Morita-Equivalence Between Strongly Non-Singular Rings and the Structure of the Maximal Ring of Quotients

by

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Abstract

This dissertation focuses on extending certain notions from Abelian group theory and module theory over integral domains to modules over non-commutative rings. In particular, we investigate generalizations of torsion-freeness and characterize rings for which torsion-freeness and non-singularity coincide under a Morita-equivalence. Here, a right R-module M is *non-singular* if xI is nonzero for every nonzero $x \in M$ and every essential right ideal I of R, and a right R-module M is *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for every $r \in R$. Incidentally, we find that this is related to characterizing rings for which the $n \times n$ matrix ring $Mat_n(R)$ is a Baer-ring. A ring is *Baer* if every right (or left) annihilator is generated by an idempotent. Strongly non-singular and semi-hereditary rings play a vital role, and we consider relevant examples and related results.

This leads to a discussion of divisible modules and two-sided submodules of the maximal ring of quotients Q. As with torsion-freeness, there are various notions of divisibility in the general setting, and we consider rings for which these various notions coincide. More specifically, we consider the structure of Q/R in the case that its projective dimension is ≤ 1 and R is a right and left duo domain. A ring R is a *right (left) duo ring* if $Ra \subseteq aR$ ($aR \subseteq Ra$) for every $a \in R$. In this setting, we find that h-divisibility and classical divisibility coincide, and Q/R can be decomposed into a direct sum of countably-generated two-sided R-submodules. We consider related results, as well as examples of such rings.

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Chapter 1

Introduction

The structure of modules over integral domains has been widely researched and has seen several advancements in recent years. This dates back to the 1930s with R. Baer's work in Abelian group theory, and since then several familiar notions from Abelian group theory have been extended to modules over integral domains. However, the degree to which this theory can be extended to general associative rings is not known. For instance, the classical notions of torsion-freeness and divisibility can be defined in several ways which are equivalent for modules over integral domains but not for modules over arbitrary rings. In this dissertation, we look to classify the rings for which some of these notions and results can be extended to non-commutative rings.

Throughout our discussions, the maximal ring of quotients will play a pivotal role. The theory of quotient rings has become an integral part of non-commutative ring theory. It has its origins in the 1930s with the development of the classical ring of fractions by Ø. Ore and K. Asano. The general theory, however, began seeing development in the 1950s through the work of Y. Utumi, A.W. Goldie, and several others. K.R. Goodearl and B. Stenström consolidated and expanded many of these results in the 1970s, and use of the maximal ring of quotients remains extensive in non-commutative ring theory to this day.

For an integral domain R, there exists a commutative ring Q^r containing R as a subring such that every non-zero element of R is a unit in Q^r. Moreover, every non-zero element in Q^r is of the form rs^{-1} for some $r, s \in R$. Here, rs^{-1} represents an equivalence class (r, s), where $(r, s) \sim (a, b)$ if and only if $rb = sa$ (see Chapter 9). The ring Q^r is unique and we will refer to it as the *classical right ring of quotients of* R. The *classical left ring of quotients* Q^l is similarly defined, and $Q^r = Q^l$ for an integral domain. A more general construction will be defined in Chapter 9, with units in Q being taken from a multiplicatively closed subset $S \subseteq R$ of non-zero divisors.

The formal construction of the classical ring of quotients can fail in the case that R is non-commutative, and such an over-ring may not exist, even in the case that R is a non-commutative domain. Moreover, the right and left ring of quotients of an arbitrary associative ring do not necessary coincide. It is well-known that a ring R has a classical right ring of quotients if it satisfies the *right Ore condition*: given $a, s \in R$ with s regular, there exists $b, t \in R$ with t regular such that $at = sb$. In Section 5.2, we consider a more general construction of the *maximal right ring of quotients*, which depends on finding an essential extension of R, as opposed to the classical construction of forming fractions. This construction results in a ring Q which coincides with the classical ring of quotients in the case that R is an integral domain.

The first part of the dissertation deals with extending the classical notion of torsion-freeness to the general setting. If R is an integral domain and M is an R-module, we define the *torsion submodule of* M to be $tM = \{x \in M \mid ann_r(x)$ contains some regular element of $R\}$, where $ann_r(x) = \{r \in R \mid xr = 0\}$ is the *right annihilator of* R and $r \in R$ is *regular* if it is not a right or left zero-divisor. We say that M is *torsion-free in the classical sense* if $tM = \{0\}$ and *torsion* if $tM = M$. Unfortunately, problems arise in the non-commutative setting since tM is not necessarily a submodule of M. There are various ways to extend the notion of torsion-freeness to the general setting. Following Hattori [18], we say that a right R-module M over a ring R is *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for every $r \in R$. This is based on homological properties of modules and coincides with the classical definition in the case R is commutative. Goodearl takes a different approach in [17] by considering the *singular submodule*

$$Z(M) = \{x \in M \mid xI = 0 \textit{ for some essential right ideal } I \textit{ of } R\}$$

of M. The module M is *singular* if $Z(M) = M$ and *non-singular* if $Z(M) = 0$. A ring R is *right non-singular* if it is non-singular as a right R-module. Determining when these two notions coincide is of great interest, and we look at relevant background information on torsion-freeness and non-singularity in Chapters 4 and 5.

In 2005, Professors U. Albrecht, J. Dauns, and L. Fuchs were able to classify the non-commutative rings for which torsion-freeness and non-singularity coincide. This significant development in module theory was published in the Journal of Algebra [3] along with related results and applications, and has led to several follow-up results. However, at the time, they were unable to classify the rings R for which the classes of torsion-free and non-singular right S-modules coincide for every ring S Morita-equivalent to R. Two rings are *Morita-equivalent* if their module categories are equivalent. One complication that arises is the fact that torsion-freeness is not preserved under a Morita-equivalence, whereas non-singularity is in fact a Morita-invariant property [13, Example 5.4].

It turns out that the question regarding coincidence of torsion-freeness and non-singularity under a Morita-equivalence is closely related to the problem of classifying the rings for which the $n \times n$ matrix ring $Mat_n(R)$ is a Baer-ring. A ring is a *Baer-ring* if every right (or left) annihilator ideal is generated by an idempotent. We are able to find necessary and sufficient conditions for a ring R so that $Mat_n(R)$ is a Baer-ring. Incidentally, these conditions also provide us with rings for which the classes of torsion-free and non-singular modules coincide under a Morita-equivalence. The characterization of these rings is provided in Theorem 6.5, which states that the classes of torsion-free and non-singular S-modules coincide for every ring S Morita equivalent to a ring R if and only if R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents.

A ring is *right strongly non-singular* if its maximal right ring of quotients is a perfect left localization. These rings will play a pivotal role throughout all of our discussions and will be explored in Section 5.2. Semi-hereditary rings will be defined and explored in Chapter 2, and we define Utumi rings in Section 5.3. In determining these conditions, we make use of the existence of a Morita-equivalence between R and $Mat_n(R)$ (Proposition 6.2), as well as the fact that $Mat_n(R)$ is isomorphic to the endomorphism ring of any free right R-module with basis $\{x_i\}_{i=1}^n$ (Lemma 2.6). The *endomorphism ring* $End_R(M)$ of a right R-module M is the set of all R-homomorphisms $f : M \to M$, which is a ring under point-wise addition and composition of functions.

After classifying the rings for which torsion-freeness and non-singularity coincide under a Morita-equivalence, we continue our discussion of strongly non-singular, semi-hereditary rings. In particular, we consider how these rings are related to two-sided essential submodules of Q^r. In the case that R is right strongly non-singular, right semi-hereditary with finite Goldie-dimension, [2] provides some information about direct summands of A^n whenever A is a two-sided essential submodule of Q^r. Moreover, we show that in this case every epimorphism $A^n \to A \to 0$ splits. This leads to a discussion of direct summands of Q/R, which is in part motivated by the integral domain case.

An integral domain R is called a *Matlis domain* if the projective dimension of its maximal ring of quotients Q is at most 1. For a Matlis domain, we can find a direct sum decomposition of $(Q/R)_R$ into countably generated summands [15]. We extend this result to the general setting and find that several complications arise. One of the primary obstacles relates to the set $R^\times$ of regular elements of R. In the general setting, we find that the localization R_S/R_T over two submonoids $T \subset S$ is not necessarily countably generated even in the case that S is countably generated over T. To overcome this difficulty, we introduce a filtration on $R^\times$ similar to the third axiom of countability developed by P. Griffith and P. Hill [19]. Furthermore, we extend the notion of a normal series of subgroups to provide a normal series of submonoids in our filtration. We discuss duo rings and localizations in Chapter 9, and our filtration is discussed in Chapter 10.

In constructing our filtration, we have to be careful in ensuring that we have suitable chains of direct summands. In particular, we must make sure that the projective dimension of the summands does not surpass that of Q, and we must ensure that each summand is countably generated. Several examples are provided. In particular, we provide a ring for which Q/R cannot be decomposed into a direct sum with countably generated factors even though it has projective dimension ≤ 1. This particular ring does not have our desired filtration. We discuss projective dimension, and resolve some general issues regarding projectivity of certain submodules of Q in Section 9.2. Divisible modules will also play a role in our discussion and main theorem, and we find that these modules have a nice connection to torsion-free modules (see Chapter 8). There are several notions of divisibility in the general setting, one of which is dual to Hattori's definition of torsion-freeness. Moreover, as with torsion-freeness, we are interested in determining when the various notions of divisibility coincide. We find that this is closely related to our generalization of Matlis domains (see Theorem 10.12).

Unless noted otherwise, all rings are assumed to be associative with unit and are not necessarily commutative. The term *domain* will refer to a ring that does not contain zero divisors and is not necessarily commutative, while *integral domain* will be used for commutative domains.

Chapter 2

Semi-hereditary Rings and p.p.-rings

We begin by looking at projective modules. A right R-module P is *projective* if given right R-modules A and B, an epimorphism $\pi : A \to B$, and a homomorphism $\varphi : P \to$ B, there exists a homomorphism $\psi :$ P $\to$ A such that $\pi\psi = \varphi$. In other words, the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\psi"'] \arrow[d, "\varphi"] & \\ A \arrow[r, "\pi"'] & B \arrow[r] & 0 \end{tikzcd}$$

In particular, every free right R-module is projective [26, Theorem 3.1]. We make use of the following well-known characterization of projective modules:

Theorem 2.1. *[26] Let R be a ring. The following are equivalent for a right R-module P:*

(a) P is projective

(b) P is isomorphic to a direct summand of a free right R-module. In other words, there is a free right R-module $F = Q \bigoplus N$, where N is a right R-module and $Q \cong P$.

(c) For any right R-module M and epimorphism $\varphi : M \to P$, $M = \ker(\varphi) \bigoplus N$.

Let Mod_R be the category of all right R-modules for a ring R. A *complex* in Mod_R is a sequence of right R-modules and R-homomorphisms in Mod_R,

$$... \to A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \to ...$$

such that $\alpha_{k+1}\alpha_k = 0$ for every $k \in \mathbb{Z}$. Observe $\alpha_{k+1}\alpha_k = 0$ implies that $im(\alpha_{k+1}) \subseteq \ker(\alpha_k)$. The sequence is called *exact* if $im(\alpha_{k+1}) = \ker(\alpha_k)$ for every $k \in \mathbb{Z}$. An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules is referred to as a *short exact sequence*. Such an exact sequence is said to *split* if there exists an R-homomorphism $\gamma : C \to B$ such that $\beta\gamma = 1_C$, where 1_C is the identity map on C.

Lemma 2.2. *[26] Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be a sequence of right R-modules. If this sequence is split exact, then* $B \cong A \bigoplus C$*.*

Proof. If the exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules splits, then there exists an R-homomorphism $\gamma : C \to B$ such that $\beta\gamma \cong 1_C$. Observe that since α is a monomorphism, $im(\alpha) \cong A$. Moreover, if $x \in \ker(\gamma)$, then $\gamma(x) = 0$. However, $\beta(0) = \beta\gamma(x) = x$ since $\beta\gamma = 1_C$. Thus, $x = 0$ and γ is also a monomorphism. Hence, $im(\beta) \cong C$. Therefore, to show that $B \cong A \bigoplus C$, it suffices to show that $B \cong im(\alpha) \bigoplus im(\gamma)$.

Let $b \in B$. Then $\beta(b) \in C$ and $\gamma\beta(b) \in im(\gamma)$. Furthermore, $b-\gamma\beta(b) \in \ker(\beta) = im(\alpha)$ since $\beta(b - \gamma\beta(b)) = \beta(b) - \beta\gamma\beta(b) = \beta(b) - \beta(b) = 0$. Hence, $b = [b - \gamma\beta(b)] + \gamma\beta(b) \in im(\alpha) + im(\gamma)$. Suppose, $x \in im(\alpha) \cap im(\gamma)$. Then, there exists some $a \in A$ such that $\alpha(a) = x$, and there exists some $c \in C$ such that $\gamma(c) = x$. Now, $\alpha(a) \in im(\alpha) = \ker(\beta)$, which implies $\beta(x) = \beta\alpha(a) = 0$. However, it is also the case that $\beta(x) = \beta\gamma(c) = c$. Hence, $c = 0$ and it follows that $x = \gamma(c) = \gamma(0) = 0$. Thus, $im(\alpha) \cap im(\gamma) = 0$. Therefore, $B \cong im(\alpha) \bigoplus im(\gamma) \cong A \bigoplus C$. □

Proposition 2.3. *[26] The following are equivalent for a right R-module P:*

(a) P is projective.

(b) The sequence $0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\mathrm{Hom}_R(P,\varphi)} \mathrm{Hom}_R(P, B) \xrightarrow{\mathrm{Hom}_R(P,\psi)} \mathrm{Hom}_R(P, C) \to 0$ *is exact whenever* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is a an exact sequence of right R-modules.*

Proof. $(a) \Rightarrow (b)$: Suppose P is projective. Observe that the functor $\mathrm{Hom}_R(P, \_)$ is left exact [26, Theorem 2.38]. Thus, if $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is exact, then

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\mathrm{Hom}_R(P,\varphi)} \mathrm{Hom}_R(P, B) \xrightarrow{\mathrm{Hom}_R(P,\psi)} \mathrm{Hom}_R(P, C)$$

is exact. Therefore, it remains to be shown that $\mathrm{Hom}_R(P, \psi)$ is an epimorphism. Let $\alpha \in \mathrm{Hom}_R(P, C)$. Since P is projective, there exists a homomorphism $\beta : P \to B$ such that $\alpha = \psi\beta$. Hence, $\mathrm{Hom}_R(P, \psi)(\beta) = \psi\beta = \alpha$. Therefore, $\mathrm{Hom}_R(P, \psi)$ is an epimorphism.

$(b) \Rightarrow (a)$: Let P be a right R-module and assume exactness of $0 \to A \overset{\varphi}{\to} B \overset{\psi}{\to} C \to 0$ implies exactness of $0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\mathrm{Hom}_R(P,\varphi)} \mathrm{Hom}_R(P, B) \xrightarrow{\mathrm{Hom}_R(P,\psi)} \mathrm{Hom}_R(P, C) \to 0$. This implies $\mathrm{Hom}_R(P, \psi)$ is an epimorphism. Thus, if $\alpha \in \mathrm{Hom}_R(P, \psi)$, then there exists some $\beta \in \mathrm{Hom}_R(P, B)$ such that $\mathrm{Hom}_R(P, \psi)(\beta) = \psi\beta = \alpha$. That is, given an epimorphism $\psi : B \to C$ and a homomorphism $\alpha : P \to C$, there exists a homomorphism $\beta : P \to B$ such that $\alpha = \psi\beta$. Therefore, P is projective. $\square$

A ring R is a *right p.p.-ring* if every principal right ideal is projective as a right R-module. A ring R is *right semi-hereditary* if every finitely generated right ideal is projective as a right R-module. For a right R-module M and any subset $S \subseteq M$, define the *right annihilator of* S in R as $ann_r(S) = \{r \in \mathrm{R} \mid xr = 0 \text{ for every } x \in S\}$. The right annihilator of S is a right ideal of R. Similarly, the *left annihilator of* S in R can be defined for a left R-module M as $ann_l(S) = \{r \in \mathrm{R} \mid rx = 0 \text{ for every } x \in S\}$. The left annihilator of S is a left ideal of R. The following proposition shows that right p.p.-rings can be defined in terms of annihilators of elements and idempotents, where an idempotent is an element $e \in R$ such that $e^2 = e$.

Proposition 2.4. *A ring R is a right p.p.-ring if and only if for every $x \in R$ there exists some idempotent $e \in R$ such that $ann_r(x) = eR$.*

Proof. For $x \in R$, consider the function $f_x : R \to xR$ given by $r \mapsto xr$. This is a well-defined epimorphism. Then R is a right p.p.-ring if and only if the principal right ideal xR is projective for for every $x \in R$ if and only if $\ker(f_x)$ is a direct summand of R for every $x \in R$. Observe that for each $x \in R$, $\ker(f_x) = ann_r(x)$. Hence, R is a right p.p.-ring if and only if $ann_r(x)$ is a direct summand of R. Note that every direct summand of R is generated by an idempotent since $R \cong eR \bigoplus (1 - e)R$ for any idempotent $e \in R$. Thus, as a direct summand, $ann_r(x) = eR$ for some idempotent $e \in R$. Therefore, R is a right p.p.-ring if and only if for every $x \in R$ there is some idempotent $e \in R$ such that $ann_r(x) = eR$. $\square$

Let $Mat_n(R)$ denote the set of all $n \times n$ matrices with entries in R. Under standard matrix addition and multiplication, $Mat_n(R)$ is a ring. A useful characterization of semi-hereditary rings is that such rings are precisely those for which $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$. To show this, the following two lemmas will be needed:

Lemma 2.5. *[26] A ring R is right semi-hereditary if and only if every finitely generated submodule U of a projective right R-module P is projective.*

Proof. Suppose R is right semi-hereditary and let U be a submodule of a projective right R-module P. By Theorem 2.1, $P \bigoplus N$ is free for some right R-module N. Hence, P is a submodule of a free module, and it follows that any submodule of P is also a submodule of a free module. Thus, without loss of generality, it can be assumed that P is a free right R-module. Moreover, since U is finitely generated, it can be assumed that P is finitely generated with basis $X = \{x_1, x_2, ..., x_n\}$ for some $0 < n < \omega$.

Inductively, it will be shown that U is a finite direct sum of finitely generated right ideals. If $n = 1$, then $P = x_1R \cong R$. Since submodules of the right R-module R are right ideals, U is a finitely generated right ideal. Suppose $n > 1$ and assume U is a finite direct sum of finitely generated right ideals for $k < n$. Let $V = U \cap (x_1R + x_2R + ... + x_{n-1}R)$. Then, V is a finitely generated submodule of a free right R-module with basis $\{x_1, x_2, ..., x_{n-1}\}$. By assumption, V is a finite direct sum of finitely generated right ideals. Note that if $u \in U$, then $u = v + x_nr$ with $v \in V$ and $r \in R$. This expression for u is unique since X is a linearly independent spanning set. Thus, the map $\varphi : U \to R$ defined by $\varphi(u) = \varphi(v + x_nr) = r$ is a well-defined homomorphism.

Now, $im(\varphi)$ is a finitely generated right ideal of R since it is the epimorphic image of the finitely generated right R-module U. Hence, $im(\varphi)$ is projective since R is right semi-hereditary. Consider the short exact sequence $0 \to K \xrightarrow{\iota} U \xrightarrow{\varphi} im(\varphi) \to 0$, where

$K = \ker \varphi$ and ι is the inclusion map. This sequence splits since $im(\varphi)$ is projective, and thus $U \cong K \bigoplus im(\varphi)$ by Lemma 2.2. Hence, U is a finite direct sum of finitely generated right ideals since both K and $im(\varphi)$ are finitely generated right ideals. Since R is right semi-hereditary, each of these right ideals is projective. Therefore, U is projective as the direct sum of projective right ideals.

Conversely, suppose that if P is a projective right R-module, then every finitely generated submodule U of P is projective. Let I be a finitely generated right ideal of R. Note that R is a free right R-module and thus projective. Hence, I is a finitely generated submodule of R, and by assumption I is projective. Therefore, R is right semi-hereditary. $\square$

Lemma 2.6. *Let R be a ring, and F a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$ for $0 < n < \omega$. Then, $Mat_n(R) \cong End_R(F)$.*

Proof. Let $S = End_R(F)$ and take $f \in S$. Then, $f(x_k) \in F$ for each $k = 1, 2, ..., n$. Hence, $f(x_k)$ is of the form $\sum_{i=1}^{n} x_i a_{ik}$, where $a_{ik} \in R$ for every i and every k. Let $A = \{a_{ik}\}$ be the $n \times n$ matrix whose i-kth entry is a_{ik}, and let $\varphi : S \to Mat_n(R)$ be defined by $f \mapsto A$. If $f, g \in S$ are such that $f = g$, then $f(x_k) = g(x_k)$ for every $k = 1, 2, ..., n$. Hence, φ is well-defined. Furthermore, if $f(x_k) = \sum_{i=1}^{n} x_i a_{ik}$ and $g(x_k) = \sum_{i=1}^{n} x_i b_{ik}$ for $k = 1, 2, ..., n$, then $(f + g)(x_k) = f(x_k) + g(x_k) = \sum_{i=1}^{n} x_i(a_{ik} + b_{ik})$. Thus, if $A = \{a_{ik}\}$ and $B = \{b_{ik}\}$ are the $n \times n$ matrices with entries determined by f and g respectively, then $A + B = \{a_{ik} + b_{ik}\}$ is the $n \times n$ matrix with entries determined by $f + g$. Hence, $\varphi(f + g) = A + B = \varphi f + \varphi g$.

To see that φ is a ring homomorphism, it remains to be seen that $\varphi(fg) = \varphi(f)\varphi(g) = AB$. In other words, it needs to be shown that the entries of the matrix AB are determined by $fg(x_j)$ for $j = 1, 2, ..., n$. Observe that if $A = \{a_{ik}\}$ and $B = \{b_{ik}\}$ are $n \times n$ matrices, then under standard matrix multiplication AB is the $n \times n$ matrix whose i-jth entry is $\sum_{k=1}^{n} a_{ik} b_{kj}$. This is indeed the matrix determined by the endomorphism fg since the following holds:

$$fg(x_j) = f(\sum_{k=1}^{n} x_k b_{kj}) = \sum_{k=1}^{n} f(x_k) b_{kj} = \sum_{k=1}^{n} \sum_{i=1}^{n} x_i a_{ik} b_{kj} = \sum_{i=1}^{n} x_i \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Finally, note that if $A = \{a_{ik}\} \in Mat_n(R)$, then $\sum_{i=1}^{n} x_i a_{ik} \in F$ and $\hat{f} : x_j \mapsto \sum_{i=1}^{n} x_i a_{ik}$ is an R-homomorphism from $\{x_i\}_{i=1}^n$ into F. This can be extended to an endomorphism $f \in F$. It readily follows that $\psi : Mat_n(R) \to S$ defined by $\{a_{ik}\} \mapsto f$ is a well-defined ring homomorphism. Moreover, $\varphi\psi(\{a_{ik}\}) = \varphi(f) = \{a_{ik}\}$ and $\psi\varphi(f) = \psi(\{a_{ik}\}) = f$. Thus, φ and ψ are inverses, and therefore φ is an isomorphism between $S = End_R(F)$ and $Mat_n(R)$. □

Theorem 2.7. *[11] A ring R is right semi-hereditary if and only if $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$.*

Proof. Suppose R is right semi-hereditary. For $0 < n < \omega$, let F be a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$. By Lemma 2.6, $Mat_n(R) \cong End_R(F)$. Therefore, it suffices to show that $S = End_R(F)$ is a right p.p.-ring. Take $s \in S$. Since F is finitely-generated, sF is a finitely generated submodule of F. Free modules are projective, and thus sF is projective by Lemma 2.5. Since sF is an epimorphic image of F, Theorem 2.1 shows that $F \cong \ker s \bigoplus N$ for some right R-module N. Thus, $\ker s = eF$ for some nonzero idempotent $e \in S$. Suppose $r \in ann_r(s) = \{t \in S \mid st(f) = 0 \text{ for every } f \in F\}$. Then, $sr = 0$ and $r \in \ker s = eF \subseteq eS$. On the other hand, suppose $et \in eS$. Since $sef = 0$ for every $f \in F$, $set(f) = 0$ for every $f \in F$. Hence, $et \in ann_r(s)$. Therefore, $ann_r(s) = eS$ and $S = End_R(F) \cong Mat_n(R)$ is a right p.p.-ring.

Suppose $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$. Let I be a finitely generated right ideal of R with generating set $\{a_1, a_2, ..., a_k\}$, and take F to be a free right R-module with basis $\{x_1, x_2, ..., x_k\}$. Note that there exists a submodule K of F which is isomorphic to I. Hence, K is also generated by k elements, say $b_1, b_2, ..., b_k$. Let $S = Mat_k(R) \cong End_R(F)$. For any $f \in F$, there exists $r_1, r_2, ..., r_k \in R$ such that $f = x_1r_1 + x_2r_2 + ... + x_kr_k$. Let $s \in S$ be the well-defined homomorphism defined by $s(f) = s(x_1r_1 + x_2r_2 + ... + x_nr_n) = b_1r_1 + b_2r_2 + ... + b_nr_k$. Note that $im(s) = K$ and thus $s : F \to K$ is an epimorphism.

It will now be shown that $\ker(s) = ann_r(s)F$. Here, as before, $ann_r(s)$ refers to the annihilator in S. If $y = \sum_{i=1}^{n} t_i f_i \in ann_r(s)F$, then $st_i f_i = 0$ for every $i = 1, 2, ..., n$. Hence, $y \in \ker(s)$. On the other hand, let $f \in \ker(s)$. Now, fR is a submodule of F, and so we can find some $t \in S$ such that $t : F \to fR$ is an epimorphism and $tf = f$. Then, for any $x \in F$, $s[t(x)] = s(fr)$ for some $r \in R$. However, $s(fr) = (sf)r = 0$. Thus, $t \in ann_r(s)$ and $f = tf \in ann_r(s)F$. Therefore, $\ker(s) = ann_r(s)F$. Moreover, since $Mat_k(R) \cong End_R(F)$ is a right p.p.-ring by assumption, $ann_r(s) = eS$ for some idempotent $e \in S$. Observe that $SF = F$ since $\sum_{i=1}^{n} s_i f_i \in F$ for $s_i \in S$ and $f_i \in F$, and $f = 1_F(f) \in SF$ for any $f \in F$. Hence, $\ker(s) = ann_r(s)F = eSF = eF$. Thus, $\ker(s)$ is a direct summand of F. It then follows from Theorem 2.1 that $I \cong K$ is projective since $s : F \to K$ is a an epimorphism. Therefore, R is a right semi-hereditary ring. □

Two idempotents e and f are called *orthogonal* if $ef = 0$ and $fe = 0$. If R contains only finite sets of orthogonal idempotents, then being a p.p.-ring is right-left-symmetric. Moreover, if R is a right (or left) p.p.-ring not containing an infinite set of orthogonal idempotents, then it satisfies both the ascending and descending chain conditions on annihilators (Theorem 2.11). A ring R satisfies the *ascending chain condition* on annihilators if given any ascending chain $I_0 \subseteq I_1 \subseteq ... \subseteq I_n \subseteq ...$ of annihilators, there exists some $k < \omega$ such that $I_n = I_k$ for every $n \geq k$. Similarly, R satisfies the *descending chain condition* on annihilators if every descending chain of annihilators terminates for some $k < \omega$. Before proving Theorem 2.11, we look at some basic results regarding annihilators and the chain conditions.

Lemma 2.8. *Let S and T be subsets of a ring R such that $S \subseteq T$. Then, $ann_r(T) \subseteq ann_r(S)$ and $ann_l(T) \subseteq ann_l(S)$.*

Proof. For $r \in ann_r(T)$ and $t \in T$, $tr = 0$. Let $s \in S \subseteq T$. Then, $sr = 0$ and hence $r \in ann_r(S)$. Thus, $ann_r(T) \subseteq ann_r(S)$. A similar computation shows the theorem holds for left annihilators. □

Lemma 2.9. *Let U be a subset of a ring R, and let $A = ann_r(U) = \{r \in R \mid ur = 0$ for every $u \in U\}$. Then, $ann_r(ann_l(A)) = A$.*

Proof. Suppose $r \in ann_r(ann_l(A))$, and let $u \in U$. Then, $ua = 0$ for every $a \in A$. Hence, $u \in ann_l(A)$, and thus $ur = 0$. Therefore, $ann_r(ann_l(A)) \subseteq A$. Conversely, suppose $a \in A$. Then, $ba = 0$ for every $b \in ann_l(A)$. Hence, $a \in ann_r(ann_l(A))$. Therefore, $A \subseteq ann_r(ann_l(A))$. □

Lemma 2.10. *R satisfies the ascending chain condition on right annihilators if and only if R satisfies the descending chain condition on left annihilators.*

Proof. Suppose R satisfies the ascending chain condition on right annihilators. Let $ann_l(U_1) \supseteq ann_l(U_2) \supseteq ...$ be a descending chain of left annihilators. Note that if $ann_l(U_i) \supseteq ann_l(U_j)$, then $ann_r(ann_l(U_1)) \subseteq ann_r(ann_l(U_2)) \subseteq ...$ is an ascending chain of right annihilators by Lemma 2.8. By the ascending chain condition on right annihilators, there is some $k < \omega$ such that $ann_r(ann_l(U_n)) = ann_r(ann_l(U_k))$ for every $n \geq k$. Therefore, $ann_l(ann_r(ann_l(U_n))) = ann_l(ann_r(ann_l(U_k)))$ for every $n \geq k$, and by a symmetric version of Lemma 2.9 it follows that $ann_l(U_n) = ann_l(U_n)$ for every $n \geq k$. A similar argument shows that the descending chain condition on left annihilators implies the ascending chain condition for right annihilators. □

Theorem 2.11. *[11] Let R be a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Then R is also a left p.p.-ring, every right or left annihilator in R is generated by an idempotent, and R satisfies both the ascending and descending chain condition for right annihilators.*

Proof. Let $A = ann_r(U)$ for some subset U of R and consider B = $ann_l(A)$. Suppose B contains nonzero orthogonal idempotents $e_1, ..., e_n$, and let e = $e_1 + ... + e_n$. Note that e is also an idempotent since $e^2 = (e_1 + ... + e_n)(e_1 + ... + e_n) = e_1^2 + ... + e_n^2 + e_1e_2 + ... + e_{n-1}e_n = e_1 + ... + e_n = e$. Suppose $B = Re$. The claim is that $A = (1 - e)R$, and hence A is generated by an idempotent. To see this, first note that $ann_r(B) = ann_r(ann_l(A)) = A$ by Lemma 2.9. Thus, it needs to be shown that $ann_r(B) = (1-e)R$. If $b \in B = Re$, then $b = se$ for some $s \in R$. For all $r \in R$, we obtain $b(1-e)r = se(1-e)r = (se-se^2)r = (se-se)r = 0$. Hence, $(1-e)R \subseteq ann_r(B)$. On the other hand, suppose $r \in ann_r(B)$. Then, $r = r - er + er = (1-e)r + er$. Note that $e \in B = ann_l(A)$, and so $er = 0$ since $r \in ann_r(B) = A$. Thus, $r = (1-e)r \in (1-e)R$, and hence $ann_r(B) \subseteq (1-e)R$. Therefore, if $B = Re$, then A is generated by an idempotent.

If $B \neq Re$, then select $b \in B \backslash Re$, and observe $ba = 0$ for every $a \in A$ since $b \neq re$ for any $r \in R$. Therefore, $B \neq Be$, which implies $B(1-e) \neq 0$. Let $0 \neq y \in B(1-e)$, say $y = s(1-e)$ for some $s \in B$. Since R is a right p.p.-ring, $ann_r(y) = (1-f)R$ for some idempotent $f \in R$. Observe that f is nonzero. For otherwise, $ann_r(y) = R$ and $y = 0$, which is a contradiction. If $0 \neq a \in A$, then $ya = s(1-e)a = sa - sea = 0 - s \cdot 0 = 0$. Thus, $a \in ann_r(y) = (1-f)R$, and so $A \subseteq (1-f)R$. Hence, $fA \subseteq f(1-f)R = 0$ and $f \in ann_l(A) = B$. Observe that $e \in ann_r(y) = (1-f)R$ since $ye = s(1-e)e = 0$, and so $e = (1-f)t$ for some $t \in R$. Thus, $(1-f)e = (1-f)(1-f)t = (1-f)t = e$, and so $fe = f(1-f)t = (f-f^2)t = 0$. Note also that $fe_i = 0$ for $i = 1, ..., n$, since $ye_i = s(1-e)e_i = s(e_i - ee_i) = s(e_i - e_i) = 0$ and hence $e_i \in ann_r(y)$.

Let $e_{n+1} = (1-e)f = f - ef$. Note e_{n+1} is an idempotent since $fe = 0$ and thus $(f-ef)(f-ef) = f - fef - ef + efef = f - 0 - ef + 0 = f - ef$. Consider e_i for some $i = 1, ..., n$. Then, $e_{n+1}e_i = (1-e)fe_i = (1-e) \cdot 0 = 0$, and $e_ie_{n+1} = e_i(1-e)f = (e_i - e_ie)f = (e_i - e_i)f = 0 \cdot f = 0$. Thus, e_{n+1} is orthogonal to $e_1, ..., e_n$. Furthermore, e_{n+1} is nonzero, since otherwise we have $f = ef$. This would imply $f = f^2 = efef = e \cdot 0 \cdot f = 0$, which is a contradiction. Note also that $e_{n+1} \in B$ since both e and f are in B.

Then, $e_1, ..., e_n, e_{n+1}$ are nonzero orthogonal idempotents contained in B. As before, if $e = e_1 + ... + e_{n+1}$ and $B \neq Re$, then there is a nonzero idempotent $e_{n+2} \in B$ orthogonal to $e_1, ..., e_{n+1}$. Since R does not contain any infinite set of orthogonal idempotents, this process must stop for $e_1, ..., e_k$. Thus, for $e = e_1 + ... + e_k$, $B = Re$ and $A = (1-e)R$. Therefore, each right and left annihilator is generated by an idempotent. From a symmetric version of Proposition 2.4, it follows that R is a left p.p.-ring.

Finally, it needs to be shown that R satisfies the ascending and descending chain conditions for right annihilators. Let $C \subseteq D$ be right annihilators. Then, there are idempotents e and f such that $C = eR$ and $D = fR$. Hence, $eR \subseteq fR$, and it follows that $e = fe$. Thus, $g = f - ef$ is a nonzero idempotent. Furthermore, g and e are orthogonal, since $eg = e(f - ef) = ef - e^2 f = ef - ef = 0$ and $ge = (f - ef)e = fe - efe = e - e^2 = 0$. Note that $fR = eR + gR$. For, if $er + gs \in eR + gR$, then $er + gs = er + (f - ef)s = er + fs + efs \in fR$, and conversely, if $fr \in fR$, then $fr = (f + ef - ef)r = efr + (f - ef)r = efr - gr \in eR + gR$.

Let $I_1 \subseteq I_2 \subseteq ...$ be a chain of right annihilators. Then, for $I_1 \subseteq I_2$, there are idempotents e and f such that $I_1 = eR$ and $I_2 = fR$, and there is an idempotent g orthogonal to e such that $I_2 = I_1 + gR$. It then follows that $I_3 = I_1 + gR + hR$ for some idempotent h orthogonal to both e and g. Since R does not contain an infinite set of orthogonal idempotents, this must terminate with some $k < \omega$ so that $I_n = I_k$ for every $n \geq k$. Therefore, R satisfies the ascending chain condition on right annihilators. The descending chain condition on right annihilators follows from Lemma 2.10. □

Chapter 3

Homological Algebra

Before discussing torsion-freeness and non-singularity of modules, we need some basic results in Homological Algebra regarding tensor products, flat modules, and functors.

3.1 Tensor Products

Let A be a right R-module, B a left R-module, and G any Abelian group. A function $f : A \times B \to G$ is called *R-biadditive*, or *R-bilinear*, if the following conditions are satisfied:

(i) For each $a, a' \in A$ and $b \in B$, $f(a + a', b) = f(a, b) + f(a', b)$,

(ii) For each $a \in A$ and $b, b' \in B$, $f(a, b + b') = f(a, b) + f(a, b')$,

(iii) For each $a \in A$, $b \in B$, and $r \in R$, $f(ar, b) = f(a, rb)$.

Note that in general $f(a + a', b + b') \neq f(a, b) + f(a', b')$. The *tensor product* of A and B, denoted $A \bigotimes_R B$, is an Abelian group and an R-biadditive function $h : A \times B \to A \bigotimes_R B$ having the universal property that whenever G is an Abelian group and $g : A \times B \to G$ is R-biadditive, there is a unique map $f : A \bigotimes_R B \to G$ such that $g = fh$.

Proposition 3.1. *[26] Let R be a ring. Given a right R-module A and a left R-module B, the tensor product $A \bigotimes_R B$ exists.*

Proof. Let F be a free Abelian group with basis $A \times B$, and let U be a subgroup of F generated by all elements of the form $(a + a', b) - (a, b) - (a', b)$, $(a, b + b') - (a, b) - (a, b')$, or $(ar, b) - (a, rb)$, where $a, a' \in A$, $b, b' \in B$, and $r \in R$. Define $A \bigotimes_R B$ to be F/U, and denote $(a, b) + U \in F/U$ as $a \otimes b$. In addition, let $h : A \times B \to A \bigotimes_R B$ be defined by

$(a, b) \mapsto a \otimes b$. Observe that h is a well-defined R-biadditive map. For if $a, a' \in A$ and $b \in B$, then $h(a + a', b) = (a + a', b) + U = (a + a', b) - [(a + a', b) - (a, b) - (a', b)] + U = [(a, b) + U] + [(a', b) + U] = h(a, b) + h(a', b)$. Similarly, $h(a, b + b') = h(a, b) + h(a, b')$ for $b, b' \in B$, and $h(ar, b) = (ar, b) + U = (ar, b) - [(ar, b) - (a, rb)] + U = (a, rb) + U = h(a, rb)$ for $r \in R$.

Let G be any Abelian group and $g : A \times B \to G$ any R-biadditive map. For F/U to be a tensor product, it needs to be shown that there is a function $\varphi : A \bigotimes_R B = F/U \to G$ such that $g = \varphi h$. Define $\hat{f} : A \times B \to G$ by $(a, b) \mapsto g(a, b)$. Each element of F is of the form $\sum_{A\times B} (a, b)n_{(a,b)}$, where $n_{(a,b)} = 0$ for all but finitely many $(a, b) \in A \times B$. Let f be defined by $\sum_{A\times B} (a, b)n_{(a,b)} \mapsto \sum_{A\times B} \hat{f}[(a, b)]n_{(a,b)}$. This is clearly well-defined since $\hat{f}$ is well-defined. Moreover, $f[(a, b)] = \hat{f}[(a, b)]$ for $(a, b) \in A \times B$, and thus f extends $\hat{f}$ to a function on F. Note that if k is another extension of $\hat{f}$, then k must equal f since they are equal on the generating set $A \times B$. Hence, f is a unique extension. Also observe that f is a homomorphism since, given $x, y \in F$, $f(x + y) = f(\sum_{A\times B} (a, b)n_{(a,b)} + \sum_{A\times B} (a', b')m_{(a,b)})$ $= \sum_{A\times B} \hat{f}[(a, b)]n_{(a,b)} + \sum_{A\times B} \hat{f}[(a', b')]m_{(a,b)} = f(x) + f(y)$.

It readily follows from g being R-biadditive that the homomorphism $f : F \to G$ which we have just constructed is also R-biadditive. To see this, observe that if $a, a' \in A$ and $b \in B$, then $f[(a + a', b)] - f[(a, b)] - f[(a', b)] = g[(a + a', b)] - g[(a, b)] - g[(a', b)] = 0$. The other two conditions are satisfied with similar computation. Thus, we have that $f(U) = 0$. Define $\varphi : F/U = A \bigotimes_R B \to G$ by $\varphi(x + U) = f(x)$. If $x + U = x' + U$, then $x - x' \in U$ and hence $f(x - x') \in f(U) = 0$. Thus, $f(x) = f(x')$ and φ is well-defined. Furthermore, $\varphi h(a, b) = \varphi[a \otimes b] = \varphi[(a, b) + U] = f[(a, b)] = g[(a, b)]$. Therefore $A \bigotimes_R B = F/U$ is a tensor product. $\square$

Proposition 3.2. *Let R be a ring, A a right R-module, and B a left R-module. Then, the tensor product $A \bigotimes_R B$ is unique up to isomorphism.*

Proof. It has already been shown that $A \bigotimes_R B$ exists. Suppose H and H' are both tensor products, and let $h : A \times B \to H$ and $h' : A \times B \to H'$ be the respective R-biadditive functions having the universal property. Then, there exists a function $f : H \to H'$ such that $h' = fh$ and a function $f' : H' \to H$ such that $h = f'h'$. Hence, $h = f'fh$ and $h' = ff'h'$. That is, $f'f \cong 1_H$ and $ff' \cong 1_{H'}$. Therefore, $f : H \to H'$ is an isomorphism. $\square$

Each element of $A \bigotimes_R B$ is a finite sum of the form $\sum_{i=1}^{n}(a_i \otimes b_i)$. These elements are referred to as *tensors*. The elements $a \otimes b$ that generate $A \bigotimes_R B$ are referred to as *elementary tensors*. Given $a, a' \in A$, $b, b' \in B$, and $r \in R$, the following properties hold for tensors:

(i) $(a + a') \otimes b = a \otimes b + a' \otimes b$,

(ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$,

(iii) $ar \otimes b = a \otimes rb$.

These properties can be proved in a method similar to that used in the proof of Proposition 3.1 to show that $h : A \times B \to A \bigotimes_R B$ defined by $(a, b) \mapsto a \otimes b$ is R-biadditive.

Proposition 3.3. *[26, Prop. 2.46] Let R be a ring, $A, A' \in Mod_R$, and $B, B' \in {}_RMod$. If $f : A \to A'$ and $g : B \to B'$ are R-homomorphisms, then there is an induced map $f \otimes g : A \bigotimes_R B \to A' \bigotimes_R B'$ such that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.*

Proof. Let $h : A \times B \to A \bigotimes_R B$ and $h' : A' \times B' \to A' \bigotimes_R B'$ be the respective R-biadditive maps with the universal tensor property. Define $\varphi : A \times B \to A' \times B'$ by $\varphi(a, b) = (f(a), g(b))$. It then follows that $h'\varphi : A \times B \to A' \bigotimes_R B'$ is R-biadditive. For if $a, a' \in A$ and $b \in B$, then $h'\varphi(a + a', b) = h'(f(a + a'), g(b)) = h'[f(a) + f(a'), g(b)] = h'[f(a), g(b)] + h'[f(a'), g(b)] = h'\varphi(a, b) + h'\varphi(a', b)$. Similarly, $h'\varphi(a, b + b') = h'\varphi(a, b) + h'\varphi(a, b')$ and $h'\varphi(ar, b) = h'\varphi(a, rb)$ for $b' \in B$ and $r \in R$. By the universal property of the R-biadditive map h, there exists a map $\hat{\varphi} : A \bigotimes_R B \to A' \bigotimes_R B'$ such that $h'\varphi = \hat{\varphi}h$. Hence, $\hat{\varphi}(a \otimes b) = \hat{\varphi}h(a, b) = h'\varphi(a, b) = h'[f(a), g(b)] = f(a) \otimes g(b)$. Therefore, $f \otimes g = \hat{\varphi}$ is an induced map satisfying $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$. $\square$

The following lemmas will be needed in a later section:

Lemma 3.4. *[15, Ch. I, Lemma 6.1] Let R be a ring, A a right R-module, and B a left R-module. If $a \otimes b$ is a tensor in $A \bigotimes_R B$, then $a \otimes b = 0$ if and only if there exists $a_1, a_2, ..., a_k \in A$ and $r_1, r_2, ..., r_k \in R$ such that $a = a_1r_1 + a_2r_2 + ... + a_kr_k$ and $r_jb = 0$ for $j = 1, 2, ..., k$.*

Lemma 3.5. *For a left R-module M, there is an R-module isomorphism $\varphi : R \bigotimes_R M \to M$ given by $\varphi(r \otimes m) = rm$. Here, R is viewed as a right R-module. Similarly, $N \bigotimes_R R \cong N$ for a right R-module N.*

Proof. First, observe that $R \times M \xrightarrow{\psi} M$ given by $\psi((r, m)) = rm$ is R-biadditive. Thus, we can define an R-module homomorphism $R \bigotimes_R M \xrightarrow{\varphi} M$ that sends each $r \otimes m \in R \bigotimes_R M$ to rm. In other words, $\varphi(r \otimes m) = \psi(r, m)$. Note that for every $s \in R$, $\varphi(s(r \otimes m)) = \varphi(sr \otimes m) = (sr)m = s(rm) = s\varphi(r \otimes m)$.

Let $\alpha : M \to R \bigotimes_R M$ be defined by $\alpha(m) = 1 \otimes m$. Clearly α is a well-defined R-module homomorphism since $\alpha(m + n) = 1 \otimes (m + n) = 1 \otimes m + 1 \otimes n = \alpha(m) + \alpha(n)$, and $\alpha(rm) = 1 \otimes rm = 1r \otimes m = 1 \otimes m$. It follows that $\alpha\varphi(r \otimes m) = \alpha(rm) = 1 \otimes rm = 1r \otimes m = r \otimes m$, and $\varphi\alpha(m) = \varphi(1 \otimes m) = 1m = m$. Thus, φ is a bijection and hence an R-module isomorphism. □

Lemma 3.6. *[26, Theorem 2.63] If $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is an exact sequence of left R-modules, then for any right R-module M, $M \bigotimes_R A \xrightarrow{1\otimes i} M \bigotimes_R B \xrightarrow{1\otimes p} M \bigotimes_R C \to 0$ is an exact sequence.*

Proof. For $M \bigotimes_R A \xrightarrow{1\otimes i} M \bigotimes_R B \xrightarrow{1\otimes p} M \bigotimes_R C \to 0$ to be exact, it needs to be shown that $im(1 \otimes i) = \ker(1 \otimes p)$ and $1 \otimes p$ is surjective. Since $im(i) = \ker(p)$ and hence $pia = 0$ for every $a \in A$, it readily follows that $im(1 \otimes i) \subseteq \ker(1 \otimes p)$. For if $\sum(m_j \otimes a_j) \in M \bigotimes_R A$, then $(1 \otimes p)(1 \otimes i)[\sum(m_j \otimes a_j)] = (1 \otimes p)[\sum(1 \otimes i)(m_j \otimes a_j)] = (1 \otimes p)[\sum(m_j \otimes ia_j)] = \sum(1 \otimes p)(m_j \otimes ia_j) = \sum(m_j \otimes pia_j) = \sum(m_j \otimes 0) = 0$. To see that $im(1 \otimes i) = \ker(1 \otimes p)$, first note that since $im(1 \otimes i)$ is contained in the kernel of $1 \otimes p$, there is a uniqe homomorphism

$\varphi : M \bigotimes_R B/im(1\otimes i) \to M \bigotimes_R C$ such that $\varphi[(m\otimes b)+im(1\otimes i)] = (1\otimes p)(m\otimes b) = m\otimes pb$ [21, Ch. IV, Theorem 1.7].

It can be shown that φ is an isomorphism, and from this it will follow that $im(1 \otimes i) = \ker(1 \otimes p)$. Note that since the sequence $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is exact and hence p is surjective, for every $c \in C$ there exists an element $b \in B$ such that $pb = c$. Let the function $f : M \times C \to M \bigotimes_R B/im(1 \otimes i)$ be defined by $(m, c) \mapsto p \otimes b$. If there is another element $b_0 \in B$ such that $pb_0 = c$, then $p(b - b_0) = pb - pb0 = c - c = 0$. Hence, $b - b_0 \in \ker(p) = im(i)$. Thus, there is an $a \in A$ such that $ia = b - b_0$, and it then follows that $m\otimes b - m\otimes b_0 = m\otimes(b-b_0) = m\otimes ia \in im(1\otimes i)$. Hence, $(m\otimes b - m\otimes b_0) + im(1\otimes i) = 0$, and therefore f is well-defined. Furthermore, it is easily seen that f is an R-biadditive function. Thus, if $h : (m, c) \mapsto m \otimes c$ is the biadditive function of the tensor product, then there is a homomorphism $\psi : M \bigotimes_R C \to M \bigotimes_R B/im(1 \otimes i)$ such that $\psi h = f$. In other words, $\psi(m \otimes c) = (m \otimes b) + im(1 \otimes i)$.

Observe that $\psi\varphi[(m\otimes b)+im(1\otimes i)] = \psi(m\otimes pb) = \psi(m\otimes c) = (m\otimes b)+im(1\otimes i)$ and $\varphi\psi(m\otimes c) = \varphi[(m\otimes b)+im(1\otimes i)] = m\otimes pb = m\otimes c$. Thus, φ is an isomorphism with inverse ψ. Now, let $\pi : M \bigotimes_R B \to M \bigotimes_R B/im(1 \otimes i)$ be the canonical epimorphism given by $m\otimes b \mapsto m\otimes b+im(1\otimes i)$. Then, $\varphi\pi(m\otimes b) = \varphi[(m\otimes b)+im(1\otimes i)] = m\otimes pb = (1\otimes p)(m\otimes b)$. Hence, $\varphi\pi = 1 \otimes p$. Therefore, since φ is an isomorphism, $\ker(1 \otimes p) = \ker(\varphi\pi) = \ker(\pi) = im(1 + i)$.

Finally, it needs to be shown that $1 \otimes p$ is surjective. Let $\sum(m_j \otimes c_j) \in M \bigotimes_R C$. Since p is surjective, for each j, there exists an element $b_j \in B$ such that $pb_j = c_j$. Thus, $(1 \otimes p)[\sum(m_j \otimes b_j)] = \sum(1 \otimes p)(m_j \otimes b_j) = \sum(m_j \otimes pb_j) = \sum(m_j \otimes c_j)$. Therefore, $1 \otimes p$ is surjective and the sequence $M \bigotimes_R A \xrightarrow{1\otimes i} M \bigotimes_R B \xrightarrow{1\otimes p} M \bigotimes_R C \to 0$ is exact. $\square$

A right R-module M is *flat* if $0 \to M \bigotimes_R A \xrightarrow{1_M\otimes\varphi} M \bigotimes_R B \xrightarrow{1_M\otimes\psi} M \bigotimes_R C \to 0$ is an exact sequence of Abelian groups whenever $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact sequence of left R-modules.

Proposition 3.7. *[26, Prop. 3.46] Let R be a ring and let $\{M_i\}_{i\in I}$ be a collection of right R-modules for some index set I. Then, the direct sum $\bigoplus_I M_i$ is flat if and only if M_i is flat for every $i \in I$. Moreover, R is flat as a right R-module, and any projective right R-module P is flat.*

Proof. First note that if $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact sequence of left R-modules, then $M \bigotimes_R A \xrightarrow{1_M\otimes\varphi} M \bigotimes_R B \xrightarrow{1_M\otimes\psi} M \bigotimes_R C \to 0$ is exact by Lemma 3.6. Thus, M is flat if and only if $1_M \otimes \varphi$ is a monomorphism whenever φ is a monomorphism.

Suppose A and B are left R-modules and let $\varphi : A \to B$ be a monomorphism. For $\bigoplus_I M_i$ to be flat, it needs to be shown that $1 \otimes \varphi : (\bigoplus_I M_i) \bigotimes_R A \to (\bigoplus_I M_i) \bigotimes_R B$ is a monomorphism. By [26, Theorem 2.65], there exist isomorphisms $f : (\bigoplus_I M_i) \bigotimes_R A \to (\bigoplus_I M_i \bigotimes_R A)$ and $g : (\bigoplus_I M_i) \bigotimes_R B \to (\bigoplus_I M_i \bigotimes_R B)$ defined by $f : (x_i) \otimes a \mapsto (x_i \otimes a)$ and $g : (x_i)\otimes b \mapsto (x_i\otimes b)$. Furthermore, since $1_{M_i}\otimes\varphi$ is a homomorphism for each $i \in I$, there is a homomorphism $\psi : \bigoplus_I(M_j \bigotimes_R A) \to \bigoplus_I(M_j \bigotimes_R B)$ such that $(x_i \otimes a) \mapsto (x_i \otimes \varphi(a))$. Observe that ψ is a monomorphism if and only if $1_{M_i} \otimes \varphi$ is a monomorphism for each $i \in I$. It then follows that $\psi f = g(1\otimes\varphi)$ since $\psi f[(x_i)\otimes a] = \psi(x_i\otimes a) = x_i\otimes\varphi(a) = g[(x_i)\otimes\varphi(a)] = g(1 \otimes \varphi)[(x_i \otimes a)]$. Therefore, $\bigoplus_I M_i$ is flat if and only if $1 \otimes \varphi$ is a monomorphism if and only if ψ is a monomorphism if and only if $1_{M_i} \otimes \varphi$ is a monomorphism for each i if and only if M_i is flat for each i.

To see that R is flat as a right R-module, note that Lemma 3.5 gives isomorphisms $f : A \to R \bigotimes_R A$ and $g : B \to R \bigotimes_R B$ defined by $f(a) = 1_R \otimes a$ and $g(b) = 1_R \otimes b$. Observe that $(1_R \otimes \varphi)f(a) = (1_R \otimes \varphi)(1_R \otimes a) = 1_R \otimes \varphi(a) = g\varphi(a)$. Hence, $(1_R \otimes \varphi) = g\varphi f^{-1}$, which is a monomorphism. Therefore, R is flat as a right R-module.

Let P be a projective right R-module. Then there is a free right R-module F and an R-module N such that $F = P \bigoplus N$. As a free module, F is a direct sum of copies of R, which is flat. Hence, F is also flat. Therefore, P is flat as a direct summand of F. □

3.2 Bimodules and the Hom and Tensor Functors

Let A be a right R-module. Consider the functor $T_A : {}_RMod \to Ab$ defined by $T_A(B) = A \bigotimes_R B$ with induced map $T_A(\varphi) = 1_A \otimes \varphi : A \bigotimes_R B \to A \bigotimes_R B'$, where Ab is the category of all Abelian groups and $\varphi \in \mathrm{Hom}_R(B, B')$ for left R-modules B and B'. Observe that $T_A(\varphi)(a \otimes b) = a \otimes \varphi(b)$. T_A is sometimes denoted $T_A(\_) = A \bigotimes_R \_$. Similarly, the functor $T_B(A) = A \bigotimes_R B$ with induced map $\psi \otimes 1_B$ can be defined for a left R-module B and $\psi \in \mathrm{Hom}_R(A, A')$. We also consider the functor $\mathrm{Hom}_R(A, \_) : Mod_R \to Ab$ with induced map $f_* : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, C)$ defined by $f_*(h) = fh$, where $f : B \to C$ is a homomorphism for right R-modules B and C.

Let R and S be rings and let M be an Abelian group which has both a left R-module structure and a right S-module structure. Then, M is an (R, S)-*bimodule* if $(rx)s = r(xs)$ for every $r \in R$, $s \in S$, and $x \in M$. This is sometimes denoted ${}_RM_S$. In particular, if A is a right R-module and $E = End_R(A)$, then M is an (E, R)-bimodule. Note that for $x \in M$ and $\alpha \in E$, scalar multiplication αx is defined as $\alpha(x)$.

Proposition 3.8. *Let R and S be rings. Suppose M is an (R, S)-bimodule and N is a right S-module. Then,* $\mathrm{Hom}_S(M_S, N_S)$ *is a right R-module and* $\mathrm{Hom}_S(N_S, M_S)$ *is a left R-module.*

Proof. First, observe that $\mathrm{Hom}_S(M_S, N_S)$ is an Abelian group. For if $f, g \in \mathrm{Hom}_S(M_S, N_S)$, then $f(xr) = f(x)r$ and $g(xr) = g(x)r$ for every $r \in R$. Hence, $f + g \in \mathrm{Hom}_S(M_S, N_S)$ since $(f+g)(xr) = f(xr)+g(xr) = f(x)r+g(x)r = (f+g)(x)r$. Moreover, if $h \in \mathrm{Hom}_S(M_S, N_S)$, then $[f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + g(x) + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x)$. Hence, $\mathrm{Hom}_S(M_S, N_S)$ is associative. Furthermore, the map $\alpha : a \mapsto 0$ acts as the zero element. Finally, note that if $f \in \mathrm{Hom}_S(M_S, N_S)$, then $g : M \to N$ defined by $g(x) = -f(x)$ is such that $(f+g)(x) = f(x)+g(x) = f(x)-f(x) = 0$. Hence, every element of $\mathrm{Hom}_S(M_S, N_S)$ has an inverse. Therefore, $\mathrm{Hom}_S(M_S, N_S)$ is an Abelian group.

Now, let $\varphi \in \mathrm{Hom}_S(M_S, N_S)$, $r, r' \in R$, and $x \in M$. Define the right R-module structure on $\mathrm{Hom}_S(M_S, N_S)$ by $(\varphi r)(x) = \varphi(rx)$. Then, $(\varphi + \psi)(r)(x) = (\varphi r + \psi r)(x) =$

$(\varphi r)(x) + (\psi r)(x) = \varphi(rx) + \psi(rx) = (\varphi + \psi)(rx)$ for $\psi \in \mathrm{Hom}_S(M_S, N_S)$. Moreover, $[\varphi(r + r')](x) = \varphi[(r + r')x] = \varphi[rx + r'x] = \varphi(rx) + \varphi(r'x) = (\varphi r)(x) + (\varphi r')(x)$ for $r' \in R$. Finally, observe that $[\varphi(rr')](x) = \varphi[(rr')(x)] = \varphi[r(r'x)] = (\varphi r)(r'x)$. Therefore, $\mathrm{Hom}_S(M_S, N_S)$ satisfies the conditions of a right R-module. Similarly, $\mathrm{Hom}_S(N_S, M_S)$ is a left R-module with $(r\pi)(x) = r\pi(x)$ for any $\pi \in \mathrm{Hom}_S(N_S, M_S)$. □

Proposition 3.9. *[26] Let R be a subring of S. Suppose M is an (R, S)-bimodule and A is a right R-module. Then, $A \bigotimes_R M$ is a right S-module. In particular, S is an (R, S)-bimodule and hence $A \bigotimes_R S$ is a right S-module.*

Proof. Let $y = \sum_{i=1}^{n}(a_i \otimes x_i) \in A \bigotimes_R M$ and let $s \in S$. Define the right S-module structure on $A \bigotimes_R M$ by $(\sum_{i=1}^{n}(a_i \otimes x_i))s = \sum_{i=1}^{n}(a_i \otimes x_i s)$. To see that this does define a right S-module, consider the well-defined map $\mu_s : M \to M$ defined by $\mu_s(x) = xs$. By the bimodule structure of M, $r\mu_s(x) = r(xs) = (rx)s = \mu_s(rx)$ for $r \in R$. Hence, $\mu_s \in \mathrm{Hom}_R(M, M)$. Consider the functor $T_A(\_) = A \bigotimes_S \_$. By Proposition 3.3, there is a well-defined homomorphism $T_A(\mu_s) = 1_A \otimes \mu_s : A \bigotimes_R M \to A \bigotimes_R M$ such that $(1_A \otimes \mu_s)(a \otimes x) = a \otimes \mu_s(x) = a \otimes xs$. If the element ys is defined by $ys = (1_A \otimes \mu_s)(y) = (1_A \otimes \mu_s)(\sum_{i=1}^{n}(a_i \otimes x_i)) = \sum_{i=1}^{n}(1_A \otimes \mu_s)(a_i \otimes x_i) = \sum_{i=1}^{n}(a_i \otimes x_i s)$, then the S-module structure is well-defined since $(1_A \otimes \mu_s)$ is a well-defined homomorphism and $\sum_{i=1}^{n}(a_i \otimes x_i s) \in A \bigotimes_R M$. The remaining right S-module conditions follow readily. Moreover, it is easy to see that S satisfies the conditions of an (R, S)-bimodule. Therefore, given any right R-module A, $A \bigotimes_R S$ is a right S-module. □

Proposition 3.10. *Let $R \leq S$ be rings and let M be an (R, S)-bimodule. Then, the following hold:*

(a) The functor $T_M(\_) = \_ \bigotimes_R M : Mod_R \to Ab$ is actually a functor $Mod_R \to Mod_S$.

(b) The functor $\mathrm{Hom}_S(M, \_) : Mod_S \to Ab$ is actually a functor $Mod_S \to Mod_R$.

Proof. (a): It has already been shown in Proposition 3.9 that $T_M(A) = A \bigotimes_R M$ is a right S-module for any right R-module A. It needs to be shown that if $\psi \in \mathrm{Hom}_R(A, A')$ for $A' \in Mod_R$, then $T_M(\psi) = \psi \otimes 1_M \in \mathrm{Hom}_S(A \bigotimes_R M, A' \bigotimes_R M)$. In other words, it needs to be shown that $\psi \otimes 1_M$ is an S-homomorphism. Let $s \in S$. Then, $(\psi \otimes 1_M)(a \otimes x)s = (\psi(a) \otimes x)s = \psi(a) \otimes xs = (\psi \otimes 1_M)(a \otimes xs) = (\psi \otimes 1_M)[(a \otimes x)s]$. Thus, $T_M(\psi)$ is a morphism in Mod_S, and therefore $T_M(\_)$ is a functor with values in Mod_S.

(b): Given any right S-module N, $\mathrm{Hom}_S(M, N)$ is a right R-module by Proposition 3.8. It needs to be shown that if $f : N \to N'$ is a homomorphism for $N, N' \in Mod_S$, then the induced map $f_* = \mathrm{Hom}_R(M, f) : \mathrm{Hom}_S(M, N) \to \mathrm{Hom}_S(M, N')$ defined by $f_*(\varphi) = f\varphi$ is an R-homomorphism. Note that if $\varphi, \psi \in \mathrm{Hom}_S(M, N)$, then $f(\varphi + \psi) = f\varphi + f\psi$. Hence, f_* is a homomorphism since $f_*(\varphi + \psi) = f(\varphi + \psi) = f\varphi + f\psi = f_*\varphi + f_*\psi$. Let $r \in R$. Observe that $(\varphi r)(x) = \varphi(rx)$ by Proposition 3.8. Moreover, since M has a left R-module structure and $f\varphi$ is an element of the right R-module $\mathrm{Hom}_S(M, N')$, Proposition 3.8 also shows that $[f\varphi(x)]r = f[\varphi r](x) = f\varphi(rx)$ for $x \in M$. Thus, $[f_*(\varphi(x))]r = [f\varphi(x)]r = f\varphi(rx) = f_*[\varphi(rx)] = f_*[(\varphi r)(x)]$. Hence, f_* is an R-homomorphism, and therefore $\mathrm{Hom}_S(M, \_)$ is a functor with values in Mod_R. $\square$

The following lemmas will be used later to show $Mod_R \cong Mod_{Mat_n}(R)$. The proofs are omitted and can be found in *Rings and Categories of Modules* by Frank Anderson and Kent Fuller.

Lemma 3.11. *[5, Proposition 20.10] Let R and S be rings, M a right R-module, N a right S-module, and P an (S, R)-bimodule. If M is finitely generated and projective, then $\mu : N \bigotimes_S \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, N \bigotimes_S P)$ defined by $\mu(y \otimes f)(x) = y \otimes f(x)$ is a natural isomorphism. Here, $x \in M$, $y \in N$, and $f \in \mathrm{Hom}_R(M, P)$.*

Lemma 3.12. *[5, Proposition 20.11] Let R and S be rings, M a right R-module, N a left S-module, and P an (S, R)-bimodule. If M is finitely generated and projective, then $\nu : \mathrm{Hom}_R(P, M) \bigotimes_S N \to \mathrm{Hom}_R(\mathrm{Hom}_S(N, P), M)$ defined by $\nu(f \otimes y)(g) = fg(y)$ is a natural isomorphism. Here, $f \in \mathrm{Hom}_R(P, M)$, $g \in \mathrm{Hom}_S(N, P)$, and $y \in N$.*

3.3 The Tor and Ext Functors

Consider the exact sequence $P = \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ of right R-modules, where P_j is projective for every j. Such an exact sequence is called a *projective resolution* of the right R-module A. Note that a projective resolution can be formed for any projective right R-module A since every right R-module is the epimorphic image of a projective right R-module. Define the *deleted projective resolution*, denoted P_A, by removing the morphism ϵ and the right R-module A. Note that the projective resolution is an exact sequence, and hence $im(d_{i+1}) = \ker(d_i)$. Therefore, $d_i d_{i+1} = 0$ for every $i \in \mathbb{Z}^+$, and thus the projective resolution P and the deleted projective resolution P_A are both complexes. However, P_A is not necessarily exact since $im(d_1) = \ker(\epsilon)$, which may not equal the kernel of the morphism $P_0 \to 0$. Now, if $T : Mod_R \to Ab$ is an additive covariant functor then we can form the *induced complex* TP_A, which is defined as $\cdots \to T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \to 0$.

For $n \in \mathbb{Z}$, the n^{th} *homology* is $H_n(C) = Z_n(C)/B_n(C)$, where C is a complex, $Z_n(C) = \ker(d_n)$, and $B_n(C) = im(d_{n+1})$. Hence, $H_n(C) = \ker(d_n)/im(d_{n+1})$. If we consider the deleted projective resolution P_A as defined above, then $\cdots \to P_2 \bigotimes_R B \xrightarrow{d_2 \otimes 1_B} P_1 \bigotimes_R B \xrightarrow{d_1 \otimes 1_B} P_0 \bigotimes B \to 0$ is the induced complex $T_B P_A$ of the functor $T_B(\_) = \_ \bigotimes_R B$. The *Tor functor* $\text{Tor}_n^R(A, \_) : {}_R Mod \to Ab$ is defined by

$$\text{Tor}_n^R(A, B) = H_n(T_B P_A) = \ker(d_n \otimes 1_B)/im(d_{n+1} \otimes 1_B).$$

Note that $\text{Tor}_n^R(A, B)$ does not depend on the choice of projective resolution [26]. The functor $\text{Tor}_n^R(A, \_)$ is referred to as the *left derived functor* of $A \bigotimes_R B$ since it makes up for the loss of exactness from applying the tensor functor to an exact sequence. The following two well-known propositions will be useful later:

Proposition 3.13. *[26] If $M \in Mod_R$ and $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, then the induced sequence $\cdots \to \text{Tor}_{n+1}^R(M, C) \to \text{Tor}_n^R(M, A) \to \text{Tor}_n^R(M, B) \to \text{Tor}_n^R(M, C) \to \cdots \to \text{Tor}_1^R(M, C) \to M \bigotimes_R A \to M \bigotimes_R B \to M \bigotimes_R C \to 0$ is exact.*

Proposition 3.14. *[26] A right R-module M is flat if and only if* $\mathrm{Tor}_n^R(M,X)=0$ *for every left R-module X and every* $n\geq 1$.

Dual to the notion of the left derived functor Tor is the right derived functor Ext. Given a right R-module B, we choose an injective resolution $E = 0\to B \xrightarrow{\nu} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \cdots$, where each E^j is injective. As with Tor, we form the deleted injective resolution E^B and apply a functor T to this new complex to form the induced complex TE^B. Define the n^{th} homology of TE^B to be $H^n(TE^B) = \ker(Td^n)/im(Td^{n-1})$. If $T = \mathrm{Hom}_R(A, \_)$ is the Hom functor, we have $0\to \mathrm{Hom}_R(A,E^0) \xrightarrow{d^0_*} \mathrm{Hom}_R(A,E^1) \xrightarrow{d^1_*} \mathrm{Hom}_R(A,E^2) \xrightarrow{d^2_*} \cdots$, and the *Ext functor* $\mathrm{Ext}^n_R(A,\_) : Mod_R \to Ab$ is given by $\mathrm{Ext}^n_R(A,B) = H^n(\mathrm{Hom}_R(A,E^B)) = \ker(d^n_*)/im(d^{n-1}_*)$. The functor $\mathrm{Ext}^n_R(A,\_)$ is referred to as the *right derived functor* of $\mathrm{Hom}_R(A,B)$ since it makes up for the loss of exactness from applying the Hom functor to an exact sequence:

Proposition 3.15. *[26] If* $M\in Mod_R$ *and* $0\to A\to B\to C\to 0$ *is an exact sequence of right R-modules, then the following induced sequences are exact:*

$$a)\quad 0\to \mathrm{Hom}_R(C,M)\to \mathrm{Hom}_R(B,M)\to \mathrm{Hom}_R(A,M)\to \mathrm{Ext}^1_R(C,M)\to \mathrm{Ext}^1_R(B,M)$$

$$\to \mathrm{Ext}^1_R(A,M)\to \mathrm{Ext}^2_R(C,M)\to \mathrm{Ext}^2_R(B,M)\to \mathrm{Ext}^2_R(A,M)\to \cdots$$

$$b)\quad 0\to \mathrm{Hom}_R(M,A)\to \mathrm{Hom}_R(M,B)\to \mathrm{Hom}_R(M,C)\to \mathrm{Ext}^1_R(M,A)\to \mathrm{Ext}^1_R(M,B)$$

$$\to \mathrm{Ext}^1_R(M,C)\to \mathrm{Ext}^2_R(M,A)\to \mathrm{Ext}^2_R(M,B)\to \mathrm{Ext}^2_R(M,C)\to \cdots$$

Moreover, we have the following useful connection between the Tor and Ext functors:

Proposition 3.16. *[14, Theorem 3.2.1] Let R and S be rings, M a left R-module and N an* (S,R)*-bimodule. If E is an injective left S-module, then for every* $i<\omega$,

$$\mathrm{Ext}^i_R(M, \mathrm{Hom}_S(N,E)) \cong \mathrm{Hom}_S(\mathrm{Tor}_i^R(N,M),E).$$

Chapter 4

Torsion-freeness

In 1960, Hattori used the homological properties of classical torsion-free modules over integral domains to give a more general definition of torsion-freeness. He defines a right R-module M to be *torsion-free* if $\operatorname{Tor}_1^R(M, R/Rr) = 0$ for every $r \in R$, and he defines a left R-module N to be *torsion-free* if $\operatorname{Tor}_1^R(R/sR, N) = 0$ for every $s \in R$ [18]. The following equivalent definition of torsion-freeness is also given by Hattori in [18, Proposition 1]:

Proposition 4.1. *[18] The following are equivalent for a right R-module M.*

(a) M is torsion-free

(b) For each $x \in M$ and $r \in R$, $xr = 0$ implies the existence of $x_1, x_2, ..., x_k \in M$ and $r_1, r_2, ...r_k \in R$ such that $x = \sum_{j=1}^{k} x_j r_j$ and $r_j r = 0$ for every $j = 1, 2, ..., k$.

Proof. Consider the exact sequence $0 \to Rr \xrightarrow{\iota} R \xrightarrow{\pi} R/Rr \to 0$ of left R-modules, where ι is the inclusion map and π is the epimorphism $r \mapsto r + Rr$. This induces a long exact sequence $X = ... \to \operatorname{Tor}_1^R(M, R/Rr) \xrightarrow{f} M \bigotimes_R Rr \xrightarrow{1_M \otimes \iota} M \bigotimes_R R \cong M \xrightarrow{1_M \otimes \pi} M \bigotimes_R R/Rr \to 0$ [26, Corollary 6.30]. Observe that condition (b) is equivalent to $1_M \otimes \iota$ being a monomorphism. For if $1_M \otimes \iota : x \otimes r \mapsto xr$ is a monomorphism, then $xr = 0$ implies $x \otimes r = 0$. Hence, there exists $x_1, x_2, ..., x_k \in M$ and $r_1, r_2, ..., r_k \in R$ such that $x = x_1r_1 + x_2r_2 + ... + x_kr_k$ and $r_jr = 0$ for $j = 1, 2, ..., k$ by Lemma 3.4. On the other hand, if $xr = 0$ implies $x = x_1r_1 + x_2r_2 + ... + x_kr_k$ and $r_jr = 0$, then $x \otimes r = x_1r_1 + x_2r_2 + ... + x_kr_k \otimes r = x_1 \otimes r_1r + x_2 \otimes r_2r + ... + x_k \otimes r_kr = 0$. Hence, $\ker(1_M \otimes \iota) = 0$ and $1_M \otimes \iota$ is a monomorphism.

To complete the proof, it needs to be shown that M is torsion-free if and only if $1_M \otimes \iota$ is a monomorphism. If M is torsion-free, then $\operatorname{Tor}_1^R(M, R/Rr) = 0$. Thus, $0 \to M \bigotimes_R Rr \xrightarrow{1_M \otimes \iota}$

$M \bigotimes_R R \cong M \xrightarrow{1_M \otimes \pi} M \bigotimes_R R/Rr \to 0$ is exact and so $1_M \otimes \iota$ is a monomorphism. Conversely, if $1_M \otimes \iota$ is a monomorphism, then $im(f) = \ker(1_M \otimes \iota) = 0$ in the induced sequence X. However, f is a monomorphism. Hence, $0 = im(f) \cong \mathrm{Tor}_1^R(M, R/Rr)$. □

A ring R is *torsion-free* if every finitely generated right (or left) ideal is torsion-free as a right (or left) R-module. Hattori shows in [18] that a ring R is torsion-free if and only if every principal left ideal of R is flat. To see this, observe that if $0 \to J \xrightarrow{i} R \xrightarrow{p} R/J \to 0$ is an exact sequence of right R-modules with J finitely generated, then $0 \to J \bigotimes_R Rr \xrightarrow{i \otimes 1_{Rr}} R \bigotimes_R Rr \xrightarrow{p \otimes 1_{Rr}} R/J \bigotimes_R Rr \to 0$ is an exact sequence whenever Rr is flat. This is the case if and only if $\mathrm{Tor}_1^R(R/J, Rr) = 0$. Hattori gives a natural isomorphism in [18, Proposition 7] showing that $\mathrm{Tor}_1^R(R/J, Rr) \cong \mathrm{Tor}_1^R(J, R/Rr)$. Hence, $\mathrm{Tor}_1^R(J, R/Rr) = 0$ if and only if Rr is flat for every $r \in R$. That is, every finitely generated right ideal is torsion-free if and only if every principal left ideal is flat.

In 2004, John Dauns and Lazlo Fuchs provided the following useful characterization of torsion-free rings:

Theorem 4.2. *[13]The following are equivalent for a ring R:*

(a) R is torsion-free.

(b) For every $s, r \in R$, $sr = 0$ if and only if $s \in s \cdot ann_l(r)$. In other words, $sr = 0$ if and only if $s = su$ and $ur = 0$ for some $u \in R$.

Proof. (a) ⇒ (b): Suppose R is a torsion-free ring. For $s \in R$, sR is torsion-free as a right R-module. By Proposition 4.1, if $a \in sR$ and $r \in R$ with $ar = 0$, then there exists $u \in R$ so that $a = su$ and $ur = 0$. Hence, if $sr = 0$, we have $s = su$ and $ur = 0$ for some $u \in R$, since $s = s \cdot 1 \in sR$. Conversely, if there is some $u \in R$ such that $s = su$ and $ur = 0$, then $sr = (su)r = s(ur) = s \cdot 0 = 0$. Therefore, $sr = 0$ if and only if $s = su$ and $ur = 0$ for some $u \in R$.

(b) ⇒ (a): Assume that $sr = 0$ for every $s, r \in R$ if and only if $s = su$ and $ur = 0$ for some $u \in R$. Let Rr be a finitely generated left ideal of R. Assume that the sequence

$0 \to J \to R \to R/J \to 0$ is exact with J finitely generated. Then, R is a torsion-free ring if $0 \to J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is exact. By Lemma 3.6, it follows that $J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is exact. In order for the entire sequence to be exact, it needs to be shown that φ is a monomorphism.

Note that $R \bigotimes_R Rr \cong Rr$ by Lemma 3.5. Consider $j \otimes sr \in J \bigotimes_R Rr$. Since $j \otimes sr = js \otimes r$ and $js \in J$, tensors in $J \bigotimes_R Rr$ can be written as $k \otimes r$ for some $k \in J$. Thus, it needs to be shown that $J \bigotimes_R Rr \xrightarrow{\varphi} Rr$ given by $\varphi(k \otimes r) = kr$ is a monomorphism. Let $k \otimes r \in \ker \varphi$. Then $\varphi(k \otimes r) = kr = 0$. By assumption, there exists some $u \in R$ such that $k = ku$ and $ur = 0$. Then, $k \otimes r = ku \otimes r = k \otimes ur = k \otimes 0 = 0$. Thus, $\ker \varphi = 0$ and φ is a monomoprhism. Therefore, $0 \to J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is an exact sequence, and hence R is a torsion-free ring. □

Proposition 4.3. *[18, Proposition 7] A ring R is torsion-free if and only if every submodule of a torsion-free right R-module is torsion-free.*

Proof. Suppose R is torsion-free and let N be a submodule of a torsion-free right R-module M. Consider the exact sequence $0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to 0$, where ι is the inclusion map and π is the canonical epimorphism. As noted above, if R is torsion-free, then the principal left ideal Rr is flat for every $r \in R$. Hence, $0 \to N \bigotimes_R Rr \to M \bigotimes_R Rr \to M/N \bigotimes_R Rr \to 0$ is exact and so $\mathrm{Tor}_1^R(M/N, Rr) \cong 0$. Observe that $\mathrm{Tor}_1^R(M, R/Rr) \cong 0$ since M is torsion-free. If we consider the long exact sequence derived from the functor $\mathrm{Tor}_n^R(\_, R/Rr)$, then $0 \cong \mathrm{Tor}_1^R(M/N, Rr) \cong \mathrm{Tor}_2^R(M/N, R/Rr) \to \mathrm{Tor}_1^R(N, R/Rr) \to \mathrm{Tor}_1^R(M, R/Rr) \cong 0$ is exact. Therefore, $\mathrm{Tor}_1^R(N, R/Rr) = 0$ and N is torsion-free. On the other hand, if every submodule of a torsion-free right R-module is torsion-free, then every finitely generated right ideal of R is torsion-free since R itself is torsion-free as a right R-module. □

Theorem 4.4. *[13]A ring R is a right p.p.-ring if and only if R is torsion-free and, for each $x \in R$, $ann_r(x)$ is finitely generated.*

Proof. Suppose R is a right p.p.-ring. Then, for each $r \in R$, $ann_r(r) = eR$ for some idempotent $e \in R$. Let $s \in R$ be such that $rs = 0$. Then, $s \in ann_r(r)$, and hence $s = es^{'}$ for some $s^{'} \in R$. It follows that $es = e^2s^{'} = es^{'} = s$. Furthermore, $e = e^2 \in eR = ann_r(r)$ and hence $re = 0$. Note also that if $s = es$ and $re = 0$, then $s \in eR = ann_r(r)$ and hence $rs = 0$. Thus, $rs = 0$ if and only if $s = es$ and $re = 0$. Therefore, R is a torsion-free ring by a symmetric version of Theorem 4.2. Moreover, since R is a right p.p.-ring, $ann_r(r)$ is generated by an idempotent and thus finitely generated.

Conversely, suppose R is a torsion-free ring and the right annihilator of every element of R is finitely generated. Let $s \in R$ and let $\{s_1, ..., s_n\}$ be the finite set of generators for $ann_r(s)$. Note that each $s_i \in ann_r(s)$, and so $ss_i = 0$ for each $i = 1, ..., n$. Let $S = \bigoplus^{n} R$ be the direct sum of n copies of R, and consider S as a left R-module. Let $s^{'} = (s_1, ...s_n) \in S$. Note that S is a torsion-free left R-module since it is the direct sum of copies of R, which is torsion-free as a left R-module. Thus, the submodule $Rs^{'}$ of S is torsion-free by Proposition 4.3. Hence, Proposition 4.1 gives some $u \in R$ such that $s^{'} = us^{'}$ and $su = 0$, and thus $u \in ann_r(s)$. Note that $s_i = us_i$ for each $i = 1, ..., n$. This implies that $s_i \in uR$ for each i, and so $\{s_1, ..., s_n\} \subseteq uR$. It follows that $ann_r(s) = s_1R + ... + s_nR \subseteq uR$. Suppose $x \in uR$. Then, $x = ut$ for some $t \in R$. Thus, $sx = sut = 0 \cdot t = 0$, and so $x \in ann_r(s)$. Therefore, $ann_r(s) = uR$.

Now, since R is a torsion-free ring, uR is torsion-free as a finitely generated right ideal of R. By a symmetric version of Theorem 4.2, since $su = 0$, there exists an $e \in uR = ann_r(s)$ such that $u = eu$ and $se = 0$. Let $x \in uR$. Then, $x = ut = eut \in eR$ for some $t \in R$. Hence, $uR \subseteq eR$. On the other hand, suppose $y \in eR$. Then, for some $v \in R$, $y = ev$ and $sy = sev = 0 \cdot v = 0$. Thus, $y \in ann_r(s)$ and $eR \subseteq ann_r(s) = uR$. Hence, $ann_r(s) = uR = eR$ and $e = ur$ for some $r \in R$. It then follows that e is an idempotent since $e^2 = eux = ux = e$. Therefore, $ann_r(s)$ is generated by an idempotent and so R is a right p.p.-ring. $\square$

Lemma 4.5. *If R is a right p.p.-ring and $e \in R$ is a nonzero idempotent, then $eR = ann_r(x)$ for some $x \in R$. In particular, $eR = ann_r(1 - e)$.*

Proof. If $er \in eR$, then $(1-e)er = (e-e^2)r = (e-e)r = 0$. Hence, $er \in ann_r(1-e)$ and $eR \subseteq ann_r(1-e)$. On the other hand, if $s \in ann_r(1-e)$, then $(1-e)s = 0$. Hence, $s - es = 0$, and so $s = es \in eR$. Therefore, $eR = ann_r(1-e)$. $\square$

Proposition 4.6. *[3] If R is a right and left p.p.-ring which does not contain an infinite set of orthogonal idempotents and M is a torsion-free right R-module, then $ann_r(x)$ is generated by an idempotent for every $x \in M$.*

Proof. Let R be a right and left p.p.-ring which does not contain an infinite set of orthogonal idempotents. Take M to be a torsion-free right R-module and let $A = ann_r(x)$ for some nonzero $x \in M$. Suppose $r_0 \in R$ is such that $xr_0 = 0$. Note that the cyclic submodule xR is torsion-free since R is a right p.p.-ring. Moreover, $ann_l(r_0) = Re_0$ for some idempotent $e_0 \in R$ since R is a left p.p.-ring. By Proposition 4.1, there exists $xs_1, xs_2, ..., xs_n \in xR$ and $t_1e_0, t_2e_0..., t_ne_0 \in Re_0 = ann_l(r_0)$ such that $x = xs_1t_1e_0 + xs_2t_2e_0 + ... + xs_nt_ne_0$. Hence, $xe_0 = xs_1t_1e_0^2 + xs_2t_2e_0^2 + ... + xs_nt_ne_0^2 = x$. Thus, $0 = x - xe_0 = x(1-e_0)$. Therefore, if $(1-e_0)r \in (1-e_0)R$, then $x(1-e_0)r = 0$ and $(1-e_0)R \subseteq A$.

Now, if there exists some $r_1 \in A \backslash (1-e_0)R$, then $r_1 \neq (1-e_0)r_1$ and hence $e_0r_1 \neq 0$. However, $xe_0r_1 = xr_1 = 0$. Since R is a left p.p.-ring, $ann_l(e_0r_1) = R(1-f)$ for some idempotent $1-f$. Note that as before it follows from Proposition 4.1 that $x = x(1-f)$ since $xe_0r_1 = 0$. Furthermore, $1 - e_0 \in ann_l(e_0r_1) = R(1-f)$ since $(1-e_0)e_0r_1 = e_0r_1 - e_0r_1 = 0$. Hence, there is some $r \in R$ such that $(1-e_0)f = r(1-f)f = r(f-f) = 0$. Thus, $e_0f = f$. Let $e_1 = (1-f)e_0 = e_0 - fe_0$. Then, $e_1^2 = (e_0 - fe_0)(e_0 - fe_0) = e_0 - e_0fe_0 - fe_0 + fe_0fe_0 = e_0 - fe_0 - fe_0 + fe_0 = e_0 - fe_0 = e_1$. Thus, e_1 is an idempotent. Moreover, e_1 is nonzero, since otherwise $e_0 = fe_0$ and hence $e_0 = 0$.

Now, $e_1e_0 = (1-f)e_0e_0 = (1-f)e_0 = e_1$, and Lemma 4.5 shows that $(1-e_0)R = ann_r(e_0)$ and $(1-e_1)R = ann_r(e_1)$. Thus, if $r \in ann_r(e_0)$, then $e_1r = e_1e_0r = 0$. Hence, $r \in ann_r(e_1) = (1-e_1)R$, and so $(1-e_0)R \subseteq (1-e_1)R$. Moreover, $e_1e_0r_1 = e_1r_1 = (1-f)e_0r_1 = 0$ since $1-f \in ann_r(e_0r_1)$. Thus, $e_0r_1 \in ann_r(e_1) = (1-e_1)R$. However, e_0r_1 is nonzero and hence $e_0r_1 \notin ann_r(e_0) = (1-e_0)R$. Thus, $(1-e_0)R \subset (1-e_1)R$ is a proper inclusion. By supposing there is some $r_2 \in A \backslash (1-e_1)R$ and repeating these steps, and then supposing there is some $r_3 \in A \backslash (1-e_2)R$ and so on, we can construct an ascending chain $(1-e_0)R \subset (1-e_1)R \subset (1-e_2)R \subset ....$ However, this chain must terminate at some point since R only contains finite sets of orthogonal idempotents. Therefore, there is some idempotent $e \in R$ such that $A = (1-e)R$. $\square$

Proposition 4.7. *[3] If R is a right and left p.p.-ring not containing an infinite set of orthogonal idempotents, then a cyclic submodule of a torsion-free right R-module is projective.*

Proof. Let M be a torsion-free right R-module, and take N to be a cyclic submodule of M. Then, N is of the form xR for some $x \in N \leq M$. By Proposition 4.6, $ann_r(x) = eR$ for some idempotent $e \in R$. If $f : R \to xR$ is the epimorphism defined by $r \mapsto xr$, then $xR \cong R/\ker(f) = R/ann_r(x)$ by the First Isomorphism Theorem. It then follows that $xR \cong R/ann_r(x) \cong [eR \bigoplus (1-eR)]/ann_r(x) \cong [eR \bigoplus (1-e)R]/eR \cong (1-e)R$. Therefore, N is a principal right ideal of R, and thus projective, since R is a right p.p.-ring. $\square$

A ring R is a *Baer-ring* if $ann_r(A)$ is generated by an idempotent for every subset A of R. Note that if R is Baer, then $ann_r(ann_l(A)) = eR$ for some idempotent $e \in R$. Hence, $ann_l(A) = ann_l(ann_r(ann_l(A))) = ann_l(eR) = R(1-e)$ by Lemma 4.5. Thus, $ann_r(A)$ is generated by an idempotent if and only if $ann_l(A)$ is generated by an idempotent. Therefore, the property that R is a Baer ring is right-left-symmetric. The following theorem from Dauns and Fuchs [13] gives conditions for which a ring R is Baer:

Theorem 4.8. *[13] If R is a torsion-free ring and right annihilators of elements are finitely generated and satisfy the ascending chain condition, then R is a Baer-ring.*

Proof. It follows from *Theorem* 4.4 that R is a right p.p.-ring since $ann_r(x)$ is finitely generated for every $x \in R$. Thus, for each $x \in R$, there is some idempotent $e \in R$ such that $ann_r(x) = eR$. Suppose R contains an infinite set E of orthogonal idempotents. Consider two idempotents e_1 and e_2 in E, and let $e_1 r \in e_1 R$. Note that since e_1 and e_2 are orthogonal idempotents, $e_1 r = (e_1 + 0)r = (e_1^2 + e_2 e_1)r = (e_1 + e_2)e_1 r \in (e_1 + e2)R$. Therefore, $e_1 R \subseteq (e_1 + e_2)R$. Inductively, we can construct an ascending chain of principal ideals generated by idempotents. For if $e_1, ..., e_n, e_{n+1}$ are orthogonal idempotents in the infinite set and $(e_1 + ... + e_n)r \in (e_1 + ... + e_n)R$, then $(e_1 + e_2 + ... + e_n)r = (e_1^2 + e_2^2 ... + e_n^2 + 0)r = [(e_1^2 + e_1 e_2 + ... e_1 e_n) + (e_2 e_1 + e_2^2 + ... + e_2 e_n) + ... + (e_n e_1 + ... + e_n^2) + (e_{n+1} e_1 + ... + e_{n+1} e_n)]r = (e_1 + ... + e_{n+1})(e_1 + ... + e_n)r \in (e_1 + ... + e_{n+1})R$.

Hence, $e_1 R \subseteq (e_1 + e_2)R \subseteq ... \subseteq (e_1 + ... + e_n)R \subseteq (e_1 + ... + e_{n+1})R \subseteq ...$ is an ascending chain of principal ideals generated by idempotents. Furthermore, this will be an infinite chain since there are an infinite number of idempotents in E. Note that by Lemma 4.5, for each $n \in \mathbb{Z}^+$, $(e_1 + ... + e_n)R = ann_r(x)$ for some $x \in R$. Thus, an infinite ascending chain of right annihilators has been constructed, contradicting the ascending chain condition on right annihilators. Therefore, R does not contain an infinite set of orthogonal idempotents. Since R is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents, by Theorem 2.11 every right annihilator in R is generated by an idempotent. Therefore, R is a Baer-ring. $\square$

Chapter 5

Non-singularity

5.1 Essential Submodules and the Singular Submodule

Let R be a ring and consider a submodule A of a right R-module M. If $A \cap B$ is nonzero for every nonzero submodule B of M, then A is said to be an *essential* submodule of M. This is denoted $A \leq_e M$. In other words, $A \leq_e M$ if and only if $B = 0$ whenever $B \leq M$ is such that $A \cap B = 0$. A monomorphism $\alpha : A \to B$ is called *essential* if $im(A) \leq_e B$.

Proposition 5.1. *[5, Corollary 5.13] A monomorphism $\alpha : A \to B$ is essential if and only if, for every right R-module C and every $\beta \in \operatorname{Hom}_R(B, C)$, β is a monomorphism whenever $\beta\alpha$ is a monomorphism.*

The *singular submodule of* M is defined as $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of R$\}$. Equivalently, $Z(M) = \{x \in M \mid ann_r(x) \leq_e R\}$. For if $I \leq_e R$ and $x \in M$ is such that $xI = 0$, then for any nonzero right ideal J of R, there is an element $a \in I \cap J$. Since $a \in I$, $xa = 0$. Hence, $a \in ann_r(x) \cap J$ and so $ann_r(x) \leq_e R$. On the other hand, note that $ann_r(x)$ is a right ideal of R such that $x \cdot ann_r(x) = 0$. A right R-module M is called *singular* if $Z(M) = M$ and *non-singular* if $Z(M) = 0$. If R is viewed a right R-module, then the *right singular ideal of* R is $Z_r(R) = Z(R_R)$. The ring R is *right non-singular* if it is non-singular as a right R-module.

Proposition 5.2. *[17] A right R-module A is non-singular if and only if $\operatorname{Hom}_R(C, A) = 0$ for every singular right R-module C.*

Proof. Suppose A is a non-singular right R-module and C is a singular right R-module. Let $f \in \operatorname{Hom}_R(C, A)$. If it can be shown that $f(Z(C)) \leq Z(A)$, then the proof follows readily since $f(C) = f(Z(C))$ and $Z(A) = 0$. Suppose $x \in Z(C)$. Then, $ann_r(x) \leq_e R$. Hence, if I is any nonzero right ideal of R, then there exists some $y \in I$ such that $xy = 0$. Then, $f(x)y = f(xy) = f(0) = 0$ and $y \in ann_r(f(x)) \cap I$. Thus, $ann_r(f(x)) \leq_e R$ and so $f(x) \in Z(A)$. Therefore, $f(Z(C)) \leq Z(A)$.

Conversely, suppose A is a right R-module and $\mathrm{Hom}_R(C, A) = 0$ for every singular right R-module C. Then, $\mathrm{Hom}_R(Z(A), A) = 0$ since the singular submodule $Z(A)$ is singular. Hence, the inclusion map $\iota : Z(A) \to A$ given by $\iota(x) = x$ is a zero map. Thus, $Z(A) = \iota(Z(A)) = 0$. Therefore, A is a non-singular right R-module. □

Proposition 5.3. *[17] The following are equivalent for a right R-module C:*

(a) C is singular.

(b) There exists an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that f is essential.

Proof. $(a) \Rightarrow (b)$: Suppose C is a right R-module. Let $0 \to A \xrightarrow{\iota} B \xrightarrow{g} C \to 0$ be an exact sequence of right R-modules such that B is free and ι is the inclusion map. Let $\{x_\alpha\}_{\alpha \in K}$ be a basis for B for some index K. Then, for each $\alpha \in K$, $g(x_\alpha) \in C = Z(C)$. Hence, there exists an essential right ideal I_α of R such that $g(x_\alpha I_\alpha) = g(x_\alpha)I_\alpha = 0$. Thus, for each $\alpha \in K$ and each $i_\alpha \in I_\alpha$, $x_\alpha i_\alpha \in \ker g = A$. That is, $x_\alpha I_\alpha \leq A$ for each $\alpha \in K$, and it follows that $\bigoplus_K x_\alpha I_\alpha \leq A$. If $x_\alpha J$ is a nonzero right ideal of $x_\alpha R$, then J is a nonzero right ideal of R, and there is a nonzero element $y \in I_\alpha \cap J$. Then it readily follows that $x_\alpha y \in x_\alpha I_\alpha \cap x_\alpha J$ is nonzero. Hence, $x_\alpha I_\alpha \leq_e x_\alpha R$ for each $\alpha \in K$. Thus, $\bigoplus_K x_\alpha I_\alpha \leq_e \bigoplus_K x_\alpha R = B$. Therefore, A is also essential in B since $\bigoplus_K x_\alpha I_\alpha \leq A$. It then follows from the exactness of the sequence that $im(A) \cong A \leq_e B$.

$(b) \Rightarrow (a)$: Assume $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of right R-modules such that $im(A) \leq_e B$. For each $b \in B$, define $h_b : R \to B$ by $h_b(r) = br$, and let
$I_b = \{r \in R \mid br \in im(A)\}$. Note that I_b is a nonzero right ideal of R. Suppose I_b is not essential in R. Then there is a nonzero right ideal J of R such that $I_b \cap J = 0$. Moreover, if $s \in \ker(h_b)$, then $h_b(s) = bs = 0 \in im(A)$ and it follows that $\ker(h_b) \subseteq I_b$. Hence,

$\ker(h_b) \cap J = 0$. Thus, $h_b|_J$ is a monomorphism. This implies that $h_b(J)$ must be a nonzero right ideal of B since J is a nonzero right ideal of R. Thus, $h_b(J) \cap im(A) \neq 0$ by the assumption that $im(A) \leq_e B$. Then for some nonzero $j \in J$, $bj = h_b(j) \in im(A)$. Hence, $j \in I_b \cap J$, which is a contradiction. Therefore, I_b is an essential right ideal of R. Note that for every $b \in B$, if $bi \in bI_b$, then $bi \in im(A)$. Then by exactness of the sequence, $bI_b \subseteq im(A) = \ker g$. Hence, $g(b)I_b = g(bI_b) = 0$, which implies $g(b) \in Z(C)$. Since this is the case for every $b \in B$, $g(B) \subseteq Z(C)$. Furthermore, since the sequence is exact, $C = g(B) \subseteq Z(C)$. Therefore, $C = Z(C)$. □

Proposition 5.4. *If R is a right p.p.-ring, then R is a right non-singular ring.*

Proof. Let R be a right p.p.-ring and take any $x \in R$. Suppose $ann_r(x) \leq_e R$. Since R is a right p.p.-ring, $ann_r(x) = eR$ for some idempotent $e \in R$. Observe that $R = eR \bigoplus (1-e)R$. Hence, $ann_r(x) \cap (1-e)R = 0$. However, this implies that $(1-e)R = 0$ since $ann_r(x) \leq_e R$. Hence, $1 - e = 0$, and so $ann_r(x) = 1R = R$. Thus, $xr = 0$ for every $r \in R$, which implies $x = 0$. Therefore, R is right non-singular. □

5.2 The Maximal Ring of Quotients and Right Strongly Non-singular Rings

The maximal ring of quotients and strongly non-singular rings will play an important role in determining which rings satisfy the condition that the classes of torsion-free and non-singular modules coincides. We explore these concepts in this section. If R is a subring of a ring Q, then Q is a *classical right ring of quotients* of R if every regular element of R is a unit in Q and every element of Q is of the form rs^{-1}, where $r, s \in R$ with s regular [21]. We will discuss this construction in more detail in Chapter 9 when we discuss general localizations. For a ring which is not necessarily commutative, such a Q may not exist. Thus, we consider a more general way to define the right ring of quotients which guarantees its existence for any ring R.

This general construction of the right ring of quotients is based on the property that $R_R \leq_e Q_R$ whenever R is a ring with classical right ring of quotients Q. However, we actually need a property slightly stronger than essentiality. Let A be a submodule of a right R-module B. If $\mathrm{Hom}_R(M/A, B) = 0$ for every right R-module M satisfying $A \leq M \leq B$, then B is a *rational extension* of A. This is denoted $A \leq_r B$.

Lemma 5.5. *[17] Let B be a non-singular right R-module and take any submodule A of B. Then, $A \leq_r B$ if and only if $A \leq_e B$.*

Proof. Suppose $A \leq_r B$ and let $M \leq B$ be such that $M \cap A = 0$. Now, $M \bigoplus A$ is a right R-module satisfying $A \leq M \bigoplus A \leq B$. Hence, $\mathrm{Hom}_R([M \bigoplus A]/A, B) = 0$. Consider $f : (M \bigoplus A)/A \to M$ defined by $(m+a)+A \mapsto m$ for $m \in M$ and $a \in A$. If $m, m_0 \in M$ and $a, a_0 \in A$ are such that $(m+a)+A = (m_0)+a_0)+A$, then $(m-m_0)+(a-a_0) \in A$. Hence, $m-m_0 \in A$. However, $M \cap A = 0$ and so $m-m_0 = 0$. Thus, f is well-defined. Moreover, f is an isomorphism. For if $m \in M$, then $f[(m+a)+A] = m$ for any $a \in A$, and $f[(m+a)+A] = 0$ implies that $(m+a)+A = m+A = 0$. Observe that $f \in \mathrm{Hom}_R([M \bigoplus A]/A, B) = 0$ since $M \leq B$. Thus, $M = im(f) = 0$ and therefore $A \leq_e B$. Note that this implication does not require B to be right non-singular.

On the other hand, suppose $A \leq_e B$ and take M to be a right R-module such that $A \leq M \leq B$. Then, any nonzero submodule N of B is such that $A \cap N \neq 0$. Hence, any nonzero submodule K of M is such that $A \cap K \neq 0$ since any such submodule is also a submodule of B. Thus, $A \leq_e M$. Consider the exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{\pi} M/A \to 0$, where ι is the inclusion map and π is the canonical epimorphism. Observe that $im(\iota) = A \leq_e M$. Hence, M/A is singular by Proposition 5.3. It then follows from Proposition 5.2 that $\mathrm{Hom}_R(M/A, B) = 0$ since B is nonsingular. Therefore, B is a rational extension of A. $\square$

Let R be a subring of a ring Q. If $R_R \leq_r Q_R$, then Q is a *right ring of quotients of R*. Observe that R is a right ring of quotients of itself since $R_R \leq_r R_R$. Similarly, if ${}_RR \leq_r {}_RQ$, then Q is a *left ring of quotients of R*. Let Q be a right ring of quotients of R such that given any other right ring of quotients P of R, the inclusion map $\mu : R \to Q$ extends to a monomorphism $\nu : P \to Q$. Here, Q is called a *maximal right ring of quotients of R*. This is denoted Q^r when there is no confusion as to which ring the maximal quotient ring applies, and $Q^r(R)$ otherwise. The *maximal left ring of quotients Q^l* is similarly defined. In general, $Q^r \neq Q^l$.

A right R-module E is called *injective* if, given any two right R-modules A and B, a monomorphism $\alpha : A \to B$, and a homomorphism $\varphi : A \to E$, there exists a homomorphism $\psi : B \to E$ such that $\varphi = \psi\alpha$. If M is a right R-module and E is an injective right R-module such that $M_R \leq_e E_R$, then E is called an *injective hull* of M. Every right R-module M has an injective hull, which is unique up to isomorphism [17, Theorems 1.10, 1.11].

Theorem 5.6. *[17] For any ring R, the maximal right ring of quotients $Q^r(R)$ exists. In particular, if E is the injective hull of R_R and $T = End_R(E)$, then $Q = \cap\{\ker\delta \mid \delta \in T$ and $\delta R = 0\}$ is a maximal right ring of quotients.*

Proof. If E is the injective hull of R, then $\tau x = \tau(x)$ defines a left T-module structure on E for $\tau \in T$ and $x \in E$. Let $T_0 = End_T(E)$ and define $\omega(x) = x\omega$ for $\omega \in T_0$ and $x \in E$. Consider the homomorphisms $\psi : T \to E$ and $\varphi : T_0 \to E$ defined by $\psi\tau = \tau 1$ and $\varphi\omega = 1\omega$. It is easily seen that ψ is an epimorphism and φ is a monomorphism. Let $x \in E$ and consider the homomorphism $\sigma : R \to xR$ defined by $\sigma(r) = xr$. Since R is a subring of E, σ can be extended to a homomorphism $\tau : E \to E$. Thus, $\tau(1) = \sigma(1) = x$ and so $\psi(\tau) = \tau(1) = x$. Therefore, ψ is an epimorphism. Now, suppose $\omega \in \ker\varphi$. Then $1\omega = \varphi(\omega) = 0$. If $x \in E$, then $\tau 1 = x$ for some $\tau \in T$ since ψ is an epimorphism. Hence, $\omega(x) = x\omega = (\tau 1)\omega = \tau(1\omega) = \tau(0) = 0$. Therefore, $\omega = 0$ and φ is a monomorphism.

If $\delta \in T$ is such that $\delta R = 0$, then $\delta(1\omega) = (\delta 1)\omega = 0$ for every $\omega \in T_0$. Hence, $1\omega \in Q$. Therefore, φ can actually be defined as a map $T_0 \to Q$. It readily follows that φ maps onto Q and hence $\varphi : T_0 \to Q$ is an isomorphism. To see this, let $x \in Q$ and consider $\nu : E \to E$ defined by $(\tau 1)\nu = \tau x$. This can be defined for every $\tau \in T$ since

φ is a well-defined epimorphism onto E. Thus, if $1_E \in T$ is the identity map on E, then $\varphi(\nu) = 1\nu = [1_E(1)]\nu = 1_E(x) = x$. Therefore, φ is onto.

We now define multiplication on Q. For $x, y \in Q$, let $x \cdot y = \varphi[(\varphi^{-1}x)(\varphi^{-1}y)] = 1(\varphi^{-1}x)(\varphi^{-1}y)$. Clearly $x \cdot y \in Q$ and it is easily seen to be associative. Since φ is an isomorphism, if $r \in R$, then there exists some $\omega \in T_0$ such that $\varphi(\omega) = 1\omega = r$. Thus, if $x \in Q$, then $x \cdot r = 1(\varphi^{-1}x)(\varphi^{-1}r) = (\varphi\varphi^{-1}x)(\omega) = x\omega = (x1)\omega = x(1\omega) = xr$. It follows from [17, Theorem 2.26] that this multiplication defines a unique ring structure on Q which is consistent with the R-module structure..

To see that Q is a right ring of quotients, suppose $R \leq M \leq Q$ for some right R-module M and let $\alpha \in \mathrm{Hom}_R(M/R, Q)$. Consider the epimorphism $\pi : M \to M/R$ given by $x \mapsto x+R$, and define $\gamma = \alpha\pi : M \to Q$. Observe that $\gamma R = 0$ since $\gamma(r) = \alpha\pi(r) = \alpha(r+R) = 0$ for any $r \in R$. Moreover, γ can be extended to a map $\beta \in T$ such that $\beta R = 0$. Since Q is the intersection of the kernels of all homomorphisms $\delta \in T$ satisfying $\delta R = 0$, $M \subseteq Q \subseteq \ker \beta$. Thus, $\gamma M = \beta M = 0$ and so $\alpha(x + R) = \gamma(x) = 0$ for any $x \in M$. Therefore, $R \leq_r Q$ and Q is a right ring of Quotients.

To see that Q^r is a maximal right ring of quotients, let P be another right ring of quotients. Then $R_R \leq_r P_R$ by definition, and hence $R_R \leq_e P_R$ by Lemma 5.5. If $\iota : R \to P$ and $\mu : R \to E$ are the inclusion maps, then by injectivity of E, there exists a homomorphism $\nu : P \to E$ such that $\nu\iota = \mu$. Observe that $R \cap \ker \nu = \ker \mu = 0$. This implies $\ker \nu = 0$ since R is essential in P and $\ker \nu$ is a submodule of P. Therefore, the inclusion map $\mu : R \to E$ can be extended to a monomorphism $\nu : P \to E$. Moreover, [17, Theorem 2.26] shows that νP is contained in Q, and hence the inclusion map $R \to Q$ can be extended to a monomorphism $\nu : P \to Q$. Finally, note that since $R \leq \nu P \leq Q$ and $R_R \leq_r Q_R$, $\mathrm{Hom}_R(\nu P/R, Q) = 0$. Hence, given $x \in P$, the homomorphism $\sigma : \nu P/R \to Q$ defined by $\sigma(\nu y + R) = \nu(xy) - (\nu x)(\nu y)$ is the zero map. Therefore, ν is a ring homomorphism and Q is a maximal right ring of quotients of R. $\square$

Goodearl shows in [17, Corollary 2.31] that Q^r is injective as a right R-module. Therefore, $Q^r(R)$ is an injective hull of R since $R_R \leq_e Q^r_R$ by Lemma 5.5. Moreover, since the injective hull is unique up to isomorphism, we can refer to $Q^r(R)$ as the injective hull of R. The following results about maximal quotient rings will be needed later. The proofs are omitted.

Proposition 5.7. *[3, Proposition 2.2] For a right non-singular ring R, R is a left p.p.-ring such that $Q^r(R)$ is torsion-free as a right R-module if and only if all non-singular right R-modules are torsion-free.*

Theorem 5.8. *[28, Ch. XII, Proposition 7.2] If R is a right non-singular ring and M is a finitely generated non-singular right R-module, then there exists a monomorphism $\varphi : M \to \oplus_n Q^r$ for some $n < \omega$. In other words, M is isomorphic to a submodule of a free Q^r-module.*

For a ring R, its maximal right ring of quotients Q^r is a *perfect left localization of R* if Q^r is flat as a right R-module and the multiplication map $\varphi : Q^r \bigotimes_R Q^r \to Q^r$, defined by $\varphi(a \otimes b) = ab$, is an isomorphism. If R is a right non-singular ring for which Q^r is a perfect left localization, then R is called *right strongly non-singular.* Goodearl provides the following useful characterization of right strongly non-singular rings:

Theorem 5.9. *[17, Theorem 5.17] Let R be a right non-singular ring. Then, R is right strongly non-singular if and only if every finitely generated non-singular right R-module is isomorphic to a finitely generated submodule of a free right R-module.*

Corollary 5.10. *[17, Theorem 5.18] Let R be a right non-singular ring. Then, R is right semi-hereditary, right strongly non-singular if and only if every finitely generated non-singular right R-module is projective.*

Proof. For a right non-singular ring R, suppose R is right semi-hereditary, right strongly non-singular. Let M be a finitely generated non-singular right R-module. By Theorem 5.9,

M is isomorphic to a finitely generated submodule of a free right R-module F. Therefore, since R is right semi-hereditary, M is projective by Lemma 2.5.

Conversely, assume every finitely generated non-singular right R-module is projective. Since R is right non-singular, every finitely generated right ideal of R is non-singular. Hence, every finitely generated right ideal is projective and R is right semi-hereditary. Furthermore, every finitely generated non-singular right R-module is a direct summand, and hence a submodule, of a free right R-module. Therefore, R is right strongly non-singular by Theorem 5.9. $\square$

5.3 Coincidence of Torsion-freeness and Non-singularity

We know turn our attention to rings for which the classes of torsion-free and non-singular right R-modules coincide, which is investigated in [3] by Albrecht, Dauns, and Fuchs. A few definitions are needed before stating their theorems in full. A ring is *right semi-simple* if it can be written as a direct sum of modules which have no proper nonzero submodules, and a ring is *right Artinian* if it satisfies the descending chain condition on right ideals. Assume *semi-simple Artinian* to mean *right semi-simple, right Artinian*. The following results from Stenström consider rings with semi-simple right maximal ring of quotients.

Proposition 5.11. *[28, Ch. XI, Proposition 5.4] Let R be a ring whose maximal right ring of quotients is semi-simple. Then, $Q^r = Q^l$ if and only if Q^r is flat as a right R-module.*

Theorem 5.12. *[28, Ch. XII, Corollaries 2.6,2.8] Let R be a ring and suppose $Q^r(R)$ is semi-simple. Then:*

(a) Q^r is a perfect right localization of R. In other words, if R is left non-singular, then it is left strongly non-singular.

(b) If M is any non-singular right R-module, then $M \bigotimes_R Q^r$ is the injective hull of M.

A ring R is *von Neumann regular* if, given any $r \in R$, there exists some $s \in R$ such that $r = rsr$. These rings are of interest because R is von Neumann regular if and only if every right R-module is flat [26, Theorem 4.9]. The following lemmas will be needed in the next chapter.

Lemma 5.13. *[26] If R is a semi-simple Artinian ring, then R is von Neumann regular.*

Proof. The Wedderburn-Artin Theorem states that R is semi-simple Artinian if and only if it is isomorphic to a finite direct product of matrix rings over division rings. For any division ring D, $Mat_n(D) \cong End_D(\bigoplus^n D)$ is von Neumann regular [26]. Therefore, R is von Neumann regular since direct products of regular rings are regular. $\square$

Lemma 5.14. *[28] A ring R is right non-singular if and only if Q^r is von Neumann regular.*

Proof. Stenström shows in [28, Ch. XII] that if R is right non-singular, then $Q^r \cong End_R(E)$, where $E \cong Q^r$ is the injective hull of R. In [28, Ch. V, Proposition 6.1], it is shown that such rings are regular.

Conversely, assume Q^r is von Neumann regular. Let I be an essential right ideal of R and take $x \in R$ to be nonzero. Suppose $xI = 0$. Since Q^r is regular, there exists some $q \in Q$ such that $xqx = x$. Hence, qxR is a nonzero right ideal of R, and so $I \cap qxR \neq 0$. Thus, $0 \neq qxr \in I$ for some nonzero $r \in R$. However, $xr = xqxr \in xI = 0$. This implies $qxr = 0$, which is a contradiction. Therefore, $xI \neq 0$ and R is right non-singular. $\square$

Let R be a ring and M a right R-module. A submodule U of M is ***S**-closed* if M/U is non-singular. The following lemma shows that annihilators of elements are **S**-closed for non-singular rings.

Lemma 5.15. *If R is a right non-singular ring, then for any $x \in R$, $ann_r(x)$ is **S**-closed.*

Proof. Let R be right non-singular. It needs to be shown that $R/ann_r(x)$ is non-singular for any $x \in R$. That is, for $x \in R$, $Z(R/ann_r(x)) = \{r + ann_r(x) \mid (r + ann_r(x))I = 0$ for some $I \leq_e R\} = 0$. Let $0 \neq r + ann_r(x) \in R/ann_r(x)$ and I be a nonzero essential right ideal of R such that $(r + ann_r(x))I = 0$. Then, for any $a \in I$, $ra + ann_r(x) = 0$. Hence, $ra \in ann_r(x)$ and $xra = 0$ for every $a \in I$. In other words, $(xr)I = 0$. If $xr \neq 0$, then there is a contradiction since $I \leq_e R$ and $Z(R) = 0$. Thus, $xr = 0$ and $r \in ann_r(x)$. Therefore, $r + ann_r(x) = 0$, and it follows readily that $Z(R/ann_r(x)) = 0$. □

If R is a right non-singular ring and every **S**-closed right ideal of R is a right annihilator, then R is referred to as a *right Utumi ring*. Similarly, R is a *left Utumi ring* if R is left non-singular and every **S**-closed left ideal of R is a left annihilator. The following result from Goodearl characterizes non-singular rings which are both right and left Utumi.

Theorem 5.16. *[17, Theorem 2.38] If R is a right and left non-singular ring, then $Q^r = Q^l$ if and only if every R is both right and left Utumi.*

For a ring R, if every direct sum of nonzero right ideals of R contains only finitely many direct summands, then R is said to have *finite right Goldie-dimension*. Denote the Goldie-dimension of R as G-dim R_R. If a ring R with finite right Goldie-dimension also satisfies the ascending chain condition on right annihilators, then R is a *right Goldie-ring*. The maximal right quotient ring Q^r is a *semi-perfect left localization of R* if Q^r_R is torsion-free and the multiplication map $Q^r \bigotimes_R Q^r \to Q^r$ is an isomorphism. The following is a useful characterization of rings with finite right Goldie-dimension:

Theorem 5.17. *[28, Ch. XII, Theorem 2.5] If R is a right non-singular ring, then Q^r is semi-simple if and only if R has finite right Goldie-dimension.*

We are now ready to state two key results from U. Albrecht, L. Fuchs, and J. Dauns, which consider rings for which the classes of torsion-free and non-singular modules coincide. These will be needed in the next chapter to prove the main theorem of this thesis. The proof of Theorem 5.18 is omitted.

Theorem 5.18. *[3, Theorem 3.7] The following are equivalent for a ring R:*

(a) R is a right Goldie right p.p.-ring and Q^r is a semi-perfect left localization of R.

(b) R is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents.

(c) R is a right non-singular ring which does not contain an infinite set of orthogonal idempotents, and every finitely generated non-singular right R-module is torsion-free.

(d) A right R-module M is torsion-free if and only if M is non-singular.

Furthermore, if R satisfies any of the equivalent conditions, then R is a Baer-ring and Q^r is semi-simple Artinian.

Theorem 5.19. *[3] The following are equivalent for a ring R:*

*(a) R is a right and left non-singular ring which does not contain an infinite set of orthogonal idempotents, and every **S**-closed left or right ideal is generated by an idempotent.*

(b) R is a right or left p.p.-ring, and $Q^r = Q^l$ is semi-simple Artinian.

(c) R is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents.

(d) R is right strongly non-singular, and a right R-module is torsion-free if and only if it is non-singular.

(e) For a right R-module M, the following are equivalent:

(i) M is torsion-free

(ii) M is non-singular

(iii) If $E(M)$ is the injective hull of M, then $E(M)$ is flat.

Proof. $(a) \Rightarrow (b)$: Assume R is right and left non-singular, contains no infinite set of orthogonal idempotents, and every **S**-closed right or left ideal is generated by an idempotent. Let I be an **S**-closed right ideal of R. Then, $I = eR$ for some idempotent $e \in R$. As shown in the proof of Lemma 4.5, $eR = ann_r(1-e)$. Thus, $I = eR$ is the right annihilator of $1-e$. Note that a symmetric argument shows that if J is an **S**-closed left ideal of R, then $J = Rf$ is a left annihilator of $1-f$ for some idempotent $f \in R$. Hence, R is both a right and left Utumi ring. By Lemma 5.15, since R is a right non-singular ring, $ann_r(x)$ is **S**-closed for every $x \in R$. This implies that $ann_r(x)$ is generated by an idempotent for every $x \in R$. Therefore, R is a right p.p-ring. A symmetric argument shows that R is also a left p.p.-ring since condition (a) applies to both right and left ideals. Note that R satisfies condition (b) of Theorem 5.18 since it is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents. Hence, Q^r is semi-simple Artinian by Theorem 5.18. Furthermore, since every right and left **S**-closed ideal is an annihilator, R is right and left Utumi. Therefore, $Q^r = Q^l$ by Theorem 5.16.

$(b) \Rightarrow (c)$: Suppose R is a right p.p.-ring and $Q^r = Q^l$ is semi-simple Artinian. Since R is a right p.p.-ring, it is also a right non-singular ring. Hence, R has finite right Goldie-dimension by Theorem 5.17. Suppose R contains an infinite set of orthogonal idempotents. Consider two orthogonal idempotents e and f, and let $x \in eR \cap fR$. Then, $x = er = fs$ for some r, $s \in r$. This implies that $x = 0$ since $er = e^2r = efs = 0$. Thus, $eR \cap fR = 0$ for any two orthogonal idempotents e and f in the infinite set, and $eR \bigoplus fR$ is direct. Hence, R contains an infinite direct sum of nonzero right submodules, which contradicts R having finite right Goldie-dimension. Therefore, R does not contain an infinite set of orthogonal idempotents.

By Theorem 5.12, since R is semi-simple Artinian, R is a left strongly non-singular ring. Hence, the multiplication map $\varphi : Q^r \bigotimes_R Q^r \to Q^r$, defined by $\varphi(q \otimes p) = qp$, is an isomorphism. Note that this also implies that Q^r is flat as a left R-module. However, in order for R to be right strongly non-singular, it needs to be shown that Q^r is flat as a right R-module. By Proposition 5.11, Q^r is indeed flat as a right R-module since $Q^r = Q^l$ is assumed to be semi-simple Artinian. Therefore, R is a right strongly non-singular ring which does not contain an infinite set of orthogonal idempotents. Note that Theorem 2.11 shows that R is also a left p.p.-ring. Thus, if we had instead assumed that R is a left p.p.-ring, then a symmetric argument could be used to show that R is also a right p.p.-ring, and the latter part of the proof would remain the same.

$(c) \Rightarrow (d)$: Assume R is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Then, Q^r is flat as a right R-module, which follows from R being right strongly non-singular. Since flat R-modules are torsion-free, this implies that Q^r is torsion-free. By Theorem 2.11, since R is a right p.p.-ring and does not contain an infinite set of orthogonal idempotents, R is also a left p.p.-ring. Hence, every non-singular right R-module is torsion-free by Proposition 5.7. Thus, R satisfies condition (c) of Theorem 5.18, which implies that a right R-module M is torsion-free if and only if M is non-singular.

$(d) \Rightarrow (e)$: Suppose R is right strongly non-singular, and a right R-module is torsion-free if and only if it is non-singular. Then, conditions (i) and (ii) of (e) are clearly equivalent, and it suffices to show that a right R-module is non-singular if and only if its injective hull is flat. Suppose M is a non-singular right R-module. Note that R satisfies condition (d) of Theorem 5.18, and hence Q^r is semi-simple Artinian. By Theorem 5.12, $M \bigotimes_R Q^r$ is an injective hull of M. Thus, if $E(M)$ denotes the injective hull of M, then $E(M) \cong M \bigotimes_R Q^r$, since an injective hull of a right R-module is unique up to isomorphism. This implies that $E(M)$ is a right Q^r-module, since $M \bigotimes_R Q^r$ is a right Q^r-module by Proposition 3.9. Furthermore, since Q^r is semi-simple Artinian, every Q^r-module is projective. Hence, $E(M)$ is projective and thus isomorphic to a direct summand of a free Q^r-module F. Note that Q^r is flat as a right R-module since R is right strongly non-singular. Thus, Proposition 3.7 shows that any free Q^r-module is flat since such modules can be written as $\bigoplus_{i \in I} M_i$ for some index set I, where M_i is isomorphic to Q^r for every $i \in I$. This implies that $E(M)$ is flat by Proposition 3.7 since it is a direct summand of the flat right R-module $F = \bigoplus_{i \in I} M_i$.

On the other hand, assume that the injective hull $E(M)$ of some right R-module M is flat. Noting again that R satisfies condition (d) of Theorem 5.18, it follows that R is a right p.p.-ring. Thus, R is a torsion-free ring by Theorem 4.4. Since flat R-modules are torsion-free, $E(M)$ is torsion-free as a right R-module. Hence, M is a submodule of a torsion-free right R-module. Thus, M is a torsion free right R-module by Proposition 4.3. Therefore,

M is non-singular since a right R-module is torsion-free if and only if it is non-singular by assumption.

$(e) \Rightarrow (a)$: For a right R-module M, assume that M is torsion-free if and only if M is non-singular if and only if the injective hull $E(M)$ is flat. By Theorem 5.18, R is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. It then follows from Proposition 5.4 that R is a right non-singular ring. Hence, R is also a left p.p.-ring by Proposition 5.7, since every non-singular right R-module is torsion-free, and a symmetric argument for Proposition 5.4 shows that R is left non-singular.

The injective hull $E(R)$ is flat as a right R-module since R is assumed to be right non-singular. Hence, Q^r is flat as a right R-module, since Q^r is the injective hull of R. We've already shown that R satisfies the equivalent conditions of Theorem 5.18, which implies that Q^r is a semi-simple Artinian ring. Thus, it follows from Proposition 5.11 that $Q^r = Q^l$. Since R is both right and left non-singular, every **S**-closed right ideal of R is a right annihilator and every **S**-closed left ideal of R is a left annihilator by Theorem 5.16. Furthermore, note that Theorem 5.18 shows that R is a Baer-ring. Hence, every annihilator is generated by an idempotent. Therefore, every **S**-closed right ideal and every **S**-closed left ideal is generated by an idempotent. $\square$

Chapter 6

Morita Equivalence

Before proving the main theorem, we discuss Morita equivalences. In particular, we show that there is a Morita equivalence between R and $Mat_n(R)$ for any $0 < n < \omega$. This is then used to show that the classes of torsion-free and non-singular $Mat_n(R)$-modules coincide for certain conditions placed on R.

Let R and S be rings. The categories Mod_R and Mod_S are *equivalent* (or *isomorphic*) if there are functors $F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$ such that $FG \cong 1_{Mod_S}$ and $GF \cong 1_{Mod_R}$. Note that these are natural isomorphisms. In other words, if $\eta : GF \to 1_{Mod_R}$ denotes the natural isomorphism, then for each $M, N \in Mod_R$, there exist isomorphisms $\eta_M : GF(M) \to M$ and $\eta_N : GF(N) \to N$ such that $\beta\eta_M = \eta_N GF(\beta)$ whenever $\beta \in \mathrm{Hom}_R(M, N)$. Here, $GF(\beta)$ denotes the induced homomorphism. The functors F and G are referred to as an *equivalence* of Mod_R and Mod_S. If such an equivalence exists, then R and S are said to be *Morita-equivalent*. In [28, Ch. IV, Corollary 10.2], Stenström shows that R and S are Morita-equivalent if and only if there are bimodules ${}_SP_R$ and ${}_RQ_S$ such that $P \bigotimes_R Q \cong S$ and $Q \bigotimes_S P \cong R$. A property P is referred to as *Morita-invariant* if for every ring R satisfying P, every ring S Morita-equivalent to R also satisfies P.

A *generator* of Mod_R is a right R-module P satisfying the condition that every right R-module M is a quotient of $\bigoplus_I P$. Note that R and any free right R-module are generators of Mod_R. A *progenerator* of Mod_R is a generator which is finitely generated and projective.

Lemma 6.1. *[5] Let R be a ring, P a progenerator of Mod_R, and $S = End_R(P)$. Then, there is an equivalence $F : Mod_R \to Mod_S$ given by $F(M) = \mathrm{Hom}_R(P, M)$ with inverse $G : Mod_S \to Mod_R$ given by $G(N) = N \bigotimes_S P$.*

Proof. As a projective generator of Mod_R, P is a right R-module. P also has a left S-module structure with $(f * g)(x) = f(g(x))$ for $f, g \in S$ and $x \in P$, where multiplication in the endomorphism ring is defined as composition of functions. It then readily follows that P is an (S, R)-bimodule since $f(xr) = f(x)r$ for any $f \in S$ and $r \in R$. Thus, $F = \mathrm{Hom}_S(P, \_)$ is a functor $Mod_S \to Mod_R$ and $G = \_ \bigotimes_R P$ is a functor $Mod_R \to Mod_S$ by Proposition 3.10.

It needs to be shown that $GF \cong 1_{Mod_R}$ and $FG \cong 1_{Mod_S}$ are natural isomorphisms. Since P is a progenerator of Mod_R, it is finitely generated and projective as a right R-module. Thus, it follows from Lemma 3.12 that if M is any right R-module, then $GF(M) = G(\mathrm{Hom}_R(P, M)) = \mathrm{Hom}_R(P, M) \bigotimes_S P \cong \mathrm{Hom}_R(\mathrm{Hom}_S(P, P), M) \cong \mathrm{Hom}_R(End_S(P), M)$ $\cong \mathrm{Hom}_R(R, M) \cong M$. Similarly, given any right S-module N, $FG(N) = F(N \bigotimes_S P) = \mathrm{Hom}_R(P, N \bigotimes_S P) \cong N \bigotimes_S \mathrm{Hom}_R(P, P) = N \bigotimes_S S \cong N$ by Lemma 3.11. Therefore, F is an equivalence with inverse G. $\square$

Proposition 6.2. *Let R be a ring. For every $0 < n < \omega$, R is Morita-equivalent to $Mat_n(R)$.*

Proof. Let P be a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$ for $0 < n < \omega$. Then, P is a progenerator of Mod_R and $Mat_n(R) \cong End_R(P)$ by Lemma 2.6. Therefore, the equivalence of Lemma 6.1 is a Morita-equivalence between R and $Mat_n(R)$. $\square$

Lemma 6.3. *[28, Ch. X, Proposition 3.2] If R and S are Morita-equivalent, then the maximal ring of quotients, $Q^r(R)$ and $Q^r(S)$, are also Morita equivalent.*

Proposition 6.4. *Let R and S be Morita-equivalent rings with equivalence*
$F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$.

(i) If U is an essential submodule of a right R-module M, then $F(U)$ is an essential submodule of the right S-module $F(M)$.

(ii) If M is a non-singular right R-module, then $F(M)$ is a non-singular right S-module.

In other words, essentiality and non-singularity are Morita-invariant properties.

Proof. (i): Let U be an essential submodule of $M \in Mod_R$. Then, the inclusion map $\iota : U \to M$ is an essential monomorphism. Consider the induced homomorphism $F(\iota) : F(U) \to F(M)$. Note that since ι is a monomorphism, $F(\iota)$ is a monomorphism [5, Proposition 21.2]. Let W be any right S-module and take $\beta \in \mathrm{Hom}_S(F(M), W)$ to be such that $\beta F(\iota) : F(U) \to W$ is a monomorphism. There is a natural isomorphism $\Phi_{U,W} : \mathrm{Hom}_S(F(U), W) \to \mathrm{Hom}_R(U, G(W))$ defined by $\gamma \mapsto G(\gamma)\eta_U^{-1}$, where η_U denotes the isomorphism $GF(U) \to U$ [5, 21.1]. Hence, $\Phi_{U,W}(\beta F(\iota))$ is a monomorphism. Moreover, $\Phi_{U,W}(\beta F(\iota)) = G(hF(\iota))\eta_U^{-1} = G(h)GF(\iota)\eta_U^{-1} = G(h)\eta_M^{-1}\eta_M GF(\iota)\eta_U^{-1} = \Phi_{M,W}(\beta)\iota\eta_U\eta_U^{-1} = \Phi_{M,W}(\beta)\iota$. Thus, $\Phi_{M,W}(\beta)\iota$ is a monomorphism and it follows from Proposition 5.1 that $\Phi_{M,W}(\beta)$ is a monomorphism since ι is essential. Furthermore, $\Phi_{M,W}(\beta)$ is a monomorphism if and only if β is a monomorphism [5, Lemma 21.3]. Hence, $F(\iota)$ is an essential monomoprhism by Proposition 5.1. Therefore, $F(U) \cong im(F(\iota))$ is an essential submodule of $F(M)$.

(ii): Let M be a non-singular right R-module. It needs to be shown that $F(M)$ is a non-singular right S-module and in view of Proposition 5.2 it suffices to show that $\mathrm{Hom}_S(C, F(M)) = 0$ for any singular right S-module C. By Proposition 5.3, there is an exact sequence $0 \to A \xrightarrow{f} F \to C \to 0$ of right S-modules such that $f(A) \leq_e F$ and F is free. Then, $G(f(A)) \leq_e G(B)$ by (i). Hence, $0 \to G(A) \xrightarrow{G(f)} G(B) \to G(C) \to 0$ is an exact sequence of right R-modules such that $G(f(A)) \leq_e G(B)$. Thus, $G(C)$ is a singular right R-module by Proposition 5.3. Since $G(C)$ is singular and M is non-singular, $\mathrm{Hom}_R(G(C), M) = 0$ by Proposition 5.2. Therefore, $\mathrm{Hom}_S(C, F(M)) \cong \mathrm{Hom}_R(G(C), M) = 0$. Observe that in this proof, it is also shown that singularity is Morita-invariant since we show that $G(C)$ is singular for an arbitrary singular module C. □

We now prove the main theorem of this portion of the dissertation, which characterizes rings for which torsion-freeness and non-singularity coincide under a Morita-equivalence.

Theorem 6.5. *The following are equivalent for a ring R:*

(a) R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents.

(b) Whenever S is Morita-equivalent to R, then the classes of torsion-free right S-modules and non-singular right S-modules coincide.

(c) For every $0 < n < \omega$, $Mat_n(R)$ is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents.

Moreover, if R is such a ring, then the corresponding left conditions are also satisfied.

Proof. $(a) \Rightarrow (b)$: Assume R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents. Let R and S be Morita equivalent, and let $F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$ be an equivalence. Also, take N to be a finitely generated non-singular right R-module. Since R is right strongly non-singular, N is isomorphic to finitely generated submodule V of a free right R-module by Theorem 5.9. Furthermore, since R is right semi-hereditary and free R-modules are projective, $V \cong N$ is projective by Lemma 2.5. Thus, since projective modules are torsion-free, it follows that finitely generated non-singular right R-modules are torsion-free. Therefore, R satisfies condition (c) of Theorem 5.18, which implies that the maximal ring of quotients $Q^r(R)$ is a semi-simple Artinian ring. Note that $Q^r(R)$ and $Q^r(S)$ are Morita-equivalent by Lemma 6.3. Hence, $Q^r(S)$ is also semi-simple Artinian, since these properties are preserved under a Morita-equivalence [5]. Furthermore, $Q^r(S)$ is a regular ring by Lemma 5.13. Therefore, Lemma 5.14 shows that S is right non-singular.

Let M be a finitely generated non-singular right S-module. Then, $G(M)$ is a finitely generated non-singular right R-module since non-singularity and being finitely generated are both Morita-invariant properties [5]. Thus, since R is a right strongly non-singular ring, $G(M)$ is isomorphic to a finitely generated submodule of a free right R-module P by Theorem 5.9. Note that as a free right R-module, P is projective, which is also a Morita-invariant property [5]. Hence, $F(P)$ is a projective right S-module. Furthermore, since $G(M)$

is isomorphic to a finitely generated submodule of P, $FG(M) \cong M$ is isomorphic to a finitely generated submodule U of $F(P)$. Now, $F(P)$ is projective and hence a submodule of a free right S-module, which implies $U \cong M$ is a submodule of a free right S-module. Therefore, M is isomorphic to a finitely generated submodule of a free right S-module, and S is right strongly non-singular by Theorem 5.9.

It has been shown that S is a right non-singular ring with a semi-simple Artinian maximal right ring of quotients. Thus, S has finite right Goldie-dimension by Theorem 5.17. Hence, S cannot contain an infinite set of orthogonal idempotents. Moreover, S is a right p.p.-ring since R is right semi-hereditary. For if P is a principal right ideal of S, then $G(P)$ is a finitely generated right ideal of the right semi-hereditary ring R, which implies that $G(P)$ is projective. Hence, $FG(P) \cong P$ is projective, which again follows from projectivity being Morita-invariant. Then, S is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Therefore, a right S-module is torsion-free if and only if it is non-singular by Theorem 5.19.

$(b) \Rightarrow (a)$: Assume that the classes of torsion-free and non-singular S-modules coincide for every ring S Morita-equivalent to R. Thus, since $Mat_n(R)$ is Morita-equivalent to R for every $0 < n < \omega$, the classes of torsion-free right $Mat_n(R)$-modules and non-singular right $Mat_n(R)$-modules coincide for every $0 < n < \omega$. Hence, $Mat_n(R)$ is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents by Theorem 5.18. Thus, R is right semi-hereditary by Theorem 2.7. In particular, since these conditions are satisfied for every $0 < n < \omega$, they are satisfied for $n = 1$. Hence, $R \cong Mat_1(R)$ is a right semi-hereditary right Utumi ring not containing an infinite set of orthogonal idempotents.

It needs to be shown that R is right strongly non-singular. Let M be a finitely generated non-singular right R-module. By Corollary 5.10, R is right strongly non-singular if M is projective. Let $0 \to U \to F = \bigoplus^{n} R \to M \to 0$ be an exact sequence of right R-modules. Since F is a finitely generated free right R-module, it is a progenerator of Mod_R. Hence, $0 \to \mathrm{Hom}_R(F, U) \to \mathrm{Hom}_R(F, F) = End_R(F) \to \mathrm{Hom}_R(F, M) \to 0$ is exact by

Proposition 2.3. Moreover, if $S = End_R(F) \cong Mat_n(R)$, then $F : Mod_R \to Mod_S$ given by $F(M) = \mathrm{Hom}_R(F, M)$ and $G : Mod_S \to Mod_R$ given by $G(N) = N \bigotimes_S F$ is an equivalence by Lemma 6.1. Thus, $\mathrm{Hom}_R(F, M)$ is a non-singular right S-module by Proposition 6.4 (ii). Furthermore, since S is Morita-equivalent to R, the S-module $\mathrm{Hom}_R(F, M)$ is torsion-free by assumption. Note that since the sequence is exact, $\mathrm{Hom}_R(F, M) \cong S/\mathrm{Hom}_R(F, U)$. Thus, $\mathrm{Hom}_R(F, M)$ is cyclic as an S-module since $\mathrm{Hom}_R(F, U)$ is a right ideal of the right S-module S. Note also that S is a left p.p.-ring by Theorem 2.11 since S is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Thus, the cyclic torsion-free right S-module $\mathrm{Hom}_R(F, M)$ is projective by Proposition 4.7. Therefore, $M \cong GF(M) = G(\mathrm{Hom}_R(F, M))$ is a projective right R-module and R is right strongly non-singular.

$(a) \Rightarrow (c)$: Assume R is right strongly non-singular, right semi-hereditary, right Utumi, and does not contain an infinite set of orthogonal idempotents. It has been shown that any ring S Morita-equivalent to such a ring is right strongly non-singular and the classes of torsion-free and non-singular right S-modules coincide. Thus, $Mat_n(R)$ is right strongly non-singular and a right $Mat_n(R)$-module is torsion-free if and only if it is non-singular, which follows from $Mat_n(R)$ being Morita-equivalent to R for any $0 < n < \omega$. By Theorem 5.19, $Mat_n(R)$ is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. It then follows from Theorem 2.11 that $Mat_n(R)$ satisfies the ascending chain condition on right annihilators. Furthermore, Theorem 4.4 shows that since $Mat_n(R)$ is a right p.p.-ring, $Mat_n(R)$ is a torsion-free ring such that right annihilators of elements are finitely generated. Hence, $Mat_n(R)$ is a Baer-ring by Theorem 4.8. Moreover, Theorem 5.19 shows that every **S**-closed one-sided ideal of $Mat_n(R)$ is generated by an idempotent. Thus, every right ideal of $Mat_n(R)$ is a right annihilator and every left ideal of $Mat_n(R)$ is a left annihilator. Hence, $Mat_n(R)$ is a right and left Utumi ring.

$(c) \Rightarrow (a)$: Suppose $Mat_n(R)$ is a right and left Utumi Baer-ring for every $0 < n < \omega$ and does not contain an infinite set of orthogonal idempotents. Then, $Mat_n(R)$ is a right p.p.-ring, and so R is right semi-hereditary by Theorem 2.7. Furthermore, since $Mat_n(R)$

satisfies these conditions for every $0 < n < \omega$, $R \cong Mat_1(R)$ is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents. Thus, every **S**-closed one-sided ideal of R is an annihilator and hence generated by an idempotent. Therefore, since R is a right and left p.p.-ring and hence right and left non-singular, R is right strongly non-singular by Theorem 5.19. □

Corollary 6.6. *The following are equivalent for a ring R which does not contain an infinite set of orthogonal idempotents:*

(a) R is a right and left Utumi, right semi-hereditary ring.

(b) For every $0 < n < \omega$, $Mat_n(R)$ is a Baer-ring, and $Q^r(R)$ is torsion-free as a right R-module.

Proof. $(a) \Rightarrow (b)$: Suppose R is right and left Utumi and right semi-hereditary. Then, R is a right p.p.-ring and hence right non-singular. Moreover, R is a left p.p.-ring by Theorem 2.11, which implies that R is also a left non-singular ring. Since R is both right and left Utumi, $Q^r(R) = Q^l(R)$ by Theorem 5.16. Furthermore, since R is a right Utumi right p.p.-ring which does not contain an infinite set of orthogonal idempotents, $Q^r(R) = Q^l(R)$ is semi-simple Artinian and torsion-free by Theorem 5.18. Therefore, R is right strongly non-singular by Theorem 5.19.

Since R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents, the classes of torsion-free and non-singular right $Mat_n(R)$-modules coincide by Theorem 6.5. Moreover, the proof of Theorem 6.5 shows that $Mat_n(R)$ is right strongly non-singular. Thus, $Mat_n(R)$ is a right strongly non-singular, right p.p.-ring not contain an infinite set of orthogonal idempotents by Theorem 5.19. It then follows from Theorem 2.11 that $Mat_n(R)$ satisfies the ascending chain condition on right annihilators. Since $Mat_n(R)$ is a right p.p.-ring, Theorem 4.4 shows that $Mat_n(R)$ is a torsion-free ring such that right annihilators of elements are finitely generated. Hence, $Mat_n(R)$ is a Baer-ring by Theorem 4.8.

$(b) \Rightarrow (a)$: Assume $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$, and $Q^r(R)$ is torsion-free as a right R-module. Since $Mat_n(R)$ is a Baer-ring, it is both a right and left p.p.-ring. Hence, R is both right and left semi-hereditary by Theorem 2.7. It then readily follows that R is right and left non-singular. Note also that $R \cong Mat_1(R)$ is a Baer-ring since $Mat_n(R)$ is Baer for every $0 < n < \omega$. Let I be a proper **S**-closed right ideal of R. Then, R/I is non-singular as a right R-module. Furthermore, R/I is cyclic and thus finitely generated. Hence, R/I is isomorphic to a submodule of a free Q^r-module by Theorem 5.8. Since Q^r is assumed to be torsion-free as a right R-module, it follows from Proposition 4.6 that I is generated by an idempotent $e \in R$. Hence, $I = ann_r(1-e)$ by Lemma 4.5 and R is right Utumi. Observe that the argument works for **S**-closed left ideals as well, and so R is also left Utumi. □

The next example illustrates why it is necessary to consider right semi-hereditary rings in Theorem 6.5.

Example 6.7. *Let $R = \mathbb{Z}[x]$. As an integral domain, R is a strongly non-singular p.p.-ring not containing an infinite set of orthogonal idempotents [3, Corollary 3.10]. By Theorem 5.19, the classes of torsion-free and non-singular right R-modules coincide, and by Theorem 5.18 R is right Utumi. However, R is not semi-hereditary since the ideal $x\mathbb{Z}[x] + 2\mathbb{Z}[x]$ of $\mathbb{Z}[x]$ is not projective. As seen in the proof of Theorem 2.7, this implies $S = Mat_2(R)$ is not a right or left p.p.-ring, and hence not a Baer ring. Therefore, Theorem 6.5 does not hold if R is not assumed to be right semi-hereditary.*

Moreover, this example shows that the classes of torsion-free and non-singular S-modules do not necessarily coincide, even if R has this property and S is Morita-equivalent to R.

In [9, Theorem 4.3.5], Birkenmeier, Park, and Rizvi show that $Mat_n(R)$ is a Baer-ring precisely when every finitely generated torsionless right R-module is projective. A right R-module is *torsionless* if it is isomorphic to a submodule of R^I for some set I. In case that R has finite right Goldie-dimension, this condition is equivalent to R being right semi-hereditary:

Corollary 6.8. *The following are equivalent for a ring R with finite right Goldie-dimension:*

a) R is right semi-hereditary.

b) Every finitely generated torsionless right R-module is projective.

Proof. In view of [9, Theorem 4.3.5], it needs to be shown that a ring R with finite right Goldie-dimension is right semi-hereditary if and only if $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$. Now, R is right semi-hereditary if and only if $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$ [27]. Hence, R is right semi-hereditary whenever $Mat_n(R)$ is a Baer-ring. On the other hand, note that $Mat_n(R)$ has finite right Goldie-dimension since every ring Morita-equivalent to R also has finite dimension. Thus, $Mat_n(R)$ does not contain an infinite set of orthogonal idempotents. Therefore, if R is right semi-hereditary, $Mat_n(R)$ is a right p.p-ring not containing an infinite set of orthogonal idempotents, and it follows from [27, Theorem 1] that $Mat_n(R)$ is a Baer-ring. □

Clearly, the conditions in part *a*) of Theorem 6.5 imply that every finitely generated torsionless module is projective since these conditions imply that $Mat_n(R)$ is a Baer-ring. However, the condition on the torsionless modules in [9] is not enough to ensure that the coincidence of torsion-freeness and non-singularity is preserved by Morita-equivalence. The following examples provide rings for which the conditions of Theorem 6.5 fail, even though every finitely generated torsionless module is projective.

Example 6.9. *Let $R = F^I$ for some field F and an infinite index-set I. Then R is a commutative semi-hereditary ring which is its own maximal ring of quotients. Thus, R is strongly non-singular, and all finitely generated torsionless R-modules are projective. Therefore, $Mat_n(R)$ is a Baer-ring for all $n < \omega$, but R does not satisfy Theorem 6.5 since it has infinite Goldie-dimension.*

The next example provides a ring with finite right Goldie-dimension but infinite left Goldie-dimension. In the context of this thesis, this example provides a right Utumi Baer-ring which is not left Utumi. Hence, the conditions of Theorem 6.5 fail. However, it is easily seen that every finitely generated torsionless module is projective.

Example 6.10. *[11] Let $K = F(y)$ for some field F and consider the endomorphism f of K determined by $y \mapsto y^2$. The ring we consider is $R = K[x]$ with coefficients written on the right and multiplication defined according to $kx = xf(k)$ for any $k \in K$. Observe that $yx = xy^2$. It can be shown that $Rx \cap Rxy = 0$, and hence $Rxy \oplus Rxyx \oplus Rxyx^2 \oplus ... \oplus Rxyx^k \oplus ...$ is an infinite direct sum of left ideals of R. Thus, R has infinite left Goldie-dimension. On the other hand, every right ideal of R is a principal ideal [11], and thus R is right Noetherian. It then follows from Theorem 5.18 that R is a right Utumi Baer ring and Q^r is semi-simple Artinian. However, R having infinite left Goldie-dimension but finite right Goldie-dimension implies that $Q^r \neq Q^l$ [3, Proposition 4.1]. Therefore, Theorem 5.16 shows that R cannot be left Utumi.*

Thus, we have a right Utumi Baer-ring which is not left Utumi, and so this ring fails to satisfy the conditions of Theorem 6.5. However, since R is a Baer-ring and every right ideal is principal, R is right semi-hereditary. Therefore, every finitely generated torsionless right R-module is projective by Corollary 6.8. Observe that Example 6.10 also illustrates why it is necessary in Theorem 6.5 to include the requirement $Mat_n(R)$ is both right and left Utumi.

Chapter 7

The Baer-Splitting Property

In the previous chapter, we saw that the strongly non-singular and semi-hereditary properties ensure the preservation of coincidence of torsion-freeness and non-singularity under a Morita-equivalence. In [2], U. Albrecht shows that these properties are also related to two-sided essential submodules of Q^r:

Proposition 7.1. *[2] The following are equivalent for a ring R with finite right Goldie-dimension:*

i) R is right strongly non-singular, right semi-hereditary.

ii) If A is a two-sided R-submodule of Q^r such that A_R is essential in Q^r, then every $\boldsymbol{S}$-closed submodule of A^n is a direct summand.

iii) If A is a two-sided R-submodule of Q^r such that A_R is essential in Q^r and $n < \omega$, then every right non-singular epimorphic image of A^n is a direct summand.

This is dual to a property known as the finite Baer-splitting property. An R-module A has the *finite Baer-splitting property* if every epimorphism $A^n \to A \to 0$ splits. In other words, a module A has this property when finite direct sums of copies of A behave like projectives. Furthermore, in [6], R. Baer shows that for a finite rank completely decomposable Abelian group, every pure subgroup is a direct summand. A submodule N of an R-module M is *pure* if R/I is projective with respect with the exact sequence $0 \to N \to M \to M/N \to 0$ whenever I is a finitely-generated right ideal of R.

Let R be a ring with finite right Goldie-dimension. We show that the submodules of Proposition 7.1 have the finite Baer-splitting property. In Chapter 10, we will see that under similar conditions $(Q/R)_R$ can be decomposed into a direct sum of countably-generated two-sided submodules of the form A_i/R, where each A_i has the finite Baer-splitting property.

Proposition 7.2. *Let R be a right strongly non-singular, right semi-hereditary ring with finite right Goldie-dimension. If A is a two-sided R-submodule of Q such that A_R is essential in Q, then A has the finite Baer-splitting property.*

Proof. Consider an exact sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} A \to 0$ of right R-modules, and apply $()^* = \mathrm{Hom}_R(\_, A)$ to get the induced sequence $0 \to A^* \xrightarrow{\beta^*} (A^n)^* \to U^*$ of left E-modules, where $E = End_R(A_R)$. As shown in [2], E is a subring of Q^r containing R precisely when A is a two-sided essential submodule of Q^r. Since R is a right strongly non-singular, right semi-hereditary ring with finite right Goldie-dimension, the classes of torsion-free and non-singular right R-modules coincide by Theorem 5.19. Hence, R is left strongly non-singular by [3, Corollary 4.3]. Moreover, [3, Theorem 5.2] shows that R is left semi-hereditary and $Q^r = Q^l$ is semi-simple Artinian.

Now, every intermediate ring S satisfying $R \subseteq S \subseteq Q^r$ is also left strongly non-singular, left semi-hereditary by [2, Theorem 3.2], and hence $E \cong A^*$ is a left strongly non-singular, left semi-hereditary ring. Thus, $coker\ \beta^* = (A^n)^*/im\ \beta^*$ is projective since it is a finitely generated non-singular left E-module. Consequently, β^* splits, and applying the $()^*$ functor again leads to β^{**} splitting as well [22]. Thus, there exists $\gamma : A^{**} \to (A^n)^{**}$ such that $\beta^{**}\gamma = 1_{A^{**}}$. Furthermore, there are natural homomorphisms $A \to A^{**}$ and $A^n \to (A^n)^{**}$, and the following diagram commutes:

$$\begin{tikzcd}
(A^n)^{**} \arrow[r, shift left, "\beta^{**}"] \arrow[d, "\sigma"'] & A^{**} \arrow[l, shift left, "\gamma"] \arrow[d, "\sigma"] & \\
A^n \arrow[r, "\beta"] & A \arrow[r] & 0
\end{tikzcd}$$

A diagram chase shows that there exists $\delta : A \to A^n$ such that $\beta\delta = 1_A$, and therefore β splits. ∎

Chapter 8

Divisible Modules

Related to the concept of torsion-freeness is the notion of divisibility. As with torsion-freeness, issues arise when trying to extend the concept of divisibility from integral domains to general non-commutative rings. As such, there are various definitions of divisibility in the general setting. We refer to $D \in Mod_R$ as *divisible in the classical sense* if $Dc = D$ for every regular element $c \in R$. In other words, D is divisible in the classical sense if right multiplication by c on D is an epimorphism for every regular element $c \in R$. A slightly stronger notion of divisibility, which was developed by E. Matlis in [24], is that of h-divisibility. We say that a right R-module D is *h-divisible* if it is an epimorphic image of a direct sum of copies of Q^r.

Finally, we say that a right R-module D is *divisible* if $\mathrm{Ext}^1_R(R/Rr, D) = 0$ for every $r \in R$. Observe that this is similar to Hattori's definition of torsion-freeness based on the Tor functor, and it generalizes the notion of divisibility in the classical sense. As such, we find that the notions of torsion-freeness and divisibility are related through their characters modules. The *character module* of a right (left) R-module M is the left (right) R-module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Proposition 8.1. *Let R be a ring. A right R-module M is torsion-free if and only if M^* is divisible.*

Proof. Suppose M is torsion-free, and consider any $r \in R$. Using Proposition 3.16, we see that $\mathrm{Ext}^1_R(R/Rr, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(M, R/Rr), \mathbb{Q}/\mathbb{Z})$ since M is a $(\mathbb{Z}, R)$-bimodule and $\mathbb{Q}/\mathbb{Z}$ is injective. Since M is torsion-free, $\mathrm{Tor}^R_1(M, R/Rr) = 0$ and hence $\mathrm{Ext}^1_R(R/Rr, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = 0$. Therefore, M^* is divisible. On the other hand, suppose

M^* is divisible. We again use Proposition 3.16 to find that $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(M, R/Rr), \mathbb{Q}/\mathbb{Z}) = 0$, implying that $\mathrm{Tor}_1^R(M, R/Rr) = 0$. Therefore, M is torsion-free. $\square$

Furthermore, [4, Section 2] shows that if R is a right p.p.-ring, then a right R-module N is divisible if and only if N^* is torsion-free.

As with torsion-freeness, it is of interest when the various notions of divisibility coincide. It is always the case that h-divisibility implies classic divisibility. Moreover, divisibility and classic divisibility coincide for domains. L. Fuchs and L. Salce show in [15] that all three notions of divisibility coincide in the case that R is a countable integral domain. In the general setting, we have the following from U. Albrecht in [1]:

Theorem 8.2. *[1, Theorem 5.5] Let R be a semi-prime right and left Goldie-ring such that Q_R is countably generated.*

a) An R-module is h-divisible if and only if it divisible in the classical sense.

b) If a right R-module D is divisible in the classical sense, then $Z(D)$ is a direct summand of D.

c) R is a right p.p.-ring if and only if the classes of h-divisible and divisible modules coincide.

Thus, all three notions of divisibility coincide in the case that R is a semi-prime right and left Goldie p.p.-ring for which Q is countably generated as a right R-module. In Chapter 10, we will characterize rings for which this holds without requiring Q to be countably generated. Finally, if R is a semi-prime right Goldie-ring, then [1, Corollary 4.5] shows that a non-singular module D is divisible if and only if it is divisible in the classical sense if and only if it is injective.

A right R-module M is *weakly cotorsion* if $\mathrm{Ext}_R^1(Q^r, M) = 0$. The proof of [1, Theorem 5.5] shows that for a semi-prime right and left Goldie-ring for which Q_R is countably generated, every classically divisible module is weakly cotorsion. Furthermore, the following theorem from [1] gives a nice characterization of weakly cotorsion modules:

Proposition 8.3. *[1, Theorem 4.3] If R is a right Utumi p.p.-ring not containing an infinite set of orthogonal idempotents, then*

i) A right R-module D is divisible if and only if the singular submodule $Z(D)$ is divisible and $D/Z(D)$ is injective.

ii) A right R-module D is weakly cotorsion if and only if $Z(D)$ is a direct summand whenever D is divisible.

Related to condition *ii*) of Proposition 8.3, we have the following for a ring with finite Goldie-dimension:

Theorem 8.4. *The following are equivalent for a right non-singular ring R:*

a) R has finite right Goldie dimension.

b) For any $M \in Mod_R$, $M \otimes_R Q^r/R = 0$ if and only if $M/Z(M)$ is injective.

Proof. *a*) $\Rightarrow$ *b*) Let M be a right R-module and assume $M \otimes_R Q^r/R = 0$. Note that $M/Z(M) \otimes_R Q^r/R = 0$ as well. Since R has finite right Goldie dimension, $\mathcal{K} = \text{Ker}\,\text{Tor}_1^R(-, Q^r/R)$ coincides with the class of non-singular right R-modules [4, Theorem 3.4]. We first show that if E is the injective hull of $M/Z(M)$, then $E/(M/Z(M)) \in \mathcal{K}$. Observe that $M/Z(M)$ is non-singular, and hence E is non-singular since $M/Z(M) \leq_e E$ and non-singularity is closed under essential extensions [17, Proposition 1.22]. Thus, $E \in \mathcal{K}$ and $\text{Tor}_1^R(E, Q^r/R) = 0$. By exactness of $0 \to M/Z(M) \to E \to E/(M/Z(M)) \to 0$, we obtain the exact sequence

$$0 = \text{Tor}_1^R(E, Q^r/R) \to \text{Tor}_1^R(E/(M/Z(M)), Q^r/R) \to M/Z(M) \otimes_R Q^r/R = 0,$$

implying that $\text{Tor}_1^R(E/(M/Z(M)), Q^r/R) = 0$ and $E/(M/Z(M)) \in \mathcal{K}$. Thus, $E/(M/Z(M))$ is non-singular. However, this implies $E = M/Z(M)$ since E is non-singular and $M/Z(M) \leq_e E$ [17, Proposition 1.21]. Therefore, $M/Z(M)$ is injective.

Conversely, assume $M/Z(M)$ is injective. Observe that since R has finite right Goldie dimension and $Z(M)$ is singular, $Z(M) \otimes_R Q^r/R = 0$ by [4, Theorem 3.4]. Thus, if $M/Z(M) \otimes_R Q^r/R = 0$, we obtain the exact sequence

$$0 = Z(M) \otimes_R Q^r/R \to M \otimes_R Q^r/R \to M/Z(M) \otimes_R Q^r/R = 0.$$

So, it suffices to show that $M \otimes_R Q^r/R = 0$ for any injective, non-singular right R-module M. Let U be a finitely generated submodule of such an M. Since M is injective, there is a direct summand V of M which is the injective hull of U. Thus, we may assume that M is the injective hull of the finitely generated non-singular right R-module U. Hence, $U \leq_e M$ and the inclusion map $\beta : U \to M$ is an essential monomorphism. By [28, Ch. XII, Proposition 7.2], there exists a monomorphism $\alpha : U \to \oplus_n Q^r$ for some $n < \omega$. Moreover, since $\oplus_n Q^r$ is injective and the inclusion map $\beta : U \to M$ is an essential monomophism, there exists a monomorphism $\gamma : M \to \oplus_n Q^r$ with $\gamma\beta = \alpha$ [28, Ch. V, Lemma 2.2]. Therefore, the injective module M is a direct summand of $\oplus_n Q^r$ [26, Corollary 3.27]. We obtain the exact sequence $\oplus_n Q^r \otimes_R Q^r/R \to M \otimes_R Q^r/R \to 0$ from the canonical projection map, and thus the result follows provided $\oplus_n Q^r \otimes_R Q^r/R = 0$.

Observe that if $Q^r \otimes_R Q^r/R = 0$, then $\oplus_n Q^r \otimes_R Q^r/R \cong \oplus_n(Q^r \otimes_R Q^r/R) = 0$ since Q^r is an (R, R)-bimodule [26]. So it remains to be seen that $Q^r \otimes_R Q^r/R = 0$. We obtain the following commutative diagram, where h and g are multiplication maps:

$$\begin{tikzcd}
Q^r \otimes_R R \arrow[r, "f"] \arrow[d, "h"'] & Q^r \otimes_R Q^r \arrow[r] \arrow[d, "g"] & Q^r \otimes_R Q^r/R \arrow[r] & 0 \\
Q^r \arrow[r, "1_{Q^r}"'] & Q^r & &
\end{tikzcd}$$

Now, R has finite right Goldie dimension and thus Q^r is a perfect left localization of R

[28, Ch. XII, Corollary 2.6]. Hence, the multiplication map $g : Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. It readily follows that f is an isomorphism, and therefore $Q^r \otimes_R Q^r/R = 0$.

$b) \Rightarrow a)$ Assume that for any right R-module M, $M \bigotimes_R Q^r/R = 0$ if and only if $M/Z(M)$ is injective. We show that R has finite right Goldie dimension by showing that $\oplus_I Q^r$ is injective for any set I [28, Ch. XIII, Proposition 3.3]. By assumption, if $\oplus_I Q^r \otimes_R Q^r/R = 0$, then $\oplus_I Q^r/Z(\oplus_I Q^r)$ is injective. Since R is right non-singular, Q^r is right non-singular, and hence $\oplus_I Q^r$ is right non-singular [17, Propositions 1.22, 1.32]. Therefore, $\oplus_I Q^r/Z(\oplus_I Q^r) \cong \oplus_I Q^r$ is injective whenever $\oplus_I Q^r \otimes_R Q^r/R = 0$. However, $Q^r \otimes_R Q^r/R = 0$ by assumption since Q^r is right non-singular and injective, and it readily follows that $\oplus_I Q^r \otimes_R Q^r/R = 0$. $\square$

Chapter 9

Duo Rings

9.1 Localizations and Duo Rings

We find that the problem concerning coincidence of the various notions of divisibility is closely related to rings for the which the projective dimension of Q is at most 1. If R is an integral domain and $pd_R(Q) \leq 1$, then R is called a *Matlis domain.* E. Matlis [24], S.B. Lee [23], and L. Fuchs and L. Salce [15, Ch. VII, Theorem 2.8] characterize Matlis domains by showing that an integral domain R is a Matlis domain if and only if divisible R-modules are h-divisible if and only if Q/R is a direct sum of countably generated (divisible) submodules. We look to extend this result to the non-commutative setting. In particular, we find that several related results hold for semi-prime right and left Goldie-rings, an important class of non-commutative rings. We begin with a discussion on general localizations, duo rings, and projective dimension.

For a commutative ring R and multiplicatively closed subset $T \subseteq R$, the *localization of R at T*, denoted by R_T, is the set of equivalence classes of pairs (r, t), with $r \in R$ and $t \in T$, under the equivalence relation $(r, t) \sim (r', t')$ if and only if $s(rt' - r't) = 0$ for some $s \in R$. Typically, (r, t) is denoted as the fraction r/t, and R_T is a ring under fraction addition and multiplication. For an integral domain, the classical ring of quotients Q is the localization at the monoid of non-zero elements. As mentioned previously, R does not necessarily have a right or left classical ring of quotients in the general setting. However, for semi-prime Goldie-rings, we have the following:

Theorem 9.1. *[17] A ring R has a classical right ring of quotients which is semi-simple if and only if R is a semi-prime right Goldie-ring.*

Theorem 9.2. *[17] If R is a semi-prime right and left Goldie-ring, then there exists a semi-simple ring Q which is both the classical right and left ring of quotients of R, as well as the right and left maximal ring of quotients of R.*

Of special note is the localization of R at $R\backslash P$, where P is a prime ideal of R. This is referred to as the localization at P and is denoted by R_P. In the general setting, $R\backslash P$ is not necessarily multiplicatively closed. However, it is multiplicatively closed in the case that R is a duo ring. A ring R is a *right duo ring* if $Ra \subseteq aR$ for every $a \in R$, and it is a *left duo ring* if $aR \subseteq Ra$ for every $a \in R$. We call R a *duo ring* if it is both a right and left duo ring. A prime ideal P is *completely prime* if $xy \in P$ implies that $x \in P$ or $y \in P$ for every $x, y \in R$. It is clear that if P is completely prime, then $R\backslash P$ is multiplicative since $x, y \in R\backslash P$ implies $xy \notin P$. The following shows that if R is a duo ring, then every prime ideal is completely prime, from whence it follows that $R\backslash P$ is multiplicative and the localization at P is defined.

Proposition 9.3. *If R is a duo ring, then every prime ideal is completely prime.*

Proof. Let P be a prime ideal of R, and let $x, y \in R$ with $xy \in P$. Then, $yR = Ry$ since R is a duo ring, and hence $(xR)(yR) = xyR \subseteq P$. Since P is prime, $xR \subseteq P$ or $yR \subseteq P$. Therefore, $x \in P$ or $y \in P$ and P is completely prime. $\square$

In some cases, it is convenient for the localization R_P of a duo ring R at a prime ideal P to again be a duo ring. The following give two instances when this occurs. As H.H. Brungs shows in [10], Lemma 9.4 implies that R_P is duo in the case that R is a Noetherian duo ring.

Lemma 9.4. *[10] Let R be a duo ring. If R_P satisfies the ascending chain condition for principal right and left ideals, then R_P is a duo ring.*

Lemma 9.5. *Let R be a duo ring and consider any prime ideal P of R. If $xP = Px$ for every $x \in R$, then R_P is a duo ring.*

Proof. Let P be a prime ideal of R, and let $0 \neq r \in R$ and $t \in R\backslash P$. We show that $rt^{-1}(R_P) = (R_P)rt^{-1}$. Take any $as^{-1} \in R_P$ and consider $(rt^{-1})(as^{-1}) \in rt^{-1}(R_P)$. Since

R is a duo ring, $at = ta_1$ for some $a_1 \in R$, and hence $t^{-1}a = a_1t^{-1}$. Moreover, there exists $a_2 \in R$ such that $ra_1 = a_2r$. Next, we show that $t(R\backslash P) = (R\backslash P)t$. Once this is shown, we can find $s_1 \in R\backslash P$ such that $st = ts_1$ and hence $t^{-1}s^{-1} = s_1^{-1}t^{-1}$. Let $tx \in t(R\backslash P)$. Since R is duo, $tx = x_1t$ for some $x_1 \in R$. If $x_1 \in P$, then $x_1t = tx_2$ for some $x_2 \in P$ since R is strong duo. Thus, $tx = x_1t = tx_2$ and hence $t(x - x_2) = 0$. However, this implies $t = 0$ or $x = x_2 \in P$, which is a contradiction since t is regular and $x \in R\backslash P$. Therefore, $x_1 \in R\backslash P$ and $t(R\backslash P) \subseteq (R\backslash P)t$. The other inclusion is similar.

Finally, we use a similar process to show there exists $s_2 \in R\backslash P$ such that $rs_1 = s_2r$ and hence $rs_1^{-1} = s_2^{-1}r$. From the duo condition on R, there exists $s_2 \in R$ such that $rs_1 = s_2r$. Suppose $s_2 \in P$. Since R is strong duo, there exists $s_3 \in P$ such that $r(s_1 - s_3) = 0$. If r is regular, this leads to $s_1 = s_3 \in P$, which is a contradiction. If r is not regular, there may exist $0 \neq y \in R$ such that $s_1 - s_3 = y$. If $y \in P$, then $s_1 = y + s_3 \in P$, which is a contradiction. However, if $y \in R\backslash P$, then $y = s_1 - s_3 = 0$ since elements of $R\backslash P$ cannot be zero divisors. This again leads to $s_1 \in P$, which is a contradiction. Therefore, $s_2 \in R\backslash P$.

Putting everything together, we have the following:

$$rt^{-1}as^{-1} = ra_1t^{-1}s^{-1} = a_2rt^{-1}s^{-1} = a_2rs_1^{-1}t^{-1} = a_2s_2^{-1}rt^{-1} \in (R_p)rt^{-1}$$

Thus, $rt^{-1}(R_P) \subseteq (R_P)rt^{-1}$. The other inclusion is similar since we assume R to be both right and left duo. Therefore, $rt^{-1}(R_P) = (R_P)rt^{-1}$ and R_P is a duo ring. □

We also consider right and left duo rings which do not contain any zero-divisors. These rings are of interest because they are right and left Ore domains with finite Goldie-dimension. A domain R is a *right Ore domain (left Ore domain)* if $aR \cap bR \neq 0$ $(Ra \cap Rb \neq 0)$ for all non-zero $a, b \in R$. A ring R satisfies the *right Ore condition (left Ore condition)* if given $a, s \in R$ with s regular, there exists $b, t \in R$ with t regular such that $at = sb$ $(ta = bs)$. A ring R has a classical right (left) ring of quotients if and only if it satisfies the right (left) Ore condition [17].

Lemma 9.6. *If R is a right (left) duo ring, then it satisfies the right (left) Ore condition.*

Proof. Let $a, s \in R$ with s regular. If R is a right duo ring then $Rs \subseteq sR$, and thus $as = sb$ for some $b \in R$. The left condition is similar. $\square$

Lemma 9.7. *If R is a right and left duo ring not containing zero-divisors, then it is a right and left Ore domain. In Particular, R has right and left Goldie-dimension 1.*

Proof. Let $0 \neq a, b \in R$. Since R has no zero-divisors, $ab \neq 0$, and we can find $0 \neq c \in R$ such that $0 \neq ab = bc$ using the duo property. Thus, $0 \neq ab \in aR \cap bR$, and therefore R is a right Ore domain. Similarly, $Ra \cap bR \neq 0$. By [17, Theorem 3.30], R has finite right and left Goldie-dimension. Now, R has a classical ring of quotients Q which is a division ring. Hence, R is uniform as both a right and left R-module by [17, Corollary 3.25], and therefore R has right and left Goldie-dimension 1. $\square$

Lemma 9.8. *If R is a right and left duo ring not containing zero-divisors, then it is a semi-prime right and left Goldie-ring.*

Proof. If R is a duo ring not containing zero-divisors, then it is a right and left Ore domain with finite right and left Goldie-dimension by Lemma 9.6 and Lemma 9.7. Moreover, R satisfies the ACC on right and left annihilator ideals since $\{0\}$ and R are the only such ideals in a domain [17]. Therefore, R is a right and left Goldie-ring. Finally, if R does not contain any zero-divisors, then $aRa = 0$ implies $a = 0$, and thus R is semi-prime. $\square$

9.2 Projective Dimension

We now discuss the projective dimension of a module over a right and left duo domain. For any ring R, we say that a right R-module A has *projective dimension* $\leq n$, denoted $pd_R(A) \leq n$, if there exists a finite projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$

If the length of the shortest projective resolution is n then $pd_R(A) = n$, and $pd_R(A) = \infty$ if every projective resolution of A has infinite length. The following two lemmas will be useful in determining the projective dimension of a module:

Lemma 9.9. *[26, Prop. 8.6] The following are equivalent for a right R-module A:*

i) $pd_R(A) \leq n$.

ii) $\mathrm{Ext}_R^k(A, M) = 0$ *for every right R-module M and every $k \geq n+1$.*

iii) $\mathrm{Ext}_R^{n+1}(A, M) = 0$ *for every right R-module M.*

Lemma 9.10. *Let R be a ring, and let $0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules.*

i) If any two of $pd_R(A)$, $pd_R(B)$, or $pd_R(C)$ is finite, then so is the third.

ii) Only the following can occur:

a) $pd_R(A) < pd_R(B) = pd_R(C)$;

b) $pd_R(B) < pd_R(A) = pd_R(C) - 1$;

c) $pd_R(A) = pd_R(B) \geq pd_R(C) - 1$.

Proof. If $pd_R(A) = pd_R(B) = \infty$, then $pd_R(C) \leq \infty = pd_R(B) + 1$ and we are in case *c*). Suppose $pd_R(A) = pd_R(C) = \infty$. If $pd_R(B) < \infty = pd_R(A)$, then $pd_R(C) = \infty = pd_R(A) + 1$ and we are in case *b*). If $pd_R(B) = \infty = pd_R(A)$, then $pd_R(C) = \infty \leq pd_R(A) + 1$ and we are in case *c*). Finally, suppose $pd_R(B) = pd_R(C) = \infty$. If $pd_R(A) < \infty = pd_R(B)$, then $pd_R(C) = \infty = pd_R(B)$ and we are in case *a*). If $pd_R(A) = \infty = pd_R(B)$, then $pd_R(C) = \infty \leq pd_R(A) + 1$ and we are in case *c*).

Assume that neither A, B nor C is projective. We will deal with those cases at the end. Suppose that $pd_R(A) < pd_R(B) = n$ for any $1 \leq n < \omega$. Then, given any right R-module M, we have $\mathrm{Ext}_R^k(A, M) = 0$ for every $k \geq n$ and $\mathrm{Ext}_R^m(B, M) = 0$ for every

$m \geq n+1$. Hence, the sequence $0 = \text{Ext}_R^n(A, M) \to \text{Ext}_R^{n+1}(C, M) \to \text{Ext}_R^{n+1}(B, M) = 0$ is exact by Proposition 3.15, from whence it follows $\text{Ext}_R^{n+1}(C, M) = 0$. Therefore, $pd_R(C) \leq n$. Now, suppose $pd_R(C) < n$. Then, $\text{Ext}_R^n(C, M) = 0$ for every right R-module M, and thus $0 = \text{Ext}_R^n(C, M) \to \text{Ext}_R^n(B, M) \to \text{Ext}_R^n(A, M) = 0$ is exact. However, this implies that $\text{Ext}_R^n(B, M) = 0$ and $pd_R(B) \leq n-1$, which is a contradiction. Therefore, $pd_R(C) = n$.

The other cases are similar. Assume $pd_R(B) < pd_R(A) = n$ for any $1 \leq n < \omega$. Then, given any right R-module M, we have $\text{Ext}_R^k(A, M) = 0$ for every $k \geq n+1$ and $\text{Ext}_R^m(B, M) = 0$ for every $m \geq n$. Hence, the sequence $0 = \text{Ext}_R^{n+1}(A, M) \to \text{Ext}_R^{n+2}(C, M) \to \text{Ext}_R^{n+2}(B, M) = 0$ is exact by Proposition 3.15, from whence it follows $\text{Ext}_R^{n+2}(C, M) = 0$. Therefore, $pd_R(C) \leq n+1$. Now, suppose $pd_R(C) < n+1$. Then, $\text{Ext}_R^{n+1}(C, M) = 0$ for every right R-module M, and thus $0 = \text{Ext}_R^n(B, M) \to \text{Ext}_R^n(A, M) \to \text{Ext}_R^{n+1}(C, M) = 0$ is exact. However, this implies that $\text{Ext}_R^n(A, M) = 0$ and $pd_R(A) \leq n-1$, which is a contradiction. Therefore, $pd_R(C) = n+1$.

Finally, assume $pd_R(B) = pd_R(A) = n$ for any $1 \leq n < \omega$. Then, given any right R-module M, we have $\text{Ext}_R^k(A, M) = \text{Ext}_R^k(B, M) = 0$ for every $k \geq n+1$. Hence, the sequence $0 = \text{Ext}_R^{n+1}(A, M) \to \text{Ext}_R^{n+2}(C, M) \to \text{Ext}_R^{n+2}(B, M) = 0$ is exact by Proposition 3.15, from whence it follows $\text{Ext}_R^{n+2}(C, M) = 0$. Therefore, $pd_R(C) \leq n+1$. In this case, there is no contradiction in assuming $pd_R(C) < n+1$.

If C is projective, then the sequence splits and $B \cong A \oplus C$. This implies that $pd_R(B) = \sup\{pd_R(A), pd_R(C)\} = pd_R(A)$ [26], and we are in case *c*). Suppose B is projective. If $pd_R(B) < pd_R(A) = n$ for some $n > 0$, then $pd_R(C) = pd_R(A) + 1$ using the same long exact sequence used above to prove case *b*). If $0 = pd_R(B) = pd_R(A)$, then we have a projective resolution of C of length 1, and hence $pd_R(C) \leq 1 = pd_R(A) + 1$. Finally, suppose A is projective. If $pd_R(A) < pd_R(B) = n$ for some $n > 0$, then $pd_R(C) = pd_R(B)$ using the same long exact sequence used above to prove case *a*). If $0 = pd_R(A) = pd_R(B)$, then we again have a projective resolution of C of length 1 and are in case *c*). $\square$

We are interested in the relationship between the projective dimensions over the rings R and R/rR, where $r \in R$ is a non-zero divisor. In the case that R is commutative, Kaplansky found that $pd_{R/rR}(M/rM) \leq pd_R(M)$ whenever $r \in R$ is a non-zero divisor such that $xr \neq 0$ for every $0 \neq x \in M$ [15, Lemma VI.2.10]. This is one of Kaplansky's *Change of Rings Lemmas*, and it can be extended to right and left duo rings not containing zero-divisors. The proof used in [15, Lemma VI.2.10] carries over to this setting.

Lemma 9.11. *Let R be a right and left duo ring without zero-divisors, and let $0 \neq s \in R$. If M is a left R-module such that $sx \neq 0$ for every $0 \neq x \in M$, then $pd_{R/sR}(M/sM) \leq pd_R(M)$.*

We now look to extend another of Kaplansky's Change of Rings Lemmas to the case that R is a duo ring without zero-divisors. First, we need the following from [25]:

Lemma 9.12. *[25, Theorem 9.32] If $\varphi : R \to R^*$ is a ring homomorphism and A^* is a right R^*-module, then $pd_R(A^*) \leq pd_{R*}(A^*) + pd_R(R^*)$.*

Observe that if R is any ring and $\sigma : R \to R$ is an automorphism of rings, then every right R-module M carries another R-module structure induced by σ: For $x \in M$ and $r \in R$, define $x * r = x\sigma(r)$. Let M^* denote the R-module M when using the structure induced by σ. Since $1 * r = 1\sigma(r)$, we have that R^* is a free right R-module. Hence, $pd_R(R^*) = 0$ and $pd_R(M) = pd_R(M^*) \leq pd_{R^*}(M^*)$ by Lemma 9.12. Since σ is an isomorphism, we can use σ^{-1} to get the reverse inequality, and therefore $pd_R(M) = pd_{R^*}(M^*)$.

Proposition 9.13. *Let R be a duo ring without zero-divisors, and let $0 \neq s \in R$. If $\sigma : R \to R$ denotes the automorphism defined by $\sigma(r) = s^{-1}rs$, then $\overline{\sigma} : R/sR \to R/sR$ defined by $\overline{\sigma}(r + sR) = \sigma(r) + sR$ is an automorphism of R/sR.*

Proof. If $r' = r + st$ for some $t \in R$, then $s^{-1}r's = s^{-1}rs + s^{-1}sts = s^{-1}rs + ts$. Since $sR = Rs$, we have $\overline{\sigma}(r') = \overline{\sigma}(r)$ and hence $\overline{\sigma}$ is well-defined. It is easily seen that $\overline{\sigma}$ is an epimorphism and an R-map. To see that $\overline{\sigma}$ is a monomorphism, observe that $\overline{\sigma}(r) = 0$ and the duo condition yield $s^{-1}rs = ts$ for some $t \in R$. Hence $t = s^{-1}r \in Q$, and $r = st \in sR$. Therefore, $\overline{\sigma}$ is an automorphism of R/sR. □

Let U be any right R/sR-module. Then, U can be viewed as a right R-module with $Us = 0$. Moreover, using $\overline{\sigma}$ as defined in Proposition 9.13, we have $u*(r+sR) = u\overline{\sigma}(r+sR) = u(\sigma(r) + sR) = u\sigma(r) + 0 = u \times r$, where $*$ and $\times$ denote the module structures induced by $\overline{\sigma}$ and σ, respectively. We are now ready to extend Kaplansky's change of rings result.

Theorem 9.14. *Let R be a duo ring without zero-divisors, and let $0 \neq s \in R$. If M is a right R/sR-module such that $pd_{R/sR}(M) = 1$, then $pd_R(M) = 2$.*

Proof. Assume, for a contradiction, that $pd_R(M) \leq 1$, and consider an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ of right R-modules with P_0 and P_1 projective. Applying $\_ \bigotimes_R R/sR$ induces the exact sequence of right R/sR-modules:

$$0 \to \mathrm{Tor}^1_R(M, R/sR) \to P_1 \bigotimes_R R/sR \to P_0 \bigotimes_R R/sR \to M \bigotimes_R R/sR \to 0.$$

However, $M \bigotimes_R R/sR \cong M$ since M is an R/sR-module. Furthermore, since $P_i \bigotimes_R R/sR$ is a projective R/sR-module for $i = 0, 1$, and $pd_{R/sR}(M) = 1$, we have that $\mathrm{Tor}^1_R(M, R/sR)$ is a projective R/sR-module.

Now, the sequence $0 \to sR \xrightarrow{i} R \to R/sR \to 0$ is an exact sequence of R-R-bimodules. We consider the induced sequence $0 \to \mathrm{Tor}^1_R(M, R/sR) \xrightarrow{\partial} M \bigotimes_R sR \xrightarrow{i^*} M \bigotimes_R R$. Note that in $M \bigotimes_R R$ we have $x \otimes st = xs \otimes t = 0$ since M is a right R-module satisfying $Ms = 0$. Thus, $im\ i^* = 0$ and ∂ is an isomorphism. It then follows that $A = M \bigotimes_R sR$ is isomorphic to $\mathrm{Tor}^1_R(M, R/sR)$ as an R-module, and hence as an R/sR-module. Therefore, A is a projective R/sR-module.

Let A^* denote the R-module A with the module structure induced by $\overline{\sigma}$ as defined in Proposition 9.13. For $x \otimes ts \in A$, we have $(x \otimes ts) * r = x \otimes tss^{-1}rs = x \otimes trs$. However, $\lambda : A^* \to M$ defined by $\lambda(x \otimes ts) = xt$ is an isomorphism of R-modules, and hence also of R/sR-modules. As previously shown, Lemma 9.12 implies that A and A^* have the same projective dimension as both R and R/sR-modules since σ and $\overline{\sigma}$ are automorphisms of R

and R/sR, respectively. Thus, we have a contradiction since this leads to

$$1 = pd_{R/sR}(M) = pd_{R/sR}(A^*) = pd_{R/sR}(A) = pd_{R/sR}(\mathrm{Tor}_R^1(M, R/sR)) = 0.$$

Therefore, $pd_R(M) > 1$ and $pd_{R/sR}(M) < pd_R(M)$. $\square$

Chapter 10

Generalizations of Matlis Domains

10.1 Tight Systems and G($\aleph_0$) Families

As we will see in Theorem 10.11, R_T/R is a direct summand of Q/R in the case that $pd_R(Q) \leq 1$ and $pd_R(Q/R_T) \leq 1$. We will eventually consider a direct sum decomposition of Q/R whose construction depends on these conditions. In order to ensure the second condition is satisfied, as well as to ensure we have the desired direct sum decomposition, we need a family of submodules which satisfy the following conditions. For a right R-module M, a set $\mathcal{S}$ of submodules of M is a *$G(\aleph_0)$-family* if the following are satisfied:

(i) $0, M \in \mathcal{S}$.

(ii) $\mathcal{S}$ is closed under unions of chains.

(iii) Given $A \in \mathcal{S}$ and a countable subset X of M, there exists $B \in \mathcal{S}$ such that $A, X \subseteq B$ and B/A is countably generated.

A submodule N of a right R-module M is called *tight* if $pd_R(M/N) \leq pd_R(M)$. For a right R-module M of projective dimension $\leq m$, a family $\mathcal{T} = \{M_i \mid i \in I\}$ of tight submodules of M is called a *tight system* if

(i) $0, M \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under unions of chains;

(iii) if $M_i, M_j \in \mathcal{T}$ with $M_i < M_j$, then $pd_R(M_j/M_i) \leq pd_R(M) \leq m$;

(iv) For every $M_i \in \mathcal{T}$ and every subset S of M of cardinality $\leq \aleph_{m-1}$, there exists $M_j \in \mathcal{T}$ such that $M_i \leq M_j$, $S \subseteq M_j$ and M_j/M_i is $\leq \aleph_{m-1}$-generated.

For our purposes, we have that M_j/M_i is countably-generated in condition (iv) since we consider a tight system for Q/R in the case that $pd_R(Q/R) \leq 1$. The following result ensures the existence of a tight system in the case that $pd_R(M) \leq 1$. The proof is the same as the integral domain case found in [15, Prop. 5.1].

Lemma 10.1. *[15] Let R be a semi-prime right and left Goldie-ring and M a right R-module. If $pd_R(M) \leq 1$, then M admits a tight system.*

Once we have an appropriate $G(\aleph_0)$-family of tight submodules, we will use the following lemma to extract a well-ordered ascending chain of direct summands.

Lemma 10.2. *[20] Let R be a ring and let M be a right R-module. Let $\mathcal{U}$ be a family of submodules of M, and take $\mathcal{U}_0$ to be a subset of $\mathcal{U}$. Suppose there exists a chain $\{M_\gamma\}_{\gamma\leq\beta}$ such that*

- *for every $\gamma < \beta$, $M_{\gamma+1} = M_\gamma \oplus U_\gamma$ for some $U_\gamma \in \mathcal{U}_0$,*
- *$M_0 = 0$, and $M_\gamma = \bigcup_{\nu<\gamma} M_\nu$ for every $\gamma \leq \beta$.*

Then, $M = \bigoplus_{\gamma<\beta} U_\gamma$ is a direct sum of modules with $U_\gamma \in \mathcal{U}_0$ for every $\gamma < \beta$.

The following variation developed in [7] by Bazzoni, Eklof, and Trlifaj of a *tight system* is sometimes useful in producing factors which have generating sets of higher cardinality. For a right R-module M of projective dimension ≤ 1 and a regular uncountable cardinal κ, a set $\mathcal{T} = \{M_i \mid i \in I\}$ of submodules of M is a *κ-tight system* if the following hold:

(i) $0 \in \mathcal{T}$ and each $M_i \in \mathcal{T}$ is $< \kappa$-generated.

(ii) $\mathcal{T}$ is closed under unions of well-ordered chains of length $< \kappa$.

(iii) Every $M_i \in \mathcal{T}$ is a tight submodule of M. That is, $pd_R(M/M_i) \leq 1$.

(iv) For every $M_i \in \mathcal{T}$ and every subset S of M of cardinality $< \kappa$, there exists $M_j \in \mathcal{T}$ such that $M_i \leq M_j$, $S \subseteq M_j$ and M_j/M_i is $< \kappa$-generated.

It is of note that if M is a right R-module of a semi-prime right Goldie-ring R, then M admits such a system. For any right R-module M, define

$$M^{\perp} = \text{Ker Ext}^1_R(M, \_) = \{X \in Mod_R \mid \text{Ext}^1_R(M, X) = 0\}.$$

Using the notation in [7], for any index set I and cardinal κ, let $E^{[I,<\kappa]}$ denote the submodule of $E(R)^I$ consisting of elements with support of cardinality $< \kappa$.

Lemma 10.3. *[7, Lemma 3.1] Let M be a right R-module and consider any regular uncountable cardinal $\kappa \leq gen\ M$. If $pd_R(M) \leq 1$ and $M^{\perp}$ contains $E^{[I,<\kappa]}$ for every index set I, then M admits a κ-tight system.*

Lemma 10.4. *Let R be a semi-prime right Goldie-ring and M a right R-module. If $pd_R(M) \leq 1$, then M admits a κ-tight system for any regular uncountable cardinal $\kappa \leq gen\ M$.*

Proof. Let I be any index set and κ any regular uncountable cardinal of cardinality $\leq$ gen M. As a submodule of a direct sum of non-singular modules, $E^{[I,<\kappa]}$ is a non-singular right R-module. Let $x = (x_\alpha)_{\alpha \in I} \in E^{[I,<\kappa]}$, and take c to be any regular element of R. Since R is a semi-prime right Goldie-ring, the non-singular, injective module $E(R)$ is divisible in the classical sense [1, Corollary 4.5]. Hence, for every $\alpha \in I$, there exists $y_\alpha \in E(R)$ such that $x_\alpha = y_\alpha c$. Let $y = (y_\alpha)_{\alpha \in I}$ so that $x = yc$, and observe that $|\text{supp } (y)| < \kappa$ since $|\text{supp } (x)| < \kappa$ and c is regular. Thus, $y \in E^{[I,<\kappa]}$ and we conclude that $E^{[I,<\kappa]}$ is divisible in the classical sense. Furthermore, as a non-singular, divisible right R-module of a semi-prime right Goldie-ring, $E^{[I,<\kappa]}$ must also be injective [1, Corollary 4.5]. Therefore, $E^{[I,<\kappa]}$ is contained in $M^{\perp}$ and Lemma 10.3 shows that M admits a κ-tight system. $\square$

10.2 Pre-Matlis Duo Domains

We are now ready to turn our attention to modules of projective dimension ≤ 1. In particular, we consider rings for which the maximal right ring of quotients Q^r has projective dimension ≤ 1. Observe that it follows immediately from Lemma 9.10 that $pd_R(Q) \leq 1$ precisely when $pd_R(Q/R) \leq 1$. As previously mentioned, we look to decompose $(Q/R)_R$ into a direct sum of countably generated submodules and extend the characterization of Matlis domains to a more general setting. We start with the following result from U. Albrecht in [1]:

Theorem 10.5. *[1] Let R be a semi-prime right and left Goldie-ring. If Q/R is a direct sum of countably generated submodules, then $pd_R(Q) \leq 1$.*

We now consider the converse of this result. We begin our discussion with an example which provides a ring for which $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q. However, this particular ring is hereditary and hence $pd_R(Q_R) \leq 1$. We first consider the following lemmas from Bessenrodt, Brungs, and Törner in [8]:

Lemma 10.6. *[8, Lemma 3.1] The following are eqivalent for a ring R:*

a) R is a right Noetherian, right chain ring.

b) R is a local principal right ideal ring.

c) The lattice of right ideals of R is inversely well-ordered by inclusion.

Lemma 10.7. *[8, Lemmas 1.4, 3.2] Let R be a right Noetherian right chain ring. Then R is a right duo ring. In particular, every right ideal is two-sided.*

Proof. Let I be a right ideal of R. Since R is a right Noetherian right chain ring, I is a principal ideal, say $I = aR$ for some $a \in R$. As a right chain ring, R is a local ring whose unique maximal right ideal $J(R)$ consists of all non-unit elements of R. Let $U = R \backslash J(R)$ denote the group of units, and suppose there exists $u \in U$ such that $aR \subsetneq uaR$. Then, $uaR \subsetneq u^2aR \subsetneq ... \subsetneq u^n aR \subsetneq ...$, $n < \omega$, is a strictly ascending chain of right ideals of R. This contradicts R being right Noetherian, and thus $UaR \subseteq aR$.

We now show that R is right duo by showing that $Ra \subseteq UaR \subseteq aR$. Take $ra \in Ra$. If $r \in U$ or if $ra \in aR$, then clearly $ra \in UaR$. Otherwise, $r \notin U$ and there exists $x \in J(R)$ such that $rax = a$. Hence, $ra(1+x) = ra + rax = ra + a = a + ra = (1+r)a$. Observe that both $1+x$ and $1+r$ are units in R, since otherwise $1 = (1+x) - x = (1+r) - r \in J(R)$. Thus, $ra = (1+r)a(1+x)^{-1} \in UaR$. □

Theorem 10.8. *Let R be a right Noetherian, right chain domain whose lattice of right ideals is inversely order isomorphic to an ordinal σ of uncountable cardinality. Then, R is a right hereditary right duo ring with classical right ring of quotient Q such that $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q.*

Proof. Observe that R is a right duo ring by Lemma 10.7, and hence every right ideal of R is two-sided. Moreover, R is a right hereditary ring since every right ideal of R is principal by Lemma 10.6, and R has a classical right ring of quotients Q since every right Noetherian domain is a right Ore domain.

We first show that ${}_RQ$ is not countably generated. If it were, then we could find $\{c_n \mid n < \omega\}$ such that $Q = \sum_{n<\omega} Rc_n^{-1}$. We consider the right ideals c_nR of R , and observe that $\cap_{n<\omega} c_nR \neq 0$ since σ is of uncountable cardinality. We pick a non-zero $d \in \bigcap_{n<\omega} c_nR$, say $d = c_nr_n$. For all $q \in Q$, we have $qd \in R$. Specifically, $c^{-1}d \in R$ for all $0 \neq c \in R$. Thus, $d \in \bigcap_{c\neq 0} cR$. In particular, $0 \neq d^2$ and $d^2R \subseteq dR \subseteq \bigcap_{c\neq 0} cR \subseteq d^2R$, and we can find $r \in R$ such that $d = d^2r$. Since R is a domain, $1 = dr$. Hence, $d \notin J(R)$ and d is a unit, from whence it follows $R = Qd = Q$, a contradiction. Thus, ${}_RQ$ is not countably generated.

Now assume $(Q/R)_R \cong \bigoplus_I A_i/R$ for some index set I, where A_i/R is countably generated and A_i is a two-sided submodule of R. Note that if A_i is a subring of Q, then A_i is indeed a two-sided submodule R. Pick a countable subset $J_0 \subseteq I$, and write $\sum_{J_0} A_j = \sum_{n<\omega}(r_nc_n^{-1})R$. Then, $r_nc_n^{-1} \in \sum_m Rc_m^{-1}$. However, Rc_m^{-1} is also an R-submodule of Q_R. To see this, let $r \in R$ and pick $s \in R$ such that $rc_m = c_ms$. This is possible since a right Noetherian, right chain ring is right duo by Lemma 10.7. Then, $c_m^{-1}r = sc_m^{-1}$ and thus $\sum_{J_0} A_j \subseteq \sum_m Rc_m^{-1}$. Since ${}_RQ$ is not countably generated, we may assume that this inclusion is proper. Otherwise, we can add Rd^{-1} to the sum on the right-hand side, and proceed with $\sum_m Rc_m^{-1} + Rd^{-1}$ such that $d^{-1} \notin \sum_m Rc_m^{-1}$.

We can now find a countable subset J_1 of I such that $J_0 \subseteq J_1$ and $c_m^{-1} \in \sum_{J_1} A_j$. Since each A_j is two-sided, $\sum_{J_0} A_j \subsetneq \sum_m Rc_m^{-1} \subseteq \sum_{J_1} A_j$. Inductively, we obtain an ascending chain $J_0 \subseteq J_1 \subseteq ...$ of countable subsets of I and a countable family $\{d_n \mid n < \omega\} \subseteq R$ such that $J = \bigcup_{n<\omega} J_n$ is a countable subset of I with $\sum_J A_j = \sum_{n<\omega} Rd_n^{-1}$. If ${}_RQ \neq \sum_{n<\omega} Rd_n^{-1}$, then there exists $0 \neq c \in R$ such that $c^{-1} \notin \sum_{n<\omega} Rd_n^{-1}$. Since R is a right chain ring, either $cR \subseteq d_nR$ or $d_nR \subseteq cR$. If the latter occurs, then $d_n = ct_n$ for some $t_n \in R$ and $c^{-1} = t_n d_n^{-1}$, a contradiction. Thus, $c = d_n s_n$ for some $s_n \in R$ and $d_n^{-1} = s_n c^{-1}$. It readily follows that $\sum_{n<\omega} Rd_n^{-1} \subseteq Rc^{-1}$. However, $R \subseteq Rc^{-1}$, so that $\sum_J A_j \subseteq \sum_{n<\omega} Rd_n^{-1} \subseteq Rc^{-1}$ implies $\bigoplus_J A_j/R \subseteq Rc^{-1}$. Thus, $Rc^{-1} = \bigoplus_J (A_j/R) \oplus U/R$ for some $R \subseteq U \subseteq Rc^{-1}$ since $(\bigoplus_J A_j/R) \sqsubseteq Q/R$. Observe that $Q/R = \bigoplus_I A_i/R$ is a decomposition of both $(Q/R)_R$ and ${}_R(Q/R)$ since A_i is a two-sided submodule for each $i \in I$. Moreover, $\bigoplus_J (A_j/R)$ is not finitely generated since $\sum_{J_n} A_j \subseteq \sum_{J_{n+1}} A_j$ for every $n < \omega$, and we have a contradiction. Thus, $Q = \sum_{n<\omega} Rd_n^{-1}$, contradicting the fact that ${}_RQ$ is not countably generated. Therefore, $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q. $\square$

Let $R^\times$ denote the multiplicative monoid of regular elements of R. In trying to fully extend the characterization of Matlis domains to the general non-commutative setting, a few issues arise involving the localization of R at a submonoid T of $R^\times$ and the formal construction of the ring of quotients. A primary complication that arises is related to the filtration properties of the multiplicative monoid of non-zero elements of an integral domain. We say that $R^\times$ has a filtration if it is the union of a continuous (or smooth) well-ordered ascending chain

$$\{1\} = T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...T_\kappa = R^\times$$

of submonoids. The chain is well-ordered if the index set α runs over the ordinals $\alpha < \kappa$ for some ordinal κ, and the chain is continuous (or smooth) if $T_\beta = \bigcup_{\gamma<\beta} T_\gamma$ for every limit ordinal $\beta < \kappa$.

In the general setting, $R^\times$ does not necessarily have a filtration with the same properties that we find in integral domains. In particular, if we consider a submonoid T of $R^\times$ and a countable subset S of $R^\times$, then it is not guaranteed that the localization at the submonoid generated by T and S is countably generated over the localization at T. If R is an integral domain, then the monoid of non-zero elements does in fact have this property. As we will see, filtrations which have this characteristic will be essential in decomposing Q/R into countably generated summands. The example in Theorem 10.8 provides a ring for which $R^\times$ does not have our desired filtration, and we see that in this instance Q/R is not a direct sum of countably generated submodules. To counter this issue, we introduce a filtration similar to the third axiom of countability introduced by P. Griffith and P. Hill in [19]. A monoid T satisfies the *third axiom of countability* if there exists a family $\mathcal{C} = \{T_i \mid i \in I\}$ of submonoids of T such that

(i) $1 \in \mathcal{C}$.

(ii) $\mathcal{C}$ is closed under unions of chains.

(iii) If $i \in I$ and $X \subseteq T$ is countable, then there exists $i_0 \in I$ such that $T_i, X \subseteq T_{i_0}$ and T_{i_0} is countably generated over T_i.

We refer to the family $\mathcal{C}$ as an *Axiom III* family of T.

Moreover, we introduce notions similar to that of a normal subgroup and a normal series of a group. Define a *normal series* of a submonoid T of $R^\times$ to be an ascending chain

$$\{1\} = T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...T_\kappa = T$$

of submonoids of T such that $T_\alpha \triangleleft T$ for every $\alpha < \kappa$. In other words, $tT_\alpha = T_\alpha t$ for every $t \in T$ and every $\alpha < \kappa$. Combining a variation of the third axiom of countability with this notion of normality, we develop a filtration for $R^\times$ which will allow us to produce a chain of direct summands of Q/R. We say that a domain R is a *pre-Matlis domain* if $R^\times$ is the union of a smooth filtration

$$\{1\} = T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...T_\kappa = R^\times$$

of submonoids with the following properties:

(i) $T_\alpha \triangleleft R^\times$ for every $\alpha < \kappa$.

(ii) If $\alpha < \kappa$ and $X \subseteq R^\times$ is countable, then there exists $\beta < \kappa$ such that $T_\alpha, X \subseteq T_\beta$ and T_β is countably generated over T_α.

We consider an example from Bessenrodt, Brungs, and Törner in [8] of a ring whose monoid of regular elements has the desired filtration of normal submonoids. For an ordered group $(G, \leq)$ with identity e, let $G^+ = \{g \in G \mid e \leq g\}$ denote the positive cone of G. Let K be a division ring and consider all power series of the form $a = \sum_{g \in G} g a_g$, with $a_g \in K$. Define the support of a to be $supp(a) = \{g \in G \mid a_g \neq 0\}$, and refer to a as a generalized power series if $supp(a)$ is a well-ordered subset of G. If $ag = ga$ for every $a \in K$ and $g \in G$, then the set of all generalized power series, denoted $K[[G]]$, is a ring with normal power series addition and multiplication. Moreover, $K[[G]]$ is a division ring and the following proposition from [8] shows that $K[[G^+]]$ is a duo chain domain with quotient ring $K[[G]]$.

Proposition 10.9. *[8, Prop. 1.24] Let $(G, \leq)$ be an ordered group and K a division ring. Then the subring $R = \{a \in K[[G]] \mid e \leq \min supp(a)\} \cup \{0\}$ of $K[[G]]$ is a duo chain domain satisfying the following properties:*

i) The set of non-zero principal right ideals is given by $\{gR \mid e \leq g\}$.

ii) The two sided ideals of R correspond to the upper classes of G^+, the prime ideals to the convex semigroups of G^+.

iii) The residue field $R/J(R)$ is isomorphic to K.

Theorem 10.10. *Let $(G, \leq)$ be an ordered group which has an Axiom III family of normal subgroups, and let $R = K[[G^+]]$. Then R is a pre-Matlis domain.*

Proof. Suppose G has an Axiom III family $C = \{N_\alpha \mid \alpha < \kappa\}$ of normal subgroups. Since $G^+ \cap N_\alpha$ is a normal subroup of G^+ for each $\alpha < \kappa$, it is easily seen that $C' = \{G^+ \cap N_\alpha \mid \alpha < \kappa\}$ is an Axiom III family of G^+:

i) $\{e\} = G^+ \cap \{e\} \in C'$ since $\{e\} \in C$.

ii) If $\{G^+ \cap N_\beta\}_{\beta<\gamma}$ is a chain in C', then $\{N_\beta\}_{\beta<\gamma}$ is a chain in C. Hence, $\bigcup_{\beta<\gamma} N_\beta \in C$, from whence it follows $G^+ \cap (\bigcup_{\beta<\gamma} N_\beta) \in C'$.

iii) Let $\alpha < \kappa$ and let $X \subseteq G^+ \subseteq G$ be countable. Since C is an Axiom III family, there exists $\beta < \kappa$ such that $N_\alpha, X \subseteq N_\beta$ and N_β is countably generated over N_α. Therefore, $G^+ \cap N_\alpha, X \subseteq G^+ \cap N_\beta$ and $G^+ \cap N_\beta$ is countably generated over $G^+ \cap N_\alpha$.

For each $\alpha < \kappa$, define $T_\alpha = K[[G^+ \cap N_\alpha]] \backslash \{0\}$ to be the set of all non-zero generalized power series $\sum g a_g$ over $G^+ \cap N_\alpha$ and K. By Proposition 10.9, $K[[G^+ \cap N_\alpha]]$ is a duo ring, and hence $rT_\alpha = T_\alpha r$ for every $r \in R^\times$. By extending property *iii*) of the Axiom III family of G^+ to $\{T_\alpha\}_{\alpha<\kappa}$, we have the second condition of our filtration satisfied. Therefore, $K[[G^+]]$ is a pre-Matlis domain. □

It is of note that a right and left chain ring is a strongly non-singular, semi-hereditary ring with finite Goldie-dimension [2]. Hence, $K[[G^+]]$ satisfies the conditions of Proposition 7.1, and every two-sided essential submodule of Q has the finite Baer-splitting property. In 1952, C.G. Chehata showed in [12] that there exists a simple, totally ordered group G. If we take this group G and its positive cone G^+, then $K[[G^+]]$ is a duo domain which has no non-trivial normal submonoids coming from G, and whose monoid of regular elements has our desired filtration.

We are now ready for the main theorems, which extend the characterization of Matlis domains to duo rings not containing zero-divisors. The first result shows that in our setting R_T/R is a direct summand of Q/R whenever T is a normal submonoid of $R^{\times}$, and both $pd_R(Q)$ and $pd_R(Q/R_T)$ are ≤ 1.

Theorem 10.11. *Let R be a right and left duo ring not containing zero-divisors. If $T \triangleleft S$ are submonoids of $R^{\times}$ such that $pd_R(R_S) \leq 1$ and $pd_R(R_S/R_T) \leq 1$, then R_T/R is a direct summand of R_S/R.*

Proof. As a first step, we show that $(R_T/R)_P$ is S-divisible for all prime ideals P of R. Since R is a duo ring, P is completely prime, and $R \backslash P$ is multiplicatively closed. If $T \cap P = \emptyset$, then $(R_T)_P = R_P$ since $T \subseteq R \backslash P$ in this case. Thus, $t^{-1}p^{-1} \in R_P$ for all $t \in T$ and $p \in R \backslash P$. Since localizing at P is a flat functor, $(R_T/R)_P = (R_T)_P/R_P = R_P/R_P = 0$ is S-divisible.

Now, assume $T \cap P \neq \emptyset$. We first show that given any $s \in S$, R_T/sR_T is projective as a left R/sR-module. Clearly, R_T is a left R-module since $r(at^{-1}) = (ra)t^{-1} \in R_T$ for any $r \in R$ and any $at^{-1} \in R_T$. Since R is a duo ring, $sR = Rs$. Thus, for any $r \in R$, we can find $r_1 \in R$ such that $rs = sr_1$. Thus, $r(sat^{-1}) = sr_1(at^{-1})$, and sR_T is a submodule of ${}_RR_T$. Since R is duo, sR is a two-sided ideal of R, and we can view R_T/sR_T as a left R/sR-module.

Consider the exact sequence $0 \to R_T \to R_S \to R_S/R_T \to 0$. By assumption, $pd_R(R_S) \leq 1$ and $pd_R(R_S/R_T) \leq 1$. If $pd_R(R_T) > 1$, then $pd_R(R_T) > pd_R(R_S)$ and hence $pd_R(R_S/R_T) = pd_R(R_T) + 1 > 1$ by Lemma 9.10. However, this is a contradiction and thus $pd_R(R_T) \leq 1$. Consequently, $pd_{R/sR}(R_T/sR_T) \leq pd_R(R_T) \leq 1$ by Lemma 9.11. Now, consider the exact sequence $0 \to R_T/sR_T \to R_S/sR_T \to R_S/R_T \to 0$ of left R-modules. Since R_S is s-divisible and hence $sR_S = R_S$, we have $R_S/R_T \cong sR_S/sR_T = R_S/sR_T$, and thus $pd_R(R_S/R_T) =$

$pd_R(R_S/sR_T)$. Therefore, we have

$$pd_R(R_T/sR_T) < pd_R(R_S/R_T) = pd_R(R_S/sR_T) \leq 1$$

by Lemma 9.10. However, Theorem 9.14 shows that if $pd_{R/Rs}(R_T/sR_T) = 1$ then $pd_R(R_T/sR_T)$ must be 2, which is a contradiction. Therefore, $pd_{R/sR}(R_T/sR_T) = 0$, and hence R_T/sR_T is projective as a left R/sR-module.

We now show that $(R_T)_P = (R_S)_P$ whenever $T \cap P \neq \emptyset$. Observe that R_P is a local ring since P is completely prime. Hence, R_P/sR_P is a local ring. Moreover, since R is a duo ring and T is a normal submonoid of $R^\times$, we can view $(R_T)_P$ as a left R_P-module. To see this, take $(at^{-1})m^{-1} \in (R_T)_P$ and $bn^{-1} \in R_P$ where $n, m \in R \setminus P$. The duo condition provides $a_1 \in R$ such that $n^{-1}an = na_1$. Since T is normal, we can find $t_1 \in T$ such that $tn = nt_1$. Thus,

$$bn^{-1}(at^{-1}m^{-1}) = ba_1n^{-1}t^{-1}m^{-1} = (ba_1t_1^{-1})(n^{-1}m^{-1}) = (ba_1t_1^{-1})(mn)^{-1} \in (R_T)_P.$$

Since localization at P is an exact functor, $(R_T)_P/s(R_T)_P$ is projective as a left $(R_P)/s(R_P)$-module by what was shown in the preceding paragraph. Since projective modules over local rings are free [26, 4.58], $(R_T)_P/s(R_T)_P$ is a free $(R_P)/s(R_P)$-module.

Now assume $(R_T)_P/s(R_T)_P \neq 0$, and consider $t \in T \cap P \neq \emptyset$. If t were a unit of R_P, then $t^{-1} \in R_P$ would imply $t \in R \backslash P$. Furthermore, if $(au^{-1})m^{-1} \in (R_T)_P$, then the duo condition provides $a_1 \in R$ such that

$$au^{-1}m^{-1} = tt^{-1}au^{-1}m^{-1} = ta_1(ut)^{-1}m^{-1} \in t(R_T)_P.$$

Hence, $t(R_T)_P = (R_T)_P$.

Since $(R_T)_P/s(R_T)_P$ is a free $(R_P)/s(R_P)$-module, there exists some index set I such that $(R_T)_P/s(R_T)_P \cong \oplus_I(R_P)/s(R_P)$. Moreover, since $(R_T)_P$ is divisible by t, it must also be the case that $\oplus_I(R_P)/s(R_P) = t \oplus_I (R_P)/s(R_P)$. However, this implies $R_P/sR_P = t(R_P/sR_P)$. But $t \in PR_P$, which is a contradiction since t is not a unit in R_P. Therefore, given any $s \in S$, $(R_T)_P/s(R_T)_P = 0$ and hence $(R_T)_P = s(R_T)_P$. Thus, if $(rm^{-1})u^{-1} \in (R_P)_S$, we can use the duo and normality conditions to find $r_1 \in R$ and $u_1 \in S$ such that $(rm^{-1})u^{-1} = u_1^{-1}r_1m^{-1} \in u_1^{-1}(R_T)_P = (R_T)_P$. Furthermore, it is easily seen that $(R_T)_P \subseteq (R_P)_S$ since $xT = Tx$ for every $x \in R^\times$ and $T \subseteq S$. For if $rt^{-1}m^{-1} \in (R_T)_P$, then there exists $t_1 \in T \subseteq S$ such that $rt^{-1}m^{-1} = rm^{-1}t_1^{-1} \in (R_P)_S$. Thus, $(R_T)_P = (R_P)_S$. Finally, observe that there exists $s_1 \in S$ such that $(rs^{-1})m^{-1} = rm^{-1}s_1^{-1} \in (R_P)_S$ whenever $(rs^{-1})m^{-1} \in (R_S)_P$, whence it follows $(R_P)_S = (R_S)_P$. Therefore, $(R_T)_P = (R_P)_S = (R_S)_P$, and it readily follows from the S-divibility of R_S that $(R_T/R)_P = (R_T)_P/R_P = (R_S)_P/R_P$ is S-divisible. Consequently, R_T/R is S-divisible.

Suppose $s \in S$. By the S-divisibility of R_T/R, we have $s(R_T/R) = R_T/R$, and hence $sR_T + R = R_T$. Furthermore, R_T/sR_T is projective as a left R/sR-module. Hence $R/(R \cap sR_T) \cong (sR_T{+}R)/sR_T = R_T/sR_T$ is projective as left R/sR-module. The canonical epimorphism $\pi : R/sR \to R/(R \ \cap \ sR_T)$ defined by $\pi(r + sR) = r + (R \ \cap \ sR_T)$ induces the exact sequence

$$0 \to (R \ \cap \ sR_T)/sR \to R/sR \to R/(R \ \cap \ sR_T) \to 0$$

which splits since $R/(R \ \cap \ sR_T)$ is projective as a R/sR-module. However, multiplication by s induces isomorphisms

$$s^{-1}R/R \cong R/sR$$

and

$$(s^{-1}R \ \cap \ R_T)/R \cong (R \ \cap \ sR_T)/sR.$$

Hence

$$[s^{-1}R/R]/[(s^{-1}R \ \cap \ R_T)/R] \cong s^{-1}R/[(s^{-1}R\cap \ R_T)] \cong R/(R \ \cap \ sR_T)$$

is a projective R/sR-module. Thus, $[s^{-1}R/R] = [(s^{-1}R \cap R_T)/R] \oplus C/R$ for some submodule C of $s^{-1}R$ containing R Observe that $C/R \cong R_T/sR_T$. Using the notation of Fuchs and Salce, let $B = \bigcap_{P \in \mathcal{W}}(R_P \cap R_S)$ where $\mathcal{W}$ is the set of maximal ideals P with $T \cap P \neq \emptyset$. We have $(R_T)_P = (R_S)_P$ in the case that $T \cap P \neq \emptyset$. Hence,

$$(C/R)_P \cong (R_T/sR_T)_P = (R_T)_P/s(R_T)_P = (R_S)_P/s(R_S)_P = (R_S)_P/(R_S)_P = 0$$

from which we obtain $C_P = R_P$. Since $C \subseteq R_S$ and $(s^{-1}R \cap R_T)/R \subseteq R_T/R$, we have $s^{-1}R/R \leq R_T/R + B/R$ for every $s \in S$. Thus $R_S/R = R_T/R + B/R$

It remains to be seen that $(R_T/R) \cap (B/R) = 0$. Once this is established, we have shown that $R_S/R = (R_T/R) \oplus (B/R)$. Again using the notation of Fuchs and Salce, let $A = \bigcap_{P \in \mathcal{V}}(R_P \cap R_S)$, where $\mathcal{V}$ is the set of maximal ideals with $T \cap P = \emptyset$. Since R_T is clearly contained in A and $R_T \cap B \leq A \cap B$, it suffices to show that $A \cap B = R$. It is easily seen that $R \subseteq A \cap B$. For if $x \in R$, then $x \in R_T$ for any submonoid T of $R^{\times}$. Hence, $x \in R_P \cap R_S$ for every maximal ideal P and thus $x \in A \cap B$.

To see that $A \cap B \subseteq R$, it suffices to show that $R = [\cap_{P \in \text{m-Spec}} R_P] \cap R_S$ where m-Spec is the set of all maximal ideals of R. Let $x = us^{-1} \in R_S \backslash R$ and consider the right ideal $I_x = \{r \in R \mid xr \in R\}$. It is non-zero since $xs = us^{-1}s = u \in R$ yields $s \in I_x$. Moreover, I_x is a proper right ideal since $1 \notin I_x$. Hence, it follows that there exists a maximal right ideal P containing I_x. Since R is duo, P is a two-sided ideal. If $x \in R_P$, then $x = rm^{-1}$ for some $r \in R$ and $m \in R \backslash P$. However, $xm = r \in R$ implies that $m \in I_x \subseteq P$, which is a contradiction. Thus, given $x \in R_S \backslash R$, there exists some maximal ideal P of R such that $x \notin R_P$. Hence, $x \in R$ whenever $x \in R_P$ for every maximal ideal P of R. Therefore, $R = [\cap_{P \in \text{m-Spec}} R_P] \cap R_S$ and $A \cap B = R$. □

Theorem 10.12. *Consider the following conditions for a semi-prime right and left Goldie-ring R with classical right and left ring of quotients Q, and let $K = Q/R$:*

a) $K_R \cong \oplus_I A_i/R$ where each A_i is a subring of Q such that $(A_i)_R$ is countably generated.

b) K_R is a direct sum of countably generated submodules.

c) Every divisible module is h-divisible.

d) All divisible modules are weakly cotorsion.

e) $Z(D)$ is a direct summand of D whenever D is divisible.

f) $pd_R(Q/R) \leq 1$.

Then a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) $\Rightarrow$ e) $\Rightarrow$ f), and f) $\Rightarrow$ a) if R is a right and left duo pre-Matlis domain.

Proof. Since $a) \Rightarrow b)$ is obvious, we turn to $b) \Rightarrow c)$. Since R is a duo ring not containing zero-divisors, it is a semi-prime right and left Goldie-ring by Lemma 9.8. Hence, every element of Q can be written as $c^{-1}r$. Let D be a divisible module, and consider $a \in Z(D)$. We select a regular element s_0 of R such that $as_0 = 0$. Using a standard back and forth argument, we may find a countable subset $\{s_n | n < \omega\}$ of R^x such that $E = [\Sigma_{n<\omega} s_n^{-1} R]/R$ is a direct summand of Q/R.

We now show that we can find regular elements t_n of R with $t_0 = s_0$ and $t_{n+1} = r_n t_n$ for all $n < \omega$ such that $\Sigma_{n<\omega} s_n^{-1} R \subseteq \cup_{n<\omega} t_n^{-1} R$. Assume that we have already constructed $t_0, \ldots, t_n$ with the desired properties such that $s_0^{-1}, \ldots, s_n^{-1} \in t_n^{-1} R$. Since R is a semi-prime right and left Goldie ring, Rt_n and Rs_{n+1} are essential left ideals of R. We hence obtain a regular element $t_{n+1} \in R$ such that $t_{n+1} = ts_{n+1}$ and $t_{n+1} = r_{n+1} t_n$ for some $r_n, t \in R$. Thus, $s_{n+1}^{-1} = t_{n+1}^{-1} t$ and $t_n^{-1} = t_{n+1}^{-1} r_{n+1}$. Observe that each r_n is regular in R. Let $U = \cup_{n<\omega} t_n^{-1} R$, and observe $E \subseteq U/R$. We let $a_0 = a$ and $r_0 = s_0$, and select $\{a_n \in D | n < \omega\}$ such that $a_{n+1} r_{n+1} = a_n$ for $n < \omega$. Since $t_n^{-1} R$ is a free right R-module, setting $\alpha_n(t_n^{-1}) = a_n$ defines a map $\alpha_n : t_n^{-1} R \to D$. Moreover,

$$\alpha_{n+1}(t_n^{-1}) = \alpha_{n+1}(t_{n+1}^{-1}) r_{n+1} = a_{n+1} r_{n+1} = a_n = \alpha_n(t_n^{-1})$$

and $\alpha_0(1) = \alpha_0(t_0^{-1}s_0 = a_0 s_0 = 0$. Thus, the α_n induce a map $\alpha : U/R \to D$ with $\alpha(t_0^{-1} + R) = a$. However, $t_0^{-1} = s_0^{-1} \in E$, and so $\alpha|E : E \to D$ contains a in its image. Hence, there is a map $\beta : Q_R \to D$ such that $a \in im\ \beta$.

Moreover, the last arguments of the last paragraph show that any countable direct summand of K_R can be embedded into a submodule of K_R of projective dimension 1. Thus, K_R has projective dimension 1 and the same holds for Q_R. Since R is a semi-prime right and left Goldie-ring, every non-singular module, which is divisible in the classical sense, is actually a Q-module, and hence injective and has projective dimension 1. This holds in particular for $D/Z(D)$. By [1, Corollary 4.6] $Z(D)$ is weakly cotorsion, and so $\mathrm{Ext}^1_R(D/Z(D), Z(D)) = 0$. This shows that D is h-divisible.

c) $\Rightarrow$ *d)*: By Theorem 4.1 of [1], $Z(D)$ is a direct summand of D whenever D is a divisible module because D is h-divisible by c). Since divisible modules are divisible in the classical sense, all modules which are divisible in the classical sense are weakly cotorsion by Corollary 4.6 of [1]. Thus all divisible modules are weakly cotorsion. However, if all such modules are weakly cotorsion, then their singular submodule is a direct summand. Thus, *d)* $\Rightarrow$ *e)* holds. Finally, *e)* $\Rightarrow$ *f)* follows from [1, Proposition 5.1].

f) $\Rightarrow$ *a)*: Assume $pd_R(Q/R) = 1$, and assume that R is a pre-Matlis domain with the desired filtration

$$\{1\} = T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...T_\kappa = R^\times.$$

As a semi-prime right and left Goldie-ring, R has a maximal right ring of quotients Q which is also its maximal left ring of quotients, as well as its classical right and left ring of quotients [17, Theorem 3.37]. Thus, every regular element of R is invertible in Q and $Q = \{ab^{-1} \mid a, b \in R \text{ with } b \text{ regular}\} = \{c^{-1}d \mid c, d \in R \text{ with } c \text{ regular}\}$. Let $\mathcal{U} = \{R_{T_\alpha}/R \mid \alpha \leq \kappa\}$. Observe that for each $\alpha < \kappa$, R_{T_α}/R is a submodule of Q/R. We show that $\mathcal{U}$ is a $G(\aleph_0)$-family of Q/R. Clearly, condition *i)* is satisfied since $\{0\} = R_{\{1\}}/R \in \mathcal{U}$ and $Q/R = R_{R^\times}/R \in \mathcal{U}$. Moreover, $\mathcal{U}$ is closed under unions of chains since $\{T_\alpha\}_{\alpha \leq \kappa}$ forms a smooth chain and includes $R^\times = \bigcup_{\alpha<\kappa} T_\alpha$.

To see that condition *iii*) is satisfied, take $R_{T_\alpha}/R \in \mathcal{U}$ and let

$$X = \{r_j s_j^{-1} + R \mid r_j, s_j \in R \text{ with } s_j \text{ regular}, \ j < \omega\}$$

be a countable subset of Q/R. Using condition *ii*) of the filtration, there exists $\beta < \kappa$ such that $T_\alpha \subseteq T_\beta$, $\{s_j \mid j < \omega\} \subseteq T_\beta$, and T_β is countably generated over T_α. Hence, $R_{T_\alpha}/R, X \subseteq R_{T_\beta}/R$ and there exists a countable subset $S_\alpha \subseteq T_\beta$ such that $T_\beta = S_\alpha T_\alpha = T_\alpha S_\alpha$. Thus, if $t \in T_\beta$, there exists $s_{\alpha_1}, s_{\alpha_2}, ..., s_{\alpha_n} \in S_\alpha$ and $t_{\alpha_1}, t_{\alpha_2}, ..., t_{\alpha_n} \in T_\alpha$ such that $t = s_{\alpha_1} t_{\alpha_1} s_{\alpha_2} t_{\alpha_2} ... s_{\alpha_n} t_{\alpha_n}$. Then if $rt^{-1} + R_{T_\alpha} \in R_{T_\beta}/R_{T_\alpha}$, we have $rt^{-1} = rt_{\alpha_n}^{-1} s_{\alpha_n}^{-1} ... t_{\alpha_2}^{-1} s_{\alpha_2}^{-1} t_{\alpha_1}^{-1} s_{\alpha_1}^{-1}$. Therefore, $(R_{T_\beta}/R)/(R_{T_\alpha}/R) \cong R_{T_\beta}/R_{T_\alpha}$ is countably generated by $\{s^{-1} \mid s \in S_\alpha \backslash T_\alpha\}$ and $\mathcal{U}$ is a $G(\aleph_0)$-family of Q/R.

It follows from Lemma 10.1 that Q/R admits a tight system $\mathcal{T}$. It is clear that $\mathcal{T}$ is also a $G(\aleph_0)$-family of Q/R, and it is easily seen that $\mathcal{U} \cap \mathcal{T}$ is a $G(\aleph_0)$-family of tight submodules of Q/R of the form R_{T_α}/R for $\alpha < \kappa$. Thus, given any $R_{T_\alpha}/R \in \mathcal{U} \cap \mathcal{T}$,

$$pd_R(Q/R_{T_\alpha}) = pd_R((Q/R)/(R_{T_\alpha}/R)) \leq pd_R(Q/R) \leq 1.$$

It then follows from Theorem 10.11 that R_{T_α}/R is a direct summand of Q/R for every $\alpha < \kappa$. Since $R^\times = \bigcup_{\alpha<\kappa} T_\alpha$, we have $Q/R = \bigcup_{\alpha<\kappa} R_{T_\alpha}/R$. Moreover, the smooth filtration ensures that $R_{T_\beta}/R = \bigcup_{\gamma<\beta} R_{T_\gamma}/R \in \mathcal{U} \cap \mathcal{T}$, and hence there exists $\beta \leq \kappa$ and a continuous well-ordered ascending chain $\{R_{T_\gamma}/R \mid \gamma < \beta\} \subseteq \mathcal{U} \cap \mathcal{T}$ of submodules of Q/R such that R_{T_γ}/R is a direct summand of Q/R and $R_{T_{\gamma+1}}/R_{T_\gamma}$ is countably generated. Hence, $Q/R = \bigoplus_{\gamma<\beta} A_\gamma/R$ where each A_γ is a countably generated. Finally, since R is right and left duo and R_{T_γ} is a subring of Q for each γ, we have that each A_γ is a two-sided submodule of Q. □

As mentioned, the example in Theorem 10.8 provides a ring for which $R^\times$ does not have our desired filtration of normal submonoids. Moreover, this example is such that $pd_R(Q) \leq 1$ even though Q/R cannot be written as a direct sum of countably generated submodules. Theorem 8.2 shows that if Q_R is countably generated, then a module is h-divisible if and only if it is divisible in the classical sense. However, the ring in Theorem 10.8 has a maximal ring of quotients which is not countably generated. In the case that $(Q/R)_R$ is generated by $\aleph_1$-many elements, we can find the following filtration of countable submonoids of $R^\times$:

Corollary 10.13. *Suppose R is a semi-prime right and left Goldie-ring such that $(Q/R)_R$ is a direct sum of $\aleph_1$ many countable modules, then there exists a smooth ascending chain $T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...$, $\alpha < \aleph_1$, of countable submonoids of $R^\times$ such that $R^\times = \bigcup_{\alpha<\aleph_1} T_\alpha$.*

Proof. Let $T_0 = \{1\}$ and let $T_\sigma = \bigcup_{\beta<\sigma} T_\beta$ for each limit ordinal $\sigma < \aleph_1$. Note that each T_σ is countable as the countable union of a countable set. Let $\alpha < \aleph_1$ and suppose that for each $\beta \leq \alpha$, T_β has been defined so that R_{T_β}/R is a direct sum of countably many A_ν/R. Then, $R_{T_\alpha}/R = \bigoplus_{I_\alpha} A_\nu/R \sqsubseteq Q/R$ for some countable set I_α. If $R_{T_\alpha} = Q$, then we are done. Otherwise, there exists $\mu < \aleph_1$ with $A_\mu \nsubseteq R_{T_\alpha}$. Let $A_\mu = \langle r_n t_n^{-1} \mid n < \omega \rangle$ and define $T_\alpha^1 = \langle T_\alpha, t_n \mid n < \omega \rangle$. Observe that T_α^1 is countable since it is countably generated by countable sets. Since $R_{T_\alpha^1}/R \subseteq Q/R = \bigoplus_{\nu<\aleph_1} A_\nu/R$, we can find a countable subset $I_\alpha^1 \supseteq I_\alpha$ such that $R_{T_\alpha^1}/R \subseteq \bigoplus_{I_\alpha^1} A_\nu/R$.

If $R_{T_\alpha^1} = Q$, then we are done. Otherwise, there exists $\mu_2 < \aleph_1$ with $A_{\mu_2} \nsubseteq R_{T_\alpha^1}$. As before, let $A_{\mu_2} = \langle r_n t_{n,2}^{-1} \mid n < \omega \rangle$ and define $T_\alpha^2 = \langle T_\alpha^1, t_{n,2} \mid n < \omega \rangle$. Then, T_α^2 is countable and we can find a countable subset $I_\alpha^2 \supseteq I_\alpha^1$ such that $R_{T_\alpha^2}/R \subseteq \bigoplus_{I_\alpha^2} A_\nu/R$. Note that $R_{T_\alpha^1}/R \subseteq \bigoplus_{I_\alpha^1} A_\nu/R \subseteq R_{T_\alpha^2}/R \subseteq \bigoplus_{I_\alpha^2} A_\nu/R$. Continue this process to find $I_\alpha \subseteq I_\alpha^1 \subseteq I_\alpha^2 \subseteq ... \subseteq I_\alpha^n \subseteq ...$ and $T_\alpha \subseteq T_\alpha^1 \subseteq T_\alpha^2 \subseteq ... \subseteq T_\alpha^n \subseteq ...$ satisfying $R_{T_\alpha^n}/R \subseteq \bigoplus_{I_\alpha^n} A_\nu/R \subseteq R_{T_\alpha^{n+1}}/R \subseteq \bigoplus_{I_\alpha^{n+1}} A_\nu/R$.

Let $T_{\alpha+1} = \bigcup_{n<\omega} T_\alpha^n$ and let $I = \bigcup_{n<\omega} I_\alpha^n$. Observe that both $T_{\alpha+1}$ and I are countable since each T_α^n and each I_α^n are countable. If $rt^{-1} + R \in R_{T_{\alpha+1}}/R$, then $t \in T_\alpha^n$ for some $n < \omega$. Hence, $rt^{-1} + R \in \bigoplus_{I_\alpha^n} A_\nu/R \subseteq \bigoplus_I A_\nu/R$ and so $R_{T_{\alpha+1}}/R \subseteq \bigoplus_I A_\nu/R$. On the other hand, if $x \in \bigoplus_I A_\nu/R = \bigcup_n \bigoplus_{I_\alpha^n} A_\nu/R$, then $x \in \bigoplus_{I_\alpha^n} A_\nu/R$ for some $n < \omega$, and thus $x \in R_{T_\alpha}^{n+1}/R \subseteq R_{T_{\alpha+1}}/R$. Hence, $R_{T_{\alpha+1}}/R = \bigoplus_I A_\nu/R \sqsubseteq Q/R$. Therefore, T_α is defined for every $\alpha < \aleph_1$ and $T_0 \leq T_1 \leq ... \leq T_\alpha \leq ...$, $\alpha < \aleph_1$ is a smooth ascending chain of countable submonoids of $R^\times$ such that $R^\times = \bigcup_{\alpha<\aleph_1} T_\alpha$. $\square$

We conclude by once again considering rings which are right strongly non-singular and right semi-hereditary. Recall that if R is a right strongly non-singular, right semi-hereditary ring with finite right Goldie-dimension, then any two-sided essential submodule of Q has the finite Baer-splitting property (Proposition 7.2). In the context of Theorem 10.12, we see that for this class of rings our decomposition of $(Q/R)_R$ results in submodules A_i of Q which have the finite Baer-splitting property:

Corollary 10.14. *Let R be a right and left duo pre-Matlis domain such that $pd_R(Q) \leq 1$. If R is right semi-hereditary, then $(Q/R)_R \cong \oplus_I A_i/R$ where each A_i is an R-submodule of Q satisfying the finite Baer-splitting property.*

Proof. The proof of Theorem 10.12 shows that $(Q/R)_R \cong \oplus_I A_i/R$, where each A_i is a two-sided R-submodule of Q. Moreover, each A_i is of the form R_{T_i}, where T_i is a submonoid of $R^{\times}$. Since $R \subseteq R_{T_i} \subseteq Q$ and $R_R \leq_e Q_R$, we have $R_R \leq_e (R_{T_i})_R \leq_e Q_R$ by [17, Prop. 1.1]. Hence, A_i is a two-sided R-submodule of Q such that $(A_i)_R$ is essential in Q. As a semi-prime right and left Goldie-ring, R is strongly non-singular by [28, Ch. XI, Proposition 5.4] and [28, Ch. XII, Corollary 2.6]. Therefore, each A_i has the finite Baer-splitting property by Proposition 7.2. $\square$

Bibliography

[1] Albrecht, U.; *Right Utumi p.p.-rings*; Rendiconti del Seminario Matematico della Universit di Padova 124 (2010); 25-42.

[2] Albrecht, U; *Two-sided Essential Submodules of $Q^r(R)$*; Houston Journal of Mathematics 33(1) (2005); 103-123.

[3] Albrecht, U., Dauns, J. and Fuchs, L.; *Torsion-freeness and non-singularity over right p.p.-rings.* Journal of Algebra, Vol. 285, no. 1 (2005), 98-119.

[4] Albrecht, U. and Trilifaj, J.; *Cotilting Classes of Torsion-free Modules*; J. Alg and Appl. 5(6) (2006); 747-764.

[5] Anderson, F.; Fuller, K. *Rings and Categories of Modules.* Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York (1992).

[6] Baer, R.*Abelian Groups Without Elements of Finite Order.* Duke Mathematical Journal 3 (1937), 68-122.

[7] Bazzoni, S., Eklof, P. and Trlifaj, J.; *Tilting Cotorsion Pairs*; Bull. London Math. Soc. 37 (5) (2005); 683-696.

[8] Bessenrodt, C., Brungs, H.H. and Törner, G.; *Right Chain Rings, Part 1*; Schriftenreihe des Fachbereichs Mathematik der Universitt Duisburg, Heft 74 (1985).

[9] Birkenmeier, G., Park, J. and Rizvi, S.; *Extensions of Rings and Modules*; Springer, New York, Heidelberg, Dordrecht, London (2013).

[10] Brungs, H.H.; *Three Questions on Duo Rings*; Pacific Journal of Mathematics Vol. 58, No. 2 (1975); 345-349.

[11] Chatters, A.; Hajarnavis, C. *Rings with Chain Conditions.* Research Notes in Mathematics, Vol. 44, Pitman Advanced Publishing Program, London, Melbourne (1980).

[12] Chehata, C.G.*An Algebraically Simple Ordered Group.* Proceedings of the London Mathematical Society vol. s3-2, issue 1 (1952); 183-197.

[13] Dauns, J.; Fuchs, L. *Torsion-freeness in rings with zero divisors.* Journal of Algebra and Its Applications, Vol. 3, no. 3 (2004), 221-237.

[14] Enochs, E.; Jenda, O.; *Relative Homological Algebra.* de Gruyter, Berlin - New York (2000).

[15] Fuchs, L.; Salce, L. *Modules over Non-Noetherian Domains.* Math. Surveys and Monographs, Vol. 84, American Mathematical Society (2001).

[16] Göbel, R. and Trlifaj, J.; *Approximations and Endomorphism Algebras of Modules: Volume 1 - Approximations/Volume 2 - Predictions*; Walter de Gruyter (2012).

[17] Goodearl, K.; *Ring Theory: Nonsingular Rings and Modules.* Marcel Dekker, New York, Basel (1976).

[18] Hattori, A.; *A foundation of torsion theory for modules over general rings.* Nagoya Math. J., Vol. 17 (1960), 147-158.

[19] Hill, P.; *The Third Axiom of Countability for Abelian Groups*; Proc. Amer. Math. Soc. 82 (1981); 347-350.

[20] Hügel, L.A., Herbera, D. and Trlifaj, J.; *Divisible modules and localization*; Journal of Algebra 294 (2005); 519-551.

[21] Hungerford, T.; *Algebra.* Graduate Texts in Mathematics, Vol. 73, Springer, New York (2003).

[22] Jans, J.; *Rings and Homology.* Holt, Rinehart, and Winston (1964).

[23] Lee, S.B.; *On divisible modules over domains*; Arch. Math. 53 (1989); 259-262.

[24] Matlis, E.; *Divisible Modules*; Proc. Amer. Math. Soc. 11 (1960); 385-391.

[25] Rotman, J.; *An Introduction to Homological Algebra, 1st Edition.* Academic Press (1979).

[26] Rotman, J.; *An Introduction to Homological Algebra, Second Edition.* Springer, New York (2009).

[27] Small, L.; *Semihereditary Rings*; Bull. Amer. Math. Soc. 73 (1967); 656-658.

[28] Stenström, B.; *Ring of Quotients.* Lecture Notes in Mathematics, Vol. 217, Springer-Verlag, Berlin, Heidelberg, New York (1975).