Defining Chaos

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
August 5, 2017

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Abstract

This thesis is a study of the paper "Defining Chaos" by Brian Hunt and Edward Ott [1] with added details to make arguments easier to follow. They introduced a new entropy-based definition of chaos called expansion entropy. A system is defined to be chaotic when the expansion entropy is positive. Some benefits of this definition of chaos are that it is applicable to attractors, repellers, and non-periodically forced systems; autonomous and nonautonomous systems; and discrete and continuous time. We will explore the different properties of expansion entropy as well as calculate the expansion entropy of different examples. Expansion entropy is compared with topological entropy under certain conditions. Lastly, the limitations of nonentropy-based definitions of chaos are analyzed.

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1. Introduction

Chaos is studied in many fields of science and mathematics, and in each field there exist different definitions for this phenomenon. The purpose of Hunt and Ott's paper [1] is to study the limitations of previous definitions of chaos as well as define an entropy-based definition of chaos that does not possess the same restrictions as these other definitions.

When studying chaos, typically scientists look for two characteristics in a system. The first is sensitivity to small perturbations which was studied by Lorenz [2] in his research on weather patterns. Li and Yorke [3] studied the second characteristic which is a complex orbit structure.

Hunt and Ott [1] believed that a sufficient definition of chaos should be general, simple, and computable as well as contain the properties listed above. The desire for a general definition is so that it applies to systems containing *repellers* [4](non-attracting sets) as well as nonautonomous (time-dependent) time inputs. Repellers often occur in fields of study such as fluid dynamics [5], celestial mechanics [6], chemistry [7], and atomic physics [8]. Since chaos occurs in these fields of study, it is also important to have a definition of chaos that includes nonautonomous time inputs. These inputs have the potential to be quasi-periodic [9], stochastic [10], or chaotic. When studying externally forced systems, we are looking to see if a particular *realization* (observed output) of the system is chaotic instead of determining if the external forcing is chaotic.

The following section introduces the new entropy-based definition for chaos that Hunt and Ott [1] proposed and is called *expansion entropy*. It applies to an n-dimensional dynamical system with restraining region S and can be written as the difference between two asymptotic exponential rates. The first rate is the maximum over $d \leq n$ of the rate that the system expands d-dimensional volume in S. The second is the rate that n-dimensional volume leaves S (if the restraining region S is invariant, then this rate is 0). *A system is defined to be chaotic if the expansion entropy on a restraining region is positive.*

Section 5 contains examples for the application of expansion entropy to different systems as well as numerical evaluations in some examples. In Section 7, the definitions of expansion entropy and topological entropy are compared. Lastly, Section 8 presents other definitions of chaos, and the limitations of these definitions are discussed. It is important to note that the problem of detecting chaos in experimental data was not discussed in Hunt and Ott's paper [1], and likewise will not be discussed in this paper. The study of dynamics in experimental data supplies extra problems, especially in the presence of externally forced systems or non-attracting sets.

2. Expansion Entropy

Let the dynamical system be smooth and the *state space* be a finite-dimensional manifold M. In general, for $S \subseteq M$, the volume of S is denoted $\mu(S)$ and $d\mu(x)$ is integration with respect to the volume. In reference to expansion entropy, the *restraining region* $S \subseteq M$ must be a closed set with finite volume. Note that S is not required to be invariant. Let f be a family of maps $f_{t',t} : M \to M$ such that if x and x' are the states of the system at time t and t' respectively, then $f_{t',t}(x) = x'$. These maps must satisfy the identities $f_{t,t}(x) = x$ and $f_{t'',t}(x) = f_{t'',t'}(f_{t',t}(x))$. These maps are defined for discrete time as well as continuous time. If the system is noninvertible, then the restriction $t' \geq t$ is necessary. The system can either be nonautonomous or autonomous, and in either case, the maps $f_{t',t}$ are assumed to be differentiable functions of x. If the system is autonomous (i.e. $f_{t',t}(x) = f_{u'u}(x)$ when $t' - t = u' - u$), then $f_{t',t}$ will be denoted as f_T where $T = t' - t$.

The *singular values* of a matrix A are defined to be the square roots of the eigenvalues of $A^T A$. If the linear transformation A is applied to the unit ball, the resulting image will be an ellipse. The semiaxes of the ellipses are the singular values of A. Define $G(A)$ to be the product of the singular values that are greater than 1; if none exist, then define $G(A) = 1$.

In order to better understand this G value, suppose that A is an $n \times n$ matrix, and for $d \leq n$, let P_d be a d-dimensional plane in n-dimensional Euclidean space. Let $W \subseteq P_d$ be a d-dimensional ball. Suppose that $A(W)$ is the image of W under A and μ_d is the d-dimensional volume. Then $G(A)$ is the maximum over all orientations of P_d and the maximum over all

$d \le n$ of $\frac{\mu_d(A(W))}{\mu_d(W)}$. Therefore, $G(A)$ is viewed as the largest possible growth ratio of d-dimensional volumes under A. For the definition of expansion entropy, G is applied to the Jacobian $Df_{t',t}$ and will therefore express a local volume growth ratio for $f_{t',t}$.

Let $S_{t',t}$ be the set of $x \in S$ such that $f_{t'',t}(x) \in S$ for all $t'' \in [t, t']$. Define

$$E_{t',t}(f,S) = \frac{1}{\mu(S)} \int_{S_{t',t}} G(Df_{t',t}(x))d\mu(x). \tag{1}$$

Definition 2.1. *Expansion entropy, denoted H_0, is defined as*

$$H_0(f,S) = \lim_{t'\to\infty} \frac{1}{t'-t} \ln E_{t',t}(f,S). \tag{2}$$

The limits considered are well-defined only if the limit exists. Note that if the restraining region S is not invariant and if the system f is nonautonomous, then H_0 could depend on the initial time t as well as f and S.

In order to better understand the definition of H_0, we will express it in a slightly different manner. To do this, make the substitution $\frac{1}{\mu(S)} = \frac{1}{\mu(S_{t',t})}\frac{\mu(S_{t',t})}{\mu(S)}$ in the definition of $E_{t',t}(f,S)$. Then (2) can be written as

$$H_0(f,S) = \lim_{t'\to\infty} \frac{1}{t'-t} \ln \widetilde{E}_{t',t}(f,S) - \frac{1}{\tau_+} \tag{3}$$

where

$$\widetilde{E}_{t',t}(f,S) = \frac{1}{\mu(S_{t',t})} \int_{S_{t',t}} G(Df_{t',t}(x))d\mu(x) \tag{4}$$

and

$$\frac{1}{\tau_+} = \lim_{t'\to\infty} \frac{1}{t'-t} \ln \frac{\mu(S)}{\mu(S_{t',t})}. \tag{5}$$

Therefore, H_0 can be viewed as the difference of two exponential rates (the limits in Equations (3) and (5)) with the following explanation. Let N initial conditions be uniformly spread throughout S at time t; also assume that N is large ($N \to \infty$). Equation (4) is the average (over trajectories remaining in S from time t to t') of the maximum local d-dimensional volume growth ratio along the trajectory. Therefore, the first term in Equation (3) is the exponential growth rate of this average. As t' increases, Equation (5) is the exponential decay rate of the number of trajectories of initial conditions that remain in S for all $t'' \in [t, t']$.

As mentioned earlier, it is required that in order for a system to be considered chaotic by the definition of expansion entropy, $H_0 > 0$. This implies that for a chaotic system, the exponential volume growth ratio must strictly exceed the exponential decay rate at which trajectories leave S.

A few of the initial benefits of this definition for chaos is that it applies to nonautonomous systems, it assigns an entropy value to all restraining regions in the manifold, and it implies a computational method for numerically estimating H_0 exists (Section 4). The last important property of expansion entropy to note for now is that:

$$S' \subseteq S \implies H_0(f, S') \leq H_0(f, S). \tag{6}$$

If $S' \subseteq S$, then clearly $S'_{t',t} \subseteq S_{t',t}$, and as a result, $E_{t',t}(f, S') \leq E_{t',t}(f, S)$. This result is important to note because if expansion entropy detects chaos in a set S', then it will also detect chaos in any set containing S'. This property is demonstrated in Example 5.2 where the system contains a nonchaotic attractor as well as a chaotic repeller.

3. Expansion Entropy of the Inverse System

This section provides a proof that for an autonomous, invertible system, the expansion entropy of the original function is the same as the expansion entropy of the inverse system. As will be discussed in Section 7, this property is also true for topological entropy. In order to prove this property for expansion entropy, first note the validity of the following corollary of the Singular Value Decomposition Theorem.

Corollary 3.1. *For a square matrix, the product of the singular values is the absolute value of its determinant.*

Proof. Let A be an $n \times n$ matrix. The singular value decomposition of A is $U\Sigma V^T$ where U and V are unitary matrices. Then:

$$\begin{aligned}
|\det A| &= |\det(U\Sigma V^T)| \\
&= |\det U \det \Sigma \det V^T| \\
&= |(\pm 1)(\sigma_1 \cdots \sigma_n)(\pm 1)| \\
&= \sigma_1 \cdots \sigma_n.
\end{aligned}$$

□

Theorem 3.2. *If f is an autonomous, invertible system, then $H_0(f, S) = H_0(f^{-1}, S)$.*

Proof. First, note that $f_{t,t'} = f^{-1}_{t',t}$ (specifically, $f_{t',t}(S_{t',t}) = S_{t,t'}$ and $f_{t,t'}(S_{t,t'}) = S_{t',t}$). It is obvious then that $Df_{t',t}(x)$ and $Df_{t,t'}(x')$ are inverses, which implies that the singular values of $Df_{t,t'}(x')$ are the inverses of the singular values of $Df_{t',t}(x)$. By Corollary 3.1, if A is an invertible $n \times n$ matrix, then: $|\det A| = \frac{G(A)}{G(A^{-1})}$. For this problem, that means:

$$|\det(Df_{t',t}(x))| = \frac{G(Df_{t',t}(x))}{G(Df_{t,t'}(x'))}. \tag{7}$$

To prove this first part, only assume that $f_{t',t}$ is invertible.

$$\begin{aligned}
E_{t,t'}(f, S) &= \frac{1}{\mu(S)} \int_{S_{t,t'}} G(Df_{t,t'}(x'))d\mu(x') \\
&= \frac{1}{\mu(S)} \int_{S_{t',t}} G(Df_{t,t'}(x'))|\det Df_{t',t}(x)|d\mu(x) \\
&= \frac{1}{\mu(S)} \int_{S_{t',t}} G(Df_{t',t}(x))d\mu(x) \\
&= E_{t',t}(f, S).
\end{aligned}$$

Now assume that $f_{t',t}$ is also autonomous. This means that $f_{t',t} = f_{t'-t} = f_T$ and $f_T^{-1} = f_{-T}$. Then:

$$H_0(f^{-1}, S) = \lim_{T\to\infty} \frac{1}{T} \ln E_{T,0}(f^{-1}, S)$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{T} \ln E_{-T,0}(f,S) \\
&= \lim_{T\to\infty} \frac{1}{T} \ln E_{0,-T}(f,S) \\
&= \lim_{T\to\infty} \frac{1}{T} \ln E_{T,0}(f,S) \\
&= H_0(f,S).
\end{aligned}
$$

Therefore, for an autonomous, invertible system, the expansion entropy of a function is equal to the expansion entropy of the inverse function. □

Since it is possible to use either $f_{t',t}$ or $f_{t,t'}$ to determine the expansion entropy, one may wonder if it matters which function is used. This question is explored in the next section.

4. Numerical Evaluation of Expansion Entropy

As mentioned earlier, a definition of chaos should have a computational method available in order to test for chaos in a model. This section discusses the method of calculating expansion entropy in this type of situation. To start, take a large number of initial conditions $\{x_1, ..., x_N\}$ that are spread uniformly on S. Next, take the trajectory of $f_{T,0}(x_i)$ and the tangent map $Df_{T,0}(x_i)$ and evolve them forward in time as long as the trajectory is contained in S. Next use a discrete sequence of times T and evaluate

(8) $$\widehat{E}_T(f,S) = \frac{1}{N} \sum_{i=1}^{N}{}' G(Df_{T,0}(x_i)).$$

The prime after the summation indicates computing the sum for only the i values where $f_{T,0}(x_i)$ stays in S up to time T. Based on the definition of E given in Equation (1), we can see that $\widehat{E}$ is an estimate of E. If N and T are large, we expect to find an approximately linear relationship when plotting $\ln \widehat{E}_T(f,S)$ against T. Therefore, H_0 can be estimated as the slope of the straight line fitted to the data ([11] is similiar for two dimensions).

As one may imagine, judgement and experimentation are used to determine a sufficient choice for N and T in order to gain reliable calculations from a computer. According to Hunt and Ott [1], using 100 samples of size N is a good choice for the sample size. They also recommend computing the mean and standard deviation of $\ln \widehat{E}_T(f, S)$. This method supplies a sampling error and a more reliable mean estimate than just computing $\ln \widehat{E}_T(f, S)$ for a single sample of $100N$ points. Examples 5.2 and 5.3 provide illustrations using this numerical approach.

We have shown for an autonomous, invertible system that $H_0(f, S) = H_0(f^{-1}, S)$, so naturally it is important to wonder when it is better to use f or f^{-1} to calculate expansion entropy in terms of computational cost and accuracy. In order to determine this, first generalize the definition of the exponential decay rate to incorporate backwards time as well as forwards as follows:

$$\frac{1}{\tau_\pm} = \lim_{\pm T \to \pm\infty} \frac{1}{\pm T} \ln \frac{\mu(S)}{\mu(S_{\pm T,0})}$$

where $S_{T,0}$ is as defined earlier and $S_{-T,0}$ is the set of initial conditions whose trajectories remain in S from time 0 to $-T$.

In reference to the numerical evaluation of expansion entropy, $\frac{1}{\tau_+}$ (respectively $\frac{1}{\tau_-}$) is the exponential temporal decay rate of the number of initial conditions uniformly spread on S at time 0 that have trajectories remaining in S up to time T (respectively $-T$).

Theorem 4.1. *As $T \to \infty$, the forward calculation of H_0 is computationally more efficient if $\frac{1}{\tau_+} < \frac{1}{\tau_-}$.*

Proof. Assume that $f_{t',t}$ is autonomous and invertible. First note that $S_{-T,0} = f_{T,0}(S_{T,0})$. Subtract $\frac{1}{\tau_+} - \frac{1}{\tau_-}$:

$$\begin{aligned}
\frac{1}{\tau_+} - \frac{1}{\tau_-} &= \lim_{T\to\infty} \frac{1}{T} \ln \frac{\mu(S)}{\mu(S_{T,0})} - \lim_{T\to\infty} \frac{1}{T} \ln \frac{\mu(S)}{\mu(S_{-T,0})} \\
&= \lim_{T\to\infty} \frac{1}{T} \ln \frac{\mu(S_{-T,0})}{\mu(S_{T,0})} \\
&= \lim_{T\to\infty} \frac{1}{T} \ln \frac{\mu(f_{T,0}(S_{T,0}))}{\mu(S_{T,0})}.
\end{aligned}$$

Assume that $f_{T,0}$ is volume contracting. Then the right hand side is negative:

$$\frac{1}{\tau_+} - \frac{1}{\tau_-} < 0 \implies \frac{1}{\tau_+} < \frac{1}{\tau_-}.$$

What we have is that if $f_{T,0}$ is volume contracting, then $\frac{1}{\tau_+} < \frac{1}{\tau_-}$. It is obvious this is more computationally efficient since the forward time function is decreasing in volume. $\square$

The general idea of this is to calculate H_0 with the function that has the highest decay rate. Even if the system is still chaotic, it is still better to do the calculations with the function that has less points leaving S on the time interval $[t, t']$.

Some common examples considered in chaos theory are the Hénon map and the Lorenz system. Both systems are uniformly volume contracting for all points in the state space, specifically this means that $\frac{1}{\tau_+} < \frac{1}{\tau_-}$. Contrarily, Hamiltonian systems are volume preserving which means $\frac{1}{\tau_+} = \frac{1}{\tau_-}$.

5. Examples

Example 5.1. (Attracting and Repelling Fixed Points) A function f has an attracting (respectively repelling) fixed point x_0 if x_0 is a fixed point and $|Df(x_0)| < 1$ (respectively $|Df(x_0)| > 1$). Similarly, a function f has an attracting n-periodic point x_0 if x_0 is an n-periodic point and $|Df^n(x_0)| < 1$ (respectively $|Df^n(x_0)| > 1$). Let f be a one-dimensional continuously differentiable map with a fixed point x_0. Assume that $Df(x_0) \neq \pm 1$. Let S be an interval such that $x_0 \in S$ and for all $x \in S$, $Df(x) \neq \pm 1$. Then $H_0(f, S) = 0$.

Case 1: Attracting Fixed Point

Note that $S_{t',t} = S$ in this case. Since $|Df(x_0)| < 1$ for all $x \in S$, then $G(Df_{t',t}(x)) = 1$ for all $x \in S$ and $t' > t$. Therefore,

$$E_{t',t}(f, S) = \frac{1}{\mu(S)} \int_S 1 d\mu(x) = 1$$

$$\implies H_0(f, S) = \lim_{t' \to \infty} \frac{\ln 1}{t' - t} = 0.$$

Case 2: Repelling Fixed Point

Suppose that $|Df(x_0)| > 1$. In this case, $S_{t',t}$ is a subinterval of S whose endpoints map to the endpoints of S under $f_{t',t}$. Specifically, this means that $\mu(f_{t',t}(S_{t',t})) = \mu(S)$. Then,

$$\begin{aligned} E_{t',t}(f,S) &= \frac{1}{\mu(S)} \int_{S_{t',t}} G(Df_{t',t}(x)) d\mu(x) \\ &= \frac{1}{\mu(S)} \int_{S_{t',t}} |Df_{t',t}(x)| d\mu(x) \\ &= \frac{1}{\mu(S)} \left| \int_{S_{t',t}} Df_{t',t}(x) d\mu(x) \right| \\ &= \frac{1}{\mu(S)} \mu(f_{t',t}(S_{t',t})) = 1 \end{aligned}$$

$$\implies H_0(f,S) = \lim_{t' \to \infty} \frac{1}{t'-t} = 0.$$

Therefore, isolated periodic points are not chaotic.

Example 5.2. (One-Dimensional Map with a Chaotic Repeller and an Attracting Fixed Point) Let f be the function displayed in Figure 1 of [1]. Suppose that $S = [-1, 1.5]$ and $S' = [0,1]$ are the restraining regions. This function has an invariant Cantor set in S' which will classify this example as chaotic by the definition of expansion entropy. The attracting fixed point is $x = -\frac{1}{2}$ and it attracts almost every initial condition with respect to the Lebesgue measure in S.

In order to prove this system is chaotic, first, calculate the expansion entropy of S'. Note that f is linear on S' with a derivative of 3 on $\left[0, \frac{1}{3}\right]$ and -2 on $\left[\frac{1}{2}, 1\right]$. The invariant Cantor set is made of all initial conditions in S' that never land in $\left(\frac{1}{3}, \frac{1}{2}\right)$.

Let L represent the iterations of the interval $\left[0, \frac{1}{3}\right]$ and R represent the iterations of the interval $\left[\frac{1}{2}, 1\right]$. Then $S'_{T,0}$ is a set made of 2^T intervals that correspond to all possible strings of length T of the letters L and R. Suppose that a string has k L's and $T-k$ R's, then the length of the interval will be $3^{-k}(-2)^{k-T}$ and the derivative will be $(3)^k(-2)^{T-k}$. The integral of $G(Df^T)$ on each interval will be 1, therefore $E_{T,0}(f,S') = 2^T$. This means $H_0(f,S') = \ln 2$, and therefore f is chaotic on S'. By Property (6), f is also chaotic on S.

The numerical computation of H_0 is computed next by first choosing a sample of size N and range of T values. For each T, use Equation (8) to compute the estimate $\widehat{E}_T$ of $E_{T,0}$ for 100 samples of N points. Next, compute the mean (solid curve) and standard deviation (vertical bars) of these 100 samples. Figure 2 of [1] shows the results for N= 1000 and N= 100000 on S', and Figure 3 of [1] shows the results for the same N on S. The approximated value of H_0 is the slope of the solid curve in an appropriate scaling interval. For N, the scaling interval can be judged by consistence of the results with a larger value of N as well as the shortness of the error bars and straightness of the curve. Note that the scaling interval of S is approximately the same as for S'. Lastly, note that the slope of the solid curve is $\ln 2$ which is the same result from the earlier calculation of H_0.

Example 5.3. (Random One-Dimensional Map) Let $f : [0, 2\pi) \to [0, 2\pi)$ be defined as $f(\theta_t) = \theta_{t+1} = [\theta_t + \alpha_t + K \sin\theta_t] \bmod 2\pi$, where $K > 0$ and α_i are independent random variables uniformly distributed on $[0, 2\pi)$. This example is chaotic if $K > 1$.

Let S be the unit circle. Note that:

$$\left|\frac{d\theta_T}{d\theta_0}\right| = \prod_{t=0}^{T-1} |1 + K\cos\theta_t|,$$

and

$$E_{T,0}(f, S) = \left\langle \max\left(\left|\frac{d\theta_T}{d\theta_0}\right|, 1\right)\right\rangle_{\theta_0}$$

where $\langle \cdots \rangle_x$ denotes the expected value of x. If θ_0 is uniformly distributed, then $\theta_0, \theta_1, ...$ are independent and uniformly distributed. Then:

$$\begin{aligned}\left\langle\left|\frac{d\theta_T}{d\theta_0}\right|\right\rangle_{\theta_0,\ldots,\theta_{T-1}} &= \prod_{t=0}^{T-1} \langle |1 + K\cos\theta_t| \rangle_{\theta_t} \\ &= \langle |1 + K\cos\theta| \rangle_\theta^T .\end{aligned}$$

If $\left|\frac{d\theta_T}{d\theta_0}\right| < 1$ then clearly we get the result $H_0(f, S) = 0$. Therefore, suppose that $\left|\frac{d\theta_T}{d\theta_0}\right| > 1$. Then:

$$H_0(f, S) = \lim_{T\to\infty} \frac{1}{T} \ln E_{T,0}(f, S)$$

$$= \lim_{T\to\infty} \frac{1}{T} \ln \langle |1 + K\cos\theta| \rangle_\theta^T$$

$$= \ln \langle |1 + K\cos\theta| \rangle_\theta = \lambda.$$

(In this case where the θ_0's are uniformly distributed, for simplicity we will call the expansion entropy λ).

Case 1: $0 < K \leq 1$

$$\langle |1 + K\cos\theta| \rangle_\theta = \langle 1 + K\cos\theta \rangle_\theta = 1 + K \langle \cos\theta \rangle_\theta$$

$$= 1 + \frac{K}{2\pi} \int_0^{2\pi} \cos\theta d\theta = 1$$

$$\implies \ln \langle |1 + K\cos\theta| \rangle_\theta = \ln 1 = 0.$$

Case 2: $K > 1$

For simplicity, let $a = \cos^{-1}(-\frac{1}{k})$. Note that the angle a is located on the interval $(\frac{\pi}{2}, \pi)$ and $\sin a = \frac{\sqrt{K^2-1}}{K}$.

$$\langle |1 + K\cos\theta| \rangle_\theta = \frac{1}{\pi} \left[\int_0^a (1 + K\cos\theta) d\theta - \int_a^\pi (1 + K\cos\theta) d\theta \right]$$

$$= \frac{1}{\pi} \left[(\theta + K\sin\theta)\Big|_0^a - (\theta + K\sin\theta)\Big|_a^b \right]$$

$$= \frac{1}{\pi} [2a + 2K\sin a - \pi]$$

$$= \frac{1}{\pi} [2a + 2\sqrt{K^2 - 1} - \pi].$$

If $K = 1$, then $a = \pi$:

$$\implies \langle |1 + K\cos\theta| \rangle_\theta = 1.$$

If $K \to \infty$, then $a = \frac{\pi}{2}$:

$$\implies \langle |1 + K\cos\theta| \rangle_\theta = \infty.$$

Since $\frac{1}{2\pi}[2a+2\sqrt{K^2 - 1}-\pi]$ is monotonically increasing, then for $K > 1$, $\langle |1 + K\cos\theta| \rangle_\theta > 1$. Therefore, $\lambda = \ln \langle |1 + K\cos\theta| \rangle_\theta > \ln 1 = 0$.

This next part is about calculating $E_{T,0}$ when θ_0 is not uniformly distributed. The maps $\theta_{t+1} = [\theta_t + \alpha_t + K \sin\theta_t] \bmod 2\pi$ where $K \leq 1$ are all diffeomorphisms. If $\theta_t = 0$, then $\theta_{t+1} = \alpha_t$. If $\theta_t = 2\pi$, then $\theta_{t+1} = 2\pi + \alpha_t$. Since θ_{t+1} is a diffeomorphism, then θ_{t+1} is one-one and onto. Therefore, it is clear that $\frac{d\theta_T}{d\theta_0} > 0$. This means that:

$$\begin{aligned}
E_{T,0}(f,S) &= \left\langle \max\left(\frac{d\theta_T}{d\theta_0}, 1\right)\right\rangle_{\theta_0} < \left\langle \frac{d\theta_T}{d\theta_0} + 1 \right\rangle_{\theta_0} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta_T}{d\theta_0} + 1 d\theta_0 = \frac{1}{2\pi}(\theta_T + \theta_0)\Big|_0^{2\pi} \\
&= \frac{1}{2\pi}(\theta_T\Big|_0^{2\pi} + 2\pi) = \frac{1}{2\pi}(4\pi) = 2.
\end{aligned}$$

Therefore, $E_{T,0}$ is not exponentially increasing. This means that $H_0 = 0$ for $0 < K \leq 1$.

Numerical experiments also agree with the argument that $H_0 > 0$ for $K > 1$, but establishing the transition to chaos (using the definition of expansion entropy) occurring exactly at $K = 1$ requires further study. Figure 4 of [1] shows the results for the numerical calculation at each T of $\widehat{E}$ for 100 samples of size N= 1000000. The dashed line has slope $\ln \langle |1 + 1.5\cos\theta| \rangle_\theta$ which is slightly larger than the slope of the computational data.

Example 5.4. (Shear Map on the 2-Torus) Let $\theta_{t+1} = [\theta_t + \omega] \bmod 2\pi$ where $\frac{\omega}{2\pi}$ is irrational [12] and $\phi_{t+1} = [\phi_t + \theta_t] \bmod 2\pi$. This is a shear map and it is not chaotic. There are two types of shears,

Horizontal Shear:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + my \\ y \end{bmatrix}.$$

Vertical Shear:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ mx + y \end{bmatrix}.$$

If $m > 0$ ($m < 0$), then any points above the x-axis will shift to the right (left) and any points below the x-axis will shift to the left (right). For this example, θ is the horizontal variable and ϕ is the vertical variable. This means that our problem looks like:

$$f_{t+1,t}\begin{bmatrix}\theta_t\\ \phi_t\end{bmatrix}=\begin{bmatrix}\theta_{t+1}\\ \phi_{t+1}\end{bmatrix}=\begin{bmatrix}\theta_t+\omega\\ \phi_t+\theta_t\end{bmatrix},$$

i.e. this problem is a vertical shear. This means $f_{t',t}$ needs a constant in front of the horizontal variable θ, specifically $t'-t$. Note that in order for this shear to make sense, it must be that $(t'-t)\in\mathbb{Z}$. Specifically,

$$f_{t',t}\begin{bmatrix}\theta_t\\ \phi_t\end{bmatrix}=\begin{bmatrix}\theta_{t'}\\ \phi_{t'}\end{bmatrix}=\begin{bmatrix}\theta_t+\omega\\ \phi_t+(t'-t)\theta_t\end{bmatrix}=\begin{bmatrix}1 & 0\\ t'-t & 1\end{bmatrix}\begin{bmatrix}\theta_t\\ \phi_t\end{bmatrix}+\begin{bmatrix}\omega\\ 0\end{bmatrix}.$$

$$\implies Df_{t',t}\begin{bmatrix}\theta_t\\ \phi_t\end{bmatrix}=\begin{bmatrix}1 & 0\\ t'-t & 1\end{bmatrix}.$$

To find the singular values, we must find the eigenvalues of $(Df_{t',t})^T(Df_{t',t})$.

$$(Df_{t',t})^T(Df_{t',t})=\begin{bmatrix}1 & t'-t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ t'-t & 1\end{bmatrix}=\begin{bmatrix}1+(t'-t)^2 & t'-t\\ t'-t & 1\end{bmatrix}$$

$$\begin{aligned}0=\begin{vmatrix}1+(t'-t)^2-\lambda & t'-t\\ t'-t & 1-\lambda\end{vmatrix}&=(1+(t'-t)^2-\lambda)(1-\lambda)-(t'-t)^2\\ &=\lambda^2+(-2-(t'-t)^2)\lambda+1\end{aligned}$$

$$\begin{aligned}\implies \lambda&=\frac{1}{2}\left[2+(t'-t)^2\pm\sqrt{(-2-(t'-t)^2)^2-4}\right]\\ &=\frac{1}{2}\left[2+(t'-t)^2\pm\sqrt{(t'-t)^2(t'-t)^2+4}\right]\\ &=\frac{1}{2}\left[2+(t'-t)^2\pm(t'-t)\sqrt{(t'-t)^2+4}\right].\end{aligned}$$

These are the eigenvalues of $(Df_{t',t})^T(Df_{t',t})$. If we graph these two eigenvalues along with $(t'-t)^2$ and $(t'-t)^{-2}$, we find that as $t' \to \infty$:

$$\lambda_1 = \frac{1}{2}\left[2 + (t'-t)^2 + (t'-t)\sqrt{(t'-t)^2+4}\right] \longrightarrow (t'-t)^2$$
$$\lambda_2 = \frac{1}{2}\left[2 + (t'-t)^2 - (t'-t)\sqrt{(t'-t)^2+4}\right] \longrightarrow (t'-t)^{-2}.$$

It is a little clearer in the graphs of the singular values that these functions are approximately the same:

$$\sigma_1 = \left(\frac{1}{2}\left[2 + (t'-t)^2 + (t'-t)\sqrt{(t'-t)^2+4}\right]\right)^{\frac{1}{2}} \longrightarrow (t'-t)$$
$$\sigma_2 = \left(\frac{1}{2}\left[2 + (t'-t)^2 - (t'-t)\sqrt{(t'-t)^2+4}\right]\right)^{\frac{1}{2}} \longrightarrow (t'-t)^{-1}.$$

Assume (without loss of generality) that $(t'-t) > 1$ (if $(t'-t) < 1$, then $(t'-t)^{-1} > 1$ and the calculations would then be the same).

$$\begin{aligned} E_{t',t}(f,S) &= \frac{1}{\mu(S)} \int_{S_{t',t}} G(Df_{t',t}(x))d\mu(x) \\ &= \frac{1}{\mu(S)} \int_{S_{t',t}} (t'-t)d\mu(x) \\ &= \frac{1}{\mu(S)}(t'-t)\mu(S_{t',t}) = t'-t. \end{aligned}$$

Therefore:

$$H_0(f,S) = \lim_{t'\to\infty} \frac{1}{t'-t} \ln(t'-t) = 0.$$

Therefore this example is not chaotic. Specifically, this example illustrates a situation where orbits are dense and nearby orbits separate over time (as in chaos), but the rate of separation is linear instead of exponential.

Example 5.5. (Horseshoe Map) The process of the Horseshoe map in $\mathbb{R}^2$ on the unit square is drawn in Figure 6 of [1]. For this example, the restraining region is the unit square. From step (a) to (b), a uniform horizontal compression is applied to the restraining region. A bending formation is applied from step (b) to (c). Assume this bending only occurs in the shaded region. Step (c) to (d) shows the portion of the horseshoe that remains in S after one iteration.

Let $\rho \in \left(0, \frac{1}{2}\right)$ be the factor by which S is compressed horizontally, then $\frac{1}{\rho}$ is the factor that S is stretched vertically, i.e. ρ and $\frac{1}{\rho}$ are the singular values, but $\frac{1}{\rho}$ is the only singular value greater than 1. Specifically, $G(Df_{t',t}) = \frac{1}{\rho}$, and after $t' - t$ iterates, $G(Df_{t',t}) = \left(\frac{1}{\rho}\right)^{t'-t}$. After one iteration, the portion of the horseshoe inside the square is 2ρ, and after $t' - t$ iterations, $(2\rho)^{t'-t}$ remains in the square, i.e. $\mu(S_{t',t}) = (2\rho)^{t'-t}$. Then $E_{t',t}(f, S) = 2^{t'-t}$ and $H_0(f, S) = \ln 2$. Therefore, the Horseshoe map is chaotic in the unit square.

6. Q-Order Expansion Entropy

Previous work on fractal dimensions determined that the box-counting dimension can be generalized to a spectrum of dimensions (denoted D_q where $q \in \mathbb{N}_0$). The box-counting dimension is the case when $q = 0$. Additionally, a spectrum of entropy-like values were introduced by Grassberger and Procaccia [13] and these also depend on $q \in \mathbb{N}_0$. The previous research on this topic implies that there may be a similar spectrum of q-order expansion entropies (denoted H_q where $q \in \mathbb{N}_0$). With this idea in mind, this section will discuss a definition for this q-order expansion entropy and explore its reliability.

Definition 6.1. *Q-order expansion entropy is defined as*

$$H_q(f, S) = \frac{1}{1-q} \lim_{t' \to \infty} \frac{1}{t'-t} \ln \left[\frac{\int_{S_{t',t}} [G(Df_{t',t}(x))]^{1-q} d\mu(x)}{\mu(S_{t',t})^q \mu(S)^{1-q}} \right]$$

where G is still the product of the singular values of $Df_{t',t}(x)$ that are greater than 1.

Note that this definition of q-order expansion entropy matches the definition of expansion entropy when $q = 0$. The following example is used to determine if H_q is as reliable as the definition of H_0.

Example 6.2. This example calculates the q-order expansion entropy for the one-dimensional map with a chaotic repeller and an attracting fixed point (Example 5.2). Recall that $S = [-1, 1.5]$ and $S' = [0, 1]$. First, calculate the expansion entropy of the set S'. $S'_{T,0}$ is made up of 2^T intervals which have length $3^{-k}2^{k-T}$ (for $k = 0, ..., T$). This implies that $G(Df^T) = 3^k 2^{T-k}$. Note that the number of intervals with a given k is the binomial coefficient: $C(T,k) = \frac{T!}{k!(T-k)!}$. Then:

$$\begin{aligned}\int_{S'_{T,0}} G(Df^T)^{1-q} d\mu &= \sum_{k=0}^{T} C(T,k)[3^k 2^{T-k}]^{1-q}[3^{-k}2^{k-T}] \\ &= \sum_{k=0}^{T} C(T,k)[3^{-kq}2^{q(k-T)}] \\ &= \sum_{k=0}^{T} C(T,k)[(3^{-q})^k (2^{-q})^{T-k}] \\ &= (3^{-q} + 2^{-q})^T.\end{aligned}$$

The length of the points in S' that remain in S' through time T is $(\frac{5}{6})^T$. Therefore:

$$\begin{aligned}H_q(f,S') &= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{\int_{S'_{T,0}} G(Df)^{1-q} d\mu}{\mu(S'_{T,0})^q \mu(S')^{1-q}} \right] \\ &= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{(3^{-q} + 2^{-q})^T}{\left(\frac{5}{6}\right)^{Tq} (1)^{1-q}} \right] \\ &= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{3^{-q}}{\left(\frac{5}{6}\right)^{-q}} + \frac{2^{-q}}{\left(\frac{5}{6}\right)^{-q}} \right]^T \\ &= \frac{1}{1-q} \ln \left[\left(\frac{2}{5}\right)^q + \left(\frac{3}{5}\right)^q \right]. \qquad (9)\end{aligned}$$

For this next part, take this evaluation of the q-order expansion entropy on S' and calculate the results when $q = 0, \infty$.

Case 1: $q = 0$

$$H_{q=0}(f,S') = \ln \left[\left(\frac{2}{5}\right)^0 + \left(\frac{3}{5}\right)^0 \right] = \ln 2.$$

Case 2: $q = \infty$

$$\begin{aligned}
\lim_{q\to\infty} H_q(f,S') &= \lim_{q\to\infty} \frac{1}{1-q} \ln\left[\left(\frac{2}{5}\right)^q + \left(\frac{3}{5}\right)^q\right] \\
&= \lim_{q\to\infty} \frac{q}{1-q}\frac{1}{q} \ln\left[\left(\frac{2}{5}\right)^q + \left(\frac{3}{5}\right)^q\right] \\
&= \lim_{q\to\infty} \frac{q}{1-q} \lim_{q\to\infty} \frac{1}{q} \ln\left[\left(\frac{2}{5}\right)^q + \left(\frac{3}{5}\right)^q\right] \\
&= -\lim_{q\to\infty} \frac{1}{\left(\frac{2}{5}\right)^q + \left(\frac{3}{5}\right)^q} \left[\left(\frac{2}{5}\right)^q \ln\frac{2}{5} + \left(\frac{3}{5}\right)^q \ln\frac{3}{5}\right] \\
&= -\lim_{q\to\infty} \frac{\ln\frac{2}{5} + \left(\frac{3}{5}\right)^q \ln\frac{3}{5}}{1+\left(\frac{3}{2}\right)^q} \\
&= -\lim_{q\to\infty} \frac{\left(\frac{3}{2}\right)^{-q} \ln\frac{2}{5} + \ln\frac{3}{5}}{\left(\frac{3}{2}\right)^{-q} + 1} \\
&= -\ln\frac{3}{5} = \ln\frac{5}{3}.
\end{aligned}$$

Since H_q is monotonically decreasing, $\ln 2 = H_0 \geq H_q \geq H_\infty = \ln\frac{5}{3}$. Specifically, $H_q(f,S') \in [\ln 2, \ln\frac{5}{3}]$.

Next, calculate the expansion entropy of S. First note that $\mu(S_{T,0}) = 2.5$ since all points stay inside S for all $T \geq 0$ and $S = [-1, 1.5]$. Also, since $S' \subseteq S$, then:

$$(3^{-q} + 2^{-q})^T \leq \int_{S_{T,0}} G(Df^T)^{1-q} d\mu.$$

Note also that $G(Df^T) = 1$ for initial conditions on the interval near $x = -\frac{1}{2}$ (since these points have $Df < 1$). These initial conditions contribute at least $c > 0$ to the integral (where c is the length of the contracting interval). This means:

$$c + (3^{-q} + 2^{-q})^T \leq \int_{S_{T,0}} G(Df^T)^{1-q} d\mu.$$

There exists a constant C that is independent of T such that the contribution to the integral of $G(Df^T)^{1-q}$ from points in $S_{T,0} \setminus S'_{T,0}$ is bounded above by $CT \max[(3^{-q}+2^{-q})^T, 1]$. The T in this upper bound comes from the trajectories that initially leave S' at time $t =$

$0, ..., T-1$. Therefore, the following bounds exist for the integral of $G(Df^T)^{1-q}$ on $S_{T,0}$:

$$c + (3^{-q} + 2^{-q})^T \leq \int_{S_{T,0}} G(Df^T)^{1-q} d\mu \leq CT \max\left[(3^{-q} + 2^{-q})^T, 1\right].$$

This implies for $q \neq 1$:

$$\begin{aligned}
H_q(f,S) &= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{\int_{S_{T,0}} G(Df^T)^{1-q} d\mu}{\mu(S_{T,0})^q \mu(S)^{1-q}} \right] \\
&\leq \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{CT \max\left[(3^{-q} + 2^{-q})^T, 1\right]}{\mu(S_{T,0})^q \mu(S)^{1-q}} \right] \\
&= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{CT \max\left[(3^{-q} + 2^{-q})^T, 1\right]}{\mu(S)} \right] \\
&= \frac{1}{1-q} \lim_{T\to\infty} \frac{1}{T} \ln \left[\frac{C^{\frac{1}{T}} T^{\frac{1}{T}} \max\left[(3^{-q} + 2^{-q}), 1\right]}{\mu(S)^{\frac{1}{T}}} \right]^T \\
&= \frac{1}{1-q} \lim_{T\to\infty} \ln \left[\frac{C^{\frac{1}{T}} T^{\frac{1}{T}} \max\left[(3^{-q} + 2^{-q}), 1\right]}{\mu(S)^{\frac{1}{T}}} \right] \\
&= \frac{1}{1-q} \ln \lim_{T\to\infty} \left[\frac{C^{\frac{1}{T}} T^{\frac{1}{T}} \max\left[(3^{-q} + 2^{-q}), 1\right]}{\mu(S)^{\frac{1}{T}}} \right] \\
&= \frac{1}{1-q} \ln \left[\max\left(3^{-q} + 2^{-q}, 1\right)\right]. \qquad (10)
\end{aligned}$$

It is important to note that there exists a critical number $q_c \in (0,1)$ such that $3^{-q_c} + 2^{-q_c} = 1$. Then for $q > q_c$, $\max(3^{-q} + 2^{-q}, 1) = 1 \implies H_q(f,S) = 0$.

To summarize the results so far, according to Equations (8) and (9), $H_0(f,S') = H_0(f,S) = \ln 2$ (as were the results in Example 5.2), but $H_q(f,S') > H_q(f,S)$ for $q > 0$ (see Figure 9 of [1]). Lastly, note that if the slopes 3 and/or (-2) were increased, then the critical number q_c could be made arbitrarily close to 0. Therefore, this example demonstrates how H_q may not always detect chaos.

7. Topological Entropy

The notion of topological entropy was first introduced by Adler, Konheim, and McAndrews [14]; it applied to a continuous function f on a compact topological space X. Another definition of topological entropy was introduced if X is a metric space by Dinaburg and Bowen [15] as follows. Let $\epsilon > 0$ and $x, y \in X$. These points are (T, ϵ)-*separated* if $d(f^i(x), f^i(y)) > \epsilon$ for some $0 \leq i < T$. Another way to use this distance is to introduce the distance:

$$d_{T,f}(x, y) = \sup_{0 \leq j < T} d(f^j(x), f^j(y)).$$

A set $P \subseteq X$ with $\mu(P) < \infty$ is said to (T, ϵ)-*span* X if there does not exist a point in X that is (T, ϵ)-separated from every point in P. The minimum number of points needed to (T, ϵ)-span X is notated as $n(T, \epsilon)$. $N(T, \epsilon)$ represents the maximum number of points in X that are pairwise (T, ϵ)-separated. Define:

$$h_n(T, \epsilon) = \limsup_{T \to \infty} \frac{1}{T} \ln n(T, \epsilon)$$

and

$$h_N(T, \epsilon) = \limsup_{T \to \infty} \frac{1}{T} \ln N(T, \epsilon).$$

There is an interesting relationship between $N(T, \epsilon)$ and $n(T, \epsilon)$ as will be observed in the following theorem.

Theorem 7.1. *For $\epsilon > 0$ and $T \in \mathbb{N}$,*

$$N(T, 2\epsilon) \leq n(T, \epsilon) \leq N(T, \epsilon).$$

Proof. Let $E_N(T, \epsilon)$ be a maximal (T, ϵ)-separated set for X and let $x \in X$. There is some $y \in E_N(T, \epsilon)$ such that $d_{T,f}(x, y) \leq \epsilon$, because otherwise $E_N(T, \epsilon) \cup \{x\}$ would be a (T, ϵ)-separated set for X and $E_N(T, \epsilon)$ would not be maximal. Therefore $E_N(T, \epsilon)$ (T, ϵ)-spans X, and

$$N(T, \epsilon) = \#(E_N(T, \epsilon)) \geq n(T, \epsilon).$$

Let $E_N(T, 2\epsilon)$ be a maximal $(T, 2\epsilon)$-separated set for X, and $E_n(T, \epsilon)$ be a minimal (T, ϵ)-spanning set for X. Using the fact that $E_n(T, \epsilon)$ spans, we are going to define a map $T : E_N(T, 2\epsilon) \to E_n(T, \epsilon)$. For $x \in E_N(T, 2\epsilon)$ there is a $y = T(x) \in E_n(T, \epsilon)$ with $d_{T,f}(x, y) \leq \epsilon$. If $T(x_1) = T(x_2)$ for $x_1, x_2 \in E_N(T, 2\epsilon)$, then

$$d_{T,f}(x_1, x_2) \leq d_{T,f}(x_1, y) + d_{T,f}(y, x_2) \leq 2\epsilon.$$

Because $E_N(T, 2\epsilon)$ is a $(T, 2\epsilon)$-separated set, $x_1 = x_2$. This shows that T is one to one, and:

$$\begin{aligned} N(T, 2\epsilon) &= \#(E_N(T, 2\epsilon)) \\ &\leq \#(E_n(T, \epsilon)) \\ &= n(T, \epsilon). \end{aligned}$$

Therefore:

$$N(T, 2\epsilon) \leq n(T, \epsilon) \leq N(T, \epsilon).$$

□

It is important to note the results from the previous theorem because this means that $N(T, \epsilon)$ and $n(T, \epsilon)$ have the same limit as $\epsilon \to 0$. Therefore, *topological entropy* is defined as:

$$h(f, X) = \lim_{\epsilon \to 0} h_n(f, \epsilon) = \lim_{\epsilon \to 0} h_N(f, \epsilon).$$

Topological entropy and expansion entropy are actually the same when f is considered to be a smooth, autonomous system on a compact manifold M with the restraining region $S = M$. In order to verify this idea, first define $\widetilde{N}(T, \epsilon)$ as the maximum number of trajectories at either time 0 or T that are distance ϵ apart.

Lemma 7.2. *Let f be a smooth, autonomous system on M which is a compact manifold. Let the restraining region $S = M$. Then for $\epsilon > 0$,*

$$E_{T,0}(f, S) \approx \frac{\widetilde{N}(T, \epsilon)}{N(0, \epsilon)}.$$

Proof. Assume $\epsilon > 0$ is small enough that the remainder term in the first order Taylor expansion of $f_{T,0}$ is much smaller than ϵ for points within ϵ of each other, i.e.:

$$|f_{T,0}(y) - f_{T,0}(x) - Df_{T,0}(x)(y-x)| \ll \epsilon$$

for $x, y \in S$ with $|y - x| \leq \epsilon$.

Cover S with N_0 boxes whose diameters are ϵ. This means that N_0 has the same order of magnitude as the maximum number $N(0, \epsilon)$ of ϵ-separated points in S. Each box B is contained in a ball of radius ϵ, and contains a ball whose radius has the same order of magnitude as ϵ. Note that $\mu(B) \approx \frac{\mu(S)}{N_0}$ for small ϵ. Let χ_B be the center of B, and $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of $Df_{T,0}(\chi_B)$. Then $f_{T,0}(B)$ is contained in an ellipses whose semiaxes are $\sigma_1\epsilon, \cdots, \sigma_n\epsilon$. Also $f_{T,0}(B)$ contains an ellipses whose semiaxes have the same order of magnitude as $\sigma_1\epsilon, \cdots, \sigma_n\epsilon$.

Let d be the largest index such that $\sigma_d > 1$. Then the maximum number of ϵ-separated points in $f_{T,0}(B)$ has the same order of magnitude as $\sigma_1 \cdots \sigma_d = G(Df_{T,0}(\chi_B))$. Summing over all B, the maximum number $\widetilde{N}(T, \epsilon)$ of trajectories that are ϵ-separated at either time 0 or T has the same order of magnitude as:

$$\begin{aligned}
\sum_B G(Df_{T,0}(\chi_B)) &\approx \frac{1}{\mu(B)} \int_S G(Df_{T,0}(x)) d\mu(x) \\
&\approx \frac{N_0}{\mu(S)} \int_S G(Df_{T,0}(x)) d\mu(x) \\
&= N_0 E_{T,0}(f, S)
\end{aligned}$$

$$\implies E_{T,0}(f, S) \approx \frac{\widetilde{N}(T, \epsilon)}{N(0, \epsilon)}$$

□

The above lemma shows that:

$$E_{T,0}(f, S) = \lim_{\epsilon \to 0} \frac{\widetilde{N}(T, \epsilon)}{N(0, \epsilon)}.$$

Now we would like to revisit the claim that, under specific circumstances, topological entropy equals expansion entropy. In order to expand on the definition of topological entropy, notice that normalizing by $N(0, \epsilon)$ does not change the limit.

$$\begin{aligned} h(f, S) &= \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \ln N(T, \epsilon) \\ &= \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \ln[N(T, \epsilon) - \ln N(0, \epsilon)] \\ &= \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \ln \frac{N(T, \epsilon)}{N(0, \epsilon)}. \end{aligned}$$

It is important to note that $\widetilde{N}(t, \epsilon)$ is a lower bound of $N(T, \epsilon)$. Stated below are the definitions we have from our calculations of $H_0(f, S)$ and $h(f, S)$.

$$H_0(f, S) = \lim_{T \to \infty} \lim_{\epsilon \to 0} \frac{1}{T} \ln \frac{\widetilde{N}(T, \epsilon)}{N(0, \epsilon)}$$

and

$$h(f, S) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \ln \frac{N(T, \epsilon)}{N(0, \epsilon)}.$$

The difference between the definition of topological entropy and expansion entropy is that they take the same limits but in reverse order, and expansion entropy uses $\widetilde{N}(f, S) \leq N(f, S)$.

8. Other Definitions of Chaos

This section presents other definitions for chaos as well as some examples that show the limits of these definitions. A general idea that is often an indicator of chaos is called *sensitive dependence* (or sometimes "weak sensitive dependence"). This characteristic takes two nearby initial conditions and states that at some point in time, their orbits will be far apart.

Definition 8.1. *Let M be a compact metric space. A continuous map $f : M \to M$ has sensitive dependence if there exists $\rho > 0$ such that for all $\delta > 0$ and all $x \in M$, there exists $y \in M$ with $|x - y| < \delta$ such that $|f^t(x) - f^t(y)| > \rho$ where $t \geq 0$.*

This definition does not consider the rate of separation, which implies the definition incorrectly detects chaos if the rate of separation is linear (as in Example 5.4).

It is also possible to define sensitive dependence on a compact invariant set when the space is not necessarily compact.

Definition 8.2. *Let M be a metric space and $J \subseteq M$ be a compact invariant set. A continuous map $f : J \to J$ has sensitive dependence if there exists a $\rho > 0$ such that for every $\delta > 0$ and every $x \in J$, there exists $y \in J$ with $|x - y| < \delta$ such that $|f^t(x) - f^t(y)| > \rho$ where $t \geq 0$.*

The next definition was introduced by Devaney [16].

Definition 8.3. *(Devaney Chaos) Let M be a compact metric space. A continuous map $f : M \to M$ is chaotic if it satisfies the following conditions.*
(i) f has sensitive dependence on M,
(ii) f has dense periodic points in M,
(iii) f has a dense orbit (i.e., there exists an initial condition x^* such that for every $y \in M$ and every $\delta > 0$, $|f^t(x^*) - y| < \delta$ where $t \geq 0$).

It is also possible to define Devaney chaos on a compact invariant set $J = f(J)$ by replacing all the M's with J's in the above definition. Banks et al. [17] claimed that conditions (ii) and (iii) implied (i).

Devaney's definition does not detect chaos in some situations that most consider chaotic. For example, the following map has quasi-periodic forcing:

$$z_{t+1} = G(z_t, \theta_t), \quad \theta_{t+1} = [\theta + \omega] \mod 2\pi \tag{11}$$

where $\frac{\omega}{2\pi}$ is an irrational number. This is a dynamical system with state $x = (z, \theta)$. Since θ is quasi-periodic, there are no periodic points in the system, which implies this example fails condition (ii) of Devaney chaos. Therefore, according to Devaney's definition of chaos, quasi-periodic functions can never be chaotic, but quasi-periodic functions are of practical interest and can have a positive Lyapunov exponent on attractors.

Suppose that G does not depend on θ, i.e. $z_{t+1} = G(z_t)$. It is possible that z can satisfy the definition of Devaney chaos. By considering the state $x = (z, \theta)$ with θ still quasi-periodic, the system still does not satisfy condition (ii) of Devaney chaos even though the chaotic dynamics of z do not change.

Robinson [12] believes that in reference to Devaney's definition for chaos, the requirement of a dense set of periodic orbits does not appear "central to the idea of chaos." Therefore he (and another mathematician Wiggins [18]) proposed the following definition.

Definition 8.4. *(Robinson-Chaos) Let M be a compact metric space. A continuous map $f : M \to M$ is chaotic if it satisfies conditions (i) and (iii) of Devaney Chaos.*

Since Robinson's definition of chaos does not use condition (ii) of Devaney chaos, then it is possible that systems like (11) could be considered chaotic. The problem with Robinson's definition of chaos is seen with the example of the shear map on the torus discussed in Example 5.4. This example was considered by Robinson [12]. The results discussed earlier showed that orbits in this system were dense and nearby points had a linear rate of separation in time. Robinson's definition of chaos classifies this example as chaotic, but the two Lyapunov exponents of the system are zero. Therefore, according to the Lyapunov exponents, this system is not chaotic. Since the separation of points is linear, this example is considered nonchaotic (for further reading on strange nonchaotic attractors, see [9]).

The idea of a "scrambled set" was introduced by Li and Yorke [3] as a definition of chaos. This definition works well for one-dimensional maps (which are the examples they considered), but we will look at an example of higher dimensions where the definition will fail.

Definition 8.5. *Let M be a compact metric space and $J \subseteq M$ be uncountably infinite. Suppose that $f : M \to M$. J is a scrambled set if for all $x, y \in J$ with $x \neq y$,*

$$\limsup_{t\to\infty} |f^t(x) - f^t(y)| > 0, \quad \liminf_{t\to\infty} |f^t(x) - f^t(y)| = 0.$$

The first conclusion says that the distance between orbits of two distinct points will be larger than a fixed positive number infinitely many times. The second conclusion states that the distance between these distinct orbits will be arbitrarily close an infinite number of times.

One benefit to using the definition of a scrambled set is it takes some nonchaotic examples with sensitive dependence and classifies them as not chaotic. For example, consider the shear map on the torus from Example 5.4 again. Since the θ-distance between two orbits stays constant, the second condition of Li and York's definition is not satisfied. Therefore, the shear map does not have a scrambled set. Note that if the θ-distance is zero, then the ϕ-distance would supply the same result.

A limitation of the uncountable scrambled set definition is that it includes examples that are regarded generally as nonchaotic just as Robinson's definition of chaos. One example of this problem is considered by Robinson [12] and Ott and Yorke [19] in Figure 1 of both papers. The example is a two-dimensional flow with an attracting homoclinic orbit. A finite piece of a trajectory that converges to the homoclinic orbit forms an uncountable scrambled set. Therefore, the compact invariant set created by the homoclinic orbit (along with its interior) shows scrambling.

With the information provided so far in this section, it is clear that using definitions based on sensitive dependence are problematic when trying to provide a definition of chaos that is generally applicable. The remaining portion of this section will be discussing another widely used definition of chaos. It is quite beneficial to define a chaotic attractor using Lyapunov exponents. The problematic case would be using Milnor's definition [20] of an attractor (see below), so first is stated the better definition of a chaotic attractor.

Definition 8.6. *(Attractor of a Map) Let A be a bounded set with a dense orbit such that there exists an ϵ-neighborhood A_ϵ such that:*

$$\bigcap_{t=0}^{\infty} f^t(A_t) = A.$$

Then a chaotic attractor of f is an attractor with positive Lyapunov exponent.

Definition 8.7. *(Milnor's Definition of an Attractor) Let* $f : M \to M$ *be a map.* A *is an attractor of* f *if there exists a positive Lebesgue measure of points* $x \in M$ *such that* A *is the forward time limit of* A*.*

To see why Milnor's definition of a chaotic attractor can be problematic, consider the function $f(x)$ given in Figure 8 of [1]. This function is 0 at $x = \pm 1$ and remains 0 for $x \in (-\infty, -1] \cup [1, \infty)$. Let $x_0 \in [a, b]$, then $x_1 > 1$ and $x_i = 0$ for $i \geq 2$. Specifically, any point on $[a, b]$ (this interval has positive Lebesgue measure) will have a trajectory that remains at 0 after two iterations. Therefore, the unstable fixed point $x = 0$ is a Milnor attractor. Since $\frac{df}{dx} > 1$ at $x = 0$, then the Lyapunov exponent is positive, however, it seems odd to classify the set $x = 0$ as chaotic.

This example is theoretical as opposed to practical, so in general, it is typically sufficient to use the presence of a positive Lyapunov exponent as a definition of a chaotic attractor. The main issue with Lyapunov exponents is found in dealing with repellers. It is possible to have a fixed point that is a repeller with positive Lyapunov exponent, but that situation is not considered chaotic. If expansion entropy is used as the definition for chaos, then the fixed point repeller and Milnor example problems are resolved (see Example 5.1).

9. Conclusion

Expansion entropy possesses several desirable properties for defining chaos [Section 2-4] including being defined in a general manner so that it applies to many types of systems (i.e. autonomous, nonautonomous, discrete/continuous time, etc.). Some previous definitions of chaos required the use of an invariant set which can be difficult when the sets are unknown or do not exist, but expansion entropy only requires a bounded restraining region. Sections 6-8 discuss the limitations of these other definitions of chaos and prove that in those instances, expansion entropy detects chaos better than previous definitions of chaos.

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