A Peano Continuum which is homogeneous but not bihomogeneous

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Kevin Brian Gammon, son of William and Brenda Gammon, was born on August 16, 1982. He attended Gordon Lee High School in Chickamauga, Georgia where he graduated seventh in his class in May, 2000. He entered Berry College under an academic scholarship and graduated Magna cum Laude in May, 2004. He was awarded with a Bachelor of Science degree in Mathematics. He then entered Auburn University in August, 2004 to pursue a Master of Science degree in Mathematics.

Thesis Abstract

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The author describes in detail the construction of a Peano continuum which is homogeneous and non-bihomogeneous. This continuum was originally constructed by G. Kuperberg in the paper Another homogeneous, non-bihomogeneous Peano continuum [5].

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Chapter 1

Introduction

The purpose of this thesis is to describe in detail the construction of a Peano continuum which is homogeneous but not bihomogeneous. The example was originally constructed by G. Kuperberg [5]. The question whether every homogeneous space is bihomogeneous was originally raised by B. Knaster approximately around 1921. The question was restated to continua in 1930 by D. van Dantzig. K. Kuperberg constructed a locally connected example in [6]. G. Kuperberg constructed the example considered in this thesis in order to make an example of a homogeneous, nonbihomogeneous Peano continuum which is both simpler and of lower dimension than that described by K. Kuperberg in [6]. The example constructed by G. Kuperberg uses the notion developed in [8] that certain Cartesian products with the Menger manifolds as one of the factors has a certain rigidity which must be preserved by homeomorphisms. Several of these results depend on the characterization of the Menger Curve developed by R.D. Anderson and k-dimensional Menger compacta developed by M. Bestvina in [1] and [2], respectively. Another example was given by Minc in [10] of a homogeneous, non-bihomogeneous continuum. However, this example is not locally connected.

In order to describe the continuum constructed by G. Kuperberg, several background theorems regarding covering spaces are required. It is the intention of the writer to prove these theorems or collect this information from several different sources and write it with a unified notation which can easily be followed by the reader. While many elementary definitions and theorems are presented, a basic knowledge of topological spaces and the fundamental group is assumed.

Chapter 2 provides the background information needed to understand the example. This is where the reader may find the common definitions and theorems that will be used throughout the other chapters. The majority of theorems and definitions can be found in James Munkres' Topology book [12]. Theorems from other sources are modified to fit the notation of this thesis. The proofs given in the text are primarily from course work done during the second semester of an Introduction to Algebraic Topology course and a Research and Thesis course taught at Auburn University under the direction of Krystyna Kuperberg.

Chapter 3 introduces the semidirect product of $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$. Several facts about this group which are essential to the example presented in this thesis are described in this chapter. A normal subgroup $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ used in the creation of the homogeneous, non-bihomogeneous Peano continuum is identified and some properties about the automorphisms of $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ are examined. The chapter also includes a group table of $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ for reference purposes.

Chapter 4 is devoted to the construction of the example of a Peano continuum which is homogeneous and non-bihomogeneous. Two constructions of the space are given in detail. The first construction of the space considered relies heavily on the concepts of covering spaces and fundamental groups developed in Chapter 2 along with basic knowledge of group theory and fiber bundles. The second, equivalent construction uses a group of homeomorphism to induce a quotient space with the desired properties.

In Chapter 5 some of the properties of the continuum constructed in Chapter 4 are examined. This chapter will also include the proofs required to see that the continuum is homogeneous and not bihomogeneous. These proofs also rely heavily on the basic theorems presented in Chapter 2 regarding covering spaces and lifts of maps.

Chapter 2

Preliminary Information and Theorems

The following definitions and theorems are essential background information in the construction of the desired continuum. Throughout this section X, E, and B will be topological spaces. The symbol $*$ will denote the group operation associated with fundamental groups. Given a group G and an element $x \in G$, the symbol $\langle x \rangle$ will denote the subgroup of G generated by x.

Definition 2.1. *If G is a group then the collection of automorphisms of G along with the operation of composition forms a group. This group is called the* ***group of automorphisms of*** *G and will be denoted $Aut(G)$ [4].*

Definition 2.2. *Let G be a group and H a subgroup of G. The* ***normalizer*** *of H in G, denoted $N(H)$, is defined as $N(H) = \{x \in G \mid xH = Hx\}$ [4].*

Note that if G is an abelian group, $N(H) = G$.

Definition 2.3. *Given groups H and G and a homomorphism $\theta : H \to Aut(G)$, the* ***semidirect product*** *denoted $G \rtimes H$ is the group whose elements are the elements of $G \times H$ along with the group operation given by $(a, b)(c, d) = (a(\theta(b)(c)), bd)$ [4].*

The following three definitions may be found in [12].

Definition 2.4. *A* ***path*** *between points x and y in a space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. If f is a path, denote the reverse of f by $\bar{f}$. In particular, $\bar{f}(t) = f(1 - t)$.*

Definition 2.5. *A space is **path connected** if any two points can be connected by a path.*

Definition 2.6. *A space is **locally path connected** if it has a basis consisting of path connected sets.*

Definition 2.7. *A space is **locally connected** if it has a basis consisting of connected sets.*

The next three definitions may be found in G. Kuperberg's paper [5].

Definition 2.8. *A space is **homogeneous** if given any two points x and y in X there is a homeomorphism taking x to y.*

Definition 2.9. *A space is **bihomogeneous** if given any two points x and y there is a homeomorphism exchanging x and y.*

Definition 2.10. *A space X is **strongly locally homogeneous** if for every x in X and for every open neighborhood U of x, there exists an open set V such that $x \in V \subset U$ and for every $y \in V$ there is a homeomorphism h such that $h(x) = y$ which is the identity outside of U.*

The following two definitions may be found in [13].

Definition 2.11. *A **continuum** is a compact, connected metric space.*

Definition 2.12. *A **Peano continuum** is a locally connected continuum.*

Definition 2.13. *Let X be a space with a given triangulation. Given a vertex x of the triangulation, let C be the collection of those simplices which contain x as a vertex. Then the* ***link of*** *x is the union of the simplices contained in some simplex of C which do not contain x.*

Definition 2.14. *A* ***n-dimensional Piecewise-Linear manifold*** *is a manifold that admits a triangulation such that the link of every vertex is piecewise-linear homeomorphic to a n-sphere. A piecewise-linear manifold will be referred to as a PL-manifold.*

The following construction of a Menger manifold is given in [2].

Definition 2.15. *Given n, let K be a PL-manifold of dimension $2n+1$. Let $X_1 = K$. For $i > 1$ define X_i to be a regular neighborhood of the n-skeleton of a triangulation of X_{i-1}. Then $\mu^n_K = \cap_i X_i$ is called an* ***n-dimensional Menger manifold****.*

The remainder of the definitions and theorems, unless otherwise stated, may be found in [12].

Theorem 2.1. *A space X is locally connected if for every open set U of X, each component of U is open in X.*

Proof. Let U be an open set of X and let C_U denote the collection of components of U. Let $C = \cup_U C_U$ for U open in X. It will now be shown that C is a basis for X. Let V be an open set of X and let $x \in V$. Let C_x denote the component of V containing x. Then $C_x \in C$ and C_x is open. Moreover, $x \in C_x \subset V$. Hence C is a basis for the topology on X. $\square$

Definition 2.16. *Let f and h be maps from $X \to Y$. A **homotopy** $F : X \times I \to Y$ is a continuous map such that $F(x,0) = f(x)$ and $F(x,1) = h(x)$ for all $x \in X$. If f and h are paths from x_0 to x_1, F is called a **path homotopy** if $X = I$ and for $t \in I$ it follows that $F(0,t) = x_0$ and $F(1,t) = x_1$.*

While other equivalent definitions of fiber bundle exists, the following definition found in [3] is sufficient for the purposes of this paper.

Definition 2.17. *A **fiber bundle** over a space B with fiber F is a space E along with a map $p : E \to B$ such that for every $b \in B$, there is a neighborhood U of b and homeomorphism $h : p^{-1}(U) \to U \times F$ such that $p(a) = \pi \circ h(a)$ where π is the projection map onto the first coordinate and $a \in p^{-1}(U)$.*

The following definition may be found in [3].

Definition 2.18. *Given a fiber bundle $p : E \to B$ with fiber F and a continuous map $f : X \to B$, the **pullback bundle** induced by f, over X with fiber F is defined by $E' = \{(x,e) \in X \times E : f(x) = p(e)\}$.*

Definition 2.19. *A continuous surjective map $p : E \to B$ is said to be a **covering map** if given any $b \in B$ there exists a neighborhood U of b such that*

$$p^{-1}(U) = \bigcup_{\alpha \in A} U_\alpha$$

*where the $\{U_\alpha\}_{\alpha \in A}$ is a collection of pairwise disjoint open sets, often called **slices**, with the property that given $a \in A$ the restriction $p|U_a$ is a homeomorphism onto U.*

*In this case, E is called a **covering space** of B and the neighborhood U is said to be **evenly covered** by p.*

In the context above, a covering space E with map p is will sometimes be written as an ordered pair (E, p).

Definition 2.20. *Given a covering map $p : E \to B$ such that $p(e) = b$, the induced homomorphism $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is defined by $p_*([\alpha]) = [p \circ \alpha]$.*

Theorem 2.2. *Given a covering map $p : E \to B$ such that $p(e) = b$, the function p_* is a well defined homomorphism.*

Proof. Let $[\alpha] \in \pi_1(E, e)$. Then $p \circ \alpha$ is a loop based at b; therefore the map is well defined. Let β be another element of $\pi_1(E, e)$. Then $p_*([\alpha]) * p_*([\beta]) = [p \circ \alpha] * [p \circ \beta] = [(p \circ \alpha) * (p \circ \beta)] = [p \circ (\alpha * \beta)] = p_*([\alpha * \beta])$. Therefore the map is a homomorphism. $\square$

Definition 2.21. *Let $p : E \to B$ be a covering map. Then an **automorphism** of E with respect to p is a homeomorphism $\psi : E \to E$ such that $p \circ \psi = p$.*

Definition 2.22. *Given a covering space (E, p) of a space B, the collection of automorphisms of E along with the operation of composition forms a group. This group will called the **group of automorphisms** of E and will be denoted $A(E, p)$.*

While the above term is similar to Definition 2.1, the context in which the definition is used and the notation will avoid ambiguity.

Theorem 2.3. *Assume that E, B and Y are locally path connected and path connected spaces. Let $p : E \to B$ be a covering map such that $p(e) = b$. Let $f : Y \to B$ be a* *continuous map such that $f(y) = b$. Then f can be lifted to a map $\tilde{f} : Y \to E$ such that $f(y) = e$ if and only if $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e))$.*

Definition 2.23. *A covering space E of B with map $p : E \to B$ such that $p(e) = b$ is said to be a **regular covering space** if $p_*(\pi_1(E, e))$ is a normal subgroup of $\pi_1(B, b)$.*

Definition 2.24. *A covering space (E, p) of B is a **n-fold covering** if $p^{-1}(x)$ has exactly n points for each $x \in B$.*

Definition 2.25. *A space X is **semi-locally simply connected** if for every $x \in X$ there exists an open set such that $x \in U$ and the inclusion map $i : \pi_1(U, x) \to \pi_1(X, x)$ is trivial.*

Theorem 2.4. *Let B be a path connected, locally path connected, and semi-locally simply connected. Let $b \in B$. Given a subgroup H of $\pi_1(B, b)$, there exists a covering map $p : E \to B$ and a point $e \in p^{-1}(b)$ such that $p_*(\pi_1(E, e)) = H$.*

Theorem 2.5. *Let $p : E \to B$ be a covering map with $p(e_0) = b$ and let $H = p_*(\pi_1(E, e_0))$. If E is path connected, the lifting correspondence ϕ induces a bijection $\Phi : \pi_1(B, b)/H \to p^{-1}(b)$, where $\pi_1(B, b)/H$ denotes the left cosets of H.*

Proof. For a given path h in $\pi_1(B, b)$, let $\widetilde{h}$ denote the lifting of h beginning at e_0. Recall that the lifting correspondence $\phi : \pi_1(B, b) \to p^{-1}(b)$ is defined as follows: Given an element $[f]$ of $\pi_1(B, b)$, let $\widetilde{f}$ be the lifting of f to a path in E beginning at e_0 and define $e_0 \cdot [f] = \widetilde{f}(1)$.

Claim 1. *The lifting correspondence is well defined.*

Proof. Suppose that f is homotopic to g, where f and g are loops based at b, and let H be the path homotopy between them. Then H lifts to a continuous map $\widetilde{H} : I \times I \to E$ such that $\widetilde{H}(0,0) = e_0$ and $\widetilde{H}(1,1) = e$ for some e. However, $p(\widetilde{H}(t,0)) = H(t,0) = f(t)$ so that $H(t,0)$ is a lifting of f. Likewise $p(\widetilde{H}(t,1)) = H(t,1) = g(t)$ is a lifting of g. From the uniqueness of liftings, this implies that $\widetilde{H}(t,0) = \widetilde{f}$ and $\widetilde{H}(t,1) = \widetilde{g}$; hence the liftings of f and g have the same endpoint, e, so that the lifting correspondence is well defined. $\square$

Now define $\Phi : \pi_1(B,b)/H \to p^{-1}(b)$ by $\Phi([g] * H) = \widetilde{g}(1)$. To see that Φ is well defined, suppose that for two loops based at b, say f and g, f is homotopic to g. Then from the above, it follows that $\Phi([f] * H) = \Phi([g] * H)$. Next, to check that Φ is injective, suppose that $\Phi([f'] * H) = \Phi([g'] * H)$. Then $\widetilde{f}'(1) = \widetilde{g}'(1)$, which implies that $\widetilde{f}' * (\bar{\widetilde{g}}')$ is a loop based at e_0. Hence $[\widetilde{f}' * (\bar{\widetilde{g}}')] \in \pi_1(E, e_0)$ so that $p_*([\widetilde{f}' * (\bar{\widetilde{g}}')]) = [p \circ (\widetilde{f}' * (\bar{\widetilde{g}}'))] = [f' * \bar{g}'] \in H$. Since $[f' * \bar{g}'] \in H$ it follows that $[f'] * [\bar{g}'] * H = [f' * \bar{g}'] * H = H$ which is true if and only if $[f'] * H = [g'] * H$; hence Φ is injective. In order to see that Φ is surjective, let $b_0 \in p^{-1}(b)$. Since E is path connected, there exists a path h from e_0 to b_0. Hence $p \circ h$ is a loop based at b and, by construction, $\Phi([p \circ h] * H) = b_0$. Therefore, Φ is a bijection. $\square$

The following theorem is stated in [9].

Theorem 2.6. *Let $p : E \to B$ be a covering map and let Y be a connected space. Given any two maps f and g from $Y \to E$ such that $p \circ f = p \circ g$, the set $\{y \in Y |\ f(y) = g(y)\}$ is either empty or all of Y.*

Proof. It will be shown that the set $\mathcal{C} = \{y \in Y | \, f(y) = g(y)\}$ is both closed and open. To see that $\mathcal{C}$ is open, let $y \in \mathcal{C}$. Let $U \subset B$ be an evenly covered neighborhood of $p(f(y)) = p(g(y))$ and let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be the partition of $p^{-1}(U)$ into slices. Let U_i be the slice containing $f(y) = g(y)$. Then $y \in f^{-1}(U_i)$ and $y \in g^{-1}(U_i)$ are open; hence $f^{-1}(U_i) \cap g^{-1}(U_i)$ is open and non-empty. Suppose that $x \in f^{-1}(U_i) \cap g^{-1}(U_i)$. Then $p(f(x)) = p(g(x))$ and $f(x) \in U_i$ and $g(x) \in U_i$. However, $p|U_i$ is a homeomorphism which implies that $f(x) = g(x)$. Hence $f^{-1}(U_i) \cap g^{-1}(U_i) \subset \mathcal{C}$ and thus $\mathcal{C}$ is open.

Next, let $x \in Y - \mathcal{C}$ and let U be an evenly covered neighborhood of $p(f(x)) = p(g(x))$. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be the partition of $p^{-1}(U)$ into slices. Let $U_{f(x)}$ and $U_{g(x)}$ be the slices containing $f(x)$ and $g(x)$ respectively. Then $f^{-1}(U_{f(x)}) \cap g^{-1}(U_{g(x)})$ is an open neighborhood of x. Moreover, given any $y \in f^{-1}(U_{f(x)}) \cap g^{-1}(U_{g(x)})$ it follows that $f(y) \in U_{f(x)}$ and $g(y) \in U_{g(x)}$. Since the two sets are disjoint, it follows that $f(y) \neq g(y)$. Hence $Y - \mathcal{C}$ is open and therefore $\mathcal{C}$ is a closed and open set. Thus $\mathcal{C} = \emptyset$ or $\mathcal{C} = Y$. □

Theorem 2.7. *Let $p : E \to B$ with $p(e) = b$ be a covering map with E path connected and let $A(E, p)$ denote the group of automorphisms of E with respect to p. Then the group $A(E, p)$ is isomorphic to $N(H)/H$ where $H = p_*(\pi_1(E, e))$ and $N(H)$ denotes the normalizer of H in $\pi_1(B, b)$.*

Proof. Define $\psi : A(E, p) \to p^{-1}(b)$ by $\psi(h) = h(e)$ for each $h \in A(E, p)$. From Theorem 2.6, ψ is injective since if h and $g \in A(E, p)$ such that $h(e) = g(e)$ it follows that $h = g$. Let $\Phi : \pi_1(B, b)/H \to p^{-1}(b)$ be the bijection as defined in

Theorem 2.5. Then $\Phi^{-1} \circ \psi$ is a map from $A(E, p)$ to $N(H)/h$ if given $h \in A(E, p)$ it follows that $\Phi^{-1} \circ \psi(h) = [\alpha] * H$ where $[\alpha] \in N(H)$. Since h is a homeomorphism, $\pi_1(E, e) = \pi_1(E, h(e))$. Let $p_*(\pi_1(E, h(e))) = H_1$. Then $H = H_1$ and it is therefore sufficient to show that $[\alpha] * H_1 * [\alpha]^{-1} = H$. Let $\widetilde{\alpha}$ be the lift of α beginning at e and $[\widetilde{\beta}] \in H_1$ with $p_*([\beta]) = [\widetilde{\beta}]$ for $\beta \in \pi_1(E, h(e))$. Then $\widetilde{\alpha} * \beta * \bar{\widetilde{\alpha}}$ is a loop based at e so that $p_*([\widetilde{\alpha} * \beta * \bar{\widetilde{\alpha}}]) \in H$. Hence $[\alpha] * H_1 * [\alpha]^{-1} \subset H$. A similar argument shows that $[\alpha] * H * [\alpha]^{-1} \subset H_1$. Since $H = H_1$ it follows that $[\alpha] \in N(H)$.

To see that $\Phi^{-1} \circ \psi$ is a homomorphism, let $f, g \in A(E, p)$ with $f(e) = e_1$ and $g(e) = e_2$. Since E is path connected, there is a path α from e to e_1 and a path β from e to e_2. By definition, $\Phi([\alpha] * H) = e_1$ and $\Phi([\beta] * H) = e_2$. Hence $\Phi^{-1} \circ \psi(f) = [\alpha] * H$ and $\Phi^{-1} \circ \psi(g) = [\beta] * H$. It remains to be shown that $\Phi^{-1} \circ \psi(f \circ g) = [\alpha] * [\beta] * H = [\alpha * \beta] * H$.

Let $f \circ g(e) = e_3$. Then the above holds if $\Phi([\alpha * \beta] * H) = e_3$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be lifts of α and β respectively beginning at e. Then $\widetilde{\beta}$ is a path from e to e_2. Then $f \circ \widetilde{\beta}$ is a path from e_1 to $f(e_2) = e_3$ and $\widetilde{\alpha} * (f \circ \widetilde{\beta})$ is a path from e to e_3. Moreover, $p \circ (\widetilde{\alpha} * (f \circ \widetilde{\beta})) = (p \circ \widetilde{\alpha}) * (p \circ (f \circ \widetilde{\beta})) = (p \circ \widetilde{\alpha}) * (p \circ \widetilde{\beta}) = \alpha * \beta$. In particular, $\widetilde{\alpha} * (f \circ \widetilde{\beta})$ is the lift of $\alpha * \beta$ beginning at e with endpoint e_3. Hence $\Phi([\alpha * \beta] * H) = e_3$ and therefore $\Phi^{-1} \circ \psi$ is a homomorphism.

Suppose that $\Phi^{-1} \circ \psi(f) = \Phi^{-1} \circ \psi(g)$ for some $f, g \in A(E, p)$. Then since Φ^{-1} is a bijection $\psi(f) = \psi(g)$ which implies that $f(e) = g(e)$. However, from Theorem 2.6, this implies that $f = g$. Therefore $\Phi^{-1} \circ \psi$ is injective. Next, let $[\alpha] * H \in N(H)/H$. Let $\widetilde{\alpha}$ be the lifting of α beginning at e and let $e_1 = \alpha(1)$. Then $e_1 \in p^{-1}(b)$. Since

$p_*(\pi_1(E, e_1)) = p_*(\pi_1(E, e))$ there is a automorphism F such that $F(e) = e_1$. Then $\Phi^{-1} \circ \psi(F) = \Phi^{-1}(e_1) = [\alpha] * H$. Therefore $\Phi^{-1} \circ \psi$ is surjective. $\square$

The following useful theorem is stated as an exercise in [12].

Theorem 2.8. *Let $p : E \to B$ with $p(e_0) = b$ be a covering map with E and B path connected and locally path connected. Then $H_0 = p_*(\pi_1(E, e_0))$ is a normal subgroup of $\pi_1(B, b)$ if and only if for every pair of points $e_1, e_2 \in p^{-1}(b)$, there is an automorphism of E with $h(e_1) = e_2$.*

Proof. Assume that H_0 is a normal subgroup of $\pi_1(B, b)$ and that e_1 and e_2 are points of $p^{-1}(b)$. Then denote $H_i = p_*(\pi_1(E, e_i))$ for $i = 1, 2$. Since H_0 is normal and each of the groups H_i for $i = 0, 1, 2$ are conjugate, it follows that $H_1 = H_0 = H_2$. Hence, $p_*(\pi_1(E, e_1)) \subset p_*(\pi_1(E, e_2))$. Therefore, p lifts to a map $\tilde{p} : E \to E$ such that $p \circ \tilde{p} = p$ and $p(e_1) = e_2$. Then $\tilde{p}$ is continuous and surjective. To see that $\tilde{p}$ is injective, note that $\tilde{p}^{-1} \circ \tilde{p}$ is a map such that $\tilde{p}^{-1} \circ \tilde{p}(e_1) = e_1$ and such that $p \circ (\tilde{p}^{-1} \circ \tilde{p}) = p$. Note that this implies $(\tilde{p}^{-1} \circ \tilde{p})$ is a lifting to E of the map p. From the uniqueness of liftings, $(\tilde{p}^{-1} \circ \tilde{p}) = id$, where id denotes the identity map. In particular, if $p(a) = p(b)$, then $a = b$.

Next, assume that for any two points $e_1, e_2 \in p^{-1}(b)$, there is an automorphism f of E such that $f(e_1) = e_2$. Consider $\pi_1(E, e_1)$ for some $e_1 \neq e_0$. By assumption, there exists an automorphism h such that $h(e_1) = e_0$. Then $h_*(\pi_1(E, e_1)) = \pi_1(E, e_0)$ and $p \circ h = p$. From Theorem 2.3, this implies that $\pi_1(E, e_1) \subset \pi_1(E, e_0)$. Likewise, using the fact that $h^{-1}(e_0) = e_1$ and $p \circ h^{-1} = p$, it follows that $\pi_1(E, e_0) \subset \pi_1(E, e_1)$. In particular, since $\pi_1(E, e_0) = \pi_1(E, e_1)$, the subgroup H_0 is normal. $\square$

Chapter 3

Properties of the group $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$

In this chapter, addition modulo n will be denoted $+_n$. Define $\alpha : \mathbb{Z}/7 \to \mathbb{Z}/7$ by $\alpha(i) = 2i \bmod(7)$. Next, define the homomorphism $\phi : \mathbb{Z}/3 \to \mathrm{Aut}(\mathbb{Z}/7)$ by first defining $\phi(1) = \alpha$. Note that, as given in [4], since 7 is prime it follows that $\mathrm{Aut}(\mathbb{Z}/7) \cong \mathbb{Z}/6$. Since 1 is a generator for $\mathbb{Z}/3$, given any $b = n(1) \in \mathbb{Z}/3$ for some n, set $\phi(b) = \alpha^n$. In particular, given $i \in \mathbb{Z}/7$, the following hold:

$$\phi(1)(i) = \alpha(i) = 2i \bmod(7) \tag{3.1}$$

$$\phi(2)(i) = \alpha^2(i) = 4i \bmod(7) \tag{3.2}$$

$$\phi(0)(i) = \text{identity}(i) = i. \tag{3.3}$$

Then from Definition 2.3, for (x, y) and (c, d) in $\mathbb{Z}/7 \times \mathbb{Z}/3$, the group $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ is defined by $(x, y) *_{\rtimes} (c, d) = (x +_7 (\phi(y)(c)), y +_3 d)$. Using equations 3.1, 3.2, and 3.3, $(x, y) *_{\rtimes} (c, d)$ may be written as follows for particular values of y:

$$(x, 1) *_{\rtimes} (c, d) = (x +_7 (2c \bmod(7)), 1 +_3 d) \quad \text{or} \tag{3.4}$$

$$(x, 2) *_{\rtimes} (c, d) = (x +_7 (4c \bmod(7)), 2 +_3 d) \quad \text{or} \tag{3.5}$$

$$(x, 0) *_{\rtimes} (c, d) = (x +_7 c, d). \tag{3.6}$$

The group $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ is denoted by G, the element $(0, 1)$ is denoted by a, and the element $(1, 0)$ is denoted by b. In the following group table, $*_{\rtimes}$ denotes the group operation defined in Definition 2.3. Due to the size of the group, the table is split into multiple pages. Following the group table, several facts about the group used in the following chapters are discussed.

$*_{\rtimes}$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(3,0)
(0,0)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(3,0)
(0,1)	(0,1)	(0,2)	(0,0)	(2,1)	(2,2)	(2,0)	(4,1)	(4,2)	(4,0)	(6,1)
(0,2)	(0,2)	(0,0)	(0,1)	(4,2)	(4,0)	(4,1)	(1,2)	(1,0)	(1,1)	(5,2)
(1,0)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(3,0)	(3,1)	(3,2)	(4,0)
(1,1)	(1,1)	(1,2)	(1,3)	(3,1)	(3,2)	(3,0)	(5,1)	(5,2)	(5,0)	(0,1)
(1,2)	(1,2)	(1,0)	(0,1)	(5,2)	(5,0)	(5,1)	(2,2)	(2,0)	(2,1)	(6,2)
(2,0)	(2,0)	(2,1)	(2,2)	(3,0)	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)
(2,1)	(2,1)	(2,2)	(2,0)	(4,1)	(4,2)	(4,0)	(6,1)	(6,2)	(6,0)	(1,1)
(2,2)	(2,2)	(2,0)	(2,1)	(6,2)	(6,0)	(6,1)	(3,2)	(3,0)	(3,1)	(0,2)
(3,0)	(3,0)	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	(6,0)
(3,1)	(3,1)	(3,2)	(3,0)	(5,1)	(5,2)	(5,0)	(0,1)	(0,2)	(0,0)	(2,1)
(3,2)	(3,2)	(3,0)	(3,1)	(0,2)	(0,0)	(0,1)	(4,2)	(4,0)	(4,1)	(1,2)
(4,0)	(4,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	(6,0)	(6,1)	(6,2)	(0,0)
(4,1)	(4,1)	(4,2)	(4,0)	(6,1)	(6,2)	(6,0)	(1,1)	(1,2)	(1,0)	(3,1)
(4,2)	(4,2)	(4,0)	(4,1)	(1,2)	(1,0)	(1,1)	(5,2)	(5,0)	(5,1)	(2,2)
(5,0)	(5,0)	(5,1)	(5,2)	(6,0)	(6,1)	(6,2)	(0,0)	(0,1)	(0,2)	(1,0)
(5,1)	(5,1)	(5,2)	(5,0)	(0,1)	(0,2)	(0,0)	(2,1)	(2,2)	(2,0)	(4,1)
(5,2)	(5,2)	(5,0)	(5,1)	(2,2)	(2,0)	(2,1)	(6,2)	(6,0)	(6,1)	(3,2)
(6,0)	(6,0)	(6,1)	(6,2)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)
(6,1)	(6,1)	(6,2)	(6,0)	(1,1)	(1,2)	(1,0)	(3,1)	(3,2)	(3,0)	(5,1)
(6,2)	(6,2)	(6,0)	(6,1)	(3,2)	(3,0)	(3,1)	(0,2)	(0,0)	(0,1)	(4,2)

$*_\rtimes$	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	(6,0)
(0,0)	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	(6,0)
(0,1)	(6,2)	(6,0)	(1,1)	(1,2)	(1,0)	(3,1)	(3,2)	(3,0)	(5,1)
(0,2)	(5,0)	(5,1)	(2,2)	(2,0)	(2,1)	(6,2)	(6,0)	(6,1)	(3,2)
(1,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	(6,0)	(6,1)	(6,2)	(0,0)
(1,1)	(0,2)	(0,0)	(2,1)	(2,2)	(2,0)	(4,1)	(4,2)	(4,0)	(6,1)
(1,2)	(6,0)	(6,1)	(3,2)	(3,0)	(3,1)	(0,2)	(0,0)	(0,1)	(4,2)
(2,0)	(5,1)	(5,2)	(6,0)	(6,1)	(6,2)	(0,0)	(0,1)	(0,2)	(1,0)
(2,1)	(1,2)	(1,0)	(3,1)	(3,2)	(3,0)	(5,1)	(5,2)	(5,0)	(0,1)
(2,2)	(0,0)	(0,1)	(4,2)	(4,0)	(4,1)	(1,2)	(1,0)	(1,1)	(5,2)
(3,0)	(6,1)	(6,2)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)
(3,1)	(2,2)	(2,0)	(4,1)	(4,2)	(4,0)	(6,1)	(6,2)	(6,0)	(1,1)
(3,2)	(1,0)	(1,1)	(5,2)	(5,0)	(5,1)	(2,2)	(2,0)	(2,1)	(6,2)
(4,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(3,0)
(4,1)	(3,2)	(3,0)	(5,1)	(5,2)	(5,0)	(0,1)	(0,2)	(0,0)	(2,1)
(4,2)	(2,0)	(2,1)	(6,2)	(6,0)	(6,1)	(3,2)	(3,0)	(3,1)	(0,2)
(5,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(3,0)	(3,1)	(3,2)	(4,0)
(5,1)	(4,2)	(4,0)	(6,1)	(6,2)	(6,0)	(1,1)	(1,2)	(1,0)	(3,1)
(5,2)	(3,0)	(3,1)	(0,2)	(0,0)	(0,1)	(4,2)	(4,0)	(4,1)	(1,2)
(6,0)	(2,1)	(2,2)	(3,0)	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)
(6,1)	(5,2)	(5,0)	(0,1)	(0,2)	(0,0)	(2,1)	(2,2)	(2,0)	(3,1)
(6,2)	(4,0)	(4,1)	(1,2)	(1,0)	(1,1)	(5,2)	(5,0)	(5,1)	(2,2)

$*_{\rtimes}$	(6,1)	(6,2)
(0,0)	(6,1)	(6,2)
(0,1)	(5,2)	(5,0)
(0,2)	(3,0)	(3,1)
(1,0)	(0,1)	(0,2)
(1,1)	(6,2)	(6,0)
(1,2)	(4,0)	(4,1)
(2,0)	(1,1)	(1,2)
(2,1)	(0,2)	(0,0)
(2,2)	(5,0)	(5,1)
(3,0)	(2,1)	(2,2)
(3,1)	(1,2)	(1,0)
(3,2)	(6,0)	(6,1)
(4,0)	(3,1)	(3,2)
(4,1)	(2,2)	(2,0)
(4,2)	(0,0)	(0,1)
(5,0)	(4,1)	(4,2)
(5,1)	(3,2)	(3,0)
(5,2)	(1,0)	(1,1)
(6,0)	(5,1)	(5,2)
(6,1)	(3,2)	(3,0)
(6,2)	(2,0)	(2,1)

Proposition 3.1. *G has no automorphism which sends a to a^{-1}.*

Proof. Suppose to the contrary that f were an automorphism such that $f(a) = a^{-1} = a^2$. Then since each element of G can be written in terms of a and b, the automorphism is completely determined by $f(b)$. Note that since $\langle b\rangle$ has 7 elements, $\langle f(b)\rangle$ must have seven elements. The group G only has one subgroup with seven elements, hence $f(b)$ must be some power of b. Therefore, there are six choices for $f(b)$. The following facts can be verified by referencing the group table of G given in Chapter 3. Assume $f(b) = b$, then since f is a homomorphism $f(aba^{-1}) = a^2ba$ and $f(b^2) = b^2$ but $a^2ba \neq b^2$ which would give a contradiction. Likewise, if $f(b) = b^2$ then $f(aba^{-1}) = a^2b^2a \neq b^4 = f(b^2)$. Were $f(b) = b^3$, then $f(aba^{-1}) = a^2b^3a \neq b^6 = f(b^2)$. Continuing in this fashion, if $f(b) = b^4$ it would follow that $f(aba^{-1}) = a^2b^4a \neq b^8 = f(b^2)$. Given $f(b) = b^5$ then $f(aba^{-1}) = a^2b^5a \neq b^{10} = b^3 = f(b^2)$. In the last case, if $f(b) = b^6$, then $f(aba^1) = a^2b^6a \neq b^{12} = b^5 = f(b^2)$. Hence G has no automorphism which sends a to a^{-1}. $\square$

In the following chapters information about the left cosets of the subgroup $\langle b\rangle$ in $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ is required. An essential property of the group G is that $\langle b\rangle$ is a normal subgroup of G.

Proposition 3.2. *$\langle b\rangle$ is a normal subgroup of G.*

Proof. First, note that $a\langle b\rangle = \langle b\rangle a$ if and only if $a\langle b\rangle a^{-1} = \langle b\rangle$. Recall that $aba^{-1} = b^2$ so that, for $b^n \in \langle b\rangle$, $ab^na^{-1} = a(a^{-1}b^2a)(b^{n-1})a^{-1} = (b^2a)(b^{n-1}a^{-1})$. Continuing in this fashion, it follows that $ab^na^{-1} = b^{2n}$. Since b^{2n} is an element of $\langle b\rangle$, the inclusion $a\langle b\rangle a^{-1} \subset \langle b\rangle$ holds.

Next, let $b^n \in \langle b\rangle$. If n is even, then $b^n = ab^{\frac{n}{2}}a^{-1} \in a\langle b\rangle a^{-1}$. If n is odd, then $7+n$ is even and $b^n = b^{7+n} = ab^{\frac{7+n}{2}}a^{-1} \in a\langle b\rangle a^{-1}$. Hence $\langle b\rangle \subset a\langle b\rangle a^{-1}$. Thus $a\langle b\rangle = \langle b\rangle a$.

Since $a\langle b\rangle = \langle b\rangle a$ and clearly $\langle b\rangle = \langle b\rangle$, it follows that $a^2\langle b\rangle$ must equal $\langle b\rangle a^2$. Hence $\langle b\rangle$ is a normal subgroup of G. $\square$

The Proposition below will discuss the number of left cosets of $\langle b\rangle$ in G.

Proposition 3.3. *$G/\langle b\rangle$ contains exactly three elements.*

Proof. Note that if $x \in \langle b\rangle$, then $x\langle b\rangle = \langle b\rangle$ where $\langle b\rangle = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(0,0)\}$. Moreover $a\langle b\rangle = \{(2,1),(4,1),(6,1),(1,1),(3,1),(5,1),(0,1)\}$ and $a^2\langle b\rangle = \{(4,2),(1,2),(5,2),(2,2),(6,2),(3,2),(0,2)\}$. Thus the left cosets $\langle b\rangle$, $a\langle b\rangle$, and $a^2\langle b\rangle$ are disjoint. Also note that $\langle b\rangle \cup a\langle b\rangle \cup a^2\langle b\rangle = G$. Since cosets must either be disjoint or equal, given any $y \in G$ it follows that $y\langle b\rangle$ must be equal to either $\langle b\rangle, a\langle b\rangle$, or $a^2\langle b\rangle$. Hence $G/\langle b\rangle$ has exactly three elements. $\square$

Proposition 3.4. *Let $i \in \{1,2\}$. Given any two elements x and y in $a^i\langle b\rangle$, there exists an automorphism of G which takes x to y.*

Proof. To begin, let $x = a^ib^k$ and $y = a^ib^m$ be two elements of $a^i\langle b\rangle$ for $1 \leq k, m < 7$. Note that since 7 is prime, b^k is also a generator of $\langle b\rangle$. Hence $b^m = (b^k)^j$ for some $1 \leq j < 7$. Define $f : G \to G$ by $f(a) = a$ and $f(b) = b^j$. In order to show that f

is a monomorphism, assume that $f(x) = e$ for some $x = a^i b^k$ in G. It follows that $f(x) = f(a^i)f(b^k) = a^i b^{kj}$. Hence $(a^i)^{-1} = b^{kj}$. However, the only element of $\langle a \rangle$ in $\langle b \rangle$ is e. Hence $a^i = b^{kj} = e$. Moreover, since $j < 7$ this implies that $b^k = e$. Hence the kernel of f is the set $\{e\}$ and f is a monomorphism. To see that f is an epimorphism, let $a^i b^k$ be an element of G. Then there exists an element $b^{k'}$ such that $(b^{k'})^j = b^k$. Then $f(a^i b^{k'}) = a^i b^k$. Hence f is an automorphism. Also, $f(a^i b^k) = a^i (b^k)^j = a^i b^m$ and $f(aba^{-1}) = f(a)f(b)f(a^{-1}) = ab^j a^{-1}$. Also, $f(b^2) = b^{2j}$. As shown in Proposition 3.2, $b^{2j} = ab^j a^{-1}$. Therefore, $f(aba^{-1}) = f(b^2)$.

It will now be shown that for any $y = ab^i$ for $1 \leq i < 7$, there exists an automorphism taking a to ab^i. Note that ab^i has order 3. Define $g : G \to G$ by $g(a) = ab^i$ and $g(b) = b$. To show that g is an automorphism, first assume that $f(a^k b^j) = e$ for $1 \leq k \leq 3$ and $1 \leq j \leq 7$. Then $(ab^i)^k b^j = e$ which implies that $b^j = ((ab^i)^k)^{-1}$. In particular, $(ab^i)^k \in \langle b \rangle$ and $k = 3$. Hence $a^k = e$ and $b^j = e$ so that $a^k b^j = e$. The argument that g is an epimorphism is similar to the proceeding paragraph. Also note that $g(aba^{-1}) = (ab^i)(b)(ab^i)^{-1} = ab^{i+1}(b^i)^{-1}a^{-1} = aba^{-1} = b^2 = g(b^2)$. Next, if x and y are in $a^2\langle b \rangle$, then x and y are equal to $(x')^2$ and $(y')^2$ for some x' and y' in $a\langle b \rangle$. Hence there exists an automorphism taking x' to y' and as a result, the automorphism also takes x to y. $\square$

The following Corollary to Proposition 3.4 will used in Chapter 5.

Corollary 3.1. *There is no automorphism of G which takes $a\langle b \rangle$ to $a^2\langle b \rangle$.*

Proof. Assume such an automorphism exists. Then apply Proposition 3.4 to get an automorphism which takes a to a^2, contradicting Proposition 3.1. $\square$

This concludes the exploration of the necessary facts about the group G used in the following chapters.

Chapter 4

Construction of the Continuum X

Let $G = \mathbb{Z}/7 \rtimes \mathbb{Z}/3$, which is generated by elements a and b with the relations $a^3 = b^7 = 1$ and $aba^{-1} = b^2$. Some fundamental properties of this group, including the group table itself, can be found in Chapter 3. Let W be a closed, PL 5-manifold with fundamental group G. The method for creating such a space is given in [5]. From Theorem 2.4, W has a connected covering space $\widetilde{W}$ and a covering map p' such that $p'_*(\pi_1(\widetilde{W})) \cong \langle b \rangle \cong \mathbb{Z}/7$.

Denote $p'_*(\pi_1(\widetilde{W}, e))$ by H. Then $H \cong \langle b \rangle$. Note that since $\widetilde{W}$ is path connected, Theorem 2.5 gives bijection $\Phi : \pi_1(W, x)/H \cong G/\langle b \rangle \to p'^{-1}(x)$ where $G/\langle b \rangle$ represents the left cosets of $\langle b \rangle$ in G. However, as shown in Chapter 3 Proposition 3.3, $G/\langle b \rangle$ has three elements: $\langle b \rangle$, $a\langle b \rangle$, and $a^2\langle b \rangle$. Therefore $\widetilde{W}$ is a 3-fold cover of W. Also note that from Proposition 3.2, $\langle b \rangle$ is a normal subgroup of G. Hence $\widetilde{W}$ is a regular cover of W.

Using the fact given in [5] that $\pi_k(\mu^n_M) \cong \pi_k(M)$ for $k < n$, it follows that $\mu^2_{\widetilde{W}}$ and μ^2_W have the same fundamental groups as $\widetilde{W}$ and W, respectively. Then $\mu^2_{\widetilde{W}}$ is a regular 3-fold covering of μ^2_W. Let $p : \mu^2_{\widetilde{W}} \to \mu^2_W$ be the covering map.

Proposition 4.1. *$\mu^2_{\widetilde{W}}$ has an automorphism ϕ of order three with no fixed points.*

Proof. Note that from Theorem 2.7 the group of automorphisms $A(E, p)$ is isomorphic to $N(\langle b \rangle)/\langle b \rangle$. However $\langle b \rangle$ is a normal subgroup of G and thus $N(\langle b \rangle) = G$. Hence

$A(E,p) \cong G/\langle b \rangle$. As shown in Chapter 3, $G/\langle b \rangle$ has exactly three elements; $\langle b \rangle, a\langle b \rangle$, and $a^2\langle b \rangle$. The element $a\langle b \rangle$ has order three. Hence there is a automorphism ϕ of order 3. If ϕ had any fixed points, Theorem 2.6 would imply that ϕ would be equal to the identity map and would not have order 3. Hence ϕ has no fixed points. □

4.1 Construction using Fiber Bundles

Throughout this construction, I will denote the unit interval and a circle will refer to any homeomorphic image of S^1. The construction uses the representation of the Menger Curve μ^1 given by R.D. Anderson in [1]. In particular, to construct μ^1, let $N = I^3$. Then for $i \in \mathbb{N}^+$, define the sets $D_i(x)$ to be the collection of all open intervals on the x-axis contained in I with length $\frac{1}{3^i}$ which have rational endpoints contained in $(0,1)$ whose denominator is 3^i when simplified into lowest terms. Define the sets $D_i(y)$ and $D_i(z)$ in a similar fashion on the y-axis and z-axis respectively. Then the Menger Curve μ_1 is defined to be the subset of N consisting of the points $(x,y,z) \in I^3$ such that for any i, at most one of the projections onto the x, y, and z-axis is contained in the set $D_i(x) \cup D_i(y) \cup D_i(z)$ (see Figure 4.1). Another way to say this is that the projection of $(x,y,z) \in \mu^1$ to the xy, xz or yz-plane is contained in the Sierpiński Curve. By a result of R.D. Anderson [1], every 1-dimensional Peano continuum with no open subset embeddable in the plane which has no local cut points is homeomorphic to the Menger Curve.

Next, let $C = \{(x,y,0) |\ x,y \in I$ and at least one of x or y is 0 or 1$\}$ and note that C is a circle such that $C \subset \mu^1$ (see Figure 4.2). Now define $f_1 : \mu^1 \to S$,

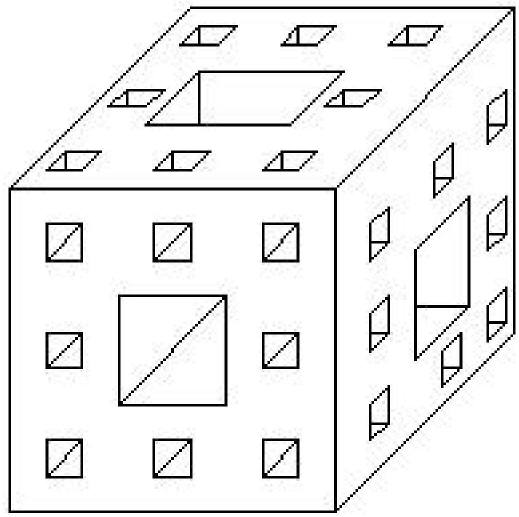

Figure 4.1: The Menger Curve μ^1.

where $S = \{(x, y, z)|\ (x, y, z) \in \mu^1 \text{ and } z = 0\}$, by $f_1((x, y, z)) = (x, y, 0)$. Then f_1 continuous and for $x \in C$, $f_1(x) = x$.

To define $f_2 : S \to C$, first consider the point $q = (\frac{1}{2}, \frac{1}{2}, 0)$. Then $q \notin S$ and, given any point $a \in S$, the ray r_a beginning at q and passing through a intersects C at exactly one point, say x_a. Define the map $f_2(a) = x_a$ for $a \in S$. Then for $a \in S$ such that $a \in C$, $f_2(a) = a$.

Now define $f : \mu^1 \to C$ by $f(x) = (f_2 \circ f_1)(x)$. Then $f(\mu^1) = C$ and for $x \in C$ it follows that $f(x) = x$. Hence f is a retraction of μ^1 onto C, a circle contained in μ^1 (see Figure 4.2).

Note that C is homeomorphic to the quotient space I^* of I obtained by the partition of I which consists of all one element sets $\{x\}$ for $x \in (0, 1)$ and the two element set $\{0, 1\}$. Call this homeomorphism η and for $x \in I$ let $[x]$ denote the member of the partition of I containing x. Let q_1 be the quotient map from I to I^*.

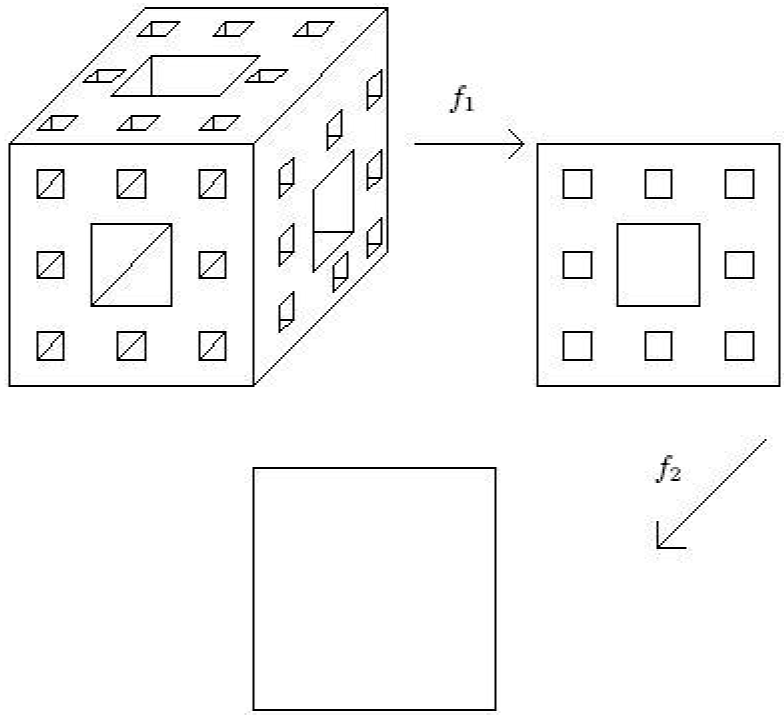

Figure 4.2: A retraction to a circle contained in the Menger Curve.

Consider $Y = I \times \mu^2_{\widetilde{W}}$ and let X' be the quotient space of Y obtained by the partition of Y into the one element sets $\{(x, y)\}$ for $x \in (0, 1)$ and the two element sets $\{(0, y), (1, \phi(y))\}$. Let q_2 be the quotient map from Y to X' and let $[a, b]$ denote the element of the partition of Y containing (a, b). It will now be shown that X' is a bundle over I^* and hence over C. Define $\pi_C : X' \to I^*$ by $\pi_C([a, b]) = [a]$. This is well defined since $[0] = [1]$ and surjective since, given any $[a] \in I^*$ select $b \in \mu^2_{\widetilde{W}}$, then $\pi_C([a, b]) = [a]$. It remains to be shown that given any $[a] \in I^*$ there exists an open set $U \subset I^*$ and a homeomorphism $h_x : \pi_C^{-1}(U) \to U \times \mu^2_{\widetilde{W}}$ such that

$$\pi_C|\pi_C^{-1}(U) = p_x \circ h_x \tag{4.1}$$

where p_x is projection onto the first coordinate.

Proposition 4.2. *Let $[x] \in I^*$ and suppose that $[x] \neq [0]$. Then $[x]$ has a neighborhood U and there is a homeomorphism $h_x : \pi_C^{-1}(U) \to U \times \mu^2_{\widetilde{W}}$ satisfying equation 4.1.*

Proof. Let $[x] \in I^*$ and suppose that $[x] \neq [0]$. Let $U \subset I$ be a open ball about x which does not contain the points 0 and 1. Then $q_1(U)$ is open in I^* since $q_1^{-1}(q_1(U)) = U$. Note that $\pi_C^{-1}(q_1(U)) = \{[a, b] \in X' : [a] \in q_1(U) \text{ and } b \in \mu^2_{\widetilde{W}}\}$. Define the homeomorphism $h_x : \pi_C^{-1}(q_1(U)) \to q_1(U) \times \mu^2_{\widetilde{W}}$ by $h_x([a, b]) = ([a], b)$. Since U does not contain 0 or 1, this map is well defined.

It will now be shown that h_x is continuous. Let $U' \times V'$ be a basic open set in $U \times \mu^2_{\widetilde{W}}$. Note that $q_2(q_1^{-1}(U') \times V')$ is open in X' since $q_2^{-1}(q_2(q_1^{-1}(U') \times V')) = q_1^{-1}(U') \times V'$. In order to show that h_x is continuous, it is sufficient to show that $h_x^{-1}(U' \times V') = q_2(q_1^{-1}(U') \times V')$. Let $[a, b] \in h_x^{-1}(U' \times V')$. Then $[a] \in U'$ and $b \in V'$. Hence $q_1^{-1}([a]) = a \in q_1^{-1}(U')$ so that $q_2((a, b)) = [a, b] \in q_2(q_1^{-1}(U') \times V')$. Therefore $h_x^{-1}(U' \times V') \subset q_2(q_1^{-1}(U') \times V')$. Next, assume that $[a, b] \in q_2(q_1^{-1}(U') \times V')$. Note that since $q_2^{-1}(q_2(q_1^{-1}(U') \times V')) = q_1^{-1}(U') \times V'$, it follows that $q_1(a) = [a] \in U'$ and $b \in V'$. Hence $([a], b) \in U' \times V'$ so that $[a, b] \in h_x^{-1}(U' \times V')$. Hence $q_2(q_1^{-1}(U') \times V') \subset h_x^{-1}(U' \times V')$. Therefore, h_x is continuous.

Assume that $h_x([a, b]) = h_x([c, d])$. Then $([a], b) = ([c], d)$ so that $[a] = [c]$ and $b = d$. Since $[a] = [c]$, the choice of U ensures that $a = c$. Hence $[a, b] = [c, d]$ so that h_x is injective. Also observe that if $([a], b) \in q_1(U) \times \mu^2_{\widetilde{W}}$, then $\pi_C([a, b]) = [a] \in q_1(U)$. Hence $[a, b] \in \pi_C^{-1}(q_1(U))$ and $h_x([a, b]) = ([a], b)$. Therefore h_x is surjective. Next, note that for $[a,b] \in \pi_C^{-1}(q_1(U))$ it follows that $p_x \circ h_x([a,b]) = p_x(([a],b)) = [a]$. Hence $\pi_C = p_x \circ h_x$ on $\pi_C^{-1}(q_1(U))$, satisfying equation 4.1.

It will now be shown that h_x^{-1} is continuous. Once this is done, it will follow that h_x is a homeomorphism and $q_1(U)$ is an open set satisfying the required conditions for the definition of a fiber bundle. Let U' be an open set in $\pi_C^{-1}(U)$. Then $U' = q_2(V')$ for some open set V' in $I \times \mu^2_{\widetilde{W}}$. It is necessary to show that $h_x(U') = (h_x^{-1})^{-1}(U')$ is open. Let $([a],b) \in h_x(U')$. Choose a basic open set of the form $U \times V$ such that $(a,b) \in U \times V \subset V'$. Note that $q_2(U \times V)$ is open and that $q_2(U \times V) \subset q_2(V') = U'$. Then $q_1(U)$ is an open set containing $[a]$ and V is an open set containing b. Hence, $([a],b) \in q_1(U) \times V$. However, from the above information, $q_1(U) \times V \subset h_x(q_2(V')) = h_x(U')$. Therefore, h_x^{-1} is continuous. $\square$

Proposition 4.3. *Let $[x] \in I^*$ and suppose that $[x] = [0]$. Then $[x]$ has a neighborhood U and there is a homeomorphism $h_x : \pi_C^{-1}(U) \to U \times \mu^2_{\widetilde{W}}$ satisfying equation 4.1.*

Proof. Let $U = q_1([0, \frac{1}{3}) \cup (\frac{2}{3}, 1])$ which is an open neighborhood of $[0]$ in I^*. Then $\pi_C^{-1}(U) = \{[a,b] : b \in \mu^2_{\widetilde{W}} \text{ and } [a] \in U\}$. Define $h_x : \pi_C^{-1}(U) \to U \times \mu^2_{\widetilde{W}}$ as follows:

$$h_x([a,b]) = \begin{cases} ([a], \phi(b)) & \text{if } a \in [0, \frac{1}{3}) \\ ([a], b) & \text{if } a \in (\frac{2}{3}, 1]. \end{cases}$$

The function is well defined since $h_x([0,y]) = h_x([1, \phi(y)])$.

It will now be shown that the function is continuous. Let $A \times B$ be a basic open subset of $U \times \mu^2_{\widetilde{W}}$. Then A is an open subset of U in I^* and $h_x^{-1}(A \times B) = \{[a,b] : [a] \in A, a \in (\frac{2}{3}, 1], \text{ and } b \in B\} \cup \{[a,b] : [a] \in A, a \in [0, \frac{1}{3}), \text{ and } b \in \phi^{-1}(B)\}$. Let

$[a, b] \in h_x^{-1}(A \times B)$. If $[a] \neq [0]$ then finding an open set about $[a, b]$ contained in $h_x^{-1}(A \times B)$ is similar to the argument in the preceding proposition. Therefore, suppose $[a] = [0]$. Let $[0, n)$ and $(m, 1]$ be open subset of I such that $q_1([0, n) \cup (m, 1]) \subset A$. Then $V = q_2([0, n) \times \phi^{-1}(B)) \cup q_2((m, 1] \times B)$ is an open neighborhood of $[a, b]$ in X'. Moreover, V is a subset of $h_x^{-1}(A \times B)$. Therefore $h_x^{-1}(A \times B)$ is open and h_x is continuous.

Next, let $V \subset \pi_C^{-1}(U)$ be open and let $([a], b) \in h_x(V)$. Suppose that $a = 1$; then $[a, b]$ is of the form $\{(0, \phi^{-1}(b)), (1, b)\}$. Choose open subsets of I of the form $[0, m)$ and $(n, 1]$ and open subsets B^1 and B^2 of $\mu^2_{\widetilde{W}}$ such that $q_2(([0, m) \times B^1) \cup ((n, 1] \times B^2))$ is an open subset of V containing $[a, b]$. Then the set $\phi(B^1) \cap B^2$ is nonempty since b is in both sets. Since ϕ is a homeomorphism the intersection is also open. Then set $q_1([0, m) \cup (n, 1]) \times (\phi(B^1) \cap B^2)$ is open in $I^* \times \mu^2_{\widetilde{W}}$ containing $([a], b)$. Moreover, for any $([c], d)$ in $q_1([0, m) \cup (n, 1]) \times (\phi(B^1) \cap B^2)$, if $c \in (n, 1]$ then $[c, d] \in V$ and if $c \in [0, m)$ then $[c, \phi^{-1}(d)] \in V$. In either case $([c], d)$ is an element of $h_x(V)$. If $a \neq 0$ or 1 then an argument similar to the argument in Proposition 4.2 gives an open set about $([a], b)$ contained in $h_x(V)$. Thus h_x^{-1} is a continuous function and h_x is a homeomorphism. $\square$

From Propositions 4.2 and 4.3, X' is a fiber bundle over C with fiber $\mu^2_{\widetilde{W}}$. Let X be the pullback of X' induced by the map f. In particular $X = \{([c, b], a) \in X' \times \mu^1$ such that $\eta(\pi_C([c, b])) = f(a)\}$. Note that $([c, b], a) \in X$ if and only if $\eta([c]) = f(a)$.

4.2 Construction using a Quotient Map

For the alternative construction of X let h be a regular covering of μ^1 with itself of order 3 with automorphism α. To see that the Menger Curve has such a cover, let μ_A^1 and μ_B^1 be two copies of the Menger Curve as constructed previously. Denote $(x, y, z) \in \mu_A^1$ by $(x, y, z)_A$ and $(x, y, z) \in \mu_B^1$ by $(x, y, z)_B$ to avoid ambiguity. Similar notation will be used when constructing other Menger Curves. Glue the two copies of the Menger Curve together by combining points of the form $(x, y, 0)_A$ to the corresponding point in μ_B^1 of the form $(x, y, 0)_B$. By the result of R.D. Anderson [1], the resulting space is a representation of the Menger Curve. Denote this representation by μ_C^1 (see Figure 4.3). Denote points $(x, y, z) \in \mu_C^1$ by $(x, y, z)_{C-A}$ if $(x, y, z) \in \mu_A^1$ and $(x, y, z)_{C-B}$ if $(x, y, z) \in \mu_B^1$.

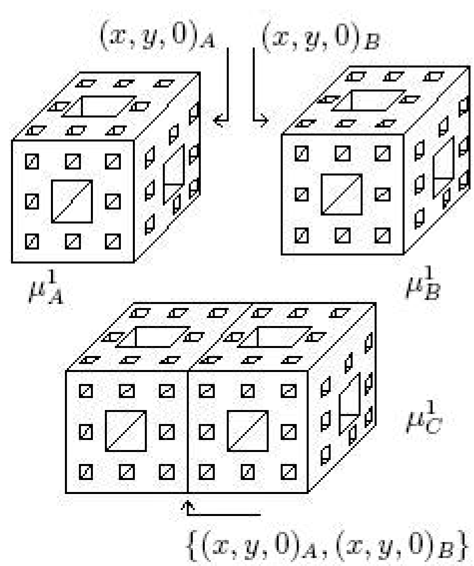

Figure 4.3: Combing two Menger Curves.

Next, take μ_C^1 and create another Menger curve by gluing points of the form $(x, y, 1)_A$ to the corresponding point $(x, y, 1)_B$. Denote this space by μ_D^1. Again, by the result of Anderson in [1], this space is a representation of the Menger Curve which closely resembles the shape of a torus (see Figure 4.4). It will be shown that there exists a regular 3-fold covering of μ_D^1 and hence of μ^1.

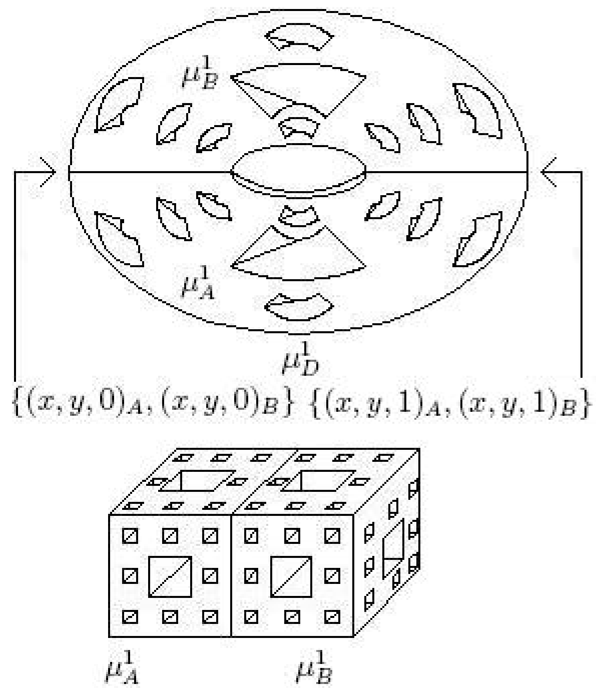

Figure 4.4: Creating a circular Menger curve.

In order to create the regular 3-fold covering, begin with three copies of μ_C^1. Denote these copies by μ_{C1}^1, μ_{C2}^1, and μ_{C3}^1. Glue the three copies of μ_C^1 together as follows: Glue points of the form $(x, y, 1)_{C1-A}$ to the corresponding point $(x, y, 1)_{C2-B}$, points of the form $(x, y, 1)_{C2-A}$ to the corresponding point $(x, y, 1)_{C3-B}$, and points of the form $(x, y, 1)_{C3-A}$ to the corresponding point $(x, y, 1)_{C1-B}$ (see Figure 4.5). Then this is a Menger Curve once more by Anderson's result in [1]. Denote this copy of the Menger Curve by μ^1_E.

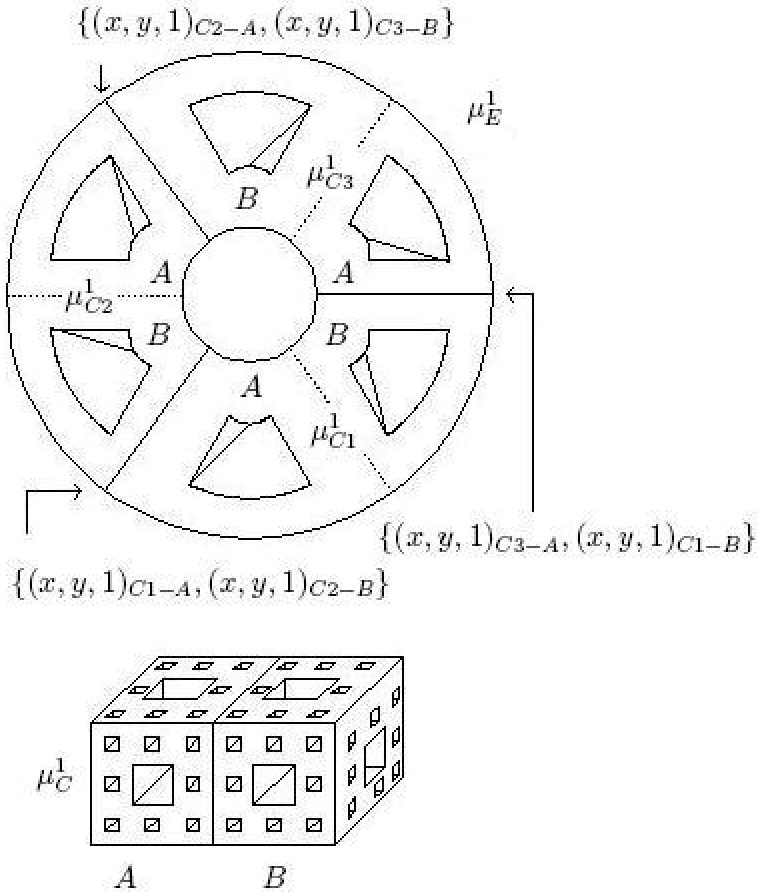

Figure 4.5: Creating a 3-fold cover of the Menger Curve.

To show that μ^1_E is a regular 3-fold cover of μ^1_D and hence of μ^1, define a map $k : \mu^1_E \to \mu^1_D$ by $k((x,y,z)_W) = (x,y,z)_D$ for $W = C1-A$, $C1-B$, $C2-A$, $C2-B$, $C3-A$, or $C3-B$. Then k is a 3-fold covering map of μ^1_D (see Figure 4.5). Then there exists an automorphism α of order 3 corresponding to rotating μ^1_E about its central hole. In particular, for $V = A$ or B:

$$\alpha((x, y, z)_{C1-V}) = (x, y, z)_{C2-V}, \tag{4.2}$$

$$\alpha((x, y, z)_{C2-V}) = (x, y, z)_{C3-V}, \text{and} \tag{4.3}$$

$$\alpha((x, y, z)_{C3-V}) = (x, y, z)_{C1-V}. \tag{4.4}$$

Note that α has order 3. Moreover, from Theorem 2.8, this implies that μ^1_E is a regular cover of μ^1_D and hence of μ^1.

Consider the set G of homeomorphisms of $\mu^2_{\widetilde{W}} \times \mu^1$ given by $g = \{id, (\phi(x), \alpha(y)), (\phi^2(x), \alpha^2(y))\}$ where id denotes the identity map. Then let X be the quotient space of $\mu^2_{\widetilde{W}} \times \mu^1$ obtained by taking a point x in $\mu^2_{\widetilde{W}} \times \mu^1$ to the orbit of x. Denote this map by q and let $\widetilde{(a, b)}$ denote the orbit of the point (a, b).

Proposition 4.4. *X is a regular 3-fold covering of $\mu^2_W \times \mu^1$.*

Proof. Let p and k be the regular 3-fold covering maps defined previously from $\mu^2_{\widetilde{W}}$ to μ^2_W and from μ^1 to μ^1, respectively. Define $\pi : X \to \mu^2_W \times \mu^1$ by $\pi(\widetilde{(a, b)}) = (p(a), k(b))$. Since the maps ϕ and α are automorphisms, the map π is well defined. Let the covering map $\pi' : \mu^2_{\widetilde{W}} \times \mu^1 \to \mu^2_W \times \mu^1$ be defined by $\pi'((a, b)) = (p(a), k(b))$.

Let $(a,b) \in \mu_W^2 \times \mu^1$ and let U be an neighborhood evenly covered by π' containing (a,b) and let $C = \{U_\alpha\}_{\alpha \in A}$ be the partition of $\pi'^{-1}(U)$ into slices. Define the open sets $C_{\widetilde{(x,y)}}$ to be the union of those elements $U_\alpha \in C$ such that $(\phi^i(x), \alpha^i(y)) \in U_\alpha$ for some i. Then $C_{\widetilde{(x,y)}}$ is the union of exactly three elements of C since both α and ϕ have order three. Moreover, the set $\{C_{\widetilde{(x,y)}}\}$ for $\widetilde{(x,y)} \in \pi^{-1}((a,b))$ is a collection of disjoint open sets in $\mu_W^2 \times \mu^1$. It will now be shown that the pairwise disjoint collection $\{q(C_{\widetilde{(x,y)}})\}$ for $\widetilde{(x,y)} \in \pi^{-1}((a,b))$ is a collection of open sets in X. Let $(c,d) \in q^{-1}(q(C_{\widetilde{(x,y)}}))$ for some $\widetilde{(x,y)}$. Then $(c,d) = (\phi^i(x), \alpha^i(y))$ for some i; hence $\pi((c,d)) \in U$. Therefore, (c,d) is in some member of the set C, say U'. Then U' is open and U' is contained inside of $q^{-1}(q(C_{\widetilde{(x,y)}}))$ so that $q^{-1}(q(C_{\widetilde{(x,y)}}))$ is open. Therefore $q(C_{\widetilde{(x,y)}})$ is open. Hence $q(C_{\widetilde{(x,y)}})$ is open for each $\widetilde{(x,y)} \in \pi^{-1}((a,b))$.

It will now be shown that $\pi^{-1}(U) = \cup q(C_{\widetilde{(x,y)}})$ for $\widetilde{(x,y)} \in \pi^{-1}((a,b))$. Let $\widetilde{(x,y)} \in \pi^{-1}(U)$. Then $(p(x), f(y)) \in U$ so that $(x,y) \in \pi'^{-1}(U)$. Therefore, $(x,y) \in C_{\widetilde{(x',y')}}$ for some (x',y') so that $\widetilde{(x,y)} \in q(C_{\widetilde{(x',y')}})$. Thus $\pi^{-1}(U) \subset \cup q(C_{\widetilde{(x,y)}})$. Next, let $\widetilde{(x,y)} \in q(C_{\widetilde{(a',b')}})$ for some $\widetilde{(a',b')}$. Then $(x,y) = (\phi^i(x'), \alpha^i(y'))$ for some $(x',y') \in q^{-1}(q(C_{\widetilde{(a',b')}})$ so that $(p(x), k(y)) \in U$. Hence $\widetilde{(x,y)} \in \pi^{-1}(U)$. Thus U is a neighborhood about (a,b) which is evenly covered by π.

Now let $(a,b) \in \mu_W^2 \times \mu^1$ and let x_1, x_2 and x_3 be the three elements of $p^{-1}(a)$. Assume without loss of generality that $\phi(x_1) = x_2$. Then $\phi^2(x_1) = \phi(x_2) = x_3$. Moreover, for $y \in h^{-1}(b)$, the orbits of (x_i, y) for $i \in \{1,2,3\}$ are as follows:

$$\{(x_1, y), (x_2, \alpha(y)), (x_3, \alpha^2(y))\},$$

$$\{(x_2, y), (x_3, \alpha(y)), (x_1, \alpha^2(y))\}, \text{and}$$

$$\{(x_3, y), (x_1, \alpha(y)), (x_2, \alpha^2(y))\}.$$

These three orbits are disjoint and given that $(c, d) \in \pi'^{-1}(a, b)$ then $c = x_i$ for some $i = 1, 2,$ or 3 and $d = \alpha^j(b)$ for some $j = 1, 2,$ or 3. Hence (a, b) is in one of the three orbits. Thus $\pi^{-1}(x, y)$ has exactly three elements and π is a 3-fold covering.

In order to establish that X is a regular covering of $\mu_W^2 \times \mu^1$, let $y \in \mu_W^2 \times \mu^1$ and $(\widetilde{a, b}), (\widetilde{c, d}) \in \pi^{-1}(y)$. Then $c = \phi^i(a)$ for some $i = 1, 2$ or 3 and $d = \alpha^j(b)$ for some $j = 1, 2$ or 3. Then the map β which takes $(\widetilde{a', b'}) \mapsto (\widetilde{\phi^i(a'), \alpha^j}(b'))$ is a well defined homeomorphism. Moreover, since both ϕ and α are automorphisms, it follows that β is an automorphism. Then from Theorem 2.8, X is a regular covering space of $\mu_W^2 \times \mu^1$. $\square$

To show that X is a Peano continuum, note that since it is the continuous image of a compact and connected space, X is compact and connected. Moreover, since X is Hausdorff and the continuous image of a compact metric space, X itself is a metric space [13]. Therefore X is a continuum. To see that X is locally connected, let U be an open set in X and let C be a component of U. Then for $x \in p^{-1}(C)$, let C_x be the component of $p^{-1}(U)$ containing x. Then $p(C_x)$ is connected and intersects C, so that C_x is contained in $p^{-1}(C)$. Thus $p^{-1}(C)$ can be written as the union of components of $p^{-1}(U)$. In particular, these components are open so that $p^{-1}(C)$ is open. Hence C is open in X. Therefore, X is locally connected from Theorem 2.1.

Chapter 5

Properties of the Continuum X

It will first be shown that the continuum X constructed in Chapter 4 is a homogeneous continuum.

Proposition 5.1. *Given any two points in $\mu^2_W \times \mu^1$, there exists a homeomorphism taking one to the other which lifts to X.*

Proof. Let (a, b) and (c, d) be elements of $\mu^2_W \times \mu^1$. Since both μ^2_W and μ^1 are arcwise connected, there exists an arc $A_{a,c}$ in μ^2_W connecting a to c and an arc $A_{b,d}$ in μ^1 connecting b to d. For each x in $A_{a,c}$ there exists an open set of the form $U'_x \times V_x$ such that U'_x is open in μ^2_W and V_x open in μ^1 with the property that $U'_x \times V_x$ is evenly covered by the map π and V_x contains b. Furthermore, since μ^2_W is strongly locally homogeneous, there exists an open set $x \in U_x \subset U'_x$ with the property that given any $y \in U_x$, there exists a homeomorphism taking x to y which is the identity outside of U_x. Then the collection $\{U_x \times V_x\}$ for $x \in A_{a,c}$ is an open cover for the compact set $A_{a,c} \times \{b\}$. Hence there exists a finite subcover which will be denoted $\{\hat{U}_i \times \hat{V}_i\}_{i=1 \ldots n}$ with the properties that given any i, $\hat{U}_i \times \hat{V}_i$ is evenly covered by π and given any (a', b) and (c', b) in $(A_{a,c} \times \{b\}) \cap (\hat{U}_i \times \hat{V}_i)$ there is a homeomorphism taking (a', b) to (c', b). Likewise, there is a finite cover $C_2 = \{\tilde{U}_j \times \tilde{V}_j\}_{j=1 \ldots m}$ of $\{c\} \times A_{b,d}$ with the property that each set is evenly covered by π and given any (c, a') and (c, b') in $(\{c\} \times A_{b,d}) \cap (\tilde{U}_i \times \tilde{V}_i)$ there is a homeomorphism taking (c, a') to (c, b'). Set

$C = C_1 \cup C_2$ and enumerate C by $U_k \times V_k = \hat{U_k} \times \hat{V_k} \in C_1$ for $1 \le k \le n$ and $U_k \times V_k = \tilde{U_{k-n}} \times \tilde{V_{k-n}} \in C_2$ for $n < k \le n+m$. Assume that C is a minimal such cover in the sense that $(U_k \times V_k) \cap (U_j \times V_j) \neq \emptyset$ if and only if $|j-k| = 1$. Moreover, assume that $(a,b) \in U_1 \times V_1$ and that $(c,d) \in U_{n+m} \times V_{n+m}$.

Let $x_0 = (a,b)$ and $x_{n+m} = (c,d)$ and suppose x_{i-1} is defined for some $0 \le i < n+m-1$. Define x_i to be any element of $(A_{a,c} \times \{b\}) \cap (U_i \times V_i) \cap (U_{i+1} \times V_{i+1})$ if $i \le n$ and any element of $(\{c\} \times A_{b,d}) \cap (U_i \times V_i) \cap (U_{i+1} \times V_{i+1})$ otherwise. Then for $0 \le i < n$ the elements x_i and x_{i+1} are of the form (a',b) and (c',b). Then there exists a homeomorphism h' of μ_W^2 taking a' to c' which is the identity outside of U_i. Define the homeomorphism $h_i = (h', id)$ where id denotes the identity map. Likewise, for $n \le i < n+m$ define a similar homeomorphism h_i which acts on the second coordinate of x_i. Define $h = h_{n+m-1} \circ h_{n+m-2} \circ .... \circ h_0$. Then h is a homeomorphism such that $h(a,b) = (c,d)$.

Next, it will now be shown that by the construction of h, the homeomorphism can be lifted to a map $\tilde{h} : \mu_W^2 \times \mu^1 \to X$ such that $h = \pi \circ \tilde{h}$. Let $e \in X$ such that $\pi(e) = h((a,b))$. Since $U_1 \times V_1$ is evenly covered by π, the map h_0 lifts to a map $\tilde{h}_0 : \mu_W^2 \times \mu^1 \to X$ such that $\tilde{h}_0((a,b))) = e$. Likewise, h_1 lifts to a map with $\tilde{h}_0(x_1)$ in the image of h_1. In general, suppose that $\tilde{h}_{i-1}$ is already defined. Then h_i lifts to a map $\tilde{h}_i$ with $\tilde{h}_{i-1}(x_i)$ in the image of h_i. Then $\tilde{h} = \tilde{h}_{n+m-1} \circ \tilde{h}_{n+m-2} \circ .... \circ \tilde{h}_0$ is a lift of h.

$\square$

Note that since X is a regular 3-fold covering of $\mu_W^2 \times \mu^1$, a similar argument as in Chapter 4 Proposition 4.1 indicates that X has an automorphism ϕ_X of order three with no fixed points. As such, a π fiber may be given a cyclic ordering. For $x \in \mu_W^2 \times \mu^1$, select an element k in $\pi^{-1}(x)$. Then k is sent to $\phi_X(k)$ and $\phi_X(k)$ is sent to $\phi_X^2(k)$. Likewise, $\phi_X^3(k) = k$. The element $\phi_X^i(k)$ for a given $k \in X$ will be denoted k^i. In particular the following hold:

$$k^{0 \bmod 3} = k,$$

$$k^{1 \bmod 3} = \phi_X(k), \text{and}$$

$$k^{2 \bmod 3} = \phi_X^2(k).$$

Let $\pi^{-1}(c)$ and $\pi^{-1}(d)$ be two π fibers. From Proposition 5.1, there exists a homeomorphism which which takes c to d that lifts to X. Denote this homeomorphism by h and the lift of h by $\tilde{h}$. As such, the cyclic ordering of a π fiber along with this lift induces a homeomorphism $H : X \to X$ defined as follows: Given some $x \in \pi^{-1}(c')$ for some c', $x = \alpha^i$ for some $\alpha \in \text{Image}(\tilde{h}) \cap \pi^{-1}(c')$ and $i \in \{1, 2, 3\}$. Define

$$H(x) = (\tilde{h}(\pi(x)))^i. \tag{5.1}$$

Proposition 5.2. *The map H in Equation 5.1 is a homeomorphism from X to X.*

Proof. The map H is a bijection. In order to show that H is continuous, let U be an open set in X and let $c \in H^{-1}(U)$. Since U is open, choose an open set $V \subset U$ about $H(c)$ with the following properties:

1. $\pi(V)$ and $h^{-1}(\pi(V))$ are evenly covered by π.

2. V is connected.

3. V^i is contained inside of Image$(\tilde{h})$ for some $i = \{1, 2, 3\}$.

Then the slice of $\pi^{-1}(h^{-1}(\pi(V)))$ containing c must be contained inside of $H^{-1}(U)$. Hence $H^{-1}(U)$ is open. Since H is a continuous bijection from a compact space to a metric space, H is a homeomorphism. $\square$

Since any π fiber can be taken to another by a homeomorphism, it follows that X is homogeneous if given any two elements a and b of the same π fiber there exists a homeomorphism taking a to b. However, since X is a regular covering space of $\mu_W^2 \times \mu^1$, Theorem 2.8 implies there is an automorphism which takes a to b. Therefore the space X is homogeneous.

It will now be shown that X is not bihomogeneous. First, let $f : \mu_W^2 \times \mu^1 \to \mu_W^2$ and $g : \mu_W^2 \times \mu^1 \to \mu^1$ be projection maps. Define a horizontal fiber of X to be the preimage of a point under the map $f \circ \pi$ and a vertical fiber to be the preimage of a point under the map $g \circ \pi$.

Proposition 5.3. *A π fiber can be characterized as the intersection of a vertical and a horizontal fiber.*

Proof. Let $(a, b) \in \mu^2_W \times \mu^1$. Let $x \in \pi^{-1}((a, b))$. Then $\pi(x) = (a, b)$ and thus $(f \circ \pi)(x) = a$ and $(g \circ \pi)(x) = b$. Hence $x \in (f \circ \pi)^{-1}(a)$ and $x \in (g \circ \pi)^{-1}(b)$. Thus $\pi^{-1}((a, b)) \subset (f \circ \pi)^{-1}(a) \cap (g \circ \pi)^{-1}(b)$. Next, suppose that $x \notin \pi^{-1}((a, b))$. Then $\pi(x) = (c, d)$ where either $c \neq a$ or $d \neq b$. Without loss of generality, assume that $c \neq a$. Then $(f \circ \pi)(x) \neq a$ so that $x \notin (f \circ \pi)^{-1}(a)$. Therefore $x \notin (f \circ \pi)^{-1}(a) \cap (g \circ \pi)^{-1}(b)$ so that $(f \circ \pi)^{-1}(a) \cap (g \circ \pi)^{-1}(b) \subset \pi^{-1}((a, b))$. Therefore $\pi^{-1}((a, b)) = (f \circ \pi)^{-1}(a) \cap (g \circ \pi)^{-1}(b)$. □

Proposition 5.4. *A homeomorphism $\psi : X \to X$ must preserve horizontal and vertical fibers.*

Proof. Let ψ be a homeomorphism of X and let $a \in \mu^2_W$. Then for $x \in \pi^{-1}(\{a\} \times \mu^1)$ pick neighborhoods U_x of $\pi(x)$ and $U_{\psi(x)}$ about $\pi(\psi(x))$ which are evenly covered by π. Let U'_x be the slice of $\pi^{-1}(U_x)$ containing x and $U'_{\psi(x)}$ the slice containing $\psi(x)$. Choose an open subset $\psi(x) \in V_{\psi(x)} \subset U'_{\psi(x)}$ such that $\psi^{-1}(V_{\psi(x)}) \subset U'_x$. Then the map $\pi|V_{\psi(x)} \circ (\psi|\psi^{-1}(V_{\psi(x)})) \circ (\pi|\psi^{-1}(V_{\psi(x)}))^{-1}$ is an open embedding from a subset of U_x into $U_{\psi(x)}$. Hence, by a Lemma of K. Kuperberg introduced in [6], $\{a\} \times \mu^1$ is mapped into some set of the form $\{b\} \times \mu^1$ for some $b \in \mu^2_{\widetilde{W}}$. Hence $\psi(x) \in \pi^{-1}(\{b\} \times \mu^1)$ so that $\psi(\pi^{-1}(\{a\} \times \mu^1)) \subset \pi^{-1}(\{b\} \times \mu^1)$. A similar argument gives that $\pi^{-1}(\{b\} \times \mu^1) \subset \psi(\pi^{-1}(\{a\} \times \mu^1))$. Thus ψ must preserve horizontal fibers. Likewise ψ preserves vertical fibers. □

Since a π fiber is the intersection of a vertical and a horizontal fiber, ψ must also preserve π fibers.

Next, it will be shown that if x and y are points of X which are both in the same π fiber, then the homeomorphism ψ can not exchange them. Since X is a three fold cover and preserves π fibers, if ψ exchanged x and y then ψ would leave the third point of the π fiber unchanged. Denote this unchanged element by m. Let ψ' denote the restriction of ψ to the vertical fiber containing x and y. This vertical fiber is of the form $\pi^{-1}(\mu_W^2 \times \{a\})$ for some a in μ^1. Since ψ preserves vertical fibers, ψ' maps $\pi^{-1}(\mu_W^2 \times \{a\})$ into a set of the form $\pi^{-1}(\mu_W^2 \times \{b\})$ for some b in μ^1.

Proposition 5.5. *The map* $\bar{\psi} = \pi \circ \psi' \circ \pi^{-1} : \mu_W^2 \times \{a\} \to \mu_W^2 \times \{b\}$ *obtained by restricting* π *is a homeomorphism between two copies of* μ_W^2*.*

Proof. In order to show that $\bar{\psi}$ is continuous, let U be an open set in $\mu_W^2 \times \{b\}$. Since π is continuous, then $\pi^{-1}(U)$ is an open set contained in $\pi^{-1}(\mu_W^2 \times \{b\})$. Then ψ' is a homeomorphism so that $(\psi')^{-1}(\pi^{-1}(U))$ is an open set in $\mu_W^2 \times \{a\}$. Lastly, since π is an open map $\pi((\psi')^{-1}(\pi^{-1}(U)))$ is an open set inside of $\mu_W^2 \times \{a\}$. Therefore $\bar{\psi}$ is a continuous map.

To see that the map $\bar{\psi}$ is injective, suppose that $\bar{\psi}((c, a)) = \bar{\psi}((d, a))$ for some c and d in μ_W^2. Then $\psi'(\pi^{-1}((c, a)))$ and $\psi'(\pi^{-1}((d, a)))$ lie in the same π fiber. Since a homeomorphism must preserve π fibers $\pi^{-1}((c, a))$ and $\pi^{-1}((d, a))$ must lie in the same π fiber. This implies that $(c, a) = (d, a)$. Therefore $\bar{\psi}$ is injective.

Next, suppose that $(c, b) \in \mu_W^2 \times \{b\}$. Then $\pi^{-1}((c, b)) \in \pi^{-1}(\mu_W^2 \times \{b\})$. Since ψ' preserves π fibers, there exists an element $(d, a) \in \mu_W^2 \times \{a\}$ such that

$\psi'(\pi^{-1}((d,a))) = \pi^{-1}((c,b))$. Then it follows that $\bar{\psi}((d,a)) = (c,b)$. Since $\bar{\psi}$ is a continuous bijection from a compact space to a metric space, $\bar{\psi}$ is a homeomorphism.

□

Recall that by selecting x, the deck translation ϕ_X gives a cyclic order to the π fiber $\pi^{-1}(\pi(x))$ given by x^i. Assume, without loss of generality, that $\phi_X(m) = x$. Then $m^2 = y$ and $m^3 = m$. Note that since ψ exchanges x and y, the map ψ alters this cyclic ordering so that $\phi_X(\psi(m)) = y$ and $\psi(m)^2 = x$. Note that $\pi^{-1}(\mu_W^2 \times \{a\})$ or $\pi^{-1}(\mu_W^2 \times \{b\})$, which shall be denoted $\tilde{X}_a$ and $\tilde{X}_b$ respectively, along with the map obtained by restricting π may be regarded as a covering space over μ_W^2. Denote this restriction by $\bar{\pi}$. Denote $\bar{\pi}_*(\pi_1((X_a, x)))$ by H. Since $\bar{\pi}$ is a 3-fold covering over μ_W^2 and from Theorem 2.5 there is a bijection between $\pi_1(\mu_W^2, \pi(x))/H$ and $\bar{\pi}^{-1}(\pi(x))$, there are three elements in $\pi_1(\mu_W^2, \pi(x))/H$. Since μ_W^2 has 21 elements, H must have 7 elements. Since the group G has only one subgroup with 7 elements, $H = \langle b \rangle \cong \mathbb{Z}/7$. Likewise $\bar{\pi}_*(\pi_1((X_b, x)) = \langle b \rangle$.

Proposition 5.6. *Let x, y, and m be in the same π fiber as above. If the map ψ changes the cyclic ordering of a π fiber and leaves the point m fixed, then the induced map $\bar{\psi}_*$ from Proposition 5.5 sends $a\langle b \rangle$ to $a^{-1}\langle b \rangle$.*

Proof. Suppose without loss of generality that $m \cdot a = x$ and $m \cdot a^2 = y$, where $m \cdot a^i$ is the lifting correspondence as defined in the proof of Theorem 2.5. Since ψ changes the cyclic ordering of a π fiber, $m \cdot \bar{\psi}(a) = y$ and $m \cdot \bar{\psi}(a^2) = x$. Hence $\bar{\psi}_*(a) \in a^2\langle b \rangle$. Hence $\bar{\psi}_*$ maps $a\langle b \rangle$ to $a^2\langle b \rangle$. □

From the comments following Proposition 3.4, the group G has no such automorphism which would imply a contradiction. Therefore ψ can not exchange elements of the same π fiber. Hence, X can not be a bihomogeneous continuum.

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