given by

$$(u, v)_{M_\sigma^o} = \sum_k \eta_\delta(k) u_k^o v_k^o \quad \forall u,\, v \in M_\sigma^o.$$

In addition, given an exponent s, one can define the general space M_σ^{so} by

$$M_\sigma^{so} = \Big\{u \in L^2 : \|u\|_{M_\sigma^{so}} = \Big(\sum_{k=1}^{\infty} \eta_\delta^s(k) u_k^{o2}\Big)^{1/2} < \infty\Big\}.$$

Remark. We can see that M_σ^{so} is a Hilbert space (refer to [21, Lemma 2.3] for the proof) and varied with different conditions of σ.

2.2 The relations between M_σ^o and Sobolev spaces

Let H_o^s denote the standard fractional order Sobolev space on $(0, \pi)$ for $s \in [0, 1)$. In our circumstance, one can characterize the space and its norm as the following equivalent form[3]:

$$H_o^s := \big\{v = \sum_{k=1}^{\infty} v_k^o \sin(kx) \,|\, \|v\|_s^2 = \sum_{k=1}^{\infty} |v_k^o|^2 k^{2s} < \infty\big\}.$$

[3]The norm is equivalent to the regular Sobolev's norm by Parseval formula.

Note that as $s = 0$, $H_o^0 = M_\sigma^{0o} = L_o^2$.

In virtue of this representation of fractional order Sobolev space, we are able to show the following natural relations of spaces:

Lemma 2.0.1. *With the assumption of σ in (2.4), the space M_σ^o satisfies*[4]

$$H_o^1 \hookrightarrow M_\sigma^o \hookrightarrow L_o^2,$$

with $\eta_\delta(k)$ satisfying

$$0 < \inf_{k\geq 1} c_\delta \int_{-\delta}^{\delta} (1-\cos(ky))\rho(|y|)dy \leq \eta_\delta(k) \leq \frac{\tau_\delta}{2}k^2 \quad k \geq 1. \tag{2.12}$$

Proof. First, by (2.4), we have

$$\eta_\delta(k) \geq c_\delta \int_{-\delta}^{\delta} (1-\cos(ky))\rho(|y|)dy > 0 \quad \forall k \geq 1.$$

Also by the *Riemann lemma*,

$$\lim_{k\to\infty} \int_{-\delta}^{\delta} \cos(ky)\rho(|y|)dy = 0,$$

which gives

$$\lim_{k\to\infty} \int_{-\delta}^{\delta} \big(1-\cos(ky)\big)\rho(|y|)dy = \int_{-\delta}^{\delta} \rho(|y|)dy > 0. \tag{2.13}$$

Thus the infimum in (2.12) is attainable and remains positive, which impiles that $M_\sigma^o \hookrightarrow L_o^2$. □

As to other assumptions of σ, we have

Lemma 2.0.2. *(i) Assume σ is such that, for some constant $\gamma_1 > 0$ and $\alpha \in (0,2)$,*

$$\sigma(|y|) \geq \gamma_1 |y|^{3+\alpha} \qquad \forall |y| \leq \delta. \tag{2.14}$$

[4] The symbol "$\hookrightarrow$" is the conventional notation for the continuous embedding between spaces.

Then for some constant $C_1^\delta(\alpha)$,

$$0 \le \eta_\delta(k) \le C_1^\delta(\alpha)^2 k^\alpha \qquad \forall k \ge 1. \tag{2.15}$$

Moreover,

$$\|u\|_{M_\sigma^o} \le C_1^\delta(\alpha)\|u\|_{\alpha/2} \qquad \forall u \in H_o^{\alpha/2}, \tag{2.16}$$

i.e., the space M_σ^o *satisfies* $\qquad H_o^{\alpha/2} \hookrightarrow M_\sigma^o$.

(ii) Assume σ *is such that, for some constant* $\gamma_2 > 0$ *and* $\beta \in [0, 2)$,

$$\sigma(|y|) \le \gamma_2 |y|^{3+\beta} \qquad \forall |y| \le \delta. \tag{2.17}$$

Then for some constant $C_2^\delta(\beta)$,

$$\eta_\delta(k) \ge C_2^\delta(\beta)^2 k^\beta \qquad \forall k \ge 1. \tag{2.18}$$

Moreover,

$$C_2^\delta(\beta)\|u\|_{\beta/2} \le \|u\|_{M_\sigma^o} \qquad \forall u \in M_\sigma^o, \tag{2.19}$$

i.e., the space M_σ^o *satisfies* $\qquad M_\sigma^o \hookrightarrow H_o^{\beta/2}$.

Proof. For (i), the coefficient $\eta_\delta(k)$ as defined in (2.9) satisfies

$$\begin{aligned} 0 \le \eta_\delta(k) &\le \frac{c_\delta}{\gamma_1} \int_{-\delta}^{\delta} \big(1 - \cos(ky)\big) \frac{|y|^2}{|y|^{3+\alpha}} dy \\ &= \frac{k^\alpha}{\gamma_1}\ c_\delta \int_{-k\delta}^{k\delta} \frac{1-\cos(z)}{|z|^{1+\alpha}} dz \\ &\le \frac{k^\alpha}{\gamma_1}\ c_\delta \int_{-\infty}^{\infty} \frac{1-\cos(z)}{|z|^{1+\alpha}} dz, \qquad (\divideontimes) \end{aligned}$$

let

$$\omega_\delta(\alpha, \chi) = c_\delta \int_{-\chi}^{\chi} \frac{1-\cos(z)}{|z|^{1+\alpha}} dz,$$

and $C_1^\delta(\alpha) = (\omega_\delta(\alpha,\infty)/\gamma_1)^{\frac{1}{2}}$, then

$$(\divideontimes) = k^\alpha \frac{\omega_\delta(\alpha,\infty)}{\gamma_1} = k^\alpha C_1^\delta(\alpha)^2.$$

Note that the improper integral $\omega_\delta(\alpha,\infty) < \infty$, as $\alpha \in (0,2)$. Thus, we get done with (2.15). The space relation of (2.16) then follows immediately.

Similarly for (ii),

$$\begin{aligned}
\eta_\delta(k) &\geq \frac{c_\delta}{\gamma_2}\int_{-\delta}^{\delta}\big(1-\cos(ky)\big)\frac{|y|^2}{|y|^{3+\beta}}dy \\
&= \frac{k^\beta}{\gamma_2}\ c_\delta \int_{-k\delta}^{k\delta}\frac{1-\cos(z)}{|z|^{1+\beta}}dz \\
&\geq \frac{k^\beta}{\gamma_2}\ c_\delta \int_{-\delta}^{\delta}\frac{1-\cos(z)}{|z|^{1+\beta}}dz \\
&= k^\beta\ \frac{\omega_\delta(\beta,\delta)}{\gamma_2}.
\end{aligned}$$

Let $C_2^\delta(\beta) = (\omega_\delta(\beta,\delta)/\gamma_2)^{\frac{1}{2}}$. We obtain (2.18) which is followed by (2.19). □

2.3 Properties of solutions

Fourier expansions (2.5) and (2.7) enable us to convert the PD equation in (2.11) to an algebraic equation:

$$\sum_{k=1}^{\infty}\eta_\delta(k)u_k^o\sin(kx) = \sum_{k=1}^{\infty}f_k^o\sin(kx),$$

from which we derive

$$u_k^o = \frac{f_k^o}{\eta_\delta(k)}.$$

Thus,

$$u(x) = \sum_{k=1}^{\infty}\frac{f_k^o}{\eta_\delta(k)}\sin(kx). \tag{2.20}$$

Lemma 2.0.3. *Under the assumption (2.4) on σ, (2.20) is the unique solution of nonlocal BVP (2.11) in space M_σ^o.*

Proof. First, (2.20) is a solution of the nonlocal problem (2.11) because of

$$\|u\|^2_{M^o_\sigma} = \sum_{k=1}^{\infty} \eta_\delta(k) u_k^{o2} = \sum_{k=1}^{\infty} \frac{f_k^{o2}}{\eta_\delta(k)} \leq C\|f\|_0 < \infty,$$

where constant C can be obtained according to (2.13).

For uniqueness, we consider the variational problem:

find $u \in M^o_\sigma$ such that $J(u) = \min_{v \in M^o_\sigma} J(v)$, where

$$J(v) = \frac{1}{2}(-\mathcal{L}^o_\delta v, v) - (f, v).$$

We claim this variational problem admits the only solution by the facts that the PD operator $-\mathcal{L}^o_\delta$ is a self-adjoint operator and M^o_σ is a Hilbert space. Hence, we conclude (2.20) is the unique solution of nonlocal BVP (2.11) in M^o_σ. □

A very important and useful result that can be derivable immediately from (2.20) is the property of *regularity*:

Lemma 2.0.4. *If σ satisfies both (2.4) and (2.17) with $\beta \in [0, 2)$, then for $f \in H^m_o$ ($m \geq -\beta$), the unique solution $u \in H^{m+\beta}_o$. Moreover,*

$$\|u\|_{m+\beta} \leq C^\delta_2(\beta)^{-2}\|f\|_m. \tag{2.21}$$

That is, an enhancing in regularity of order β for solutions comparing with data functions.

Proof.

$$\|f\|^2_m = \sum_k k^{2m} f_k^{o2} = \sum_k k^{2m} \eta^2_\delta(k) u_k^{o2},$$

on the other hand,

$$\sum_k k^{2m} \eta^2_\delta(k) u_k^{o2} \geq \sum_k k^{2m} \cdot k^{2\beta} C^\delta_2(\beta)^4 \cdot u_k^{o2} = C^\delta_2(\beta)^4 \cdot \|u\|^2_{m+\beta}$$

due to (2.18). □

The lemma above implies that the smoother the data function is, the smoother the solution will be. This observation directly leads to the consequence about analyticity of solutions, i.e., analytic data functions bring about analytic solutions.

Before giving out our result, we shall learn about something on analytic functions.

Definition 2.3. *A function f, with domain a set U and range either the real or complex numbers, is said to be* ***analytic at $\boldsymbol{x_0}$*** *$\in U$ if its Taylor expansion at x_0 converges to itself nearby x_0, i.e.,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad \text{in } B_{r_{x_0}}(x_0).$$

The function is said to be ***analytic on $\boldsymbol{U}$*** *if it is analytic at each point of U.*

i) If the set U is an interval on $\mathbb{R}$, then f is said to be ***(real) analytic on the interval $\boldsymbol{U}$****.*

ii) If the set U is a region of the complex plane and f is a function with complex variable z defined in U, then f is said to be ***analytic in the region $\boldsymbol{U}$****.*

Remark. $f(x)$ analytic on $[a, b]$ implies $f(x) \in C^\infty[a, b]$.

The relationship between functions analytic on an interval and functions analytic in a region is shown in the following propostion:

Proposition. *The function $f(x)$ is analytic on some interval I, if and only if, there exists a region D containing I in which $f(z)$ is analytic.*

Proof. For each point $x_0 \in I$, there is a quantity r_{x_0} and an expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{valid in } |x - x_0| < r_{x_0}.$$

When x is replaced by $z = x + iy$, the expression above defines a complex function analytic in the circle $|z - x_0| < r_{x_0}$. Let x_0 run through the interval, the circles $|z - x_0| < r_{x_0}$ will cover I. Let D be the union of these circles. Then D is an open set and is arcwise connected. In fact, if $p, q \in D$, join p to x_1 and q to x_2, the centers of their respective circles, then the arc px_1x_2q lies in D. D is therefore a region and $f(x)$ can be **continued analytically into it** such that $f(z)$ is analytic inside. $\square$

Analytic functions on an interval can be completely characterized by the growth of their derivatives, which is shown in [14]:

Lemma 2.0.5. *([14, Theorem 1.9.3]) The function $f(x)$ is analytic on an interval I if and only if there exist $C_f,\ r > 0$ depending only on f such that for $\forall x \in I$,*

$$|f^{(n)}(x)| \leq C_f\, r^n n!, \qquad \forall n \in \mathbb{N}, \tag{2.22}$$

Proof. "$\Longrightarrow$" . Suppose x_0 is a fixed point in I and (2.22) holds. By Taylor expansion we have for $x \in I$,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x - x_0) \qquad \text{for all } n, \tag{2.23}$$

where ξ is between x and x_0.

According to (2.22),

$$\left|\frac{f^{(n)}(\xi)}{n!}(x - x_0)^n\right| \leq C_f\, r^n |x - x_0|^n < (dr)^n |x - x_0|^n, \quad \text{where } d = \max(1, C_f),$$

so that if $|x - x_0| < \dfrac{1}{dr + 1}$, the remainder in (2.23) will converge to 0. The function f possesses a power series expansion valid in a neiborhood of x_0, which means f is analytic on I.

"$\Longleftarrow$". Assume f is analytic on I, then by proposition, we can find a simply connected region R containing I in which $f(z)$ is analytic. Let C be a curve surrounding I and lying in R. Then for $x \in I$, we have *Cauchy integral formula*:

$$|f^{(n)}(x)| \leq \frac{n!}{2\pi} \int_C \frac{|f(z)|}{|z - x_0|^{n+1}} ds.$$

If $L(C)$ denotes the length of C and δ is the minimum distance from C to I, then

$$|f^{(n)}(x)| \leq \frac{\max\limits_{z \in C} |f(z)|\, L(C) n!}{2\pi\, \delta^{n+1}} = \frac{Mn!}{\delta^n} \qquad \forall x \in I,$$

where M and δ are the constants depending only on f. Here we derive (2.22). □

Theorem 2.1 (Analyticity of solutions)**.** *If σ satisfies (2.4), (2.14) and (2.17) with $0 \leq \beta \leq \alpha \in (0, 2)$, then for $f(x) = \sum_{k=1}^{\infty} f_k^o \sin(x)$ analytic on $[0, \pi]$ and admiting an analytic continuation to the strip $|Imz| < \tau_0$ in the complex plane $\mathbb{C}$, the solution $u(x)$ is analytic on $[0, \pi]$.*

Proof. Note that $f(z)$ is analytic on the rectangular region $R:\ [0, \pi] \times (-\tau_0, \tau_0)$ and can be transformed into the formal Fourier expansion:

$$\begin{aligned} f(z) = \sum_{k=1}^{\infty} f_k^o \sin(kz) &= \sum_{k=1}^{\infty} f_k^o \cdot \frac{e^{ikz} - e^{-ikz}}{2i} \\ &= \sum_{k=1}^{\infty} \Big(\frac{f_k^o}{2i}\, e^{ikz} - \frac{f_k^o}{2i}\, e^{-ikz} \Big) \\ &= \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i}\, e^{ikz} \qquad (0 \leq \text{Re} z \leq \pi). \end{aligned}$$

Making the change of variables $\zeta = e^{iz}$ (i.e., $z = \arg \zeta - i \ln |\zeta|$) yields

$$v(\zeta) = f(\arg \zeta - i \ln |\zeta|) = \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i}\, \zeta^k.$$

Therefore, $f(z)$ analytic on R corresponds with $v(\zeta)$ analytic in the upper annulus $e^{-\tau_0} < |\zeta| < e^{\tau_0}$.

Next, we consider the similar transformation of $u(z)$:

$$u(z) = \sum_{k=1}^{\infty} u_k^o \sin(kz) = \sum_{k=1}^{\infty} \frac{f_k^o}{\eta_\delta(k)} \sin(kz)$$
$$= \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i \cdot \eta_\delta(k)} e^{ikz} \qquad (0 \leq \text{Re } z \leq \pi).$$

Let $\zeta = e^{iz}$, then

$$w(\zeta) = u(\arg\zeta - i\ln|\zeta|) = \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i \cdot \eta_\delta(k)} \zeta^k.$$

Combining (2.15) with (2.18) obtains $C_1^\delta(\alpha)^{-2}k^{-\alpha} \leq 1/\eta_\delta(k) \leq C_2^\delta(\beta)^{-2}k^{-\beta}$. Thus, the series $w(\zeta)$ has the same maximal convergent annulus as $v(\zeta)$ because the relations of their coefficients satisfy

$$r \leq \overline{\lim_{k\to\infty}} \left| \frac{f_k^o}{2C_2^\delta(\beta)^2 \ k^\beta} \right|^{-1/k} \leq \overline{\lim_{k\to\infty}} \left| \frac{f_k^o}{2i \ \eta_\delta(k)} \right|^{-1/k} \leq \overline{\lim_{k\to\infty}} \left| \frac{f_k^o}{2C_1^\delta(\alpha)^2 \ k^\alpha} \right|^{-1/k} \leq r,$$

where r is $\overline{\lim_{k\to\infty}} |f_k^o|^{-1/k}$.

$w(\zeta)$ is hence analytic in annulus $e^{-\tau_0} < |\zeta| < e^{\tau_0}$ as well and $u(z)$ follows being analytic on R. By proposition, $u(x)$ is analytic on the interval $[0, \pi]$. □

Chapter 3

Finite-dimensional approximations to solutions

3.1 Fourier spectral method

Making use of (2.21), Du and Zhou [20] obtained an error estimate by replacing u with $u - u_n$:

Theorem 3.1. *If σ satisfies both (2.4) and (2.17) with $\beta \in [0, 2)$, then for $f \in H_o^m$, we have*

$$\|u - u_n\|_\gamma \leq C_2^\delta(\beta)^{-2} \frac{\|f\|_m}{n^{m+\beta-\gamma}} \quad \text{for any } \gamma \in [0, m], \tag{3.1}$$

where $u_n \in V_n$, the finite-dimensional subspace of M_σ^o which is spanned by the first n Fourier sine modes.

This approach to derive error estimates is *Fourier spectral method.*

As we pointed out, the error estimates (3.1) can be improved if f is assumed to be analytic:

Theorem 3.2 (Exponential approximations by Fourier spectral method)**.** *If σ satisfies (2.4) and (2.17) with $\beta \in [0, 2)$, then for $f(x) = \sum_{k=1}^{\infty} f_k^o \sin(x)$ analytic on $[0, \pi]$ and admiting an analytic continuation to the strip $|Imz| < \tau_0$ in the complex plane $\mathbb{C}$, and any τ with $0 < \tau < \tau_0$, we have*

$$\|u - u_n\|_\gamma \leq \frac{2\sqrt{2} M(\tau) C_2^\delta(\beta)^{-2}}{(1 - e^{-2\tau})^{1/2}} \cdot n^{\gamma - \beta} e^{-\tau n} \qquad \forall \gamma \geq 0,$$

where $u_n \in V_n$, $M(\tau) = \max_{|Imz| \leq \tau} |f(z)|$.

Proof. From the proof of Theorem 2.1, we know that $f(z)$ takes the form of

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i}\, e^{ikz} \qquad (0 \leq \mathrm{Re}z \leq \pi)$$

which is analytic on the rectangular region $R: \ [0,\pi] \times (-\tau_0, \tau_0)$. Making the change of variable $\zeta = e^{iz}$, we derive

$$v(\zeta) = f(\arg\zeta - i\ln|\zeta|) = \sum_{k=-\infty}^{\infty} \frac{f_k^o}{2i}\, \zeta^k$$

which is found to be analytic in the upper annulus $e^{-\tau_0} < |\zeta| < e^{\tau_0}$. The right-hand side is a Laurent expansion whose coefficients are given by

$$\frac{f_k^o}{2i} = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{v(\zeta)}{\zeta^{k+1}} d\zeta, \quad \text{where } e^{-\tau_0} < r < e^{\tau_0},$$

from which we solve for f_k^o to obtain

$$f_k^o = \frac{1}{\pi}\int_{|\zeta|=r} \frac{v(\zeta)}{\zeta^{k+1}} d\zeta,\ \ k = 0, \pm 1, \pm 2, \cdots .$$

For convenience, we choose $r = e^{\tau}$. So $r > 1$.

On the other hand, Lemma (2.0.4) implies that both

$$u(x) = \sum_{k=1}^{\infty} \frac{f_k^o}{\eta_\delta(k)}\, \sin(kx) \quad \text{and} \quad u_n(x) = \sum_{k=1}^{n} \frac{f_k^o}{\eta_\delta(k)}\, \sin(kx)$$

are infinitely differentiable on $[0,\pi]$. Thus, for $\forall\gamma \geq 0$, we obtain with (2.18)

$$\begin{aligned}
\|u-u_n\|_\gamma^2 &= \sum_{k>n} k^{2\gamma} \frac{|f_k^o|^2}{|\eta_\delta(k)|^2} \\
&\leq \sum_{k\geq n} k^{2\gamma} \frac{|f_k^o|^2}{|\eta_\delta(k)|^2} \\
&\leq \ C_2^\delta(\beta)^{-4} n^{-2\beta} \sum_{k\geq n} k^{2\gamma} |f_k^o|^2.
\end{aligned}$$

Next we focus on the term $\sum\limits_{k\geq n} k^{2\gamma}|f_k^o|^2$;

$$\begin{aligned}
\sum_{k\geq n} k^{2\gamma}|f_k^o|^2 &= \frac{1}{\pi^2} \ \sum_{k\geq n} k^{2\gamma} \Big| \int_{|\zeta|=r} \frac{v(\zeta)}{\zeta^{k+1}} d\zeta \Big|^2 \\
&\leq \frac{M^2(\tau)}{\pi^2} \Big(\int_{|\zeta|=r} \frac{1}{|\zeta|} ds \Big)^2 \cdot \sum_{k\geq n} k^{2\gamma} r^{-2k} \\
&= 4M^2(\tau) \cdot \sum_{k\geq n} k^{2\gamma} r^{-2k} \\
&= 4M^2(\tau) \Big[\sum_{k\geq n} k^{2\gamma} r^{-2k} \Big]. \qquad (3.2)
\end{aligned}$$

It is left to estimate $\sum\limits_{k\geq n} k^{2\gamma} r^{-2k}$. Let $S_\gamma(x) = \sum\limits_{k\geq n} k^{2\gamma} x^k$. The following recursion formula can be found:

$$S_{\gamma+1}(x) = x^2 S_\gamma^{''}(x) + x S_\gamma^{'}(x).$$

With $S_1(x) = \sum\limits_{k\geq n} k^2 x^k = \dfrac{n^2\, x^n}{1-x} \cdot \Big[1 + O(\dfrac{1}{n})\Big]$, we conclude for $\forall\gamma \geq 0$,

$$S_\gamma(x) = \frac{n^{2\gamma} x^n}{1-x} \cdot \Big[1 + O(\frac{1}{n})\Big] \leq \frac{2n^{2\gamma} x^n}{1-x}, \quad \text{as } n \text{ is large enough.}$$

Therefore,

$$\sum_{k\geq n} k^{2\gamma} r^{-2k} = S_\gamma(r^{-2}) \leq \frac{2n^{2\gamma} r^{-2n}}{1-r^{-2}}.$$

Substituting it into (3.2) yields

$$\sum_{k\geq n} k^{2\gamma}|f_k^o|^2 \leq \frac{8M^2(\tau)}{1-r^{-2}} \cdot n^{2\gamma} r^{-2n}.$$

To sum up, we have shown

$$\begin{aligned}
\|u-u_n\|_\gamma^2 &\leq \sum_{k\geq n} k^{2\gamma} \cdot \frac{|f_k^o|^2}{|\eta_\delta(k)|^2} \\
&\leq C_2^\delta(\beta)^{-4} n^{-2\beta} \cdot \sum_{k\geq n} k^{2\gamma}|f_k^o|^2 \\
&\leq C_2^\delta(\beta)^{-4} n^{-2\beta} \cdot \frac{8M^2(\tau)}{1-r^{-2}} \cdot n^{2\gamma} r^{-2n} \\
&= 8M^2(\tau)\frac{C_2^\delta(\beta)^{-4}}{1-r^{-2}} \cdot n^{2(\gamma-\beta)} r^{-2n}.
\end{aligned}$$

Replacing r with e^τ and taking squre root of both sides, we finally acquire

$$\|u-u_n\|_\gamma \leq 2\sqrt{2}M(\tau)\frac{C_2^\delta(\beta)^{-2}}{(1-e^{-2\tau})^{1/2}} \cdot n^{\gamma-\beta} e^{-\tau n}.$$

□

3.2 Finite element method

Recall that in Lemma 2.0.3 we prove the uniqueness by applying the following principle: nonlocal BVP (2.11) is equivalent to the variational problem:
find $u = u(x) \in M_\sigma^o$ which is a minimizer of the functional

$$J(v) = \frac{1}{2}(v,v)_{M_\sigma^o} - (f,v) \qquad \text{for } \forall v \in M_\sigma^o.$$

Since $\left(M_\sigma^o,\ (\cdot,\cdot)_{M_\sigma^o}\right)$ is a Hilbert space, the variational problem has an unique solution $u_* \in M_\sigma^o$. Thus (2.11) possess the only solution u_* (namely, (2.20)) as well.

Now let V_n be the finite-dimensional subspace of M_σ^o which is made up by continuous piecewise polynomials satisfying (2.1) with degree $m(\geq 1)$, on a regular and quasiuniform mesh having n grid points with meshwidth parameter h ($\to 0$, as $n \to \infty$).

We can form the *approximation problem*: find $u_n = u_n(x) \in V_n$ minimizing the functional

$$J(v) = \frac{1}{2}(v, v)_{M_\sigma^o} - (f, v) \qquad \text{for } \forall v \in V_n.$$

Similar to the variational problem, there exists a sole $u_n \in V_n$ that solves it, by noticing that $\big(V_n,\ (\cdot,\cdot)_{M_\sigma^o}\big)$ is a Hilbert space in its own right.

It can be proved [12] that u_n is the *best approximation* of u_* in V_n measured with the norm of M_σ^o, i.e.,

$$\|u_* - u_n\|_{M_\sigma^o} = \min_{v \in V_n} \|u_* - v\|_{M_\sigma^o}. \tag{3.3}$$

Du and Zhou [20] gained the following error estimate by applying finite element method:

Theorem 3.3. *Let σ satisfies (2.4), (2.14) and (2.17) with $0 \leq \beta \leq \alpha \in (0, 2)$. Then for $f \in H^{m'-\beta}$ with $\beta \leq m' \leq m + 1$, we have*

$$\|u - u_n\|_{\beta/2} \leq cC_1^\delta(\alpha) C_2^\delta(\beta)^{-3}\, h^{m'-\alpha/2} \|f\|_{m'-\beta}, \tag{3.4}$$

where $u_n \in V_n$; the constant c is independent of h, δ and f.

Notice that if f is smooth, arbitrary high *algebraic rate of convergence* is obtainable as the polynomial degree m is raised. Nevertheless, the exponential decay is untouchable. In view of this, we turn to other method.

3.3 $P-$version finite element method

The finite element method used above to generate approximate solution u_n is more classical, called h-version FEM where convergence is brought by mesh refinement (that is, letting the meshwidth h tend to zero) and the polynomial degree is constant and fixed; In contrast, there exists an "opposit" approach named p-version FEM, consisting in keeping the mesh fixed and letting the polynomial degrees $p \to \infty$.

In what follows, we will see that the p-version FEM can achieve *exponential convergence* provided that data function is analytic.

By convention of notation, we use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. Moreover, $C(I)$ represents the space of continuous functions on an interval I and $\mathcal{P}_n$ the space of polynomials of degree less than or equal to n.

Definition 3.1 (Gauss-Lobatto points). *Given* Legendre polynomials *defined on [-1,1] with degree* $n \in \mathbb{N}_0$*:*

$$L_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2 - 1)^n \tag{3.5}$$

The ***Gauss-Lobatto points*** *on* $[0, \pi]$ *are the* zeros *of the polynomial*

$$g(x) := x(\pi - x)\tilde{L}_n'(x) \quad \textit{on } [0, \pi],$$

where $\tilde{L}_n(x) := L_n\left(\frac{2}{\pi}x - 1\right)$.

It is a well known fact that this polynomial has $n + 1$ distinct zeros lying in $[0, \pi]$. Clearly, both endpoints 0 and π are included, and by symmetry properties of the Legendre polynomials, those points are symmetric with respect to the midpoint $\pi/2$ of the interval. We denote the set of Gauss-Lobatto points by $\mathcal{GL}_n := \{x_i | i = 0, 1, \ldots, n\}$.

Notice that (3.5) is a basis (complete orthogonal system [33]) of the space $L^2[-1, 1]$; that is, every funcion $f \in L^2[-1, 1]$ can be represented as $f(x) \overset{L^2}{=} \sum_{n=0}^{\infty} a_n L_n(x)$, where

$$a_n = \frac{2n + 1}{2} \int_{-1}^{1} f(x)L_n(x)dx \tag{3.6}$$

and the convergence is understood in the sense of the norm on the space. However, if the function is **analytic**, the convergence should be **pointwise** because f can be expanded by

Taylor series (definition 2.3) and hence by a linear combinition of (3.5). In fact, there is a more general result towards this assertion:

Theorem 3.4 (K. Neumann). *Let $f(z)$ be analytic in the interior of $\mathcal{E}_\rho$ (an ellipse in the complex plane with foci ± 1 and sum of semi-axes ρ (> 1), i.e.,*

$$\mathcal{E}_\rho = \{z \in \mathbb{C} | |z-1| + |z+1| < \rho + \rho^{-1}\}),$$

but not in the interior of any $\mathcal{E}_{\rho'}$ with $\rho' > \rho$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n L_n(z) \tag{3.7}$$

with (3.6).

The series converges absolutely and uniformly on any closed set in the interior of $\mathcal{E}_\rho$. The series diverges exterior to $\mathcal{E}_\rho$. Moreover,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{\rho}. \tag{3.8}$$

The complete proof of the theorem can be found at [14, p.312]. Here we will only focus on deriving an estimate (shown in that proof) of the coefficents (3.6) for future use.

Proposition. *For $\forall n \in \mathbb{N}_0$, a_n defined in (3.6) satisfies*

$$|a_n| \leq C(2n+1)(\rho')^{-n} \qquad 1 < \rho' < \rho, \tag{3.9}$$

where C is a constant that depends upon ρ' but not on n.

Proof. According to the assumption on the theorem, $f(z)$ is analytic in and on $\mathcal{E}_{\rho'}$. Thus we may write for $t \in [-1, 1]$ by *Cauchy integral formula* [3]:

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{E}_{\rho'}} \frac{f(z)}{z-t} dz.$$

Combining it with (3.6),

$$
\begin{aligned}
a_n &= \frac{2n+1}{4\pi i}\int_{-1}^{1} P_n(t)\int_{\mathcal{E}_{\rho'}} \frac{f(z)}{z-t}dz\,dt \\
&= \frac{2n+1}{4\pi i}\int_{\mathcal{E}_{\rho'}} f(z)\int_{-1}^{1} \frac{P_n(t)}{z-t}dt\,dz = \frac{2n+1}{2\pi i}\int_{\mathcal{E}_{\rho'}} f(z)Q_n(z)dz,
\end{aligned}
$$

where the function

$$
Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{z-t}dt \qquad n \in \mathbb{N}_0
$$

is known as the *Legendre function of the second kind*, which is linearly independent of $P_n(z)$ [14, p.311]. Therefore

$$
|a_n| \leq \frac{2n+1}{2\pi} L(\mathcal{E}_{\rho'}) \max_{z\in\mathcal{E}_{\rho'}} |f(z)| \max_{z\in\mathcal{E}_{\rho'}} |Q_n(z)| \tag{3.10}
$$

where $L(\mathcal{E}_{\rho'})$ designates the length of $L(\mathcal{E}_{\rho'})$. From the inequality in [14, p.311],

$$
\max_{z\in\mathcal{E}_{\rho'}} |Q_n(z)| \leq \frac{\pi(\rho')^{-n}}{\rho'-1}.
$$

(3.10) hence pass to

$$
|a_n| \leq C(2n+1)(\rho')^{-n}
$$

where C is a constant depending only on ρ'. □

We define the **Gauss-Lobatto interpolation operator $\boldsymbol{i_n}$** for the function $u \in C[0,\pi]$ by interpolation in the $n+1$ Gauss-Lobatto points, i.e.,

$$
(i_n u)(x) := \sum_{i=0}^{n} u(x_i) l_i^{(n)}(x),
$$

where the Lagrange polynomials $l_i^{(n)}(x)$(of degree n) are defined as

$$l_i^{(n)}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

For an analytic function defined on the interval $[0, 1]$, there is an approximation result in terms of its Gauss-Lobatto interpolation which is the key to our theorem of exponential approximations with p-version finite element method. The conclusion remains valid if the interval is replaced with $[0, \pi]$, which can be seen by applying the transformation formula $y = \frac{x}{\pi}$ to the function considered.

With regard to the proof of this result, we introduce two important inequalities beforehand.

Proposition. *1. ([38, Lemma 3.2.1]) There is $C > 0$ independent of n such that*

$$\|i_n f\|_{L^\infty(I)} \le C(1 + \ln n)\|f\|_{L^\infty(I)} \qquad \forall f \in C(I). \tag{3.11}$$

2. ([39, Theorem 3.92]) For every $v \in \mathcal{P}_n$ it holds that

$$\|v'\|_{L^\infty(I)} \le 2n^2\|v\|_{L^\infty(I)}. \tag{3.12}$$

(3.12) is refered to as Markov's inequality*.*

Now we state the approximation result. Because of its fundamental role in deriving our theorem of exponential approximations, we also present its proof in what follows.

Lemma 3.4.1. *([38, Lemma 3.2.6]) Let u be analytic on the interval $I = [0, 1]$ and satisfy for some C_u, $\gamma > 0$ and $h \in (0, 1]$*

$$\|D^m u\|_{L^\infty(I)} \le C_u\,(\gamma h)^m m! \qquad \forall m \in \mathbb{N}.$$

Then there are C, $\sigma > 0$ depending only on γ such that the Gauss-Lobatto interpolant $i_n u$ satisfies

$$\|u - i_n u\|_{L^\infty(I)} + \|(u - i_n u)'\|_{L^\infty(I)} \leq C C_u \Big(\frac{h}{h+\sigma}\Big)^{n+1} \qquad \forall n \in \mathbb{N}.$$

Proof. The proof proceeds in three steps.

Step 1: Let $\bar{u} \in \mathbb{R}$ be the average of u, i.e., $\bar{u} = \int_I u dx$. By the mean value theorem, there is $\xi \in I$ with $\bar{u} = u(\xi)$. Thus, $\tilde{u}(x) := u(x) - \bar{u}$ satisfies

$$\|D^m \tilde{u}\|_{L^\infty(I)} \leq \max\{1, 2\gamma\} C_u (\gamma h)^m m! \qquad m \in \mathbb{N}_0.$$

These bounds on the derivative of $\tilde{u}(x)$ imply the existence of σ, $C > 0$ depending only on γ such that $\tilde{u}(z)$ is holomorphic on $\mathcal{E}_\rho$ with $\rho \geq 1 + \sigma/h$; additionally, it satisfies on $\mathcal{E}_\rho$

$$\|\tilde{u}\|_{L^\infty(\mathcal{E}_\rho)} \leq C C_u.$$

Step 2: From theorem 3.4 and (3.9), we get the existence of C such that (after appropriately adjusting σ)

$$\tilde{u}(x) = \sum_{i=0}^{\infty} u_i \tilde{L}_i(x) \qquad \text{uniformly on } I,$$

$$|u_i| \leq C C_u (1 + \sigma/h)^{-i} \qquad \forall i \in \mathbb{N}_0,$$

where $\tilde{L}_i(x) := L_i(2x - 1)$. Now we define $u_n(x) := \sum_{i=0}^{n} u_i \tilde{L}_i(x) + \bar{u}$ $(\in \mathcal{P}_n(I))$. Markov's inequality (3.12) yields

$$\begin{aligned}\|(u - u_n)'\|_{L^\infty(I)} &\leq \|(\tilde{u} - \sum_{i=0}^{n} u_i \tilde{L}_i)'\|_{L^\infty(I)} \leq \sum_{i=n+1}^{\infty} |u_i| \|\tilde{L}_i'\|_{L^\infty(I)} \\ &\leq C C_u \sum_{i=n+1}^{\infty} i^2 (1 + \sigma/h)^{-i} \leq C C_u \Big(\frac{h}{h+\sigma'}\Big)^{n+1}\end{aligned}$$

for some $\sigma' < \sigma$ and $C > 0$. An analogous result holds for $\|u - u_n\|_{L^\infty(I)}$. Thus, we have proved

$$\|u - u_n\|_{L^\infty(I)} + \|(u - u_n)'\|_{L^\infty(I)} \leq CC_u \Big(\frac{h}{h+\sigma}\Big)^{n+1}. \tag{3.13}$$

Step 3: We employ inequality (3.11) in order to obtain bounds for $u - i_n u$:

$$\begin{aligned}
\|u - i_n u\|_{L^\infty(I)} &\leq \|u - u_n\|_{L^\infty(I)} + \|u_n - i_n u\|_{L^\infty(I)} \\
&\leq \|u - u_n\|_{L^\infty(I)} + \|i_n(u_n - u)\|_{L^\infty(I)} \\
&\leq C(1 + \ln n)\|u - u_n\|_{L^\infty(I)}, \\
\|(u - i_n u)'\|_{L^\infty(I)} &\leq \|(u - u_n)'\|_{L^\infty(I)} + \|(u_n - i_n u)'\|_{L^\infty(I)} \\
&\leq \|(u - u_n)'\|_{L^\infty(I)} + 2n^2\|u_n - i_n u\|_{L^\infty(I)} \\
&\leq \|(u - u_n)'\|_{L^\infty(I)} + Cn^2(1 + \ln n)\|u - u_n\|_{L^\infty(I)}.
\end{aligned}$$

Inserting (3.13) gives the desired bounds on $u - i_n u$ after appropriately adjusting the constant σ. □

Slightly modifying the preconditions and conclusion, we derive a more convenient version of the lemma:

Lemma 3.4.2. *Let u be analytic on the interval $I = [0, \pi]$ and satisfy for some C_u, $r > 0$*

$$\|D^m u\|_{L^\infty(I)} \leq C_u r^m m! \qquad \forall m \in \mathbb{N}.$$

Then there are C, $\tau > 0$ depending only on r such that the Gauss-Lobatto interpolant $i_n u$ satisfies

$$\|u - i_n u\|_{L^\infty(I)} + \|(u - i_n u)'\|_{L^\infty(I)} \leq CC_u e^{-\tau n} \qquad \forall n \in \mathbb{N}.$$

A combination of this lemma and lemma 2.0.5 will produce a very concise state of the result above:

Lemma 3.4.3. *Assume u is analytic on the interval $I = [0, \pi]$. Then there are c, $\tau > 0$ depending only on u such that the Gauss-Lobatto interpolant $i_n u$ satisfies*

$$\|u - i_n u\|_{L^\infty(I)} + \|(u - i_n u)'\|_{L^\infty(I)} \leq ce^{-\tau n} \qquad \forall n \in \mathbb{N}.$$

So far it has been ready to show our theorem of exponential convergence with p-version FEM. We denote V_n as the space composed of continuous piecewise polynomials that of degree from 0 to n and yield to (2.1).

Theorem 3.5 (Exponential approximations by p-finite element method)**.** *If σ satisfies (2.4),(2.14) and (2.17) with $0 \leq \beta \leq \alpha \in (0, 2)$, then for $f(x) = \sum_{k=1}^{\infty} f_k^o \sin(x)$ analytic on $[0, \pi]$ and admiting an analytic continuation to the strip $|Imz| < \tau_0$ in the complex plane $\mathbb{C}$, we have*

$$\|u - u_n\|_{\beta/2} \leq cC_1^\delta(\alpha)C_2^\delta(\beta)^{-1}e^{-\tau n},$$

where $u_n \in V_n$; c, $\tau > 0$ are some constants independent of n and δ.

Proof. As $0 \leq \beta \leq \alpha \in (0, 2)$, lemma 2.0.2 with interpolation theory of Sobolev space demonstrate the following relations of spaces:

$$V_n \subset H_o^1 \subset H_o^{\alpha/2} \subset M_\sigma^o \subset H_o^{\beta/2} \subset L^2.$$

Thus V_n is a finite-dimensional subspace of M_σ^o.

Since $\big(V_n,\ (\cdot, \cdot)_{M_\sigma^o}\big)$ is a Hilbert space, the following variational problem:
find $u_n \in V_n$ such that $E(u_n) = \min\limits_{v \in V_n} E(v)$, where

$$E(v) = \frac{1}{2}(v, v)_{M_\sigma^o} - (f, v),$$

has a unique solution $u_n \in V_n$ which is also the *best approximation* of the solution u (2.20) in V_n by M_σ^o's norm, i.e.,

$$\|u - u_n\|_{M_\sigma^o} = \min_{v \in V_n} \|u - v\|_{M_\sigma^o}.$$

Thus, with (2.16) and (2.19) we have

$$\begin{aligned}\|u - u_n\|_{\beta/2} &\le C_2^\delta(\beta)^{-1}\|u - u_n\|_{M_\sigma^o} = C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \|u - v_n\|_{M_\sigma^o} \\ &\le C_1^\delta(\alpha) C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \|u - v_n\|_{\alpha/2}. \end{aligned} \tag{3.14}$$

On the other hand, theorem 2.1 indicates the solution u is analytic on $[0, \pi]$. By lemma 3.4.3, there exists a polynomial p_n of degree n such that

$$\|u - p_n\|_{L^\infty(I)} + \|(u - p_n)'\|_{L^\infty(I)} \le ce^{-\tau n},$$

where $c,\ \tau > 0$ are independent of n. It follows that[1]

$$\begin{aligned}\min_{v_n \in V_n} \|u - v_n\|_{\alpha/2} &\le \|u - p_n\|_{\alpha/2} \\ &\le c\|u - p_n\|_1 \le c\big(\|u - p_n\|_{L^\infty(I)} + \|(u - p_n)'\|_{L^\infty(I)}\big) \\ &\le ce^{-\tau n},\end{aligned}$$

where c is some constant independent of n and δ.

After incorporating the derivation into (3.14), we obtain the ultimate result. □

It can be seen that our theorems of exponential approximations are based upon assorted conditions. In the next chapter, we will provide several examples to verify these conclusions are practically ture.

[1]Note that p_n is the $i_n u$ and the endpoints $0, \pi$ are sampling points to the interpolation, thus $p_n(0) = u(0) = 0$ and $p_n(\pi) = u(\pi) = 0$. Extending p_n oddly at left up to $-\delta$ and right up to $\pi + \delta$ gains $p_n^* \in V_n$ satisfying $p_n^*\big|_{[0,\pi]} = p_n$.

Chapter 4

Numerical experiments

In both Theorems 3.2 and 3.5, data functions f are supposed to be analytic on the interval $[0, \pi]$ and provided with the form of Fourier sine series, i.e.,

$$f(x) = \sum_{k=1}^{\infty} f_k^o \sin(kx), \qquad x \in [0, \pi].$$

In practice, for convenience we instead seek a function in the same fashion that is analytic and periodic on the whole real line[1]. Such a function is selected in light of the following theorem.

Theorem 4.1. *Suppose $f(z)$ is defined on the line $Imz = 0$ (x-axis) in the complex plane.*

(i) Assume f(z) is analytic in a strip S: $-\infty \leq \tau_1 < Imz < \tau_2 \leq \infty$ and have period 2π. Then

$$f(z) = \sum_{k=-\infty}^{\infty} f_k e^{ikz}, \tag{4.1}$$

where

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z) e^{-ikz} dz. \tag{4.2}$$

The series (4.1) converges uniformly and absolutely in every substrip $S' : \tau_1 < \tau_1' \leq Imz \leq \tau_2' < \tau_2$.

Conversely,

[1] Such a function must be analytic on both endpoints of the interval.

(*ii*) *Assume* $f(z)$ *satisfies (4.1), where*

$$\limsup_{k\to\infty} \frac{\ln|f_k|}{k} = A, \tag{4.3}$$

$$\liminf_{k\to\infty} \frac{\ln|f_{-k}|}{k} = B \tag{4.4}$$

and

$$\infty \leq A < B \leq \infty.$$

Then $f(z)$ *is analytic and periodic in the strip* $A < Imz < B$. *This is also the maximum strip of analyticity of* $f(z)$.

Proof. (i) Make the change of variable:

$$\zeta = e^{iz}, \tag{4.5}$$

then $z = \arg\zeta - i\ln|\zeta|$. (4.5) maps the strip S into the annulus

$$A : e^{-\tau_2} < |\zeta| < e^{-\tau_1}$$

in the $\zeta-$plane.

In view of the analyticity and periodicity of $f(z)$, the function $g(\zeta) = f(\arg\zeta - i\ln|\zeta|)$ will be single valued and analytic in the annulus. It therefore has a Laurent expansion

$$g(\zeta) = \sum_{k=-\infty}^{\infty} g_k\zeta^k \tag{4.6}$$

with

$$g_k = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(\zeta)}{\zeta^{k+1}}d\zeta \qquad k = 0, \pm1, \pm2, \cdots$$

where $e^{-\tau_2} < r < e^{-\tau_1}$. The series (4.6) converges uniformly and absolutely in any sub-annulus.

Passing back to the variable ζ,

$$f(z) = g(e^{iz}) = \sum_{k=-\infty}^{\infty} g_k e^{ikz}.$$

The integrals for g_k become

$$g_k = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z)}{e^{i(k+1)z}} \cdot ie^{iz} dz.$$

This is identical to (4.2).

As for (ii), consider the series $\sum_{k=0}^{\infty} f_k \zeta^k + \sum_{k=1}^{\infty} f_{-k} \zeta^{-k} \equiv g_1(\zeta) + g_2(\zeta)$. The radius of convergence r_1 of g_1 is given by

$$r_1 = \limsup_{k \to \infty} |g_k|^{-1/k} = e^{-A}.$$

The radius of convergence r_2 of g_2 is

$$r_2 = \liminf_{k \to \infty} |g_{-k}|^{-1/k} = e^{-B}.$$

Since $A < B$, $g_1 + g_2$ is analytic in the annulus $e^{-B} < |\zeta| < e^{-A}$ and can not be continued analytically into any larger annulus. Applying (4.5), $f(z)$ is analytic in the strip

$$A < \mathrm{Im} z < B,$$

but in no larger strip. □

According to (4.3) and (4.4), it is not difficult to see that, for any $r > 1$, the real-valued function

$$f(x) = \sum_{k=1}^{\infty} r^{-k} \sin(kx)$$

is periodic and analytic in $\mathbb{R}$.

4.1 Exponential convergence via Fourier spectral method

Let $\sigma(|y|) = y^4$ and $c_\delta = 1/\delta^2$, then

$$\frac{|y|^2}{\sigma(|y|)} = \frac{1}{|y|^2} > \frac{1}{\delta^2}, \qquad \forall y \in (-\delta, \delta)$$

and

$$\tau_\delta = c_\delta \int_{-\delta}^{\delta} \frac{|y|^4}{\sigma(|y|)} dy = \frac{1}{\delta^2} \int_{-\delta}^{\delta} 1 dy = \frac{2}{\delta} < \infty,$$

i.e., the kernel function σ satisfies the condition (2.4).

Next, take $\beta = 0$, $\gamma_2 = 1$; also assume δ is small enough (at least less than 1). Then

$$\sigma(|y|) \leq |y|^3, \qquad \forall |y| \leq \delta.$$

Thus σ satisfies condition (2.17) as well.

Choose the data function

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} \sin(kx). \tag{4.7}$$

To sum up, the nonlocal BVP (2.11) is

$$\begin{cases} -\dfrac{1}{\delta^2} \displaystyle\int_{x-\delta}^{x+\delta} \frac{u(x') - u(x)}{(x' - x)^2} dx' = f(x), & x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

with its unique solution expressed as

$$u(x) = \sum_{k=1}^{\infty} \frac{2^{-k}}{\eta_\delta(k)} \sin(kx), \tag{4.8}$$

where

$$
\begin{aligned}
\eta_\delta(k) &= c_\delta \int_{-\delta}^{\delta} \big(1-\cos(ky)\big)\frac{y^2}{\sigma(|y|)}dy \\
&= \frac{1}{\delta^2}\int_{-\delta}^{\delta} \frac{1-\cos(ky)}{y^2}dy \\
&= \frac{1}{\delta^2}\int_{-\delta}^{\delta} \frac{2\sin^2(\frac{ky}{2})}{y^2}dy \\
&= \frac{4}{\delta^2}\int_{0}^{\delta} \frac{\sin^2(\frac{ky}{2})}{y^2}dy \\
&= \frac{2k}{\delta^2}\int_{0}^{\frac{k\delta}{2}} \frac{\sin^2(\theta)}{\theta^2}d\theta \\
&= \frac{2k}{\delta^2}\Big[-\frac{2\sin^2(k\delta/2)}{k\delta} + \int_0^{k\delta}\frac{\sin(\theta)}{\theta}d\theta\Big].
\end{aligned}
$$

Applying Sobolev norm $\|\cdot\|_1$ yields error estimates of Fourier spectral method, i.e.,

$$
||u(x)-u_n(x)||_1 = \sum_{k=n+1}^{\infty}\left|\frac{f_k^o}{\eta_\delta(k)}\right|^2 k^2 = \frac{\delta^4}{4}\sum_{k=n+1}^{\infty}\frac{2^{-2k}}{\Big[-\frac{2\sin^2(k\delta/2)}{k\delta}+\int_0^{k\delta}\frac{\sin(\theta)}{\theta}d\theta\Big]^2}.
$$

Take $n=1$ to 10. For comparison, we consider δ in different values: 0.5, 0.05 and 0.005, respectively.

The numberical results are presented in Fig. 4.1. From the one on the top, we can see that the value of error drops sharply as n becomes larger. The decay is so steep at first that the order of magnitude of data changes quickly. To facilitate the observation, we apply instead the semi-log graph where the y-axis is plotted on a logarithmic (to base 10) scale and x-axis maintains the linear scale. On such a semi-log graph the y value is supposed to be the logarithm of the number but represented by the number itself.

As we expected, the error shown in Fig. 4.1 on the bottom is decreasing proportionally to n. Moreover, it can be seen that the orders of convergence (i.e., τ) stay constant through varied δs.

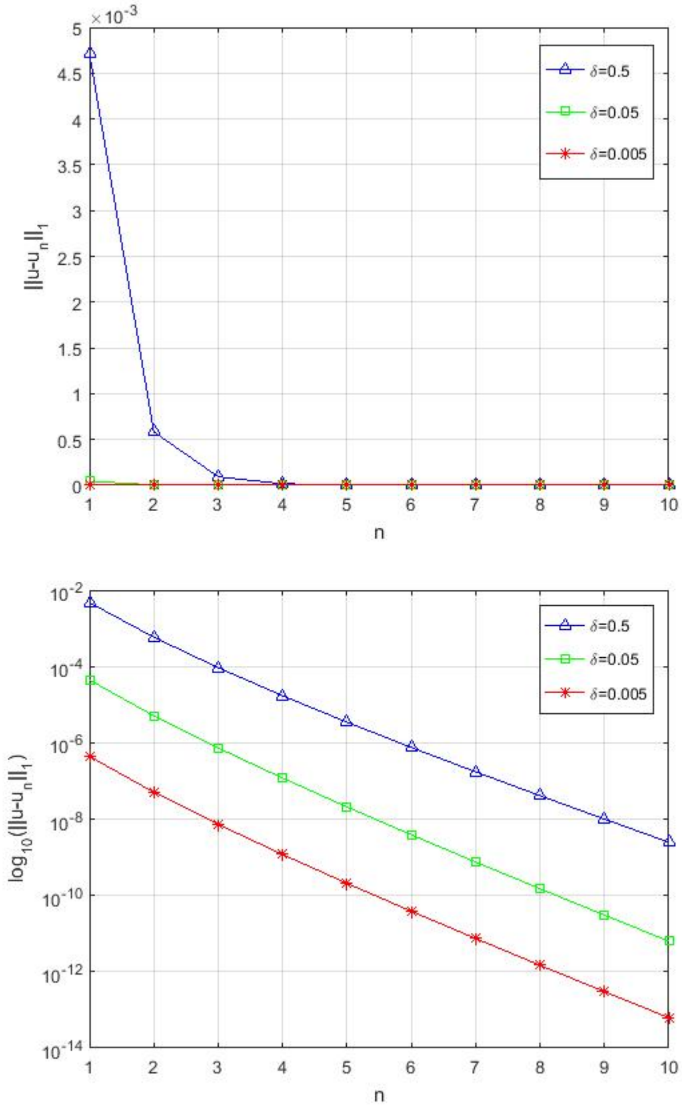

Figure 4.1: Exponential convergence via Fourier spectral method.

4.2 Exponential convergence via p-version finite element method

Retain $\sigma(|y|) = y^4$ and $c_\delta = 1/\delta^2$; $\beta = 0$, $\gamma_2 = 1$. Also assume δ is a small number less than 1. Thus the kernel function σ satisfies both conditions (2.4) and (2.17).

Furthermore, take $\alpha = 1$ and $\gamma_1 = 1$ so that σ satisfies condition (2.14).

We are still using the data function given in (4.7), then the solution (4.8) will be written as

$$u(x) = \frac{\delta^2}{2} \sum_{k=1}^{\infty} \frac{2^{-k}}{k\big(-\frac{2\sin^2(k\delta/2)}{k\delta} + \int_0^{k\delta} \frac{\sin(\theta)}{\theta} d\theta\big)} \sin(kx). \tag{4.9}$$

Recall in the proof of theorem 3.5, the finite element solution $u_n \in V_n$ was such that

$$\|u - u_n\|_{\beta/2} \le C_1^\delta(\alpha) C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \|u - v_n\|_{\alpha/2};$$

on the other hand,

$$\begin{aligned} \min_{v_n \in V_n} \|u - v_n\|_{\alpha/2} &\le \|u - p_n\|_{\alpha/2} \\ &\le c\big(\|u - p_n\|_{L^\infty(I)} + \|(u - p_n)'\|_{L^\infty(I)}\big) \\ &\le c e^{-\tau n} \end{aligned}$$

Thus, to sum up, we can see that

$$\begin{aligned} \|u - u_n\|_{\beta/2} &\le c\big(\|u - p_n\|_{L^\infty(I)} + \|(u - p_n)'\|_{L^\infty(I)}\big) \\ &\le c e^{-\tau n}. \end{aligned}$$

Note that $p_n = i_n u$, i.e., the Gauss-Lobatto interpolation of solution u, and $\beta = 0$. Hence, in fact we derive

$$\begin{aligned}\|u - u_n\|_0 &\leq c\big(\|u - i_n u\|_{L^\infty(I)} + \|(u - i_n u)'\|_{L^\infty(I)}\big)\\ &\leq ce^{-\tau n}.\end{aligned}$$

That is to say, for verifing the exponential convergence of the error estimate, it suffices to numerically demonstrate

$$\|u - i_n u\|_{L^\infty(I)} + \|(u - i_n u)'\|_{L^\infty(I)} \leq ce^{-\tau n}.$$

The following table lists $n+1$ Gauss-Lobatto points in the interval $[-1, 1]$:

Number of points $(n+1)$	Gauss-Lobatto Points
3	$0, \quad \pm 1$
4	$\pm\sqrt{\frac{1}{5}}, \quad \pm 1$
5	$0, \quad \pm\sqrt{\frac{3}{7}}, \quad \pm 1$
6	$\pm\sqrt{\frac{1}{3} - \frac{2\sqrt{7}}{21}}, \quad \pm\sqrt{\frac{1}{3} + \frac{2\sqrt{7}}{21}}, \quad \pm 1$
7	$0, \quad \pm\sqrt{\frac{5}{11} - \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \pm\sqrt{\frac{5}{11} + \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \pm 1$

Table 4.1: Gauss-Lobatto points in the interval $[-1, 1]$.

Applying transformation formula: $y = \frac{\pi}{2}(x+1)$, we will obtain the corresponding $n+1$ Gauss-Lobatto points in the interval $[0, \pi]$.

Denote the set of Gauss-Lobatto points in $[0, \pi]$ by $\{x_i | i = 0, 1, \ldots, n\}$. Then the Lagrange interpolation polynomial of the solution (4.9) is expressed as

$$(i_n u)(x) := \sum_{i=0}^{n} u(x_i) l_i^{(n)}(x), \tag{4.10}$$

where the polynomials $l_i^{(n)}(x)$ (of degree n) are defined as

$$l_i^{(n)}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Moreover, the derivative of (4.10) is computed as

$$\frac{d}{dx}\big[(i_n u)(x)\big] = \sum_{i=0}^{n} u(x_i)\frac{d}{dx}\big[l_i^{(n)}(x)\big], \tag{4.11}$$

where

$$\begin{aligned}
\frac{d}{dx}\big[l_i^{(n)}(x)\big] &= \frac{d}{dx}\Bigg[\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}\Bigg] \\
&= \frac{1}{x_i - x_0}\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} + \frac{1}{x_i - x_1}\prod_{\substack{j=0 \\ j\neq 1,i}}^{n} \frac{x - x_j}{x_i - x_j} + \cdots \\
&+ \frac{1}{x_i - x_{i-1}}\prod_{\substack{j=0 \\ j\neq i-1,i}}^{n} \frac{x - x_j}{x_i - x_j} + \frac{1}{x_i - x_{i+1}}\prod_{\substack{j=0 \\ j\neq i,i+1}}^{n} \frac{x - x_j}{x_i - x_j} + \cdots \\
&+ \frac{1}{x_i - x_n}\prod_{\substack{j=0 \\ j\neq i,n}}^{n} \frac{x - x_j}{x_i - x_j} \\
&= \sum_{\substack{k=0 \\ k\neq i}}^{n}\left(\frac{1}{x_i - x_k}\prod_{\substack{j=0 \\ j\neq i,k}}^{n} \frac{x - x_j}{x_i - x_j}\right) \\
&= \sum_{\substack{k=0 \\ k\neq i}}^{n}\left(\frac{1}{x - x_k}\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}\right) \\
&= \sum_{\substack{k=0 \\ k\neq i}}^{n} \frac{1}{x - x_k}\; l_i^{(n)}(x) \\
&= l_i^{(n)}(x)\left(\sum_{\substack{k=0 \\ k\neq i}}^{n} \frac{1}{x - x_k}\right).
\end{aligned}$$

Substituting into (4.11), we finally derive

$$\frac{d}{dx}\left[(i_n u)(x)\right] = \sum_{i=0}^{n}\left(u(x_i)l_i^{(n)}(x)\sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x-x_k}\right).$$

For convenience we let

$$S(n) = \max_{x\in[0,\pi]}|u(x) - i_n u(x)| + \max_{x\in[0,\pi]}|u'(x) - (i_n u)'(x)|.$$

As before, we set $\delta = 0.5,\ 0.05$ and 0.005 respectively. Note that the number "n" corresponds to the case of "$n+1$" Gauss-Lobatto points.

Observing Fig. 4.2 (top), we find the value of $S(n)$ falls quickly in the beginning, but then slowly down all the while. Furthermore, the semi-log graph in the Fig. 4.2 (bottom) demonstrates that the orders of convergence are identical for different δs.

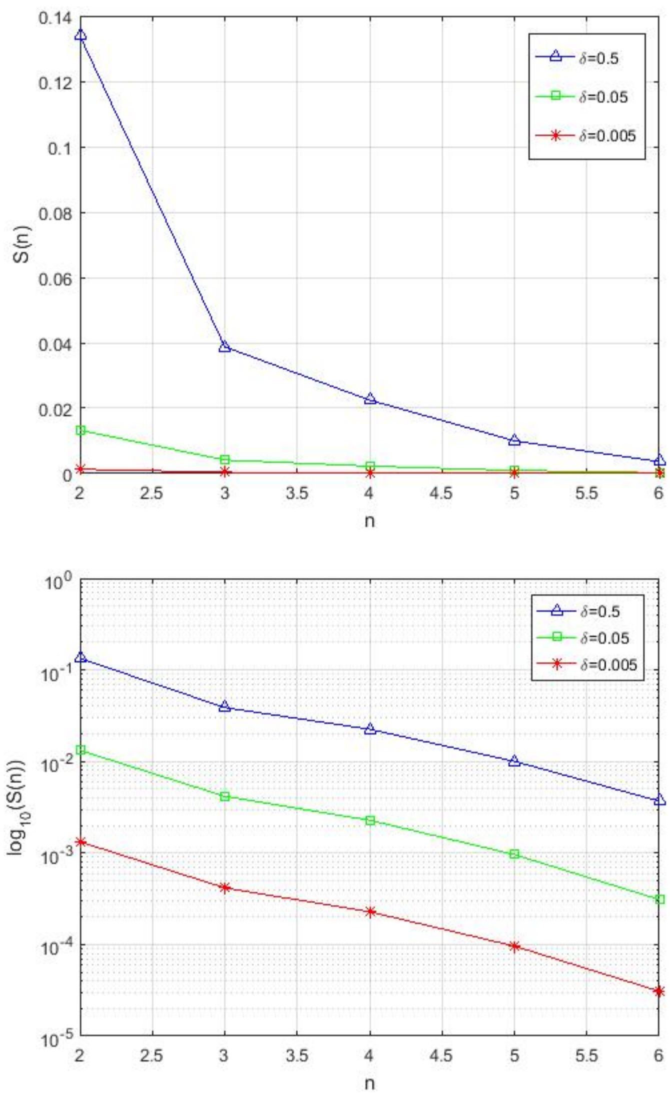

Figure 4.2: Exponential convergence via p-version finite element method.

Chapter 5

Conclusion

We demonstrated through both theoretical analysis and numerical experiments the exponential convergence (EC) of Fourier spectral and $p-$finite element approximations to the solution of linear bond-based peridynamic BVP, provided that data function was analytic.

Some different techniques can also be considered to acquire EC. The hp-version finite element method [39], equipped with proper combination of mesh refinement and increasing polynomial degree, is shown to be superior over $h-$and $p-$finite element methods. In addition, the recently developed Fourier continuation (or termed Fourier extension [11, 31]) techniques have demonstrated highly accurate approximations for non-periodic functions. The analysis [36] indicates that EC can be obtained with this method on evenly spaced points until the parameter dependent accuracy threshold is reached. The applications of all these methods to linear peridynamic BVPs remain to be investigated in the future.

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