Nonlinear Dynamics and Control of Spacecraft Relative Motion

by

Ayansola Daniel Ogundele

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Approved by

Andrew J. Sinclair, Chair, Associate Professor of Aerospace Engineering
Subhash C. Sinha, Co-Chair, Alumni Professor Emeritus of Mechanical Engineering
David A. Cicci, Professor of Aerospace Engineering
John E. Cochran Jr., Alumni Professor Emeritus of Aerospace Engineering
George Flowers, Dean of the Graduate School

Abstract

The description and control of the relative motion of spacecraft has attracted a great deal of attention over the last five decades. This is as a result of its numerous applications in rendezvous and proximity operations, spacecraft formation flying, distributed spacecraft missions etc. Generally, the linearized, simplified model of the relative motion is described using time-invariant, Hill-Clohessy-Wiltshire (HCW) equations developed in 1960s. This model was based on the assumptions that, the chief and deputy spacecraft are in close proximity and the chief spacecraft is in a near circular orbit.

The HCW equations have the disadvantage of not being able to capture the relative dynamics over a long period of time or large separations. Hence, several new models and equations of motion have been developed. In this work, two new linearized models of the relative motion based on the harmonic balance and averaging methods are developed. Numerical solutions show that the models can provide better approximations to the relative motion than the HCW model.

Another innovative contribution of this dissertation is the development of closed-form solutions of Riccati-type and Abel-type nonlinear spacecraft relative motion arising from the second and third order approximation of the variation of the true latitude rate. The results are new, closed-form analytical solutions of the true anomaly variation with time which give a better understanding of the relative motion than using Cartesian coordinates.

Feedback controllers are designed for the relative motion via State Dependent Riccati Equation (SDRE) control strategy. The key interest in the use of SDRE strategy is its ability to provide an effective algorithm for synthesizing nonlinear feedback controls by allowing for nonlinearities in the system states, while offering design flexibility through state dependent weighting matrices.

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A human being is part of the whole, called by us the “Universe”, a part limited in time and space. He experiences himself, his thoughts and feelings as something separated from the rest, a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest us. Our task must be to free ourselves from this prison by widening our circles of compassion to embrace all living creatures and the whole of nature in its beauty.

~ Albert Einstein ~

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Chapter 1

Introduction

Insight into the orbital dynamics of spacecraft in close proximity can be gained by linearizing the spacecraft relative equation of motion and ignoring the long-term effects of the natural perturbation forces such as solar radiation pressure, atmospheric drag, nonsphericity of the Earth (J_2 effects) etc. Relative motion studies have focused on linearized equations and this approach has been used for decades to analyze very close proximity operations [1,2]. The Hill-Clohessy-Wiltshire (HCW) equations describe a simplified model of orbital relative motion and make two key explicit assumptions: deputy and chief spacecraft in close proximity and a circular chief orbit. They are used for near-field rendezvous and proximity operations. The assumption of a circular orbit presents a problem in modeling formation of satellites in an elliptical orbit. Nonetheless, HCW-like techniques are still used.

Generally, the relative orbit is described using Hill frame coordinates with which it is difficult to obtain precise orbit geometry [3]. Parameterizing the relative motion using the Keplerian orbital elements simplifies the orbit description better than the use of Hill frame coordinates. Rather than using position and velocity, the use of orbital elements has benefit of having only one term (anomaly) that changes with time out of the six orbital elements, and this reduces the number of terms to be tracked from six to one.

In this dissertation, third order (cubic) polynomial approximation of spacecraft relative motion is developed. Two new linearization models are obtained by applying harmonic balance method and averaging technique to the cubic approximation. Also, this work presents third order (Abel-type) and second order (Riccati-type) approximation of nonlinear differential equations describing the dynamics of the relative motion of deputy spacecraft with respect to the chief spacecraft in terms of the orbit element differences. From this, two new analytical solutions are developed. In addition, feedback controllers are designed via state dependent Riccati equation (SDRE) technique.

This dissertation is organized as follows. Chapter 1 gives introductions on coordinate systems and transformations, conversions between orbital elements to position and velocity, spacecraft relative equations of motion, harmonic balance method, averaging method and state dependent riccati equation. The derivation of the cubic approximation model of spacecraft relative motion is given in Chapter 2. Chapter 3 describes harmonic linearization of spacecraft relative motion, while Chapter 4 describes the development of averaging model of spacecraft relative motion. In chapter 5, closed-form solutions of Abel-type and Riccati-type equations of relative motion, and the design of SDRE controllers are described in Chapter 6. The conclusions are given in Chapter 7.

1.1 Coordinate Systems and Transformations

1.1.1 Coordinate Systems

In principle, any coordinate system can be used for space mission geometry problems. However, selection of the proper reference coordinates is often critical to developing a good physical understanding, obtaining analytic expressions for key mission parameters, and reducing the probability of error. Coordinate axes consist of three mutually perpendicular axes that describe the three-dimensional coordinates of all points in space. For space applications, coordinate systems are often labeled using the location of a origin and the directions of the coordinate axes. Typically, we choose the Earth's center as the origin for problems in orbit analysis or geometry on the Earth's surface and the spacecraft's position for problems concerning the apparent motion of objects as seen from the spacecraft. The fundamental plane (that is the Equator) contains the x-axis. The reference direction defines the x-axis. The z-axis is in the direction normal to the fundamental plane and the y-axis completes the right-hand orthogonal system.

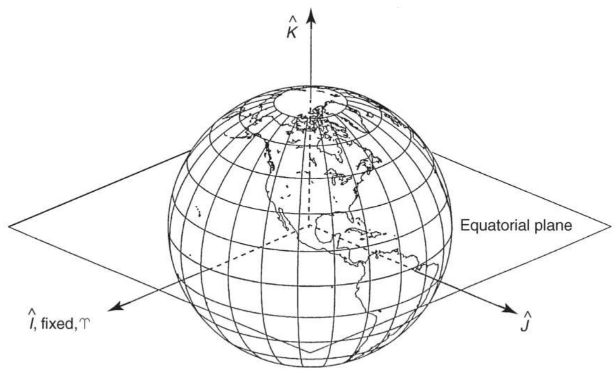

Figure 1.1: Earth Centered Inertial (ECI) Frame

Occasionally, coordinates are centered on a specific spacecraft instrument when we are interested not only in viewing the outside world, but also in obstructions of the field of view by other spacecraft components. Typical ways to fix the rotational orientation of a coordinate system are with respect to inertial space, with respect to axes fixed in the Earth or some other object being viewed, to the spacecraft, or to an instrument on the spacecraft. Several coordinate systems are further described below.

(a) Earth Centered Inertial Frame (ECI)

The Earth Centered Inertial (ECI) frame, also referred to as the geocentric equatorial coordinate system, is assumed to be inertially fixed in space but, in practice, it is slowly shifting over time. Since a truly inertial system is impossible to realize, the standard J2000 system is adopted as the best representation of an ideal, inertial frame at a fixed epoch [3,4]. The shift of this frame is so slow relative to the motion of interest that it can reasonably be neglected.

The fundamental plane is the Earth's equatorial plane and the positive X-axis points in the vernal equinox direction. The Z-axis points in the direction of the celestial north-pole while the Y axis completes the orthogonality. The XYZ system is not fixed to the earth and turning with it; rather, the geocentric equatorial frame is nonrotating with respect to the stars (except for precession of the equinoxes) and the earth turns relative to it. The unit vectors $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$ lie along the X,Y and Z axes, respectively, and will be useful in describing

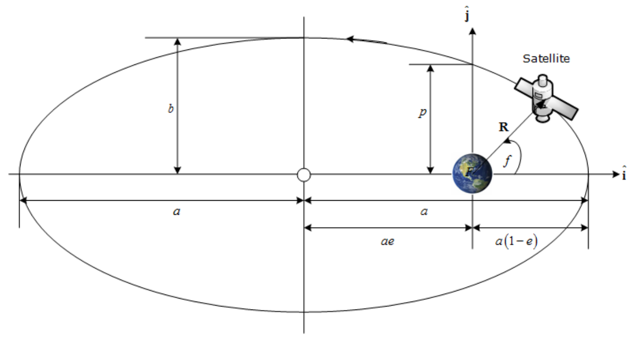

Figure 1.2: Perifocal Coordinate System

vectors in the geocentric-equatorial system. Figure 1.1 shows Earth Centered Inertial Frame. The position of a point in the ECI frame can be specified by either Cartesian coordinates or inertial spherical coordinates. The scalar R is the distance between the center of the Earth and the spacecraft. The Cartesian coordinates X,Y,Z are defined as

$$\mathbf{R} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}} \tag{1.1}$$

(b) Perifocal Coordinate System

In the perifocal reference frame, the x axis points at periapse, the z axis is normal to the orbit plane, and the y axis completes the right-hand system. The perifocal unit vectors attached to inertial coordinate axes can be represented as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and can be defined as follows:

$$\begin{aligned} \hat{\mathbf{i}} &= \tfrac{\mathbf{e}}{e} \\ \hat{\mathbf{k}} &= \tfrac{\mathbf{h}}{h} \\ \hat{\mathbf{j}} &= \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \tfrac{\mathbf{h}\times\mathbf{e}}{|\mathbf{h}\times\mathbf{e}|} \end{aligned} \tag{1.2}$$

Here, $\mathbf{e}$ is the eccentricity vector and $\mathbf{h}$ is the angular momentum vector. The direction cosine matrix that defines the orientation of the perifocal system with respect to the inertial system is

$$[\mathbf{C}_{PI}] = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \tfrac{\mathbf{e}}{e} & \tfrac{\mathbf{h}\times\mathbf{e}}{|\mathbf{h}\times\mathbf{e}|} & \tfrac{\mathbf{h}}{h} \end{bmatrix} \tag{1.3}$$

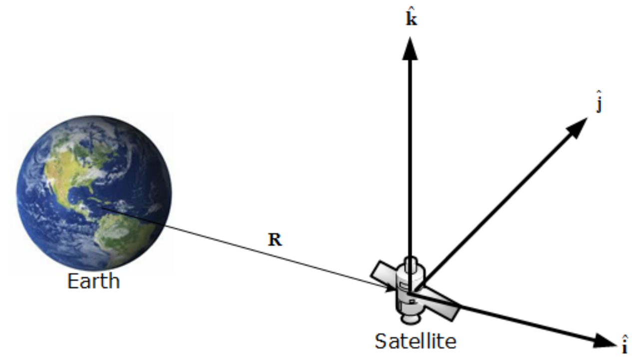

Figure 1.3: Hill Reference Frame

Figure 1.2 shows the perifocal coordinate system. The position and velocity vectors can be expressed as

$$\begin{aligned} \mathbf{r}_{PER} &= R\cos f\hat{\mathbf{i}} + R\sin f\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \\ \mathbf{v}_{PER} &= \sqrt{\tfrac{\mu}{p}}\left[-\sin f\hat{\mathbf{i}} + (e+\cos f)\,\hat{\mathbf{j}}\right] \end{aligned} \tag{1.4}$$

where $\mathbf{r}_{PER}$ and $\mathbf{v}_{PER}$ are the position and velocity vectors in perifocal frame, f is the true anomaly, μ is the gravitational parameter, p is the semi-latus rectum and $e = |\mathbf{e}|$.

(c) Hill Reference Frame

This is a local rotating frame whose origin coincides with the motion of the reference trajectory. This frame, shown in Figure 1.3, is used to define the relative motion of a spacecraft about a reference trajectory. The coordinates $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ can be expressed as

$$\begin{aligned} \hat{\mathbf{i}} &= \tfrac{\mathbf{R}}{|\mathbf{R}|} \\ \hat{\mathbf{k}} &= \tfrac{\mathbf{R}\times\mathbf{V}}{|\mathbf{R}\times\mathbf{V}|} = \tfrac{\mathbf{h}}{|\mathbf{h}|} \\ \hat{\mathbf{j}} &= \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \tfrac{\mathbf{h}\times\mathbf{R}}{|\mathbf{h}\times\mathbf{R}|} \end{aligned} \tag{1.5}$$

where, $\mathbf{R}$ and $\mathbf{V}$ are the position and velocity vectors of the reference trajectory. This frame is also known as Local-Vertical Local-Horizontal (LVLH) frame.

1.1.2 Reference Frame Conversions

Rotation matrices are used to rotate a set of coordinates about an axis with a certain angle θ. The following 3-by-3 matrices describe rotations about the $x-$, $y-$ and $z-$ axes, respectively.

$$\mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \mathbf{C}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$\mathbf{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6}$$

(a) Transformation from Perifocal frame to ECI frame

The Perifocal coordinates can be reached from ECI via three rotations:

i) Rotate Ω about $\hat{\mathbf{k}}$

ii) Rotate i about $\hat{\mathbf{i}}$

iii) Rotate ω about $\hat{\mathbf{k}}$

where Ω is the right ascension of ascending node, i is the inclination and ω is the argument of perigee. Rotating coordinates has the opposite effect of rotating the vector. Thus, in ECI frame we have

$$\mathbf{r}_{ECI} = \mathbf{C}_3(\Omega)\,\mathbf{C}_1(i)\,\mathbf{C}_3(\omega)\,\mathbf{r}_{PER} \tag{1.7}$$

where $\mathbf{r}_{ECI}$ is position vector in ECI frame.

The transformation matrix from Perifocal to ECI is

$$
\begin{aligned}
[\mathbf{C}_{IP}] &= \mathbf{C_3}(\Omega)\,\mathbf{C_1}(i)\,\mathbf{C_3}(\omega) \\
&= \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega \mathrm{sin} i \\ \sin\Omega\sin\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega \mathrm{sin} i \\ \sin\omega \mathrm{sin} i & \cos\omega \mathrm{sin} i & \cos i \end{bmatrix}
\end{aligned}
\tag{1.8}
$$

while the transformation matrix from ECI to Perifocal is

$$
[\mathbf{C}_{PI}] = [\mathbf{C}_{IP}]^{-1} = [\mathbf{C}_{IP}]^{T} \tag{1.9}
$$

Therefore, position and velocity in ECI frame can be expressed as

$$
\begin{aligned}
\mathbf{r}_{ECI} &= [\mathbf{C}_{IP}] \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \\
\mathbf{v}_{ECI} &= [\mathbf{C}_{IP}] \begin{bmatrix} -\left(\sqrt{\frac{\mu}{p}}\right)\sin f \\ \left(\sqrt{\frac{\mu}{p}}\right)(e + \cos f) \\ 0 \end{bmatrix}
\end{aligned}
\tag{1.10}
$$

The transformation from the perifocal frame to ECI is achieved using the transpose of $[\mathbf{C}_{IP}]$.

(b) Transformation from Orbital frame to ECI frame

A satellite orbit is typically defined using six Keplerian orbital elements which, with the exception of the true anomaly angle, have the advantage of being constant in time (assuming the satellites are modeled as point masses and are not subjected to orbital perturbations). If orbital propagation is conducted in the ECI frame it is necessary to convert the satellite orbits from orbital elements into Cartesian position and velocity vectors in the ECI frame at the start of any simulation.

If the Keplerian (i.e. unperturbed) initial position and velocity of a satellite is given in the orbital frame as

$$\mathbf{r}_o = r\begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix}$$
$$\mathbf{v}_o = \sqrt{\frac{\mu}{p}}\begin{bmatrix} -\sin f \\ e+\cos f \\ 0 \end{bmatrix} \tag{1.11}$$

where $r = p/\left(1 + e\cos f\right)$, then the rotation of these vectors into the inertial frame is

$$\mathbf{R}_I = \left[\mathbf{C}_{IO}\right]\mathbf{r}_O = \left[\mathbf{C}_{OI}\right]^T\mathbf{r}_O = \left[\mathbf{C_3}\left(\omega\right)\mathbf{C_1}\left(i\right)\mathbf{C_3}\left(\Omega\right)\right]\mathbf{r}_O$$
$$\mathbf{V}_I = \left[\mathbf{C}_{IO}\right]\mathbf{v}_O \tag{1.12}$$

where $\mathbf{R}_I$ and $\mathbf{V}_I$ are the inertial position and velocity.

(c) Conversion between Inertial and HCW Coordinates

The deputy spacecraft's position vector can be determined from the inertial position using the expression

$$\mathbf{R}_D = \left[\mathbf{C}_{HI}\right]\rho_I \tag{1.13}$$

where, $\rho_I = \mathbf{R}_D - \mathbf{R}_C$, $\mathbf{R}_D = \begin{bmatrix} X_D & Y_D & Z_D \end{bmatrix}^T$ and $\mathbf{R}_C = \begin{bmatrix} X_C & Y_C & Z_C \end{bmatrix}^T$ are the deputy and chief spacecraft inertial position vector and $\left[\mathbf{C}_{HI}\right]$ is the rotation matrix from the ECI frame to the Hill frame given by

$$\left[\mathbf{C}_{HI}\right] = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{R}_C}{R_C} & \frac{\mathbf{H}_C\times\mathbf{R}_C}{\left|\mathbf{H}_C\times\mathbf{R}_C\right|} & \frac{\mathbf{H}_C}{H_C} \end{bmatrix} \tag{1.14}$$

Here, $\mathbf{H}_C$ is the chief's angular momentum per unit mass. The subscripts "D" and "C" represent deputy and chief spacecraft. The rotation matrix from the Hill reference frame to the ECI frame is

$$[\mathbf{C}_{IH}] = [\mathbf{C}_{HI}]^{-1} = [\mathbf{C}_{HI}]^T \tag{1.15}$$

In the Hill frame, the relative velocity can be expressed as

$$\dot{\rho}_H = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T \tag{1.16}$$

while in the inertial frame it can be expressed as

$$\dot{\rho}_I = \dot{\mathbf{R}}_D - \dot{\mathbf{R}}_C = \mathbf{V}_D - \mathbf{V}_C \tag{1.17}$$

The conversion of the relative position and velocity from ECI frame to HCW frame is

$$\begin{aligned} \rho_H &= [\mathbf{C}_{HI}]\, \rho_I \\ \mathbf{V}_H = \dot{\rho}_H &= \left[\dot{\mathbf{C}}_{HI}\right] \rho_I + [\mathbf{C}_{HI}]\, \dot{\rho}_I \end{aligned} \tag{1.18}$$

and the inverse transformation is given as

$$\begin{aligned} \rho_I &= [\mathbf{C}_{HI}]^T \rho_H \\ \mathbf{V}_I = \dot{\rho}_I &= \left[\dot{\mathbf{C}}_{HI}\right]^T \rho_H + [\mathbf{C}_{HI}]^T \dot{\rho}_H \end{aligned} \tag{1.19}$$

1.2 Conversion from Orbital Elements to Position and Velocity and Vice-Versa

In this section, procedures for the conversion of orbital elements to position and velocity and vice-versa are shown.

1.2.1 Conversion from Orbital Elements to Position and Velocity

The steps highlighted below can be used to solve for the position vector $\mathbf{R}$ and velocity vector $\mathbf{V}$ in geocentric equatorial coordinate system given the orbital elements: semi-major axis, a, eccentricity, e, inclination, i, argument of perigee, ω, right ascension of ascending node, Ω and true anomaly, f.

a) Calculate the position R

$$R = \frac{a\left(1 - e^2\right)}{1 + e\cos f} \tag{1.20}$$

b) Calculate the position and velocity in perifocal frame using Eq. (1.4).

c) Calculate the transformation matrix using Eq. (1.8).

d) Calculate the ECI position and velocity vectors

$$\begin{aligned} \mathbf{R} &= \left[\mathbf{C}_{IP}\right]\mathbf{r}_{PER} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}} \\ \mathbf{V} &= \left[\mathbf{C}_{IP}\right]\mathbf{v}_{PER} = \dot{X}\hat{\mathbf{I}} + \dot{Y}\hat{\mathbf{J}} + \dot{Z}\hat{\mathbf{K}} \end{aligned} \tag{1.21}$$

1.2.2 Conversion from Position and Velocity to Orbital Elements

Tracking stations provide **R** and **V** in geocentric equatorial coordinate system and they can be used to solve for the classical orbital elements $a, e, i, \Omega, \omega, f$ using the steps highlighted below.

a) Calculation of eccentricity

$$\begin{aligned} \mathbf{e} &= \left(\tfrac{V^2}{\mu} - \tfrac{1}{R}\right)\mathbf{R} - \tfrac{1}{\mu}\left(\mathbf{R}\cdot\mathbf{V}\right)\mathbf{V} \\ \mathbf{e} &= e_x\hat{\mathbf{I}} + e_y\hat{\mathbf{J}} + e_z\hat{\mathbf{K}} \end{aligned} \tag{1.22}$$

where $\mathbf{e}$ is the eccentricity vector, position $R = |\mathbf{R}|$, and e_x, e_y, e_z are the x, y and z components of $\mathbf{e}$.

b) Calculation of semi-major axis

$$\begin{aligned} \mathbf{h} = \mathbf{R}\times\mathbf{V} &= h_x\hat{\mathbf{I}} + h_y\hat{\mathbf{J}} + h_z\hat{\mathbf{K}}, h = |\mathbf{h}| \\ p = \tfrac{h^2}{\mu}&, a = \tfrac{p}{(1-e^2)} \end{aligned} \tag{1.23}$$

where h_x, h_y, h_z are the x, y and z components of the angular momentum vector $\mathbf{h}$.

c) Calculation of inclination angle

Since i is the angle between $\mathbf{h}$ and $\hat{\mathbf{K}}$

$$\begin{gathered}\mathbf{h}\cdot\hat{\mathbf{K}} = |\mathbf{h}|\left|\hat{\mathbf{K}}\right|\cos i = h_z \\ i = \cos^{-1}\left(\frac{h_z}{h}\right), i \leq \pi\end{gathered} \tag{1.24}$$

d) Calculation of longitude of the ascending node

Since Ω is the angle between $\mathbf{N}$ and $\hat{\mathbf{I}}$

$$\begin{gathered}\mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = N_x\hat{\mathbf{I}} + N_y\hat{\mathbf{J}} = -h_y\hat{\mathbf{I}} + h_x\hat{\mathbf{J}} \\ N = |\mathbf{N}|, \mathbf{N}\cdot\hat{\mathbf{I}} = |\mathbf{N}|\left|\hat{\mathbf{I}}\right|\cos\Omega = N_x, \Omega = \cos^{-1}\left(\frac{N_x}{N}\right)\end{gathered} \tag{1.25}$$

If $N_y > 0, \Omega < \pi$ and if $N_y < 0, \Omega > \pi$.

e) Calculation of argument of periapsis

Since ω is the angle between $\mathbf{N}$ and $\mathbf{e}$

$$\begin{gathered}\mathbf{N}\cdot\mathbf{e} = |\mathbf{N}|\,|\mathbf{e}|\cos\omega \\ \omega = \cos^{-1}\left(\frac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right)\end{gathered} \tag{1.26}$$

If $e_z > 0, \omega < \pi$, and if $e_z < 0, \omega > \pi$.

f) Calculation of true anomaly

Since f is the angle between $\mathbf{e}$ and $\mathbf{R}$

$$\begin{gathered}\mathbf{e}\cdot\mathbf{R} = |\mathbf{e}|\,|\mathbf{R}|\cos f \\ f = \cos^{-1}\left(\frac{\mathbf{e}.\mathbf{R}}{eR}\right)\end{gathered} \tag{1.27}$$

If $\mathbf{R}\cdot\mathbf{V} < 0, f > \pi$ and if $\mathbf{R}\cdot\mathbf{V} > 0, f < \pi$. All angles are in radian.

1.3 Spacecraft Relative Equations of Motion

The relative orbital motion problem of a deputy spacecraft with respect to a chief spacecraft is, usually, described using a set of differential equations of motion governing the motion of the spacecraft relative to each other instead of describing their motion, separately, relative to the Earth. A simplified model of the relative motion dynamics, valid only for a chief in a circular orbit, is given by the time-invariant Hill-Clohessy-Wilshire (HCW) equations in a Cartesian, rotating, Local Vertical, Local Horizontal (LVLH) coordinates.

As shown in Figure 1.4, a local-vertical/local-horizontal frame (LVLH) with unit vectors $\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right)$, referred to as Hill frame, is attached to the chief. The in-plane motion is defined by x and y while the out-of-plane motion is defined by the z axis. The inertial equation of motion is given as

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^3}\mathbf{R} \tag{1.28}$$

Expressed in the Hill frame, the deputy's position relative to the chief is given by $\rho = (x, y, z)^T$ and the angular velocity of the frame is $\omega = \dot{f}\hat{\mathbf{k}} = \frac{h}{r^2}\hat{\mathbf{k}}$. The position of the deputy spacecraft relative to the center of the gravitational field is given by

$$\mathbf{r} = \mathbf{R} + \rho = (R + x)\,\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1.29}$$

Using the gravitational force, we have

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\left[(R + x)\,\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right] \tag{1.30}$$

Differentiating Eq. (1.29) twice with respect to time in the inertial frame we have

$$\ddot{\mathbf{r}} = \left(\ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2x + \left(\ddot{R} - \dot{f}^2R\right)\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2y + \left(R\ddot{f} + 2\dot{R}\dot{f}\right)\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{1.31}$$

The chief spacecraft position vector relative to the Earth (orbit radius) can be written as

$$\mathbf{R} = R\hat{\mathbf{i}} \tag{1.32}$$

Upon substitution of Eq. (1.32) into the relative equation of motion in Eq. (1.28) we can express acceleration of the chief spacecraft as

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^2}\hat{\mathbf{i}} \tag{1.33}$$

Twice differentiation of Eq. (1.32) gives

$$\ddot{\mathbf{R}} = \left(\ddot{R} - \dot{f}^2 R\right)\hat{\mathbf{i}} + \left(\dot{f}^2 R + 2\dot{f}\dot{R}\right)\hat{\mathbf{j}} \tag{1.34}$$

Using the chief position vector (Eq. 1.32) and velocity $\mathbf{v} = \dot{R}\hat{\mathbf{i}} + R\dot{f}$, and taking the cross product of $\mathbf{R}$ and $\mathbf{V}$ the angular momentum yields

$$\mathbf{h} = \mathbf{R} \times \mathbf{v} = R^2\dot{f}\hat{\mathbf{k}} \tag{1.35}$$

The angular momentum $\mathbf{h}$ is a constant, and the orbital motion lies in a plane perpendicular to $\mathbf{h}$. Differentiating the scalar form of Eq. (1.35),$h = R^2\dot{f}$, we have

$$\frac{d}{dt}(h) = \frac{d}{dt}\left(R^2\dot{f}\right) = R\ddot{f} + 2\dot{R}\dot{f} = 0 \tag{1.36}$$

Using Eqs. (1.33), (1.34) and (1.36) then Eq. (1.31) can be rewritten as

$$\ddot{\mathbf{r}} = \left(\ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x - \frac{\mu}{R^2}\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{1.37}$$

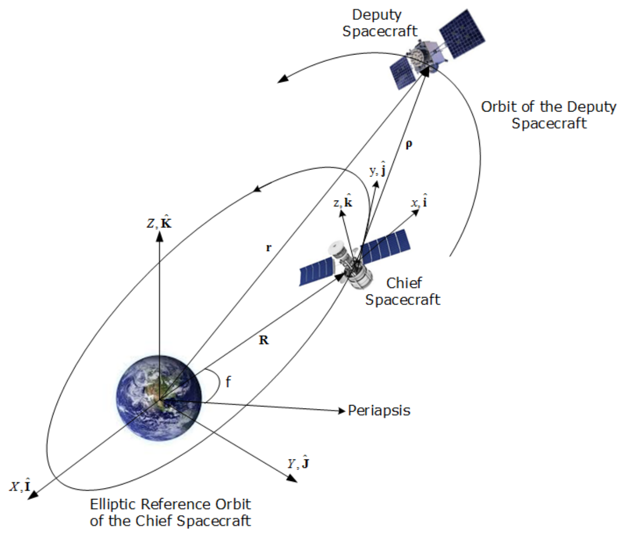

Figure 1.4: Relative Motion of Deputy with respect to Chief Spacecraft in Elliptical Orbit

Equating Eq. (1.30) and Eq. (1.37) gives

$$\begin{aligned} &\ddot{x} - 2\dot{f}y - \ddot{f}y - \dot{f}^2 x - \tfrac{\mu}{R^2} = -\tfrac{\mu}{r^3}(R + x) \\ &\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y = -\tfrac{\mu}{r^3} y \\ &\ddot{z} = -\tfrac{\mu}{r^3} z \end{aligned} \tag{1.38}$$

Eqs. (1.38) are the three second-order nonlinear differential equations of relative motion for a chief in an elliptical orbit.

1.3.1 Spacecraft Linearized Relative Equation of Motion

The deputy spacecraft orbital radius is

$$r = R\left(1 + \frac{2x}{R} + \frac{x^2 + y^2 + z^2}{R^2}\right)^{1/2} \tag{1.39}$$

If we assume that the distance of the deputy from the chief orbit is small compared to the chief orbit radial distance, i.e. $(x^2 + y^2 + z^2) \ll R^2$, then neglecting higher order terms gives

$$r \approx R\left(1 + \frac{2x}{R}\right)^{1/2} \tag{1.40}$$

We can express $\frac{\mu}{r^3}$, using a binomial series expansion and neglect second-order and higher order terms, as

$$\frac{\mu}{r^3} \approx \frac{\mu}{R^3}\left(1 + \frac{2x}{R}\right)^{-3/2} \approx \frac{\mu}{R^3}\left(1 - 3\frac{x}{R}\right) \tag{1.41}$$

Therefore, retaining only linear terms, the following three approximations are made

$$\begin{aligned} &-\tfrac{\mu}{r^3}(R+x) \approx -\tfrac{\mu}{R^3}(R-2x) \\ &-\tfrac{\mu}{r^3}y \approx -\tfrac{\mu}{R^3}y \\ &-\tfrac{\mu}{r^3}z \approx -\tfrac{\mu}{R^3}z \end{aligned} \tag{1.42}$$

Substituting Eqs. (1.42) into Eqs. (1.38) yields

$$\begin{aligned} &\ddot{x} - 2\dot{f}\dot{y} - \left(\dot{f}^2 + \tfrac{2\mu}{R^3}\right)x - \ddot{f}y = 0 \\ &\ddot{y} + 2\dot{f}\dot{x} - \left(\dot{f}^2 - \tfrac{\mu}{R^3}\right)y + \ddot{f}x = 0 \\ &\ddot{z} + \tfrac{\mu}{R^3}z = 0 \end{aligned} \tag{1.43}$$

These are the spacecraft linearized relative equations of motion with respect to an elliptical reference orbit. In state-space form, these can be written as the following.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \left(\dot{f}^2 + \frac{2\mu}{R^3}\right) & \ddot{f} & 0 & 0 & 2\dot{f} & 0 \\ -\ddot{f} & \left(\dot{f}^2 - \frac{\mu}{R^3}\right) & 0 & -2\dot{f} & 0 & 0 \\ 0 & 0 & -\frac{\mu}{R^3} & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{A}(t)\mathbf{x} \tag{1.44}$$

The state matrix $\mathbf{A}(t)$ and the state vector $\mathbf{x} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$ are time-varying function. Also, the parameters$\dot{f}$, $\ddot{f}$ and R are time-varying, periodic coefficients.

1.3.2 Hill-Clohessy-Wiltshire (HCW) Equations

The relative motion of a deputy spacecraft with respect to a chief spacecraft that is in a circular orbit about a central body can be described by the Hill-Clohessy-Willshire differential equations for the relative motion. For a circular orbit, using the parameters $\ddot{f}=0$, $\dot{f}=n=\sqrt{\mu/R^3}$, $\dot{f}^2=\mu/R^3=n^2$ reduce Eqs. (1.43) to

$$
\begin{aligned}
&\ddot{x}-2n\dot{y}-3n^2x=0\\
&\ddot{y}+2n\dot{x}=0\\
&\ddot{z}+n^2z=0
\end{aligned}
\tag{1.45}
$$

where $n=\left(\mu/R^3\right)^{1/2}$ is the mean motion of the chief orbit. Eqs. (1.45) are referred to as the Hill-Clohessy-Wilshire equations. In state-space form, Eq. (1.45) can be written as

$$
\dot{\mathbf{x}}=\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
3n^2 & 0 & 0 & 0 & 2n & 0\\
0 & 0 & 0 & -2n & 0 & 0\\
0 & 0 & -n^2 & 0 & 0 & 0
\end{bmatrix}\mathbf{x}=\mathbf{A}\mathbf{x}
\tag{1.46}
$$

The general solution of these homogeneous equations is

$$
\begin{aligned}
&x(t)=\left(\tfrac{\dot{x}_0}{n}\right)\sin nt-\left(\tfrac{2\dot{y}_0}{n}+3x_0\right)\cos nt+2\left(\tfrac{\dot{y}_0}{n}+2x_0\right)\\
&y(t)=2\left(\tfrac{2\dot{y}_0}{n}+3x_0\right)\sin nt+2\left(\tfrac{\dot{x}_0}{n}\right)\cos nt+\left(y_0-\tfrac{2\dot{x}_0}{n}\right)-3\left(\dot{y}_0+2nx_0\right)t\\
&z(t)=\left(\tfrac{\dot{z}_0}{n}\right)\sin nt+z_0\cos nt
\end{aligned}
\tag{1.47}
$$

and its derivative is

$$\begin{aligned}
\dot{x}(t) &= \dot{x}_0 \cos nt + (2\dot{y}_0 + 3nx_0)\sin nt \\
\dot{y}(t) &= 2(2\dot{y}_0 + 3nx_0)\cos nt - 2\dot{x}_0 \sin nt - 3(\dot{y}_0 + 2nx_0) \\
\dot{z}(t) &= \dot{z}_0 \cos nt - nz_0 \sin nt
\end{aligned} \tag{1.48}$$

where x_0, y_0, z_0 and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are the initial relative position and velocity components. From simple harmonic oscillator theory, Eqs. (1.47) and (1.48) can be written in the magnitude phase form as [4]

$$\begin{aligned}
x(t) &= -A\cos(nt + \alpha) + 2\left(\tfrac{\dot{y}_0}{n} + 2x_0\right) \\
y(t) &= 2A\sin(nt + \alpha) + \left(y_0 - \tfrac{2\dot{x}_0}{n}\right) - 3(\dot{y}_0 + 2nx_0)\,t \\
z(t) &= B\sin(nt + \beta)
\end{aligned} \tag{1.49}$$

and

$$\begin{aligned}
\dot{x}(t) &= nA\sin(nt + \alpha) \\
\dot{y}(t) &= 2nA\cos(nt + \alpha) - 3(\dot{y}_0 + 2nx_0) \\
\dot{z}(t) &= nB\cos(nt + \beta)
\end{aligned} \tag{1.50}$$

where the amplitude and phase angle of the radial and in-track oscillation are,

$$A = \sqrt{\left(\frac{\dot{x}_0}{n}\right)^2 + \left(\frac{2\dot{y}_0}{n} + 3x_0\right)^2}, \alpha = \mathrm{atan}\left(\frac{\dot{x}_0}{2\dot{y}_0 + 3nx_0}\right) \tag{1.51}$$

Similarly, for the cross-track oscillation

$$B = \sqrt{{z_0}^2 + \left(\frac{\dot{z}_0}{n}\right)^2} \;\; , \;\; \beta = \mathrm{atan}\left(\frac{nz_0}{\dot{z}_0}\right) \tag{1.52}$$

The amplitude of the in-track motion is twice the amplitude of the radial motion, and their oscillations are 90 degrees out of phase. That is, the in-track oscillation is a quarter of a period ahead of the radial oscillation.

1.4 Harmonic Balance Method (HBM)

Most real-life physical systems that are of importance in medicine, physical sciences, and engineering researches are nonlinear. Therefore, the differential equations governing the evolution of the system's variables are nonlinear. Linear equations are easier to characterize mathematically and the tools for their analysis are well developed. The same is not true for nonlinear equations. However, there are methods for finding approximate analytical solutions. In solving nonlinear equations, quadratic and cubic approximations methods can be used to approximate complex nonlinear equations tat can further be linearized using harmonic balance, floquet, homotopy perturbation etc. Many of these methods are based on attempting to find a solution as a combination of well-known mathematical functions.

The harmonic balance method is one of the most straight forward practical methods for estimating periodic solutions. It is used to find periodic and quasi-periodic oscillations, periodic and quasi-periodic conditions in automatic control theory, as well as stationary conditions, and in the studies of the stability of solutions. The essence of the method is to replace the nonlinear forces in the oscillating systems by specially-constructed linear functions, so that the theory of linear differential equations may be employed to find approximate solutions of the nonlinear systems [9]. The harmonic balance method can be applied to both standard and truly nonlinear oscillator equations. Using this method, it is easy to formulate the functional forms for approximating the periodic solutions.

In this section, a brief review of the theory of harmonic balance method is presented. Simple dynamical systems are linearized using harmonic balance method. The conventional linearization method, often used in engineering practice and which is only valid in a narrow strip around the linearization point, is compared with the harmonically linearized system using numerical solutions.

1.4.1 General Principle of Harmonic Balance Method

The general principle of harmonic balance method described below is adapted from the Encyclopedia of Mathematics [9]. Consider a nonlinear perturbed oscillator differential equation

$$\ddot{x} + \omega^2 x + \varepsilon f\left(x, \dot{x}\right) = 0,\ \left(0 < \varepsilon \ll 1\right) \tag{1.53}$$

where ε is a small parameter. Let the nonlinear forcing function be

$$F\left(x, \dot{x}\right) = \varepsilon f\left(x, \dot{x}\right) \tag{1.54}$$

Harmonic linearization is the replacement of $F\left(x, \dot{x}\right)$ by the linear function

$$F_l\left(x, \dot{x}\right) = kx + \lambda \dot{x} \tag{1.55}$$

where the parameters k and λ are computed by the formulas

$$\begin{aligned} k\left(a\right) &= \tfrac{\varepsilon}{\pi a} \int_0^{2\pi} f\left(a\cos\psi,\ -a\omega\sin\psi\right)\cos\psi d\psi \\ \lambda\left(a\right) &= -\tfrac{\varepsilon}{\pi a\omega} \int_0^{2\pi} f\left(a\cos s\psi,\ -a\omega\sin\psi\right)\sin\psi d\psi \end{aligned} \tag{1.56}$$

where $\psi = \omega t + \theta$. If $x = a\cos\left(\omega t + \theta\right)$, $a=$ constant, $\omega =$ constant, the nonlinear force $F\left(x, \dot{x}\right)$ is a periodic function of time, and its Fourier series expansion contains an infinite number of harmonics, having the frequencies $n\omega,\ n = 1, 2, \ldots$, i.e. it is of the form

$$F\left(x, \dot{x}\right) = \sum_{n=0}^{\infty} F_n \cos\left(n\omega t + \theta_n\right) \tag{1.57}$$

The term $F_1\cos(\omega t+\theta_1)$ is called the fundamental harmonic of the expansion (1.57). The amplitude and the phase of the linear function F_l coincide with the respective characteristics of the fundamental harmonic of the nonlinear force. For the differential equation

$$\ddot{x} + \omega^2 x + F(x, \dot{x}) = 0 \tag{1.58}$$

which is typical in the theory of quasi-linear oscillations, the harmonic balance method consists in replacing $F(x, \dot{x})$ by the equivalent linear force in Eq. (1.55) to give

$$\ddot{x} + \lambda\dot{x} + k_1 x = 0 \tag{1.59}$$

where $k_1 = k + \omega^2$. It is usual to call F_l the equivalent linear force, λ the equivalent damping coefficient and k_1 the equivalent elasticity coefficient. It has been proved that if the nonlinear equation (1.53) has a solution of the form

$$x = \text{a}\cos(\omega t + \theta) \tag{1.60}$$

where,

$$\dot{\text{a}} = o(\varepsilon), \dot{\omega} = o(\varepsilon) \tag{1.61}$$

then the order of the difference between the solutions of (1.53) and (1.59) is ε^2. In the harmonic balance method the frequency of the oscillation depends on the amplitude (through the quantities k and λ). Using traditional linearization about the origin we have $F(x, \dot{x}) = 0$ and Eq. (1.53) becomes

$$\ddot{x} + \omega^2 x = 0 \tag{1.62}$$

Comparing the harmonic linearization result in Eq. (1.59) with the traditional linearization result in Eq. (1.62) it can be easily seen that harmonic linearization gave an improved result with correction terms to the frequency of the system.

The motivations for the study of harmonic balance method are to use the method to study the dynamics of satellite relative motion and to develop an improved, harmonically linearized model.

1.4.2 Illustration of Harmonic Linearization of Simple Systems

The harmonic linearization of several simple nonlinear oscillators is carried out in order to demonstrate the effectiveness of the method.

(a) Undamped, Nonlinear Oscillator with Quadratic Nonlinearity

Consider the following unforced, undamped nonlinear oscillator containing quadratic nonlinearity

$$\ddot{x} + x + \varepsilon x^2 = 0, x_0 = A,\ \dot{x}_0 = 0 \tag{1.63}$$

The conventional and harmonic linearization of Equation (1.63) is as follows.

i) Conventional Linearization

Using conventional linearization about the origin the nonlinear part is zero, i.e. $\varepsilon x^2 = 0$, and the linearized equation becomes

$$\ddot{x} + x = 0 \tag{1.64}$$

This equation has the solution

$$x(t) = a\cos t + b\sin t \tag{1.65}$$

ii) Harmonic Linearization

First-order harmonic balance approximation (HB1):

The solution in Eq. (1.65) is used as assumed approximate solution and can be expressed as $x(t) = A\cos(t + \phi_1) = A\cos\phi$ where $\phi = t + \phi_1$ and $A = \sqrt{a^2 + b^2}$. The nonlinear term is linearized using the solution, substituting the trigonometric identity $\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi)$ and ignoring higher harmonic term $\cos 2\phi$, as follows

$$\varepsilon x^2 = \varepsilon A^2 \cos^2\phi = \varepsilon\left(\frac{A^2}{2}(1 + \cos 2\phi)\right) \approx \frac{\varepsilon A^2}{2} \tag{1.66}$$

Substituting Eq. (1.66) into Eq. (1.63) gives the equivalent linear equation

$$\ddot{x} + x + \frac{\varepsilon A^2}{2} = 0 \tag{1.67}$$

This equation has the eigenvalues

$$s^2 + 1 = 0, s = \pm j \tag{1.68}$$

Using the assumed solution, at $t = 0$, $x_0 = A\cos(\phi_1)$, $\dot{x}_0 = -A\sin(\phi_1)$, $\phi_1 = \arctan\left(-\frac{\dot{x}_0}{x_0}\right)$ and $A = \left[{x_0}^2 + (\dot{x}_0)^2\right]^{1/2}$.

Assumed solution with offset in x (HB 2):

For HB 2, the assumed solution is $x(t) = A\cos\phi + A_0$. Upon substituiton into the nonlinear term we have

$$\varepsilon x^2 = \varepsilon(A\cos\phi + A_0)^2 \approx \varepsilon\left(\frac{A^2}{2} + 2A_0 A\cos\phi + A_0^2\right) = 2\varepsilon A_0 x + \varepsilon\left(\frac{A^2}{2} - A_0^2\right) \tag{1.69}$$

Substituting Eq. (1.69) into Eq. (1.63) we have the following equivalent linear equation

$$\ddot{x} + (1 + 2\varepsilon A_0)\, x + \varepsilon\left(\frac{A^2}{2} - A_0^2\right) = 0 \tag{1.70}$$

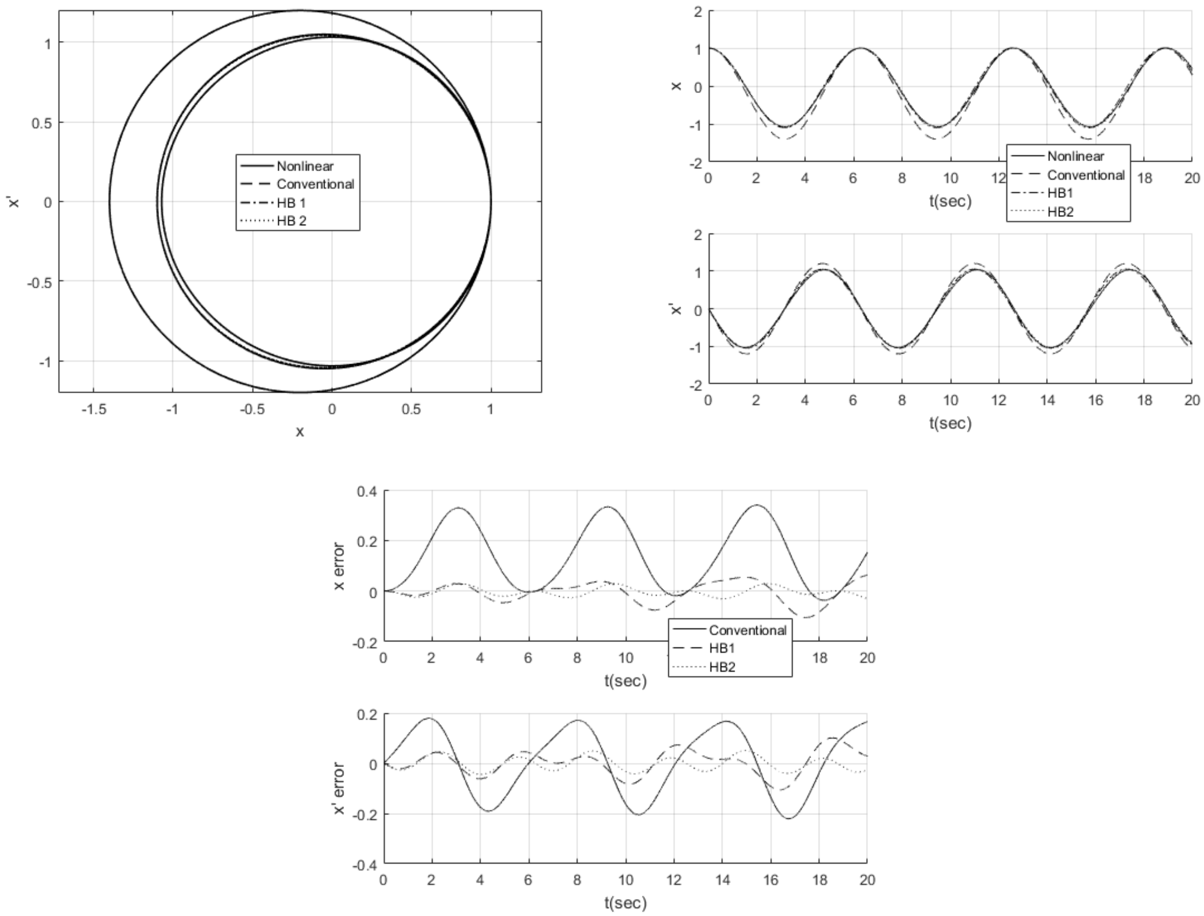

Figure 1.5: Trajectory plots for $\ddot{x} + x + \varepsilon x^2 = 0$, $\varepsilon = 0.1$, $x = 1$, $\dot{x} = 0$

Differentiating $x(t) = A\cos\phi + A_0$ twice and substituting the result and Eq. (1.69) into Eq. (1.63) yields

$$\ddot{x} + x + \varepsilon x^2 = -A\cos\phi + A\cos\phi + A_0 + \varepsilon\left(\frac{A^2}{2} + 2A_0A\cos\phi + A_0^2\right) = 0 \tag{1.71}$$

Eq. (1.71) reduces to

$$A_0 + \varepsilon A_0^2 + \frac{1}{2}\varepsilon A^2 + 2\varepsilon A_0 A\cos\phi = 0 \tag{1.72}$$

Equating the constant part of Eq. (1.72) to zero gives

$$A_0 + \varepsilon A_0^2 + \frac{1}{2}\varepsilon A^2 = 0 \tag{1.73}$$

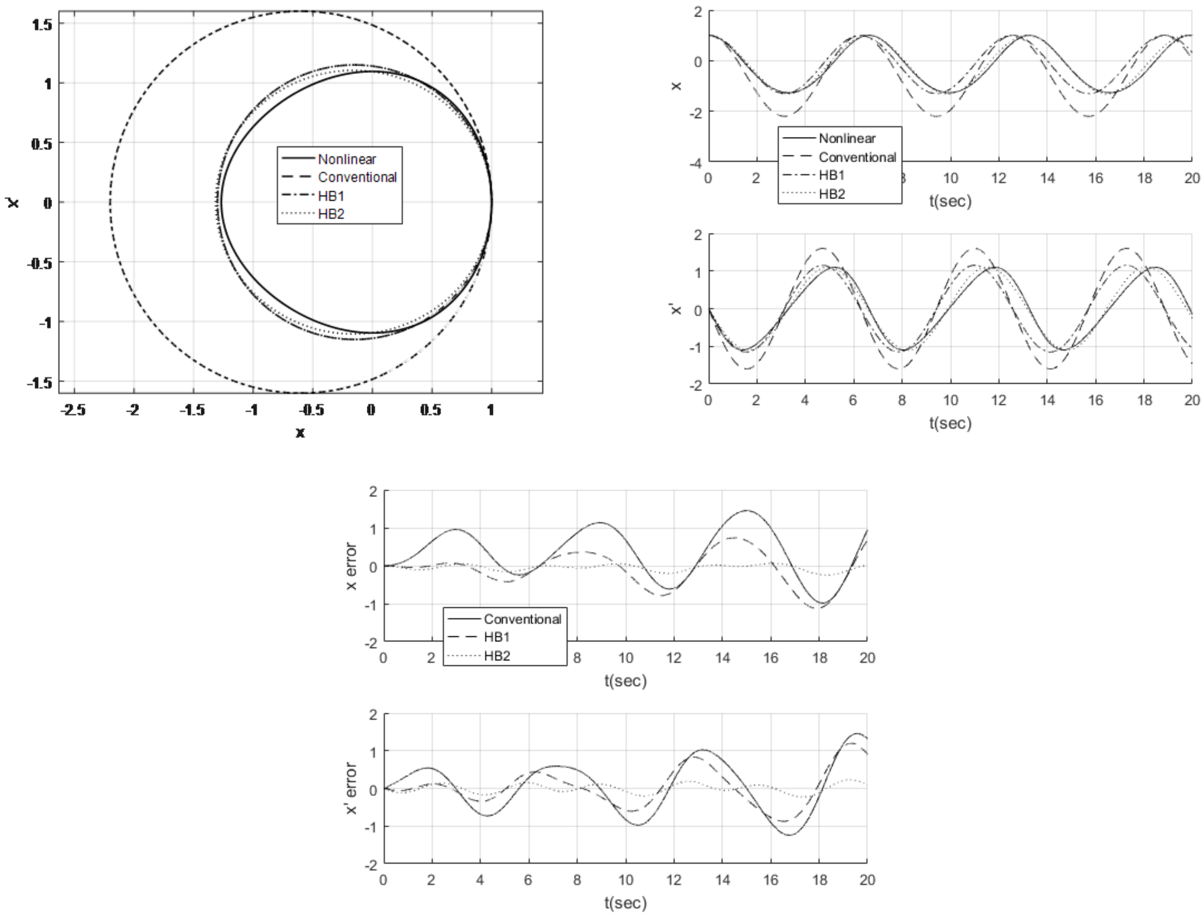

Figure 1.6: Trajectory plots for $\ddot{x} + x + \varepsilon x^2 = 0$, $\varepsilon = 0.3$, $x = 1$, $\dot{x} = 0$

When A is small, neglecting A_0^2 we have

$$A_0 = -\frac{1}{2}\varepsilon A^2 \tag{1.74}$$

Figures 1.5 and 1.6 show velocities versus positions, trajectories of x and $\dot{x}$ versus time and approximation errors of conventional linearization, HB 1 and HB 2. For $\varepsilon = 0$ and non-zero initial conditions, irrespective of the models, the system becomes the normal harmonic oscillator equation with sinusoidal solutions. As shown in the figures, the higher the value of ε, the greater the error in the linear approximation. The HB 2 has better results than the conventional and HB 1 model, with lesser approximation error, because the offset in its assumed solution contributed immensely to the better result.

Small Parameter	Initial Conditions		Root Mean Square Error (RMSE)					
			Conventional		HB 1		HB 2	
ε	x	$\dot{x}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$
0	1	0	0	0	0	0	0	0
0.1	1	0	0.1842	0.1281	0.0414	0.0489	0.0168	0.0272
0.1	3	1	0.1656	0.1591	0.0884	0.1049	0.0648	0.0768
0.3	1	0	0.17173	0.6867	0.4704	0.4980	0.0968	0.1172
0.3	3	1	0.4423	1.3081	1.2903	1.1276	0.2338	0.2751

Table 1.1: Comparison of the root mean square errors (RMSE) of each of the model for $\ddot{x} + x + \varepsilon x^2 = 0$

Table 1.1 shows the comparison of the root mean square errors (RMSE) of each model calculated using $RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \hat{x}_j)^2}$ where x_j is the true solution while $\hat{x}_j$ is the approximated solution. For $\varepsilon = 0$ (linear) all the models behaved like a simple harmonic oscillator with zero RMSE. But, as the nonlinearity increases ($\varepsilon \neq 0$) the RMSE in position (x_{RMSE}) and velocity ($\dot{x}_{RMSE}$) in the conventional model increases more than those of HB 1 and HB 2. As shown in the table, for different values of ε, the HB 2 model has the least amount of root mean square error. This shows that the model was able to capture the dynamics, in a way, better than the other models. Although, as ε increases the error in HB 2 increases but not like that of HB 1.

(b) Undamped, Nonlinear Oscillator with Cubic Nonlinearity (Duffing Equation)

Consider the following unforced, undamped nonlinear oscillator containing cubic nonlinearity

$$\ddot{x} + x + \varepsilon x^3 = 0, x_0 = A,\ \dot{x}_0 = 0 \tag{1.75}$$

The conventional and harmonic linearization methods are applied to Equation (1.75) as follows.

i) Conventional Linearization

Using conventional linearization about the origin the nonlinear part is zero, i.e. $\varepsilon x^3 = 0$, and the linearized equation becomes

$$\ddot{x} + x = 0 \tag{1.76}$$

This equation has the solution

$$x(t) = a\cos t + b\sin t \tag{1.77}$$

and eigenvalues $s^2 + 1 = 0, \quad s = \pm j$.

ii) Harmonic Linearization

First-order harmonic balance approximation (HB1):

The solution in Eq. (1.77) can be expressed as $x\,(t) = A\cos\,(t + \phi_1) = A\cos\phi$ where $\phi = t + \phi_1$ and $A = \sqrt{\mathrm{a}^2 + b^2}$. The nonlinear term is linearized using the solution, substituting the trigonometric identity $\cos^3\phi = \frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi$ and ignoring higher harmonic term $\cos 3\phi$, as follows

$$\varepsilon x^3 = \varepsilon A^3\cos^3\phi = \varepsilon A_1^3\left(\frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi\right) \approx \frac{3\varepsilon A^3}{4}\cos\phi = \frac{3\varepsilon A^2}{4}x \tag{1.78}$$

Substituting Eq. (1.78) into Eq. (1.75) we obtain harmonic balance model as

$$\ddot{x} + \left(1 + \frac{3}{4}\varepsilon A^2\right)x = 0 \tag{1.79}$$

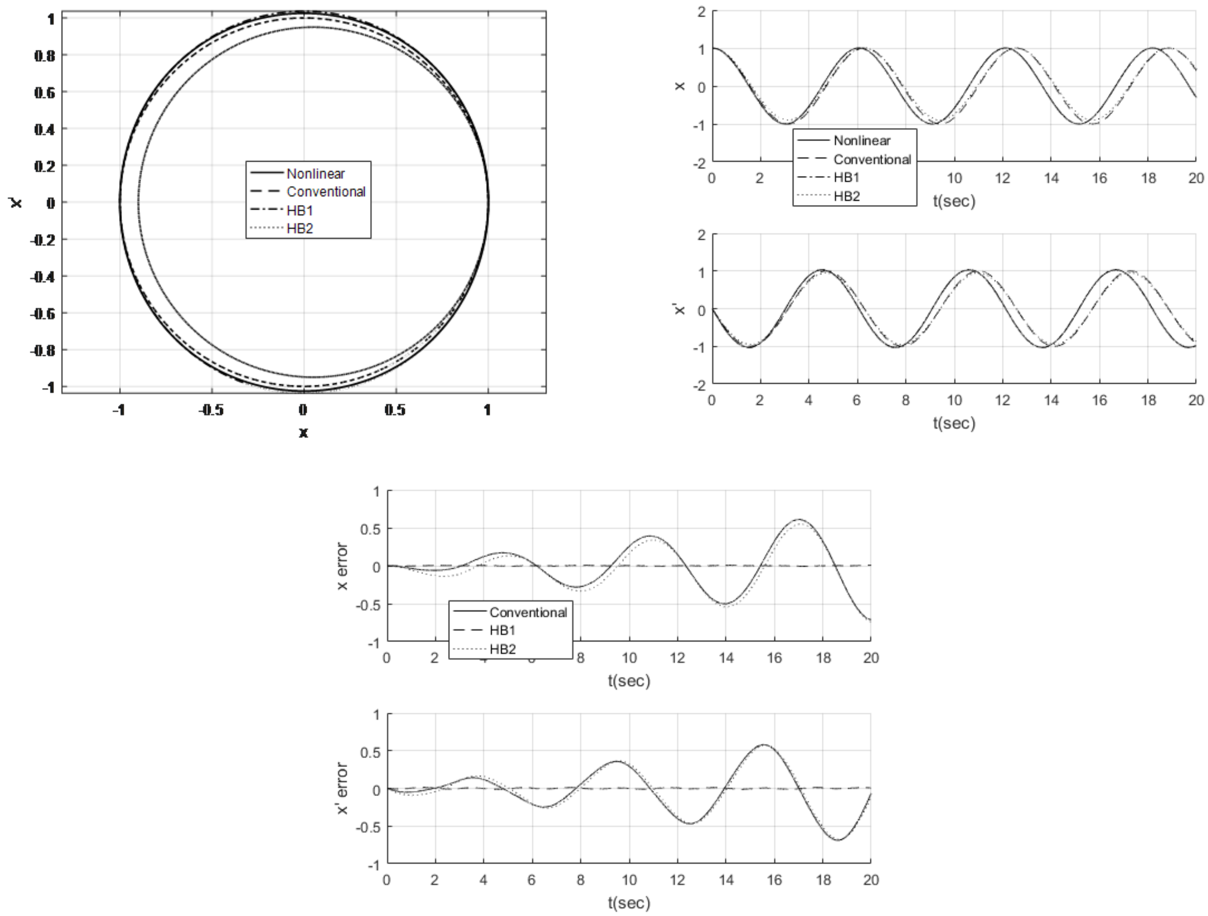

Figure 1.7: Trajectory plots for $\ddot{x} + x + \varepsilon x^3 = 0$, $\varepsilon = 0.1$, $x = 1$, $\dot{x} = 0$

where the amplitude-frequency relationship and solution x(t) are given as

$$\omega^2 = 1 + \frac{3\varepsilon A^2}{4}, x\left(t\right) = A\text{cos}\left(\sqrt{1 + \frac{3\varepsilon A^2}{4}}\right)t \tag{1.80}$$

This equation has the eigenvalues $s = \left(1 + \frac{3\varepsilon A^2}{4}\right)^{1/2}$.

Assumed solution with offset in x (HB 2):

For the HB 2, the assumed solution $x(t) = A\text{cos}\,\phi + A_0$ with offset in x is used in the harmonic linearization and the nonlinear term is linearized as follows

$$\varepsilon x^3 = \varepsilon (A\cos\phi + A_0)^3 = \varepsilon\left(A^3 \text{cos}^3\phi + 3A_0 A^2 \text{cos}^2\phi + 3AA_0^2 \text{cos}\phi + A_0^3\right) \tag{1.81}$$

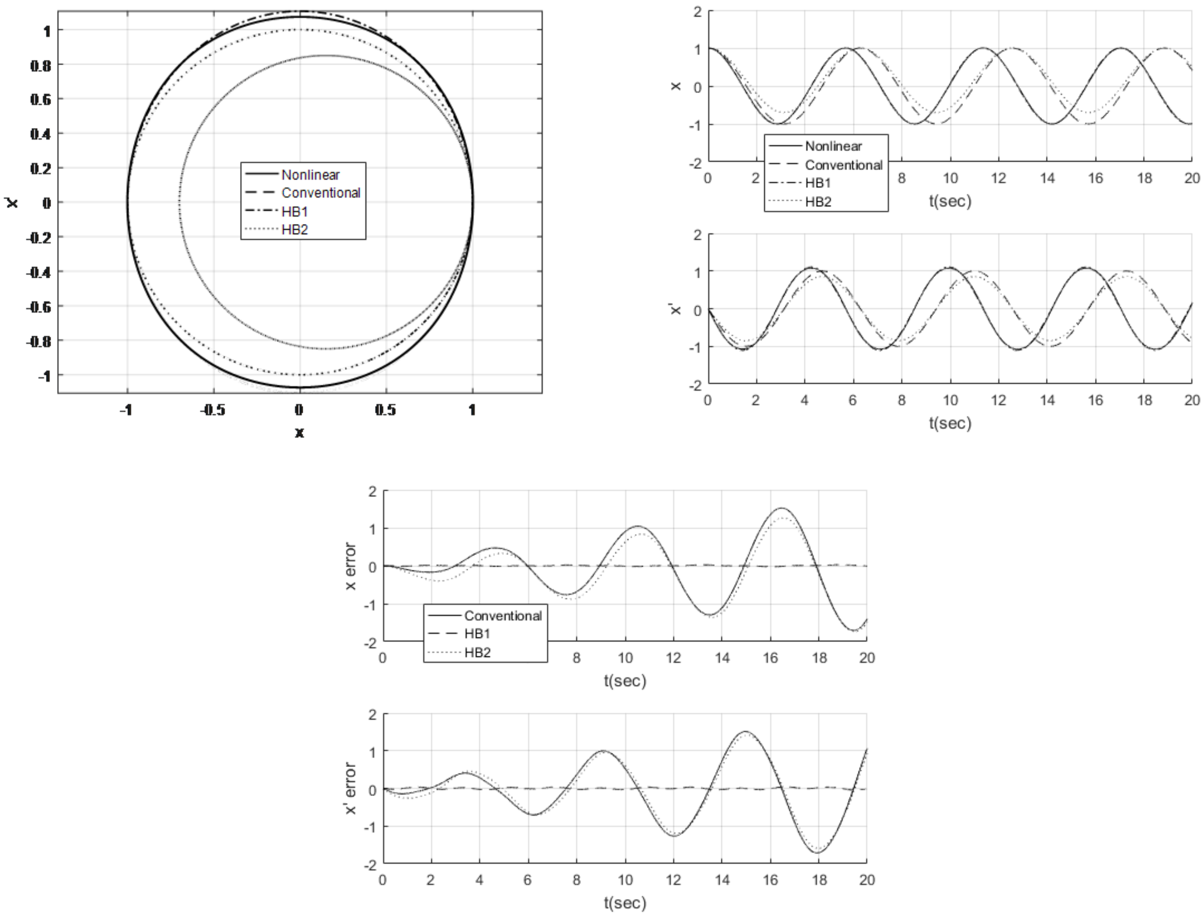

Figure 1.8: Trajectory plots for $\ddot{x} + x + \varepsilon x^3 = 0$, $\varepsilon = 0.3$, $x = 1$, $\dot{x} = 0$

Substituting the trigonometric identities $\cos^3\phi = \frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi$ and $\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi)$ into Eq. (1.81) yields

$$\varepsilon x^3 = \varepsilon \left(A^3 \left\{ \frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi \right\} + 3A_0 A^2 \frac{1}{2} \left\{ 1 + \cos 2\phi \right\} + 3AA_0^2 \cos\phi + A_0^3 \right) \tag{1.82}$$

Eliminating the higher order harmonics and arrange the result in form of the assumed solution gives

$$\begin{aligned} \varepsilon x^3 &\approx \tfrac{3}{4}\varepsilon A^2 \left(A\cos\phi + A_0 \right) + \tfrac{3}{4}\varepsilon A_0 A^2 + 3\varepsilon A_0^2 \left(A\cos\phi + A_0 \right) - 2\varepsilon A_0^2 \\ &= \left(\tfrac{3}{4}\varepsilon A^2 + 3\varepsilon A_0^2 \right) x + \tfrac{3}{4}\varepsilon A_0 A^2 - 2\varepsilon A_0^2 \end{aligned} \tag{1.83}$$

Substituting Eq. (1.83) into Eq. (1.75) gives the equivalent linear equation

$$\ddot{x} + \left(1 + \varepsilon\left\{\frac{3}{4}A^2 + 3A_0^2\right\}\right)x + \varepsilon\left(\frac{3}{4}A_0A^2 - 2A_0^2\right) = 0 \tag{1.84}$$

Differentiating the solution $x(t) = A\cos\phi + A_0$ twice and substituting the result and Eq. (1.83) into Eq. (1.75) yields

$$\ddot{x}+x+\varepsilon x^3 = -A\cos\phi + A\cos\phi + A_0 + \varepsilon\left(\frac{3}{4}A^3\cos\phi + \frac{3}{2}A_0A^2 + 3AA_0^2\cos\phi + A_0^3\right) = 0 \tag{1.85}$$

Eq. (1.85) reduces to

$$A_0 + \varepsilon\left(\frac{3}{2}A_0A^2 + A_0^3\right) + \varepsilon\left(\frac{3}{4}A^3 + 3AA_0^2\right)\cos\phi = 0 \tag{1.86}$$

Equating the constant part of Eq. (1.86) to zero and coefficient of $\cos\phi$ to zero gives

$$A_0 + \varepsilon\left(\frac{3}{2}A_0A^2 + A_0^3\right) = 0, \varepsilon\left(\frac{3}{4}A^3 + 3AA_0^2\right) = 0 \tag{1.87}$$

Eq. (1.87) gives

$$A_0^2 = -\frac{1}{4}A^2 \tag{1.88}$$

Figures 1.7 and 1.8 shows velocities versus positions, trajectories of x and $\dot{x}$ versus time and the approximation errors in the conventional linearization, HB 1, and HB 2. For $\varepsilon = 0$ and non-zero initial conditions, irrespective of the models, the system becomes the normal harmonic oscillator equation with sinusoidal solutions. As shown in the figures, the higher the value of ε, the more the conventional model diverges from the nonlinear model. The HB 1 has better results than the conventional and HB 2 model because this system is a truly nonlinear oscillator unlike the one with quadratic nonlinearity which performed better with the inclusion of offset in the assumed solution.

Small Parameter	Initial Conditions		Root Mean Square Error (RMSE)					
			Conventional		HB 1		HB 2	
ε	x	$\dot{x}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$
0	1	0	0	0	0	0	0	0
0.1	1	0	0.4955	0.3842	0.0031	0.0069	0.2877	0.3001
0.1	3	1	5.3242	5.0311	0.4275	0.6138	2.6356	3.3747
0.3	1	0	1.5025	1.2160	0.011	0.0215	0.7486	0.7417
0.3	3	1	11.1125	8.3347	1.4288	2.6678	0.9166	2.9863

Table 1.2: Comparison of the root mean square errors (RMSE) of each of the model for $\ddot{x} + x + \varepsilon x^3 = 0$

Table 1.2 shows the comparison of the root mean square errors (RMSE) of each model calculated using $RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \hat{x}_j)^2}$, where x_j is the true solution while $\hat{x}_j$ is the approximated solution. For $\varepsilon = 0$ (linear) all the models behaved like a simple harmonic oscillator with zero RMSE. But, as the nonlinearity increases ($\varepsilon \neq 0$) the RMSE in position (x_{RMSE}) and velocity ($\dot{x}_{RMSE}$) in conventional model increases more than those of HB1 and HB2.

As shown in the table, for different values of ε, the HB 1 model has the least amount of root mean square error. This shows that the model was able to capture the dynamics, in a way, better than the other models. Although, as ε increases the error in HB 1 increases but not like that of HB 2.

(c) Undamped, Nonlinear System with Quadratic Nonlinearity

Consider the following unforced, undamped nonlinear oscillator with quadratic nonlinearity

$$\ddot{x} + \varepsilon x^2 = 0, x_0 = A,\ \dot{x}_0 = 0 \tag{1.89}$$

The conventional, harmonic and rectilinear linearizations are carried out on Eq. (1.89) as follows.

i) Conventional Linearization

Using conventional linearization about the origin the nonlinear part is zero, i.e. $\varepsilon x^2 = 0$, and the linearized equation becomes

$$\ddot{x} = 0 \tag{1.90}$$

This equation has the unbounded solution

$$x\,(t) = \mathrm{a} + b\,t \tag{1.91}$$

and eigenvalues $s = \pm 0$.

ii) Harmonic Linearization

First-order harmonic balance approximation (HB 1):

Using the solution $x\,(t) = A\cos\,\phi$ the nonlinear term can be linearized as

$$\varepsilon x^2 = \varepsilon A^2 \cos^2\phi = \varepsilon\left(\frac{A^2}{2}(1 + \cos\,2\phi)\right) \approx \frac{\varepsilon A^2}{2} \tag{1.92}$$

Substituting Eq. (1.96) into Eq. (1.93) gives the equivalent linear equation

$$\ddot{x} + \frac{\varepsilon A^2}{2} = 0 \tag{1.93}$$

This equation has the eigenvalues $s = \pm 0$.

Assumed solution with offset in (HB 2):

Using the solution $x(t) = A\cos\,\phi + A_0$ the nonlinear term can be linearized as

$$\varepsilon x^2 = (A\cos\phi + A_0)^2 \approx \varepsilon\left(\frac{A^2}{2} + 2A_0 A\cos\phi + A_0^2\right) = 2\varepsilon A_0 x + \varepsilon\left(\frac{A^2}{2} - A_0^2\right) \tag{1.94}$$

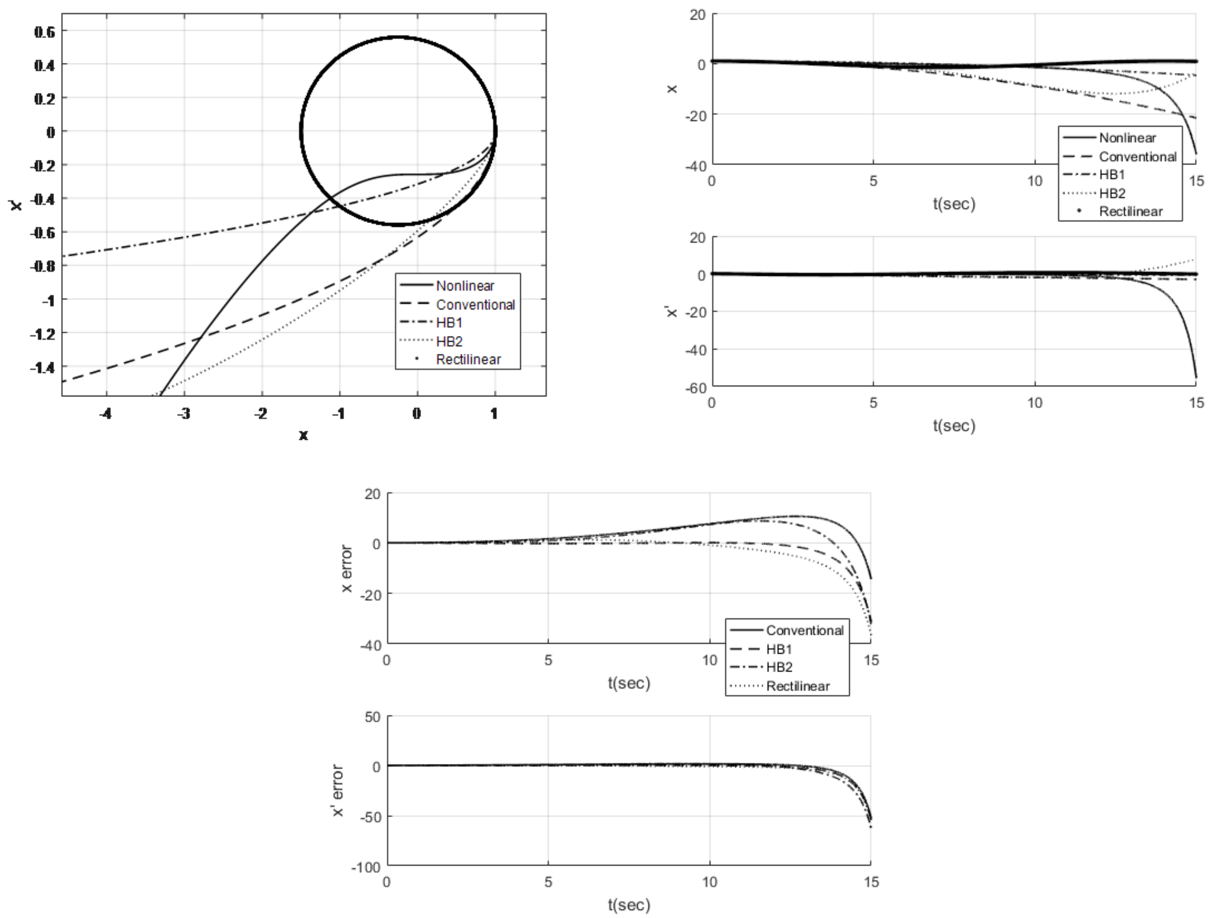

Figure 1.9: Trajectory plots for $\ddot{x} + \varepsilon x^2 = 0$,$\varepsilon = 0.1$,$x = 1$, $\dot{x} = 0$

The equivalent linear equation is

$$\ddot{x} + 2\varepsilon A_0 x + \varepsilon\left(\frac{A^2}{2} - A_0^2\right) = 0 \tag{1.95}$$

Differentiating the solution $x(t) = A\cos\phi + A_0$ twice and substituting the result and Eq. (1.94) into Eq. (1.89) yields

$$\ddot{x} + \varepsilon x^2 = -A\cos\phi + \varepsilon\left(\frac{A^2}{2} + 2A_0 A\cos\phi + A_0^2\right) = 0 \tag{1.96}$$

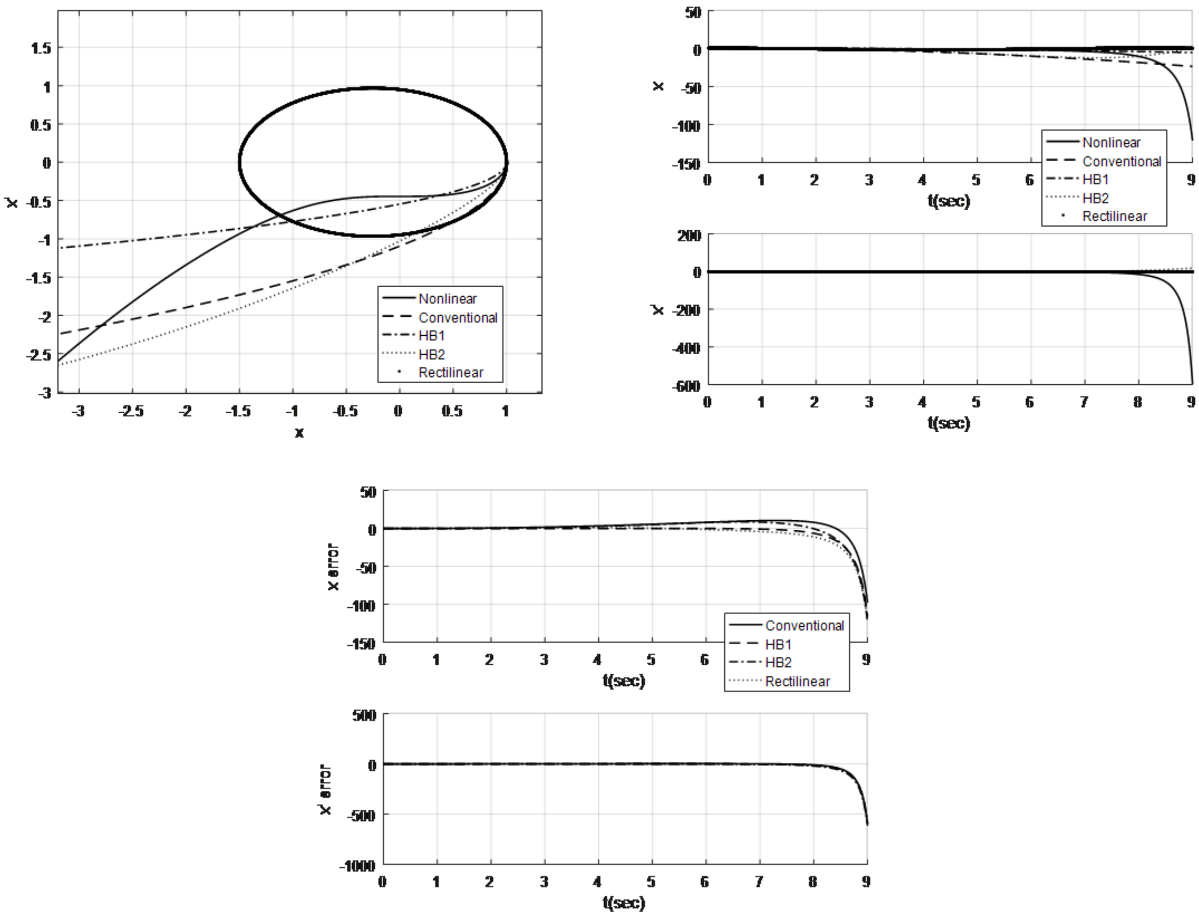

Figure 1.10: Trajectory plots for $\ddot{x} + \varepsilon x^2 = 0$, $\varepsilon = 0.3$, $x = 1$, $\dot{x} = 0$

Eq. (1.96) reduces to

$$(-A + 2\varepsilon A_0 A)\cos\phi + \varepsilon\left(\frac{A^2}{2} + A_0^2\right) = 0 \tag{1.97}$$

Equating the constant part of Eq. (1.97) to zero gives

$$A_0^2 = -\frac{1}{2}A^2 \tag{1.98}$$

iii) Rectilinear Linearization

This linearization approach is carried out using the solution $x(t) = a + b\,t$. Substitution of this solution into the nonlinear term yields

$$\varepsilon x^2 = \varepsilon\left(\mathrm{a}^2 + 2\mathrm{a}bt + b^2t^2\right) = \varepsilon\left(2\mathrm{a}\left(\mathrm{a} + bt\right) - \mathrm{a}^2 + b^2t^2\right) \approx 2\varepsilon\mathrm{a}x + \varepsilon\left(-\mathrm{a}^2 + b^2t^2\right) \quad (1.99)$$

Upon substitution of Eq. (1.99) into Eq. (1.89) we have

$$\ddot{x} + 2\varepsilon\mathrm{a}x + \varepsilon\left(-\mathrm{a}^2 + b^2t^2\right) = 0 \quad (1.100)$$

At $t = 0$ we have the initial conditions $x_0 = \mathrm{a}$ and $\dot{x}_0 = b$.

Small Parameter	Initial Conditions		Root Mean Square Error (RMSE)							
			Conventional		HB 1		HB 2		Rectilinear	
ε	x	$\dot{x}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$
0	1	0	0	0	0	0	0	0	0	0
0.1	1	0	5.7734	6.5551	4.6457	7.0861	1.5041	8.4680	6.4327	7.3036
0.1	3	1	16.2200	49.4404	19.9718	52.2268	0.85474	60.2324	27.8619	54.0442
0.3	1	0	11.1446	55.644	13.5306	56.5005	12.0384	59.9179	15.1535	56.7123
0.3	3	1	5.8840	23.6946	7.0774	24.1499	8.8482	34.5626	13.1167	27.5117

Table 1.3: Comparison of the root mean square errors (RMSE) of each of the model for $\ddot{x} + \varepsilon x^2 = 0$

Figures 1.9 and 1.10 show velocities versus positions, trajectories of x and $\dot{x}$ versus time and approximation errors of conventional, rectilinear, HB1 and HB2 models. As shown in the figures, the rectilinear and HB2 models behaved better than the HB1. The rectilinear model gave a better approximation of the model than the other two models with lesser error.

Table 1.3 shows the comparison of the root mean square errors (RMSE) of each model calculated using $RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\left(x_j - \hat{x}_j\right)^2}$ where x_j is the true solution while $\hat{x}_j$ is the approximated solution. As the nonlinearity increases ($\varepsilon \neq 0$) the root mean square error, in position (x_{RMSE}) and velocity ($\dot{x}_{RMSE}$), in rectilinear model increases more than those of conventional, HB 1 and HB 2 models.

(d) Undamped, Nonlinear System with Cubic Nonlinearity

Consider the following unforced, undamped nonlinear oscillator with cubic nonlinearity

$$\ddot{x} + \varepsilon x^3 = 0, x_0 = A,\ \dot{x}_0 = 0 \tag{1.101}$$

The conventional linearization, harmonic linearization and rectilinear linearization are carried out on Eq. (1.101) as follows.

i) Conventional Linearization

Using conventional linearization about the origin the nonlinear part is zero, i.e. $\varepsilon x^3 = 0$, and the linearized equation becomes

$$\ddot{x} = 0 \tag{1.102}$$

This equation has the unbounded solution

$$x(t) = a + b\,t \tag{1.103}$$

and eigenvalues of $s = \pm 0$.

ii) Harmonic Linearization

First-order harmonic balance approximation (HB 1):

Using the solution is $x(t) = A\cos\phi$ the nonlinear term is linearized as

$$\varepsilon x^3 = \varepsilon A^3 \cos^3\phi = \varepsilon A^3 \left(\frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi \right) \approx \frac{3\varepsilon A^3}{4}\cos\phi = \frac{3\varepsilon A^2}{4} x \tag{1.104}$$

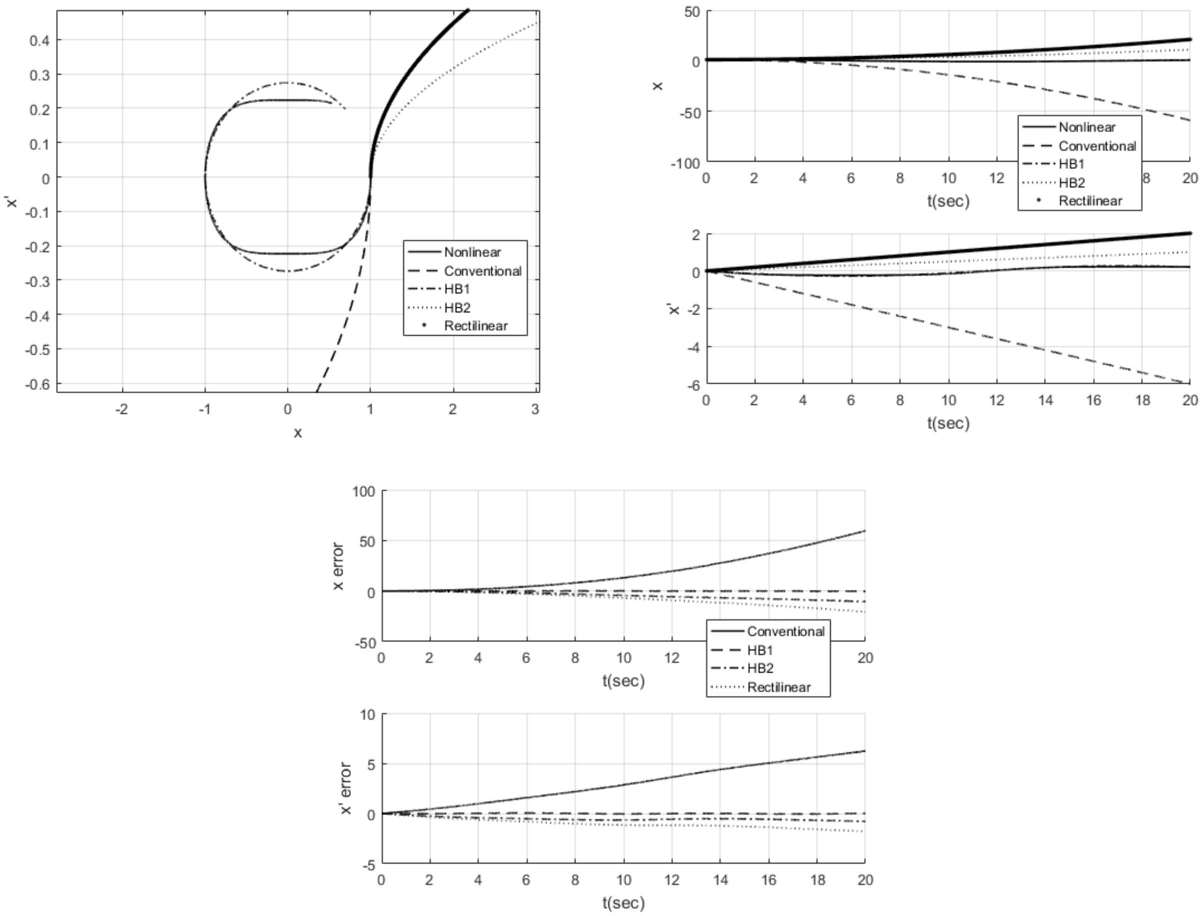

Figure 1.11: Trajectory plots for $\ddot{x} + \varepsilon x^3 = 0$, $\varepsilon = 0.1$, $x = 1$, $\dot{x} = 0$

Substituting Eq. (1.104) into Eq. (1.101) we obtain harmonic balance model

$$\ddot{x} + \frac{3}{4}\varepsilon A^2 x = 0 \tag{1.105}$$

where the amplitude frequency relationship and solution x are given as

$$\omega^2 = \frac{3\varepsilon A^2}{4}, x(t) = A\cos\left(\sqrt{\frac{3\varepsilon A^2}{4}}\right)t \tag{1.106}$$

This equation has the eigenvalues $s = \pm\left(\frac{3\varepsilon A^2}{4}\right)^{1/2}$.

Assumed solution with offset in (HB 2):

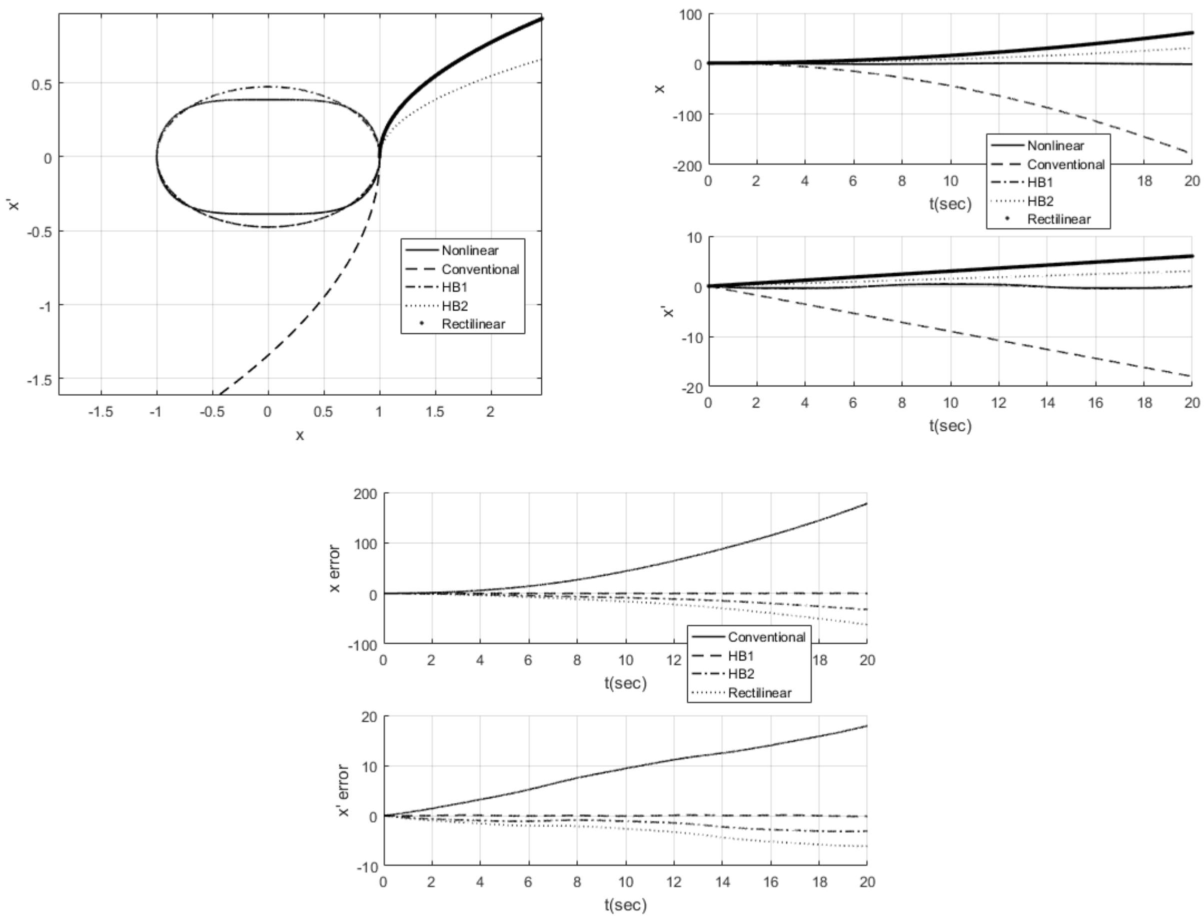

Figure 1.12: Trajectory plots for $\ddot{x} + \varepsilon x^3 = 0$, $\varepsilon = 0.3$, $x = 1$, $\dot{x} = 0$

Using the solution $x(t) = A\cos\phi + A_0$ the nonlinear term is linearized as follows

$$\varepsilon x^3 = \varepsilon(A\cos\phi + A_0)^3 = \varepsilon\left(A^3\cos^3\phi + 3A_0A^2\cos^2\phi + 3AA_0^2\cos\phi + A_0^3\right) \tag{1.107}$$

Substituting the trigonometric identities $\cos^3\phi = \frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi$ and $\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi)$ into Eq. (1.107) and eliminating the higher order terms yields

$$\begin{aligned} \varepsilon x^3 &\approx \tfrac{3}{4}\varepsilon A^2(A\cos\phi + A_0) + \tfrac{3}{4}\varepsilon A_0A^2 + 3\varepsilon A_0^2(A\cos\phi + A_0) - 2\varepsilon A_0^2 \\ &= \left(\tfrac{3}{4}\varepsilon A^2 + 3\varepsilon A_0^2\right)x + \tfrac{3}{4}\varepsilon A_0A^2 - 2\varepsilon A_0^2 \end{aligned} \tag{1.108}$$

The equivalent linear equation is

$$\ddot{x} + \varepsilon\left(\frac{3}{4}A^2 + 3A_0^2\right)x + \varepsilon\left(\frac{3}{4}A_0A^2 - 2A_0^2\right) = 0 \tag{1.109}$$

Differentiating the solution $x(t) = A\cos\phi + A_0$ twice and substituting the result and Eq. (1.108) into Eq. (1.101) yields

$$\ddot{x} + \varepsilon x^3 = -A\cos\phi + \varepsilon\left(\frac{3}{4}A^3\cos\phi + \frac{3}{2}A_0A^2 + 3AA_0^2\cos\phi + A_0^3\right) = 0 \tag{1.110}$$

Eq. (1.110) reduces to

$$\left(-A + \frac{3}{4}\varepsilon A^3 + 3\varepsilon AA_0^2\right)\cos\phi + \varepsilon\left(\frac{3}{2}A_0A^2 + A_0^3\right) = 0 \tag{1.111}$$

From Eq. (111) we have

$$A_0^2 = -\frac{3}{2}A^2 \tag{1.112}$$

Small Parameter	Initial Conditions		Root Mean Square Error (RMSE)							
			Conventional		HB 1		HB 2		Rectilinear	
ε	x	$\dot{x}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$	x_{RMSE}	$\dot{x}_{RMSE}$
0	1	0	0	0	0	0	0	0	0	0
0.1	1	0	25.8940	3.5273	0.0703	0.0296	5.4798	0.5369	9.9393	1.1104
0.1	3	1	227.7839	30.2479	0.3224	0.3859	58.5465	6.8192	20.5148	5.7623
0.3	1	0	79.7478	10.3103	0.0998	0.0650	14.1955	1.8358	27.6079	3.5639
0.3	3	1	711.1128	92.4482	1.7388	2.7145	147.6039	18.4945	20.2946	10.0462

Table 1.4: Comparison of the root mean square errors (RMSE) of each of the model for $\ddot{x} + \varepsilon x^3 = 0$

iii) Reclinear Linearization

This linearization approach is carried out using the solution $x(t) = a + bt$. Substitution of this solution into the nonlinear term yields

$$\begin{aligned} \varepsilon x^3 &= \varepsilon \left(\mathrm{a}^3 + 3\mathrm{a}^2 bt + 3\mathrm{a}b^2t^2 + b^3t^3\right) \\ &= \varepsilon \left(\mathrm{a}^3 + 3\mathrm{a}bt\left(\mathrm{a} + bt\right) + b^3t^3\right) \approx 3\varepsilon \mathrm{a}btx + \varepsilon\left(\mathrm{a}^3 + b^3t^3\right) \end{aligned} \tag{1.113}$$

Upon substitution of Eq. (1.113) into Eq. (1.101) we have

$$\ddot{x} + 3\varepsilon \mathrm{a}btx + \varepsilon\left(\mathrm{a}^3 + b^3t^3\right) = 0 \tag{1.114}$$

At $t = 0$ we have the initial conditions $x_0 = a$ and $\dot{x}_0 = b$. Figures 1.11 and 1.12 show velocities versus positions, trajectories of x and $\dot{x}$ versus time and approximation errors of conventional, rectilinear, HB1 and HB2 models. As shown in the figures, the HB1 model behaved well and gave a better approximation, with lesser error, than the other models.

Table 4 shows the comparison of the root mean square errors (RMSE) of each model calculated using $RMSE = \sqrt{\frac{1}{n}\sum\limits_{j=1}^{n}\left(x_j - \hat{x}_j\right)^2}$ where x_j is the true solution while $\hat{x}_j$ is the approximated solution. As the nonlinearity ($\varepsilon \neq 0$) increases the RMSE, (x_{RMSE}) and velocity ($\dot{x}_{RMSE}$), in conventional, HB2 and rectilinear models increases more than that of HB1. This shows that HB1 model is a better approximation model for this system.

1.5 Averaging Method

The formulation of the gravitational three-body problem as a perturbation of the two-body problem by Lagrange in the late 18th century marked the beginning of the use of averaging method. The method became one of the classical methods in analyzing nonlinear oscillations after series of researches by Krylov, Bogoliubov, Mitropolsky etc. in 1930s. The method is fairly general, thereby making it applicable to large number of nonlinear dynamical systems and very useful because it is not restricted to periodic solutions. It can be used to obtain an approximate simplified system and to investigate the stability and bifurcation of their equilibria (corresponding to periodic motions in the original system) [10-14].

1.5.1 Basic Idea of Averaging Method

The basic idea of the averaging method is as follows. Consider an equation of the form

$$\ddot{x} + \omega^2 x + \varepsilon f(x, \dot{x}) = 0 \tag{1.115}$$

where ε is a small parameter. For the case $\varepsilon = 0$, using linear theory, the solution is

$$\begin{aligned} x &= A \sin(\omega t + \phi) \\ \dot{x} &= \omega A \cos(\omega t + \phi) \end{aligned} \tag{1.116}$$

Eqs. (1.116) are used as generating solutions. Krylov and Bogoliubov suggested that, for smallε, the integration constants A and ϕ are slowly varying functions of time, that is,

$$A \to A(t), \phi \to \phi(t) \tag{1.117}$$

Using this fact, the generating solution takes the form

$$\begin{aligned} x &= A(t) \sin(\omega t + \phi(t)) \\ \dot{x} &= \omega A(t) \cos(\omega t + \phi(t)) \end{aligned} \tag{1.118}$$

Differentiating the first part of Eqs. (1.118) and equating the result to the second part of Eqs. (1.116) gives

$$\dot{\mathrm{A}} \sin(\omega t + \phi) + \mathrm{A}\dot{\phi} \cos(\omega t + \phi) = 0 \tag{1.119}$$

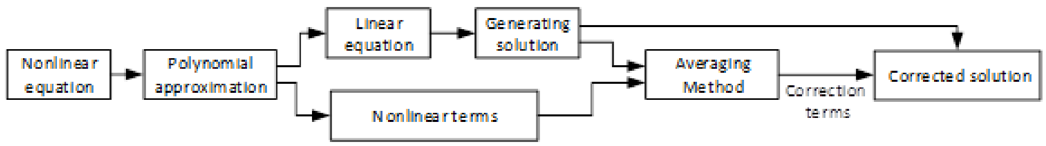

Figure 1.13: Averaging method steps

Differentiating the second part of Eqs. (1.122) and substituting the results and the second part of Eqs. (1.116) into Eq. (1.113) gives

$$\dot{\mathrm{A}}\cos\left(\omega t+\phi\right)-\mathrm{A}\dot{\phi}\sin\left(\omega t+\phi\right)=-\frac{\varepsilon}{\omega}f\left(A\sin\left(\omega t+\phi\right),\omega A\cos\left(\omega t+\phi\right)\right) \tag{1.120}$$

Averaging over one period we obtain

$$\begin{aligned}\dot{\mathrm{A}}&=-\tfrac{\varepsilon}{2\pi\omega}\int_0^{2\pi}f\left(A\sin\theta,\omega A\cos\theta\right)\cos\theta d\theta\\ \dot{\phi}&=-\tfrac{\varepsilon}{2\pi A\omega}\int_0^{2\pi}f\left(A\sin\theta,\omega A\cos\theta\right)\sin\theta d\theta\end{aligned} \tag{1.121}$$

Under the integral A and ϕ are assumed to be time independent. This method has been extensively used in plasma physics, control theory, theory of octillions etc.

1.5.2 Averaging Method Steps

The averaging method steps, diagramed in Figure 1.13, is based on attempting to find approximate solution to a nonlinear equation. The polynomial approximation of the original nonlinear equation contains two parts: linear terms and nonlinear terms. The solution of the linear equation, known as the generating solution, is employed by the averaging method to produce correction terms. The correction terms produce changes in amplitude or phase of the generating solution. Combination of the correction terms with the generating solution produces corrected solutions.

1.6 State Dependent Riccati Equation (SDRE) Theory

In the SDRE approach for nonlinear control, a state-dependent coefficient (SDC) linear structure is found and from this a stabilizing nonlinear controller is constructed.

1.6.1 Linear Quadratic Regulator (LQR)

The theory of optimal control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function is called the LQ problem. The optimal control is provided by a linear-quadratic regulator (LQR), a well known control systems design technique, used as a feedback controller to provide practical feedback gains.

Consider the linear time invariant (LTI) system

$$\dot{x} = Ax + Bu, x(t_0) = x_0 \tag{1.122}$$

and the performance index (cost functional)

$$J\left[x_0, u\right] = \int_0^{\infty} \left[x^T Q x + u^T R u\right] dt, Q \geq 0, R > 0 \tag{1.123}$$

where $\mathbf{x}(t) \in \Re^n$ are the states, $\mathbf{u}(t) \in \Re^m$ is the input (or control) vector, $m \leq n$, $\mathbf{A} \in \Re^{n \times n}$ is the system matrix, $\mathbf{B} \in \Re^{n \times m}$ is the control input matrix, $\mathbf{Q}$ is an $n \times n$ symmetric positive semidefinite matrix, $\mathbf{R}$ is an $m \times m$ symmetric positive definite matrix and n is the state dimension.

Problem: The optimal control problem is to calculate the function $u : [0, \infty] \mapsto \Re^m$ such that $J\left[u\right]$ is minimized. The LQR controller has the following form

$$u = -R^{-1} B^T P x(t) = -K x(t) \tag{1.124}$$

where $\mathbf{P} \in \Re^{n \times n}$ is given by the positive (symmetric) semi definite solution of

$$0 = PA + B^T P + Q - PBR^{-1}B^T P \tag{1.125}$$

This equation is called the algebraic Riccati equation. It is solvable if the pair (A, B) is controllable and (Q, A) is detectable.

In designing LQR controllers the following are of utmost importance for consideration:

a) LQR assumes full knowledge of the state x

b) (A, B) is given by design and cannot be modified

c) (Q, R) are the controller design parameters. Large Q penalizes transients of x, large R penalizes usage of control action u

d) $A + BK$ is Hurtwiz (asymptotically stable)

e) P is minimum in a certain sense

f) The associated J is minimized

1.6.2 State Dependent Riccati Equation (SDRE) Strategy

The LQR/LQG method is extremely powerful and widely used in applications where linearizations of the nonlinear process representations are valid over large operating areas. The State-Dependent Riccati Equation (SDRE) strategy provides an effective algorithm for synthesizing nonlinear feedback controls by allowing for nonlinearities in the system states, while offering design flexibility through state-dependent weighting matrices.

The SDRE method entails factorization of the nonlinear dynamics into the state vector and its product with a matrix-valued function that depends on the state itself. In doing so, the SDRE algorithm brings the nonlinear system to a non-unique linear structure having matrices with state-dependent coefficients. The method includes minimizing a nonlinear performance index having a quadratic-like structure. An ARE, as given by the SDC matrices, is then solved on-line to give the suboptimum control law. The coefficients of this Riccati equation vary with the given point in state space. The algorithm thus involves solving, at a given point in the state space, an algebraic state-dependent Riccati equation. The non-uniqueness of the factorization creates extra degrees of freedom, which can be used to enhance controller performance.

Extended Linearization of a Nonlinear System

Consider the deterministic, infinite-horizon nonlinear optimal regulation (stabilization) problem, where the system is full- state observable, autonomous, nonlinear in the state, and affine in the input, represented in the form

$$\dot{x} = f(x) + B(x)u, x(0) = x_0 \tag{1.126}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, $f(x) \in C^K$ and $B(x) \in C^K$ are smooth functions of approximate domain such that

(i) $B(x) \neq 0$ for all x

(ii) $f(0) = 0$

Extended Linearization is the process of factorizing a nonlinear system into a linear-like structure which contains SDC matrices. A continuous nonlinear matrix-valued function always exists such that

$$f(x) = A(x)x \tag{1.127}$$

where $A : \Re^n \mapsto \Re^{n \times n}$ is found by algebraic factorization and is clearly non-unique when $n > 1$. If $A(x)x = f(x)$, then $(A(x) + E(x))\,x = f(x)$ for any $E(x)$ such that $E(x)x = 0$ Also, given $A_1(x)x = f(x)$ and $A_2(x)x = f(x)$, then for any $\alpha \in \Re$

$$A(x, \alpha) = \alpha A_1(x) + (1 - \alpha)\,A_2(x) = \alpha f(x) + (1 - \alpha)\,f(x) = f(x) \tag{1.128}$$

is also a valid paramerization.

After extended linearization of the input-affine nonlinear system the constraint dynamics can be written with a linear structure having state dependent coefficients

$$\dot{x} = A(x)x + B(x)u, x(0) = x_0 \tag{1.129}$$

which has a linear structure with state dependent matrices $A(x)$ and $B(x)$.

SDRE Method Formulation

Consider the minimization of the infinite-time performance criterion$P(x) > 0$

$$J\left[x_0, u\right] = \int_0^\infty \left[x^T Q(x)x + u^T R(x)u\right] dt, Q(x) \geq 0, R(x) > 0 \tag{1.130}$$

The state and input weighting matrices are assumed state dependent. Under the specified conditions, the LQR method is applied pointwise for $\left(A(x), B(x)\right), \left(Q(x), R(x)\right)$ to generate a nonlinear feedback control law, accepting only $P(x) \geq 0$,

$$u = -K(x)x(t) = R(x)^{-1}B(x)^T P(x)x(t), K : \Re^n \mapsto \Re^{p\times n} \tag{1.131}$$

where $P : \Re^n \mapsto \Re^{n\times n}$ satisfies

$$P(x)A(x) + A(x)^T P(x) - P(x)B(x)R(x)^{-1}B(x)^T P(x) + Q(x) = 0 \tag{1.132}$$

By applying this method one hopes to retain the good properties of the LQR control design that the control law regulates the system to the origin, i.e.,$\lim_{t\to\infty} x(t) = 0$, while keeping cost low. Eq. (1.133) is the state-dependent algebraic Riccati equation (SDARE) associated with the nonlinear quadratic cost function. Eq. (1.132) can be solved analytically to produce an equation for each element of u, or it can be solved numerically at a sufficiently high sampling rate.

Main Stability Results

Assume that the following conditons hold [21, 22, 24]

1. The matrix valued functions $A(x), B(x), Q(x), R(x) \in C^1(\Re^n)$.

2. The pairs $(A(x), B(x))$ and $\left(A(x), Q^{1/2}(x)\right)$ are pointwise stabilizable, respectively, detectable, state dependent parameterizations of the nonlinear system for all $x \in \Re^n$. The following theorems have been used to establish stability of the system.

Theorem 1.1 *Under conditions of the Assumptions 1 and 2, the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality of the nonlinear optimal regulator problem*

$$u(x) = \arg\min H(x, \lambda, u), \lambda = P(x)x \tag{1.133}$$

Theorem 1.2 *Assume that all state dependent matrices are bounded functions along the trajectories. Then, under the conditions of Theorem 1 the Pontriaguin optimality condition*

$$\dot{\lambda} = -H(x, \lambda, u) \tag{1.134}$$

is satisfied approximately by $\lambda = P(x)x$ at a quadratic rate along the trajectory.

1.6.3 SDRE Optimality Criterion

The performance index J is convex, so any stationary point is at least locally optimal. From the performance index and constrained dynamics the Hamiltonian function can be formed

$$H(x, \lambda, u) = \frac{1}{2}x^T Q(x)x + u^T R(x)u + \lambda^T [A(x)x + B(x)u] \tag{1.135}$$

with stationary conditions

$$
\begin{aligned}
H_u &= 0 \\
\dot{\lambda} &= -H_x \\
\dot{x} &= A(x)x + B(x)u
\end{aligned}
\tag{1.136}
$$

Taking the derivative of Eq. (1.135) and using Eq. (1.131) we have

$$
\begin{aligned}
H_u = R(x)u + B^T(x)\lambda &= R(x)\left[-R(x)^{-1}B(x)^T P(x)x\right]u + B^T(x)\lambda \\
&= B^T(x)\left[\lambda - P(x)x\right]
\end{aligned}
\tag{1.137}
$$

This implies that for any choice of λ

$$
u(x) = -R(x)^{-1}B(x)^T\lambda \Rightarrow H_u = 0 \tag{1.138}
$$

where

$$
\lambda = P(x)x \tag{1.139}
$$

Satisfying Eq. (1.134) for all time will satisfy the H_u optimality condition. Differentiating Eq. (1.139) and dropping the notation (x) yields

$$
\dot{\lambda} = \dot{P}(x)x + P(x)\dot{x} \tag{1.140}
$$

Using the optimality condition in Eq. (1.140) we have

$$
\dot{\lambda} = -Qx - \frac{1}{2}x^T Q_x x - \frac{1}{2}u^T R_x u - \left(x^T A_x^T + A^T + u^T B_x^T\right)\lambda \tag{1.141}
$$

Using Eqs. (1.140) and (1.141) we have

$$
\dot{P}x + P\left(Ax - BR^{-1}B^T Px\right) = -Qx - \frac{1}{2}x^T Q_x x - \frac{1}{2}u^T R_x u - \left(x^T A_x^T + A^T + x^T PBR^{-1}B_x^T\right)Px \tag{1.142}
$$

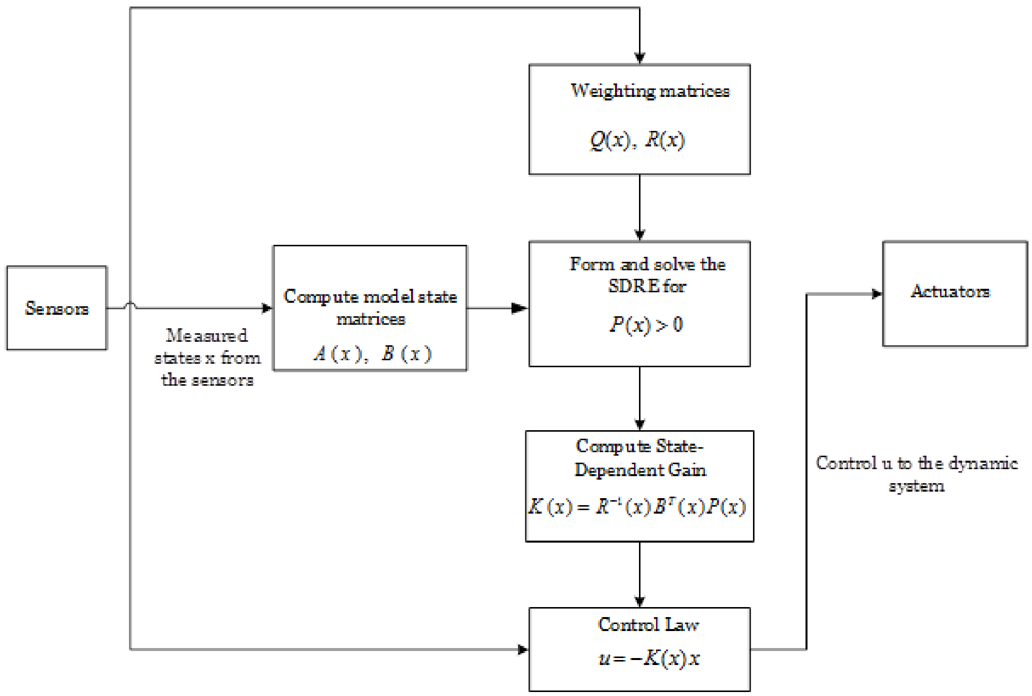

Figure 1.14: State Dependent Riccati Equation (SDRE) Method

After rearrangement we have the following form

$$\dot{P}x+\frac{1}{2}x^TQ_xx+\frac{1}{2}u^TR_xu+x^TA_x^TPx-x^TPBR^{-1}B_x^TPx+\left(PA+A^TP-PBR^{-1}B^TP+Q\right)x=0 \tag{1.143}$$

The term in bracket in Eq. (1.43) is the SDARE which equals zero and upon substituting for u we obtain

$$\dot{P}x+\frac{1}{2}x^TQ_xx+\frac{1}{2}x^TPBR^{-1}R_xR^{-1}B^TPx+x^TA_x^TPx-x^TPBR^{-1}B_x^TPx=0 \tag{1.144}$$

Equation (1.152) is the SDRE optimality criterion which, if satisfied, guarantees the closed loop solution is locally optimal and may be globally optimum. The summary of the SDRE method is shown in Figure 1.14.

In the case of scalar x (see James R. Cloutier e tal (1996)), the only SDC parameterization is given by

$$a(x)=f(x)/x \tag{1.145}$$

where $f(x) = a(x)x$. The state-dependent Riccati equation is given by

$$2\frac{f(x)}{x}p - \frac{g^2(x)}{r(x)}p^2 + q(x) = 0 \tag{1.146}$$

and its positive definite solution is

$$p = \frac{r(x)}{g^2(x)}\left[\frac{f(x)}{x} + \sqrt{\frac{f^2(x)}{x^2} + \frac{g^2(x)q(x)}{r(x)}}\right] \tag{1.147}$$

For the scalar case, the SDRE optimality criterion reduces to

$$\dot{p} + \frac{1}{2}q_x x + \frac{1}{2}\frac{g^2}{r^2}r_x p^2 x + a_x p x - \frac{g}{r}g_x p^2 x = 0 \tag{1.148}$$

There exists only one global solution for the scalar case since the performance index is convex and the differential constraint is linear in u. Therefore, the scalar nonlinear problem has the global optimal solution

$$u(t) = -\frac{1}{g(x)}\left[f(x) + \text{sgn}(x)\sqrt{f^2(x) + \frac{g^2(x)x^2q(x)}{r(x)}}\right] \tag{1.149}$$

The stabilizing solution to an algebraic Riccati equation can be found using the eigenvalues of an associated Halmitonian matrix. The associated Hamiltonian matrix is given by

$$M = \begin{bmatrix} A(x) & -B(x)R^{-1}(x)B^T(x) \\ -Q(x) & -A^T(x) \end{bmatrix} \tag{1.150}$$

The dynamics is given by the pointwise Hurwitz matrix

$$\dot{x} = \left[A(x) - B(x)R^{-1}(x)B^T(x)\right]x = A_{cl}(x)x \tag{1.151}$$

The Hamiltoniam matrix M has dimension $2n \times 2n$, with the property that its eigenvalues are symmetric about both the real and imaginary axes. A stabilizing solution exists only if M has n eigenvalues in the open left-half plane from whose corresponding eigenvectors a solution P can be found to Eq. (1.147). If the n eigenvectors are used to form a $2n \times 2n$ matrix, and we denote the $n \times n$ square blocks as X and Y, so that

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{1.152}$$

The solution to Eq. (1.152) is given by $P = XY^{-1}$.

1.6.4 Illustration of SDRE Control of Simple Systems

The examples below are used to illustrate application of the SDRE method to some nonlinear regulator problems. The details of these problems can be found in References [21, 22, 24]

a) Nonlinear Regulator Scalar Problem

Minimize

$$I = \frac{1}{2}\int_{t_0}^{\infty} \left(x^2 + u^2\right) dt \tag{1.153}$$

with respect to x and u subject to

$$\dot{x} = x - x^3 + u \tag{1.154}$$

The shortcomings of feedback linearization control against the SDRE control was shown by Freeman and Kokotovic (1994) using this example. The stabilizing controller is given by

$$u_{fl} = x^3 - 2x \tag{1.155}$$

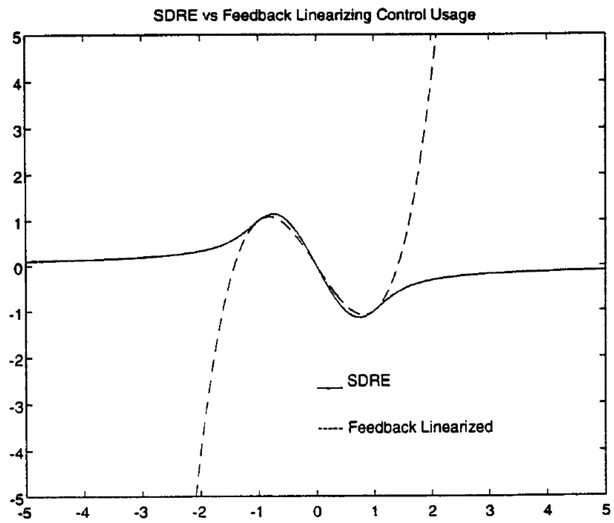

Figure 1.15: SDRE vs Feedback Linearizing Control Usage (Source: Freeman and Kokotovic (1994))

and it results in the close dynamics of the form

$$\dot{x} = -x \tag{1.156}$$

The scalar nonlinear problem has the following characteristics:

$$f(x) = x - x^3, a(x) = 1 - x^2, g(x) = 1, q = 1, r = 1 \tag{1.157}$$

Application of the SDRE method to the nonlinear problem provides the state-dependent Riccati equation is given by

$$2\left(1 - x^2\right) p - p^2 + 1 = 0 \tag{1.158}$$

that has the positive solution

$$p(x) = 1 - x^2 + \sqrt{\left(1 - x^2\right)^2 + 1} \tag{1.159}$$

Using the control $u = -p(x)x$ the optimal control derived for this problem is

$$u_{opt} = -\left(x - x^3\right) - x\sqrt{x^4 - 2x^2 + 2} \tag{1.160}$$

Despite the fact that it gives global exponential stability about $x = 0$, the feedback linearization controller requires large amount of control activity that can cause instability in the presence of actuator saturation or uncertainties. Unlike in the case of SDRE controller, the feedback linearization controller cancelled out the beneficial nonlinearity $-x^3$.

b) Multivariable Problem

This problem is extracted from the paper written by Cloutier et al [21]. Minimize

$$I = \frac{1}{2}\int_{t_0}^{\infty}\left(x^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x + u^T\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}u\right)dt \tag{1.161}$$

subject to the constraints

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 + x_2 + u_1 \\ \dot{x}_2 &= x_1 + x_1^2 x_2 - x_2 + u_2 \end{aligned} \tag{1.162}$$

Four SDC parameterizations, $\mathbf{A}_1(x)$, $\mathbf{A}_2(x)$, $\mathbf{A}_3(x)$ and $\mathbf{A}_4(x)$, of this problem are considered in [21] by the authors.

$$\begin{gathered} \mathbf{A}_1(x) = \begin{bmatrix} 1 - x_1^2 & 1 \\ 1 & x_1^2 - 1 \end{bmatrix}, \mathbf{A}_2(x) = \begin{bmatrix} 1 - x_1^2 & 1 \\ 1 + x_1 x_2 & -1 \end{bmatrix} \\ A_3(x, \alpha) = \alpha A_1(x) + (1 - \alpha)A_2(x), \mathbf{A}_4(x) = \begin{bmatrix} 1 - x_1^2 + \frac{x_2}{x_1} & 0 \\ 0 & \frac{x_1}{x_2} + x_1^2 - 1 \end{bmatrix} \end{gathered} \tag{1.163}$$

In Figures 1.16, the first figure shows the comparison between the SDRE solution and Conjugate Gradient (CG) solution of the first parameterization $\mathbf{A}_1(x)$. Near optimal state response is obrtained. If the initial states are equal the parameterization forces the initial controls to be the same. As shown in the plot, the SDRE controls rapidly converge to the optimal controls before the optimal controls reach zero. The second figure shows the comparison between the SDRE solution and CG solution of $\mathbf{A}_2(x)$. Near optimal state response is obtained and the SDRE controls converge to the optimal controls before the optimal controls reach zero. The third figure shows the optimal trajectory $\alpha^*(t)$ and the fourth shows the comparison between the SDRE solution and CG solution of $A_3\left(x, \alpha\right)$ which are identical. The fourth parameterization is considered less than desirable since it converts linear terms into nonlinear elements in the coefficient matrix which blow up if one state approaches zero faster than the other. The fifth figure shows the state response and the controls. A satisfactory stable state response is obtained despite the fact that u_1 is chattering.

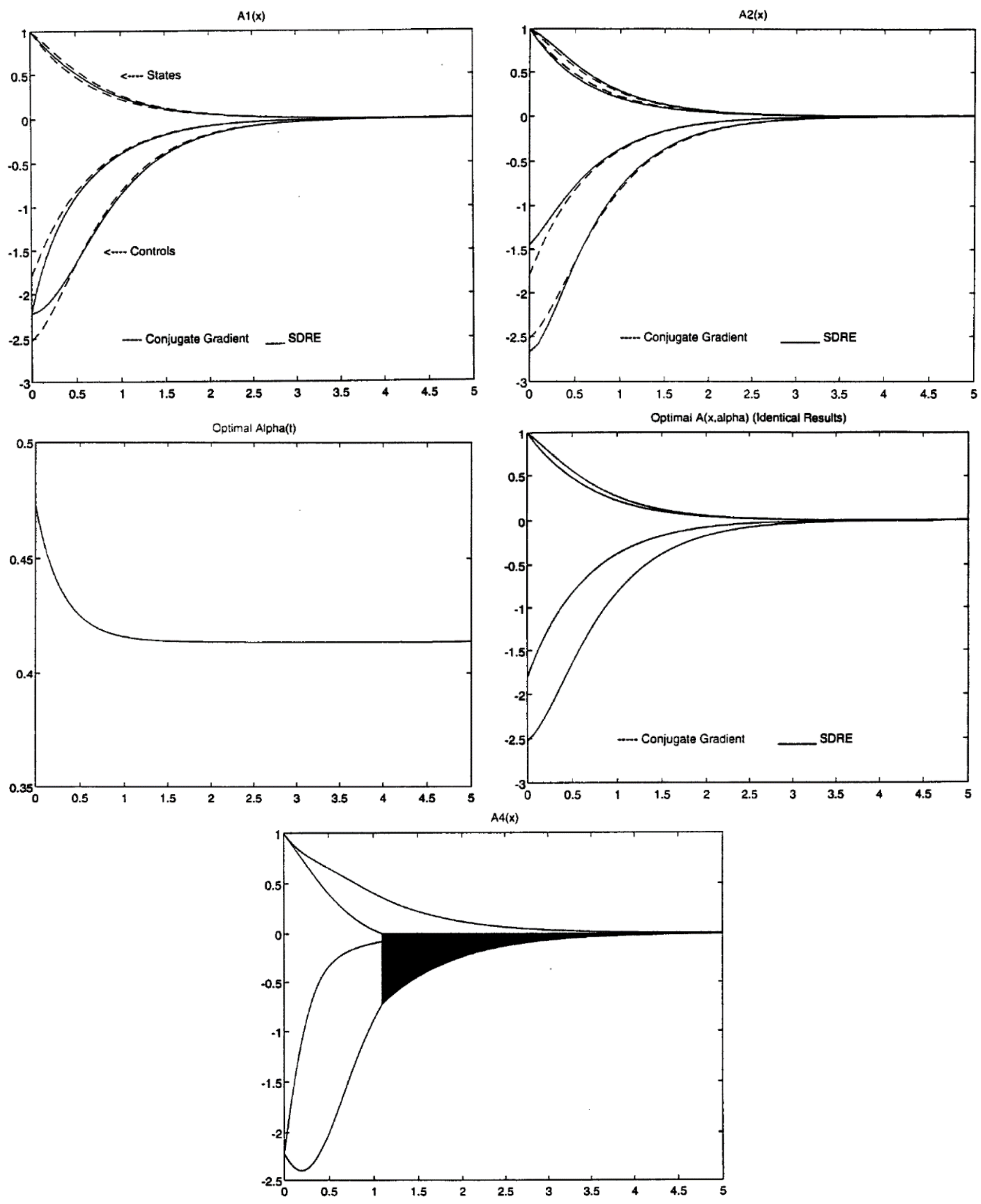

Figure 1.16: SDRE Control of the Parameterized Systems (Source: Cloutier et al [21])

Chapter 2

Development of Cubic Approximation Model of Spacecraft Relative Motion

In this chapter, a cubic approximation model of spacecraft relative motion is developed.

2.1 Local-Vertical Local-Horizontal Components of Relative Motion

The two-body vector differential equation of motion is

$$\ddot{\mathbf{r}}\,(t) = -\frac{\mu}{r(t)^3}\,\cdot\,\mathbf{r}\,(t) \tag{2.1}$$

This is an un-approximated nonlinear second order vector (differential) equation and the general solution to this equation does not admit any simple representation. The coordinate system, shown in Fig. 2.1, moves with the origin "O" in a circular path of radius R with the axes so that $\hat{\mathbf{i}}$ is along the radial direction, $\hat{\mathbf{j}}$ is along the along-track direction and $\hat{\mathbf{k}}$ is orthogonal to the orbit plane and it is along the cross track direction. The mean motion is $n = \omega = \sqrt{\mu/R^3}$ where R is the chief satellite orbital radius. The angular rate of the frame is $\omega = \omega\hat{\mathbf{k}}$. It is assumed that the distance of the deputy spacecraft from the chiefs orbit is small compared to the chief orbit radial distance, i.e. $\rho/\mathbf{R} \ll 1$ The position vector of the chief orbit, expressed in the Hill frame, is

$$\mathbf{R} = R\hat{\mathbf{i}} \tag{2.2}$$

and the distance between the chief and the deputy is $\rho = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$. The position of a chief satellite relative to the Earth is described by $\mathbf{R}$, the position of a deputy satellite relative to the Earth is described by $\mathbf{r}$, and the position of the deputy relative to the chief is ρ.

$$\mathbf{r} = \mathbf{R} + \rho \tag{2.3}$$

The acceleration kinematics are described relative to a rotating reference frame attached to the chief. The frame rotates with the chief's orbital angular velocity ω, and the chief is assumed to be in a circular orbit, resulting in a constant angular velocity. The acceleration in the LHS of Eq. (2.1) can be replaced using inertial acceleration formula. The absolute acceleration $\mathbf{a}$ of point "O" is given as

$$\mathbf{a} = \mathbf{a}_o + \dot{\omega} \times \mathbf{r}_{rel} + \omega \times (\omega \times \mathbf{r}_{rel}) + 2\omega \times \mathbf{v}_{rel} + \mathbf{a}_{rel} \tag{2.4}$$

Using Eq. (2.4), the inertial acceleration of the separation distance between the chief and the deputy satellites can be expressed as

$$\ddot{\rho} = \mathbf{a}_o + \dot{\omega} \times \rho + \overset{\circ\circ}{\rho} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) \tag{2.5}$$

Since the chief satellite is in a circular orbit with constant angular velocity then $\mathbf{a}_o = \mathbf{0}$, $\dot{\omega} = \mathbf{0}$ and Eq. (2.5) can be rewritten as

$$\ddot{\rho} = \overset{\circ\circ}{\rho} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) \tag{2.6}$$

Then, the acceleration of the deputy is

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) + \overset{\circ\circ}{\rho} \tag{2.7}$$

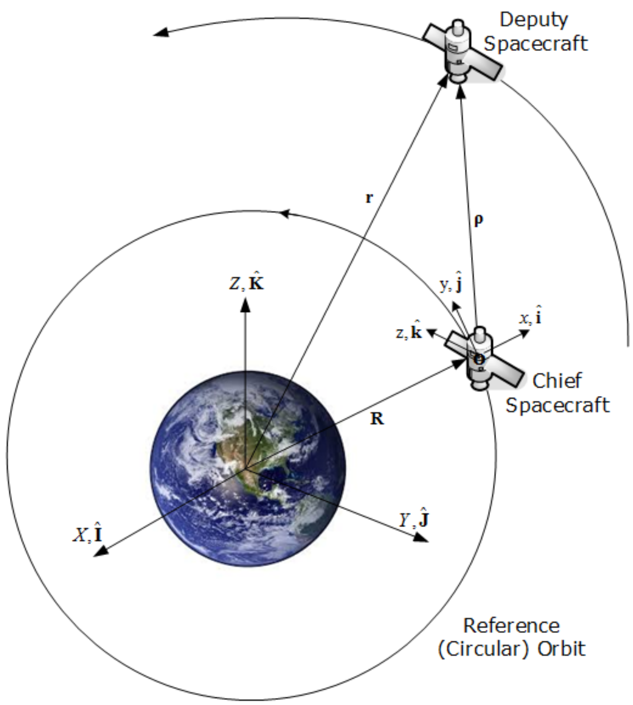

Figure 2.1: Relative Motion of Deputy Spacecraft with respect to the Chief Spacecraft in Circular Orbit

Here, $\overset{\bullet}{(\ )}$ indicates vector differentiation with respect to the inertial frame, and $\overset{\circ}{(\ )}$ indicates vector differentiation with respect to the rotating frame. Also,

$$[\rho]_L = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \left[{}^L \frac{d}{dt} (\rho) \right]_L = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \left[\overset{\circ}{\rho} \right]_L, {}^N \frac{d}{dt} (\rho) = \overset{\bullet}{\rho} \tag{2.8}$$

The chief and deputy accelerations are assumed to be Keplerian accelerations due to the Earth,

$$\begin{gathered} \ddot{R} = -\tfrac{\mu}{R^3} R \\ \ddot{\mathbf{r}} = -\tfrac{\mu}{r^3} \mathbf{r} = -\tfrac{\mu}{\|\mathbf{R}+\rho\|^3} (\mathbf{R} + \rho) \end{gathered} \tag{2.9}$$

The relative motion of deputy satellite with respect to the chief satellite is shown in the Figure 2.1. Equations (2.6), (2.8) can be used to write the vector equation,

$$\overset{\circ\circ}{\rho} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) = -\frac{\mu}{\|\mathbf{R}+\rho\|^3} (\mathbf{R}+\rho) + \frac{\mu}{R^3}\mathbf{R} \tag{2.10}$$

for the relative motion of the deputy with respect to the chief. Expressing these equations in local-vertical/local-horizontal components produces three nonlinear second-order differential equations of motion.

$$\begin{aligned} &\ddot{x} - 2n\dot{y} - n^2 x = -\frac{\mu(R+x)}{\left((R+x)^2+y^2+z^2\right)^{\frac{3}{2}}} + \frac{\mu}{R^2} \\ &\ddot{y} + 2n\dot{x} - n^2 y = -\frac{\mu y}{\left((R+x)^2+y^2+z^2\right)^{\frac{3}{2}}} \\ &\ddot{z} = -\frac{\mu z}{\left((R+x)^2+y^2+z^2\right)^{\frac{3}{2}}} \end{aligned} \tag{2.11}$$

2.2 Cubic Approximation Model of Spacecraft Relative Motion

To develop a polynomial approximation of the nonlinear equations of motion, the binomial series expansion formula shown below is used.

$$(1+\ p)^n = 1 + np + \frac{n(n-1)}{2!}p^2 + \frac{n(n-1)(n-2)}{3!}p^3 + \ldots \qquad |p| < 1 \tag{2.12}$$

Using inner products rule,

$$\begin{aligned} &\|\mathbf{R}+\rho\|^{-3} = [(\mathbf{R}+\rho)\cdot(\mathbf{R}+\rho)]^{-3/2} = \left(\mathbf{R}^T\mathbf{R} + 2\mathbf{R}^T\rho + \rho^T\rho\right)^{-3/2} \\ &= R^{-3}\left(1 + 2\frac{\mathbf{R}^T\rho}{R^2} + \frac{\rho^T\rho}{R^2}\right)^{-3/2} = R^{-3}\left(1 + \frac{2\mathbf{R}\cdot\rho}{R^2} + \frac{\rho\cdot\rho}{R^2}\right)^{-3/2} \end{aligned} \tag{2.13}$$

Upon substitution of

$$p = \frac{2\mathbf{R}\cdot\rho}{R^2} + \frac{\rho\cdot\rho}{R^2} \tag{2.14}$$

into Eq. (2.12) we have

$$(1+p)^{-3/2} = 1 - \tfrac{3}{2}\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2} + \tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15}{8}\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2} + \tfrac{\rho\cdot\rho}{R^2}\right)^2$$
$$-\tfrac{35}{16}\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2} + \tfrac{\rho\cdot\rho}{R^2}\right)^3 + \ldots$$
$$= 1 - \tfrac{3}{2}\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2} + \tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15}{8}\left[\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2}\right)\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2}\right) + \left(\tfrac{2\mathbf{R}\cdot\rho}{R^2}\right)\left(\tfrac{\rho\cdot\rho}{R^2}\right)\right. \tag{2.15}$$
$$\left. + \left(\tfrac{\rho\cdot\rho}{R^2}\right)\left(\tfrac{2\mathbf{R}\cdot\rho}{R^2}\right) + \left(\tfrac{\rho\cdot\rho}{R^2}\right)\left(\tfrac{\rho\cdot\rho}{R^2}\right)\right]$$
$$-\tfrac{35}{16}\left[4\tfrac{(\mathbf{R}\cdot\rho)^4}{R^4} + \tfrac{4(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{R^4} + \tfrac{(\rho\cdot\rho)^2}{R^4}\right]\left[\tfrac{2\mathbf{R}\cdot\rho}{R^2} + \tfrac{\rho\cdot\rho}{R^2}\right] + \ldots$$

Expanding Eq. (2.15) gives

$$(1+p)^{-3/2} = 1 - 3\left(\tfrac{\mathbf{R}\cdot\rho}{R^2}\right) - \tfrac{3}{2}\left(\tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15(\mathbf{R}\cdot\rho)^2}{2R^4} + \tfrac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4}$$
$$+\tfrac{15(\rho\cdot\rho)^2}{8R^4} - \tfrac{35}{16}\left[\tfrac{8(\mathbf{R}\cdot\rho)^3}{R^6} + \tfrac{4(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{R^6} + \tfrac{8(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{R^6}\right.$$
$$\left.+\tfrac{4(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{R^6} + \tfrac{2(\rho\cdot\rho)^2(\mathbf{R}\cdot\rho)}{R^6} + \tfrac{(\rho\cdot\rho)^3}{R^6}\right] + \ldots$$
$$= 1 - 3\left(\tfrac{\mathbf{R}\cdot\rho}{R^2}\right) - \tfrac{3}{2}\left(\tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15(\mathbf{R}\cdot\rho)^2}{2R^4} + \tfrac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4} + \tfrac{15(\rho\cdot\rho)^2}{8R^4} \tag{2.16}$$
$$-\tfrac{35(\mathbf{R}\cdot\rho)^3}{2R^6} - \tfrac{35(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6} - \tfrac{35(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{2R^6} - \tfrac{35(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{4R^6}$$
$$-\tfrac{35(\rho\cdot\rho)^2(\mathbf{R}\cdot\rho)}{8R^6} - \tfrac{35(\rho\cdot\rho)^3}{16R^6} + \ldots$$

Equation (2.16) reduces to

$$(1+p)^{-3/2} = 1 - 3\left(\tfrac{\mathbf{R}\cdot\rho}{R^2}\right) - \tfrac{3}{2}\left(\tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15(\mathbf{R}\cdot\rho)^2}{2R^4} + \tfrac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4} + \tfrac{15(\rho\cdot\rho)^2}{8R^4}$$
$$-\tfrac{35(\mathbf{R}\cdot\rho)^3}{2R^6} - \tfrac{105(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6} - \tfrac{105(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{8R^6} - \tfrac{35(\rho\cdot\rho)^3}{16R^6} + \ldots \tag{2.17}$$

Substituting Eq. (2.17) into Eq. (2.13) yields

$$\|\mathbf{R}+\rho\|^{-3} = R^{-3}(1+p)^{-3/2}$$
$$= R^{-3}\left[1 - 3\left(\tfrac{\mathbf{R}\cdot\rho}{R^2}\right) - \tfrac{3}{2}\left(\tfrac{\rho\cdot\rho}{R^2}\right) + \tfrac{15(\mathbf{R}\cdot\rho)^2}{2R^4} + \tfrac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4} + \tfrac{15(\rho\cdot\rho)^2}{8R^4}\right. \tag{2.18}$$
$$\left.-\tfrac{35(\mathbf{R}\cdot\rho)^3}{2R^6} - \tfrac{105(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6} - \tfrac{105(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{8R^6} - \tfrac{35(\rho\cdot\rho)^3}{16R^6} + \ldots\right]$$

Substituting this into the vector equation of motion produces an approximation that is a polynomial in ρ.

$$\begin{aligned}
&\overset{\circ\circ}{\rho} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) \\
&= -\frac{\mu}{R^3}\left[1 - 3\left(\frac{\mathbf{R}\cdot\rho}{R^2}\right) - \frac{3}{2}\left(\frac{\rho\cdot\rho}{R^2}\right) + \frac{15(\mathbf{R}\cdot\rho)^2}{2R^4} + \frac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4} + \frac{15(\rho\cdot\rho)^2}{8R^4} - \frac{35(\mathbf{R}\cdot\rho)^3}{2R^6}\right. \\
&\left. - \frac{105(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6} - \frac{105(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{8R^6} - \frac{35(\rho\cdot\rho)^3}{16R^6} + ...\right](\mathbf{R} + \rho) + \frac{\mu}{R^3}\mathbf{R}
\end{aligned} \tag{2.19}$$

Eq. (2.19) simplifies to

$$\begin{aligned}
&\overset{\circ\circ}{\rho} + 2\omega \times \overset{\circ}{\rho} + \omega \times (\omega \times \rho) \\
&= -\frac{\mu}{R^3}\left[\rho - 3\left(\frac{\mathbf{R}\cdot\rho}{R^2}\right)\mathbf{R} - 3\left(\frac{\mathbf{R}\cdot\rho}{R^2}\right)\rho - \frac{3}{2}\left(\frac{\rho\cdot\rho}{R^2}\right)\mathbf{R} - \frac{3}{2}\left(\frac{\rho\cdot\rho}{R^2}\right)\rho + \left(\frac{15(\mathbf{R}\cdot\rho)^2}{2R^4}\right)\mathbf{R}\right. \\
&+ \left(\frac{15(\mathbf{R}\cdot\rho)^2}{2R^4}\right)\rho + \left(\frac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4}\right)\mathbf{R} + \left(\frac{15(\mathbf{R}\cdot\rho)(\rho\cdot\rho)}{2R^4}\right)\rho + \left(\frac{15(\rho\cdot\rho)^2}{8R^4}\right)\mathbf{R} + \left(\frac{15(\rho\cdot\rho)^2}{8R^4}\right)\rho \\
&- \left(\frac{35(\mathbf{R}\cdot\rho)^3}{2R^6}\right)\mathbf{R} - \left(\frac{35(\mathbf{R}\cdot\rho)^3}{2R^6}\right)\rho - \left(\frac{105(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6}\right)\mathbf{R} - \left(\frac{105(\mathbf{R}\cdot\rho)^2(\rho\cdot\rho)}{4R^6}\right)\rho \\
&\left. - \left(\frac{105(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{8R^6}\right)\mathbf{R} - \left(\frac{105(\mathbf{R}\cdot\rho)(\rho\cdot\rho)^2}{8R^6}\right)\rho - \left(\frac{35(\rho\cdot\rho)^3}{16R^6}\right)\mathbf{R} - \left(\frac{35(\rho\cdot\rho)^3}{16R^6}\right)\rho + ...\right]
\end{aligned} \tag{2.20}$$

Substituting $\omega^2 R^3 = \mu$, $\mathbf{R} = R\hat{\mathbf{i}}$, $\omega = \omega\hat{\mathbf{k}}$ and $\rho = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ into Eq. (2.20) the equation of motion in LVLH components can be expressed as

$$\begin{aligned}
&\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} + 2\omega\hat{\mathbf{k}} \times \left(\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}\right) + \omega\hat{\mathbf{k}} \times \left(\omega\hat{\mathbf{k}} \times \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\right) \\
&= -\frac{\omega^2 R^3}{R^3}\left[\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) - 3\frac{(Rx)}{R^2}R\hat{\mathbf{i}} - 3\frac{(Rx)}{R^2}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\right. \\
&- \frac{3}{2}\frac{\left(x^2+y^2+z^2\right)}{R^2}R\hat{\mathbf{i}} - \frac{3}{2}\frac{\left(x^2+y^2+z^2\right)}{R^2}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + \frac{15}{2}\frac{R^2x^2}{R^4}R\hat{\mathbf{i}} \\
&+ \frac{15}{2}\frac{R^2x^2}{R^4}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + \frac{15}{2}\frac{Rx\left(x^2+y^2+z^2\right)}{R^4}R\hat{\mathbf{i}} \\
&+ \frac{15}{2}\frac{Rx\left(x^2+y^2+z^2\right)}{R^4}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + \frac{15}{8}\frac{\left(x^2+y^2+z^2\right)^2}{R^4}R\hat{\mathbf{i}} \\
&+ \frac{15}{8}\frac{\left(x^2+y^2+z^2\right)^2}{R^4}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) - \frac{35}{2}\frac{(Rx)^3}{R^6}R\hat{\mathbf{i}} - \frac{35}{2}\frac{(Rx)^3}{R^6}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \\
&- \frac{105}{4}\frac{R^2x^2}{R^6}\left(x^2 + y^2 + z^2\right)R\hat{\mathbf{i}} - \frac{105}{4}\frac{R^2x^2}{R^6}\left(x^2 + y^2 + z^2\right)\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \\
&- \frac{105}{8}\frac{(Rx)}{R^6}\left(x^2 + y^2 + z^2\right)^2 R\hat{\mathbf{i}} - \frac{105}{8}\frac{(Rx)}{R^6}\left(x^2 + y^2 + z^2\right)^2\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \\
&\left. - \frac{35}{16}\frac{\left(x^2+y^2+z^2\right)^3}{R^6}R\hat{\mathbf{i}} - \frac{35}{16}\frac{\left(x^2+y^2+z^2\right)^3}{R^6}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + ...\right]
\end{aligned} \tag{2.21}$$

This can be written as

$$
\begin{aligned}
&\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} + 2\omega\dot{x}\hat{\mathbf{j}} - 2\omega\dot{y}\hat{\mathbf{j}} - \omega^2 x\hat{\mathbf{i}} - \omega^2 y\hat{\mathbf{j}} \\
&= \Big[\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) - 3x\hat{\mathbf{i}} - \tfrac{3}{R}\left(x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}\right) - \tfrac{3}{2R}\left(x^2+y^2+z^2\right)\hat{\mathbf{i}} \\
&-\tfrac{3}{2R^2}\left(x^3+xy^2+xz^2\right)\hat{\mathbf{i}} - \tfrac{3}{2R^2}\left(yx^2+y^3+yz^2\right)\hat{\mathbf{j}} - \tfrac{3}{2R^2}\left(zx^2+zy^2+z^3\right)\hat{\mathbf{k}} + \tfrac{15}{2R}x^2\hat{\mathbf{i}} \\
&+\tfrac{15}{2R^2}x^3\hat{\mathbf{i}} + \tfrac{15}{2R^2}x^2y\hat{\mathbf{j}} + \tfrac{15}{2R^2}x^2z\hat{\mathbf{k}} + \tfrac{15}{2R^2}\left(x^3+xy^2+xz^2\right)\hat{\mathbf{i}} \\
&+\tfrac{15}{2R^3}\left(\left(x^4+x^2y^2+x^2z^2\right)\hat{\mathbf{i}} + \left(x^3y+xy^3+xyz^2\right)\hat{\mathbf{j}} + \left(zx^3+xzy^2+xz^3\right)\hat{\mathbf{k}}\right) \\
&+\tfrac{15}{8}\tfrac{\left(x^2+y^2+z^2\right)^2}{R^4}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + \tfrac{15}{8}\tfrac{\left(x^2+y^2+z^2\right)^2}{R^3}\hat{\mathbf{i}} - \tfrac{35}{2}\tfrac{x^3}{R^2}\hat{\mathbf{i}} - \tfrac{35}{2R^3}\left(x^4\hat{\mathbf{i}} + x^3y\hat{\mathbf{j}} + x^3z\hat{\mathbf{k}}\right) \\
&-\tfrac{105}{4R^3}\left(x^4+x^2y^2+x^2z^2\right)\hat{\mathbf{i}} - \tfrac{105}{4R^4}\left(x^4+x^2y^2+x^2z^2\right)\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \\
&-\tfrac{105}{8R^4}x(x^2+y^2+z^2)^2\hat{\mathbf{i}} - \tfrac{105}{8R^5}x(x^2+y^2+z^2)^2\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \\
&-\tfrac{35}{16R^5}(x^2+y^2+z^2)^3\hat{\mathbf{i}} - \tfrac{35}{16R^6}(x^2+y^2+z^2)^3\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) + \ldots\Big]
\end{aligned}
\tag{2.22}
$$

Eliminating the higher order terms greater than cubic power in Eq. (2.22) gives the cubic approximation model

$$
\begin{aligned}
&\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} + 2\omega\dot{x}\hat{\mathbf{j}} - 2\omega\dot{y}\hat{\mathbf{j}} - \omega^2 x\hat{\mathbf{i}} - \omega^2 y\hat{\mathbf{j}} \\
&= -\omega^2\Big[\Big\{x - 3x - \tfrac{3}{R}x^2 - \tfrac{3}{2}\tfrac{\left(x^2+y^2+z^2\right)}{R} - \tfrac{3}{2R^2}\left(x^3+xy^2+xz^2\right) \\
&+\tfrac{15}{2R}x^2 + \tfrac{15}{2R^2}x^3 + \tfrac{15}{2R^2}\left(x^3+xy^2+xz^2\right) - \tfrac{35}{2R^2}x^3\Big\}\hat{\mathbf{i}} \\
&+\Big\{y - \tfrac{3}{R}xy - \tfrac{3}{2R^2}\left(y^3+yx^2+yz^2\right) + \tfrac{15}{2R^2}x^2y\Big\}\hat{\mathbf{j}} \\
&+\Big\{z - \tfrac{3}{R}xz - \tfrac{3}{2R^2}\left(z^3+zx^2+zy^2\right) + \tfrac{15}{2R^2}x^2z\Big\}\hat{\mathbf{k}}\Big]
\end{aligned}
\tag{2.23}
$$

Eq. (2.23) contains radial, along-track and cross-track components of the relative motion.

2.3 Radial, Along-track and Cross-track Cubic Equation of Motion

Extracting the scalar components produces the following three scalar cubic equations of motion.

(a) Radial Cubic Equation of Motion

$$
\ddot{x} - 2n\dot{y} - n^2 x = -n^2\left(-2x + \frac{3}{R}x^2 - \frac{3}{2R}y^2 - \frac{3}{2R}z^2 - \frac{4}{R^2}x^3 + \frac{6}{R^2}xy^2 + \frac{6}{R^2}xz^2\right) \tag{2.24}
$$

(b) Along-track Cubic Equation of Motion

$$\ddot{y} + 2n\dot{x} - n^2 y = -n^2 \left(y - \frac{3}{R} yx + \frac{6}{R^2} yx^2 - \frac{3}{2R^2} yz^2 - \frac{3}{2R^2} y^3 \right) \tag{2.25}$$

(c) Cross-track Cubic Equation of Motion

$$\ddot{z} = -n^2 \left(z - \frac{3}{R} zx + \frac{6}{R^2} zx^2 - \frac{3}{2R^2} zy^2 - \frac{3}{2R^2} z^3 \right) \tag{2.26}$$

These expressions have nonlinear terms xy, xz,x^2, y^2, z^2, x^2z, xy^2, xz^2, y^2z, z^2y, x^3, y^3, z^3 that may be linearized using harmonic-balance model. Therefore, in state space form, Equations (2.24), (2.25) and (2.26) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\left(x, y, z\right) \tag{2.27}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \tag{2.28}$$

and the nonlinear part, $\mathbf{F}\left(x, y, z\right)$ of the equations is

$$\mathbf{F}\left(x, y, z\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -n^2 \left(-2x + \frac{3}{R}x^2 - \frac{3}{2R}y^2 - \frac{3}{2R}z^2 - \frac{4}{R^2}x^3 + \frac{6}{R^2}xy^2 + \frac{6}{R^2}xz^2 \right) \\ -n^2 \left(-2x + \frac{3}{R}x^2 - \frac{3}{2R}y^2 - \frac{3}{2R}z^2 - \frac{4}{R^2}x^3 + \frac{6}{R^2}xy^2 + \frac{6}{R^2}xz^2 \right) \\ -n^2 \left(z - \frac{3}{R}zx + \frac{6}{R^2}zx^2 - \frac{3}{2R^2}zy^2 - \frac{3}{2R^2}z^3 \right) \end{bmatrix} \tag{2.29}$$

Chapter 3

Harmonic Linearization of Spacecraft Relative Motion Using Harmonic Balance Method

The method of harmonic-balance linearization, also known as the describing-function method in control theory, is a powerful technique to approximate nonlinear dynamic systems [16,17,18]. The harmonic-balance approximation corresponds to a truncated Fourier series and allows for the systematic determination of the coefficients and frequencies of first-order harmonics. The study of the motion of a deputy spacecraft relative to a chief spacecraft has often utilized the linearized Clohessy-Wiltshire (CW) equations. The CW equations make three explicit assumptions: (1) both spacecraft obey Keplerian motion, (2) close proximity between the chief and deputy, and (3) the chief is in a circular orbit.

In this chapter, the method of harmonic balance is adapted to construct a new linear approximation for satellite relative motion to help relax the close-proximity assumption. The method is found to be useful in obtaining a more accurate linearized approximation than the traditional approach of linearization about the origin. Figure 1, adapted from Reference 5, illustrates the steps followed in finding an approximate model for a nonlinear system using harmonic balance method. In developing the harmonic-balance model for spacecraft relative motion, polynomial approximation is applied to the nonlinear equations in LVLH components and it produced cubic equations of motion containing linear and nonlinear terms. The linear equations corresponds to the Clohessy-Wiltshire equations. For the harmonic linearization, the nonlinear terms are evaluated along the generating solution to produce linear correction terms. The correction terms are combined with the original linear equations (CW equations) to produce the harmonic balance model of the spacecraft relative motion.

The harmonic linearization of the cubic approximation model of spacecraft relative motion is performed using the following assumed linear solutions [54]:

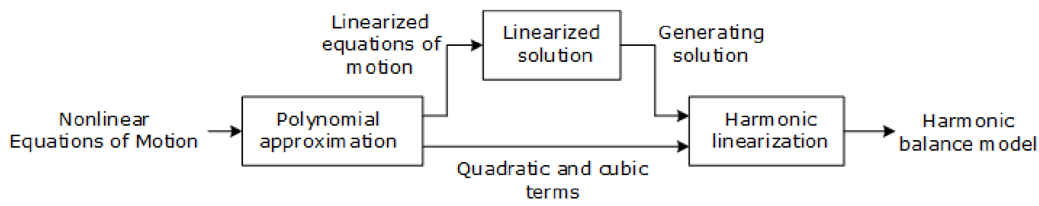

Figure 3.1: Steps followed in finding an approximate model for the nonlinear equation using harmonic balance method

$$
\begin{aligned}
x &= \tfrac{2}{n}c_1 + c_2 \sin nt + c_3 \cos nt \\
y &= -3c_1 t + 2c_2 \cos\ nt - 2c_3 \sin\ nt + c_4 \\
z &= c_5 \sin nt + c_6 \cos nt
\end{aligned}
\tag{3.1}
$$

The state vector $\mathbf{x}$ and a vector of constants $\mathbf{c}$ are defined as

$$
\mathbf{x} = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z}]^T, \mathbf{c} = [c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6]^T \tag{3.2}
$$

Equations (3.1) and their derivatives can be arranged into a fundamental matrix $\Psi(t)$.

$$
\mathbf{x} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} \frac{2}{n} & \sin nt & \cos nt & 0 & 0 & 0 \\ -3t & 2\cos nt & -2\sin nt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin nt & \cos nt \\ 0 & n\cos nt & -n\sin nt & 0 & 0 & 0 \\ -3 & -2n\sin nt & -2n\cos nt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n\cos nt & -n\sin nt \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \tag{3.3}
$$

$$
\mathbf{x}(t) = \Psi(t)\mathbf{c}
$$

The constants can be evaluated in terms of the initial states $\mathbf{x}(t_0)$.

$$
\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}(t_0) \tag{3.4}
$$

3.1 Radial Direction Linearization

The radial cubic approximation model of relative motion, from Eqs. (2.24), is

$$\ddot{x} - 2n\dot{y} - n^2x = -n^2\left(-2x + \frac{3}{R}x^2 - \frac{3}{2R}y^2 - \frac{3}{2R}z^2 - \frac{4}{R^2}x^3 + \frac{6}{R^2}xy^2 + \frac{6}{R^2}xz^2\right) \tag{3.5}$$

Each of the nonlinear terms is linearized by substituting the assumed linear solutions. For the linearization the approach in Ogundele et al [5] is followed. The following trigonometric identities are used in the linearization.

$$\begin{gathered}\cos^2 nt = \tfrac{1}{2}(1+\cos 2nt), \sin^2 nt = \tfrac{1}{2}(1-\cos 2nt)\\ \cos^3 nt = \tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt, \sin^3 nt = \tfrac{3}{4}\sin nt + \tfrac{1}{4}\sin 3nt\end{gathered} \tag{3.6}$$

The nonlinear term x^2 is linearized as follows

$$x^2 = \frac{4}{n^2}{c_1}^2 + \frac{4}{n}c_1c_2\sin nt + \frac{4}{n}c_1c_3\cos nt + 2c_2c_3\sin nt\cos nt + {c_2}^2\sin^2 nt + {c_3}^2\cos^2 nt \tag{3.7}$$

Substituting the trigonometric identities in Eq. (3.6) into Eq. (3.7) yields

$$\begin{gathered}x^2 = \tfrac{4}{n^2}{c_1}^2 + \tfrac{4}{n}c_1c_2\sin nt + \tfrac{4}{n}c_1c_3\cos nt + c_2c_3\sin 2nt + \tfrac{1}{2}{c_2}^2(1-\cos 2nt) + \tfrac{1}{2}{c_3}^2(1+\cos 2nt)\\ +\tfrac{1}{2}{c_2}^2(1-\cos 2nt) + \tfrac{1}{2}{c_3}^2(1+\cos 2nt)\end{gathered} \tag{3.8}$$

Eliminating the higher order harmonics of time we have

$$x^2 \approx \frac{4}{n^2}{c_1}^2 + \frac{1}{2}{c_2}^2 + \frac{1}{2}{c_3}^2 + \frac{4}{n}c_1c_2\sin nt + \frac{4}{n}c_1c_3\cos nt \tag{3.9}$$

Rearranging the equation in the form of the assumed linear solutions gives

$$x^2 \approx \frac{1}{2}{c_2}^2 + \frac{1}{2}{c_3}^2 - \frac{4}{n^2}{c_1}^2 + \frac{4}{n}c_1\left(\frac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) = \frac{1}{2}{c_2}^2 + \frac{1}{2}{c_3}^2 - \frac{4}{n^2}{c_1}^2 + \frac{4}{n}c_1x \tag{3.10}$$

Similarly, following the same approach as in x^2 we have

$$y^2 = 2{c_2}^2 + 2{c_3}^2 - {c_4}^2 + (2c_4 - 6c_1 t)y - 9{c_1}^2 t^2 + 6c_1 c_4 t \tag{3.11}$$

$$z^2 = \frac{1}{2}{c_5}^2 + \frac{1}{2}{c_6}^2 \tag{3.12}$$

$$x^3 = \left(-\frac{16}{n^3}{c_1}^3 + \frac{3}{2n}c_1{c_2}^2 + \frac{3}{2n}c_1{c_3}^2\right) + \left(\frac{12}{n^2}{c_1}^2 + \frac{3}{4}{c_2}^2 + \frac{3}{4}{c_3}^2\right)x \tag{3.13}$$

$$\begin{aligned} xy^2 = &\left(-\tfrac{4}{n}c_1{c_4}^2 + \tfrac{2}{n}c_1{c_3}^2 + \tfrac{2}{n}c_1{c_2}^2 + \tfrac{24}{n}{c_1}^2 c_4 t - \tfrac{54}{n}{c_1}^3 t^2 + \tfrac{18}{n}{c_1}^2 t^2\right) \\ &+ \left({c_2}^2 + {c_3}^2 + {c_4}^2 - 6c_1 c_4 t + 9{c_1}^2 t^2\right)x + \left(\tfrac{4}{n}c_1 c_4 - \tfrac{12}{n}{c_1}^2 t\right)y \end{aligned} \tag{3.14}$$

$$xz^2 = \frac{3}{4n}c_1{c_5}^2 + \frac{3}{4n}c_1{c_6}^2 + \left(\frac{1}{4}{c_5}^2 + \frac{1}{4}{c_6}^2\right)x + \left(\frac{1}{2}c_2 c_5 + \frac{1}{2}c_3 c_6\right)z \tag{3.15}$$

These linear approximations are substituted into the cubic radial equation of motion and it simplifies to

$$\begin{aligned} &\ddot{x} - 2n\dot{y} - 3n^2 x = -\tfrac{3}{R}n^2\left(-\tfrac{4}{n^2}{c_1}^2 - \tfrac{1}{2}{c_2}^2 - \tfrac{1}{2}{c_3}^2 + \tfrac{1}{2}{c_4}^2 - \tfrac{1}{4}{c_5}^2 - \tfrac{1}{4}{c_6}^2 - 3c_1 c_4 t + \tfrac{9}{2}{c_1}^2 t^2\right) \\ &+\tfrac{4}{R^2}n^2\left(-\tfrac{16}{n^3}{c_1}^3 - \tfrac{3}{2n}c_1{c_2}^2 - \tfrac{3}{2n}c_1{c_3}^2 + \tfrac{6}{n}c_1{c_4}^2 - \tfrac{9}{8n}c_1{c_5}^2 - \tfrac{9}{8n}c_1{c_6}^2\right. \\ &\left.-\tfrac{36}{n}{c_1}^2 c_4 t - \tfrac{27}{n}{c_1}^2 t^2 + \tfrac{81}{n}{c_1}^3 t^2\right) \\ &-n^2\left(\tfrac{12}{nR}c_1 + \tfrac{6}{R^2}\left(-\tfrac{8}{n^2}{c_1}^2 + \tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_3}^2 + {c_4}^2 + \tfrac{1}{4}{c_5}^2 + \tfrac{1}{4}{c_6}^2 - 6c_1 c_4 t + 9{c_1}^2 t^2\right)\right)x \\ &-n^2\left(-\tfrac{3}{2R}\left(-6c_1 t + 2c_4\right) + \tfrac{6}{R^2}\left(\tfrac{4}{n}c_1 c_4 - \tfrac{12}{n}{c_1}^2 t\right)\right)y - \tfrac{6}{R^2}n^2\left(\tfrac{1}{2}c_2 c_5 + \tfrac{1}{2}c_3 c_6\right)z \end{aligned} \tag{3.16}$$

The linearized radial equation of motion can be summarized as

$$\ddot{x} - 2n\dot{y} - 3n^2 x = b_1(c,t) + a_{11}(c,t)\,x + a_{12}(c,t)\,y + a_{13}(c)\,z \tag{3.17}$$

The additional terms in x, y and z and the nonhomogeneous term $b_1(c,t)$ are the corrections to the equation of motion for the radial direction. They are approximations of the cubic powers that were retained in the expansion of the nonlinear equations of motion.

3.2 Along-track Direction Linearization

The along-track cubic approximation model of relative motion, from Eqs. (2.17), is

$$\ddot{y} + 2n\dot{x} - n^2 y = -n^2 \left(y - \frac{3}{R} yx + \frac{6}{R^2} yx^2 - \frac{3}{2R^2} yz^2 - \frac{3}{2R^2} y^3 \right) \tag{3.18}$$

The nonlinear term xy in the in-track direction equation is linearized as follows.

$$\begin{aligned} xy = -\tfrac{6}{n}{c_1}^2 t - 3c_1c_2 t \sin nt - 3c_1c_3 t \ \cos nt + \tfrac{4}{n}c_1c_2 \cos \ nt + {c_2}^2 \sin 2nt \lim_{x\to\infty} \\ +c_2c_3\left(1+\cos 2nt\right) - \tfrac{4}{n}c_1c_3 \sin \ nt - c_2c_3\left(1-\cos 2nt\right) - {c_3}^2 \sin \ 2nt \\ +\tfrac{2}{n}c_1c_4 + c_2c_4 \sin nt + c_4c_3 \cos nt \end{aligned} \tag{3.19}$$

Eliminating the higher order harmonics of time and rearranging the remaining terms in the form of the assumed linear solutions gives

$$\begin{aligned} xy \approx -3c_1 t\left(\tfrac{2}{n}c_1 + c_2 \sin nt + c_3 \cos nt\right) + \tfrac{2}{n}c_1\left(-3c_1 t + 2c_2 cos\ nt - 2c_3 \sin \ nt + c_4\right) \\ +\tfrac{6}{n}{c_1}^2 t + c_4\left(\tfrac{2}{n}c_1 + c_2 \sin nt + c_3 \cos nt\right) - \tfrac{2}{n}c_1c_4 \\ = \left(-\tfrac{2}{n}c_1c_4 + \tfrac{6}{n}{c_1}^2 t\right) + \left(-3c_1 t + c_4\right)x + \tfrac{2}{n}c_1 y \end{aligned} \tag{3.20}$$

Similarly,

$$\begin{aligned} x^2 y = \tfrac{1}{4}{c_2}^2 c_4 + \tfrac{1}{4}{c_3}^2 c_4 - \tfrac{8}{n^2}{c_1}^2 c_4 + \tfrac{24}{n^2}{c_1}^3 t - \tfrac{3}{4}c_1{c_2}^2 t - \tfrac{3}{4}c_1{c_3}^2 t \\ + \left(\tfrac{4}{n}c_1c_4 - \tfrac{12}{n}{c_1}^2 t\right)x + \left(\tfrac{4}{n^2}{c_1}^2 + \tfrac{1}{4}{c_2}^2 + \tfrac{1}{4}\ {c_3}^2\right)y \end{aligned} \tag{3.21}$$

$$\begin{aligned} y^3 = \left(3{c_2}^2 c_4 + 3{c_3}^2 c_4 - 2{c_4}^3 + \left(-9c_1{c_2}^2 - 9c_1{c_3}^2 \ + 18c_1{c_4}^2\right)t - 54{c_1}^2 c_4 t^2 + 54{c_1}^3 t^3\right) \\ + \left(3{c_2}^2 + 3{c_3}^2 + 3{c_4}^2 - 18c_1c_4 t + 27{c_1}^2 t^2\right)y \end{aligned} \tag{3.22}$$

$$z^2 y = \frac{1}{4}\left({c_5}^2 + {c_6}^2\right)c_4 - \frac{3}{4}\left({c_5}^2 + {c_6}^2\right)c_1 t + \left(\frac{1}{4}{c_5}^2 + \frac{1}{4}{c_6}^2\right)y + \left(c_2c_6 - c_3c_5\right)z \tag{3.23}$$

These linear approximations are substituted into the cubic along-tract equation of motion and it simplifies to

$$
\begin{aligned}
&\ddot{y} + 2n\dot{x} = \tfrac{3}{R}n^2\left(-\tfrac{2}{n}c_1c_4 + \tfrac{6}{n}{c_1}^2 t\right)\\
&+\tfrac{3}{2R^2}n^2\left[\tfrac{32}{n^2}c_4{c_1}^2 + 2c_4{c_2}^2 + 2c_4{c_3}^2 + \tfrac{1}{4}c_4{c_5}^2 + \tfrac{1}{4}c_4{c_6}^2 - 2{c_4}^3\right.\\
&\left.+\left(-6c_1{c_2}^2 - 6c_1{c_3}^2 \ + 18c_1{c_4}^2 - \tfrac{3}{4}c_1{c_5}^2 - \tfrac{3}{4}c_1{c_6}^2 - \tfrac{96}{n^2}{c_1}^3\right)t - 54c_4{c_1}^2t^2 + 54{c_1}^3t^3\right]\\
&+\left[\tfrac{3}{R}n^2\left(-3c_1t + c_4\right) - \tfrac{6}{R^2}n^2\left(\tfrac{4}{n}c_1c_4 - \tfrac{12}{n}{c_1}^2 t\right)\right]x\\
&+\left[\tfrac{6}{R}nc_1 + \tfrac{3}{2R^2}n^2\left(-\tfrac{16}{n^2}{c_1}^2 + 2{c_2}^2 + 2{c_3}^2 + 3{c_4}^2 + \tfrac{1}{4}{c_5}^2\right.\right.\\
&\left.\left.+\tfrac{1}{4}{c_6}^2 - 18c_1c_4t + 27{c_1}^2t^2\right)\right]y + \tfrac{3}{2R^2}n^2\left(c_2c_6 - c_3c_5\right)z
\end{aligned}
\tag{3.24}
$$

The linearized out-of-plane in-track equation of motion can be summarized as follows

$$
\ddot{y} + 2n\dot{x} = b_2\left(c,t\right) + a_{21}\left(c,t\right)x + a_{22}\left(c,t\right)y + a_{23}\left(c\right)z \tag{3.25}
$$

The additional terms in x, y, z, and the nonhomogeneous term $b_2\left(c,t\right)$ are the corrections to the along-track direction.

3.3 Cross-track Direction Linearization

The cross-track cubic approximation model of relative motion, from Eqs. (2.17), is

$$
\ddot{z} = -n^2\left(z - \frac{3}{R}zx + \frac{6}{R^2}zx^2 - \frac{3}{2R^2}zy^2 - \frac{3}{2R^2}z^3\right) \tag{3.26}
$$

The nonlinear term xz in the cross-track direction equation is linearized as follows.

$$
\begin{aligned}
xz = \tfrac{2}{n}c_1c_5\sin\ nt + \tfrac{2}{n}c_1c_6\cos\ nt + c_2c_5\sin^2 nt + c_2c_6\cos\ nt\sin nt\\
+c_3c_5\cos nt sin\ nt + c_3c_6\cos^2 nt
\end{aligned}
\tag{3.27}
$$

Substituting the trigonometric identities in Eq. (3.6) into Eq. (3.27) gives

$$\begin{aligned} xz = \tfrac{2}{n}c_1c_5\sin\ nt + \tfrac{2}{n}c_1c_6\cos\ nt + \tfrac{1}{2}c_2c_5\left(1-\cos 2nt\right) + \tfrac{1}{2}c_2c_6\sin 2nt \\ +\tfrac{1}{2}c_3c_5\sin\ 2nt + \tfrac{1}{2}c_3c_6\left(1+\cos 2nt\right) \end{aligned} \tag{3.28}$$

Eliminating the higher order harmonics of time and rearranging the remaining terms in the form of the assumed linear solutions gives

$$xz \approx \frac{2}{n}c_1\left(c_5\sin\ nt + c_6\cos\ nt\right) + \frac{1}{2}c_2c_5 + \frac{1}{2}c_3c_6 = \frac{1}{2}c_2c_5 + \frac{1}{2}c_3c_6 + \frac{2}{n}c_1z \tag{3.29}$$

Similarly,

$$x^2z = \frac{1}{n}c_1c_2c_5 - \frac{2}{n}c_1c_3c_6 + \left(\frac{4}{n^2}{c_1}^2 + \frac{1}{4}{c_2}^2 + \frac{1}{4}{c_3}^2\right)z + \left(\frac{1}{2}c_2c_5 + \frac{1}{2}c_3c_6\right)x \tag{3.30}$$

$$\begin{aligned} y^2z = \left(c_2c_4c_6 - c_3c_4c_5 - 3c_1c_2c_6t + 3c_1c_3c_5t\right) + \left({c_2}^2 + {c_3}^2 + {c_4}^2 - 6c_1c_4t + 9{c_1}^2t^2\right)z \\ +\left(c_2c_6 - c_3c_5\right)y \end{aligned} \tag{3.31}$$

$$z^3 = \frac{3}{4}{c_5}^2\left(c_5\sin nt + c_6\cos nt\right) + \frac{3}{4}{c_6}^2\left(c_5\sin nt + c_6\cos nt\right) = \left(\frac{3}{4}{c_5}^2 + \frac{3}{4}{c_6}^2\right)z \tag{3.32}$$

These linear approximations are substituted into the cubic cross-track equation of motion and it simplifies to

$$\begin{aligned} &\ddot{z} + n^2z = \tfrac{3}{R}n^2\left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right) \\ &-\tfrac{6}{R^2}n^2\left(\tfrac{1}{n}c_1c_2c_5 - \tfrac{2}{n}c_1c_3c_6 - \tfrac{1}{4}c_2c_4c_6 + \tfrac{1}{4}c_3c_4c_5 + \tfrac{3}{4}\left(c_1c_2c_6 - c_1c_3c_5\right)t\right) \\ &-\tfrac{6}{R^2}n^2\left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right)x + \tfrac{3}{2R^2}n^2\left(c_2c_6 - c_3c_5\right)y \\ &-n^2\left(-\tfrac{6}{nR}c_1 + \tfrac{6}{R^2}\left(\tfrac{4}{n^2}{c_1}^2 - \tfrac{1}{4}{c_4}^2 - \tfrac{3}{16}{c_5}^2 - \tfrac{3}{16}{c_6}^2 - \tfrac{3}{2}c_1c_4t - \tfrac{9}{4}{c_1}^2t^2\right)\right)z \end{aligned} \tag{3.33}$$

The linearized out-of-plane cross-track equation of motion can be summarized as follows

$$\ddot{z} + n^2z = b_3\left(c,t\right) + a_{31}\left(c\right)x + a_{32}\left(c\right)y + a_{33}\left(c,t\right)z \tag{3.34}$$

The additional terms in x, y, z and the nonhomogeneous term $b_3(c,t)$ are the corrections to the cross-track direction. The expansion of radial, along-track and cross-track nonlinear terms are shown in Appendix A.

Model Summary

The harmonic-balance model developed in the previous sections is summarized below.

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2 x = b_1(\mathbf{c},t) + a_{11}(\mathbf{c},t)\,x + a_{12}(\mathbf{c},t)\,y + a_{13}(\mathbf{c})\,z \\
&\ddot{y} + 2n\dot{x} = b_2(\mathbf{c},t) + a_{21}(\mathbf{c},t)\,x + a_{22}(\mathbf{c},t)\,y + a_{23}(\mathbf{c})\,z \\
&\ddot{z} + n^2 z = b_3(\mathbf{c},t) + a_{31}(\mathbf{c})x + a_{32}(\mathbf{c})y + a_{33}(\mathbf{c},t)z
\end{aligned} \tag{3.35}$$

The model can also be expressed in state-space form as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 + a_{11}(\mathbf{c},t) & a_{12}(\mathbf{c},t) & a_{13}(\mathbf{c}) & 0 & 2n & 0 \\ a_{12}(\mathbf{c},t) & a_{22}(\mathbf{c},t) & a_{23}(\mathbf{c}) & -2n & 0 & 0 \\ a_{13}(\mathbf{c}) & a_{23}(\mathbf{c}) & a_{33}(\mathbf{c},t) - n^2 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1(\mathbf{c},t) \\ b_2(\mathbf{c},t) \\ b_3(\mathbf{c},t) \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{c},t)\,\mathbf{x} + \mathbf{B}(\mathbf{c},t) \tag{3.36}$$

Note that the harmonic balance model is linear time-varying. However, in the special case that $c_1 = 0$, the model reduces to a linear time-invariant, bounded motions of the form

$$\ddot{x} - 2n\dot{y} - 3n^2 x = -\tfrac{3}{R}n^2\left(-\tfrac{1}{2}{c_2}^2 - \tfrac{1}{2}{c_3}^2 + \tfrac{1}{2}{c_4}^2 - \tfrac{1}{4}{c_5}^2 - \tfrac{1}{4}{c_6}^2\right)$$
$$-\tfrac{6}{R^2}n^2\left(\tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_3}^2 + {c_4}^2 + \tfrac{1}{4}{c_5}^2 + \tfrac{1}{4}{c_6}^2\right)x + \tfrac{3c_4}{R}n^2 y - \tfrac{6}{R^2}n^2\left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right)z \tag{3.37}$$

$$\ddot{y} + 2n\dot{x} = \tfrac{3}{2R^2}n^2\left(2c_4{c_2}^2 + 2c_4{c_3}^2 + \tfrac{1}{4}c_4{c_5}^2 + \tfrac{1}{4}c_4{c_6}^2 - 2{c_4}^3\right)$$
$$+\tfrac{3c_4}{R}n^2 x + \tfrac{3}{2R^2}n^2\left(2{c_2}^2 + 2{c_3}^2 + 3{c_4}^2 + \tfrac{1}{4}{c_5}^2 + \tfrac{1}{4}{c_6}^2\right)y + \tfrac{3}{2R^2}n^2\left(c_2c_6 - c_3c_5\right)z \tag{3.38}$$

$$\ddot{z} + n^2 z = \tfrac{3}{R}n^2\left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right) - \tfrac{6}{R^2}n^2\left(-\tfrac{1}{4}c_2c_4c_6 + \tfrac{1}{4}c_3c_4c_5\right)$$
$$-\tfrac{6}{R^2}n^2\left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right)x + \tfrac{3}{2R^2}n^2\left(c_2c_6 - c_3c_5\right)y - \tfrac{6}{R^2}n^2\left(-\tfrac{1}{4}{c_4}^2 - \tfrac{3}{16}{c_5}^2 - \tfrac{3}{16}{c_6}^2\right)z \tag{3.39}$$

Compactly, this can be written as

$$\begin{aligned}
\ddot{x} - 2n\dot{y} - 3n^2 x &= b_1(c) + a_{11}(c)\,x + a_{12}(c)\,y + a_{13}(c)\,z \\
\ddot{y} + 2n\dot{x} &= b_2(c) + a_{21}(c)\,x + a_{22}(c)\,y + a_{23}(c)\,z \\
\ddot{z} + n^2 z &= b_3(c) + a_{31}(c)\,x + a_{32}(c)\,y + a_{33}(c)\,z
\end{aligned} \tag{3.40}$$

and in state space form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 + a_{11}(\mathbf{c}) & a_{12}(\mathbf{c}) & a_{13}(\mathbf{c}) & 0 & 2n & 0 \\ a_{12}(\mathbf{c}) & a_{22}(\mathbf{c}) & a_{23}(\mathbf{c}) & -2n & 0 & 0 \\ a_{13}(\mathbf{c}) & a_{23}(\mathbf{c}) & a_{33}(\mathbf{c}) - n^2 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1(\mathbf{c}) \\ b_2(\mathbf{c}) \\ b_3(\mathbf{c}) \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{c})\,\mathbf{x} + \mathbf{B}(\mathbf{c}) \tag{3.41}$$

Generating Solution Calibration

The harmonic-balance method is essentially a two-step linearization approach. The nonlinear system is first linearized about the state-space origin, and the solution to the resulting linear system is used to define a generating solution. The nonlinear system is then linearized about the generating solution. The motivation is that the assumption of close proximity to the generating solution may be more accurate than the original assumption of close proximity to the origin. Recently, a method of coordinate calibration has been developed to extract more accurate linearized solutions for nonlinear systems. The method takes advantage of coordinate transformations with alternative coordinates that describe the same system but enjoy a lower level of nonlinearity.

Here, the calibration of the Cartesian coordinates, $\mathbf{x}$, for the relative motion will be considered using transformations with the orbital-element differences, $\delta\mathbf{e}$. The nonlinear coordinate transformation, $\delta\mathbf{e} = \mathbf{b}(\mathbf{x})$, from Cartesian coordinates to orbital-element differences and the linearized coordinate transformation, $\mathbf{x} = \mathbf{A}\delta\mathbf{e}$, from orbital element differences to Cartesian coordinates are used. A calibrated initial condition is calculated as follows.

$$\tilde{\mathbf{x}}_0 = \mathbf{A}\mathbf{b}(\mathbf{x}_0) \tag{3.42}$$

Linear propagation of the calibrated initial condition using the traditional linear model has been seen to be more accurate than linear propagation of the true initial condition. For the purposes of the harmonic-balance model, this calibrated solution can be used as the generating solution. In forming the harmonic-balance model, this involves evaluating $\mathbf{c}(\tilde{\mathbf{x}}_0)$ as a function of the calibrated initial condition. The propagation of the harmonic-balance model, however, still uses the true initial condition, $\mathbf{x}_0$.

3.4 Numerical Simulations

3.4.1 Bounded Motion Propagation

The equations of motion in radial, along-track and cross-track directions are integrated using ode45. Table 3.1 shows the orbital elements of the chief and deputy. The subscripts C and D represent the chief and the deputy while a is the semi-major axis (in km), e is the eccentricity, i is the inclination (in degree), Ω is the right ascension of ascending node (in degree), ω is the argument of perigee (in degree) and f is the true anomaly (in degree). The orbit propagation was performed for six orbits and the simulations are carried out using two different initial conditions; true and CW calibrated initial conditions.

(a) Uncalibrated Generating Solution

In this simulation, the CW, HB, cubic and nonlinear models are propagated using true initial conditions, and the vector of constants $\mathbf{c}$ is also evaluated using true initial conditions.

Table 3.1: Chief and deputy orbital elements for a spacecraft in LEO orbit

Chief Spacecraft Orbital Elements						Deputy Spacecraft Orbital Elements					
a_C	e_C	i_C	Ω_C	ω_C	f_C	a_D	e_D	i_D	Ω_D	ω_D	f_D
7500	0	4	5	10	25	7500	0.0012	5.01	5	15	20

Table 3.2: Bounded Motion Errors over 1000 seconds using uncalibrated generating solution

Model	Average Error (km)
HCW	0.325
HB	0.040

The simulation results are illustrated in Figure 3.2. These results indicate that the HB model approximates the nonlinear dynamics more accurately than the CW model, at least over some initial time interval. To emphasize this another plot of approximation errors focusing on the first 1000 seconds is shown.

To quantify the approximation errors, the following error metric is defined for the average error.

$$\varepsilon(t) = \|\hat{\rho}(t) - \rho(t)\|, \bar{\varepsilon} = \frac{1}{t_{sim}} \int_0^{t_{sim}} \varepsilon(t) dt \tag{3.43}$$

where $\rho(t)$ is the nonlinear solution for the relative position, $\hat{\rho}(t)$ is one of the approximate linear solutions, and t_{sim} is the length of time that has been simulated. The average error for each linear solution is shown in Table 3.2.

(b) Calibrated Generating Solution

In this simulation, the HB, cubic and nonlinear models are propagated using true initial conditions, while the vector of constants c is evaluated using calibrated initial conditions. For purposes of a fair or complete comparison, the CW solution is propagated using the calibrated initial condition.

Table 3.3: Bounded Motion Errors over 1000 seconds using calibrated generating solution

Model	Average Error (km)
HCW	0.962
HB	0.048

The simulation results are illustrated in Figure 3.3. The values for each linear model is shown in Table 3.3. These results illustrate the improvement in accuracy of the HB model when using the calibrated generating solution, compared with using the uncalibrated generating solution. When using the uncalibrated generating solution, the HB model exhibited erroneous drift. When using the calibrated generating solution, the HB model maintained better accuracy over a larger time span. However, averaged over the simulation interval, the HB model using calibrated generating solution was less accurate than the calibrated CW solution.

The tradeoff, though, is the initial error introduced in the calibrated CW solution. The HB model has no such initial error. This is illustrated in the initial approximation errors shown in Figure 3.3. The HB model using the calibrated generating solution is able to provide good long term error without having to introduce initial error, unlike the calibrated CW solution.

3.4.2 Unbounded Motion Propagation

(a) Uncalibrated Generating Solution

A second example is constructed with initial conditions defined identical to Table 3.1 except the deputys semi-major axis is changed to 7505 km. The CW, HB, cubic and nonlinear models are propagated using true initial conditions, and the vector of constants c is also evaluated using true initial conditions.

Table 3.4: Unbounded Motion Errors over 1000 seconds using uncalibrated generating solution

Model	Average Error (km)
HCW	0.325
HB	0.040

Table 3.5: Unbounded Motion Errors over 1000 seconds using calibrated generating solution

Model	Average Error (km)
HCW	0.964
HB	0.046

Figure 3.4 shows the propagated solutions using the uncalibrated generating solution. The HB method provides good accuracy initially, but the error appears to grow rapidly later in the simulation interval. This may be related to how the HB model tries to approximate the nonlinear system as a linear, time varying system. To focus on the initial accuracy, the errors over the first 1000 seconds are shown in Table 3.4.

(b) Calibrated Generating Solution

The HB, cubic and nonlinear models are propagated using true initial conditions, while the vector of constants c is evaluated using calibrated initial conditions. The CW solution is propagated using the calibrated initial condition. Figure 3.5 illustrates drifting orbit propagated solutions using calibrated generating solution and Table 3.5 shows errors over 1000 seconds.

The harmonic-balance model of relative motion between chief satellite, in circular orbit, and deputy satellite has been presented. It results in an initial-condition dependent, linear model for the relative motion. The model is time varying for drifting generating solutions, but time-invariant for non-drifting generating solutions. The simulation results show that harmonic-balance model of satellite relative motion gives a more accurate linearized model because it captured the motion better than the conventional method, Clohessy-Wiltshire model. Also, the harmonic-balance model has lesser error than the Clohessy-Wiltshire model of satellite relative motion.

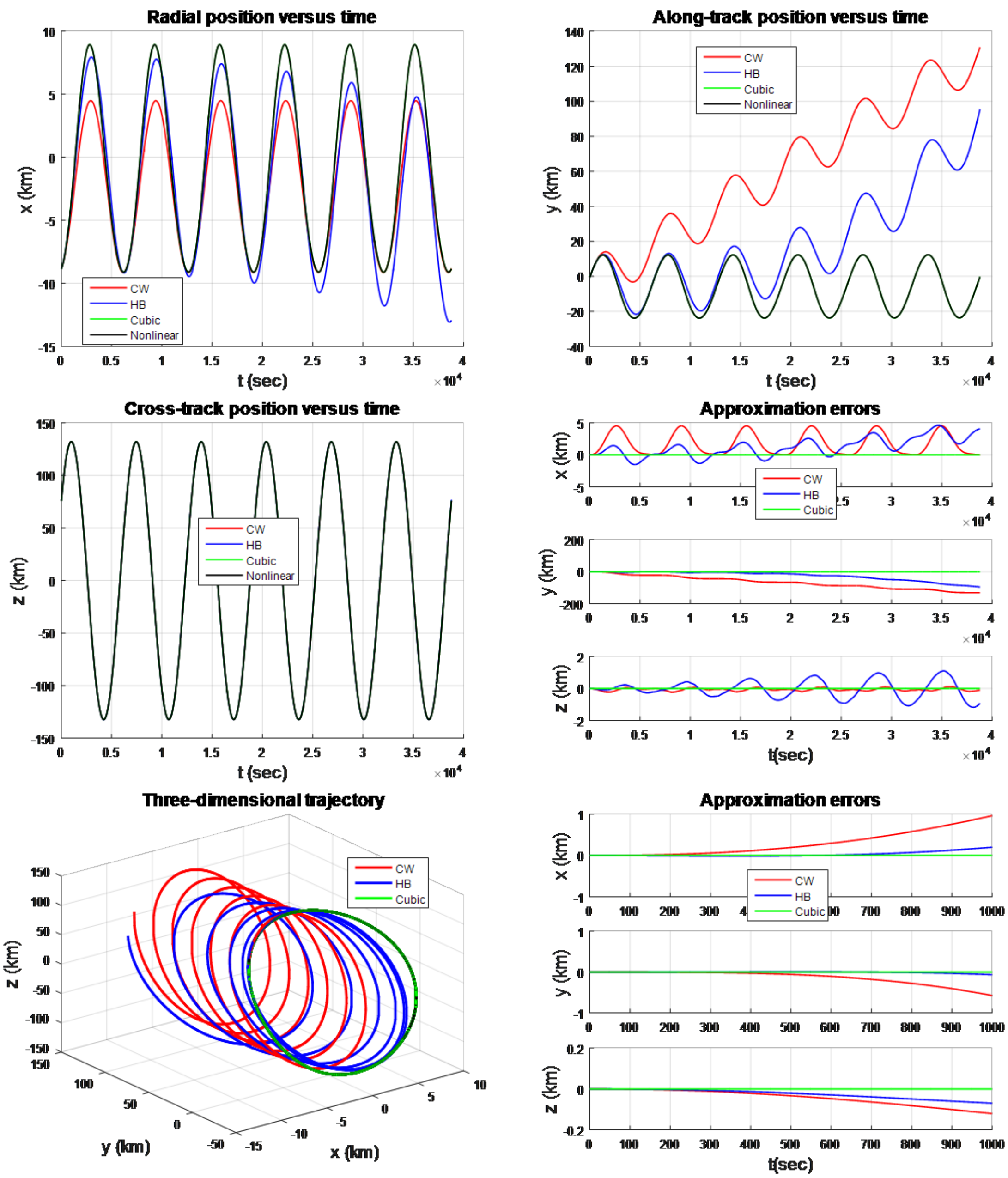

Figure 3.2: Bounded Motion Propagation Trajectories and Approximation Errors for Uncalibrated Generating Solution

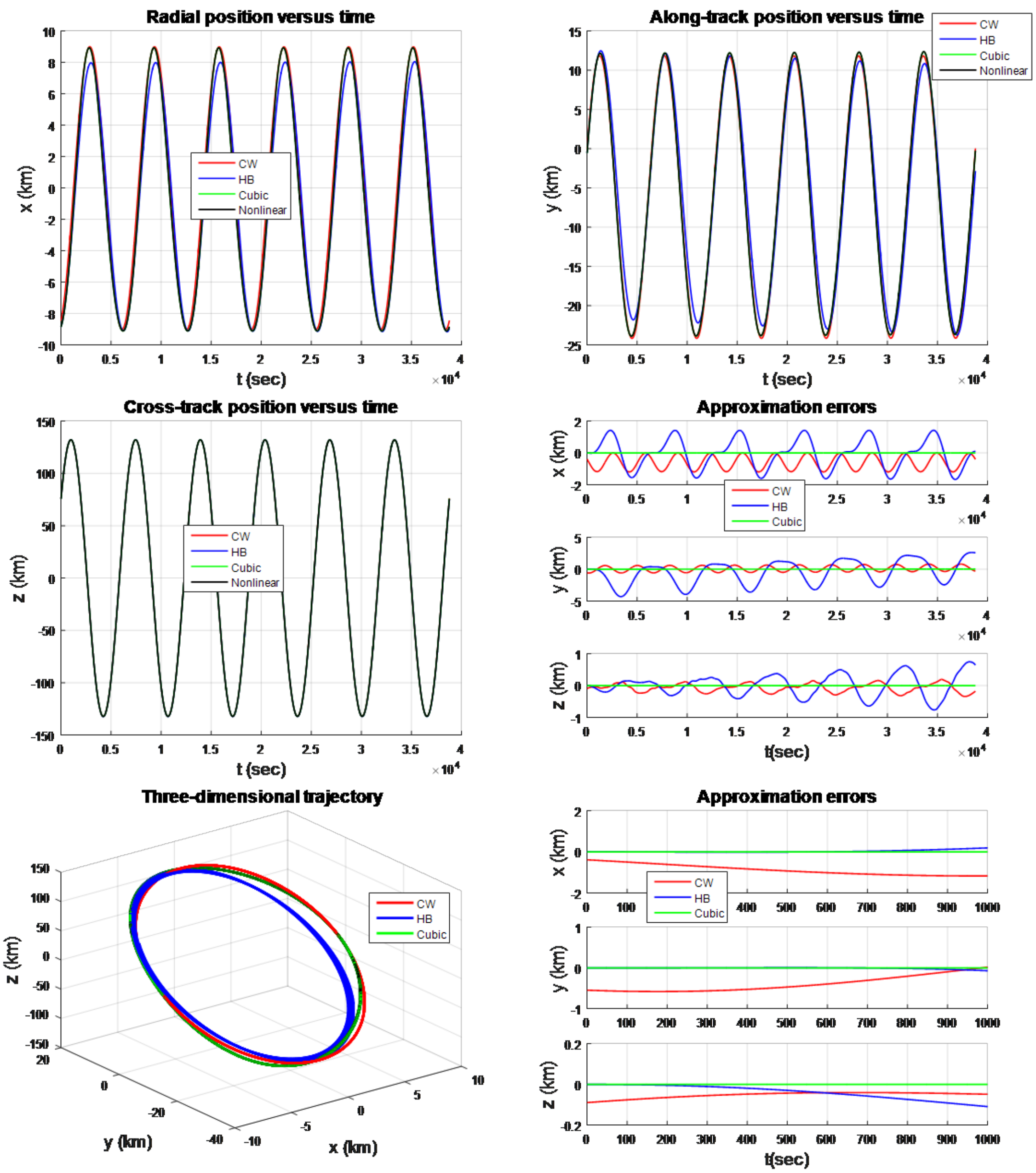

Figure 3.3: Bounded Motion Propagation Trajectories and Approximation Errors for Calibrated Generating Solution

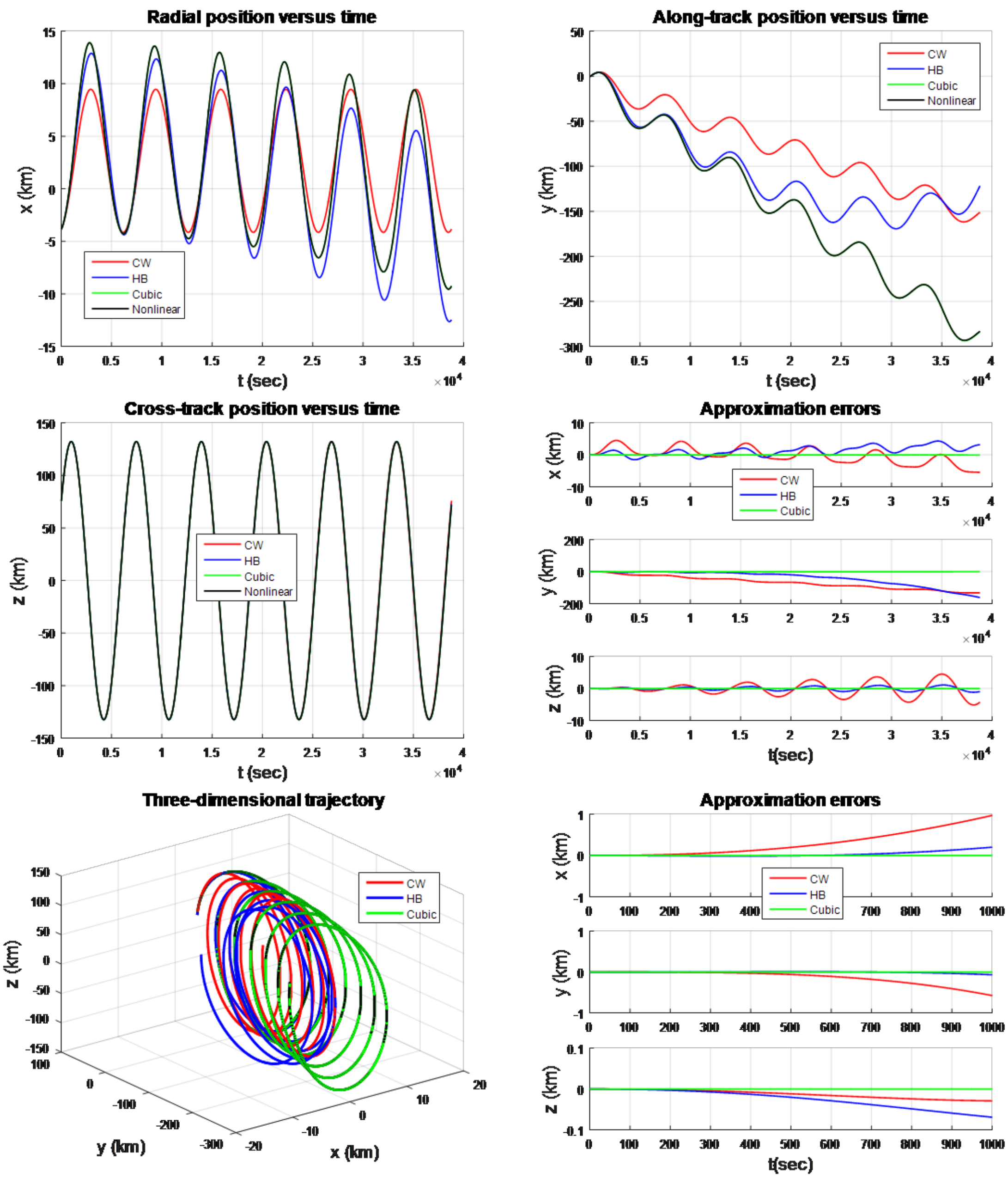

Figure 3.4: Unbounded Motion Propagation Trajectories and Approximation Errors for Uncalibrated Generating Solution

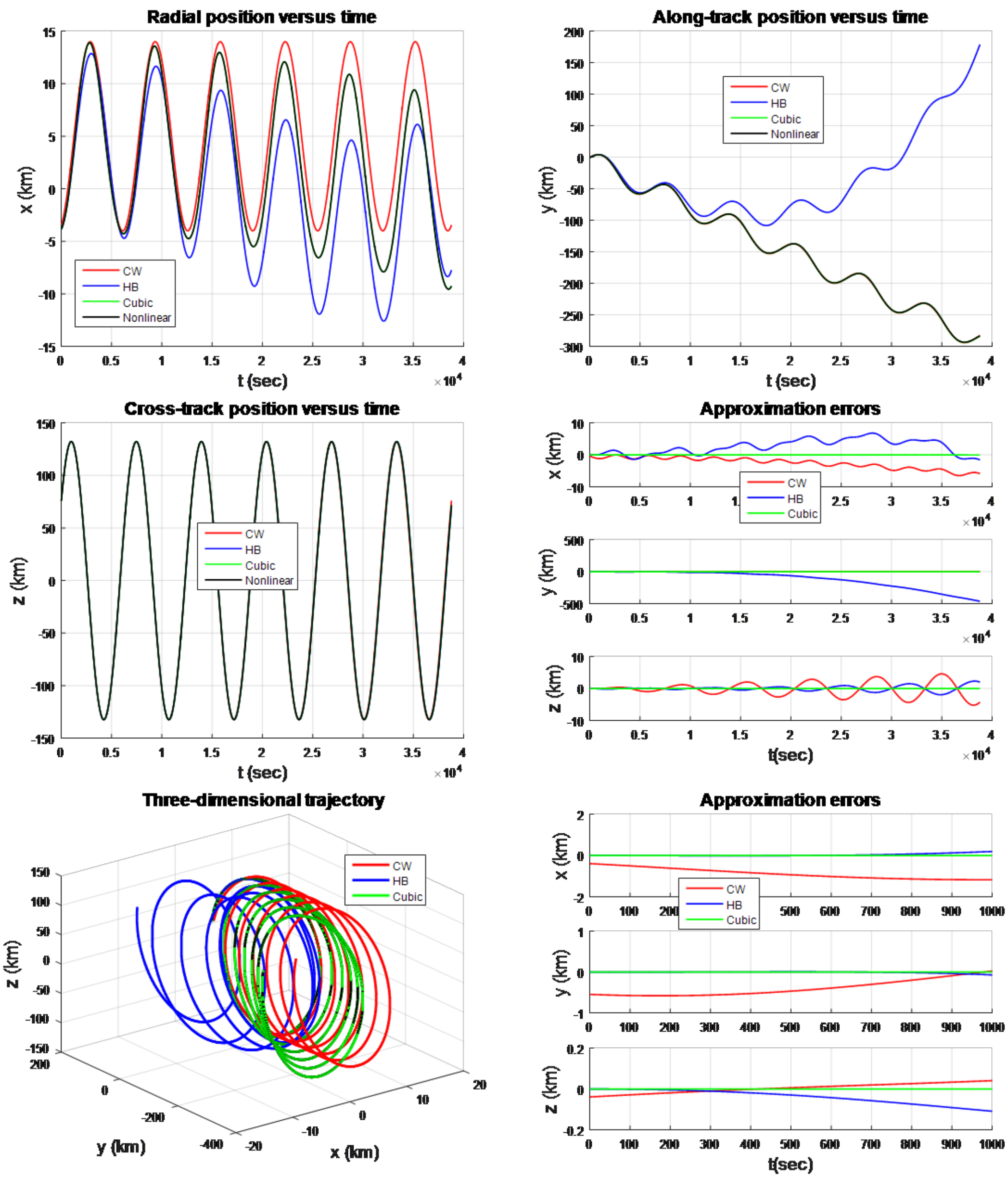

Figure 3.5: Unbounded Motion Propagation Trajectories and Approximation Errors for Calibrated Generating Solution

Chapter 4

Development of Averaging Model of Spacecraft Relative Motion Using Averaging Method

The formulation of the gravitational three-body problem as a perturbation of the two-body problem by Lagrange in the late 18th century marked the beginning of the use of averaging method. The method became one of the classical methods in analyzing nonlinear oscillations after series of researches by Krylov, Bogoliubov, Mitropolsky etc. in 1930s. The method is fairly general, thereby making it applicable to large number of nonlinear dynamical systems and very useful because it is not restricted to periodic solutions [6,13,14,15]. It can be used to obtain an approximate simplified system and to investigate the stability and bifurcation of their equilibria (corresponding to periodic motions in the original system) [19,20,21]. In this chapter, approximate analytic solutions of the cubic approximation model of the spacecraft relative equation is obtained via averaging method.

4.1 Cubic Approximation Model of Spacecraft Relative Motion

4.1.1 Cubic Model

In Cartesian coordinate the cubic approximation model of spacecraft relative motion is given by (Ref [5,6])

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = -\tfrac{3}{R}n^2x^2 + \tfrac{3}{2R}n^2y^2 + \tfrac{3}{2R}n^2z^2 - \tfrac{6}{R^2}n^2xy^2 - \tfrac{6}{R^2}n^2xz^2 + \tfrac{4}{R^2}n^2x^3 \\
&\ddot{y} + 2n\dot{x} = \tfrac{3}{R}n^2yx - \tfrac{6}{R^2}n^2yx^2 + \tfrac{3}{2R^2}n^2yz^2 + \tfrac{3}{2R^2}n^2y^3 \\
&\ddot{z} + n^2z = \tfrac{3}{R}n^2zx - \tfrac{6}{R^2}n^2zx^2 + \tfrac{3}{2R^2}n^2zy^2 + \tfrac{3}{2R^2}n^2z^3
\end{aligned} \tag{4.1}$$

Eqs. (4.1) can be expressed as

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = \varepsilon f\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right)\\
&\ddot{y} + 2n\dot{x} = \varepsilon g\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right)\\
&\ddot{z} + n^2z = \varepsilon h\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right)
\end{aligned} \tag{4.2}$$

where,

$$\begin{aligned}
&f\left(x, y, \dot{x}, \dot{y}\right) = -2Rx^2 + Ry^2 + Rz^2 - 4xy^2 - 4xz^2 - \tfrac{8}{3}x^3\\
&g\left(x, y, \dot{x}, \dot{y}\right) = 2Ryx - 4yx^2 + yz^2 - y^3\\
&h\left(x, y, \dot{x}, \dot{y}\right) = 2Rzx - 4zx^2 + zy^2 - z^3
\end{aligned} \tag{4.3}$$

and ρ is the reltive position vector, n is the mean motion and R is the chief orbital radius. The quantity $\varepsilon = 3\mu/2R^5$ is the perturbation parameter. The small parameter characterizes the closeness of the system to a linear conservative one, while f, g and h are the nonlinear functions. The nonlinear terms are: x^2, y^2,z^2, xy^2, xz^2 and x^3 for the radial, yx, yx^2, yz^2, y^3 for the along-track and zx, zx^2, zy^2, z^3 for the cross-track.

4.1.2 Bounded Solution

For $\varepsilon = 0$, that is no nonlinear terms, the system in Eqs. (4.2) reduced to the Hill-Clohessy-Wilshire Equations.

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = 0\\
&\ddot{y} + 2n\dot{x} = 0\\
&\ddot{z} + n^2z = 0
\end{aligned} \tag{4.4}$$

Using linear theory, the HCW equations have the solutions [3,4]

$$\begin{aligned}
&x = A\cos\left(nt + \alpha\right) + x_{off}\\
&y = -2A\sin\left(nt + \alpha\right) - \tfrac{3}{2}nx_{off}t + y_{off}\\
&z = B\cos\left(nt + \beta\right)
\end{aligned} \tag{4.5}$$

The derivatives of these equations are

$$\begin{aligned}
\dot{x} &= -nA\sin(nt+\alpha) \\
\dot{y} &= -2nA\cos(nt+\alpha) - \tfrac{3}{2}nx_{off} \\
\dot{z} &= -nB\cos(nt+\beta)
\end{aligned} \tag{4.6}$$

where A and B are the amplitudes,α and β are the phases, x_{off} and y_{off} are the radial and along-track offsets. In order to obtain bounded relative motion the secular growth in along-track must be eliminated, and this can be done by setting x_{off} to be zero. Using this fact, we have

$$\begin{aligned}
x &= A\cos(nt+\alpha) \\
y &= -2A\sin(nt+\alpha) + y_{off} \\
z &= B\cos(nt+\beta)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\dot{x} &= -nA\sin(nt+\alpha) \\
\dot{y} &= -2nA\cos(nt+\alpha) \\
\dot{z} &= -nB\cos(nt+\beta)
\end{aligned} \tag{4.8}$$

The bounded periodic solutions of HCW equations, shown in Eqs. (4.7) and (4.8), are suitable for formation flying missions.

4.2 Formulation of Slowly Varying Parameters

In this section, slowly varying parameters are formulated using the HCW solutions as the generating solutions. The following relationships are obtained using Eqs. (4.5) and (4.6).

$$\left.\begin{aligned}
x &= -\tfrac{\dot{y}}{2n} + \tfrac{n}{2}x_{off}, \dot{x} = \tfrac{n}{2}\left(y - y_{off} + \tfrac{3}{2}nx_{off}t\right) \\
y &= \tfrac{2\dot{x}}{n} - \tfrac{3}{2}nx_{off}t + y_{off}, \dot{y} = -2nx + \tfrac{n}{2}x_{off}
\end{aligned}\right\} \tag{4.9}$$

Substituting Eqs. (4.9) into Eqs. (4.2) we have the following three second order equations.

$$\begin{aligned} \ddot{x} + n^2 x &= \varepsilon f\left(x, y, z\right) + n^2 x_{off} \\ \ddot{y} + n^2 y &= \varepsilon g\left(x, y, z\right) + n^2 y_{off} - \tfrac{3}{2} n^3 x_{off} t \\ \ddot{z} + n^2 z &= \varepsilon h\left(x, y, z\right) \end{aligned} \tag{4.10}$$

4.2.1 Formulation of a System Amenable for Averaging

Using additive decomposition, the radial, along-track and cross-track slowly varying parameters are formulated as follows.

(i) Radial Direction

The coordinates x and y can be additively decomposed into $x = x_1 + x_2$ and $y = y_1 + y_2$. Upon substitution into the radial equation we have

$$\ddot{x} + n^2 x = \varepsilon f\left(x, y, z\right) + n^2 x_{off} \tag{4.11}$$

we have

$$\ddot{x}_1 + \ddot{x}_2 + n^2\left(x_1 + x_2\right) = \varepsilon f\left(x_1 + x_2, y_1 + y_2, z, t\right) + n^2 x_{off} \tag{4.12}$$

We can select x_1 such that

$$\ddot{x}_1 + n^2 x_1 = n^2 x_{off} \tag{4.13}$$

Assuming the particular solution of Eq. (4.12) to be $x_1(t) = c_1$ and upon substitution we have

$$n^2 c_1 = n^2 x_{off}, c_1 = x_{off}, x_1 = x_{off} \tag{4.14}$$

Using Eq. (4.5), $x_2(t)$ can be obtained as

$$x = x_1(t) + x_2(t) = A\cos\left(nt + \alpha\right) + x_{off}, x_2(t) = A\cos\left(nt + \alpha\right) \tag{4.15}$$

Similarly, from Eq. (4.12) we can make the selection

$$\ddot{x}_2 + n^2 x_2 = \varepsilon f\left(x_1 + x_2, y_1 + y_2, z, t\right) \tag{4.16}$$

(ii) Along-track Direction

Let $y = y_1 + y_2$, then

$$\begin{aligned} \ddot{y}_1 + n^2 y_1 &= n^2 y_{off} - \tfrac{3}{2} n^3 x_{off} t \\ \ddot{y}_2 + n^2 y_2 &= \varepsilon g\left(x_1 + x_2, y_1 + y_2, z, t\right) \end{aligned} \tag{4.17}$$

Let the particular solution of the first part of Eqs. (4.17) be written as $y_1 = c_1 + c_2 t$ and upon substitution we have

$$c_1 = y_{off}, c_2 = -\frac{3}{2} n x_{off} t, y_1 = y_{off} - \frac{3}{2} n x_{off} t \tag{4.18}$$

Therefore, $y_2(t)$ can be obtained as

$$y = y_1(t) + y_2(t) = -2A\sin\left(nt + \alpha\right) - \frac{3}{2} n x_{off} t + y_{off}, y_2(t) = -2A\sin\left(nt + \alpha\right) \tag{4.19}$$

(iii) Cross-track Direction

For the cross-track direction

$$\ddot{z} + n^2 z = \varepsilon h\left(x_1 + x_2, y_1 + y_2, z, t\right) \tag{4.20}$$

The homogeneous equation, $\ddot{z} + n^2 z = 0$, has the solution

$$z = B\cos\left(nt + \beta\right) \tag{4.21}$$

From Eq. (4.16), (4.17) and (4.20) we have

$$\begin{aligned}
\ddot{x}_2 + n^2 x_2 &= \varepsilon f\left(x_1 + x_2, y_1 + y_2, z, t\right) \\
\ddot{y}_2 + n^2 y_2 &= \varepsilon g\left(x_1 + x_2, y_1 + y_2, z, t\right) \\
\ddot{z} + n^2 z &= \varepsilon h\left(x_1 + x_2, y_1 + y_2, z, t\right)
\end{aligned} \tag{4.22}$$

This system of equation is amenable for averaging.

4.2.2 Development of General Solution of the Averaged Equations

Using the Krylov and Bogoliubov approach [13,15] that, for small ε, the integration constants A, B, α and θ are slowly varying functions of time, that is,

$$A \to A(t), B \to B(t), \alpha \to \alpha(t), \theta \to \theta(t) \tag{4.23}$$

then, using Eqs. (4.7) and (4.8), the generating solutions take the form

$$\begin{aligned}
x(t) &= A(t) cos\left(nt + \alpha(t)\right) + x_{off} \\
y(t) &= -2A(t)\sin\left(nt + \alpha(t)\right) - \tfrac{3}{2} n x_{off} t + y_{off} \\
z(t) &= B(t) cos\left(nt + \beta(t)\right)
\end{aligned} \tag{4.24}$$

with the derivatives

$$\begin{aligned}
\dot{x}(t) &= -nA(t)\sin\left(nt + \alpha(t)\right) \\
\dot{y}(t) &= -2nA(t)\cos\left(nt + \alpha(t)\right) - \tfrac{3}{2} n x_{off} \\
\dot{z}(t) &= -nB(t)\sin\left(nt + \beta(t)\right)
\end{aligned} \tag{4.25}$$

Using Butenin approach [14], differentiating the generating solutions yields the systems

$$\begin{aligned}
&\dot{\mathrm{A}}\cos\left(nt + \alpha\right) - A\dot{\alpha}\sin\left(nt + \alpha\right) = 0 \\
&-2\dot{\mathrm{A}}\sin\left(nt + \alpha\right) - 2\mathrm{A}\dot{\alpha}\cos\left(nt + \alpha\right) = 0 \\
&\dot{\mathrm{B}}\cos\left(nt + \beta\right) - \mathrm{B}\dot{\beta}\sin\left(nt + \beta\right) = 0
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
-n\dot{\mathrm{A}}\sin(nt+\alpha) - n\mathrm{A}\dot{\alpha}\cos(nt+\alpha) &= \varepsilon f^* \\
-2n\dot{\mathrm{A}}\cos(nt+\alpha) + 2n\mathrm{A}\dot{\alpha}\sin(nt+\alpha) &= \varepsilon g^* \\
-n\dot{\mathrm{B}}\sin(nt+\beta) - n\mathrm{B}\dot{\beta}\cos(nt+\beta) &= \varepsilon h^*
\end{aligned} \tag{4.27}$$

where,

$$\begin{gathered}
f^* = f\left(x_1+x_2, y_1+y_2, z, t\right) \\
= f\left(A\cos(nt+\alpha) + x_{off}, -2A\sin(nt+\alpha) - \tfrac{3}{2}nx_{off}t + y_{off}, B\cos(nt+\beta)\right) \\
g^* = g\left(x_1+x_2, y_1+y_2, z, t\right) \\
= g\left(A\cos(nt+\alpha) + x_{off}, -2A\sin(nt+\alpha) - \tfrac{3}{2}nx_{off}t + y_{off}, B\cos(nt+\beta)\right) \\
h^* = h\left(x_1+x_2, y_1+y_2, z, t\right) \\
= h\left(A\cos(nt+\alpha) + x_{off}, -2A\sin(nt+\alpha) - \tfrac{3}{2}nx_{off}t + y_{off}, B\cos(nt+\beta)\right)
\end{gathered} \tag{4.28}$$

The first parts of Equations (4.26) and (4.27) are the radial equations, the second parts are the along-track equations and the third parts are the cross-track equations. Substitution of the Eqs. (4.5) and (4.6) into Eq. (4.3) yields

$$\begin{aligned}
& f^* = -\tfrac{8}{3}A^3\cos^3\phi + \left(-2A^2R - 8A^2x_{off}\right)\cos^2\phi - 16A^3\cos\phi\sin^2\phi \\
& +2\left(4A^2R - 16A^2x_{off}\right)\sin^2\phi - 4AB^2\cos\phi\cos^2\theta + \left(RB^2 - 4B^2x_{off}\right)\cos^2\theta \\
& + \left(16A^2y_{off} - 24A^2ntx_{off}\right)\cos\phi\sin\phi \\
& + \left(-9An^2t^2x_{off}^2 + 12Antx_{off}y_{off} - 8Ax_{off}^2 - 4ARx_{off} - 4Ay_{off}^2\right)\cos\phi \\
& + \left(16Ax_{off}y_{off} - 4ARy_{off} - 24Antx_{off}^2 + 6ARntx_{off}\right)\sin\phi \\
& + \left(16Ax_{off}y_{off} - 4ARy_{off} - 24Antx_{off}^2 + 6ARntx_{off}\right)\sin\phi \\
& -9n^2t^2x_{off}^3 + \tfrac{9}{4}Rn^2t^2x_{off}^2 + 12ntx_{off}^2y_{off} - 3Rntx_{off}y_{off} \\
& -\tfrac{8}{3}x_{off}^3 - 2Rx_{off}^2 - 4x_{off}y_{off}^2 + Ry_{off}^2
\end{aligned} \tag{4.29}$$

$$
\begin{aligned}
g^* &= 8A^3\sin^3\phi + (6A^2ntx_{off} - 4A^2y_{off})\cos^2\phi \\
&+ (18A^2ntx_{off} - 12A^2y_{off})\sin^2\phi + \left(B^2y_{off} - \tfrac{3}{2}B^2ntx_{off}\right)\cos^2\theta \\
&-2AB^2\cos^2\theta\sin\phi + 8A^3\cos^2\phi\sin\phi + (16A^2x_{off} - 4A^2R)\cos\phi\sin\phi \\
&+ \left(2ARy_{off} - 8Ax_{off}y_{off} + 12Antx_{off}^2 - 3ARntx_{off}\right)\cos\phi \\
&+ \left(\tfrac{27}{2}An^2t^2x_{off}^2 - 18Antx_{off}y_{off} + 8Ax_{off}^2 - 4ARx_{off} + 6Ay_{off}^2\right)\sin\phi \\
&+\tfrac{27}{8}n^3t^3x_{off}^3 - \tfrac{27}{4}n^2t^2x_{off}^2y_{off} + 6ntx_{off}^3 - 3Rntx_{off}^2 + \tfrac{9}{2}ntx_{off}y_{off}^2 \\
&-4x_{off}^2y_{off} + 2Rx_{off}y_{off} - y_{off}^3
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
h^* &= -4A^2B\cos^2\phi\cos\theta + (2ABR - 8ABx_{off})\cos\phi\cos\theta + 4A^2B\sin^2\phi\cos\theta \\
&+ (6ABntx_{off} - 4ABy_{off})\sin\phi\cos\theta - B^3\cos^3\theta \\
&+ \left(\tfrac{9}{4}Bn^2t^2x_{off}^2 - 3Bntx_{off}y_{off} - 4Bx_{off}^2 + 2BRx_{off} + By_{off}^2\right)\cos\theta
\end{aligned}
\tag{4.31}
$$

Assuming that in the radial equations part of Eqs. (4.26) and (4.27), $\dot{\mathrm{A}}_r = \dot{\mathrm{A}}$,$\dot{\alpha}_r = \dot{\alpha}$, in the along-track part, $\dot{\mathrm{A}}_a = \dot{\mathrm{A}}$,$\dot{\alpha}_a = \dot{\alpha}$, and in the cross-track part, $\dot{\mathrm{B}}_c = \dot{\mathrm{B}}$,$\dot{\beta}_c = \dot{\beta}$, then Eqs. (4.26) and (4.27) are a system of equations in the variables $\dot{\mathrm{A}}_r, \dot{\mathrm{A}}_a, \dot{\mathrm{B}}_c, A\dot{\alpha}_r, A\dot{\alpha}_a, \dot{\beta}_c$ and can be represented as

$$
\begin{bmatrix}
\cos\phi & 0 & 0 & -\sin\phi & 0 & 0 \\
0 & -2\sin\phi & 0 & 0 & -2\cos\phi & 0 \\
0 & 0 & \cos\theta & 0 & 0 & -\sin\theta \\
-n\sin\phi & 0 & 0 & -n\cos\phi & 0 & 0 \\
0 & -2n\cos\phi & 0 & 0 & 2n\sin\phi & 0 \\
0 & 0 & -n\sin\theta & 0 & 0 & -n\cos\theta
\end{bmatrix}
\begin{bmatrix}
\dot{\mathrm{A}}_r \\ \dot{\mathrm{A}}_a \\ \dot{\mathrm{B}}_c \\ A\dot{\alpha}_r \\ A\dot{\alpha}_a \\ \dot{\beta}_c
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \varepsilon f^* \\ \varepsilon g^* \\ \varepsilon h^*
\end{bmatrix}
\tag{4.32}
$$

The system in Eqs. (4.32) have the determinant

$$\Delta = \begin{vmatrix} \cos\phi & 0 & 0 & -\sin\phi & 0 & 0 \\ 0 & -2\sin\phi & 0 & 0 & -2\cos\phi & 0 \\ 0 & 0 & \cos\theta & 0 & 0 & -\sin\theta \\ -n\sin\phi & 0 & 0 & -n\cos\phi & 0 & 0 \\ 0 & -2n\cos\phi & 0 & 0 & 2n\sin\phi & 0 \\ 0 & 0 & -n\sin\theta & 0 & 0 & -n\cos\theta \end{vmatrix} = -4n^3 \tag{4.33}$$

where $\phi = nt + \alpha$ and $\theta = nt + \beta$. After substituting the right hand side of the system of equations for the elements of the first to sixth columns we get

$$\begin{aligned} &\Delta_1 = 4ef^* n^2 \sin\phi, \Delta_2 = 2\varepsilon g^* n^2 \cos\phi, \Delta_3 = 4\varepsilon h^* n^2 \sin\theta \\ &\Delta_4 = 4\varepsilon f^* n^2 \cos\phi, \Delta_5 = -2\varepsilon g^* n^2 \sin\phi, \Delta_6 = 4\varepsilon h^* n^2 \cos\theta \end{aligned} \tag{4.34}$$

Therefore, for the radial direction

$$\begin{aligned} \tfrac{dA_r}{dt} &= \tfrac{\Delta_1}{\Delta} = -\tfrac{\varepsilon}{n} f^* \sin\phi \\ A\tfrac{d\alpha_r}{dt} &= \tfrac{\Delta_3}{\Delta} = -\tfrac{\varepsilon}{n} f^* \cos\phi \end{aligned} \tag{4.35}$$

for the along-track direction

$$\begin{aligned} \tfrac{dA_a}{dt} &= \tfrac{\Delta_2}{\Delta} = -\tfrac{\varepsilon}{2n} g^* \cos\phi \\ A\tfrac{d\alpha_a}{dt} &= \tfrac{\Delta_3}{\Delta} = \tfrac{\varepsilon}{2n} g^* \sin\phi \end{aligned} \tag{4.36}$$

and for the cross-track direction

$$\begin{aligned} \tfrac{dB_c}{dt} &= \tfrac{\Delta_3}{\Delta} = -\tfrac{\varepsilon}{n} h^* \sin\theta \\ B\tfrac{d\beta_c}{dt} &= \tfrac{\Delta_6}{\Delta} = -\tfrac{\varepsilon}{n} h^* \cos\theta \end{aligned} \tag{4.37}$$

Averaging the right sides of Eqs. (4.35), (4.36) and (4.37) over the periods $2\pi/n$, $2\pi/n$ and $2\pi/n$ we get approximate equations for the determination of $\mathrm{A}_r, \alpha_r, \mathrm{A}_a, \alpha_a, \mathrm{B}_c, \beta_c$ for each of the radial, along-track and cross-track equation as follows.

(i) Radial Direction

$$\begin{aligned} \frac{dA_r}{dt} &= -\frac{\varepsilon}{n}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\phi d\phi d\theta dt \\ \frac{d\alpha_r}{dt} &= -\frac{\varepsilon}{nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\phi d\phi d\theta dt \end{aligned} \tag{4.38}$$

Eq. (4.38) is the radial approximate equation. Substituting Eqs. (4.29), (4.30) and (4.31) into Eq. (4.38) and evaluating the triple integral gives

$$\begin{aligned} \dot{\mathrm{A}}_r &= \frac{A\varepsilon}{n}\left(2y_{off} - 9.4248nx_{off}\right)\left(R - 4x_{off}\right) \\ \dot{\alpha}_r &= \frac{\varepsilon}{n}\left(A^2 + B^2 + 59.218n^2x_{off}^2 - 18.85nx_{off}y_{off} - 4x_{off}^2 + 2Rx_{off} + 2y_{off}^2\right) \end{aligned} \tag{4.39}$$

Solving Eq. (4.39) yields

$$\begin{aligned} A_r &= A_0 e^{\left(\frac{\varepsilon}{n}\left(2y_{off}-9.4248nx_{off}\right)\left(R-4x_{off}\right)\right)t} \\ \alpha_r &= \frac{\varepsilon}{n}\left(A^2 + B^2 + 59.218n^2x_{off}^2 - 18.85nx_{off}y_{off} - 4x_{off}^2 + 2Rx_{off} + 2y_{off}^2\right)t + \alpha_0 \\ &= \alpha_1 t + \alpha_0 \end{aligned} \tag{4.40}$$

where A_r and α_r are the new radial amplitude and phase angle.

(ii) Along-track Direction

$$\begin{aligned} \frac{dA_a}{dt} &= -\frac{\varepsilon}{2n}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\phi d\phi d\theta dt \\ \frac{d\alpha_a}{dt} &= \frac{\varepsilon}{2nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\phi d\phi d\theta dt \end{aligned} \tag{4.41}$$

Eq. (4.41) is the along-track approximate equation. Substituting Eqs. (4.29), (4.30) and (4.31) into Eq. (4.41) and evaluating the triple integral simplifies to

$$\begin{aligned} \dot{\mathrm{A}}_a &= -\frac{0.25A\varepsilon}{n}\left(2y_{off} - 9.4248nx_{off}\right)\left(R - 4x_{off}\right) \\ \dot{\alpha}_a &= -\frac{0.25\varepsilon}{n}\left(4A^2 + B^2 + 177.65n^2x_{off}^2 - 56.549nx_{off}y_{off} - 8x_{off}^2 + 4Rx_{off} + 6y_{off}^2\right) \end{aligned} \tag{4.42}$$

and has the solution

$$\begin{aligned}
&A_a = A_0 e^{-\frac{0.25\varepsilon}{n}\left(2y_{off}-9.4248nx_{off}\right)\left(R-4x_{off}\right)t} \\
&\alpha_a = -\tfrac{0.25\varepsilon}{n}\left(4A^2 + B^2 + 177.65n^2x_{off}^2 - 56.549nx_{off}y_{off} - 8x_{off}^2 + 4Rx_{off} + 6y_{off}^2\right)t + \alpha_0 \\
&= \alpha_2 t + \alpha_0
\end{aligned} \tag{4.43}$$

where A_a and α_a are the new along-track amplitude and phase angle.

(iii) Cross-track Direction

$$\begin{aligned}
\frac{dB_c}{dt} &= -\frac{\varepsilon}{n}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \sin\theta d\phi d\theta dt \\
\frac{d\beta_c}{dt} &= -\frac{\varepsilon}{nB}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \cos\theta d\phi d\theta dt
\end{aligned} \tag{4.44}$$

Eq. (4.44) is the cross-track approximate equation. Substituting Eqs. (4.29), (4.30) and (4.31) into Eq. (4.44) and evaluating the triple integral gives

$$\begin{gathered}
\dot{\mathrm{B}}_{\mathrm{c}} = 0 \\
\dot{\beta}_c = -\tfrac{0.125\varepsilon}{n}\left(3B^2 + 118.44n^2x_{off}^2 - 37.699nx_{off}y_{off} - 16x_{off}^2 + 8Rx_{off} + 4y_{off}^2\right)
\end{gathered} \tag{4.45}$$

and has the solutions

$$\begin{gathered}
B_c = B_0 \\
\beta_c = -\tfrac{0.125\varepsilon}{n}\left(3B^2 + 118.44n^2x_{off}^2 - 37.699nx_{off}y_{off} - 16x_{off}^2 + 8Rx_{off} + 4y_{off}^2\right)t + \beta_0 \\
= \beta_1 t + \beta_0
\end{gathered} \tag{4.46}$$

where B_c and β_c are the new cross-track amplitude and phase angle. To the first approximation, the averaged solutions are

$$\begin{aligned}
x_2(t) &\approx A_r(t)\cos\left[\left(n + \alpha_r\right)t + \alpha_0\right] \\
y_2(t) &\approx -2A_a(t)\sin\left[\left(n + \alpha_a\right)t + \alpha_0\right] \\
z(t) &\approx B_c(t)\cos\left[\left(n + \beta_c\right)t + \beta_0\right]
\end{aligned} \tag{4.47}$$

Therefore, the general solution is

$$\begin{aligned} x(t) &= x_1 + x_2 \approx A_r(t)\cos\left[\left(n+\alpha_r\right)t+\alpha_0\right] + x_{off} \\ y(t) &= y_1 + y_2 \approx -2A_a(t)\sin\left[\left(n+\alpha_a\right)t+\alpha_0\right] - \tfrac{3}{2}nx_{off}t + y_{off} \\ z(t) &\approx B(t)\cos\left[\left(n+\beta\right)t+\beta_0\right] \end{aligned} \tag{4.48}$$

4.3 Equivalent Linear Equations in Radial, Along - Track and Cross-Track Directions

In this section, equivalent linear expressions are developed for radial, along-track and cross-track directions. The approach here is similar to that in Ogundele et al [6]. In section 4.2.2, it was shown that a solution

$$\begin{aligned} x(t) &= A(t)cos\left(nt+\alpha(t)\right) + x_{off} \\ y(t) &= -2A(t)\sin\left(nt+\alpha(t)\right) - \tfrac{3}{2}nx_{off}t + y_{off} \\ z(t) &= B(t)cos\left(nt+\beta(t)\right) \end{aligned} \tag{4.49}$$

of the equations

$$\begin{aligned} \ddot{x} + n^2x &= \varepsilon f\left(x,y,z\right) + n^2x_{off} \\ \ddot{y} + n^2y &= \varepsilon g\left(x,y,z\right) + n^2y_{off} - \tfrac{3}{2}n^3x_{off}t \\ \ddot{z} + n^2z &= \varepsilon h\left(x,y,z\right) \end{aligned} \tag{4.50}$$

has the equations of the first approximation in the form:

(a) Radial Direction

$$\begin{aligned} \tfrac{dA_r}{dt} &= -\tfrac{\varepsilon}{n}\left(\tfrac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^*\sin\phi d\phi d\theta dt \\ \tfrac{d\alpha}{dt} &= -\tfrac{\varepsilon}{nA}\left(\tfrac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^*\cos\phi d\phi d\theta dt \end{aligned} \tag{4.51}$$

(b) Along-track Direction

$$\begin{aligned} \tfrac{dA_a}{dt} &= -\tfrac{\varepsilon}{2n}\left(\tfrac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^*\cos\phi d\phi d\theta dt \\ \tfrac{d\alpha}{dt} &= \tfrac{\varepsilon}{2nA}\left(\tfrac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^*\sin\phi d\phi d\theta dt \end{aligned} \tag{4.52}$$

(c) Cross-track Direction

$$\begin{aligned}\frac{dB}{dt} &= -\frac{\varepsilon}{n}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \sin\theta d\phi d\theta dt \\ \frac{d\beta}{dt} &= -\frac{\varepsilon}{nB}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \cos\theta d\phi d\theta dt\end{aligned} \tag{4.53}$$

The equivalent linearized equations for radial, along-track and cross-track directions are derived as follows.

4.3.1 Radial Direction Equivalent Linear Equation

Using the notation

$$\omega_e(A_r) = n - \left(-\frac{\varepsilon}{nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\phi d\phi d\theta dt\right) \tag{4.54}$$

then A and θ must satisfy the following equations

$$\begin{aligned}\frac{dA_r}{dt} &= -\frac{\varepsilon}{n}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\phi d\phi d\theta dt \\ \frac{d\phi}{dt} &= \omega_e(A_r)\end{aligned} \tag{4.55}$$

Taking the square of the expression (4.54) and neglect the terms containing ε^2 as a factor, then we get

$$\omega_e^2(A_r) = n^2 + \frac{2\varepsilon}{A}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\phi d\phi d\theta dt \tag{4.56}$$

Defining a new function of the amplitude as

$$h(A_r) = \frac{\varepsilon}{nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\phi d\phi d\theta dt = -\frac{1}{A}\frac{dA}{dt} \tag{4.57}$$

Eqs. (4.55) can be rewritten as

$$\begin{aligned}\frac{dA_r}{dt} &= -Ah(A_r) \\ \frac{d\phi}{dt} &= \omega_e(A_r)\end{aligned} \tag{4.58}$$

By differentiating $x(t) = A(t)cos\phi + x_{off}$ we get

$$\frac{dx}{dt} = \frac{dA}{dt}\cos\phi - A\frac{d\phi}{dt}\sin\phi = h(A)A\cos\phi - A\omega_e(A)\sin\phi \tag{4.59}$$

Once again, differentiating Eq. (4.59) results to

$$\begin{aligned} \tfrac{d^2x}{dt^2} = \tfrac{dh}{dA}\tfrac{dA}{dt}\cos\phi + h\tfrac{dA}{dt}\cos\phi - hA\tfrac{d\phi}{dt}\sin\phi - \tfrac{dA}{dt}\omega_e\sin\phi \\ -A\tfrac{d\omega_e}{dA}\tfrac{dA}{dt}\sin\phi - A\omega_e\tfrac{d\phi}{dt}\cos\phi \end{aligned} \tag{4.60}$$

Making use of Eqs. (4.59) and (4.60) we have

$$\frac{d^2x}{dt^2} + 2h(A)\frac{dx}{dt} + \omega_e^2\left(x - x_{off}\right) = \mathcal{O}\left(\varepsilon^2\right) \tag{4.61}$$

where,

$$\mathcal{O}\left(\varepsilon^2\right) = -h^2(A)\left(x - x_{off}\right) + h(A)A^2\frac{d\omega_e}{dA}\sin\phi + h(A)A\frac{dh(A)}{dA}\cos\theta \tag{4.62}$$

Therefore, it can be asserted that a solution to Eq. (4.61) with an accuracy up to a quantity of order ε^2 in a neighborhood of zero will satisfy the linear differential equation of the form

$$\frac{d^2x}{dt^2} + 2h(A_r)\frac{dx}{dt} + \omega_e^2\left(x - x_{off}\right) = 0 \tag{4.63}$$

The linear oscillator in Eq. (4.63) is the radial direction equivalent system. Since both A and α are functions of time, the derivatives of x are

$$\begin{aligned} \dot{x} &= \dot{\mathrm{A}}\cos\phi - \left(n + \dot{\alpha}\right)\mathrm{A}\sin\phi \\ \ddot{x} &= \left[\ddot{\mathrm{A}} - \left(n + \dot{\alpha}\right)^2\mathrm{A}\right]\cos\phi - 2\left(n + \dot{\alpha}\right)\dot{\mathrm{A}}\sin\phi \end{aligned} \tag{4.64}$$

Using Eqs. (4.63) and (4.64) we have

$$\left[\ddot{\mathrm{A}} - (n+\dot{\alpha})^2\mathrm{A}\right]\cos\phi - 2\,(n+\dot{\alpha})\,\dot{\mathrm{A}}\sin\phi + 2h\dot{\mathrm{A}}\cos\phi - 2h\,(n+\dot{\alpha})\,\mathrm{A}\sin\phi + \omega_e^2\mathrm{A}\cos\phi = 0 \tag{4.65}$$

Collecting sine and cosine terms we have

$$\begin{aligned} &\cos\phi : \ddot{\mathrm{A}} - (n+\dot{\alpha})^2\mathrm{A} + 2h\dot{\mathrm{A}} + \omega_e^2\mathrm{A} = 0 \\ &\sin\phi : \ -2\,(n+\dot{\alpha})\,\dot{\mathrm{A}} - 2h\,(n+\dot{\alpha})\,\mathrm{A} = 0 \end{aligned} \tag{4.66}$$

Using the assumptions on the size $\dot{\mathrm{A}}$, $\dot{\alpha}$ and h that [18]

$$\begin{aligned} &\left|\ddot{\mathrm{A}}\right| \ll \left|\dot{\mathrm{A}}\right| \tfrac{n}{2\pi} \ll |\mathrm{A}| \left(\tfrac{n}{2\pi}\right)^2 \\ &|\dot{\alpha}| \ll n \\ &|h| \ll \tfrac{n}{2\pi} \end{aligned} \tag{4.67}$$

we obtain

$$h = -\left(\frac{\dot{\mathrm{A}}}{\mathrm{A}}\right), \omega_e^2 = (n+\dot{\alpha})^2 \tag{4.68}$$

Eq. (4.63) may be rewritten as

$$\frac{d^2x}{dt^2} - 2\left(\frac{\dot{A}_r}{A}\right)\frac{dx}{dt} + (n+\dot{\alpha})^2\,(x - x_{off}) = 0 \tag{4.69}$$

This is the radial equivalent linear equation which contains correction terms.

4.3.2 Along-track Direction Equivalent Linear Equation

Also, using the notation

$$\omega_e(A_a) = n - \frac{\varepsilon}{2nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\phi \, d\phi \, d\theta \, dt \tag{4.70}$$

then A and ϕ must satisfy the following equations:

$$\begin{aligned} \frac{dA_a}{dt} &= -\frac{\varepsilon}{2n}\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\phi d\phi d\theta dt \\ \frac{d\phi_a}{dt} &= \omega_e(A_a) \end{aligned} \tag{4.71}$$

Taking the square of the expression (4.70) and neglect the terms containing ε^2 as a factor, then we get

$$\omega_e^2(A_a) = n^2 - \frac{\varepsilon}{A}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\phi d\phi d\theta dt \tag{4.72}$$

Defining a new function of the amplitude we have

$$h(A_a) = \frac{\varepsilon}{2nA}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\phi d\phi d\theta dt = -\frac{1}{A}\frac{dA}{dt} \tag{4.73}$$

Eqs. (4.71) can be rewritten as

$$\begin{aligned} \frac{dA_a}{dt} &= -Ah(A_a) \\ \frac{d\phi}{dt} &= \omega_e(A) \end{aligned} \tag{4.74}$$

Differentiating $y(t) = -2A(t)\sin\phi - \frac{3}{2}nx_{off}t + y_{off}$ we get

$$\frac{dy}{dt} = -2\frac{dA}{dt}\sin\phi - 2A\frac{d\phi}{dt}\cos\phi - \frac{3}{2}nx_{off} = 2h(A)A\sin\phi - 2A\omega_e(A)\cos\phi - \frac{3}{2}nx_{off} \tag{4.75}$$

Once again, differentiating Eq. (4.75) results to

$$\begin{aligned} \frac{d^2y}{dt^2} = 2\frac{dh}{dA}\frac{dA}{dt}\sin\phi + 2h\frac{dA}{dt}\sin\phi + 2hA\frac{d\phi}{dt}\cos\phi - 2\frac{dA}{dt}\omega_e\cos\phi \\ -2A\frac{d\omega_e}{dA}\frac{dA}{dt}\cos\phi + 2A\omega_e\frac{d\phi}{dt}\sin\phi \end{aligned} \tag{4.76}$$

Using Eq. (4.74) we can rewrite Eq. (4.76) as

$$\frac{d^2y}{dt^2} + 2h\left(A_a\right)\frac{dy}{dt} + \omega_e^2\left(y + \frac{3}{2}nx_{off}t - y_{off}\right) + 3h\left(A_a\right)nx_{off} = \mathcal{O}\left(\varepsilon^2\right) \tag{4.77}$$

where,

$$\mathcal{O}\left(\varepsilon^2\right) = -h^2\left(A_a\right)\left(y + \frac{3}{2}nx_{off}t - y_{off}\right) - 2h\left(A_a\right)A\frac{dh}{dA}\sin\phi + 2h\left(A_a\right)A^2\frac{d\omega_e}{dA}\cos\phi \quad (4.78)$$

Therefore, it can be asserted that a solution to Eq. (4.77) with an accuracy up to a quantity of order ε^2 in a neighborhood of zero will satisfy the linear differential equation of the form

$$\frac{d^2y}{dt^2} + 2h\left(A_a\right)\frac{dy}{dt} + \omega_e^2\left(y + \frac{3}{2}nx_{off}t - y_{off}\right) + 3h\left(A_a\right)nx_{off} = 0 \quad (4.79)$$

Since both A and ϕ are functions of time, the derivatives of y are

$$\begin{aligned} \dot{y} &= -2\dot{A}\sin\phi - 2\left(n + \dot{\alpha}\right)A\cos\phi - \tfrac{3}{2}nx_{off} \\ \ddot{y} &= -2\left[\ddot{A} - \left(n + \dot{\alpha}\right)^2 A\right]\sin\phi - 4\left(n + \dot{\alpha}\right)\dot{A}\cos\phi \end{aligned} \quad (4.80)$$

From Eqs. (4.79) and (4.80) we have

$$\begin{gathered} -2\left[\ddot{A} - \left(n + \dot{\alpha}\right)^2 A\right]\sin\phi - 4\left(n + \dot{\alpha}\right)\dot{A}\cos\phi \\ +2h\left(A_a\right)\left(-2\dot{A}\sin\phi - 2\left(n + \dot{\alpha}\right)A\cos\phi - \tfrac{3}{2}nx_{off}\right) + \omega_e^2\left(-2A\sin\phi\right) + 3h\left(A_a\right)nx_{off} = 0 \end{gathered} \quad (4.81)$$

Collecting sine and cosine terms we have

$$\begin{aligned} \cos\phi:&\ -4\left(n + \dot{\alpha}\right)\dot{A} - 4h\left(n + \dot{\alpha}\right)A = 0 \\ \sin\phi:&\ -2\left[\ddot{\mathrm{A}} - \left(n + \dot{\alpha}\right)^2\mathrm{A}\right] - 4h\dot{\mathrm{A}} - 2\mathrm{A}\omega_e^2 = 0 \end{aligned} \quad (4.82)$$

The assumptions on the size $\dot{\mathrm{A}}$, $\dot{\alpha}$ and h are that [7]

$$\begin{aligned} &\left|\ddot{\mathrm{A}}\right| \ll \left|\dot{\mathrm{A}}\right|\tfrac{n}{2\pi} \ll \left|\mathrm{A}\right|\left(\tfrac{n}{2\pi}\right)^2 \\ &\left|\dot{\alpha}\right| \ll n \\ &\left|h\right| \ll \tfrac{n}{2\pi} \end{aligned} \quad (4.83)$$

Using these assumptions yields

$$h = -\left(\frac{\dot{\mathrm{A}}}{\mathrm{A}}\right), \omega_e^2 = (n + \dot{\alpha})^2 \tag{4.84}$$

Eq. (4.79) may be rewritten as

$$\frac{d^2y}{dt^2} - 2\left(\frac{\dot{A}}{A}\right)\frac{dy}{dt} + (n + \dot{\alpha})^2\left(y + \frac{3}{2}nx_{off}t - y_{off}\right) - 3\left(\frac{\dot{A}}{A}\right)nx_{off} = 0 \tag{4.85}$$

This is the along-track equivalent linear equation which contains correction terms.

4.3.3 Cross-track Direction Linear Equation

Using the notation

$$\omega_e(B) = n - \left(-\frac{\varepsilon}{nB}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \cos\theta d\phi d\theta dt\right) \tag{4.86}$$

and

$$\begin{aligned} \tfrac{dB}{dt} &= -\tfrac{\varepsilon}{n}\left(\tfrac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \sin\theta d\phi d\theta dt \\ \tfrac{d\theta}{dt} &= \omega_e(B) \end{aligned} \tag{4.87}$$

Taking the square of the expression (4.86) and neglect the terms containing ε^2 as a factor, then we get

$$\omega_e^2(B) = n^2 + \frac{2\varepsilon}{B}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \cos\theta d\phi d\theta dt \tag{4.88}$$

Defining a new function of the amplitude we have

$$h(B) = \frac{\varepsilon}{nB}\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h^* \sin\theta d\phi d\theta dt \frac{dB}{dt} \tag{4.89}$$

Eqs. (4.87) can be rewritten as

$$\begin{aligned} \tfrac{dB}{dt} &= -Bh(B) \\ \tfrac{d\theta}{dt} &= \omega_e(B) \end{aligned} \tag{4.90}$$

Differentiating $z(t) = B(t)cos\theta$ we get

$$\frac{dz}{dt} = \frac{dB}{dt}\cos\theta - B\frac{d\theta}{dt}\sin\theta = h(B)B\cos\theta - B\omega_e(B)\sin\theta \tag{4.91}$$

Once again, differentiating Eq. (4.91) results to

$$\frac{d^2z}{dt^2} = -\omega_e^2 z - 2h\frac{dz}{dt} - h^2 z + hB^2\frac{d\omega_e}{dA}\sin\theta + hB\frac{dh}{dB}\cos\theta \tag{4.92}$$

Therefore, it can be asserted that a solution to Eq. (4.92) with an accuracy up to a quantity of order ε^2 in a neighborhood of zero will satisfy the linear differential equation of the form

$$\frac{d^2z}{dt^2} + 2h(B)\frac{dz}{dt} + \omega_e^2 z = 0 \tag{4.93}$$

Using the assumptions on the size $\dot{B}$, $\dot{\beta}$ and h that [18]

$$\begin{aligned} &\left|\ddot{\mathrm{B}}\right| \ll \left|\dot{\mathrm{B}}\right| \tfrac{n}{2\pi} \ll |\mathrm{B}| \left(\tfrac{n}{2\pi}\right)^2 \\ &\left|\dot{\beta}\right| \ll n \\ &|h| \ll \tfrac{n}{2\pi} \end{aligned} \tag{4.94}$$

Using these assumptions yields

$$h = -\left(\frac{\dot{\mathrm{B}}}{\mathrm{B}}\right), \omega_e^2 = \left(n + \dot{\beta}\right)^2 \tag{4.95}$$

we obtained the linear equation

$$\frac{d^2z}{dt^2} - 2\left(\frac{\dot{\mathrm{B}}}{\mathrm{B}}\right)\frac{dz}{dt} + \left(n + \dot{\beta}\right)^2 z = 0 \tag{4.96}$$

This is the cross-track equivalent linear equation which contains correction terms. Therefore, the equivalent linear equations for the radial, along-track and cross-track are

$$\begin{aligned} &\frac{d^2x}{dt^2} - 2\left(\frac{\dot{A}_r}{A}\right)\frac{dx}{dt} + (n+\dot{\alpha}_r)^2(x - x_{off}) = 0 \\ &\frac{d^2y}{dt^2} - 2\left(\frac{\dot{A}_a}{A}\right)\frac{dy}{dt} + (n+\dot{\alpha}_a)^2\left(y + \frac{3}{2}nx_{off}t - y_{off}\right) - 3\left(\frac{\dot{A}}{A}\right)nx_{off} = 0 \\ &\frac{d^2z}{dt^2} - 2\left(\frac{\dot{\mathrm{B}}}{\mathrm{B}}\right)\frac{dz}{dt} + \left(n+\dot{\beta}\right)^2 z = 0 \end{aligned} \tag{4.97}$$

4.3.4 Bounded Averaging Model

For the bounded relation motion case, in which $x_{off} = 0$, we have the following for the radial, along-track and cross-track:

(a) Radial Direction

$$\begin{gathered} \frac{dA_r}{dt} = \frac{2ARy_{off}\varepsilon}{n}, \frac{d\alpha_r}{dt} = \frac{\varepsilon}{n}\left(A^2 + B^2 + 2y_{off}^2\right) \\ A_r = A_0 e^{\left(\frac{2Ry_{off}\varepsilon}{n}\right)t}, \alpha_r = \frac{\varepsilon}{n}\left(A^2 + B^2 + 2y_{off}^2\right)t + \alpha_0 \\ x(t) = A_r(t)\cos\left[(n+\alpha_r)t + \alpha_0\right] \end{gathered} \tag{4.98}$$

(b) Along-track Direction

$$\begin{gathered} \frac{dA_a}{dt} = -\frac{0.5AR\varepsilon}{n}y_{off}, \frac{d\alpha_a}{dt} = -\frac{0.25\varepsilon}{n}\left(4A^2 + B^2 + 6y_{off}^2\right) \\ A_a(t) = e^{-\frac{0.5}{n}R\varepsilon y_{off}t}, \alpha_a(t) = -\frac{0.25\varepsilon}{n}\left(4A^2 + B^2 + 6y_{off}^2\right)t + \alpha_0 \\ y(t) = -2A_a(t)\sin\left[(n+\alpha_a)t + \alpha_0\right] + y_{off} \end{gathered} \tag{4.99}$$

(c) Cross-track Direction

$$\begin{gathered} \frac{dB}{dt} = 0, \frac{d\beta}{dt} = -\frac{0.125\varepsilon}{n}\left(3B^2 + 4y_{off}^2\right) \\ B(t) = B_0, \beta(t) = -\frac{0.125\varepsilon}{n}\left(3B^2 + 4y_{off}^2\right)t + \beta_0 \\ z(t) = B(t)\cos\left[(n+\beta)t + \beta_0\right] \end{gathered} \tag{4.100}$$

and the equivalent linear equations reduced to

$$\begin{aligned} &\frac{d^2x}{dt^2} - 2\left(\frac{\dot{A}_r}{A}\right)\frac{dx}{dt} + (n+\dot{\alpha}_r)^2 x = 0 \\ &\frac{d^2y}{dt^2} - 2\left(\frac{\dot{A}_a}{A}\right)\frac{dy}{dt} + (n+\dot{\alpha})^2(y - y_{off}) = 0 \\ &\frac{d^2z}{dt^2} - 2\left(\frac{\dot{\mathrm{B}}}{\mathrm{B}}\right)\frac{dz}{dt} + \left(n+\dot{\beta}\right)^2 z = 0 \end{aligned} \tag{4.101}$$

4.4 Numerical Simulations

The solutions obtained through the averaging method and the exact solutions are compared using numerical simulations. Table 4.1 shows the orbital elements of the chief and deputy spacecraft for the First Scenario. In the table, a is the semi-major axis in km, e is the eccentricity, i is the inclination in degree, Ω is the right ascension of ascending node in degree, ω is the argument of periapsis, and f is the true anomaly in degeree.

Table 4.1: First Scenario chief and deputy spacecraft orbital elements

Orbital Elements	Chief Spacecraft	Deputy Spacecraft
a (km)	7500	7500
e	0	0.0003
i (deg)	5	7
Ω (deg)	5	5
ω (deg)	10	15
f (deg)	25	20

4.4.1 First Scenario

In the First Scenario, the chief and deputy spacecraft have the same semi-major axis 7500 km. Figure 4.1 shows relative motion trajectories of the First Scenario.

4.4.2 Second Scenario

In the Second Scenario, the chief and deputy spacecraft have different semi-major axis. The chief semi-major axis is 7500 km while the deputy semi-major axis is 7505 km.

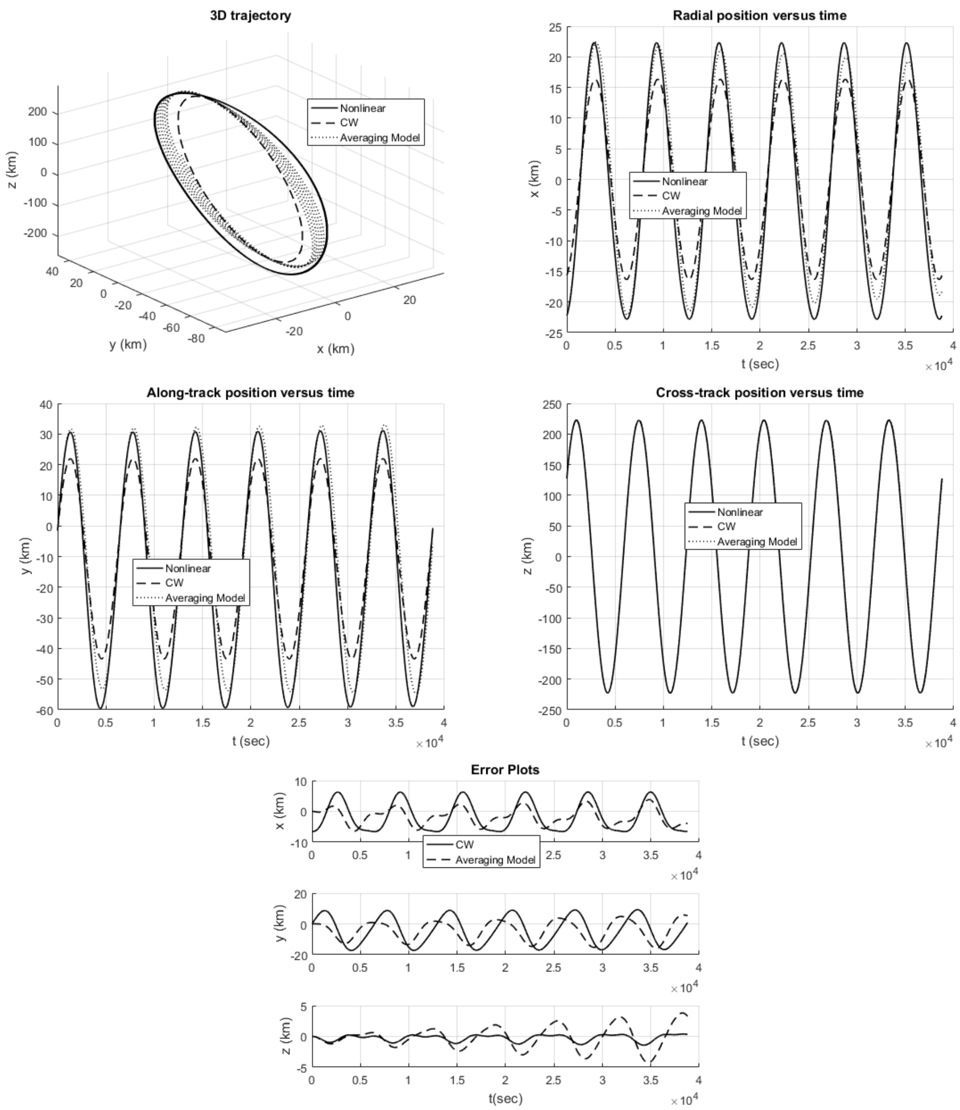

Figure 4.1: First Scenario Trajectories and Error Plots

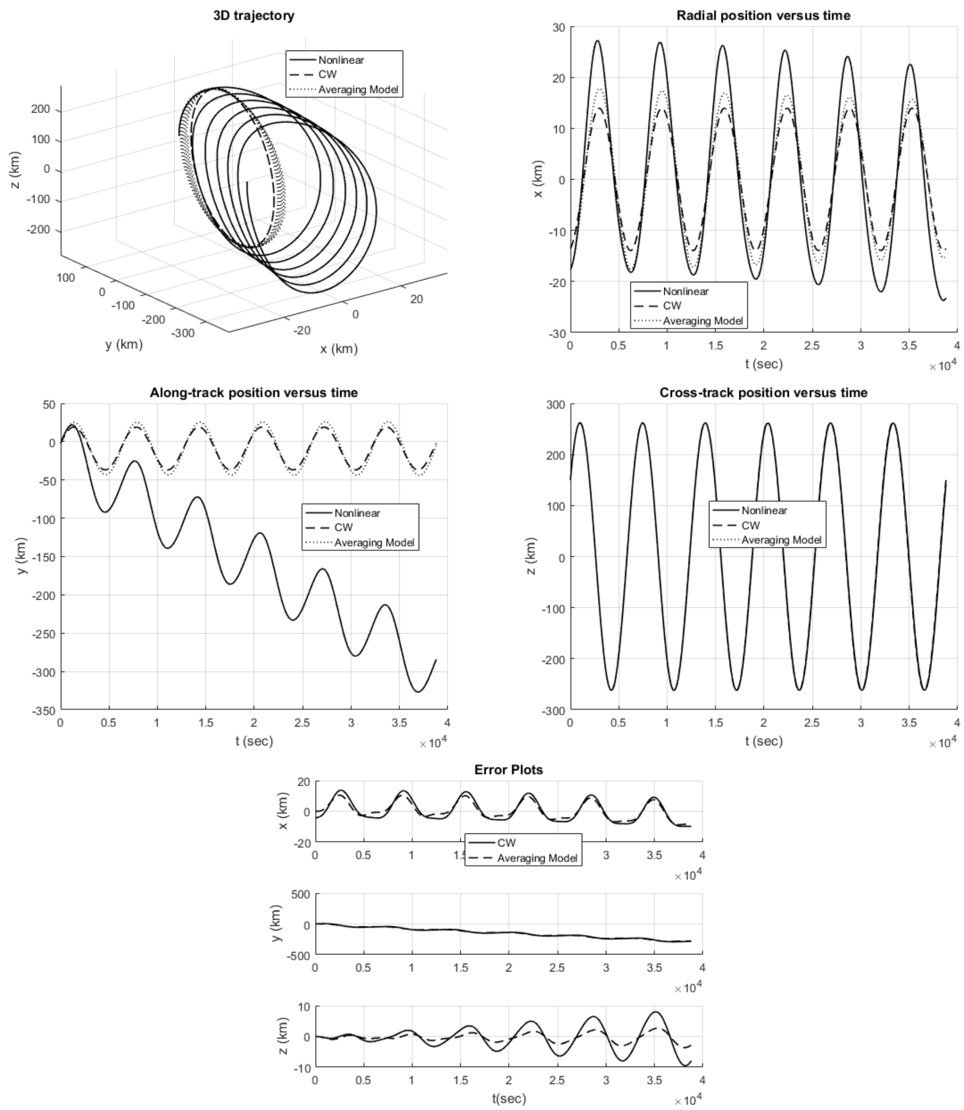

Figure 4.2: Second Scenario Trajectories and Error Plots

Chapter 5

Development of Closed Form Solutions of Nonlinear Spacecraft Relative Motion in Terms of Orbital-Element Differences

The Hill-Clohessy-Wilshire (HCW) linearized equations of motion are generally used to describe the relative motion of deputy spacecraft with respect to the chief spacecraft in circular orbit using Hill frame coordinates (x, y, z) under the assumptions that the spacecraft are very close to each other, the Earth is spherical and the nonlinear terms in the equations of motion may be neglected. The Hill frame coordinates have the disadvantages that their differential equations must be solved before the relative orbit geometry can be obtained and the HCW equations are initial condition dependent valid only if the relative orbit dimension is small in comparison to the chief orbit radius [1,2,3]. Recently, researchers have published a number of papers to show the effectiveness and simplicity of the use of orbit element differences which offers the advantage of better visualization of the relative orbit and slow time variation [9,10,11]. This approach is considered in this chaptert to obtain Abel-type and Riccati-type spacecraft equations of relative motion.

5.1 Nonlinear Equations of Motion for Orbital-Element Differences

Recently, to gain a better insight into the dynamics of the relative motion of deputy spacecraft with respect to the chief spacecraft, much work have been done using orbital elements [9,10,11]. Using Hill coordinate frames the relative orbit is determined with the Cartesian coordinates

$$\mathbf{X} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T \tag{5.1}$$

where $\begin{bmatrix} x & y & z \end{bmatrix}^T$ and $\begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$ are the position and velocity vectors. All the six variables, which vary with time per the Hill-Clohessy-Wilshire (HCW) second order differential equations that govern relative motion [3], must be determined to be able to track the location of the deputy spacecraft would be at a point in time. Rather than tracking all six variables continuously the dynamics is simplified by using Keplerian elements. This has advantage of having five constant orbital elements and one time-varying. Therefore, only one term (true anomaly) which is time varying must be tracked over time. The orbit description is simplified using orbit elements which vary slowly in the presence of perturbation forces such as third body perturbation, atmospheric and solar drag. The dynamics of the relative motion of the deputy with respect to the chief can also be described using the following six orbital element set

$$\mathbf{e} = \begin{bmatrix} a & \theta & i & q_1 & q_2 & \Omega \end{bmatrix}^T \tag{5.2}$$

where, a is the semi-major axis, $\theta = \omega + f$ is the true of latitude, i is the inclination, $q_1 = e\cos\omega$, $q_2 = e\sin\omega$, Ω is the longitude of the ascending node, ω is the argument of periapse, f is the true-anomaly and e is the eccentricity. The relative motion between the deputy and chief can be represented using the orbit element different vector as

$$\delta\mathbf{e} = \mathbf{e}_d - \mathbf{e}_c = \begin{bmatrix} \delta a & \delta\theta & \delta i & \delta q_1 & \delta q_2 & \delta\Omega \end{bmatrix}^T \tag{5.3}$$

Here, $\mathbf{e}_d$ and $\mathbf{e}_c$ are the deputy and chief spacecraft orbital element vectors, respectively. Taking the orbital element set in Eq. (5.2) as the chief spacecraft elements then the deputy spacecraft elements are $a + \delta a$, $\theta + \delta\theta$, $i + \delta i$, $q_1 + \delta q_1$, $q_2 + \delta q_2$, and $\Omega + \delta\Omega$. The linear mapping between the Hill frame coordinates and the orbit element differences is presented in References [3,11]. Using the orbit elements, the orbit radius can be expressed as

$$r = \frac{a\,(1 - q_1^2 - q_2^2)}{(1 + q_1\cos\theta + q_2\sin\theta)} \tag{5.4}$$

with the variation

$$\delta r = \frac{r}{a}\delta a + \frac{V_r}{V_t} r\delta\theta - \frac{r}{p}\left(2aq_1 + r\cos\theta\right)\delta q_1 - \frac{r}{p}\left(2aq_2 + r\sin\theta\right)\delta q_2 \tag{5.5}$$

The chief radial and transverse velocity components are defined by

$$\begin{aligned} V_r &= \dot{r} = \tfrac{h}{p}\left(q_1\sin\theta - q_2\cos\theta\right) \\ V_t &= r\dot{\theta} = \tfrac{h}{p}\left(1 + q_1\cos\theta + q_2\sin\theta\right) \end{aligned} \tag{5.6}$$

In terms of the orbit element differences, the Cartesian coordinate relative position vector components are expressed as

$$\begin{aligned} x &= \delta r \\ y &= r\left(\delta\theta + \cos i\delta\Omega\right) \\ y &= r\left(\sin\theta\delta i - \cos\theta sini\delta\Omega\right) \end{aligned} \tag{5.7}$$

while the relative velocity components are expressed as

$$\begin{aligned} \dot{x} &= -\tfrac{V_r}{2a}\delta a + \left(\tfrac{1}{r} - \tfrac{1}{p}\right)h\delta\theta + \left(V_r aq_1 + h\sin\theta\right)\tfrac{\delta q_1}{p} + \left(V_r aq_2 - h\cos\theta\right)\tfrac{\delta q_2}{p} \\ \dot{y} &= -\tfrac{3V_t}{2a}\delta a - V_r\delta\theta + \left(3V_t aq_1 + 2h\cos\theta\right)\tfrac{\delta q_1}{p} + \left(3V_t aq_2 + 2h\sin\theta\right)\tfrac{\delta q_2}{p} + V_r\cos i\delta\Omega \\ \dot{z} &= \left(V_t\cos\theta + V_r\sin\theta\right)\delta i + \left(V_t\sin\theta - V_r\cos\theta\right) sini\delta\Omega \end{aligned} \tag{5.8}$$

Since $\delta\theta$ is the only time-varying parameter in Eq. (5.3), then the rate of change of the orbit element differences vector, $\delta\mathbf{e}$, is

$$\delta\dot{\mathbf{e}} = \begin{bmatrix} 0 & \delta\dot{\theta} & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{5.9}$$

This gives equations of relative motion of the deputy with respect to the chief in terms of the orbital element differences. The true latitude rate $\dot{\theta}$, using the principle of the conservation of angular momentum h, can be expressed as

$$\dot{\theta} = \frac{h}{r^2} \tag{5.10}$$

Using Eq. (5.4) and the fact that $h = \sqrt{\mu p}$, the difference between the deputy and chief true latitude rates may be expressed as

$$\begin{aligned}\delta\dot{\theta} = &\sqrt{\frac{\mu}{\left[(a+\delta a)\left\{1-(q_1+\delta q_1)^2-(q_2+\delta q_2)^2\right\}\right]^3}}\left\{\begin{array}{l}1 + (q_1 + \delta q_1)\cos(\theta + \delta\theta) \\ + (q_2 + \delta q_2)\sin(\theta + \delta\theta)\end{array}\right\}^2 \\ &- \sqrt{\frac{\mu}{\left\{a\left(1-q_1^2-q_2^2\right)\right\}^3}}(1 + q_1\cos\theta + q_2\sin\theta)^2\end{aligned} \tag{5.11}$$

Eq. (5.11) is the nonlinear equation for the difference of latitude rate as a function of δa, $\delta\theta$, δq_1 and δq_2. The variation of Eq. (5.10) is

$$\delta\dot{\theta} = \frac{h}{r^2}\left(\frac{\delta p}{2p} - 2\frac{\delta r}{r}\right) \tag{5.12}$$

where,

$$\delta p = \frac{p}{a}\delta a - 2a\left(q_1\delta q_1 + q_2\delta q_2\right) \tag{5.13}$$

Using Eq. (5.12), Schaub and Junkins (2014) approximated $\delta\dot{\theta}$ as a linear expression of $\delta\mathbf{e}$.

$$\delta\dot{\theta} = -\frac{3h}{2ar^2}\delta a - \frac{2hV_r}{r^2V_t}\delta\theta + \left(\frac{3haq_1}{pr^2} + \frac{2h}{pr}\cos\theta\right)\delta q_1 + \left(\frac{3haq_2}{pr^2} + \frac{2h}{pr}\sin\theta\right)\delta q_2 \tag{5.14}$$

For a better accuracy than the linear model in Eq. (5.14), the nonlinear equation can be approximated into third order polynomial corresponding to Abel-type first order equation. If the expansion is done to second order only we have a Riccati-type equation.

5.2 Approximation of Orbital-Element Differences Equation of Motion

The equation of motion of the orbital element differences is nonlinear in the variation of the true of latitude rate. For elliptic orbit, to prevent the variation of the difference in true anomaly between two orbits the desired anomaly difference is usually expressed in terms of a mean anomaly difference. However, the anomaly difference equation of motion can be approximated into second, third and higher orders. The third order corresponds to the Abel-type equation while the second order corresponds to the Riccatti type equation.

In this dissertation, two models of each of the third-order and second-order nonlinear differential equations describing the dynamics of the relative motion of deputy spacecraft with respect to the chief spacecraft in terms of the orbit element differences leading to the formulation of Abel-Type and Riccati-Type differential equations are presented. Using well-known techniques and methods, analytical solutions of the equations are developed.

5.2.1 Construction of Abel-Type Nonlinear Spacecraft Relative Equation of Motion

In this section, Abel-Type orbital-element differences equations of motion are developed as functions of all the four parameters δa, $\delta\theta$, δq_1 and δq_2 using Taylor series expansion and as a nonlinear function of only one time-varying parameter, $\delta\theta$.

(a) Approximation of Orbital-Element Differences Equations of Motion as a Third-Order Function of Four Parameters

Here, the first model of the third order approximation is developed assuming that $\delta\dot{\theta}$ is a nonlinear function of all the four parameters δa, $\delta\theta$, δq_1 and δq_2 using Taylor series expansion. Rearranging Eq. (5.11) in a form to which Taylor series expansion technique can be applied we have the form [52]

$$\delta\dot{\theta} = g\left(\delta a\right) f\left(\delta q_1, \delta q_2, \delta\theta\right) - n(1 + q_1\cos\theta + q_2\sin\theta)^2\left(1 - q_1^2 - q_2^2\right)^{-3/2} \tag{5.15}$$

where, $n = \sqrt{\mu/a^3}$ is the mean motion, $g(\delta a)$ is a function of difference in semimajor axis of the deputy and the chief, and $f(\delta q_1, \delta q_2, \delta\theta)$ is a function of difference in q_1, q_2 and θ.

$$
\begin{aligned}
g(\delta a) &= n\Big(1+\tfrac{\delta a}{a}\Big)^{-3/2} \\
f(\delta q_1, \delta q_2, \delta\theta) &= \begin{bmatrix} 1+(q_1+\delta q_1)\cos(\theta+\delta\theta) \\ +(q_2+\delta q_2)\sin(\theta+\delta\theta) \end{bmatrix}^2 \begin{bmatrix} 1-(q_1+\delta q_1)^2 \\ -(q_2+\delta q_2)^2 \end{bmatrix}^{-3/2}
\end{aligned}
\tag{5.16}
$$

Application of Binomial series expansion gives

$$
g(\delta a) = n\left\{1 - \frac{3}{2a}(\delta a) + \frac{15}{8a^2}(\delta a)^2 - \frac{35}{16a^3}(\delta a)^3\right\} \tag{5.17}
$$

Therefore, Eq. (5.15) yields

$$
\begin{aligned}
\delta\dot{\theta} = n\Big(1 - \tfrac{3}{2a}(\delta a) + \tfrac{15}{8a^2}(\delta a)^2 - \tfrac{35}{16a^3}(\delta a)^3\Big) f(\delta q_1, \delta q_2, \delta\theta) \\
-n(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-3/2}
\end{aligned}
\tag{5.18}
$$

Applying Taylor series at the origin $(0,0,0)$ to $f(\delta q_1, \delta q_2, \delta\theta)$ and substituting into Eq. (5.18) results to a special type of Abel-equation of first kind of the form

$$
\begin{aligned}
\delta\dot{\theta} &= -\tfrac{3n}{2a}\delta a f\Big|_{(0,0,0)} + n\delta\theta f_{\delta\theta}\Big|_{(0,0,0)} + n\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} + n\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} \\
&-\tfrac{3n}{2a}\delta a\delta\theta f_{\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} \\
&+\tfrac{n}{2}\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_1\delta\theta f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta\theta\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+\tfrac{n}{2}\delta q_2\delta\theta f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta\theta\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+\tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)} + \tfrac{n}{2}(\delta\theta)^2 f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+\tfrac{n}{2}(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} \\
&-\tfrac{3n}{4a}\delta a\delta q_1\delta\theta f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} \\
&-\tfrac{3n}{4a}\delta a\delta q_2\delta\theta f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta\theta\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta\theta\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} \\
&-\tfrac{3n}{4a}\delta a(\delta\theta)^2 f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} \\
&+\tfrac{15n}{8a^2}(\delta a)^2\delta\theta f_{\delta\theta}\Big|_{(0,0,0)} - \tfrac{35n}{16a^3}(\delta a)^3 f\Big|_{(0,0,0)} + \tfrac{n}{6}\Big\{(\delta q_1)^3 f_{\delta q_1\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_1)^2(\delta q_2) f_{\delta q_1\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta q_1)^2(\delta\theta) f_{\delta q_1\delta q_1\delta\theta}\Big|_{(0,0,0)} + (\delta q_1)^2(\delta q_2) f_{\delta q_1\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_1)(\delta q_2)^2 f_{\delta q_1\delta q_2\delta q_2}\Big|_{(0,0,0)} + (\delta q_1)(\delta q_2)(\delta\theta) f_{\delta q_1\delta q_2\delta\theta}\Big|_{(0,0,0)} + (\delta q_1)^2(\delta\theta) f_{\delta q_1\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_1)(\delta\theta)(\delta q_2) f_{\delta q_1\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta q_1)(\delta\theta)^2 f_{\delta q_1\delta\theta\delta\theta}\Big|_{(0,0,0)} + (\delta q_2)(\delta q_1)^2 f_{\delta q_2\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_2)^2(\delta q_1) f_{\delta q_2\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)(\delta q_1)(\delta\theta) f_{\delta q_2\delta q_1\delta\theta}\Big|_{(0,0,0)} + (\delta q_2)^2(\delta q_1) f_{\delta q_2\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_2)^3 f_{\delta q_2\delta q_2\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)^2(\delta\theta) f_{\delta q_2\delta q_2\delta\theta}\Big|_{(0,0,0)} + (\delta q_2)(\delta\theta)(\delta q_1) f_{\delta q_2\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta q_2)^2(\delta\theta) f_{\delta q_2\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)(\delta\theta)^2 f_{\delta q_2\delta\theta\delta\theta}\Big|_{(0,0,0)} + (\delta\theta)(\delta q_1)^2 f_{\delta\theta\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta\theta)(\delta q_1)(\delta q_2) f_{\delta\theta\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta\theta)^2(\delta q_1) f_{\delta\theta\delta q_1\delta\theta}\Big|_{(0,0,0)} + (\delta\theta)(\delta q_2)(\delta q_1) f_{\delta\theta\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+(\delta\theta)(\delta q_2)^2 f_{\delta\theta\delta q_2\delta q_2}\Big|_{(0,0,0)} + (\delta\theta)^2(\delta q_2) f_{\delta\theta\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
&+(\delta\theta)^2(\delta q_1) f_{\delta\theta\delta\theta\delta q_1}\Big|_{(0,0,0)} + (\delta\theta)^2(\delta q_2) f_{\delta\theta\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta\theta)^3 f_{\delta\theta\delta\theta\delta\theta}\Big|_{(0,0,0)}\Big\}
\end{aligned}
\tag{5.19}
$$

Equation (5.19) is the first model of the third-order approximation of the rate of change of the true-anomaly. Rewriting Eq. (5.19) in terms of Abel-Type equation as a function of true of latitude difference we have

$$
\delta\dot{\theta} = p_3(\theta)(\delta\theta)^3 + p_2(\theta)(\delta\theta)^2 + p_1(\theta)\delta\theta + p_0(\theta) \tag{5.20}
$$

The expressions for $p_3(\theta), p_2(\theta), p_1(\theta), p_0(\theta)$ are provided in Appendix D.

(b) Approximation of Orbital-Element Differences Equations of Motion as a Third-Order Function of One Parameter

Here, $\delta\dot{\theta}$ is approximated as a third order function of only one time-varying parameter, true of latitude difference. The other three parameters, $\delta a, \delta q_1, \delta q_2$, are constants. Using series expansion technique and eliminating higher order terms above the cubic, we have the following trigonometric functions

$$\begin{aligned}
\cos(\theta+\delta\theta) &\approx \tfrac{1}{6}\sin\theta(\delta\theta)^3 - \tfrac{1}{2}\cos\theta(\delta\theta)^2 - \sin\theta\delta\theta + \cos\theta \\
\sin(\theta+\delta\theta) &\approx -\tfrac{1}{6}\cos\theta(\delta\theta)^3 - \tfrac{1}{2}\sin\theta(\delta\theta)^2 + \cos\theta\delta\theta + \sin\theta \\
\cos 2(\theta+\delta\theta) &\approx \tfrac{4}{3}\sin 2\theta(\delta\theta)^3 - 2\cos 2\theta(\delta\theta)^2 - 2\sin 2\theta\,(\delta\theta) + \cos 2\theta \\
\sin 2(\theta+\delta\theta) &\approx -\tfrac{4}{3}\cos 2\theta(\delta\theta)^3 - 2\sin 2\theta(\delta\theta)^2 + 2\cos 2\theta\,(\delta\theta) + \sin 2\theta
\end{aligned} \tag{5.21}$$

and

$$\begin{gathered}
\cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1), \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta), 2\sin\theta\cos\theta = \sin 2\theta \\
\sin\delta\theta = \delta\theta - \tfrac{(\delta\theta)^3}{3!} + \tfrac{(\delta\theta)^5}{5!} - \tfrac{(\delta\theta)^7}{7!} + \tfrac{(\delta\theta)^9}{9!} - \ldots \\
\cos\delta\theta = 1 - \tfrac{(\delta\theta)^2}{2!} + \tfrac{(\delta\theta)^4}{4!} - \tfrac{(\delta\theta)^6}{6!} + \tfrac{(\delta\theta)^8}{8!} - \ldots
\end{gathered} \tag{5.22}$$

Using Eq. (5.11), $[1+(q_1+\delta q_1)\cos(\theta+\delta\theta)+(q_2+\delta q_2)\sin(\theta+\delta\theta)]^2$ can be re-written as

$$\begin{gathered}
[1+(q_1+\delta q_1)\cos(\theta+\delta\theta)+(q_2+\delta q_2)\sin(\theta+\delta\theta)]^2 \\
= m_3(\theta)(\delta\theta)^3 + m_2(\theta)(\delta\theta)^2 + m_1(\theta)(\delta\theta) + m_0(\theta)
\end{gathered} \tag{5.23}$$

Similarly, $(1+q_1\cos\theta+q_2\sin\theta)^2$ can be expanded as

$$\begin{gathered}
(1+q_1\cos\theta+q_2\sin\theta)^2 \\
= \left(\tfrac{1}{2}q_1^2 - \tfrac{1}{2}q_2^2\right)\cos 2\theta + q_1 q_2 \sin 2\theta + 2q_1\cos\theta + 2q_2\sin\theta + 1 + \tfrac{1}{2}q_1^2 + \tfrac{1}{2}q_2^2
\end{gathered} \tag{5.24}$$

Defining M_D and M_C as

$$
\begin{aligned}
M_D &= \sqrt{\frac{\mu}{\left[(a+\delta a)\left\{1-(q_1+\delta q_1)^2-(q_2+\delta q_2)^2\right\}\right]^3}} \\
M_C &= \sqrt{\frac{\mu}{\left\{a\left(1-q_1^2-q_2^2\right)\right\}^3}}
\end{aligned} \tag{5.25}
$$

and substituting Eqs. (5.23) and (5.24) into Eq. (5.11) yields a second model as a special form of Abel-type equation of first kind

$$
\delta\dot{\theta} = k_3(\theta)(\delta\theta)^3 + k_2(\theta)(\delta\theta)^2 + k_1(\theta)\delta\theta + k_0(\theta) \tag{5.26}
$$

where, $m_3(\theta), m_2(\theta), m_1(\theta), m_0(\theta)$ and $k_3(\theta), k_2(\theta), k_1(\theta), k_0(\theta)$ are defined in Appendix E.

5.2.2 Construction of Riccati-Type Nonlinear Spacecraft Relative Equation of Motion

In a manner similar to the derivation in Section 5.2.1, Riccati-Type orbital-element differences equations of motion are developed as functions of all the four parameters δa, $\delta\theta$, δq_1 and δq_2 and as functions of only the parameter $\delta\theta$. Using Taylor series expansion (in Appendix B) with $(x, y, z) = (\delta q_1, \delta q_2, \delta\theta)$ and $(x_0, y_0, z_0) = (0, 0, 0)$, truncation after quadratic terms gives the first model of a Riccati-type second-order approximation of orbital element differences equations of motion as a function of all the four parameters as

$$
\begin{aligned}
\delta\dot{\theta} = &-\tfrac{3n}{2a}\delta a f\Big|_{(0,0,0)} + \Big(n f_{\delta\theta}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
&+\tfrac{1}{2}n\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a f_{\delta\theta}\Big|_{(0,0,0)}\Big)\delta\theta + n\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} \\
&+n\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&-\tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)} \\
&+\tfrac{1}{2}n(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} + \tfrac{1}{2}n(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} + \tfrac{1}{2}n(\delta\theta)^2 f_{\delta\theta\delta\theta}\Big|_{(0,0,0)}
\end{aligned} \tag{5.27}
$$

This can be written as

$$
\delta\dot{\theta} = p_2(\theta)(\delta\theta)^2 + p_1(\theta)\delta\theta + p_0(\theta) \tag{5.28}
$$

where,

$$
\begin{aligned}
p_0(\theta) = \tfrac{1}{2}n\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} \\
+\tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)} + \tfrac{1}{2}n(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} + \tfrac{1}{2}n(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} \\
-\tfrac{3n}{2a} f\Big|_{(0,0,0)}\delta a + n f_{\delta q_1}\Big|_{(0,0,0)}\delta q_1 + n f_{\delta q_2}\Big|_{(0,0,0)}\delta q_2 \\
p_1(\theta) = n f_{\delta\theta}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} + \tfrac{1}{2}n\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
+\tfrac{1}{2}n\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a f_{\delta\theta}\Big|_{(0,0,0)} \\
s_2(\theta) = \tfrac{1}{2}n f_{\delta\theta\delta\theta}\Big|_{(0,0,0)}
\end{aligned}
\tag{5.29}
$$

From Eq. (5.28), the second model of the Riccati-type second order approximation of orbital-element differences equations of motion as a function of only one parameter is obtained, taking $k_3(\theta) = 0$, as

$$
\delta\dot{\theta} = k_2(\theta)(\delta\theta)^2 + k_1(\theta)\delta\theta + k_0(\theta) \tag{5.30}
$$

The coefficients of Eq. (5.27) are shown in Appendix F.

5.2.3 Construction of Linearized Spacecraft Relative Equation of Motion

Using Taylor series expansion for first order approximation we have the equation

$$
\delta\dot{\theta} = -\frac{3n}{2a} f\Big|_{(0,0,0)}\delta a + n f_{\delta\theta}\Big|_{(0,0,0)}\delta\theta + n f_{\delta q_1}\Big|_{(0,0,0)}\delta q_1 + n f_{\delta q_2}\Big|_{(0,0,0)}\delta q_2 \tag{5.31}
$$

This simplifies to

$$
\begin{aligned}
&\delta\dot{\theta} = -\tfrac{3n}{2a}\left\{(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-3/2}\right\}\delta a \\
&+2n\left\{(-q_1\sin\theta+q_2\cos\theta)(1+q_1\cos\theta+q_2\sin\theta)(1-q_1^2-q_2^2)^{-3/2}\right\}\delta\theta \\
&+n\left\{\begin{array}{l} 2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)(1-q_1^2-q_2^2)^{-3/2} \\ +3q_1(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \end{array}\right\}\delta q_1 \\
&+n\left\{\begin{array}{l} 2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)(1-q_1^2-q_2^2)^{-3/2} \\ +3\,(q_2+\delta q_2)(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \end{array}\right\}\delta q_2
\end{aligned}
\tag{5.32}
$$

Eq. (5.32) can be written as

$$\delta\dot{\theta} = p_{11}(\theta)\delta\theta + p_{10}(\theta) \tag{5.33}$$

where,

$$\begin{aligned}
p_{10}(\theta) &= -\tfrac{3n}{2a}\left\{(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-3/2}\right\}\delta a \\
&+n\left\{\begin{array}{l} 2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\ +3q_1(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \end{array}\right\}\delta q_1 \\
&+n\left\{\begin{array}{l} 2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\ +3\,(q_2+\delta q_2)\,(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \end{array}\right\}\delta q_2 \\
p_{11}(\theta) &= 2n\left\{(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2}\right\}\delta\theta
\end{aligned} \tag{5.34}$$

Eq. (5.33) is the first model, linearized spacecraft relative equation of motion and it has the same form as in the linear Equation (5.14). Considering the first order approximation only, Eq. (5.30) becomes

$$\delta\dot{\theta} = k_1(\theta)\delta\theta + k_0(\theta) \tag{5.35}$$

Eq. (5.35) is the second model of linearized equation.

5.3 Closed Form Solution of Abel-Type Nonlinear Equation of Relative Motion

The approach for the formulation of closed form solution followed closed to the approach in Ogundele et al [52]. Substituting the transformation $\delta\theta = \delta\theta_p + u(\theta)E(\theta)$ into Eq. (5.20) we have the form

$$\begin{aligned}
u' = E^2 p_3 \Big[&u^3 + \tfrac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right)u^2 \\
&+\tfrac{1}{E^2p_3}\left\{3p_3(\delta\theta_p)^2 + 2p_2\delta\theta_p + p_1 - \tfrac{E'}{E}\right\}u \\
&+\tfrac{1}{E^3p_3}\left\{p_0 - \delta\theta'_p + p_3(\delta\theta_p)^3 + p_2(\delta\theta_p)^2 + p_1\left(\delta\theta_p\right)\right\}\Big]
\end{aligned} \tag{5.36}$$

where,

$$u' = \frac{du}{d\theta}, E' = \frac{dE}{d\theta}, \delta\theta'_p = \frac{d\left(\delta\theta_p\right)}{d\theta} \tag{5.37}$$

From Eq. (5.36), let

$$\frac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right) = \beta_1(\theta) \tag{5.38}$$

$$\frac{1}{E^2p_3}\left\{3p_3(\delta\theta_p)^2 + 2p_2\delta\theta_p + p_1 - \frac{E'}{E}\right\} = \beta_2(\theta) \tag{5.39}$$

$$\frac{1}{E^3p_3}\left\{p_0 - \delta\theta'_p + p_3(\delta\theta_p)^3 + p_2(\delta\theta_p)^2 + p_1\left(\delta\theta_p\right)\right\} = \beta_3(\theta) \tag{5.40}$$

Considering the case in which the system (5.38-5.40) satisfies $\beta_1(\theta) = \beta_2(\theta) = 0$ and $\beta_3(\theta) = \Phi(\theta)$we have the corresponding system

$$\frac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right) = 0 \tag{5.41}$$

$$\frac{1}{E^2p_3}\left\{3p_3(\delta\theta_p)^2 + 2p_2\delta\theta_p + p_1 - \frac{E'}{E}\right\} = 0 \tag{5.42}$$

$$\frac{1}{E^3p_3}\left\{p_0 - \delta\theta'_p + p_3(\delta\theta_p)^3 + p_2(\delta\theta_p)^2 + p_1\left(\delta\theta_p\right)\right\} = \Phi(\theta) \tag{5.43}$$

Solving for $\delta\theta_p$ in Eq. (5.41) and substituting on Eq. (5.42) and (5.43) the above system reduces to

$$\delta\theta_p = -\frac{p_2}{3p_3} \tag{5.44}$$

$$-\frac{p_2^2}{3p_3} + p_1 - \frac{E'}{E} = 0 \tag{5.45}$$

$$\frac{1}{E^3p_3}\left\{p_0 - \frac{p_1p_2}{3p_3} + \frac{2p_2^3}{3p_3^2} + \frac{1}{3}\left(\frac{p_2}{p_3}\right)'\right\} = \Phi(\theta) \tag{5.46}$$

From Eq. (5.45), E is obtained as

$$E(\theta) = \exp\left[\int\left(p_1 - \frac{p_2^2}{3p_3}\right)\partial\theta\right] \tag{5.47}$$

The condition $\beta_1(\theta) = \beta_2(\theta) = 0$ and $\beta_3(\theta) = \Phi(\theta)$ reduces Eq. (5.36) to the well-known canonical form of Abel equation of first kind

$$u'_\xi = u^3(\xi) + \Phi(\xi), \xi = \int E^2 p_3 \partial\theta \tag{5.48}$$

Expressing the canonical form as

$$u'_\xi - u^3(\xi) - \lambda u^3(\xi) = -\lambda u^3(\xi) + \Phi(\xi) \tag{5.49}$$

and imposing the right-hand side to be zero we have the system of equations

$$\begin{aligned} &u'_\xi - u^3(\xi) - \lambda u^3(\xi) = 0 \\ &-\lambda u^3(\xi) + \Phi(\xi) = 0 \end{aligned} \tag{5.50}$$

Eq. (5.50) has solution

$$u(\xi) = \frac{1}{\sqrt{C - 2\int (1+\lambda)\, d\xi}}, \Phi(\xi) = \frac{\lambda}{\left[C - 2\int (1+\lambda)\, d\xi\right]^{2/3}} \tag{5.51}$$

Using Eqs. (5.44) and (5.51) the general solution $\delta\theta = \delta\theta_p + u(\theta)E(\theta)$ can be written as

$$[\delta\theta]_{\text{Abel-Model1}} = \frac{E(\theta)}{\sqrt{C - 2\int (1+\lambda)\, d\xi}} - \frac{p_2}{3p_3} \tag{5.52}$$

and

$$\Phi(\xi) = \frac{\lambda}{\left[C - 2\int (1+\lambda)\, d\xi\right]^{2/3}} = \frac{1}{E^3 p_3}\left\{p_0 - \frac{p_1 p_2}{3p_3} + \frac{2p_2^3}{3p_3^2} + \frac{1}{3}\frac{d}{d\theta}\left(\frac{p_2}{p_3}\right)\right\} \tag{5.53}$$

Consider an analytic solution with $\lambda(\xi) = 0$, then Eq. (5.52) reduces to

$$[\delta\theta]_{\text{Abel-Model1}} = \frac{E(\theta)}{\sqrt{C - 2\int p_3 E^2(\theta) d\theta}} - \frac{p_2}{3p_3} \tag{5.54}$$

For the second Abel-Type equation model (5.26), using the same approach as above, substituting the transformation $\delta\theta = \delta\theta_p + z(\theta)V(\theta)$ we have the form

$$\begin{aligned} z' = V^2 k_3 \Big[& z^3 + \tfrac{1}{Vk_3}\left(3k_3\delta\theta_p + k_2\right) z^2 \\ & + \tfrac{1}{V^2 k_3}\left\{3k_3(\delta\theta_p)^2 + 2k_2\delta\theta_p + k_1 - \tfrac{V'}{V}\right\} z \\ & + \tfrac{1}{V^3 k_3}\left\{k_0 - \delta\theta'{}_p + k_3(\delta\theta_p)^3 + k_2(\delta\theta_p)^2 + k_1\left(\delta\theta_p\right)\right\}\Big] \end{aligned} \tag{5.55}$$

with the canonical form

$$z'_\xi = z^3(\zeta) + \Psi(\zeta) \tag{5.56}$$

where,

$$\zeta = \int V^2 k_3 d\theta, V(\theta) = \exp\left[\int \left(k_1 - \frac{k_2^2}{3k_3}\right) d\theta\right] \tag{5.57}$$

The canonical form can be written as

$$z'_\zeta - z^3(\zeta) - \eta z^3(\zeta) = -\eta z^3(\zeta) + \Psi(\zeta) \tag{5.58}$$

and imposing the right-hand side to be zero we have the system of equations

$$z'_\zeta - z^3(\zeta) - \eta z^3(\zeta) = 0, \, -\eta z^3(\zeta) + \Psi(\zeta) = 0 \tag{5.59}$$

These equations have the solutions

$$z(\zeta) = \frac{1}{\sqrt{C - 2\int\left(1+\eta\right)d\zeta}}, \Psi(\zeta) = \frac{\eta}{\left[C - 2\int\left(1+\eta\right)d\zeta\right]^{2/3}} \tag{5.60}$$

Therefore, the general solution is

$$[\delta\theta]_{\text{Abel}-\text{Model2}} = \frac{V(\theta)}{\sqrt{C - 2\int\left(1+\eta\right)\partial\zeta}} - \frac{k_2}{3k_3} \tag{5.61}$$

and

$$\Psi(\zeta) = \frac{\eta}{\left[C - 2\int(1+\eta)\,\partial\zeta\right]^{2/3}} = \frac{1}{V^3 k_3}\left\{k_0 - \frac{k_1 k_2}{3k_3} + \frac{2k_2^3}{3k_3^2} + \frac{1}{3}\left(\frac{k_2}{k_3}\right)'\right\} \tag{5.62}$$

Consider an analytic solution with $\eta(\zeta) = 0$ then Eq. (5.61) reduces to

$$[\delta\theta]_{\mathrm{Abel-Model2}} = \frac{V(\theta)}{\sqrt{C - 2\int k_3 V^2(\theta)d\theta}} - \frac{k_2}{3k_3} \tag{5.63}$$

5.4 Closed Form Solution of Riccati-Type Nonlinear Relative Equation of Motion

The approach in Polyanin and Zaitsev (2002), and Haaheim and Stein (1969) is followed for the formulation of the general solution of the Riccati Equation. It is a well-known fact that once a particular solution $\delta\theta_p = \delta\theta_p(\theta)$ of the Riccati equation is known then the general solution of the equation can be written as

$$\delta\theta = \delta\theta_p(\theta) + \frac{1}{z(\theta)} \tag{5.64}$$

Upon substitution of Eq. (5.64) into Eq. (5.28) we have linear differential equation

$$\frac{dz}{d\theta} + \{p_1(\theta) + 2p_2(\theta)\delta\theta_p\}\,z + p_2(\theta) = 0 \tag{5.65}$$

with the solution

$$z(\theta) = z_0 e^{-\Psi(\theta)} - e^{-\Psi(\theta)}\int_{\delta\theta_0}^{\delta\theta} p_2(\theta)e^{\Psi(\theta)}d\theta \tag{5.66}$$

where

$$\Psi(\theta) = \int_{\delta\theta_0}^{\delta\theta}\left[p_1(\theta) + 2p_2(\theta)\delta\theta_p\right]d\theta, z_0 = \frac{1}{\delta\theta_0 - \delta\theta_{p_0}} \tag{5.67}$$

Therefore, the general solution can be expressed as

$$\begin{gathered}\delta\theta(p_2(\theta), p_1(\theta), \delta\theta_p)_{\text{Riccati}-\text{Model1}} \\ = \delta\theta_p + e^{\Psi(\theta)}\left[\tfrac{1}{\delta\theta_0 - \delta\theta_{p0}} - \int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} p_2(\theta) d\theta\right]^{-1}\end{gathered} \tag{5.68}$$

Expanding the general solutions in a series gives

$$\begin{gathered}\delta\theta(p_2(\theta), p_1(\theta), \delta\theta_p)_{\text{Riccati}-\text{Model1}} \\ = \delta\theta_p + e^{\Psi(\theta)}\left(\delta\theta_0 - \delta\theta_{p0}\right)\Big[1 + \left(\delta\theta_0 - \delta\theta_{p0}\right)\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} p_2(\theta)\partial\theta \\ + \left(\delta\theta_0 - \delta\theta_{p0}\right)^2\left(\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} p_2(\theta)\partial\theta\right)^2 + \left(\delta\theta_0 - \delta\theta_{p0}\right)^3\left(\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} p_2(\theta)\partial\theta\right)^3 + ...\Big]\end{gathered} \tag{5.69}$$

where

$$e^{\Psi(\theta)} = 1 + \Psi(\theta) + \frac{\Psi(\theta)^2}{2} + \frac{\Psi(\theta)^3}{6} + ... \tag{5.70}$$

The general solution of the model 2 of Riccati equation in Eq. (5.30) is found in a similar manner to the approach in Model 1. Let the general solution be represented as

$$\delta\theta = \delta\theta_p(\theta) + \frac{1}{w(\theta)} \tag{5.71}$$

Substituting Eq. (5.71) into Eq. (5.30) gives differential equation

$$\frac{dw}{d\theta} + \{k_1(\theta) + 2k_2(\theta)\delta\theta_p\}\, w + k_2(\theta) = 0 \tag{5.72}$$

Eq. (5.72) has the solution

$$w(\theta) = w_0 e^{-\beta(\theta)} - e^{-\beta(\theta)} \int_{\delta\theta_0}^{\delta\theta} k_2(\theta) e^{\beta(\theta)} d\theta \tag{5.73}$$

where

$$\beta(\theta) = \int_{\delta\theta_0}^{\delta\theta} \left[k_1(\theta) + 2k_2(\theta)\delta\theta_p\right] d\theta, w_0 = \frac{1}{\delta\theta_0 - \delta\theta_{p0}} \tag{5.74}$$

Therefore, the general solution can be expressed as

$$\begin{aligned}&\delta\theta(k_2(\theta), k_1(\theta), \delta\theta_p)_{\text{Model2(Riccati)}}\\&= \delta\theta_p + e^{\beta(\theta)}\left[\tfrac{1}{\delta\theta_0 - \delta\theta_{p_0}} - \textstyle\int_{\delta\theta_0}^{\delta\theta} e^{\beta(\theta)} k_2(\theta) d\theta\right]^{-1}\end{aligned} \tag{5.75}$$

In series form, Eq. (5.75) becomes

$$\begin{aligned}&\delta\theta(g_2(\theta), g_1(\theta), \delta\theta_p)_{\text{Model2(Riccati)}}\\&= \delta\theta_p + e^{\beta(\theta)}\left(\delta\theta_0 - \delta\theta_{p_0}\right)\Big[1 + \left(\delta\theta_0 - \delta\theta_{p_0}\right) \textstyle\int_{\delta\theta_0}^{\delta\theta} e^{\beta(\theta)} k_2(\theta) d\theta\\&+ (\delta\theta_0 - \delta\theta_{p_0})^2 \left(\textstyle\int_{\delta\theta_0}^{\delta\theta} e^{\beta(\theta)} k_2(\theta) d\theta\right)^2 + (\delta\theta_0 - \delta\theta_{p_0})^3 \left(\textstyle\int_{\delta\theta_0}^{\delta\theta} e^{\beta(\theta)} k_2(\theta) d\theta\right)^3 + \ldots\Big]\end{aligned} \tag{5.76}$$

where,

$$e^{\beta(\theta)} = 1 + \beta(\theta) + \frac{\beta(\theta)^2}{2} + \frac{\beta(\theta)^3}{6} + \ldots \tag{5.77}$$

5.5 Closed Form Solution of Linearized Spacecraft Relative Equation of Motion

Using integrating factor method, the linear equation in Eq. (5.33) has the solution

$$[\delta\theta]_{\text{Model1(linear)}} = e^G \left(c_1 + \int e^{-G} p_{10}(\theta) d\theta\right), G = \int p_{11}(\theta) d\theta \tag{5.78}$$

Expanding the exponential functions in a series yields

$$\begin{aligned}&[\delta\theta(\theta)]_{\text{Model1(linear)}} = \Big\{1 + (\textstyle\int p_{11}(\theta)d\theta) + \frac{1}{2}(\int p_{11}(\theta)d\theta)^2 + \frac{1}{6}(\int p_{11}(\theta)d\theta)^3 + \ldots\Big\}\\&\Big\{c_1 + \textstyle\int \Big(1 - (\int p_{11}(\theta)d\theta) + \frac{1}{2}(\int p_{11}(\theta)d\theta)^2 - \frac{1}{6}(\int p_{11}(\theta)d\theta)^3 + \ldots\Big) p_{10}(\theta)d\theta\Big\}\end{aligned} \tag{5.79}$$

Using integrating factor method, the second model linear equation (Eq. 5.35) has the solution

$$[\delta\theta]_{\text{Model2(linear)}} = e^G \left(c_1 + \int e^{-G} k_0(\theta) \partial\theta\right), G = \int k_1(\theta) \partial\theta \tag{5.80}$$

In series form, Eq. (5.80) can be written as

$$[\delta\theta]_{\text{Model2(linear)}} = \left\{1 + (\textstyle\int k_1(\theta)d\theta) + \frac{1}{2}(\int k_1(\theta)d\theta)^2 + \frac{1}{6}(\int k_1(\theta)d\theta)^3 + ...\right\} \left\{c_1 + \textstyle\int \left(1 - (\int k_1(\theta)d\theta) + \frac{1}{2}(\int k_1(\theta)d\theta)^2 - \frac{1}{6}(\int k_1(\theta)d\theta)^3 + ...\right) k_0(\theta)d\theta\right\} \tag{5.81}$$

5.6 Numerical Simulations

In this section, numerical simulations are carried out for two scenarios. Scenario 1 is for chief spacecraft in circular orbit and Scenario 2 is for chief spacecraft in elliptical orbit.

5.6.1 Scenario 1: Chief Spacecraft in Circular Orbit

The two cases of bounded and unbounded relative motion considered are shown below.

a) Case 1: Bounded motion for Circular Chief Orbit

Table 5.1 shows chief and deputy spacecraft orbital elements for Case 1.

Table 5.1: Case 1 chief and deputy spacecraft orbital elements

Orbital Elements	Chief Spacecraft	Deputy Spacecraft
a (km)	8500	8500
e	0	0.01305
i (deg)	60	60.7
Ω (deg)	35	35.5009
ω (deg)	20	20.5
f (deg)	0	0.6

b) Case 2: Unbounded motion for Circular Chief Orbit

Here, the semi-major axis of the deputy is 8500.10 km.

5.6.2 Scenario 2: Chief Spacecraft in Elliptical Orbit

a) Case 3: Bounded motion for Elliptical Chief Orbit

Table 5.2 shows chief and deputy spacecraft orbital elements for Case 3.

Table 5.2: Case 3 chief and deputy spacecraft orbital elements

Orbital Elements	Chief Spacecraft	Deputy Spacecraft
a (km)	9500	9500
e	0.03	0.0309556
i (deg)	45	45.155
Ω (deg)	30	30.5
ω (deg)	275	275.55
f (deg)	320	320.5

b) Case 4: Unbounded motion for Elliptical Chief Orbit

Here, the semi-major axis of the deputy is 8500.10 km.

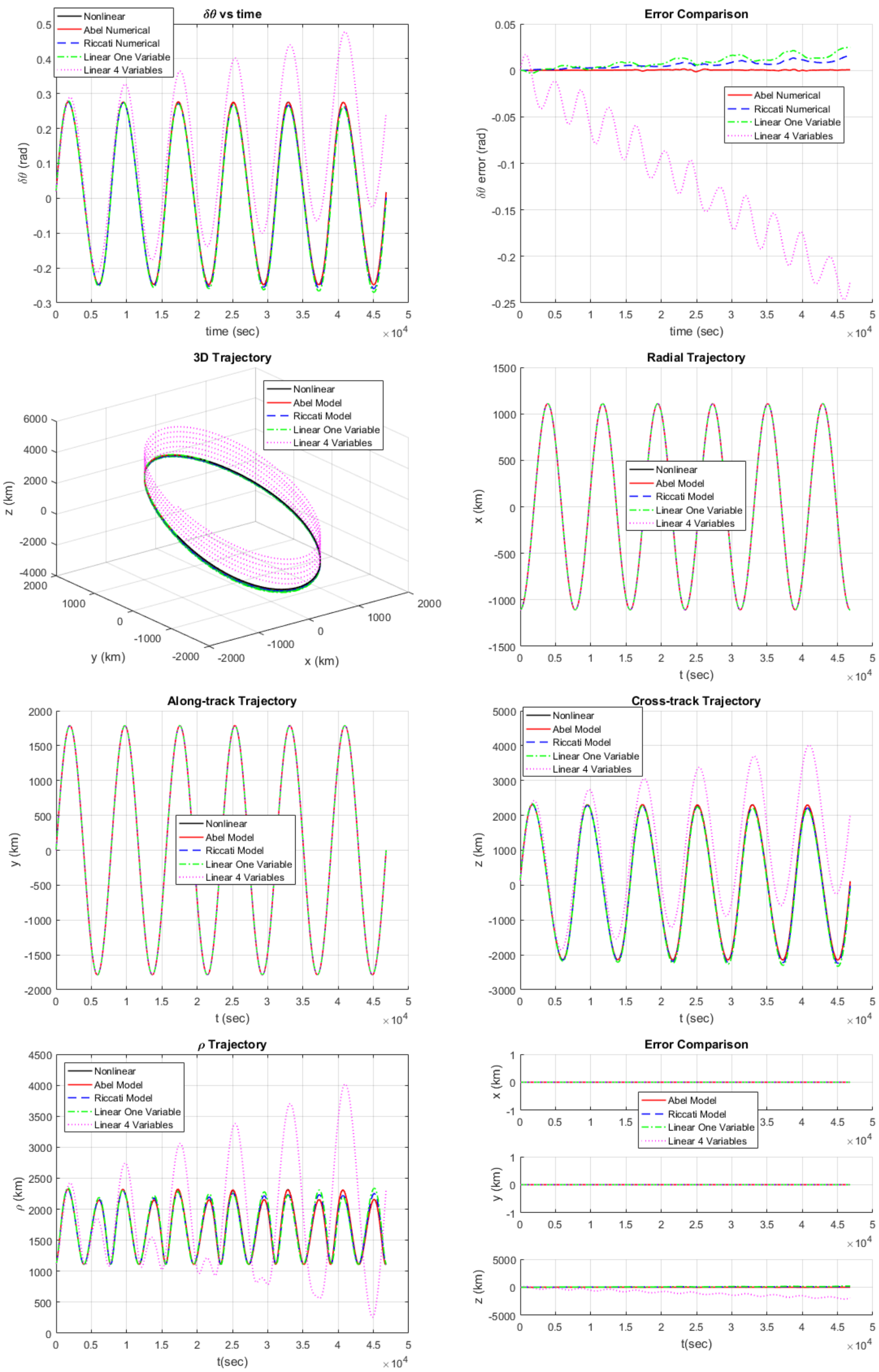

Figure 5.1: Case 1 Bounded Motion Trajectories for Circular Chief Orbit

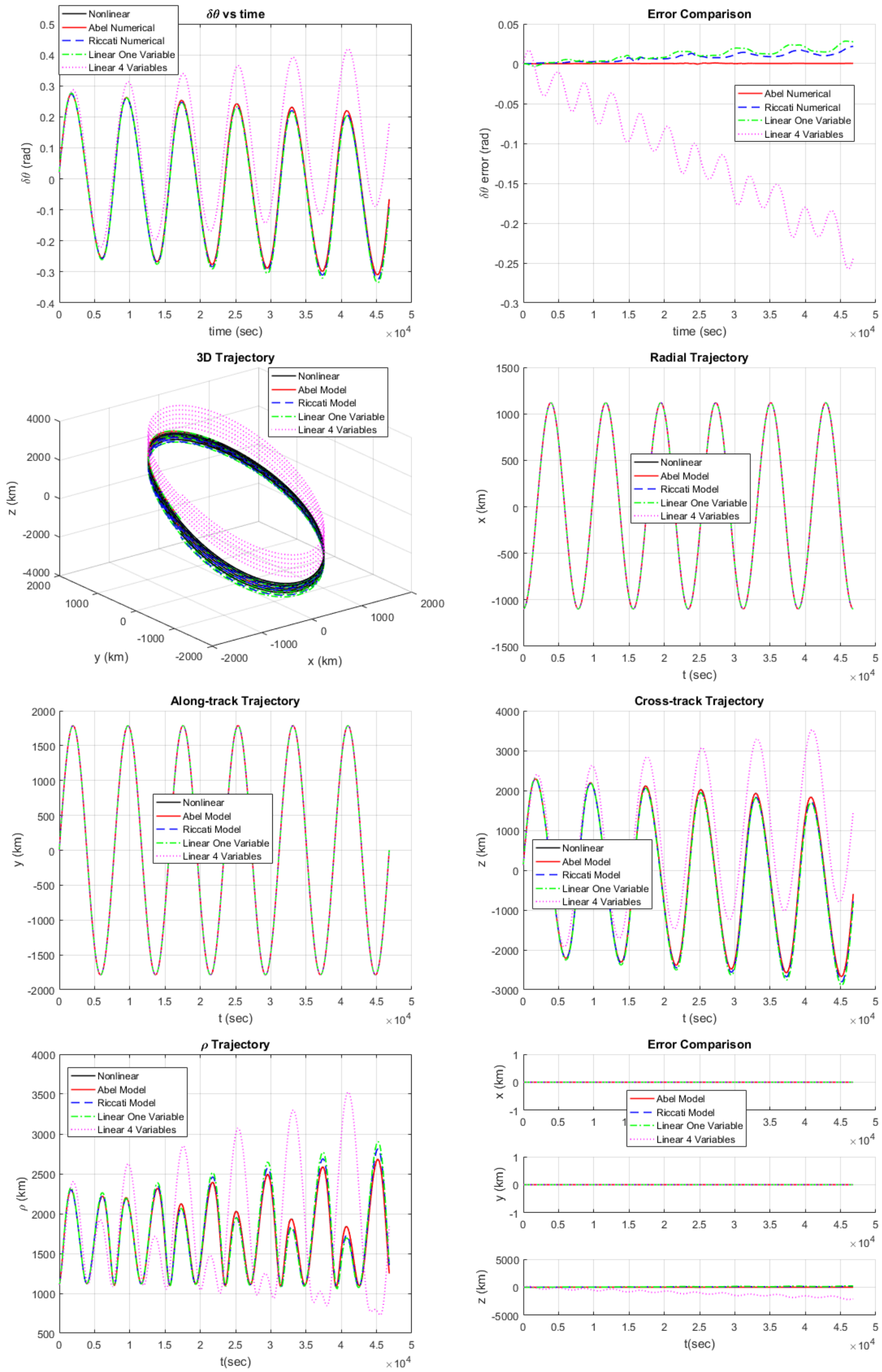

Figure 5.2: Case 2 Unbounded Motion Trajectories for Circular Chief Orbit

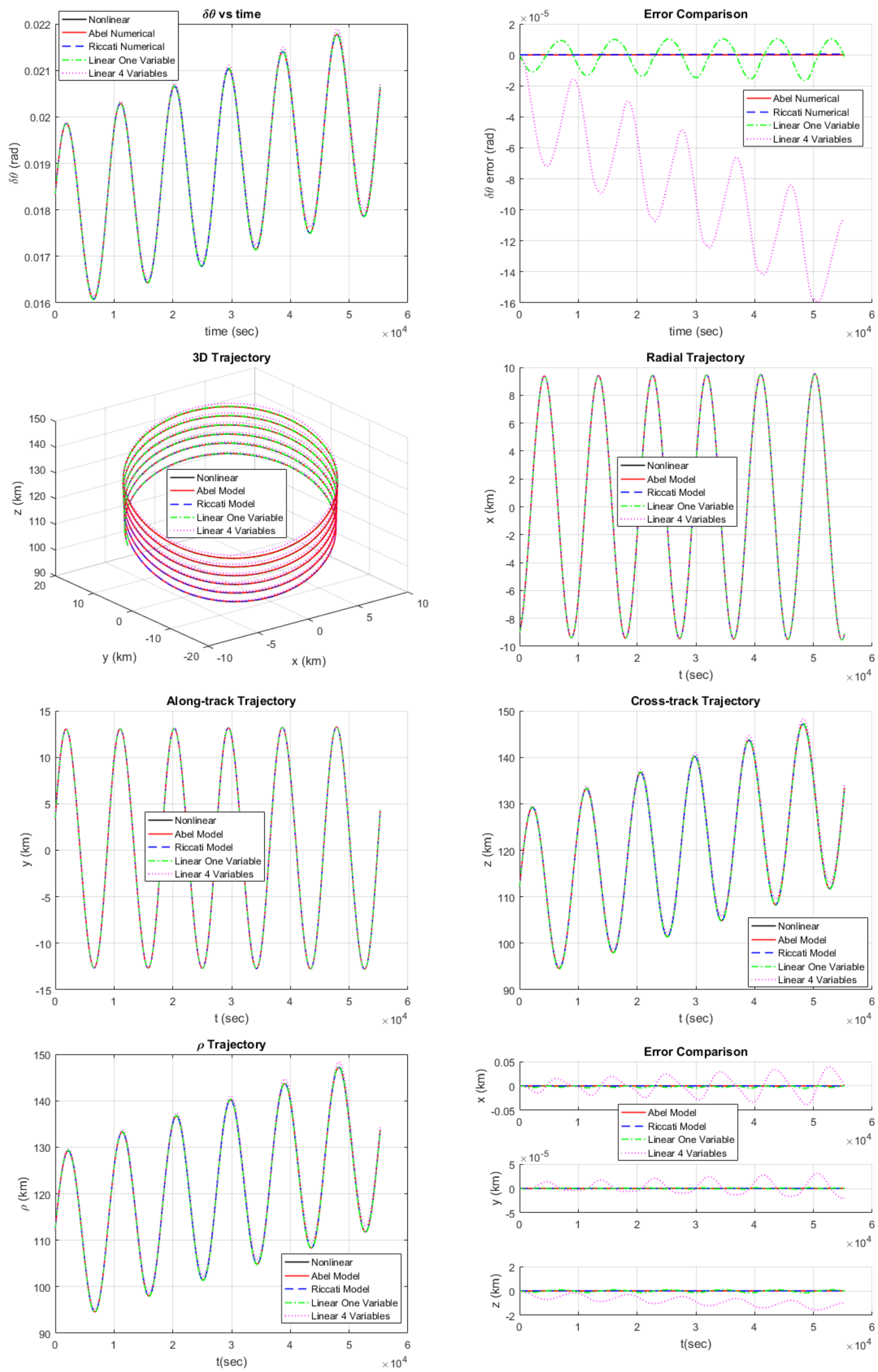

Figure 5.3: Case 3 Bounded Motion Trajectories for Elliptical Chief Orbit

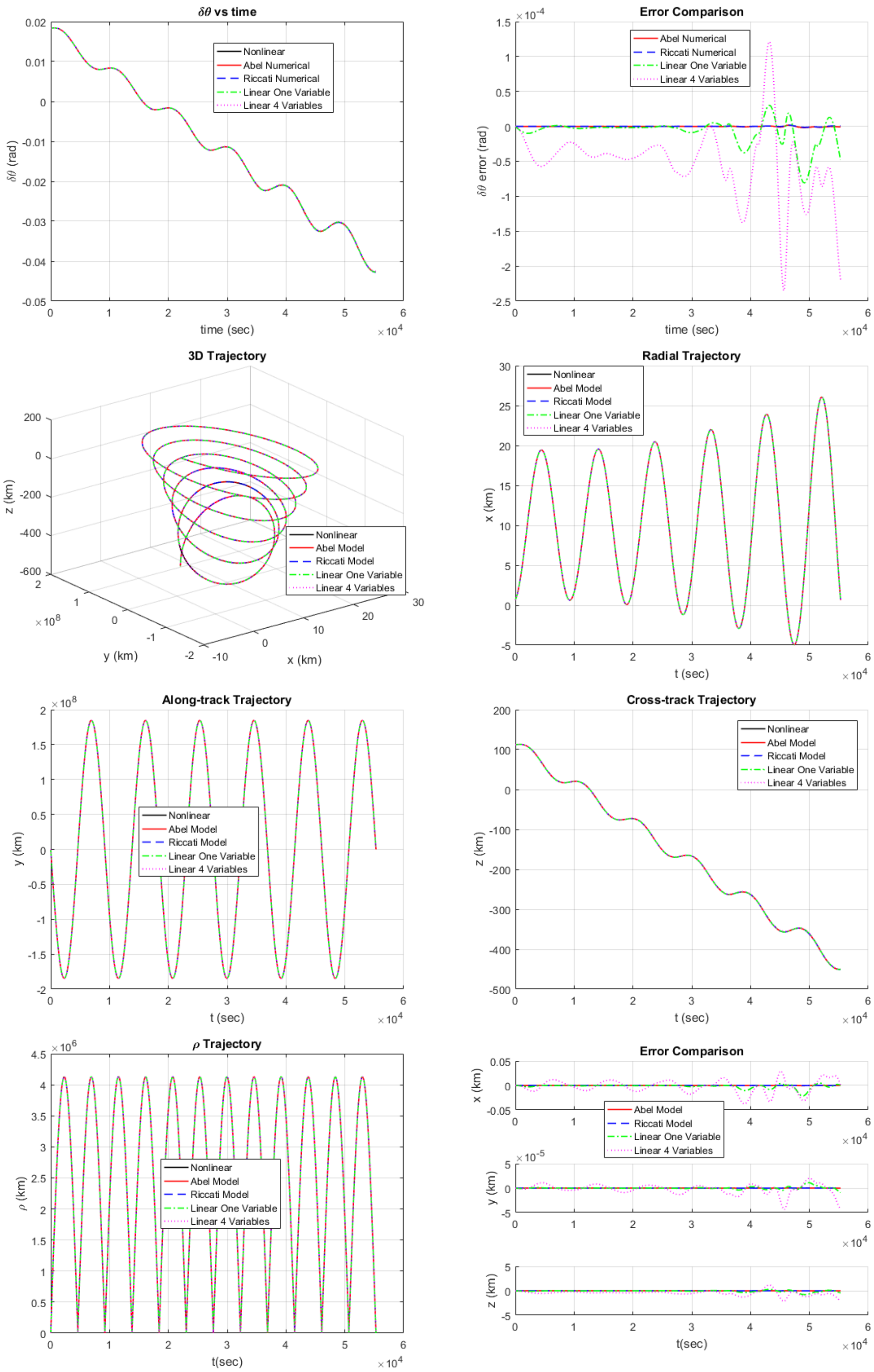

Figure 5.4: Case 4 Unbounded Motion Trajectories for Elliptical Chief Orbit

Chapter 6

Extended Linearization of Cubic Model of Spacecraft Relative Motion Using State Dependent Riccati Equation

In this chapter, extended linearization method is applied to the cubic approximation model of spacecraft relative motion to find the state-dependent coefficient (SDC) parameterization. LQR and SDRE controllers are designed for the three forms of SDC parameterization of cubic models of spacecraft relative motion previously discussed. The key interest in the SDRE approach is that, unlike LQR, it does not neglect the beneficial nonlinear terms. The simulation results show that both the LQR and SDRE methods can be used to stabilize the system. However, the LQR controller that uses the linearized model does not give good approximation of the nonlinear model in the regions that are far from the equilibrium point. On the other hand, the SDRE controller brings the system back to the equilibrium positions with less control effort. Overall, the SDRE controller has better performance compared to the LQR controller in the simulation results for the three parameterized systems.

6.1 Cubic Model of Spacecraft Relative Motion

The radial, along-track and cross-track spacecraft cubic equations of motion of deputy spacecraft with respect to the chief spacecraft in circular orbit is given as (see Ref [6,28,52])

$$
\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = \varepsilon f\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right) \\
&\ddot{y} + 2n\dot{x} = \varepsilon g\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right) \\
&\ddot{z} + n^2 z = \varepsilon h\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right)
\end{aligned}
\tag{6.1}
$$

where $\varepsilon = 3\mu/2R^5$ is a small parameter and

$$\begin{aligned} f(x, y, \dot{x}, \dot{y}) &= -2Rx^2 + Ry^2 + Rz^2 - 4xy^2 - 4xz^2 - \tfrac{8}{3}x^3 \\ g(x, y, \dot{x}, \dot{y}) &= 2Ryx - 4yx^2 + yz^2 - y^3 \\ h(x, y, \dot{x}, \dot{y}) &= 2Rzx - 4zx^2 + zy^2 - z^3 \end{aligned} \tag{6.2}$$

The deputy satellite equation of motion relative to the chief was obtained using local vertical and local horizontal (LVLH) frame with Cartesian coordinates xyz. The mean motion is n and R is the chief orbital radius. In matrix form, Eq. (6.1) can be expressed as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{bmatrix} \tag{6.3}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \varepsilon \mathbf{H}(x, y, z)$$

6.2 Extended Linearization of Cubic Model of Spacecraft Relative Motion

Extended linearization, also referred to as SDC or apparent linearization, is the process by which a nonlinear system is transformed into a pseudo linear-like system. The SDC is formulated by factorizing the nonlinear dynamics into the state vector and matrices that are state dependent [26,27,28,29,30]. Consider the nonlinear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \mathbf{x}(0) = \mathbf{x}_0 \tag{6.4}$$

which can be represented as affine in the input system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0, \mathbf{f}(0) = \mathbf{0} \tag{6.5}$$

The SDC parameterization of Eq. (6.5) leads to the following system in which the system and input matrices are explicit functions of the current state variables

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0 \tag{6.6}$$

where $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$, $\mathbf{g}(\mathbf{x}) = \mathbf{B}(\mathbf{x})$ the state vector $\mathbf{x} \in \mathbf{R}^n$, the input vector is $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{A}(\mathbf{x})$ is the $n \times n$ state-dependent matrix which can be obtained by mathematical factorization, function $\mathbf{f} : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $\mathbf{B} : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times m}$ and $\mathbf{B} \neq \mathbf{0}, \forall \mathbf{x}$. A number of approaches have been presented on how to get optimal parameterization from suboptimal parameterizations. In general, $\mathbf{A}(\mathbf{x})$ is unique only if $\mathbf{x}$ is a scalar [24,25,27,33]. If we have two distinct SDC parameterizations $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$ then $\mathbf{f}(\mathbf{x}) = \mathbf{A}_1(\mathbf{x})\mathbf{x} = \mathbf{A}_2(\mathbf{x})\mathbf{x}$ and for any $\alpha \in \mathbf{R}$ we have hyper-plane composition

$$\mathbf{A}\left(\mathbf{x}, \alpha\right) = \alpha \mathbf{A}_1(\mathbf{x})\mathbf{x} + \left(1 - \alpha\right) \mathbf{A}_2(\mathbf{x})\mathbf{x} = \alpha \mathbf{f}(\mathbf{x}) + \left(1 - \alpha\right) \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \tag{6.7}$$

where $\mathbf{A}\left(\mathbf{x}, \alpha(\mathbf{x})\right)$ is an infinite family of SDC parameterizations. Due to the fact that there are many available SDC parameterizations we can choose the most appropriate one using the state-dependent controllability matrix given by [24]

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} \mathbf{B}(\mathbf{x}) & \mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x}) & \ldots & \mathbf{A}^{(n-1)}(\mathbf{x})\mathbf{B}(\mathbf{x}) \end{bmatrix} \tag{6.8}$$

If $\mathbf{M}(\mathbf{x})$ has full-rank then the system is controllable. Consider the following representation of the cubic model of spacecraft relative motion

$$\dot{\mathbf{x}} = f(x) = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 3n^2 x_1 + 2n x_5 + \varepsilon\left(-2Rx_1^2 + Rx_2^2 + Rx_3^2 - 4x_1x_2^2 - 4x_1x_3^2 - \frac{8}{3}x_1^3\right) \\ -2nx_4 + \varepsilon\left(2Rx_1x_2 - 4x_2x_1^2 + x_2x_3^2 - x_2^3\right) \\ -n^2x_3 + \varepsilon\left(2Rx_1x_3 - 4x_3x_1^2 + x_3x_2^2 - x_3^3\right) \end{bmatrix} \tag{6.9}$$

where, $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$. Taking the gradient of Eq. (6.9) we have

$$
\nabla f(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 + \varepsilon \begin{pmatrix} -4Rx_1 - 4x_2^2 \\ -4x_3^2 - 8x_1^2 \end{pmatrix} & \varepsilon\left(2Rx_2 - 8x_1x_2\right) & \varepsilon\left(2Rx_3 - 8x_1x_3\right) & 0 & 2n & 0 \\ \varepsilon\left(2Rx_2 - 8x_1x_2\right) & \varepsilon \begin{pmatrix} 2Rx_2 - 4x_1^2 \\ +x_3^2 - 3x_2^2 \end{pmatrix} & 2\varepsilon x_2 x_3 & -2n & 0 & 0 \\ \varepsilon\left(2Rx_3 - 8x_1x_3\right) & 2\varepsilon x_2 x_3 & -n^2 + \varepsilon \begin{pmatrix} 2Rx_1 - 4x_1^2 \\ +x_2^2 - 3x_3^2 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix} \tag{6.10}
$$

Evaluating the gradient at zero yields

$$
\nabla f(0) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \tag{6.11}
$$

Equation (6.11) is the linearization of cubic approximation model about the origin. In this section, we applied extended linearization technique to obtain three SDC parameterizations of the model for use in SDRE technique. For each of the SDC parameterization the relationship between the linearization of the original nonlinear dynamical system about the origin and the SDC parameterization evaluated at zero was shown. Also, it was assumed that

$$\mathbf{B} = \begin{bmatrix} \mathbf{O}_3 \\ \mathbf{I}_3 \end{bmatrix} \tag{6.12}$$

where $\mathbf{B}$ is a 6×3 matrix, $\mathbf{O}_3$ is a 3×3 null matrix and $\mathbf{I}_3$ is a 6×3 identity matrix.

6.2.1 Cubic Model SDC Parameterization 1

In this parameterization the system dynamics is defined by

$$\dot{\mathbf{x}} = \mathbf{A}_{SDC1}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} \tag{6.13}$$

where $\mathbf{A}_{SDC1}(\mathbf{x})$ is a state dependent matrix and $\mathbf{B}$ is a constant value matrix. An SDC parameterization is

$$\mathbf{A}_{SDC1}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 - \varepsilon\left(2Rx_1 + \frac{8}{3}x_1^2 + 4x_2^2 + 4x_3^2\right) & \varepsilon R x_2 & \varepsilon R x_3 & 0 & 2n & 0 \\ \varepsilon\left(2Rx_2 - 4x_1x_2\right) & -\varepsilon x_2^2 & \varepsilon x_2 x_3 & -2n & 0 & 0 \\ \varepsilon\left(2Rx_3 - 4x_1x_3\right) & \varepsilon x_2 x_3 & -n^2 - \varepsilon x_3^2 & 0 & 0 & 0 \end{bmatrix} \tag{6.14}$$

This parameterization has state-dependent controllability matrix

$$\mathrm{M}_1(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2n & 0 \\ 0 & 1 & 0 & -2n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.15}$$

with full rank 6 for all $\mathbf{x} \in \mathbf{R}^6$.

6.2.2 Cubic Model SDC Parameterization 2

Consider the cubic model, quadratically nonlinear in the state variables, as

$$\dot{\mathbf{x}} = \mathbf{A}_{SDC2}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u}(t) = \left(\mathbf{A}_0 + \varepsilon x_1 \mathbf{A}_{NL}(\mathbf{x})\right)\mathbf{x} + \mathbf{B}\mathbf{u}(t) \tag{6.16}$$

where the matrices $\mathbf{A}_0$ and $\mathbf{B}$ are constant-valued and $\mathbf{A}_{NL}(\mathbf{x})$ is a state dependent matrix.

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & n^2 & 0 & 0 & 0 \end{bmatrix} \tag{6.17}$$

$$\mathbf{A}_{NL}(\mathbf{x}) = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\left(2R + \frac{8}{3}x_1\right) & \left(R\frac{x_2}{x_1} - 4x_2\right) & \left(R\frac{x_3}{x_1} - 4x_3\right) & 0 & 0 & 0 \\ -4x_2 & \left(2R - \frac{x_2^2}{x_1}\right) & \frac{x_2 x_3}{x_1} & 0 & 0 & 0 \\ -4x_3 & \frac{x_2 x_3}{x_1} & \left(2R - \frac{x_3^2}{x_1}\right) & 0 & 0 & 0 \end{bmatrix} \tag{6.18}$$

The state dependent matrix $\mathbf{A}_{SDC2}(\mathbf{x}) = \mathbf{A}_0 + \varepsilon x_1 \mathbf{A}_{NL}(\mathbf{x})$. This parameterization has state-dependent controllability matrix

$$\mathrm{M}_2(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2n & 0 \\ 0 & 1 & 0 & -2n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.19}$$

with full rank 6.

6.2.3 Cubic Model SDC Parameterization 3

Since there exists at least two SDC parameterizations, there are an infinite number. Given $\mathbf{f}(\mathbf{x}) = \mathbf{A}_{SDC1}(\mathbf{x})\mathbf{x}$ and $\mathbf{f}(\mathbf{x}) = \mathbf{A}_{SDC2}(\mathbf{x})\mathbf{x}$, then for any $\alpha \in \mathbf{R}$ we have hyper-plane composition

$$\mathbf{A}\left(\mathbf{x},\alpha\right) = \alpha \mathbf{A}_{SDC1}(\mathbf{x})\mathbf{x} + \left(\mathbf{1}-\alpha\right) \mathbf{A}_{SDC2}(\mathbf{x})\mathbf{x} = \alpha \mathbf{f}(\mathbf{x}) + \left(\mathbf{1}-\alpha\right) \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \tag{6.20}$$

which is a valid parameterization. The dynamics for the formulation of the control can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= 2nx_5 + 3n^2 x_1 + \varepsilon\left(-2Rx_1^2 + Rx_2^2 + Rx_3^2 - 4x_1x_2^2 - 4x_1x_3^2 - \tfrac{8}{3}x_1^3\right) + u \\
\dot{x}_5 &= -2nx_4 + \varepsilon\left(2Rx_1x_2 - 4x_2x_1^2 + x_2x_3^2 - x_2^3\right) + u \\
\dot{x}_6 &= -n^2 x_3 + \varepsilon\left(2Rx_1x_3 - 4x_3x_1^2 + x_3x_2^2 - x_3^3\right) + u
\end{aligned}
\tag{6.21}
$$

The dynamics in Eq. (6.21) can be represented as

$$
\dot{\mathbf{x}} = \mathbf{A}_{SDC3}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} \tag{6.22}
$$

where $\mathbf{A}_{SDC3}(\mathbf{x})$ is state dependent coefficient, $\mathbf{B}$ is a constant matrix and u is the control. In a similar approach to the terms with more than one multiple of the state can be represented as

$$
\begin{aligned}
4\varepsilon x_1 x_2^2 &= \alpha_1\left(4\varepsilon x_2^2\right)x_1 + (1-\alpha_1)\left(4\varepsilon x_1x_2\right)x_2 \\
4\varepsilon x_1 x_3^2 &= \alpha_2\left(4\varepsilon x_3^2\right)x_1 + (1-\alpha_2)\left(4\varepsilon x_1x_3\right)x_3 \\
2\varepsilon R x_1 x_2 &= \alpha_3\left(2\varepsilon Rx_1\right)x_2 + (1-\alpha_3)\left(2\varepsilon Rx_2\right)x_1 \\
4\varepsilon x_2 x_1^2 &= \alpha_4\left(4\varepsilon x_1^2\right)x_2 + (1-\alpha_4)\left(4\varepsilon x_1x_2\right)x_1 \\
\varepsilon x_2 x_3^2 &= \alpha_5\left(\varepsilon x_3^2\right)x_2 + (1-\alpha_5)\left(\varepsilon x_2x_3\right)x_3 \\
2\varepsilon R x_1 x_3 &= \alpha_6\left(2\varepsilon Rx_1\right)x_3 + (1-\alpha_6)\left(2\varepsilon Rx_3\right)x_1 \\
4\varepsilon x_3 x_1^2 &= \alpha_7\left(4\varepsilon x_1^2\right)x_3 + (1-\alpha_7)\left(4\varepsilon x_1x_3\right)x_3 \\
\varepsilon x_3 x_2^2 &= \alpha_8\left(\varepsilon x_2^2\right)x_3 + (1-\alpha_8)\left(\varepsilon x_2x_3\right)x_2
\end{aligned}
\tag{6.23}
$$

Therefore, choosing $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0$ and $\alpha_3 = \alpha_6 = 1$ we have the following parameterization

$$\mathbf{A}_{SDC3}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \begin{pmatrix} 3n^2 - 2\varepsilon R x_1 \\ -\frac{8}{3}\varepsilon x_1^2 \end{pmatrix} & (\varepsilon R x_2 - 4\varepsilon x_1 x_2) & (\varepsilon R x_3 - 4\varepsilon x_1 x_3) & 0 & 2n & 0 \\ -4\varepsilon x_1 x_2 & (2\varepsilon R x_1 - \varepsilon x_2^2) & \varepsilon x_2 x_3 & -2n & 0 & 0 \\ 0 & \varepsilon x_2 x_3 & \begin{pmatrix} -n^2 + 2\varepsilon R x_1 \\ -4\varepsilon x_1 x_3 - \varepsilon x_3^2 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix} \tag{6.24}$$

This parameterization has state-dependent controllability matrix

$$\mathrm{M}_3(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2n & 0 \\ 0 & 1 & 0 & -2n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.25}$$

with rank 6. Evaluating the state dependent matrices $\mathbf{A}_{SDC1}(\mathbf{x}), \mathbf{A}_{SDC2}(\mathbf{x}), \mathbf{A}_{SDC3}(\mathbf{x})$ at the origin we have

$$\nabla \mathbf{f}(0) = \mathbf{A}_{SDC1}(0) = \mathbf{A}_{SDC2}(0) = \mathbf{A}_{SDC3}(0) = \mathbf{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & n^2 & 0 & 0 & 0 \end{bmatrix} \tag{6.26}$$

Therefore,

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}_{SDC1}(\mathbf{x})\mathbf{x} = \mathbf{A}_{SDC2}(\mathbf{x})\mathbf{x} = \mathbf{A}_{SDC3}(\mathbf{x})\mathbf{x} \tag{6.27}$$

6.3 Development of SDRE Controller for the SDC Parameterized Models

The SDRE approach to nonlinear dynamical system is similar to the use of LQR design approach and it gives a suboptimal solution for the optimal control problem using a linear quadratic cost function. The SDRE controller is designed by transforming the dynamical system into a state dependent coefficient (SDC) form in which the system matrices are functions of the state variables. The design has advantage over the LQR design in the sense that it can capture the system nonlinearity at each time interval. A linear quadratic control can be stated in the following form. Consider a linear, state dependent, dynamic system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{6.28}$$

The matrix $\mathbf{A}(\mathbf{x})$, whose choice is non-unique, is a state-dependent stabilizable parameterization of the nonlinear system in Ω if the pair $(\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}))$ is stabilizable for all $\mathbf{x} \in \mathbf{R}^n$ and is a state-dependent detectable parameterization of the nonlinear system in Ω if the pair$\Big(\mathbf{A}(\mathbf{x}), Q^{1/2}(\mathbf{x})\Big)$ is detectable for all $\mathbf{x} \in \mathbf{R}^n$. The main aim is to find a state feedback control law which can minimize the performance cost function

$$\mathbf{J}\left(\mathbf{x}_0, \mathbf{u}\right) = \frac{1}{2}\int_{t_0}^{\infty}\Big\{\mathbf{x}^{\mathbf{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^{\mathbf{T}}\mathbf{R}(\mathbf{x})\mathbf{u}\Big\}\,dt \tag{6.29}$$

where $\mathbf{Q}(\mathbf{x}) \in \mathbf{R}^{n\times n}$ is symmetric positive semi-definite (SPSD) matrix, $\mathbf{R}(\mathbf{x}) \in \mathbf{R}^{m\times m}$ is symmetric positive definite (SPD) matrix and may be state dependent. The control accuracy is measured by $\mathbf{x}^{\mathbf{T}}\mathbf{Q}(\mathbf{x})\mathbf{x}$ while the control effort is measured by $\mathbf{u}^{\mathbf{T}}\mathbf{R}(\mathbf{x})\mathbf{u}$. Unlike in the case of other nonlinear control design methods, SDRE technique enables one to tradeoff between the control accuracy and control effort. Instantaneous feedback gains are calculated with the assumptions that the penalty (weighting) matrices $\mathbf{Q}$ and $\mathbf{R}$, and system matrices, $\mathbf{A}$ and $\mathbf{B}$ are constants. Similar to the case of an infinite horizon LQR controller, the feedback gain for a given state can be calculated as [24],

$$\mathbf{u}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\mathbf{x} = -\mathbf{R}(\mathbf{x})^{-\mathbf{1}}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \tag{6.30}$$

where $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}$, $\mathbf{P} : \mathbf{R}^n \mapsto \mathbf{R}^{n\times n}$ is the unique, symmetric and positive definite solution of the State-Dependent Riccati Equation (SDRE)

$$\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}){+}\mathbf{A}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-\mathbf{1}}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0} \tag{6.31}$$

Eq. (6.31) is made asymptotically stable by the control law.

For an SDC parameterization that is detectable and stabilizable, the SDRE method produces a closed loop solution that is locally asymptotically stable. The closed loop solution is given by the system

$$\dot{\mathbf{x}} = \Big[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}(\mathbf{x})^{-\mathbf{1}}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x})\Big]\,\mathbf{x} = \mathbf{A}_{\mathbf{cl}}(\mathbf{x})\mathbf{x} \tag{6.32}$$

If $A_{cl}(x)$ is Hurwitz and symmetric for all $\mathbf{x}$ then global stability holds and $\mathbf{V}(\mathbf{x}) = \mathbf{x^T x}$ is a Lyapunov function for the cost function. Using Taylor series expansion on Eq. (6.32) about zero point gives

$$\dot{\mathbf{x}} = \left[\mathbf{A(0)} - \mathbf{B(0)R(0)^{-1}(0)B^T(0)P(0)}\right]\mathbf{x} + \mathbf{O}\left(\mathbf{x^2}\right) \tag{6.33}$$

Eq. (6.34) shows that in a neighborhood about the origin, the linear term with constant stable coefficient matrix will dominate the higher order terms thereby yielding local asymptotic stability. The SDRE optimality criterion is [6]

$$\dot{\mathbf{P}}(\mathbf{x}) + \left[\frac{\partial\left(\mathbf{A}(\mathbf{x})\mathbf{x}\right)}{\partial \mathbf{x}}\right]^T \mathbf{P}(\mathbf{x}) + \left[\frac{\partial\left(\mathbf{B}(\mathbf{x})\mathbf{u}\right)}{\partial \mathbf{x}}\right]^T \mathbf{P}(\mathbf{x}) = 0 \tag{6.34}$$

Satisfaction of this criterion will enable the closed loop solution to have a local optimum and may also be global minimum.

6.4 Numerical Simulation Results

The orbital elements of the chief and deputy spacecraft used for the simulation is shown in Table 6.1.

6.4.1 SDC Prameterization 1 Numerical Results

The numerical simulations for the SDC parameterization 1 using SDRE and LQR approaches are shown below. Figures 6.1 and 6.2 show the state responses while Figure 6.3 shows the control input.

6.4.2 SDC Prameterization 2 Numerical Results

The numerical simulations for the SDC parameterization 2 using SDRE and LQR approaches are shown in the Figures 6.4, 6.5 and 6.6. Figures 6.4 and 6.5 show the state responses while Figure 6.6 shows the control input.

Table 6.1: Chief and deputy spacecraft orbital elements

Orbital Elements	Chief Spacecraft	Deputy Spacecraft
a (km)	7500	7500
e	0	0.3
i (deg)	4	35.01
Ω (deg)	5	10
ω (deg)	10	75
f (deg)	25	120

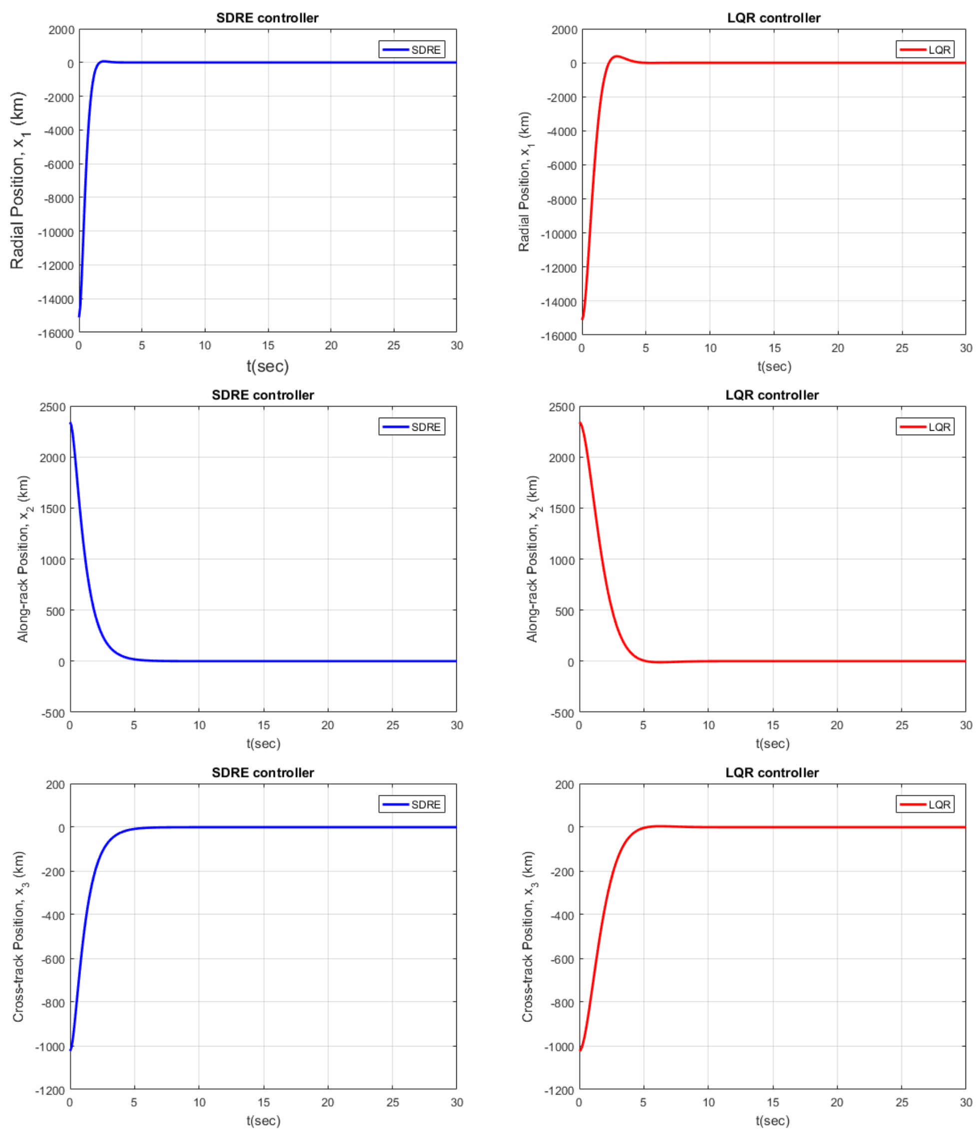

Figure 6.1: SDC Parameterization 1 x_1, x_2, x_3 states using SDRE and LQR approaches

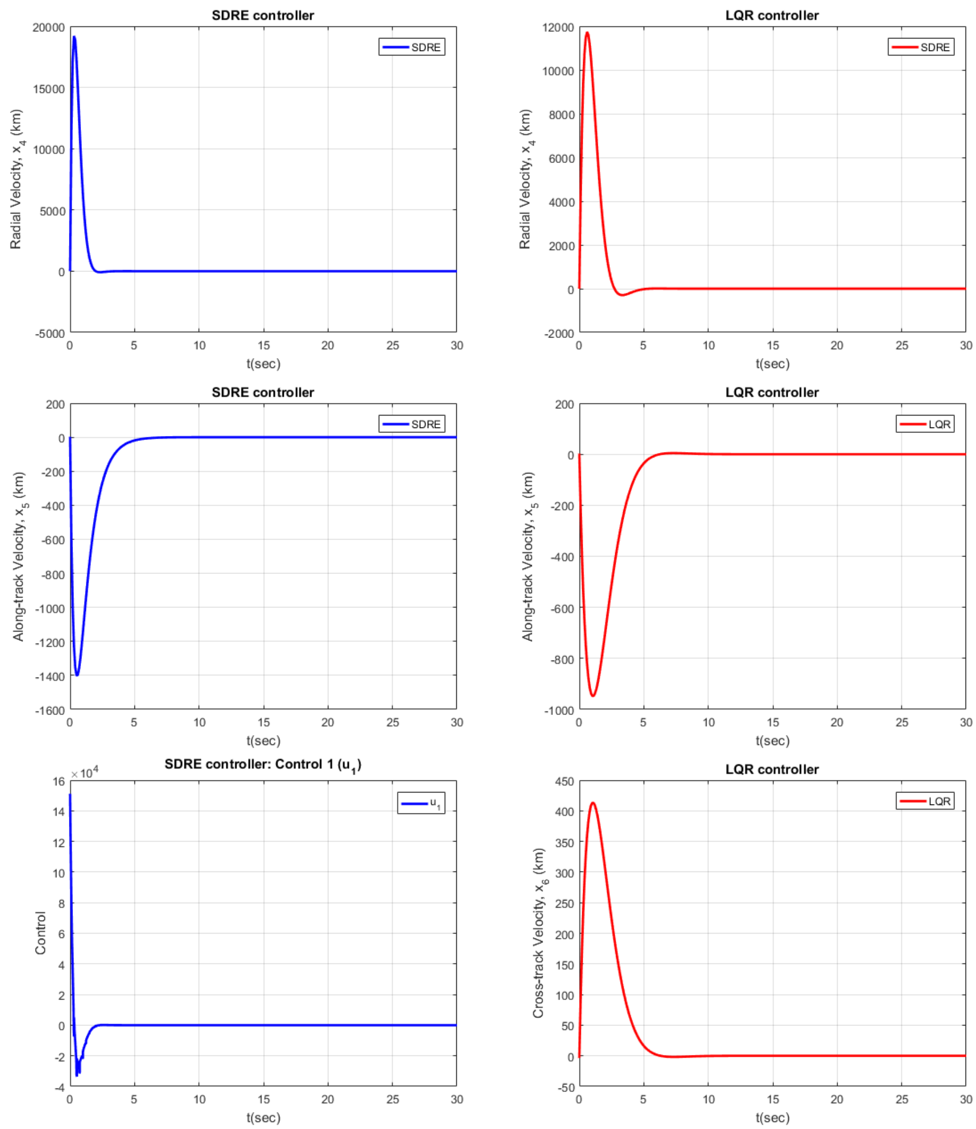

Figure 6.2: SDC Parameterization 1 x_4, x_5, x_6 states using SDRE and LQR approaches

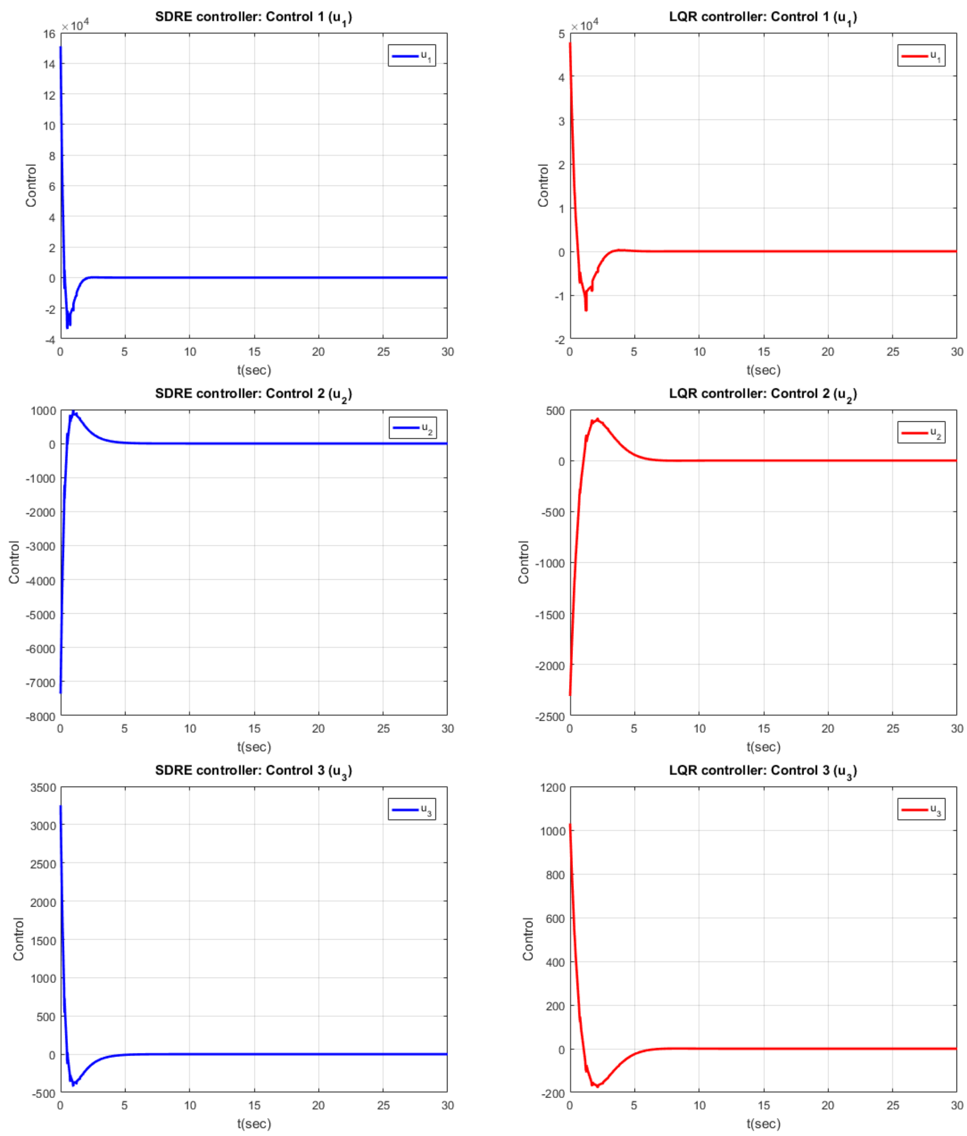

Figure 6.3: SDC Parameterization 1 control input

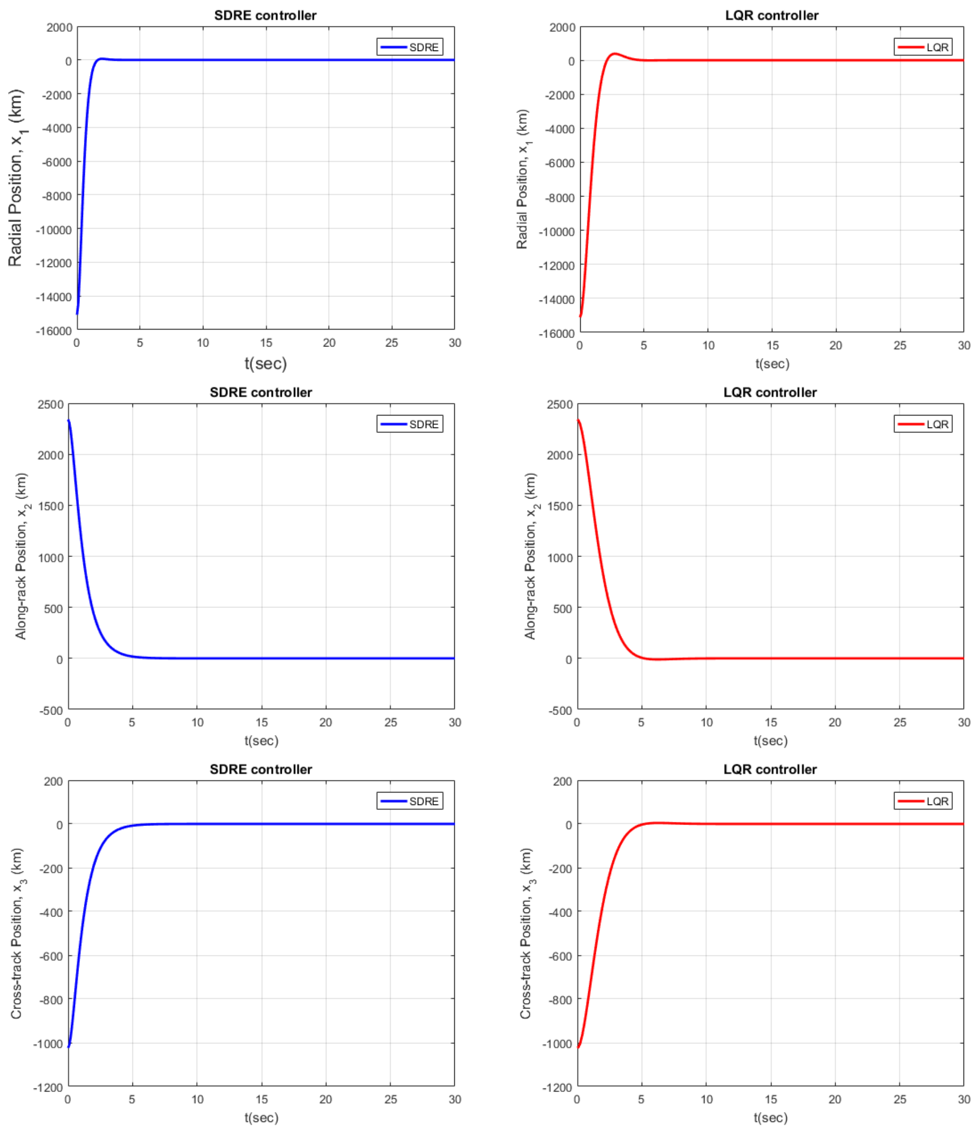

Figure 6.4: SDC Parameterization 2 x_1, x_2, x_3 states using SDRE and LQR approaches

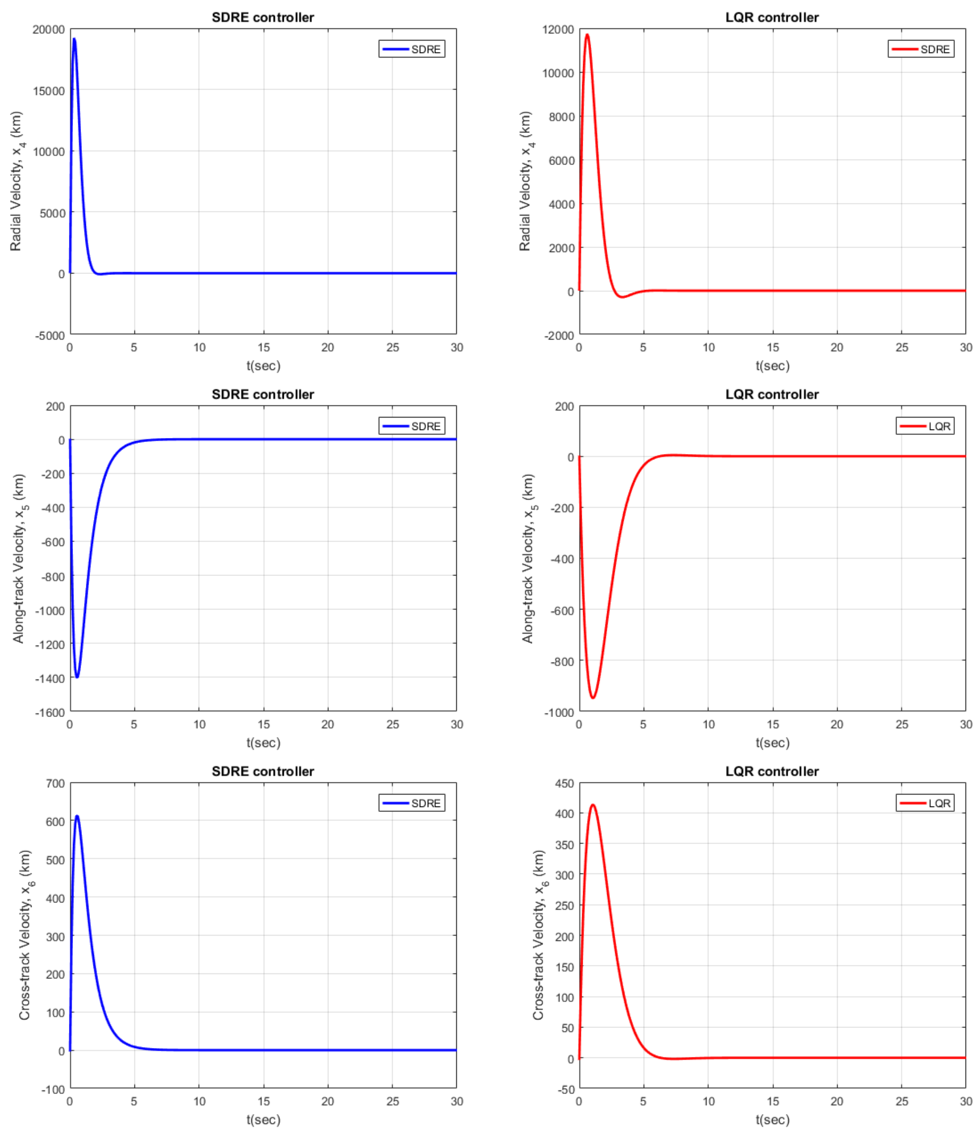

Figure 6.5: SDC Parameterization 2 x_4, x_5, x_6 states using SDRE and LQR approaches

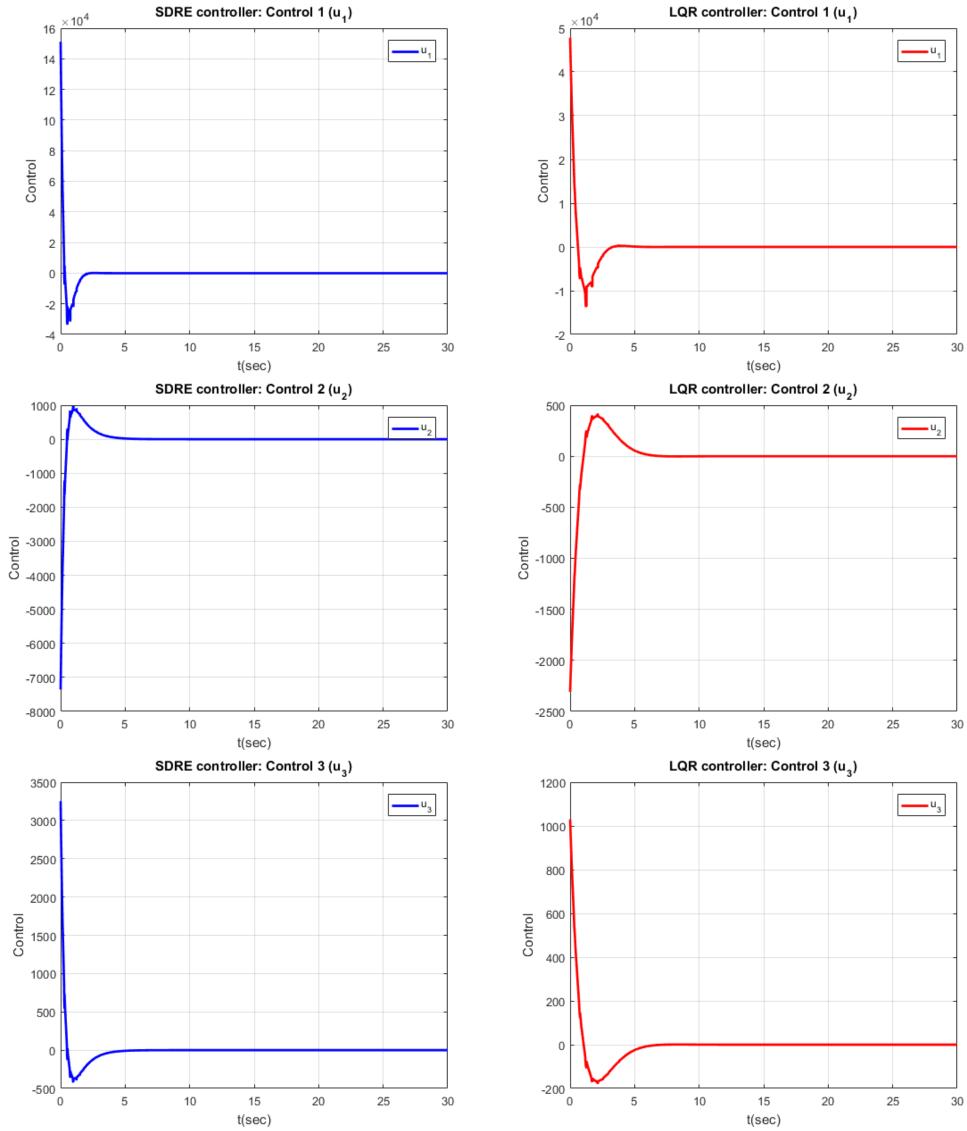

Figure 6.6: SDC Parameterization 2 control input

6.4.3 SDC Prameterization 3 Numerical Results

The numerical simulations for the SDC parameterization 1 using SDRE and LQR approaches are shown in the Figures 6.7, 6.8 and 6.9. Figures 6.7 and 6.8 show the state responses while Figure 6.9 shows the control input.

The simulation results show that both the LQR and the SDRE are able to stabilize the spacecraft. However, the LQR controller which uses the linearized model did not give good approximation of the nonlinear model in the regions that are far from the equilibrium point. On the other hand, the SDRE controller did not cancel the beneficial nonlinear terms and as a result brings the system back to the equilibrium positions with lesser control effort. In overall, the LQR controller has better and improved performance over the LQR controller as shown in the simulation results for the three parameterized systems.

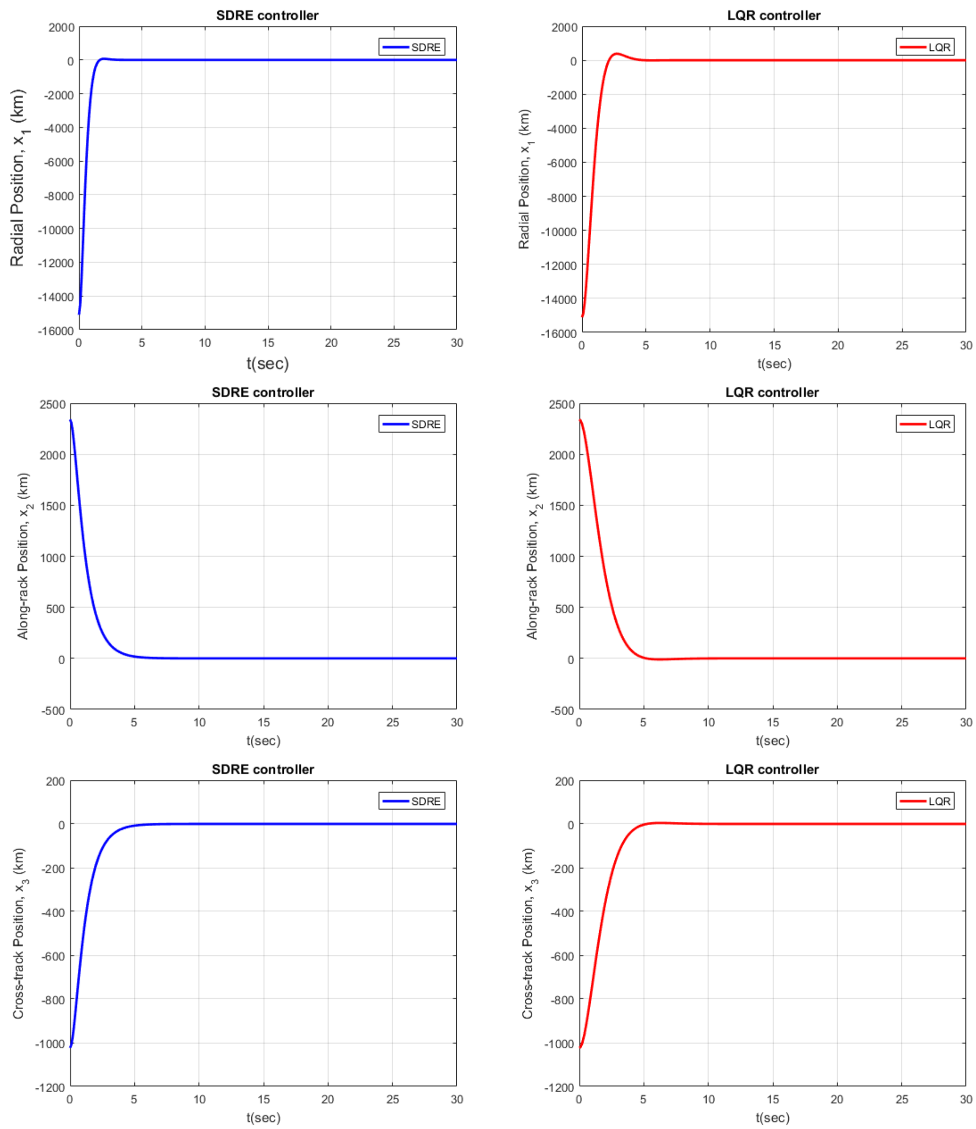

Figure 6.7: SDC Parameterization 3 x_1, x_2, x_3 states using SDRE and LQR approaches

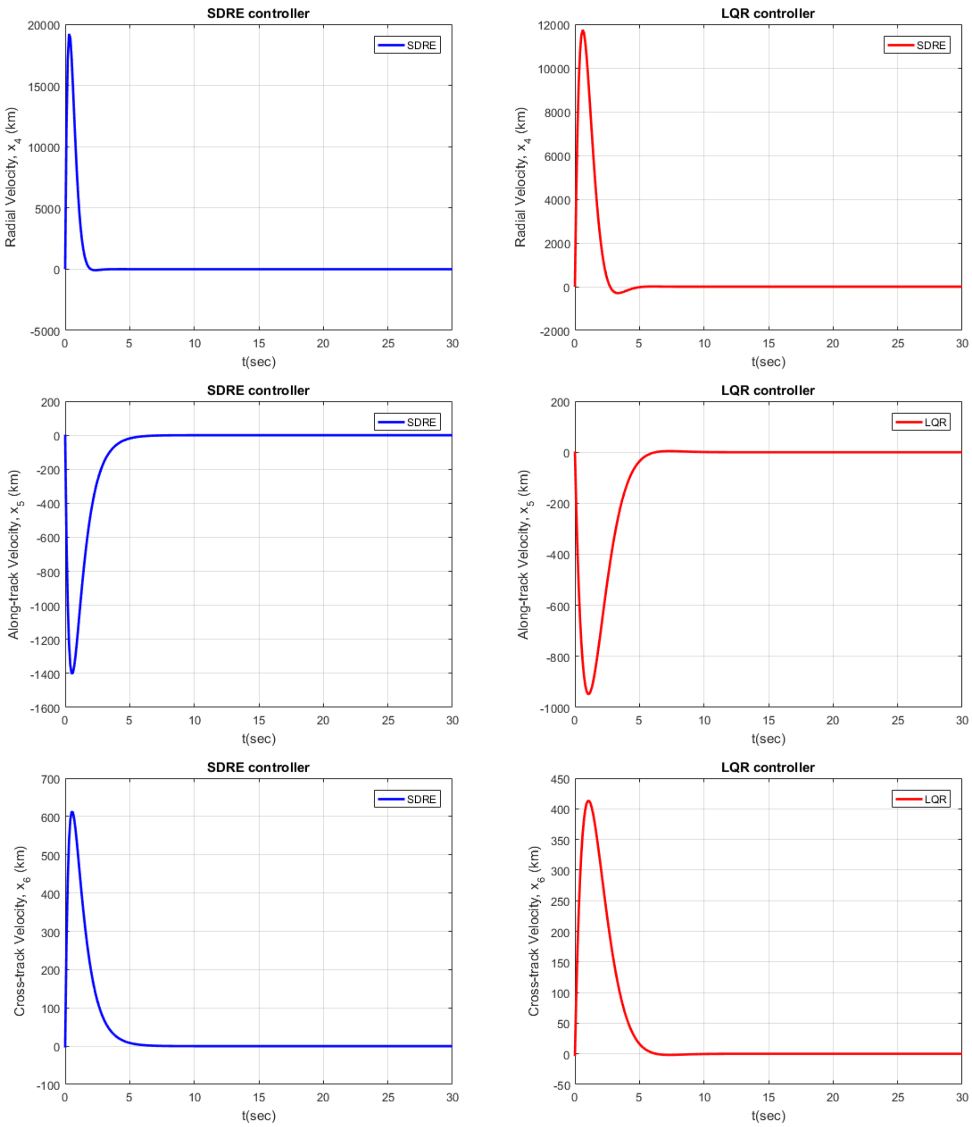

Figure 6.8: SDC Parameterization 3 x_4, x_5, x_6 states using SDRE and LQR approaches

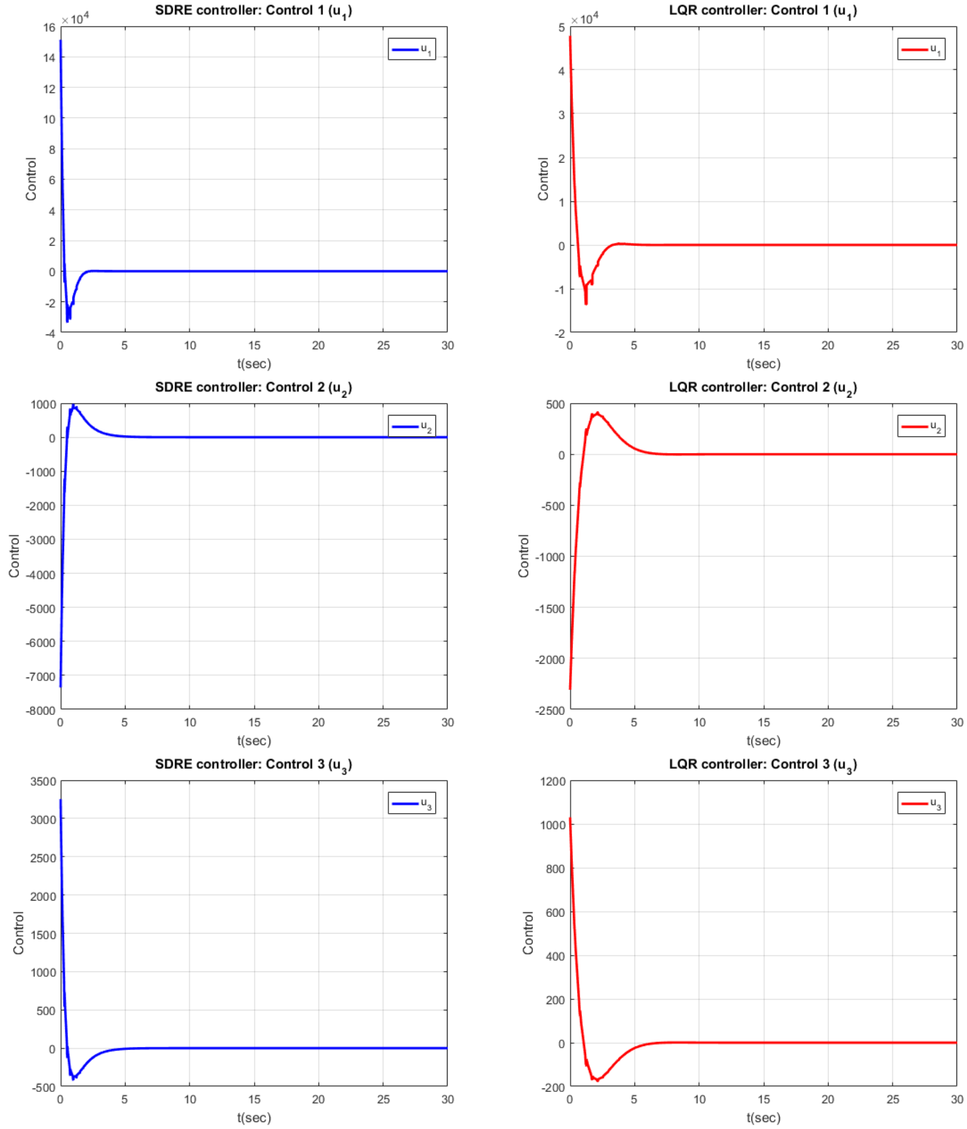

Figure 6.9: SDC Parameterization 3 control input

Chapter 7

Conclusion

The Hill-Clohessy-Wilshire (HCW) equations, a linearized model of relative motion, is generally used to desribe spacecraft relative motion due to its simplicity. This model has numerous applications in rendezvous and proximity operations, spacecraft formation flying, distributed spacecraft missions, intercept operations etc. But, the assumption of a circular orbit in the formulation of HCW equations is a problem for a formation of satellites in an elliptical orbit.

Precise orbit geometry is difficult to obtain using Hill frame coordinates to describe the relative orbit. But, parameterizing the relative motion using the Keplerian orbital elements simplifies the orbit description better than using Hill frame coordinates. The use of orbital elements is beneficial because it has only one term (true anomaly) that changes with time out of the six orbital elements and this, thereby, reduces the number of variables to be tracked from six to one.

The unapproximated differential equations governing the spacecraft relative motion are nonlinear. The simplest linear equations, HCW equations, are easier to characterize mathematically and the tools for their analysis are well developed. In solving the spacecraft nonlinear equations quadratic and cubic approximations methods can be used to approximate the equations which can be further linearized using harmonic balance method, averaging method, floquet, homotopy perturbation etc.

The State Dependent Riccati Equation (SDRE) method, based on the factorization of the nonlinear dynamics into the state vector, brings a nonlinear system to a non-unique linear structure having matrices with state dependent coefficients and gives suboptimum control law. The algorithm thus involves solving, at a given point in the state space, an algebraic state dependent Riccati equation. The non-uniqueness of the factorization creates extra degrees of freedom, which can be used to enhance controller performance. In comparison to Linear Quadratic Regulator (LQR) method, the SDRE method doe not cancel out the beneficial nonlinear terms.

In this dissertation work, cubic approximation model of spacecraft relative motion is developed. From this approximation, two new linearized models of the relative motion, using harmonic balance and averaging methods, are obtained. The numerical solutions show that the models can provide better approximations of the relative motion than the HCW model.

Another contribution of this work is the development of two basic models of Abel-type (third-order) and Riccati-type (second-order) spacecraft equations of relative motion. The models, which has only true-of latitude as time-varying, captured the dynamics better than using position and velocity in which all the six parameters vary with time. Also, using standard transformation techniques, closed form solutions of the Abel-type and Riccati-type equations are developed. These equations can be used for spacecraft control, analysis and maneuver planning.

Innovatively, feedback controllers are designed for the relative motion via State-Dependent Riccati-Equation (SDRE) control strategy. The key interest in the use of SDRE strategy is its ability to provide an effective algorithm for synthesizing nonlinear feedback controls by allowing for nonlinearities in the system states, while offering design flexibility through state dependent weighting matrices.

The future works can be extended to the case where chief spacecraft is in eccentric orbit. Cubic approximation model of the eccentric case can be developed and various nonlinear techniques such as averaging method, normal form, Lyapunov-Floquet (LF) transformation can be applied to the cubic model for better analysis of the dynamics.

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Appendices

Appendix A

Radial, Along-track and Cross-track Directions Nonlinear Terms

In this Appendix, Equations (A.1) to (A.6) are the expansions of radial nonlinear terms, Equations (A.7) to (A.10) are the expansions of along-track nonlinear terms and Equations (A.11) to (A.14) are the expansions of cross-track nonlinear terms.

Radial Direction Nonlinear Terms

$$\begin{aligned}
x^2 &= \tfrac{4}{n^2}{c_1}^2 + \tfrac{4}{n}c_1c_2\sin nt + \tfrac{4}{n}c_1c_3\cos nt + 2c_2c_3\sin nt\cos nt + {c_2}^2\sin^2 nt + {c_3}^2\cos^2 nt \\
&= \tfrac{4}{n^2}{c_1}^2 + \tfrac{4}{n}c_1c_2\sin nt + \tfrac{4}{n}c_1c_3\cos nt + c_2c_3\sin 2nt + \tfrac{1}{2}{c_2}^2(1-\cos 2nt) + \tfrac{1}{2}{c_3}^2(1+\cos 2nt) \\
&\approx \tfrac{4}{n^2}{c_1}^2 + \tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_3}^2 + \tfrac{4}{n}c_1c_2\sin nt + \tfrac{4}{n}c_1c_3\cos nt \\
&\approx \tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_3}^2 - \tfrac{4}{n^2}{c_1}^2 + \tfrac{4}{n}c_1\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&= \tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_3}^2 - \tfrac{4}{n^2}{c_1}^2 + \tfrac{4}{n}c_1 x
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
y^2 &= 9{c_1}^2t^2 - 12c_1c_2t\cos nt + 12c_1c_3t\sin nt - 6c_1c_4t + 4{c_2}^2\cos^2 nt + 4c_2c_4\cos nt \\
&\quad -4c_3c_4\sin nt - 8c_2c_3\sin nt\cos nt + 4{c_3}^2\sin^2 nt + {c_4}^2 \\
&= -6c_1t\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - 9{c_1}^2t^2 \\
&\quad +2c_4\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) + 6c_1c_4t - {c_4}^2 + 2{c_2}^2 + 2{c_3}^2 \\
&= 2{c_2}^2 + 2{c_3}^2 - {c_4}^2 + \left(2c_4 - 6c_1t\right)y - 9{c_1}^2t^2 + 6c_1c_4t
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
z^2 &= {c_5}^2\sin^2 nt + 2c_5c_6\sin nt\cos nt + {c_6}^2\cos^2 nt = \tfrac{1}{2}{c_5}^2\left(1-\cos 2nt\right) + c_5c_6\sin 2nt \\
&\quad +\tfrac{1}{2}{c_6}^2\left(1+\cos 2nt\right) \approx \tfrac{1}{2}{c_5}^2 + \tfrac{1}{2}{c_6}^2
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
x^3 &= \tfrac{8}{n^3}{c_1}^3 + \tfrac{12}{n^2}{c_1}^2c_2\sin nt + \tfrac{12}{n^2}{c_1}^2c_3\cos nt + \tfrac{6}{n}c_1c_2c_3\sin nt\cos nt + \tfrac{6}{n}c_1{c_2}^2\sin^2 nt \\
&\quad +\tfrac{6}{n}c_1{c_3}^2\cos^2 nt + 3{c_2}^2c_3\sin^2 nt\cos nt + 3c_2{c_3}^2\sin nt\cos^2 nt + {c_2}^3\sin^3 nt + {c_3}^3\cos^3 nt \\
&= \tfrac{8}{n^3}{c_1}^3 + \tfrac{12}{n^2}{c_1}^2c_2\sin nt + \tfrac{12}{n^2}{c_1}^2c_3\cos nt + \tfrac{3}{n}c_1c_2c_3\sin 2nt + \tfrac{3}{n}c_1{c_2}^2\left(1-\cos 2nt\right) \\
&\quad +\tfrac{3}{n}c_1{c_3}^2\left(1+\cos 2nt\right) + \tfrac{3}{2}{c_2}^2c_3\left(1-\cos 2nt\right)\cos nt + \tfrac{3}{2}c_2{c_3}^2\sin nt\left(1+\cos 2nt\right) \\
&\quad +{c_2}^3\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) + {c_3}^3\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) \\
&= \tfrac{12}{n^2}{c_1}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) - \tfrac{16}{n^3}{c_1}^3 + \tfrac{3}{4}{c_2}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&\quad +\tfrac{3}{2n}c_1{c_2}^2 + \tfrac{3}{4}{c_3}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{3}{2n}c_1{c_3}^2 \\
&= \left(-\tfrac{16}{n^3}{c_1}^3 + \tfrac{3}{2n}c_1{c_2}^2 + \tfrac{3}{2n}c_1{c_3}^2\right) + \left(\tfrac{12}{n^2}{c_1}^2 + \tfrac{3}{4}{c_2}^2 + \tfrac{3}{4}{c_3}^2\right)x
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
xy^2 &= \tfrac{2}{n}c_1{c_4}^2 + \tfrac{8}{n}c_1c_2c_4\cos nt - \tfrac{8}{n}c_1c_3c_4\sin nt + c_2{c_4}^2\sin nt + c_3{c_4}^2\cos nt \\
&-\tfrac{16}{n}c_1c_2c_3\sin nt\cos nt + 4{c_2}^2c_4\sin nt\cos nt - 4{c_3}^2c_4\sin nt\cos nt + \tfrac{8}{n}c_1{c_2}^2\cos^2 nt \\
&+\tfrac{8}{n}c_1{c_3}^2\sin^2 nt - 4c_2c_3c_4\sin^2 nt + 4{c_3}^3\sin^2 nt\cos nt - 8{c_2}^2c_3\sin^2 nt\cos nt \\
&+4c_2c_3c_4\cos^2 nt - 8c_2{c_3}^2\sin nt\cos^2 nt + 4{c_2}^3\sin nt\cos^2 nt + 4c_2{c_3}^2\sin^3 nt \\
&+4{c_2}^2c_3\cos^3 nt - \tfrac{12}{n}{c_1}^2c_4t - \tfrac{24}{n}{c_1}^2c_2t\sin nt + \tfrac{24}{n}{c_1}^2c_3t\sin nt - 6c_1c_2c_4t\sin nt \\
&-6c_1c_3c_4t\cos nt - 12c_1{c_2}^2t\sin nt\cos nt + 12c_1{c_3}^2t\sin nt\cos nt + 12c_1c_2c_3 t\sin^2 nt \\
&-12c_1c_2c_3t\cos^2 nt + \tfrac{18}{n}{c_1}^2t^2 + 9{c_1}^2c_2t^2\sin nt + 9{c_1}^2c_3t^2\cos nt \\
&= \tfrac{2}{n}c_1{c_4}^2 + \tfrac{8}{n}c_1c_2c_4\cos nt - \tfrac{8}{n}c_1c_3c_4\sin nt + c_2{c_4}^2\sin nt + c_3{c_4}^2\cos nt \\
&-\tfrac{8}{n}c_1c_2c_3\sin 2nt + 2{c_2}^2c_4\sin 2nt - 2{c_3}^2c_4\sin 2nt + \tfrac{4}{n}c_1{c_2}^2(1+\cos 2nt) \\
&+\tfrac{4}{n}c_1{c_3}^2(1-\cos 2nt) - 2c_2c_3c_4(1-\cos 2nt) + 2{c_3}^3(1-\cos 2nt)\cos nt \\
&-4{c_2}^2c_3(1-\cos 2nt)\cos nt + 2c_2c_3c_4(1+\cos 2nt) - 4c_2{c_3}^2\sin nt(1+\cos 2nt) \\
&+2{c_2}^3\sin nt(1+\cos 2nt) + 4c_2{c_3}^2\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) \\
&+4{c_2}^2c_3\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) \\
&-\tfrac{12}{n}{c_1}^2c_4t - \tfrac{24}{n}{c_1}^2c_2t\cos nt + \tfrac{24}{n}{c_1}^2c_3t\sin nt - 6c_1c_2c_4t\sin nt - 6c_1c_3c_4t\cos nt \\
&-6c_1{c_2}^2t\sin 2nt + 6c_1{c_3}^2t\sin 2nt + 6c_1c_2c_3t(1-\cos 2nt) - 6c_1c_2c_3t(1+\cos 2nt) \\
&+\tfrac{18}{n}{c_1}^2t^2 + 9{c_1}^2c_2t^2\sin nt + 9{c_1}^2c_3t^2\cos nt \\
&= {c_4}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{4}{n}c_1c_4(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4) \\
&-\tfrac{4}{n}c_1{c_4}^2 + {c_3}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{2}{n}c_1{c_3}^2 \\
&+{c_2}^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{2}{n}c_1{c_2}^2 \\
&-\tfrac{12}{n}{c_1}^2t(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4) - \tfrac{36}{n}{c_1}^3t^2 + \tfrac{12}{n}{c_1}^2c_4t \\
&-6c_1c_4t\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{12}{n}{c_1}^2c_4t \\
&+9{c_1}^2t^2\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) - \tfrac{18}{n}{c_1}^3t^2 + \tfrac{18}{n}{c_1}^2t^2 \\
&= \left(-\tfrac{4}{n}c_1{c_4}^2 + \tfrac{2}{n}c_1{c_3}^2 + \tfrac{2}{n}c_1{c_2}^2 + \tfrac{24}{n}{c_1}^2c_4t - \tfrac{54}{n}{c_1}^3t^2 + \tfrac{18}{n}{c_1}^2t^2\right) \\
&+({c_2}^2 + {c_3}^2 + {c_4}^2 - 6c_1c_4t + 9{c_1}^2t^2)\,x + \left(\tfrac{4}{n}c_1c_4 - \tfrac{12}{n}{c_1}^2t\right)y
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
xz^2 &= \tfrac{2}{n}c_1{c_5}^2\sin^2 nt + \tfrac{4}{n}c_1c_5c_6\sin nt\cos nt + \tfrac{2}{n}c_1{c_6}^2\cos^2 nt + c_2{c_5}^2\sin^3 nt \\
&+2c_2c_5c_6\sin^2 nt\cos nt \\
&+c_2{c_6}^2\sin nt\cos^2 nt + c_3{c_5}^2\cos nt\sin^2 nt + 2c_3c_5c_6\sin nt\cos^2 nt + c_3{c_6}^2\cos^3 nt \\
&= \tfrac{1}{n}c_1{c_5}^2(1-\cos 2nt) + \tfrac{2}{n}c_1c_5c_6\sin 2nt + \tfrac{1}{n}c_1{c_6}^2(1+\cos 2nt) \\
&+c_2{c_5}^2\left(\tfrac{3}{4}\sin nt + \tfrac{1}{4}\sin 3nt\right) + c_2c_5c_6(1-\cos 2nt)\cos nt + \tfrac{1}{2}c_2{c_6}^2\sin nt(1+\cos 2nt) \\
&+\tfrac{1}{2}c_3{c_5}^2\cos nt(1-\cos 2nt) + c_3c_5c_6\sin nt(1+\cos 2nt) + c_3{c_6}^2\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) \\
&= \tfrac{1}{n}c_1{c_5}^2 + \tfrac{1}{n}c_1{c_6}^2 + \tfrac{3}{4}c_2{c_5}^2\sin nt + \tfrac{1}{2}c_2c_5c_6\cos nt + \tfrac{1}{4}c_2{c_6}^2\sin nt + \tfrac{1}{4}c_3{c_5}^2\cos nt \\
&+\tfrac{1}{2}c_3c_5c_6\sin nt + \tfrac{3}{4}c_3{c_6}^2\cos nt \\
&= \tfrac{3}{4n}c_1{c_5}^2 + \tfrac{3}{4n}c_1{c_6}^2 + \tfrac{1}{4}{c_5}^2\left(\tfrac{1}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&+\tfrac{1}{4}{c_6}^2\left(\tfrac{1}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&+\tfrac{1}{2}c_2c_5(c_5\sin nt + c_6\cos nt) + \tfrac{1}{2}c_2c_5(c_5\sin nt + c_6\cos nt) \\
&+\tfrac{1}{2}c_3c_6(c_5\sin nt + c_6\cos nt) \\
&= \tfrac{3}{4n}c_1{c_5}^2 + \tfrac{3}{4n}c_1{c_6}^2 + \left(\tfrac{1}{4}{c_5}^2 + \tfrac{1}{4}{c_6}^2\right)x + \left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right)z
\end{aligned} \tag{A.6}$$

Along-track Direction Nonlinear Terms

$$
\begin{aligned}
&yx = -\tfrac{6}{n}{c_1}^2 t - 3c_1c_2 t\sin nt - 3c_1c_3 t\cos nt + \tfrac{4}{n}c_1c_2\cos nt + {c_2}^2\sin 2nt\\
&+c_2c_3\left(1+\cos 2nt\right) - \tfrac{4}{n}c_1c_3\sin nt - c_2c_3\left(1-\cos 2nt\right) - {c_3}^2\sin 2nt + \tfrac{2}{n}c_1c_4\\
&+c_2c_4\sin nt + c_4c_3\cos nt\\
&\approx -3c_1 t\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{2}{n}c_1\left(-3c_1 t + 2c_2\cos nt - 2c_3\sin nt + c_4\right)\\
&+\tfrac{6}{n}{c_1}^2 t + c_4\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) - \tfrac{2}{n}c_1c_4\\
&= \left(-\tfrac{2}{n}c_1c_4 + \tfrac{6}{n}{c_1}^2 t\right) + \left(-3c_1 t + c_4\right)x + \tfrac{2}{n}c_1 y
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
&yx^2 = -\tfrac{12}{n^2}{c_1}^3 t - \tfrac{12}{n}{c_1}^2c_2 t\sin nt - \tfrac{12}{n}{c_1}^2c_3 t\cos nt - 6c_1c_2c_3 t\sin nt\cos nt - 3c_1{c_2}^2 t\sin^2 nt\\
&-3c_1{c_3}^2 t\cos^2 nt + \tfrac{8}{n^2}{c_1}^2c_2\cos nt + \tfrac{8}{n}c_1{c_2}^2\sin nt\cos nt + \tfrac{8}{n}c_1c_2c_3\cos^2 nt\\
&+4{c_2}^2c_3\sin nt\cos^2 nt + 2{c_2}^3\sin^2 nt\cos nt + 2c_2{c_3}^2\cos^3 nt - \tfrac{8}{n^2}{c_1}^2c_3\sin nt - \tfrac{8}{n}c_1c_2c_3\sin^2 nt\\
&-\tfrac{8}{n}c_1{c_3}^2\sin nt\cos nt - 4c_2{c_3}^2\sin^2 nt\cos nt - 2{c_2}^2c_3\sin^3 nt - 2{c_3}^3\sin nt\cos^2 nt + \tfrac{4}{n^2}{c_1}^2c_4\\
&+\tfrac{4}{n}c_1c_2c_4\sin nt + \tfrac{4}{n}c_1c_3c_4\cos nt + 2c_2c_3c_4\sin nt\cos nt + {c_2}^2c_4\sin^2 nt + {c_3}^2c_4\cos^2 nt\\
&= -\tfrac{12}{n^2}{c_1}^3 t - \tfrac{12}{n}{c_1}^2c_2 t\sin nt - \tfrac{12}{n}{c_1}^2c_3 t\cos nt - 3c_1c_2c_3 t\sin 2nt - \tfrac{3}{2}c_1{c_2}^2 t\left(1-\cos 2nt\right)\\
&-\tfrac{3}{2}c_1{c_3}^2 t\left(1+\cos 2nt\right) + \tfrac{8}{n^2}{c_1}^2c_2\cos nt + \tfrac{4}{n}c_1{c_2}^2\sin 2nt + \tfrac{4}{n}c_1c_2c_3\left(1+\cos 2nt\right)\\
&+2{c_2}^2c_3\sin nt\left(1+\cos 2nt\right) + {c_2}^3\left(1-\cos 2nt\right)\cos nt + 2c_2{c_3}^2\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right)\\
&-\tfrac{8}{n^2}{c_1}^2c_3\sin nt - \tfrac{4}{n}c_1c_2c_3\left(1-\cos 2nt\right) - \tfrac{4}{n}c_1{c_3}^2\sin 2nt - 2c_2{c_3}^2\left(1-\cos 2nt\right)\cos nt\\
&-2{c_2}^2c_3\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) - {c_3}^3\sin nt\left(1+\cos 2nt\right) + \tfrac{4}{n^2}{c_1}^2c_4 + \tfrac{4}{n}c_1c_2c_4\sin nt\\
&+\tfrac{4}{n}c_1c_3c_4\cos nt + c_2c_3c_4\sin 2nt + \tfrac{1}{2}{c_2}^2c_4\left(1-\cos 2nt\right) + \tfrac{1}{2}{c_3}^2c_4\left(1+\cos 2nt\right)\\
&= -\tfrac{12}{n}{c_1}^2 t\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{12}{n^2}{c_1}^3 t + \tfrac{4}{n^2}{c_1}^2\left(-3c_1 t + 2c_2\cos nt - 2c_3\sin nt + c_4\right)\\
&+\tfrac{12}{n^2}{c_1}^3 t + \tfrac{1}{4}{c_2}^2\left(-3c_1 t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - \tfrac{3}{4}c_1{c_2}^2 t + \tfrac{1}{4}{c_2}^2c_4\\
&+\tfrac{1}{4}{c_3}^2\left(-3c_1 t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - \tfrac{3}{4}c_1{c_3}^2 t + \tfrac{1}{4}{c_3}^2c_4\\
&+\tfrac{4}{n}c_1c_4\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) - \tfrac{8}{n^2}{c_1}^2c_4\\
&= \tfrac{1}{4}{c_2}^2c_4 + \tfrac{1}{4}{c_3}^2c_4 - \tfrac{8}{n^2}{c_1}^2c_4 + \tfrac{24}{n^2}{c_1}^3 t - \tfrac{3}{4}c_1{c_2}^2 t - \tfrac{3}{4}c_1{c_3}^2 t + \left(\tfrac{4}{n}c_1c_4 - \tfrac{12}{n}{c_1}^2 t\right)x\\
&+\left(\tfrac{4}{n^2}{c_1}^2 + \tfrac{1}{4}{c_2}^2 + \tfrac{1}{4}{c_3}^2\right)y\\
&= \tfrac{1}{4}{c_2}^2c_4 + \tfrac{1}{4}{c_3}^2c_4 - \tfrac{8}{n^2}{c_1}^2c_4 + \tfrac{24}{n^2}{c_1}^3 t - \tfrac{3}{4}c_1{c_2}^2 t - \tfrac{3}{4}c_1{c_3}^2 t\\
&+\left(\tfrac{4}{n}c_1c_4 - \tfrac{12}{n}{c_1}^2 t\right)x + \left(\tfrac{4}{n^2}{c_1}^2 + \tfrac{1}{4}{c_2}^2 + \tfrac{1}{4}{c_3}^2\right)y
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
y^3 &= -3c_1{c_4}^2t - 12c_1c_2c_4t\cos nt + 12c_1c_3c_4t\sin nt + 24c_1c_2c_3t\sin nt\cos nt \\
&+18{c_1}^2c_4t^2 + 36{c_1}^2c_2t^2\cos nt - 36{c_1}^2c_3t^2\sin nt - 27{c_1}^3t^3 \\
&-12c_1{c_2}^2t\cos^2 nt - 12c_1{c_3}^2t\sin^2 nt + 2c_2{c_4}^2\cos nt + 8{c_2}^2c_4\cos^2 nt \\
&-24c_1{c_2}^2t\cos^2 nt + 24c_1c_2c_3t\sin nt\cos nt + 18{c_1}^2c_2t^2\cos nt \\
&+8{c_2}^3\cos^3 nt + 8c_2{c_3}^2\sin^2 nt\cos nt - 2c_3{c_4}^2\sin nt - 8c_2c_3c_4\sin nt\cos nt \\
&+8{c_3}^2c_4\sin^2 nt + 16c_2{c_3}^2\sin^2 nt\cos nt + 12c_1c_3c_4t\sin nt + 24c_1c_2c_3t\sin nt\cos nt \\
&= -3c_1{c_4}^2t - 12c_1c_2c_4t\cos nt + 12c_1c_3c_4t\sin nt + 12c_1c_2c_3t\sin 2nt \\
&+18{c_1}^2c_4t^2 + 36{c_1}^2c_2t^2\cos nt - 36{c_1}^2c_3t^2\sin nt - 27{c_1}^3t^3 \\
&-6c_1{c_2}^2t\left(1+\cos 2nt\right) - 6c_1{c_3}^2t\left(1-\cos 2nt\right) + 2c_2{c_4}^2\cos nt \\
&+4{c_2}^2c_4\left(1+\cos 2nt\right) - 4c_2c_3c_4\sin 2nt - 8{c_2}^2c_3\sin nt\left(1+\cos 2nt\right) \\
&-12c_1c_2c_4t\cos nt - 12c_1{c_2}^2t\left(1+\cos 2nt\right) + 12c_1c_2c_3t\sin 2nt \\
&+18{c_1}^2c_2t^2\cos nt + 8{c_2}^3\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) \\
&+4c_2{c_3}^2\left(1-\cos 2nt\right)\cos nt - 2c_3{c_4}^2\sin nt - 4c_2c_3c_4\sin 2nt \\
&+4{c_3}^2c_4\left(1-\cos 2nt\right) + 8c_2{c_3}^2\left(1-\cos 2nt\right)\cos nt + 12c_1c_3c_4t\sin nt \\
&+12c_1c_2c_3t\sin 2nt - 12c_1{c_3}^2t\left(1-\cos 2nt\right) - 18{c_1}^2c_3t^2\sin nt \\
&-4{c_2}^2c_3\sin nt\left(\cos 2nt + 1\right) - 8{c_3}^3\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) + {c_4}^3 \\
&+4c_2{c_4}^2\cos nt - 4c_3{c_4}^2\sin nt - 4c_2c_3c_4\sin 2nt - 6c_1{c_4}^2t \\
&-12c_1c_2c_4t\cos nt + 12c_1c_3c_4t\sin nt + 9{c_1}^2c_4t^2 + 2{c_2}^2c_4\left(1+\cos 2nt\right) \\
&+2{c_3}^2c_4\left(1-\cos 2nt\right) \\
&= -18c_1c_4t\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - 36{c_1}^2c_4t^2 + 15c_1{c_4}^2t \\
&+27{c_1}^2t^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) + 54{c_1}^3t^3 - 18{c_1}^2c_4t^2 \\
&+3{c_4}^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) + 3c_1{c_4}^2t - 2{c_4}^3 \\
&+3{c_2}^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - 3c_1{c_2}^2t - {c_2}^2c_4 \\
&+3{c_3}^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) + 3c_1{c_3}^2t + {c_3}^2c_4 - 6c_1{c_2}^2t \\
&+4{c_2}^2c_4 - 12c_1{c_3}^2t + 2{c_3}^2c_4 \\
&= \left(3{c_2}^2c_4 + 3{c_3}^2c_4 - 2{c_4}^3 + \left(-9c_1{c_2}^2 - 9c_1{c_3}^2 + 18c_1{c_4}^2\right)t - 54{c_1}^2c_4t^2 + 54{c_1}^3t^3\right) \\
&+ \left(3{c_2}^2 + 3{c_3}^2 + 3{c_4}^2 - 18c_1c_4t + 27{c_1}^2t^2\right)y
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
yz^2 &= -3c_1{c_5}^2t\sin^2 nt - 6c_1c_5c_6t\sin nt\cos nt - 3c_1{c_6}^2t\cos^2 nt + 2c_2{c_5}^2\cos nt\sin^2 nt \\
&+4c_2c_5c_6\sin nt\cos^2 nt + 2c_2{c_6}^2\cos^3 nt - 2c_3{c_5}^2\sin^3 nt - 4c_3c_5c_6\sin^2 nt\cos nt \\
&-2c_3{c_6}^2\sin nt\cos^2 nt + c_4{c_5}^2\sin^2 nt + 2c_4c_5c_6\sin nt\cos nt + c_4{c_6}^2\cos^2 nt \\
&= -\tfrac{3}{2}c_1{c_5}^2t\left(1-\cos 2nt\right) - 3c_1c_5c_6t\sin 2nt - \tfrac{3}{2}c_1{c_6}^2\mathrm{t}\left(1+\cos 2nt\right) \\
&+c_2{c_5}^2\cos nt\left(1-\cos 2nt\right) + 2c_2c_5c_6\sin nt\left(1+\cos 2nt\right) + 2c_2{c_6}^2\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) \\
&-2c_3{c_5}^2\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) - 2c_3c_5c_6\left(1-\cos 2nt\right)\cos nt \\
&-c_3{c_6}^2\sin nt\left(1+\cos 2nt\right) + \tfrac{1}{2}c_4{c_5}^2\left(1-\cos 2nt\right) + c_4c_5c_6\sin 2nt + \tfrac{1}{2}c_4{c_6}^2\left(1+\cos 2nt\right) \\
&= -\tfrac{3}{2}c_1{c_5}^2t + \tfrac{1}{2}c_2{c_5}^2\cos nt + \tfrac{1}{2}c_4{c_5}^2 - \tfrac{3}{2}c_1{c_6}^2\mathrm{t} - \tfrac{1}{2}c_3{c_6}^2\sin nt + \tfrac{1}{2}c_4{c_6}^2 \\
&-c_3c_5c_6\cos nt - \tfrac{3}{2}c_3{c_5}^2\sin nt + c_2c_5c_6\sin nt + \tfrac{3}{2}c_2{c_6}^2\cos nt \\
&= \tfrac{1}{2}{c_5}^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) + \tfrac{1}{2}{c_6}^2\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) \\
&-c_3c_5\left(c_5\sin nt + c_6\cos nt\right) + c_2c_6\left(c_5\sin nt + c_6\cos nt\right) \\
&-\tfrac{1}{4}\left({c_5}^2+{c_6}^2\right)\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - \tfrac{3}{4}\left({c_5}^2+{c_6}^2\right)c_1t + \tfrac{1}{4}\left({c_5}^2+{c_6}^2\right)c_4 \\
&= \tfrac{1}{4}\left({c_5}^2+{c_6}^2\right)c_4 - \tfrac{3}{4}\left({c_5}^2+{c_6}^2\right)c_1t + \left(\tfrac{1}{4}{c_5}^2 + \tfrac{1}{4}{c_6}^2\right)y + \left(c_2c_6 - c_3c_5\right)z
\end{aligned}
\tag{A.10}
$$

Cross-track Direction Nonlinear Terms

$$
\begin{aligned}
zx &= \tfrac{2}{n}c_1c_5\sin nt + \tfrac{2}{n}c_1c_6\cos nt + c_2c_5\sin^2 nt + c_2c_6\cos nt\sin nt + c_3c_5\cos nt\sin nt \\
&+c_3c_6\cos^2 nt \\
&= \tfrac{2}{n}c_1c_5\sin nt + \tfrac{2}{n}c_1c_6\cos nt + \tfrac{1}{2}c_2c_5\left(1-\cos 2nt\right) + \tfrac{1}{2}c_2c_6\sin 2nt \\
&+\tfrac{1}{2}c_3c_5\sin 2nt + \tfrac{1}{2}c_3c_6\left(1+\cos 2nt\right) \\
&\approx \tfrac{2}{n}c_1\left(c_5\sin nt + c_6\cos nt\right) + \tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6 = \tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6 + \tfrac{2}{n}c_1z
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
zx^2 &= \tfrac{4}{n^2}{c_1}^2c_5\sin nt + \tfrac{4}{n^2}{c_1}^2c_6cosnt + \tfrac{4}{n}c_1c_2c_5\sin^2 nt + \tfrac{4}{n}c_1c_2c_6\sin ntcosnt \\
&+\tfrac{4}{n}c_1c_3c_5\sin ntcosnt + \tfrac{4}{n}c_1c_3c_6\cos^2 nt + 2c_2c_3c_5\sin^2 nt\cos nt \\
&+2c_2c_3c_6\sin nt\cos^2 nt + c_5{c_2}^2\sin^3 nt + c_6{c_2}^2\sin^2 nt\cos nt \\
&+{c_3}^2c_5\sin nt\cos^2 nt + {c_3}^2c_6\cos^3 nt \\
&= \tfrac{4}{n^2}{c_1}^2c_5\sin nt + \tfrac{4}{n^2}{c_1}^2c_6\cos nt + \tfrac{2}{n}c_1c_2c_5\left(1-\cos 2nt\right) + \tfrac{2}{n}c_1c_2c_6\sin 2nt \\
&+\tfrac{2}{n}c_1c_3c_5\sin 2nt + \tfrac{2}{n}c_1c_3c_6\left(1+\cos 2nt\right) + c_2c_3c_5\left(1-\cos 2nt\right)\cos nt \\
&+c_2c_3c_6\sin nt\left(1+\cos 2nt\right) + c_5{c_2}^2\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) + \tfrac{1}{2}c_6{c_2}^2\left(1-\cos 2nt\right)\cos nt \\
&+\tfrac{1}{2}{c_3}^2c_5\sin nt\left(1+\cos 2nt\right) + {c_3}^2c_6\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) \\
&= \tfrac{4}{n^2}{c_1}^2c_5\sin nt + \tfrac{4}{n^2}{c_1}^2c_6\cos nt + \tfrac{2}{n}c_1c_2c_5 + \tfrac{2}{n}c_1c_3c_6 + \tfrac{1}{2}c_2c_3c_5\cos nt \\
&+\tfrac{1}{2}c_2c_3c_6\sin nt + \tfrac{3}{4}c_5{c_2}^2\sin nt + \tfrac{1}{4}c_6{c_2}^2\cos nt + \tfrac{1}{4}{c_3}^2c_5\sin nt + \tfrac{3}{4}{c_3}^2c_6\cos nt \\
&= \tfrac{4}{n^2}{c_1}^2\left(c_5\sin nt + c_6\cos nt\right) + c_2c_5\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&-\tfrac{1}{2}c_2c_5\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{1}{n}c_1c_2c_5 + c_3c_6\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) \\
&-\tfrac{2}{n}c_1c_3c_6 - \tfrac{1}{2}c_3c_6\left(\tfrac{2}{n}c_1 + c_2\sin nt + c_3\cos nt\right) + \tfrac{1}{4}{c_2}^2\left(c_5\sin nt + c_6\cos nt\right) \\
&+\tfrac{1}{4}{c_3}^2\left(c_5\sin nt + c_6\cos nt\right) \\
&= \tfrac{1}{n}c_1c_2c_5 - \tfrac{2}{n}c_1c_3c_6 + \left(\tfrac{4}{n^2}{c_1}^2 + \tfrac{1}{4}{c_2}^2 + \tfrac{1}{4}{c_3}^2\right)z + \left(\tfrac{1}{2}c_2c_5 + \tfrac{1}{2}c_3c_6\right)x
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
zy^2 &= 9c_5{c_1}^2t^2\sin nt + 9c_6{c_1}^2t^2\cos nt - 12c_1c_2c_5t\cos nt\sin nt - 12c_1c_2c_6t\cos^2 nt \\
&+12c_1c_3c_5t\sin^2 nt + 12c_1c_3c_6t\sin nt\cos nt - 6c_1c_4c_5t\sin nt - 6c_1c_4c_6t\cos nt \\
&+4c_5{c_2}^2\cos^2 nt\sin nt + 4c_6{c_2}^2\cos^3 nt + 4c_2c_4c_5\cos nt\sin nt + 4c_2c_4c_6\cos^2 nt \\
&-4c_3c_4c_5\sin^2 nt - 4c_3c_4c_6\sin nt\cos nt - 8c_2c_3c_5\sin^2 nt\cos nt - 8c_2c_3c_6\sin nt\cos^2 nt \\
&+4c_5{c_3}^2\sin^3 nt + 4{c_3}^2c_6\sin^2 nt\cos nt + c_5{c_4}^2\sin nt + c_6{c_4}^2\cos nt \\
&= 9c_5{c_1}^2t^2\sin nt + 9c_6{c_1}^2t^2\cos nt - 6c_1c_2c_5t\sin 2nt - 6c_1c_2c_6t\left(1+\cos 2nt\right) \\
&+6c_1c_3c_5t\left(1-\cos 2nt\right) + 6c_1c_3c_6t\sin 2nt - 6c_1c_4c_5t\sin nt - 6c_1c_4c_6t\cos nt \\
&+2c_5{c_2}^2\left(1+\cos 2nt\right)\sin nt + 4c_6{c_2}^2\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) + 2c_2c_4c_5\sin 2nt \\
&+2c_2c_4c_6\left(1+\cos 2nt\right) - 2c_3c_4c_5\left(1-\cos 2nt\right) - 4c_3c_4c_6\sin nt\cos nt \\
&-4c_2c_3c_5\left(1-\cos 2nt\right)\cos nt - 4c_2c_3c_6\sin nt\left(1+\cos 2nt\right) \\
&+4c_5{c_3}^2\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) + 2{c_3}^2c_6\left(1-\cos 2nt\right)\cos nt + c_5{c_4}^2\sin nt + c_6{c_4}^2\cos nt \\
&= 9{c_1}^2t^2\left(c_5\sin nt + c_6\cos nt\right) + c_2c_6\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) - 3c_1c_2c_6t \\
&+{c_2}^2\left(c_5\sin nt + c_6\cos nt\right) + c_2c_4c_6 - c_3c_5\left(-3c_1t + 2c_2\cos nt - 2c_3\sin nt + c_4\right) \\
&+3c_1c_3c_5t - c_3c_4c_5 - 6c_1c_4t(c_5\sin nt + c_6\cos nt) + {c_3}^2\left(c_5\sin nt + c_6\cos nt\right) \\
&+{c_4}^2\left(c_5\sin nt + c_6\cos nt\right) \\
&= \left(c_2c_4c_6 - c_3c_4c_5 - 3c_1c_2c_6t + 3c_1c_3c_5t\right) + \left({c_2}^2 + {c_3}^2 + {c_4}^2 - 6c_1c_4t + 9{c_1}^2t^2\right)z \\
&+\left(c_2c_6 - c_3c_5\right)y
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
z^3 &= {c_5}^3\sin^3 nt + 3c_6{c_5}^2\sin^2 nt\cos nt + 3c_5{c_6}^2\cos^2 nt\sin nt + {c_6}^3\cos^3 nt \\
&= {c_5}^3\left(\tfrac{3}{4}\sin nt - \tfrac{1}{4}\sin 3nt\right) + \tfrac{3}{2}c_6{c_5}^2\left(1-\cos 2nt\right)\cos nt + \tfrac{3}{2}{c_6}^2c_5\left(1+\cos 2nt\right)\sin nt \\
&+{c_6}^3\left(\tfrac{3}{4}\cos nt + \tfrac{1}{4}\cos 3nt\right) = \tfrac{3}{4}{c_5}^3\sin nt + \tfrac{3}{4}c_6{c_5}^2\cos nt + \tfrac{3}{4}{c_6}^2c_5\sin nt + \tfrac{3}{4}{c_6}^3\cos nt \\
&= \tfrac{3}{4}{c_5}^2\left(c_5\sin nt + c_6\cos nt\right) + \tfrac{3}{4}{c_6}^2\left(c_5\sin nt + c_6\cos nt\right) = \left(\tfrac{3}{4}{c_5}^2 + \tfrac{3}{4}{c_6}^2\right)z
\end{aligned} \tag{A.14}
$$

Appendix B

Taylor Series Expansion

First-order Approximation of Taylor Series Expansion

$$\begin{aligned} f(x,y,z) &= f(x_0,y_0,z_0) + (x-x_0)\, f_x(x_0,y_0,z_0) + (y-y_0)\, f_y(x_0,y_0,z_0) \\ &+ (z-z_0)\, f_z(x_0,y_0,z_0) \end{aligned} \tag{B.1}$$

Second-order Approximation of Taylor Series Expansion

$$\begin{aligned} f(x,y,z) &= f(x_0,y_0,z_0) + (x-x_0)\, f_x(x_0,y_0,z_0) + (y-y_0)\, f_y(x_0,y_0,z_0) \\ &+ (z-z_0)\, f_z(x_0,y_0,z_0) + \tfrac{1}{2}\Big\{(x-x_0)^2 f_{xx}(x_0,y_0,z_0) \\ &+ (x-x_0)(y-y_0)\, f_{xy}(x_0,y_0,z_0) + (x-x_0)(z-z_0)\, f_{xz}(x_0,y_0,z_0) \\ &+ (x-x_0)(y-y_0)\, f_{yx}(x_0,y_0,z_0) + (y-y_0)^2 f_{yy}(x_0,y_0,z_0) \\ &+ (y-y_0)(z-z_0)\, f_{yz}(x_0,y_0,z_0) + (z-z_0)(x-x_0)\, f_{zx}(x_0,y_0,z_0) \\ &+ (z-z_0)(y-y_0)\, f_{zy}(x_0,y_0,z_0) + (z-z_0)^2 f_{zz}(x_0,y_0,z_0)\Big\} \end{aligned} \tag{B.2}$$

Third-order Approximation of Taylor Series Expansion

$$\begin{aligned}
&f\left(x,y,z\right)\ = f\left(x_0,y_0,z_0\right) + \left(x-x_0\right)f_x\left(x_0,y_0,z_0\right) + \left(y-y_0\right)f_y\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)f_z\left(x_0,y_0,z_0\right) + \tfrac{1}{2}\Big\{\left(x-x_0\right)^2 f_{xx}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)\left(y-y_0\right)f_{xy}\left(x_0,y_0,z_0\right) + \left(x-x_0\right)\left(z-z_0\right)f_{xz}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)\left(y-y_0\right)f_{yx}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)^2 f_{yy}\left(x_0,y_0,z_0\right)\\
&+\left(y-y_0\right)\left(z-z_0\right)f_{yz}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)\left(x-x_0\right)f_{zx}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)\left(y-y_0\right)f_{zy}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)^2 f_{zz}\left(x_0,y_0,z_0\right)\Big\}\\
&+\tfrac{1}{6}\Big\{\left(x-x_0\right)^3 f_{xxx}\left(x_0,y_0,z_0\right) + \left(x-x_0\right)^2\left(y-y_0\right)f_{xxy}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)^2\left(z-z_0\right)f_{xxz}\left(x_0,y_0,z_0\right) + \left(x-x_0\right)^2\left(y-y_0\right)f_{xyx}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)\left(y-y_0\right)^2 f_{xyy}\left(x_0,y_0,z_0\right) + \left(x-x_0\right)\left(y-y_0\right)\left(z-z_0\right)f_{xyz}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)^2\left(z-z_0\right)f_{xzx}\left(x_0,y_0,z_0\right) + \left(x-x_0\right)\left(z-z_0\right)\left(y-y_0\right)f_{xzy}\left(x_0,y_0,z_0\right)\\
&+\left(x-x_0\right)\left(z-z_0\right)^2 f_{xzz}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)\left(x-x_0\right)^2 f_{yxx}\left(x_0,y_0,z_0\right)\\
&+\left(y-y_0\right)^2\left(x-x_0\right)f_{yxy}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)\left(x-x_0\right)\left(z-z_0\right)f_{yxz}\left(x_0,y_0,z_0\right)\\
&+\left(y-y_0\right)^2\left(x-x_0\right)f_{yyx}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)^3 f_{yyy}\left(x_0,y_0,z_0\right)\\
&+\left(y-y_0\right)^2\left(z-z_0\right)f_{yyz}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)\left(z-z_0\right)\left(x-x_0\right)f_{yzx}\left(x_0,y_0,z_0\right)\\
&+\left(y-y_0\right)^2\left(z-z_0\right)f_{yzy}\left(x_0,y_0,z_0\right) + \left(y-y_0\right)\left(z-z_0\right)^2 f_{yzz}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)\left(x-x_0\right)^2 f_{zxx}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)\left(x-x_0\right)\left(y-y_0\right)f_{zxy}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)^2\left(x-x_0\right)f_{zxz}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)\left(y-y_0\right)\left(x-x_0\right)f_{zyx}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)\left(y-y_0\right)^2 f_{zyy}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)^2\left(y-y_0\right)f_{zyz}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)^2\left(x-x_0\right)f_{zzx}\left(x_0,y_0,z_0\right) + \left(z-z_0\right)^2\left(y-y_0\right)f_{zzy}\left(x_0,y_0,z_0\right)\\
&+\left(z-z_0\right)^3 f_{zzz}\left(x_0,y_0,z_0\right)\Big\} + R_4\left(x,y,z\right)
\end{aligned} \tag{B.3}$$

Appendix C

Equations Containing Polynomial Functions

Frequently, there is a need to solve Abel's differential equations of first and second kind containing third-degree (cubic) polynomial and Riccati differential equation containing second-degree (quadratic) polynomial. Both Abel and Riccati equation appear in different physical and mathematical problems such as in oceanic circulation [34], in problems of magneto-statics [43], control theory [49], fluid mechanics [47], cancer therapy [40] and in solid mechanics [35]. Polyanin and Zaitsev [50] present the analytical solutions of special type of these equations.

Abel Equations

An Abel equation of first kind has the general form (Harko and Mak 2015)

$$y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \tag{C.1}$$

Here, the notations $d()/dx = ()'_x$, $d^2()/dx^2 = ()''_{xx}, ...$ denote total derivatives. Mak and Harko (2013) presented new method for generating a general solution of the nonlinear first kind Abel type differential equation from a particular one. Polyanin and Zaitsev (2002), Mancas and Rosu (2013) emphasized two connections between the dissipative nonlinear second order differential equations and the Abel equations which in its first kind form have only cubic and quadratic terms. They show how to obtain Abel solutions directly from the factorization of second-order nonlinear equations. If $y_p = y_p(x)$ is a particular solution of the Abel equation (Polyanin and Zaitsev 2002, Salinas-Hernandez, Munoz-Vega, Sosa and Lopez-Carrera 2013, Mak, Chan and Harko 2001) the substitution $y = y_p + 1/w$ reduces the equation to the Abel equation of the second kind:

$$ww'_x = -\left(3f_3y_p^2 + 2f_2y_p + f_1\right)w^2 - \left(3f_3y_p + f_2\right)w - f_3 \tag{C.2}$$

The transformation

$$\xi = \int f_3E^2dx, u = \left(y + \frac{f_2}{f_3}\right)E^{-1}, E = \exp\left[\int\left(f_1 - \frac{f_2^2}{3f_3}\right)dx\right] \tag{C.3}$$

brings the general form (C.1) to the normal (canonical) form

$$u'_\xi = u^3 + \Phi(\xi) \tag{C.4}$$

where,

$$\Phi(\xi) = \frac{1}{f_3E^3}\left[f_0 + \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right) - \frac{f_1f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}\right] \tag{C.5}$$

The analytic solution of Eq. (C.1) is given as (Polyanin and Zaitsev 2002)

$$y(x) = E\Big(C - 2\int f_3 E^2 dx\Big)^{-1/2} - \frac{f_2}{3f_3} \tag{C.6}$$

Riccati Equations

The general form of a Riccati equation is

$$\frac{dy}{dx} = f_2(x)y^2 + f_1(x)y + f_0(x) \tag{C.7}$$

where $f_2,\ f_1,\ f_0$ are arbitrary real functions of x, with $f_2, f_1, f_0 \in C^\infty(I)$ defined on a real interval $I \subseteq \Re$ is one of the most studied first order nonlinear differential equations that arises in different fields of mathematics and physics (Polyanin and Zaitsev 2002, Haaheim and Stein 1969, Bastami, Belic and Petrovic 2010, Harko, Lobo and Mak 2014, Kamke 1959, Soare, Teodorescus and Toma 2007) named after the Italian mathematician Jacopo Francesco Riccati (1724). It has the form which can be considered as the lowest order nonlinear approximation to the derivative of a function in terms of the function itself. For $f_2 \equiv 0$, we obtain a linear equation and for $f_0 \equiv 0$ we have the Bernoulli equation (Polyanin and Zaitsev 2002).

Generally, it is well-known that only special cases can be treated because solutions to the general Riccati equation are not available. To find the general solution one needs only a particular solution. Given a particular solution $y_p = y_p(x)$ of the Riccati equation, the general solution of the Riccati equation can be written as (Polyanin and Zaitsev 2002, Haaheim and Stein 1969, Bastami, Belic and Petrovic 2010, Harko, Lobo and Mak 2014)

$$y\left[f_2(x), f_1(x), y_p(x)\right] = y_p(x) + \Phi(x)\Big[C - \int \Phi(x) f_2(x) dx\Big]^{-1} \tag{C.8}$$

where

$$\Phi(x) = \exp\Big\{\int \left[2f_2(x)y_p(x) + f_1(x)y\right] dx\Big\} \tag{C.9}$$

and C is an arbitrary constant of integration. The particular solution of the Riccati equation satisfies

$$\frac{dy_p}{dx} = f_2(x)y_p^2 + f_1(x)y_p + f_0(x) \tag{C.10}$$

The substitution $u(x) = \exp\left(-\int f_2(x) y dx\right)$ reduces the general Riccati equation to a second order linear equation

$$\frac{d^2u}{dx} - \left[\frac{1}{f_2(x)}\frac{df_2(x)}{dx} + f_1(x)\right]\frac{du}{dx} + f_0(x)f_2(x) = 0 \tag{C.11}$$

If a particular solution is not known and the coefficients of the Riccati equation satisfy the following specific condition

$$f_2(x) + f_1(x) + f_0(x) = 0 \tag{C.12}$$

the Riccati equation will have the solution

$$y = \frac{K + \int [f_2(x) - f_0(x)]\, E(x) dx - E(x)}{K + \int [f_2(x) + f_0(x)]\, E(x) dx + E(x)} \tag{C.13}$$

where K is an arbitrary constant of integration. If $f_2(x) \equiv 1$, and the functions $f_1(x)$ and $f_0(x)$ are polynomials satisfying the condition (Polyanin and Zaitsev 2002, Harko, Lobo and Mak 2014, Soare, Teodorescus and Toma 2014)

$$\Delta = f_1^2(x) - 2\frac{df_1(x)}{dx} - 4f_0(x) \equiv \text{constant} \tag{C.14}$$

then

$$y_{\pm}(x) = -\frac{1}{2}\left[f_1(x) \pm \sqrt{\Delta}\right] \tag{C.15}$$

are both solutions of the Riccati equation (C.7). This paper presents two models of third-order nonlinear differential equations describing the dynamics of the relative motion of deputy spacecraft with respect to the chief spacecraft in terms of the orbit element differences leading to the formulation of Abel-type differential equations. Using well known techniques and methods, we developed analytical solutions of the Abel-type and Riccati-type spacecraft nonlinear equations of motion of the first kind obtained.

Appendix D

First Model Approximation of Abel-type Equation

In this appendix, elements of the coefficients of the first model of Abel-type equation are provided.

Coefficients of First Model of Abel-type Equation

$$\begin{aligned}
p_0(\theta) = & -\tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)} + \tfrac{n}{2}(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{15n}{8a^2}(\delta a)^2\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} - \tfrac{35n}{16a^3}(\delta a)^3 f\Big|_{(0,0,0)} + \tfrac{n}{6}\Big\{(\delta q_1)^3 f_{\delta q_1\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_1)^2(\delta q_2) f_{\delta q_1\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta q_1)^2(\delta q_2) f_{\delta q_1\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_1)(\delta q_2)^2 f_{\delta q_1\delta q_2\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)(\delta q_1)^2 f_{\delta q_2\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_2)^2(\delta q_1) f_{\delta q_2\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)^2(\delta q_1) f_{\delta q_2\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_2)^3 f_{\delta q_2\delta q_2\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{2a} f\Big|_{(0,0,0)}\delta a + n f_{\delta q_1}\Big|_{(0,0,0)}\delta q_1 + n f_{\delta q_2}\Big|_{(0,0,0)}\delta q_2
\end{aligned} \tag{D.1}$$

$$\begin{aligned}
p_1(\theta) = & \; n f_{\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a f_{\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} \\
& + \tfrac{15n}{8a^2}(\delta a)^2 f_{\delta\theta}\Big|_{(0,0,0)} + (\delta q_1)^2 f_{\delta q_1\delta q_1\delta\theta}\Big|_{(0,0,0)} + (\delta q_1)(\delta q_2) f_{\delta q_1\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
& + (\delta q_1)^2 f_{\delta q_1\delta\theta\delta q_1}\Big|_{(0,0,0)} + (\delta q_1)(\delta q_2) f_{\delta q_1\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)^2 f_{\delta q_2\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
& + (\delta q_2)(\delta q_1) f_{\delta q_2\delta\theta\delta q_1}\Big|_{(0,0,0)} + (\delta q_2)^2 f_{\delta q_2\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta q_1)^2 f_{\delta\theta\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_1)(\delta q_2) f_{\delta\theta\delta q_1\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)(\delta q_1) f_{\delta\theta\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + (\delta q_2)^2 f_{\delta\theta\delta q_2\delta q_2}\Big|_{(0,0,0)} + (\delta q_2)(\delta q_1) f_{\delta q_2\delta q_1\delta\theta}\Big|_{(0,0,0)}
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
p_2(\theta) = & \; \tfrac{n}{2} f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{6}\Big\{(\delta q_1) f_{\delta q_1\delta\theta\delta\theta}\Big|_{(0,0,0)} \\
& + (\delta q_2) f_{\delta q_2\delta\theta\delta\theta}\Big|_{(0,0,0)} + (\delta q_2) f_{\delta\theta\delta q_2\delta\theta}\Big|_{(0,0,0)} + (\delta q_1) f_{\delta\theta\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
& + (\delta q_2) f_{\delta\theta\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta q_1) f_{\delta\theta\delta\theta\delta q_1}\Big|_{(0,0,0)}\Big\}
\end{aligned} \tag{D.3}$$

$$p_3(\theta) = f_{\delta\theta\delta\theta\delta\theta}\Big|_{(0,0,0)} \tag{D.4}$$

Taylor Series Partial Terms

$$f\Big|_{(0,0,0)} = (1 + q_1\cos\theta + q_2\sin\theta)^2\left(1 - q_1^2 - q_2^2\right)^{-3/2} \tag{D.5}$$

$$\begin{aligned} f_{\delta q_1}\Big|_{(0,0,0)} &= 2\cos\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+3q_1(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \end{aligned} \tag{D.6}$$

$$\begin{aligned} f_{\delta q_2}\Big|_{(0,0,0)} &= 2\sin\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+3\,(q_2 + \delta q_2)\,(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \end{aligned} \tag{D.7}$$

$$f_{\delta\theta}\Big|_{(0,0,0)} = 2\,(-q_1\sin\theta + q_2\cos\theta)\,(1 + q_1\cos\theta + q_2\sin\theta)\left(1 - q_1^2 - q_2^2\right)^{-3/2} \tag{D.8}$$

$$\begin{aligned} f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} &= 2\cos^2\theta(1 - q_1^2 - q_2^2)^{-3/2} + 6q_1\cos\theta(1 - q_1^2 - q_2^2)^{-5/2} \\ &+3q_1(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+15q_1^2(1 - q_1^2 - q_2^2)^{-7/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_1^2\cos\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{D.9}$$

$$\begin{aligned} f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} &= 2\cos\theta\sin\theta(1 - q_1^2 - q_2^2)^{-3/2} \\ &+6\cos\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-5/2} \\ &+15q_1q_2(1 - q_1^2 - q_2^2)^{-7/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_1\sin\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{D.10}$$

$$\begin{aligned} f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} &= -2\sin\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+2\cos\theta\,(-q_1\sin\theta + q_2\cos\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+6\,(q_1 + \delta q_1)\,(1 - q_1^2 - q_2^2)^{-5/2}\,(-\sin\theta + q_2\cos\theta)\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{D.11}$$

$$\begin{aligned} f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} &= 2\sin\theta\cos\theta(1 - q_1^2 - q_2^2)^{-3/2} \\ &+6q_1\sin\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-5/2} \\ &+15q_1q_2(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_2\cos\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{D.12}$$

$$\begin{aligned} f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} &= 2\sin^2\theta(1 - q_1^2 - q_2^2)^{-3/2} + 6q_2\sin\theta(1 - q_1^2 - q_2^2)^{-5/2} \\ &+3q_2(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+15q_2^2(1 - q_1^2 - q_2^2)^{-7/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_2^2\sin\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{D.13}$$

$$\begin{aligned} f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} &= 2\cos\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+2\sin\theta\,(-q_1\sin\theta + q_2\cos\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+3\,(q_2 + \delta q_2)\,(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \end{aligned} \tag{D.14}$$

$$
\begin{aligned}
f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} &= -2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_1\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.15}
$$

$$
\begin{aligned}
f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} &= 2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\sin\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.16}
$$

$$
\begin{aligned}
f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} &= 2(-q_1\sin\theta+q_2\cos\theta)^2(1-q_1^2-q_2^2)^{-3/2} \\
&+2\,(-q_1\cos\theta-q_2\sin\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2}
\end{aligned}
\tag{D.17}
$$

$$
\begin{aligned}
f_{\delta q_1\delta q_1\delta q_1}\Big|_{(0,0,0)} &= 6q_1\cos^2\theta(1-q_1^2-{q_2}^2)^{-5/2}+6\cos\theta(1-q_1^2-{q_2}^2)^{-5/2} \\
&+30\cos\theta{q_1}^2(1-q_1^2-{q_2}^2)^{-7/2}+15q_1^2(1-q_1^2-{q_2}^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+6q_1\cos\theta(1-q_1^2-{q_2}^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30q_1(1-q_1^2-{q_2}^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+105{q_1}^3(1-q_1^2-{q_2}^2)^{-9/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+30{q_1}^2\cos\theta(1-q_1^2-{q_2}^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+12q_1\cos\theta(1-q_1^2-{q_2}^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30{q_1}^3\cos\theta(1-q_1^2-{q_2}^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6{q_1}^2\cos^2\theta(1-q_1^2-{q_2}^2)^{-5/2}
\end{aligned}
\tag{D.18}
$$

$$
\begin{aligned}
f_{\delta q_1\delta q_1\delta q_2}\Big|_{(0,0,0)} &= 6q_2\cos^2\theta(1-{q_1}^2-{q_2}^2)^{-5/2}+30q_1q_2\cos\theta(1-{q_1}^2-{q_2}^2)^{-7/2} \\
&+15q_1q_2(1-q_1^2-q_2^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+6q_1\sin\theta(1-q_1^2-q_2^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+105q_1^2q_2(1-q_1^2-q_2^2)^{-9/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+30q_1^2\sin\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30q_1^2q_2\cos\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1^2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.19}
$$

$$
\begin{aligned}
f_{\delta q_1\delta q_1\delta\theta}\Big|_{(0,0,0)} &= -4\cos\theta\sin\theta(1-{q_1}^2-{q_2}^2)^{-3/2}-6q_1\sin\theta(1-{q_1}^2-{q_2}^2)^{-5/2} \\
&+6q_1(1-{q_1}^2-{q_2}^2)^{-5/2}\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30{q_1}^2(1-{q_1}^2-{q_2}^2)^{-7/2}\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta) \\
&-6{q_1}^2\sin\theta(1-{q_1}^2-{q_2}^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6{q_1}^2\cos\theta(1-{q_1}^2-{q_2}^2)^{-5/2}\,(-q_1\sin\theta+q_2\cos\theta)
\end{aligned}
\tag{D.20}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta q_2 \delta q_1}\Big|_{(0,0,0)} &= 6q_1 \cos\theta \sin\theta (1-q_1^2-q_2^2)^{-5/2} + 6\cos^2\theta (1-q_1^2-q_2^2)^{-5/2} \\
&+30q_1 \cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-7/2} \\
&+15q_2(1-q_1^2-q_2^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+105q_1^2q_2(1-q_1^2-q_2^2)^{-9/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+30q_1q_2\cos\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6\sin\theta(1-q_1^2-q_2^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30q_1\sin\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-5/2}
\end{aligned} \tag{D.21}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta q_2 \delta q_2}\Big|_{(0,0,0)} &= 6q_2 \cos\theta \sin\theta (1-q_1^2-q_2^2)^{-5/2} + 6\cos\theta\sin\theta (1-q_1^2-q_2^2)^{-5/2} \\
&+30q_2 \cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-7/2} \\
&+15q_1(1-q_1^2-q_2^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+105q_2^2(1-q_1^2-q_2^2)^{-9/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+30q_1q_2\sin\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30q_1q_2\sin\theta(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1\sin^2\theta(1-q_1^2-q_2^2)^{-5/2}
\end{aligned} \tag{D.22}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta q_2 \delta \theta}\Big|_{(0,0,0)} &= -2\sin^2\theta (1-q_1^2-q_2^2)^{-3/2} + 2\cos^2\theta (1-q_1^2-q_2^2)^{-3/2} \\
&-6\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+30q_1q_2(1-q_1^2-q_2^2)^{-7/2}\,(1+q_1\cos\theta+q_2\sin\theta)\,(-q_1\sin\theta+q_2\cos\theta) \\
&+6q_1\cos\theta(1-q_1^2-q_2^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1\sin\theta(1-q_1^2-q_2^2)^{-5/2}\,(-q_1\sin\theta+q_2\cos\theta)
\end{aligned} \tag{D.23}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta \theta \delta q_1}\Big|_{(0,0,0)} &= -2\sin\theta\cos\theta (1-q_1^2-q_2^2)^{-3/2} \\
&-6q_1\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&-2\cos\theta\sin\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_1\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6(1-q_1^2-q_2^2)^{-5/2}\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+30(q_1+\delta q_1)^2(1-q_1^2-q_2^2)^{-7/2}\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1\sin\theta(1-q_1^2-q_2^2)^{-5/2}\,(1+q_1\cos\theta+q_2\sin\theta) \\
&+6q_1\cos\theta(1-q_1^2-q_2^2)^{-5/2}\,(-q_1\sin\theta+q_2\cos\theta)
\end{aligned} \tag{D.24}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta\theta \delta q_2}\Big|_{(0,0,0)} &= -2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&-2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&-2\cos^2\theta(1-q_1^2-q_2^2)^{-3/2}+6q_2\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+30q_1q_2(1-q_1^2-q_2^2)^{-7/2}\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right) \\
&+6q_1\cos\theta(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right) \\
&+6q_1\sin\theta(1-q_1^2-q_2^2)^{-5/2}\left(-q_1\sin\theta+q_2\cos\theta\right)
\end{aligned} \tag{D.25}
$$

$$
\begin{aligned}
f_{\delta q_1 \delta\theta \delta\theta}\Big|_{(0,0,0)} &= -2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+2\cos\theta\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+6q_1(1-q_1^2-q_2^2)^{-5/2}\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right) \\
&+6q_1(1-q_1^2-q_2^2)^{-5/2}(-q_1\sin\theta+q_2\cos\theta)^2
\end{aligned} \tag{D.26}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta q_1}\Big|_{(0,0,0)} &= -2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&-6q_1\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&-2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_1\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&-6q_1\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+30q_1^2\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}
\end{aligned} \tag{D.27}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta q_2}\Big|_{(0,0,0)} &= -2\sin^2\theta(1-q_1^2-q_2^2)^{(-3/2)} \\
&-6q_2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+2\cos^2\theta(1-q_1^2-q_2^2)^{(-3/2)}+6q_2\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+30q_1q_2\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}
\end{aligned} \tag{D.28}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta\theta}\Big|_{(0,0,0)} = &-2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2}\\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2}\\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2}\\
&+2\cos\theta\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2}\\
&+6q_1\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}\\
&+6q_1\left(-q_1\sin\theta+q_2\cos\theta\right)\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}
\end{aligned}
\tag{D.29}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_2 \delta q_1}\Big|_{(0,0,0)} = &\ 6q_1\sin^2\theta(1-q_1^2-q_2^2)^{-5/2}+30q_1q_2\sin\theta(1-q_1^2-q_2^2)^{-7/2}\\
&+15q_1q_2(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-7/2}\\
&+6q_2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}\\
&+105q_2^2q_1(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-9/2}\\
&+30q_2^2\cos\theta(1-q_1^2-q_2^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2\\
&+30q_1q_2^2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}\\
&+6q_2^2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.30}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_2 \delta q_2}\Big|_{(0,0,0)} = &\ 6q_2\sin^2\theta(1-q_1^2-q_2^2)^{-5/2}+6\sin\theta(1-q_1^2-q_2^2)^{-5/2}\\
&+30q_2^2\sin\theta(1-q_1^2-q_2^2)^{-5/2}+3(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-5/2}\\
&+15q_2^2(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-7/2}\\
&+6q_2\sin\theta(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right)\\
&+30q_2(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-7/2}\\
&+105q_2^3(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-9/2}\\
&+30\sin\theta q_2^2\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}\\
&+12q_2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}\\
&+30q_2^3\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}\\
&+6q_2^2\sin^2\theta(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.31}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_2 \delta\theta}\Big|_{(0,0,0)} = &\ 4\sin\theta\cos\theta(1-q_1^2-q_2^2)^{(-3/2)}+6q_2\cos\theta(1-q_1^2-q_2^2)^{-5/2}\\
&+6q_2(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(-q_1\sin\theta+q_2\cos\theta\right)\\
&+30q_2^2\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}\\
&+6q_2^2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}\\
&+6q_2^2\sin\theta(1-q_1^2-q_2^2)^{-5/2}\left(-q_1\sin\theta+q_2\cos\theta\right)
\end{aligned}
\tag{D.32}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta\theta \delta q_1}\Big|_{(0,0,0)} &= 2\cos^2\theta(1-q_1^2-q_2^2)^{(-3/2)} \\
&+6q_1\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&-2sin^2\theta(1-q_1^2-q_2^2)^{(-3/2)}+6q_1\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+15q_1q_2(1-q_1^2-q_2^2)^{(-7/2)}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+6q_2\cos\theta(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.33}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta\theta \delta q_2}\Big|_{(0,0,0)} &= 2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+3(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\
&+15q_2^2(1+q_1\cos\theta+q_2\sin\theta)^2(1-q_1^2-q_2^2)^{-7/2} \\
&+6q_1\sin\theta(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right)
\end{aligned}
\tag{D.34}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta q_1}\Big|_{(0,0,0)} &= -2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&-6q_1\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&-2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_1\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&-6q_1\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+30q_1^2\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}
\end{aligned}
\tag{D.35}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta q_2}\Big|_{(0,0,0)} &= -2\sin^2\theta(1-q_1^2-q_2^2)^{(-3/2)} \\
&-6q_2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+2\cos^2\theta(1-q_1^2-q_2^2)^{(-3/2)}+6q_2\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+30q_1q_2\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-7/2}
\end{aligned}
\tag{D.36}
$$

$$
\begin{aligned}
f_{\delta q_2 \delta q_1 \delta\theta}\Big|_{(0,0,0)} &= -2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&-2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+2\cos\theta\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+6q_1\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2} \\
&+6q_1\left(-q_1\sin\theta+q_2\cos\theta\right)\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}
\end{aligned}
\tag{D.37}
$$

$$
\begin{aligned}
f_{\delta\theta\delta q_2\delta q_1}\Big|_{(0,0,0)} &= 2\cos^2\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_1\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&-2\sin^2\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_1\sin\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&-6q_2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{(-5/2)} \\
&+6q_2\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+30q_2\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-7/2}
\end{aligned}
\tag{D.38}
$$

$$
\begin{aligned}
f_{\delta\theta\delta q_2\delta q_2}\Big|_{(0,0,0)} &= 2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_2\sin\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6\,(-q_1\sin\theta+q_2\cos\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6q_2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6q_2\sin\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+30q_2\,(1+q_1\cos\theta+q_2\sin\theta)\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-7/2}
\end{aligned}
\tag{D.39}
$$

$$
\begin{aligned}
f_{\delta\theta\delta q_2\delta\theta}\Big|_{(0,0,0)} &= -2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\sin\theta\,(-q_1\cos\theta-q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\,(-q_1\cos\theta-q_2\sin\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2} \\
&+6q_2\,(-q_1\sin\theta+q_2\cos\theta)\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.40}
$$

$$
\begin{aligned}
f_{\delta\theta\delta\theta\delta q_1}\Big|_{(0,0,0)} &= -2\sin\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_1(-q_1\sin\theta+q_2\cos\theta)^2(1-q_1^2-q_2^2)^{-5/2} \\
&-2\cos\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\cos\theta\,(-q_1\cos\theta-q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_1\,(-q_1\cos\theta-q_2\sin\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.41}
$$

$$
\begin{aligned}
f_{\delta\theta\delta\theta\delta q_2}\Big|_{(0,0,0)} &= 2\cos\theta\,(-q_1\sin\theta+q_2\cos\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2(-q_1\sin\theta+q_2\cos\theta)^2(1-q_1^2-q_2^2)^{-5/2} \\
&-2\sin\theta\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+2\sin\theta\,(-q_1\cos\theta-q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-3/2} \\
&+6q_2\,(-q_1\cos\theta-q_2\sin\theta)\,(1+q_1\cos\theta+q_2\sin\theta)\,(1-q_1^2-q_2^2)^{-5/2}
\end{aligned}
\tag{D.42}
$$

$$
\begin{aligned}
f_{\delta\theta\delta\theta\delta\theta}\Big|_{(0,0,0)} &= 2\left(-q_1\cos\theta - q_2\sin\theta\right)\left(-q_1\sin\theta + q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+2\left(q_1\sin\theta - q_2\cos\theta\right)\left(1+q_1\cos\theta + q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\
&+2\left(-q_1\cos\theta - q_2\sin\theta\right)\left(-q_1\sin\theta + q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2}
\end{aligned}
\tag{D.43}
$$

$$
\begin{aligned}
&f\left(\delta q_1,\delta q_2,\delta\theta\right) = f\Big|_{(0,0,0)} + \delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} + \delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} + \delta\theta f_{\delta\theta}\Big|_{(0,0,0)} \\
&+\tfrac{1}{2}\Big\{\left(\delta q_1\right)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} + \delta q_1\delta q_2\ f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \delta q_1\delta\theta f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
&+\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} + \left(\delta q_2\right)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} + \delta q_2\delta\theta f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} + \delta\theta\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+\delta\theta\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta\theta\right)^2 f_{\delta\theta\delta\theta}\Big|_{(0,0,0)}\Big\} + \tfrac{1}{6}\Big\{\left(\delta q_1\right)^3 f_{\delta q_1\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_1\right)^2\left(\delta q_2\right) f_{\delta q_1\delta q_1\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_1\right)^2\left(\delta\theta\right) f_{\delta q_1\delta q_1\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_1\right)^2\left(\delta q_2\right) f_{\delta q_1\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_1\right)\left(\delta q_2\right)^2 f_{\delta q_1\delta q_2\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_1\right)\left(\delta q_2\right)\left(\delta\theta\right) f_{\delta q_1\delta q_2\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_1\right)^2\left(\delta\theta\right) f_{\delta q_1\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_1\right)\left(\delta\theta\right)\left(\delta q_2\right) f_{\delta q_1\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_1\right)\left(\delta\theta\right)^2 f_{\delta q_1\delta\theta\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta q_1\right)^2 f_{\delta q_2\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_2\right)^2\left(\delta q_1\right) f_{\delta q_2\delta q_1\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta q_1\right)\left(\delta\theta\right) f_{\delta q_2\delta q_1\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_2\right)^2\left(\delta q_1\right) f_{\delta q_2\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_2\right)^3 f_{\delta q_2\delta q_2\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_2\right)^2\left(\delta\theta\right) f_{\delta q_2\delta q_2\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta\theta\right)\left(\delta q_1\right) f_{\delta q_2\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta q_2\right)^2\left(\delta\theta\right) f_{\delta q_2\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta\theta\right)^2 f_{\delta q_2\delta\theta\delta\theta}\Big|_{(0,0,0)} + \left(\delta\theta\right)\left(\delta q_1\right)^2 f_{\delta\theta\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta\theta\right)\left(\delta q_1\right)\left(\delta q_2\right) f_{\delta\theta\delta q_1\delta q_2}\Big|_{(0,0,0)} + \left(\delta\theta\right)^2\left(\delta q_1\right) f_{\delta\theta\delta q_1\delta\theta}\Big|_{(0,0,0)} + \left(\delta\theta\right)\left(\delta q_2\right)\left(\delta q_1\right) f_{\delta\theta\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta\theta\right)\left(\delta q_2\right)^2 f_{\delta\theta\delta q_2\delta q_2}\Big|_{(0,0,0)} + \left(\delta\theta\right)^2\left(\delta q_2\right) f_{\delta\theta\delta q_2\delta\theta}\Big|_{(0,0,0)} + \left(\delta\theta\right)^2\left(\delta q_1\right) f_{\delta\theta\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
&+\left(\delta\theta\right)^2\left(\delta q_2\right) f_{\delta\theta\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta\theta\right)^3 f_{\delta\theta\delta\theta\delta\theta}\Big|_{(0,0,0)}\Big\} + R_4\left(\delta q_1,\delta q_2,\delta\theta\right)
\end{aligned}
\tag{D.44}
$$

Appendix E
Second Model Approximation of Abel-type Equation

Coefficients of Abel-type Second Model

$$k_3(\theta) = M_D m_3(\theta) = \frac{1}{3}M_D \left\{ \begin{array}{c} -4\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\cos 2\theta + 2\left(\left(q_1+\delta q_1\right)^2 - \left(q_2+\delta q_2\right)^2\right)\sin 2\theta \\ -\left(q_2+\delta q_2\right)\cos\theta + \left(q_1+\delta q_1\right)\sin\theta \end{array} \right\} \tag{E.1}$$

$$k_2(\theta) = M_D m_2(\theta) = M_D \left\{ \begin{array}{c} \left(-\left(q_1+\delta q_1\right)^2 + \left(q_2+\delta q_2\right)^2\right)\cos 2\theta - 2\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\sin 2\theta \\ -\left(q_1+\delta q_1\right)\cos\theta - \left(q_2+\delta q_2\right)\sin\theta \end{array} \right\} \tag{E.2}$$

$$k_1(\theta) = M_D m_1(\theta) = M_D \left\{ \begin{array}{c} 2\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\cos 2\theta + \left(-\left(q_1+\delta q_1\right)^2 + \left(q_2+\delta q_2\right)^2\right)\sin 2\theta \\ +2\left(q_2+\delta q_2\right)\cos\theta - 2\left(q_1+\delta q_1\right)\sin\theta \end{array} \right\} \tag{E.3}$$

$$\begin{aligned} k_0(\theta) &= M_D m_0(\theta) - M_C \left\{ \begin{array}{c} \left(\frac{1}{2}q_1^2 - \frac{1}{2}q_2^2\right)\cos 2\theta + q_1 q_2 \sin 2\theta \\ +1 + \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 + 2q_1\cos\theta + 2q_2\sin\theta \end{array} \right\} \\ &= \left\{ M_D\left(1 + \tfrac{1}{2}\left(q_1+\delta q_1\right)^2 + \tfrac{1}{2}\left(q_2+\delta q_2\right)^2\right) - M_C\left(1 + \tfrac{1}{2}q_1^2 + \tfrac{1}{2}q_2^2\right)\right\} \\ &\quad + \left\{ \tfrac{1}{2}M_D\left(\left(q_1+\delta q_1\right)^2 - \left(q_2+\delta q_2\right)^2\right) - M_C\left(\tfrac{1}{2}q_1^2 - \tfrac{1}{2}q_2^2\right)\right\}\cos 2\theta \\ &\quad + \left\{ M_D\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right) - M_C q_1 q_2 \right\}\sin 2\theta \\ &\quad + \left\{ 2M_D\left(q_1+\delta q_1\right) - 2M_C q_1 \right\}\cos\theta \\ &\quad + \left\{ 2M_D\left(q_2+\delta q_2\right) - 2M_C q_2 \right\}\sin\theta \end{aligned} \tag{E.4}$$

Appendix F
First Model Approximation of Riccati-type Equation

Coefficients of First Model of Riccati-type Equation

$$f\Big|_{(0,0,0)} = (1 + q_1\cos\theta + q_2\sin\theta)^2\Big(1 - q_1^2 - q_2^2\Big)^{-3/2} \tag{F.1}$$

$$\begin{aligned} f_{\delta q_1}\Big|_{(0,0,0)} &= 2\cos\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+3q_1(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \end{aligned} \tag{F.2}$$

$$\begin{aligned} f_{\delta q_2}\Big|_{(0,0,0)} &= 2\sin\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+3q_2(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \end{aligned} \tag{F.3}$$

$$f_{\delta\theta}\Big|_{(0,0,0)} = 2\,(-q_1\sin\theta + q_2\cos\theta)\,(1 + q_1\cos\theta + q_2\sin\theta)\,\Big(1 - q_1^2 - q_2^2\Big)^{-3/2} \tag{F.4}$$

$$f_{\delta\theta}\Big|_{(0,0,0)} = 2\,(-q_1\sin\theta + q_2\cos\theta)\,(1 + q_1\cos\theta + q_2\sin\theta)\,\Big(1 - q_1^2 - q_2^2\Big)^{-3/2} \tag{F.5}$$

$$\begin{aligned} f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} &= 2\cos^2\theta(1 - q_1^2 - q_2^2)^{-3/2} + 6q_1\cos\theta(1 - q_1^2 - q_2^2)^{-5/2} \\ &+3q_1(1 - q_1^2 - q_2^2)^{-5/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+15q_1^2(1 - q_1^2 - q_2^2)^{-7/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_1^2\cos\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{F.6}$$

$$\begin{aligned} f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} &= 2\cos\theta\sin\theta(1 - q_1^2 - q_2^2)^{-3/2} + 6\cos\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-5/2} \\ &+15q_1q_2(1 - q_1^2 - q_2^2)^{-7/2}(1 + q_1\cos\theta + q_2\sin\theta)^2 \\ &+6q_1\sin\theta(1 - q_1^2 - q_2^2)^{-5/2}\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{F.7}$$

$$\begin{aligned} f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} &= -2\sin\theta\,(1 + q_1\cos\theta + q_2\sin\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+2\cos\theta\,(-q_1\sin\theta + q_2\cos\theta)\,(1 - q_1^2 - q_2^2)^{-3/2} \\ &+6\,(q_1 + \delta q_1)\,(1 - q_1^2 - q_2^2)^{-5/2}\,(-\sin\theta + q_2\cos\theta)\,(1 + q_1\cos\theta + q_2\sin\theta) \end{aligned} \tag{F.8}$$

$$f_{\delta q_2 \delta q_1}\Big|_{(0,0,0)} = 2\sin\theta\cos\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_1\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-5/2}$$
$$+15q_1q_2(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2$$
$$+6q_2\cos\theta(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right) \tag{F.9}$$

$$\begin{aligned} f_{\delta q_2 \delta q_2}\Big|_{(0,0,0)} &= 2\sin^2\theta(1-q_1^2-q_2^2)^{-3/2} + 6q_2\sin\theta(1-q_1^2-q_2^2)^{-5/2} \\ &+3q_2(1-q_1^2-q_2^2)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\ &+15q_2^2(1-q_1^2-q_2^2)^{-7/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \\ &+6q_2^2\sin\theta(1-q_1^2-q_2^2)^{-5/2}\left(1+q_1\cos\theta+q_2\sin\theta\right) \end{aligned} \tag{F.10}$$

$$\begin{aligned} f_{\delta q_2 \delta\theta}\Big|_{(0,0,0)} &= 2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+3\left(q_2+\delta q_2\right)\left(1-q_1^2-q_2^2\right)^{-5/2}(1+q_1\cos\theta+q_2\sin\theta)^2 \end{aligned} \tag{F.11}$$

$$\begin{aligned} f_{\delta\theta \delta q_1}\Big|_{(0,0,0)} &= 2\sin\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+2\cos\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+6q_1\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right) \end{aligned} \tag{F.12}$$

$$\begin{aligned} f_{\delta\theta \delta q_2}\Big|_{(0,0,0)} &= 2\cos\theta\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+2\sin\theta\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \\ &+6q_2\left(-q_1\sin\theta+q_2\cos\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right) \end{aligned} \tag{F.13}$$

$$\begin{aligned} f_{\delta\theta \delta\theta}\Big|_{(0,0,0)} &= 2(-q_1\sin\theta+q_2\cos\theta)^2(1-q_1^2-q_2^2)^{-3/2} \\ &+2\left(-q_1\cos\theta-q_2\sin\theta\right)\left(1+q_1\cos\theta+q_2\sin\theta\right)\left(1-q_1^2-q_2^2\right)^{-3/2} \end{aligned} \tag{F.14}$$

$$f\left(\delta q_1,\delta q_2,\delta\theta\right) = f\Big|_{(0,0,0)} + \delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} + \delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} + \delta\theta f_{\delta\theta}\Big|_{(0,0,0)}$$
$$+\frac{1}{2}\Big\{(\delta q_1)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} + \delta q_1\delta q_2\ f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \delta q_1\delta\theta f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} + \delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)}$$
$$+(\delta q_2)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} + \delta q_2\delta\theta f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} + \delta\theta\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)}$$
$$+\delta\theta\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} + (\delta\theta)^2 f_{\delta\theta\delta\theta}\Big|_{(0,0,0)}\Big\} \tag{F.15}$$