Equitable Block-Colorings of Graph-Decompositions and Tiling Generalized Petersen Graphs

by

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A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
May 5, 2018

Keywords: equitable colorings, amalgamations, graph decompositions, tilings

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Abstract

A C_4-decomposition of $K_v - F$, where F is a 1-factor of K_v and C_4 is the cycle on 4 vertices, is a partition P of $E(K_v - F)$ into sets, each element of which induces a C_4 (called a block). A function assigning a color to each block defined by P is said to be an (s,p)-equitable block-coloring if: exactly s colors are used; each vertex v is incident with blocks colored with exactly p colors; and the blocks containing v are shared out as evenly as possible among the p color classes.

We introduce the study of the structure of such colorings, defining the color vector $V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$ of an (s,p)-equitable block-coloring E of G, arranged in non-decreasing order, where $c_i(E)$ is the number of vertices in G incident with a block of color i. In all cases where $\chi'_p(v) > p$, the most interesting values of $V(E)$ are considered, namely $c_1(E)$ and $c_s(E)$. The problems of finding the value of the smallest color class when it is as large as possible, $\overline{\psi'_1}(C_4, K_v - F)$, and the value of the largest color class when it is as small as possible, $\psi'_s(C_4, K_v - F)$, are settled.

We then consider the opposite extremes, solving the problems of finding the value of the smallest color class when it is as small as possible, $\psi'_1(C_4, K_v - F)$, and the value of the largest color class when it is as large as possible, $\overline{\psi'_s}(C_4, K_v - F)$. These extreme colorings follow from another interesting problem, namely finding (s,p)-equitable edge-colorings of K_v. The most interesting values are $\psi'_1(K_2, K_{v/2})$, $\overline{\psi'_s}(K_2, K_{v/2})$, $\psi'_1(C_4, K_v - F)$ and $\overline{\psi'_s}(C_4, K_v - F)$, but as a bonus from our method of proof in Chapter 6, we settle the value of $\overline{\psi'_i}(K_2, K_{v/2})$ and $\overline{\psi'_i}(C_4, K_v - F)$ for all other values of i as well.

Finally, some work on tiling generalized Petersen graphs, $P(n,k)$, with paths consisting of two and four vertices is also presented along with our plans for future work with tiling with paths on six and eight vertices.

Acknowledgments

The work of Chapter 8 was completed with Elizabeth McInnis, who I would like to acknowledge and thank for her collaboration and friendship. For his unending help and flexibility, I would like to thank my adviser Dr. Chris Rodger. I would like to thank The Mathematics Department at Auburn University for my position as a graduate teaching assistant, which made my work possible financially. For their support from starting pre–school a year early (as I was so excited to learn), to twenty four years later as I complete this terminal degree, I would like to thank my parents, Tim and Laurie Bailey. I would like to thank my grandmother, Martha Ann Bailey, for providing a sense of home in the stress of graduate work. For falling in love with a mathematician and continually supporting me in every way, I want to thank my husband, Isaac Matson. Most of all I thank God for orchestrating every detail that allowed me to achieve my doctorate. I dedicate this work to my son, Arius Matson, may you explore to the full extent whatever field you love.

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Chapter 1

Introduction to Block Colorings of Graph Decompositions

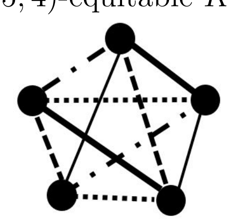
An *H-decomposition* of a graph G is an ordered pair (V, B) where V is the vertex set of G and B is a partition of the edges of G into sets, each of which induces a copy of H. The graphs induced by the elements of B are known as the *blocks* of the decomposition. (V, B) is said to have an *(s, p)-equitable block-coloring* $E : B \mapsto C = \{1, 2, \ldots, s\}$ if:

1. the blocks in B are colored with exactly s colors,
2. for each vertex $u \in V(G)$ the blocks containing u are colored using exactly p colors, and
3. for each vertex $u \in V(G)$ and for each $\{i, j\} \subset C(E, u)$,
   $|b(E, u, i) - b(E, u, j)| \leq 1$,

where $C(E, u) = \{i \mid E \text{ colors some block incident with } u \text{ with color } i\}$ and $b(E, u, i)$ is the number of blocks in B containing u that are colored i by E. For ease of notation, an (s, p)-equitable H-coloring of G is equivalent to an (s, p)-equitable block-coloring of an H-decomposition of G, where the blocks of the coloring are copies of H. For an example of such a coloring see Figure 1.1.

Such colorings were originally introduced by L. Gionfriddo, M. Gionfriddo, and Ragusa in [10]. They studied (s, p)-equitable C_4-colorings of K_v where C_4 is the cycle of length 4 and

Figure 1.1: A $(5, 4)$-equitable K_2-coloring of K_5



K_v is the complete graph on v vertices. They considered such colorings where $p \in \{2, 3, 4\}$, noting that a C_4-decomposition of K_v exists if and only if $v = 1 + 8k$ with $k \geq 1$. For some values of v, (s, p)-equitable block-colorings of H-decompositions of K_v have also been studied in the case where H is a 4-cycle in [12], a 6-cycle in [3] and an 8-cycle in [4] (necessarily v is odd in these cases).

For any C_4-decomposition $\Sigma = (V, B)$ of K_v, Gionfriddo et al. defined its spectrum to be $\Omega_p(\Sigma) = \{s \mid \text{there exists an } (s, p)\text{-equitable block-coloring of } \Sigma\}$. This definition suggests the problem of finding the p-color-spectrum $\Omega_p(v) = \bigcup \Omega_p(\Sigma)$, where the union is taken over the set of all C_4-decompositions, Σ, of K_v. Gionfriddo et al. also considered two values of interest within $\Omega_p(v)$, the lower p-chromatic index defined to be $\chi'_p(v) = \min \Omega_p(v)$, and the upper p-chromatic index defined to be $\overline{\chi'_p}(v) = \max \Omega_p(v)$; that is the least and greatest values of s for which there exists an (s, p)-equitable block-coloring of some C_4-decomposition of K_v. The specific results of Gionfriddo et al. of most interest related to our work are summarized in Theorem 2.1.

The work of Gionfriddo et al. was extended by Li and Rodger in [24] where they considered the existence of (s, p)-equitable block-colorings of C_4-decompositions of $K_v - F$ where F is a 1-factor of K_v. For any C_4-decomposition $\Sigma = (V, B)$ of $K_v - F$, Li and Rodger defined its spectrum to be $\Omega_p(\Sigma) = \{s \mid \text{there exists an } (s, p)\text{-equitable block-coloring of } \Sigma\}$. This definition again suggests the problem of finding the p-color-spectrum $\Omega_p(v) = \bigcup \Omega_p(\Sigma)$, where the union is taken over the set of all C_4-decompositions, Σ, of $K_v - F$. The values of primary interest in $\Omega_p(v)$ are namely the lower p-chromatic index, defined to be $\chi'_p(v) = \min \Omega_p(v)$, and the upper p-chromatic index, defined to be $\overline{\chi'_p}(v) = \max \Omega_p(v)$, that is the least and greatest values of s for which there exists an (s, p)-equitable block-coloring of some C_4-decomposition of $K_v - F$.

The main interest of Li and Rodger in [24] was to find $\chi'_p(v)$ and $\overline{\chi'_p}(v)$ when $p \leq 4$. In so doing, they established $\chi'_{2t}(v)$ for $t \in \{1, 2\}$ when $v \equiv 4t + 2 \pmod{8t}$ stated in Theorem 1.1, which includes a non-existence result concerning $(2t, 2t)$-equitable block-colorings. This non-existence result was the starting point of our work with equitable colorings of graph. In another result they also settled the value of $\chi'_4(v)$ for all other values of v; in such cases $\chi'_4(v) = 4$ (see Theorem 2.2).

Theorem 1.1 ([24]). *Let $v \equiv 4t + 2 \pmod{8t}$. Then*

1. *there is no C_4-decomposition of $K_v - F$ for which there exists a $(2t, 2t)$-equitable block-coloring*

2. *$\chi'_{2t}(v) = 2t + 1$ for $t \in \{1, 2\}$.*

This leaves open the interesting problem of finding $\chi'_{2t}(v)$ when $v \equiv 4t + 2 \pmod{8t}$, noting that Theorem 1.1 just shows that $\chi'_{2t}(v) > 2t$ and settles the case where $t \leq 2$. We continued Li and Rodger's work to show that there is a $(2t + 1, 2t)$-equitable block-coloring of some C_4-decomposition of $K_v - F$ when $v \equiv 4t + 2 \pmod{8t}$ (see Theorem 4.1). As a consequence, the value of $\chi'_{2t}(v)$ when $v \equiv 4t + 2 \pmod{8t}$ is established in Corollary 4.1, thereby settling the open case left in [24]. The complete results from [24] of most interest related to our work are summarized in Theorem 2.2.

Another important focus of our work is developing the study of the structure within such equitable block-colorings. Here we introduce two concepts, originally defined in [23], that provide a way to categorize such colorings. The *color vector* of an (s, p)-equitable block-coloring E of an H-decomposition $(V(G), B)$ of a graph G is the vector

$$V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$$

in which, for $1 \leq i \leq s$, $c_i(E)$ is the number of vertices in G that are incident with a block of color i. In stating results concerning the color vector, we always assume that $c_1(E) \leq c_2(E) \leq \cdots \leq c_s(E)$. If E is clear then, more simply, c_i is written instead of $c_i(E)$. Regarding the color vector, the values naturally of most interest are $c_1(E)$ and $c_s(E)$, in particular just how small or large they can be as E ranges over all possible colorings. Some basic results are also presented for $c_i(E)$ for the intermediate components of the vector and the largest c_i can be for all i is settled in Chapter 5. The following definition formalizes these natural parameters of interest. For any graphs G and H and for $1 \leq i \leq s$, define,

- $\phi(H, G; s, p, i) = \{c_i(E) \mid E$ is an (s, p)-equitable block-coloring of an H-decomposition of $G\}$.
- $\psi'(H, G; s, p, i) = \min \phi(H, G; s, p, i)$, and
- $\overline{\psi'}(H, G; s, p, i) = \max \phi(H, G; s, p, i)$.

Our work with colorings of H-decompositions of G is focused on how large or how small $c_i(E)$ can be in all cases where $\chi'_p(v) > p$. Thus we are solely considering $(2t+1, 2t)$-equitable block-colorings and for convenience define,

$$\psi'(H, G; 2t+1, 2t, i) = \psi'_i(H, G), \text{ and } \overline{\psi'}(H, G; 2t+1, 2t, i) = \overline{\psi'_i}(H, G).$$

For $v' \equiv 4t+2 \pmod{8t}$, the largest value that the smallest element of the color vector can attain, $\overline{\psi'_1}(C_4, K_{v'} - F)$, and the smallest value that the largest element of the color vector can attain, $\psi'_{2t+1}(C_4, K_{v'} - F)$, have been determined, presented here in Chapter 4. In order to find these parameters for C_4-decompositions of $K_{v'} - F$ it suffices to find K_2-decompositions of $K_{v'/2}$ as explained in Chapter 3. Thus $\overline{\psi'_1}(K_2, K_{v'/2})$ and $\psi'_{2t+1}(K_2, K_{v'/2})$ are also determined.

The value of the remaining two parameters of most interest, namely the smallest value that the smallest element of the color vector can attain, $\psi'_1(C_4, K_{v'} - F)$, and the largest value that the largest element of the color vector can attain, $\overline{\psi'_{2t+1}}(C_4, K_{v'} - F)$, are presented in Chapter 5 with an alternate proof presented in Chapter 6. Again, in the process, we determine $\psi'_1(K_2, K_{v'/2})$ and $\overline{\psi'_{2t+1}}(K_2, K_{v'/2})$, thereby also establishing the value of the remaining two parameters of most interest in regards to edge-colorings of $K_{v'/2}$.

One important facet of Chapter 5, is that the proof technique of graph amalgamations is used for the first time to obtain (s,p)-equitable block-colorings (see the proof of Propositions 5.1, 5.2, and 5.3). Formally,

- a graph H is said to be an *amalgamation* of a graph G if there exists a function ψ from $V(G)$ onto $V(H)$ and a bijection $\psi^{'}: E(G) \rightarrow E(H)$ such that,

$$e = \{u,v\} \in E(G) \iff \psi^{'}(e) = \{\psi(u), \psi(v)\} \in E(H).$$

- The function ψ is called an *amalgamation function.*

- We say that G is a *detachment* of H, where each vertex u of H splits into the vertices of $\psi^{-1}(\{u\})$.

- An η-*detachment* of H is a detachment in which each vertex u of H splits into $\eta(u)$ vertices.

The amalgamation approach has been successfully used in many graph decomposition results, especially when edge-colorings representing the decompositions are required to share the colors out fairly in quite a variety of ways, which is further detailed in Chapter 2.

The following notation will be useful in our results. Let $K[R]$ denote the complete graph defined on the vertex set R. Also, define:

- $\lceil x \rceil^{o}$ to be the smallest odd number greater than or equal to x,

- $\lfloor x \rfloor_{o}$ to be the largest odd number less than or equal to x ,

- $\lceil x \rceil^{e}$ to be the smallest even number greater than or equal to x,

- $\lfloor x \rfloor_{e}$ to be the largest even number less than or equal to x,

- $\lceil x \rceil^{d4}$ to be the smallest integer divisible by 4 and greater than or equal to x, and

- $\lfloor x \rfloor_{d4}$ to be the largest integer divisible by 4 and less than or equal to x.

In what follows, a color i is said to appear at a vertex u if at least one block incident with u is colored i.

In Chapter 2 we provide a brief history of graph decompositions . In Chapter 3 general results are presented which form the base for our work, giving parameters for the color vector, and explaining the connection between C_4-colorings of $K_v - F$ and edge-colorings of $K_{v/2}$. In Chapter 4 we settle the open case left in [24], finding the value of $\chi'_{2t}(v')$ for $v' \equiv 4t + 2 \pmod{8t}$. As a result of the construction presented, we also settle the value of $\overline{\psi'_1}(C_4, K_{v'} - F)$, $\psi'_{2t+1}(C_4, K_{v'} - F)$, $\overline{\psi'_1}(K_2, K_{v'/2})$, and $\psi'_{2t+1}(K_2, K_{v'/2})$. In Chapter 5 we settle the value of $\psi'_1(C_4, K_{v'} - F)$ and $\overline{\psi'_i}(C_4, K_{v'} - F)$ for all remaining values of i and the corresponding results in reference to edge-colorings of $K_{v'/2}$ as well. In Chapter 6 we present a simple direct construction of a coloring that is helpful if only maximizing c_{2t+1} is of concern. We also present in Chapter 6 an alternate proof for several results in Chapter 5 via another direct construction. In Chapter 7 we outline further areas of interest in block-colorings of graph-decompositions and where our work will progress from what is presented here.

Finally in Chapter 8 we introduce our work with tiling generalized Petersen graphs, $P(n, k)$, with it's own introduction and history. We detail how to tile $P(n, k)$ with paths on two and four vertices. We also introduce furthers areas of consideration with tiling $P(n, k)$ with paths on six and eight vertices. We then have a concluding summary in Chapter 9.

Chapter 2

History of Graph Decompositions

Hamiltonian decompositions of graphs have been an area of interest in mathematics since 1982 when Walecki proved that K_n has a Hamilton decomposition if and only if n is odd (see [25]). In 1976 Laskar and Auerbach [17] proved that the complete p-partite graph $K_{m,\dots,m}$ has a Hamilton decomposition when $m(p-1)$ is even. They also considered the case where $m(p-1)$ is odd, utilizing a key idea in graph decompositions, that $K_{m,\dots,m}$ has a Hamilton decomposition once a 1-factor is removed (the 1-factor is removed in order to make all vertices have even degree).

As described in the introduction, (s,p)-equitable block-colorings were originally introduced by L. Gionfriddo, M. Gionfriddo, and Ragusa in [10]. They studied (s,p)-equitable C_4-colorings of K_v with $p \in \{2,3,4\}$, noting that a C_4-decompositions of K_v exists only if $v = 1+8k$ with $k \geq 1$. The results from [10] of most interest related to our work are summarized in Theorem 2.1. For some values of v, (s,p)-equitable block-colorings of H-decompositions of K_v have also been studied in the case where H is a 4-cycle in [12], a 6-cycle in [3] and an 8-cycle in [4] (necessarily v is odd in these cases).

Theorem 2.1 ([10]). *Let $v = 1+8k$ with $k \geq 1$. Considering C_4-decompositions of K_v,*

(1) $\Omega_2(v) = \emptyset$ *if k is odd and* $\Omega_2(v) = \{2,3\}$ *is k is even,*

(2) $\chi'_3(v) = 3$,

(3) $\overline{\chi'_3}(v) \leq 8$ *if* $k \equiv 0 \pmod 3$ *or* $k = 1$, $\overline{\chi'_3}(v) \leq 9$ *if* $k \equiv 1 \pmod 3$ *or* $k \equiv 2 \pmod 3$, $v \neq 9, 17$, $\overline{\chi'_3}(v) \leq 10$ *if* $v = 17$,

(4) $\overline{\chi'_4}(v) = 4$ *if and only if* $k \equiv 0 \pmod 4$,

(5) there exists a $(9,4)$-equitable block-coloring of any C_4-decomposition of K_9,

(6) for $s \in \{6,7,8\}$, there exists a C_4-decomposition of K_9 for which there exists an $(s,4)$-equitable block-coloring,

(7) there is no C_4-decomposition of K_9 for which there exists a $(5,4)$-equitable block-coloring, and

(8) if $k = 1$, $\overline{\chi'_4}(v) = 9$, if $k = 2$, $\overline{\chi'_4}(v) \le 13$, if $k \in \{3,4,5\}$, $\overline{\chi'_4}(v) \le 14$, and if $k \ge 6$, $\overline{\chi'_4}(v) \le 15$.

Again the work of Gionfriddo et al. was extended by Li and Rodger in [24] where they considered the existence of (s,p)-equitable block-colorings of C_4-decompositions of $K_v - F$ where F is a 1-factor of K_v and C_4 is the cycle of length 4. Part of Li and Rodger's results were stated in the introduction, but the complete results from [24] of most interest related to our work are summarized in Theorem 2.2.

Theorem 2.2 ([24]). *Concerning C_4-decompositions of $K_v - F$:*

(1) If $v/2$ is even, then there exists an (s,s)-equitable block-coloring of a C_4-decomposition of $K_v - F$ if and only if $v - 2 \ge 2s$,

(2) for each $s \in \{2,3\}$, there exists an $(s,2)$-equitable block-coloring of some C_4-decomposition of $K_v - F$ if and only is v is even, $v \ge 6$, and if $s = 2$ then $v \equiv 6 \pmod 8$,

(3) if $v \equiv 0,\ 2,\ or\ 4 \pmod 8$ then $\Omega_2(v) = \{2,3\}$ and if $v \equiv 6 \pmod 8$ then $\Omega_2(v) = \{3\}$,

(4) there exists a $(3,3)$-equitable block-coloring of some C_4-decomposition of $K_v - F$ if and only if $v \ge 8$,

(5) suppose $v \ge 8$, then

- $\overline{\chi'_3}(v) \le 8$ *if* $v \equiv 2$ *or* $8 \pmod{12}$ *with* $v \ne 8$
- $\overline{\chi'_3}(v) \le 9$ *if* $v \equiv 0,\ 4,\ 6,$ *or* $10 \pmod{12}$ *with* $v \notin \{10,12,18,24,30\}$

– $\overline{\chi_3'}(v) \leq 10$ *if* $v \in \{18, 24, 30\}$

– $\overline{\chi_3'}(v) = 6$ *if* $v \in \{8, 10\}$ *and*

– $\overline{\chi_3'}(12) = 7$,

(6) *There exists a* $(4,4)$*-equitable block-coloring of* $K_v - F$ *if and only is* v *is even,* $v \geq 10$*, and* $v \not\equiv 10 \pmod{16}$*, and*

(7) *if* $v \equiv 10 \pmod{16}$*, then* $\chi_4'(v) = 5$ *, and if* $v \not\equiv 10 \pmod{16}$*, then* $\chi_4'(v) = 4$.

The amalgamation approach, as described in the introduction, has been successfully used in many graph decomposition results, especially when edge-colorings representing the decompositions are required to share the colors out fairly in quite a variety of ways. A key amalgamation result of Bahmanian and Rodger is presented in [2] and explained further in Chapter 3. Their work allows us to disentangle an edge-colored amalgamated graph in an organized equitable way, perfectly suiting our work. Hilton and Rodger [15, 16] used this technique to find embeddings of edge-colorings into Hamiltonian decompositions. Buchanan [5] used amalgamations to find Hamiltonian decompositions of $K_n - E(U)$ for any 2-regular spanning subgraph U. Buchanan's work was then extended to various multipartite graphs by Leach and Rodger [18, 21]. Leach and Rodger [20] went on to find Hamilton decompositions of complete multipartite graphs where each Hamilton cycle spreads its edges out as evenly as possible among the pairs of parts of the graph.

This notion was recently extended further by Erzurumluoğlu and Rodger [9] to (s, p)-equitable block colorings of the complete multipartite graph $K(n, r)$ (n vertices in each of r parts) where the blocks are holey 1-factors (i.e. matchings of size $n(r-1)/2$ in which each matching saturates all vertices except for those in one part called the hole of the matching); so a consequence is that $s = nr$ and $p = n(r-1)$. A similar result of Erzurumluoğlu and Rodger is found in [8] in which the blocks of the decomposition of $K(n, r)$ are cycles of length $n(r-1)$. Additional work with holey decompositions is presented in [19] and [21].

It is worth noting here that if the blocks are defined to be K_2, then we would be seeking a type of edge-coloring that generalizes the well-studied equitable edge-colorings, each of which is easily seen to be equivalent to an (s, s)-equitable edge-coloring (so each block is a copy of K_2). Edge-colorings which are proper are certainly equitable, but equitable edge-colorings become particularly interesting when the number of colors being used to color $E(G)$ is less than $\chi'(G)$ (for example, see [2, 14, 19, 31] for some results and applications). Interchanging colors along paths with alternately colored edges is a traditionally powerful technique for finding such edge-colorings, but they are rendered useless in this more general setting whenever it is required that $s > p$, as is the situation for results presented here. Not only are these edge-colorings challenging to produce in themselves, but also (s, p)-equitable edge-colorings of K_v are relevant here because of the connection to C_4-decompositions of $K_v - F$ described in Lemma 3.1.

Chapter 3

General Results for Block Colorings of Graph Decompositions

Define $G \times 2$ to be the graph with vertex set $\{(u,1),(u,2) \mid u \in V(G)\}$ and edge set $\{\{(u,i),(w,j)\} \mid 1 \leq i,j \leq 2 \text{ and } \{u,w\} \in E(G)\}$. As Lemma 3.1 suggests, when studying C_4-decompositions of $K_v - F$, edge-colorings of the graph $K_{v/2}$ are pertinent and useful.

Lemma 3.1 ([24]). *If there exists an (s,p)-equitable edge-coloring E of G then there exists an (s,p)-equitable C_4-coloring E' of $G \times 2 - F$ for some 1-factor F of $G \times 2$.*

Furthermore, note by Lemma 3.1 that $2c_i(E) = c_i(E')$ for $1 \leq i \leq s$. The specifics of forming a C_4-coloring of $G \times 2$ from a K_2-coloring of G are detailed in [24]. As was done in [23] it can be shown that there exists a $(2t+1, 2t)$-equitable block-coloring of some C_4-decompositions of $K_{v'} - F$ when $v' \equiv 4t + 2 \pmod{8t}$ by simply showing that there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v where $v = v'/2$ as a result of Lemma 3.1. This connection is naturally the source of our interest in equitable edge-colorings of K_v, but they are also of interest in their own right.

In any (s,p)-equitable block-coloring E of an H-decomposition $(V(G), B)$ of some graph G, for each $u \in V(G)$, let $b(H, G; E, u, i)$ denote the number of blocks incident with u that are colored i. An immediate consequence of this definition, a less specific version of which was first presented in [23], is the following. Note for ease of notation, let $x \approx y$ represent the fact that $\lfloor y \rfloor \leq x \leq \lceil y \rceil$.

Lemma 3.2 ([26]). *Let $v' = 2v = 8tx + 4t + 2$ for some integer x. Let E and E' be $(2t+1, 2t)$-equitable edge and C_4-colorings of K_v and $K_{v'} - F$ respectively. Then*

$$b(K_2, K_v; E, u, i) = b(C_4, K_{v'} - F; E', u', i') = 2x + 1$$

for all $u \in V(K_v)$*,* $i \in C(E,u)$*,* $u' \in V(K_{v'} - F)$*, and* $i' \in C(E', u')$*.*

Proof. Let $v' = 2v = 4t + 2 + 8tx$ for some integer x and E be a $(2t+1, 2t)$-equitable edge-coloring of K_v. Therefore since E is a $(2t+1, 2t)$-equitable edge-coloring,

$$b(K_2, K_v; E, u, i) \approx \frac{d_{K_v}(u)}{2t} = \frac{2t + 1 + 4tx - 1}{2t} = 2x + 1$$

for all $u \in V(K_v)$, and $i \in C(E, u)$.

Note for $u' \in V(K_{v'} - F)$, $d_{K_{v'}-F}(u') = 8tx + 4t$ and the number of blocks in any C_4-decomposition of $K_{v'} - F$ incident with u' is $d_{K_{v'}-F}(u')/2 = 4tx + 2t$. Let E' be a $(2t+1, 2t)$-equitable C_4-coloring of $K_{v'} - F$ and $i' \in C(E', u')$. Then

$$b(C_4, K_{v'} - F; E', u', i') = \frac{4tx + 2t}{2t} = 2x + 1.$$

□

The particular cases of interest here are $(2t+1, 2t)$-equitable edge-coloring, E of K_v where $v \equiv 2t + 1 \pmod{4t}$ and C_4-colorings of $K_{v'} - F$ where $v' = 2v$. So in this situation (and, indeed, whenever $v \equiv p + 1 \pmod{2p}$) Lemma 3.2 implies that $b(K_2, K_v; E, u, i)$ and $b(C_4, K_{v'} - F; E', u', i')$ are constant for all $u \in V(K_v)$, $u' \in V(K_{v'} - F)$, $i \in C(E, u)$, and $i' \in C(E', u')$ regardless of the choice of E or E'; so in such cases it makes sense to define,

$$b(v) = b'(v') = b(K_2, K_v; E, u, i) = b(C_4, K_{v'} - F; E', u', i') = 2x + 1.$$

We now get a series of lemmas that restrict parameters of interest in regards to equitable edge-colorings of K_v and equitable C_4-colorings of $K_{v'} - F$ where $v' \equiv 4t + 2 \pmod{8t}$ and $v = v'/2$ with some more general results as well.

Lemma 3.3. *Let* $v \equiv p + 1 \pmod{2p}$*. In any* (s,p)*-equitable edge-coloring* E *of* K_v*, for* $1 \leq i \leq s$

(i) $c_i(E)$ *must be even,*

(ii) $c_i(E) \geq b(v) + 1 = \frac{v-1}{p} + 1$*, and*

(iii) if v *is odd, then* $c_i(E) \leq v - 1$.

Proof. Let $1 \leq i \leq s$ and E be an (s,p)-equitable edge-coloring of K_v. Since $v \equiv p+1 \pmod{2p}$, let $v = p + 1 + 2px$, where x is an integer. Then by Lemma 3.2, $b(v) = \frac{v-1}{p} = \frac{p+2px}{p} = 1+2x$, so $b(v)$ is odd. If $c_i(E)$ is odd, the subgraph induced by $W = \{u \in V(K_v) | i \in C(E,u)\}$ would be an odd-regular graph with an odd number of vertices, which cannot exist, so (i) follows. For $u \in V(K_v)$ and $i \in C(E,u)$, u is joined to $b(v)$ neighbors with an edge colored i, so at least $b(v)+1$ vertices are incident with an edge colored i; so (ii) follows. Clearly $c_i \leq v$, so (iii) follows since c_i was just shown to be even and v is clearly odd. $\square$

Lemma 3.4. *Let* $v' \equiv 4t + 2 \pmod{8t}$*. In any* $(2t+1, 2t)$*-equitable* C_4*-coloring* E' *of* $K_{v'} - F$*, for* $1 \leq i \leq 2t+1$

(1) 4 *divides* $c_i(E')$*,*

(2) $c_i('E) \geq 2(b'(v') + 1)$*, and*

(3) $c_i(E') \leq v' - 2$.

Proof. Let $v' = 4t+2+8tx$ for some integer x. For $u \in V(K_{v'} - F)$, $d_{K_{v'}-F}(u) = 8tx + 4t$, so the number of blocks in any C_4-decomposition of $K_{v'} - F$ incident with u is $d_{K_{v'}-F}(u)/2 = 4tx + 2t$. Let E' be a $(2t+1, 2t)$-equitable C_4-coloring of $K_{v'} - F$ and $i \in C(E', u)$. Then

$$b(C_4, K_{v'} - F; E', u, i) = \frac{4tx + 2t}{2t} = 2x + 1.$$

Note then the number of edges in the blocks in each color class is

$$2c_i(E')b(C_4, K_{v'} - F; E', u, i)/2 = c_i(E')(2x + 1).$$

Therefore 4 divides $c_i(E')$ since E' is a C_4-coloring.

Note as well that the $2b'(v')$ neighbors of u are clearly each incident with a block colored i, so $c_i(E) \geq 2b'(v') + 1 = 4x + 3$. Therefore, since by (1) 4 divides $c_i(E)$, $c_i(E) \geq 4x + 4 = 2(b'(v') + 1)$. Also, since 4 divides $c_i(E)$ and $v' \equiv 4t + 2 \pmod{8t}$, it follows that $c_i(E) \leq v' - 2$. □

Lemma 3.5. *Let E be a $(2t+1, 2t)$-equitable block-coloring of a graph G with $v = |V(G)|$ vertices. Then,*

$$\sum_{i=1}^{2t+1} c_i(E) = 2tv.$$

Proof. Since the number of colors appearing at each vertex is $2t$ and c_i is the number of vertices where a block of color i appears, the above holds. □

Lemma 3.6. *Let $v \equiv 2t + 1 \pmod{4t}$. In any $(2t+1, 2t)$-equitable edge-coloring E of K_v,*

(i) $c_1(E) \leq \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$ *and*

(ii) $c_{2t+1}(E) \geq \left\lceil \frac{2tv}{2t+1} \right\rceil^e$.

Proof. Note by Lemma 3.5 the average of the integers $c_1, \ldots, c_{2t+1}$ is $\dfrac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \dfrac{2tv}{2t+1}$. By Lemma 3.3, c_i is even and by definition $c_1 \leq c_i$ for $1 \leq i \leq 2t+1$, so $c_1 \leq \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$. Similarly by Lemma 3.3, c_i is even and by definition $c_{2t+1} \geq c_i$ for $1 \leq i \leq 2t+1$, so it follows that $c_{2t+1} \geq \left\lceil \frac{2tv}{2t+1} \right\rceil^e$. □

Lemma 3.7. *Let $v' \equiv 4t + 2 \pmod{8t}$. In any $(2t+1, 2t)$-equitable C_4-coloring E' of $K_{v'} - F$,*

(i) $c_1(E') \leq \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$ *and*

(ii) $c_{2t+1}(E') \geq \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$.

Proof. Note by Lemma 3.5 the average of the integers $c_1, \ldots, c_{2t+1}$ is $\dfrac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \dfrac{2tv'}{2t+1}$. By Lemma 3.4 for $1 \leq i \leq 2t+1$, c_i is divisible by 4. By definition $c_1 \leq c_i$ for $1 \leq i \leq 2t+1$, so $c_1 \leq \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$. Similarly by definition $c_{2t+1} \geq c_i$ for $1 \leq i \leq 2t+1$, so it follows that $c_{2t+1} \geq \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$. $\square$

Lemma 3.8. *Let $v \equiv 2t+1 \pmod{4t}$ and let E be a $(2t+1, 2t)$-equitable edge-coloring of K_v. If $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$ then*

(i) $c_1(E) = \overline{\psi_1'}(K_2, K_v) = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$, and

(ii) $c_{2t+1}(E) = \psi_{2t+1}'(K_2, K_v) = \left\lceil \frac{2tv}{2t+1} \right\rceil^e$.

Proof. Since E is clear we let $c_i = c_i(E)$ for $1 \leq i \leq 2t+1$. First suppose that $|c_1 - c_{2t+1}| = 0$. Since we have named the colors so that $c_i \leq c_j$ for $1 \leq i < j \leq 2t+1$, it follows that $c_i = c_j$ for $1 \leq i < j \leq 2t+1$. Then by Lemma 3.5,

$$c_i = \frac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \frac{2tv}{2t+1}.$$

Therefore by Lemma 3.3, $\frac{2tv}{2t+1}$ is an even integer and $c_i = \frac{2tv}{2t+1} = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e = \left\lceil \frac{2tv}{2t+1} \right\rceil^e$. Hence by Lemma 3.6, $c_1 = \overline{\psi_1'}(K_2, K_v) = \frac{2tv}{2t+1}$ and $c_{2t+1} = \psi_{2t+1}'(K_2, K_v) = \frac{2tv}{2t+1}$.

Now suppose that $|c_1 - c_{2t+1}| = 2$. By the naming of the colors, we have that $c_1 = c_{2t+1} - 2$. Note by Lemma 3.5, $\frac{2tv}{2t+1}$ is the average of the integers $c_1, \ldots, c_{2t+1}$. So if $c_1 < c_{2t+1}$ then,

$$c_1 < \frac{2tv}{2t+1} < c_{2t+1}. \tag{3.1}$$

If $\frac{2tv}{2t+1}$ is an integer then it is clearly even; but then by (3.1) this is a contradiction, because by Lemma 3.3 c_{2t+1} is an even integer.

Therefore $\frac{2tv}{2t+1}$ is not an integer and $\lfloor \frac{2tv}{2t+1} \rfloor_e = \lceil \frac{2tv}{2t+1} \rceil^e - 2$. So by Lemma 3.6 we have that

$$\begin{aligned} c_{2t+1} - 2 = c_1 \leq \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e &= \left\lceil \frac{2tv}{2t+1} \right\rceil^e - 2, \text{ so} \\ c_{2t+1} \leq \left\lceil \frac{2tv}{2t+1} \right\rceil^e &\leq c_{2t+1}, \end{aligned}$$

so $c_{2t+1} = \psi'_{2t+1}(K_2, K_v) = \lceil \frac{2tv}{2t+1} \rceil^e$. It also follows now that

$$c_1 = c_{2t+1} - 2 = \left\lceil \frac{2tv}{2t+1} \right\rceil^e - 2 = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e,$$

so by Lemma 3.6, $c_1 = \overline{\psi'_1}(K_2, K_v) = \lfloor \frac{2tv}{2t+1} \rfloor_e$. $\square$

Lemma 3.9. *Let $v' \equiv 4t + 2 \pmod{8t}$ and let E' be a $(2t + 1, 2t)$-equitable C_4-coloring of $K_{v'} - F$. If $|c_1(E') - c_{2t+1}(E')| \in \{0, 4\}$ then*

(i) $c_1(E') = \overline{\psi'_1}(C_4, K_{v'} - F) = \lfloor \frac{2tv'}{2t+1} \rfloor_{d4}$, *and*

(ii) $c_{2t+1}(E') = \psi'_{2t+1}(C_4, K_{v'} - F) = \lceil \frac{2tv}{2t+1} \rceil^{d4}$.

Proof. Since E' is clear we let $c_i = c_i(E')$ for $1 \leq i \leq 2t+1$. First suppose that $|c_1 - c_{2t+1}| = 0$. Since we have named the colors so that $c_i \leq c_j$ for $1 \leq i < j \leq 2t+1$, it follows that $c_i = c_j$ for $1 \leq i < j \leq 2t + 1$. Then by Lemma 3.5

$$c_i = \frac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \frac{2tv'}{2t+1}.$$

Therefore by Lemma 3.4, $\frac{2tv'}{2t+1}$ is divisible by 4 and $c_i = \frac{2tv'}{2t+1} = \lfloor \frac{2tv'}{2t+1} \rfloor_{d4} = \lceil \frac{2tv'}{2t+1} \rceil^{d4}$. Hence by Lemma 3.7 $c_1 = \overline{\psi'_1}(C_4, K_{v'} - F) = \frac{2tv'}{2t+1}$ and $c_{2t+1} = c_{2t+1} = \psi'_s(C_4, K_{v'} - F) = \psi'_{2t+1}(C_4, K_{v'} - F) = \frac{2tv'}{2t+1}$.

Now suppose that $|c_1 - c_{2t+1}| = 4$. By the naming of the colors, we have that $c_1 = c_{2t+1} - 4$. Note by Lemma 3.5, $\frac{2tv'}{2t+1}$ is the average of the integers $c_1, \ldots, c_{2t+1}$. So if $c_1 < c_{2t+1}$ then

$$c_1 < \frac{2tv'}{2t+1} < c_{2t+1}. \tag{3.2}$$

If $\frac{2tv'}{2t+1}$ is an integer then it is clearly divisible by 4 since $v' \equiv 4t+2 \pmod{8t}$; but then by (3.2) this is a contradiction, because by Lemma 3.4 c_{2t+1} is divisible by 4.

Therefore $\frac{2tv'}{2t+1}$ is not an integer and $\left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4} = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4$. So by Lemma 3.7 we have that

$$\begin{aligned} c_{2t+1} - 4 = c_1 \leq \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4} &= \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4, \text{ so} \\ c_{2t+1} \leq \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} &\leq c_{2t+1}, \end{aligned}$$

so $c_{2t+1} = \psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$. It also follows now that

$$c_1 = c_{2t+1} - 4 = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4 = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4},$$

so by Lemma 3.7, $c_1 = \overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$. □

Lemma 3.10. *Let $v \equiv 2t+1 \pmod{4t}$. In any $(2t+1, 2t)$-equitable edge-coloring E of K_v, $c_1(E) \geq 2t$.*

Proof. Since v is odd, by Lemma 3.3 $c_i \leq v-1$ for $1 \leq i \leq 2t+1$. Thus $\sum_{i=2}^{2t+1} c_i \leq 2t(v-1)$. Thus by Lemma 3.5,

$$\begin{aligned} 2tv = \sum_{i=1}^{2t+1} c_i &= \sum_{i=2}^{2t+1} c_i + c_1 \\ &\leq 2t(v-1) + c_1. \end{aligned}$$

Thus $2tv - 2t(v-1) \leq c_1$ and so $2t \leq c_1$ as required. □

Lemma 3.11. *Let $v' \equiv 4t+2 \pmod{8t}$. In any $(2t+1, 2t)$-equitable block-coloring E of a C_4-decomposition of $K_{v'} - F$, $c_1(E) \geq 4t$.*

Proof. Note by Lemma 3.4 $c_i \leq v' - 2$ for $1 \leq i \leq 2t+1$. Thus $\sum_{i=2}^{2t+1} c_i \leq 2t(v'-2)$. Note as well since there are exactly $2t$ colors appearing at each vertex, $\sum_{i=2}^{2t+1} c_i = 2tv'$. Thus,

$$\begin{aligned} 2tv' = \sum_{i=1}^{2t+1} c_i &= \sum_{i=2}^{2t+1} c_i + c_1 \\ &\leq 2t(v'-2) + c_1. \end{aligned}$$

Thus $2tv' - 2t(v'-2) \leq c_1$ and $4t \leq c_1$ as required. □

Lemma 3.12. *For $v \equiv 2t+1 \pmod{4t}$ and $v' = 2v$,*

$$\overline{\psi_i'}(K_2, K_v) \leq \left\lfloor \frac{2tv - \sum_{j=1}^{i-1} \psi_j'(K_2, K_v)}{2t+2-i} \right\rfloor_e \text{ and}$$

$$\overline{\psi_i'}(C_4, K_{v'} - F) \leq \left\lfloor \frac{2tv' - \sum_{j=1}^{i-1} \psi_j'(C_4, K_{v'} - F)}{2t+2-i} \right\rfloor_e.$$

Proof. Note the elements of the color vector are listed in non-decreasing order; and since in Lemma 3.5 it is shown that for any $(2t+1, 2t)$-equitable edge-coloring E of K_v and for any $(2t+1, 2t)$-equitable C_4-coloring E' of $K_{v'} - F$, both $\sum_{i=1}^{2t+1} c_i(E) = 2tv$ and $\sum_{i=1}^{2t+1} c_i(E') = 2tv'$, the above holds. □

To describe the amalgamation result used in Chapter 5 more precisely, some notation will be needed. Again, we let $x \approx y$ represent the fact that $\lfloor y \rfloor \leq x \leq \lceil y \rceil$. Furthermore, let $\ell(u)$ denote the number of loops incident with vertex u, where loops contribute two to the degree of u, let $G(j)$ denote the subgraph of G induced by the edges colored j, and let $m(u, v)$ denote the number of edges between the pair of vertices u and v in G.

The following is a special case of Theorem 3.1 in [2] (omitting the condition that ensures color classes are connected and a balanced property on the color classes for multigraphs since in our case G is simple).

Theorem 3.1. (Bahmanian, Rodger [2, Theorem 3.1]) *Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $v \in V(H)$, $\eta(v) = 1$ implies*

$\ell_H(v) = 0$. Then there exists a loopless η-detachment G of H in which each $v \in V(H)$ is detached into $v_1, \dots, v_{\eta(v)}$, such that G satisfies the following conditions:

(i) $d_G(u_i) \approx d_H(u)/\eta(u)$ *for each* $u \in V(H)$ *and* $1 \le i \le \eta(u)$;

(ii) $d_{G(j)}(u_i) \approx d_{H(j)}(u)/\eta(u)$ *for each* $u \in V(H)$, $1 \le i \le \eta(u)$, *and* $1 \le j \le k$;

(iii) $m_G(u_i, u_{i'}) \approx \ell_H(u)/\binom{\eta(u)}{2}$ *for each* $u \in V(H)$ *with* $\eta(u) \ge 2$ *and* $1 \le i < i' \le \eta(u)$; *and*

(iv) $m_G(u_i, v_{i'}) \approx m_H(u,v)/(\eta(u)\eta(v))$ *for every pair of distinct vertices* $u, v \in V(H)$, $1 \le i \le \eta(u)$, *and* $1 \le i' \le \eta(v)$.

Chapter 4

Initial Main Result for Block Colorings of Graph Decompositions

The following theorem establishes the value of $\chi'_{2t}(v')$ for $v' \equiv 4t+2 \pmod{8t}$, settling the open case left in [24] (see Corollary 4.1). In so doing, with $v = v'/2$, an extreme equitable edge-coloring is produced for K_2-decompositions of K_v, establishing the largest value that the smallest element of the color vector can attain, $\overline{\psi'_1}(K_2, K_v)$, and the smallest value that the largest element of the color vector can attain, $\psi'_{2t+1}(K_2, K_v)$ (see Corollary 4.2). Using Lemma 3.1, this also creates an extreme equitable block-coloring for C_4-decompositions of $K_{v'} - F$, establishing the analogous extreme values of the color vector, specifically $\overline{\psi'_1}(C_4, K_{v'} - F)$ and $\psi'_{2t+1}(C_4, K_{v'} - F)$ as stated in Corollary 4.3.

Theorem 4.1. *Let $v' \equiv 4t+2 \pmod{8t}$ Then there exists a $(2t+1, 2t)$-equitable block-coloring of some C_4-decomposition of $K_{v'} - F$.*

Proof. By Lemma 3.1 it need only be shown that there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v where $v = \frac{v'}{2}$, thus we let $v = 2t+1+4tx$. We first describe the coloring E, show E is well defined, and then show E is a $(2t+1, 2t)$-equitable edge-coloring.

We begin by partitioning the vertices into $2t+1$ groups such that

1. each group has an odd number of vertices and

2. the number of vertices in any two groups differs by at most two.

Therefore the size of each group is $l(v) = \left\lceil \frac{v}{2t+1} \right\rceil^o$ or $s(v) = \left\lfloor \frac{v}{2t+1} \right\rfloor_o$. Note by construction, the number of groups with $s(v)$ vertices, which we refer to as small groups, is

$$S_v = \frac{l(v)(2t+1) - v}{2} = \frac{2l(v)t + l(v) - (2t+1+4tx)}{2} = l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{1}{2}.$$

Note, S_v is easily seen to be an integer. The number of groups with $l(v)$ vertices, which we refer to as large groups, is $L_v = 2t + 1 - S_v$. Note as well for calculation purposes,

$$b(v) = \frac{v-1}{p} = \frac{2t+1+4tx-1}{2t} = 2x+1.$$

Let the groups with $s(v)$ vertices be named $P_1, \ldots, P_{S_v}$ and the groups with $l(v)$ vertices be named $P_{S_v+1}, \ldots, P_{2t+1}$. For $1 \leq j \leq t$ and $1 \leq i \leq 2t+1$ we color all the edges between the vertices in group P_m and the vertices in group P_n with color i for $m \equiv i + j \pmod{2t+1}$ and $n \equiv i - j \pmod{2t+1}$. Clearly the coloring of all edges between the groups is well defined.

To color edges joining vertices within the groups, first note that $b(v)$, $s(v)$ and $l(v)$ are all odd (so $b(v) - s(v)$ and $b(v) - l(v)$ are even) and it is well known that there exists a 2-factorization of $K[P_i]$ for $1 \leq i \leq 2t+1$. The 2-factors in such 2-factorizations are combined as follows. For $1 \leq l \leq 2t+1$ and $1 \leq j \leq S_v$ with $l \neq j$, if the edges joining P_l to P_j are colored i then color the edges in one $\big(b(v) - s(v)\big)$-factor in $K[P_l]$ with color i; so now there are exactly $b(v)$ edges of color i incident with each vertex in P_l. Similarly, for $1 \leq l \leq 2t+1$ and $S_v + 1 \leq k \leq 2t+1$ with $l \neq k$, if the edges joining P_l to P_k are colored i, then we color the edges in one $\big(b(v) - l(v)\big)$-factor in $K[P_l]$ with color i; so it is also the case that now there are exactly $b(v)$ edges of color i incident with each vertex in P_l. So for every vertex v and for every color i on an edge incident with v:

$$\text{v is incident with exactly } b(v) \text{ edges colored } i. \tag{4.1}$$

Notice that 4.1 implies that for each v, blocks incident with v have been colored with $\deg(v)/b(v) = p$ colors as required. By considering two cases in turn, we now show that this construction is well-defined, and that all the edges in $K[P_i]$ for $1 \leq i \leq 2t+1$ have been colored.

Case 1: Suppose $l(v) = s(v)$, so the number of vertices in each group is $l(v) = \frac{v}{2t+1}$, and $K[P_i] \cong K_{l(v)}$ for $1 \leq i \leq 2t+1$. It suffices to show that each vertex in each $K_{l(v)}$ has degree equal to the sum of the degrees of factors defined in the coloring. Therefore since each P_i is joined to P_j for all $j \neq i$, the sum of the degrees of the factors is

$$\begin{aligned}
\sum_{1}^{2t} b(v) - l(v) &= 2t\big(b(v) - l(v)\big) \\
&= 2t\left(2x + 1 - \frac{v}{2t+1}\right) \\
&= 4xt + 2t - \frac{\big(4t^2 + 2t + 8xt^2\big)}{2t+1} \\
&= \frac{4xt(2t+1) + 2t(2t+1) - 4t^2 - 2t - 8xt^2}{2t+1} \\
&= \frac{8xt^2 + 4xt + 4t^2 + 2t - 4t^2 - 2t - 8xt^2}{2t+1} \\
&= \frac{4xt}{2t+1} \\
&= \frac{4xt + 2t + 1 - 2t - 1}{2t+1} \\
&= \frac{v}{2t+1} - 1
\end{aligned}$$

which is exactly the degree of each vertex in $K_{l(v)}$. Therefore the coloring of the complete graphs induced by each group is well defined.

Case 2: Suppose $l(v) \neq s(v)$. Then $\frac{v}{2t+1}$ is not an odd integer, so $s(v) = l(v) - 2$.

First we show that each vertex in $K[P_i]$ for $1 \leq i \leq S_v$ has degree equal to the sum of the degrees of its factors defined in the coloring. Since each P_i for $1 \leq i \leq S_v$ is joined to all small groups except itself and to all large groups, the sum of the degrees of the factors is

$$
\begin{aligned}
\big(S_v - 1\big)(b(v) - s(v)) + L_v\big(b(v) - l(v)\big) \\
&= \big(S_v - 1\big)\big(b(v) - (l(v) - 2)\big) + L_v\big(b(v) - l(v)\big) \\
&= \big(b(v) - l(v)\big)\big(S_v - 1 + L_v\big) + 2\big(S_v - 1\big) \\
&= \big(b(v) - l(v)\big)\big(S_v - 1 + 2t + 1 - S_v\big) + 2\big(S_v - 1\big) \\
&= \big(b(v) - l(v)\big)2t + 2\big(S_v - 1\big) \\
&= \big(2x + 1 - l(v)\big)2t + 2\left(l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{3}{2}\right) \\
&= 4tx + 2t - 2tl(v) + 2tl(v) + l(v) - 4tx - 2t - 3 \\
&= l(v) - 3 = s(v) - 1
\end{aligned}
$$

which is exactly the degree of each vertex in $K[P_i]$ for $1 \leq i \leq S_v$. Therefore the coloring of each complete graph induced on the vertices of each small group is well defined.

We now show that each vertex in $K[P_i]$ for $S_v + 1 \leq i \leq 2t + 1$ has degree equal to the sum of the degrees of its factors defined in the coloring. Since each P_i for $S_v \leq i \leq 2t+1$ is joined to all small groups and all large groups except itself, the sum of the degrees of the factors is

$$
\begin{aligned}
S_v(b(v) - s(v)) + \big(L_v - 1\big)\big(b(v) - l(v)\big)& \\
&= S_v\big(b(v) - (l(v) - 2)\big) + \big(L_v - 1\big)\big(b(v) - l(v)\big) \\
&= \big(b(v) - l(v)\big)\big(S_v + L_v - 1\big) + 2S_v \\
&= \big(b(v) - l(v)\big)\big(S_v + 2t + 1 - S_v - 1\big) + 2S_v \\
&= \big(b(v) - l(v)\big)2t + 2S_v \\
&= \big(2x + 1 - l(v)\big)2t + 2\left(l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{1}{2}\right) \\
&= 4tx + 2t - 2tl(v) + 2tl(v) + l(v) - 4tx - 2t - 1 \\
&= l(v) - 1
\end{aligned}
$$

which is exactly the degree of each vertex in $K[P_i]$ for $S_v + 1 \leq i \leq 2t + 1$. Therefore the coloring of each complete graph induced on the vertices of each large group is well defined.

In both cases the number of colors used is exactly $p = 2t + 1$, so it remains to show there are exactly $s = 2t$ colors appearing at each vertex. We will do this by showing that for $1 \leq i \leq 2t + 1$:

(a) color i is does not appear at any vertex of P_i and

(b) for each color $h \neq i$, color h appears at all vertices of P_i.

Note that the edges colored i between parts join P_{i+j} and P_{i-j} for $1 \leq j \leq t$. In particular, there are no edges colored i between parts which are incident with vertices in P_i. Therefore the construction also ensures that there is no factor colored i in $K[P_i]$, so it follows that (a) holds. Note that for $1 \leq h \leq 2t + 1$ with $h \neq i$, there exist j', $1 \leq j' \leq t$, for which $i \equiv h \pm j' \pmod{2t+1}$. So each vertex in P_i has an edge of color h incident with it by construction, so (b) holds. Finally note by 4.1, for each $u \in V(K_v)$ and $i \in C(E, u)$

there are $b(v)$ edges of color i incident with u, so the coloring is equitable. Therefore we have formed a $(2t+1, 2t)$-equitable edge-coloring of K_v. □

Corollary 4.1. *Let $v' \equiv 4t+2 \pmod{8t}$. Then $\chi'_{2t}(v') = 2t+1$.*

Proof. By Theorem 1.1 we know that $\chi'_{2t}(v') > 2t$. So, using the edge-coloring produced in the proof of Theorem 4.1, it follows by Lemma 3.1 that $\chi'_{2t}(v') = 2t+1$. □

Corollary 4.2. *Let $v \equiv 2t+1 \pmod{4t}$. Then*

(i) $\overline{\psi'_1}(K_2, K_v) = \lfloor \frac{2tv}{2t+1} \rfloor_e$, *and*

(ii) $\psi'_{2t+1}(K_2, K_v) = \lceil \frac{2tv}{2t+1} \rceil^e$.

Proof. By the proof of Theorem 4.1, there exists an edge-coloring E of K_v such that $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$. So by Lemma 3.8, $\overline{\psi'_1}(K_2, K_v) = \lfloor \frac{2tv}{2t+1} \rfloor_e$ and $\psi'_{2t+1}(K_2, K_v) = \lceil \frac{2tv}{2t+1} \rceil^e$. □

Corollary 4.3. *Let $v' \equiv 4t+2 \pmod{8t}$. Then*

(i) $\overline{\psi'_1}(C_4, K_{v'} - F) = \lfloor \frac{2tv'}{2t+1} \rfloor_{d4}$, *and*

(ii) $\psi'_{2t+1}(C_4, K_{v'} - F) = \lceil \frac{2tv'}{2t+1} \rceil^{d4}$.

Proof. Let $v = v'/2$. By the proof of Theorem 4.1, there exists an edge-coloring E of K_v such that $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$. Thus by Lemma 3.1 there exists a C_4-coloring E' of $K_{v'} - F$ and $c_i(E) = 2c_i(E')$ for $1 \le i \le s$, so $|c_1(E') - c_2(E')| = |2c_1(E) - 2c_{2t+1}(E)| = 2|c_1(E) - c_{2t+1}(E)| \in \{0, 4\}$. So by Lemma 3.9, $\overline{\psi'_1}(C_4, K_{v'} - F) = \lfloor \frac{2tv'}{2t+1} \rfloor_{d4}$ and $\psi'_{2t+1}(C_4, K_{v'} - F) = \lceil \frac{2tv'}{2t+1} \rceil^{d4}$. □

Thus in regards to equitable edge-colorings of K_v and equitable block-colorings of C_4-decompositions of $K_{v'} - F$, we have established two extreme values for each; the largest values that the smallest element of the color vector can attain and the smallest value that the largest element of the color vector can attain.

Chapter 5

Exploring the Color Vector for Block Colorings of Graph Decompositions

The largest value that the smallest element of the color vector can attain, $\overline{\psi'_1}(C_4, K_{v'} - F)$, and the smallest value that the largest element of the color vector can attain, $\psi'_{2t+1}(C_4, K_{v'} - F)$, were proved in Chapter 4. In this Chapter we obtain another extreme coloring, establishing the smallest value that the smallest element of the color vector can attain, $\psi'_1(C_4, K_{v'} - F)$ and the largest value that the largest element of the color vector can attain, $\overline{\psi'_{2t+1}}(C_4, K_{v'} - F)$; see Corollary 5.1.

These extreme colorings again follow from another interesting problem, finding $\psi'_1(K_2, K_{v'/2})$ and $\overline{\psi'_{2t+1}}(K_2, K_{v'/2})$; see Theorem 5.1. The powerful proof technique of graph amalgamations is utilized for the first time in the study of graph decompositions, applying Theorem 3.1 in the proofs of Propositions 5.1, 5.2, and 5.3. As a bonus from our method of proof, we settle $\overline{\psi'_i}(K_2, K_v)$ and $\overline{\psi'_i}(C_2, K_{v'} - F)$ for all other values of i (see Corollary 5.1 and Theorem 5.1).

Lemma 5.1. *Let $v = v'/2 = 4tx + 2t + 1$ for some integer x and $b(v) + 1 = b'(v') + 1 \geq 2t$. Then,*

$$\overline{\psi'_2}(K_2, K_v) \leq \left\lfloor v - \frac{x+1}{t} \right\rfloor_e \text{ and } \overline{\psi'_2}(C_4, K_{v'} - F) \leq \left\lfloor v' - \frac{2x+2}{t} \right\rfloor_e .$$

Proof. Let $b(v) + 1 \geq 2t$. By Lemma 3.3, $\psi'_1(K_2, K_v) \geq b(v) + 1$. Therefore by Lemma 3.12,

$$\begin{aligned}
\overline{\psi'_2}(K_2, K_v) &\leq \lfloor \frac{2tv - \psi'_1(K_2, K_v))}{2t + 2 - 2} \rfloor_e \\
&\leq \lfloor v - \frac{(b(v) + 1)}{2t} \rfloor_e \\
&= \lfloor v - \frac{2x + 2}{2t} \rfloor_e \\
&= \lfloor v - \frac{x + 1}{t} \rfloor_e .
\end{aligned}$$

By Lemma 3.4, $\psi_1'(C_4, K_{v'} - F) \geq 2b'(v') + 2$. Therefore by Lemma 3.12,

$$\begin{aligned}
\overline{\psi_2'}(C_4, K_{v'} - F) &\leq \left\lfloor \frac{2tv' - \psi_1'(C_4, K_{v'} - F))}{2t + 2 - 2} \right\rfloor_e \\
&\leq \left\lfloor v' - \frac{(2b'(v') + 2)}{2t} \right\rfloor_e \\
&= \left\lfloor v' - \frac{2x + 4}{2t} \right\rfloor_e \\
&= \left\lfloor v' - \frac{2x + 2}{t} \right\rfloor_e .
\end{aligned}$$

□

Proposition 5.1. *Let $v \equiv 2t + 1 \pmod{4t}$ with $v > 1$. Let $2t \leq b(v) + 1$. Then*

$$\overline{\psi_2'}(K_2, K_v) = \left\lfloor v - \frac{x + 1}{t} \right\rfloor_e .$$

Proof. Let $v = 4tx + 2t + 1$ for some integer x. Form a complete graph $\mathcal{G}_0$ on the set of vertices $V_0 = \{u_1, \ldots, u_{2x+2}\}$ and color all the edges of $\mathcal{G}_0$ with color $2t + 1$. So each vertex in $\mathcal{G}_0$ is incident with $2x + 1 = b(v)$ edges colored $2t + 1$ as desired. Notice that in the final edge-coloring of K_v, each vertex is missing (i.e., is not incident with any edges of) exactly one color. We will arrange for $1 \leq i \leq 2t$, color $m(i) = i$ to be missing from vertex u_i, for $2t + 1 \leq i \leq 2x + 2$ color $m(i) = \lceil \frac{i-2t}{2} \rceil \pmod{2t} \in \{1, \ldots, 2t\}$ to be missing from u_i, and color $m(\alpha_i) = 2t + 1$ to be missing from the remaining $v - 2x - 2$ vertices (which will be named $\alpha_1, \ldots, \alpha_{\eta(\alpha)}$ below). For $1 \leq i \leq 2t$ let $M(i) = \{u_j \in V_0 \mid m(j) = i\}$. Note for $1 \leq i < j \leq 2t$, $|\,|M(i)| - |M(j)|\,| \in \{0, 2\}$ and $|M(i)|$ is odd for all i.

Next form a new edge-colored graph $\mathcal{G}_0^+$ from $\mathcal{G}_0$ as follows. Add a single vertex, α. The aim now is to complete the proof by using Theorem 3.1 with $\eta(u_i) = 1$ for $1 \leq i \leq 2x + 2$ and $\eta(\alpha) = v - 2x - 2$. For $1 \leq i \leq 2x + 2$ join u_i to α with $b(v)$ edges of each color in $\{1, 2, \ldots, 2t\} \setminus \{m(i)\}$. Thus for $1 \leq i \leq 2x + 2$ the number of edges joining u_i to α is $(2t-1)(2x+1) = 4tx + 2t + 1 - (2x+1) - 1 = v - 1 - (2x+1) = \eta(\alpha)$, and $d_{\mathcal{G}_0^+}(u_i) = v - 1$.

Let $a(i)$ be the number of vertices in G_0^+ where color i appears and let $\epsilon_i = 2$ for $1 \leq i \leq x+1-t \pmod{2t}$ and $\epsilon_i = 0$ otherwise. Therefore $a(2t+1) = 2x+2$ and for $1 \leq i \leq 2t$,

$$\begin{aligned} a(i) &= 2x+3-|M(i)| \\ &= 2x+2-2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor - \epsilon_i. \end{aligned}$$

Note since $x \geq 0$ and $t \geq 1$ for $1 \leq i \leq 2t$,

$$\begin{aligned} \eta(\alpha)-(a(i)-1) &= v-2x-2-\left(2x+2-2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor - \epsilon_i - 1\right) \\ &= v-4x-3-2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor + \epsilon_i \\ &\geq 4tx+2t-2-4x-\left(\frac{2x+2-2t}{2t}\right) \\ &= 4x(t-1)+2t-1-\frac{x+1}{t} \\ &= (4x+1)(t-1)+t-\frac{x+1}{t} \geq 0. \end{aligned}$$

Thus for $1 \leq i \leq 2t$ add $(b(v)\eta(\alpha)-b(v)(a(i)-1))/2$ loops colored i to α, thus resulting in $d_{\mathcal{G}_0^+(i)}(\alpha) = b(v)\eta(\alpha)$. By the above calculations we know we will be adding a non-negative number of loops for all colors $1, \ldots, 2t$.

Let $l(\alpha)$ be the number of loops incident with α and $E(V(G_0), \alpha)$ be the set of edges from a vertex in G_0 to α. Therefore,

$$
\begin{aligned}
l(\alpha) &= \left(d_{G_0^+}(\alpha) - |E(V(G_0), \alpha)|\right)/2 \\
&= \left(\eta(\alpha)b(v)2t - (2x+2)[b(v)(2t-1)]\right)/2 \\
&= \left(\eta(\alpha)b(v)2t - (2x+2)\eta(\alpha)\right)/2 \\
&= \eta(\alpha)\left(b(v)2t - 2x - 2\right)/2 \\
&= \eta(\alpha)\left(4tx + 2t + 1 - 2x - 3\right)/2 \\
&= \eta(\alpha)(v - 2x - 2 - 1)/2 \\
&= \eta(\alpha)(\eta(\alpha) - 1)/2.
\end{aligned}
$$

Now apply Theorem 3.1 to form the detachment $\mathcal{G}$ of $\mathcal{G}_0^+$ in which α is detached into the vertices $\alpha_1, \ldots, \alpha_{\eta(\alpha)}$. For $1 \leq i \leq 2x+2$, since u_i is joined to α with $b(v)$ edges in $\mathcal{G}_0^+$, by condition (3) u_i is joined to each vertex α_j for $1 \leq j \leq \eta(\alpha)$ by exactly one edge in $\mathcal{G}$. Also, since α is incident with $\eta(\alpha)(\eta(\alpha)-1)/2$ loops in $\mathcal{G}_0^+$, by condition (4) α_i is joined to α_j by exactly one edge for $1 \leq i < j \leq \eta(\alpha)$ in $\mathcal{G}$. It follows that $\mathcal{G}$ is isomorphic to $K_{2x+2+\eta(\alpha)} = K_v$. By condition (2), for each vertex u in $\mathcal{G}$, each color which appears at u does so on $b(v)$ edges. Therefore the edge-coloring E of $\mathcal{G}$ is $(2t+1, 2t)$-equitable. Furthermore, in $\mathcal{G}$, color $2t+1$ appears at $b(v)+1 \geq 2t$ vertices and for $1 \leq i \leq 2t$, the number of vertices where color i appears is

$$
\begin{aligned}
a(i) - 1 + \eta(\alpha) &= (2x+2) - 2\left\lfloor \frac{x+1-t}{2t} \right\rfloor - \epsilon_i - 1 + v - (2x+2) \\
&= v - 1 - 2\left\lfloor \frac{x+1-t}{2t} \right\rfloor - \epsilon_i.
\end{aligned}
$$

Therefore, since $a(i)$ and $\eta(\alpha)$ are both odd integers, if $2t$ divides $(x+1-t)$, then $\epsilon_1 = 0$ and

$$\begin{aligned} a(1) - 1 + \eta(\alpha) &= v - 1 - \frac{x+1-t}{t} \\ &= v - \frac{x+1}{t} \\ &= \left\lfloor v - \frac{x+1}{t} \right\rfloor_e , \end{aligned}$$

and if $2t$ does not divide $(x+1-t)$ then $\epsilon_1 = 2$ and

$$\begin{aligned} a(1) - 1 + \eta(\alpha) &= v - 1 - \left(2\left\lfloor \frac{x+1-t}{2t} \right\rfloor + 2\right) \\ &= v - 1 - 2\left\lceil \frac{x+1-t}{2t} \right\rceil \\ &= v - 1 + 2\left\lfloor \frac{-(x+1-t)}{2t} \right\rfloor \\ &= 2\left\lfloor \frac{v-1}{2} + \frac{1}{2} - \frac{x+1}{2t} \right\rfloor \\ &= 2\left\lfloor \frac{v}{2} - \frac{x+1}{2t} \right\rfloor \\ &= \left\lfloor v - \frac{x+1}{t} \right\rfloor_e . \end{aligned}$$

Therefore by Lemma 5.1, $\overline{\psi_2'}(K_2, K_v) = \lfloor v - \frac{x+1}{t} \rfloor_e$ and the proof is complete (after renaming color $2t+1$ with 1 and renaming colors $1, 2, \ldots, 2t$ with $2, 3, \ldots, 2t+1$ respectively). $\square$

By modifying the proof of Proposition 5.1 we obtain the proof of Proposition 5.2.

Proposition 5.2. *Let $v \equiv 2t+1 \pmod{4t}$ with $v > 1$ and $b(v) + 1 \geq 2t$. Then*

1. *$\psi_1'(K_2, K_v) = b(v) + 1$ and*
2. *for $3 \leq i \leq 2t+1$, $\overline{\psi_i'}(K_2, K_v) = v - 1$.*

Proof. Let $v = 4tx + 2t + 1$ for some integer x. Form $\mathcal{G}_0$ in the same way as in Proposition 5.1. Here color $m(i) = i$ will be missing from vertex u_i for $1 \leq i \leq 2t-1$, color $m(i) = 2t$

will be missing from vertex u_i for $2t \le i \le 2x+2$, and color $m(\alpha_i) = 2t+1$ will be missing from the remaining $v - 2x - 2$ vertices (which will be named $\alpha_1, \dots, \alpha_{\eta(\alpha)}$ below).

Next form a new edge-colored graph $\mathcal{G}_0^+$ as in Proposition 5.1 and again the aim now is to complete the proof using Theorem 3.1 with $\eta(u_i) = 1$ for $1 \le i \le 2x+2$ and $\eta(\alpha) = v-2x-2$. For $1 \le i \le 2x+2$ join u_i to α with $b(v)$ edges of each color $\{1, 2, \dots, 2t\} \setminus \{m(i)\}$ as in Proposition 5.1; again the number of edges joining u_i to α is $\eta(\alpha)$, and $d_{\mathcal{G}_0^+}(u_i) = v - 1$. For $1 \le i \le 2t-1$ add $b(v)(\eta(v) - (2x+1))/2$ loops of color i to α; so α has degree $b(v)\eta(v)$ in color class i (where loops contribute 2 to the degree of the incident vertex). Also add $b(v)(\eta(v) - (2t-1))/2$ loops of color $2t$ to α; so α has degree $b(v)\eta(v)$ in color class $2t$ as well. Notice that the number of loops incident with α is

$$\begin{aligned}
l(\alpha) &= (2t-1)b(v)(\eta(\alpha) - (2x+1)/2) + b(v)(\eta(\alpha) - (2t-1))/2 \\
&= (2t(2x+1)\eta(\alpha) - (2x+1)(2t-1)(2x+2))/2 \\
&= (2x+1)(2t\eta(\alpha) - (4xt - 2x - 4t - 2))/2 \\
&= (2x+1)(2t\eta(\alpha) - (\eta(\alpha) + 2t - 1))/2 \\
&= (2x+1)(\eta(\alpha) - 1)(2t-1))/2 \\
&= \eta(\alpha)(\eta(\alpha) - 1)/2.
\end{aligned}$$

As in the proof of Proposition 5.1, Theorem 3.1 allows us to form $\mathcal{G}$ isomorphic to K_v from $\mathcal{G}_0^+$ so that the edge-coloring E of $\mathcal{G}$ is $(2t+1, 2t)$-equitable. Furthermore, in $\mathcal{G}$, color $2t+1$ appears at $b(v)+1$ vertices, color $2t$ appears at $v-2t-1$ vertices, and each other color appears at $v-1$ vertices. Since in [26] it is shown in this case that $\psi_i'(K_2, K_v) \ge b(v)+1$ and that $\overline{\psi_i'}(K_2, K_v) \le v-1$ for $1 \le i \le 2t+1$, the proof is complete (after renaming the colors $1, 2, \dots, 2t+1$ with $2t+1, 2t, \dots, 1$ respectively). $\square$

We apply a well known result stated in Lemma 5.2 at the start of the proof of Proposition 5.3.

Lemma 5.2. *There exists a simple k-regular graph on n-vertices if and only if $k \leq n-1$ and kn is even.*

Lemma 5.3 will also be useful in the proof of Proposition 5.3 and is a result of Vizing's Theorem and work presented by McDiarmid in [28]. An edge-coloring is said to be equalized if the number of edges of each color is within one of the number of edges of each other color.

Lemma 5.3. *If G is a k-regular graph, for $m > k$ there exists an equalized proper m-edge coloring of G.*

The following result is also used in the proof of Proposition 5.3 and is a well known corollary of Hall's theorem.

Lemma 5.4. *If G is a k-regular bipartite graph with $k \geq 1$ then G has a 1-factor.*

Proposition 5.3. *Let $v \equiv 2t+1 \pmod{4t}$ with $v > 1$ and $b(v)+1 \leq 2t$. Then*

1. *$\psi_1'(K_2, K_v) = 2t$ and*
2. *for $2 \leq i \leq 2t+1$, $\overline{\psi_i'}(K_2, K_v) = v-1$.*

Proof. By Lemma 3.10, $\psi_1'(K_2, K_v) \geq 2t$ and $\overline{\psi_i'}(K_2, K_v) \leq v-1$ for $1 \leq i \leq 2t+1$. Thus the result is proved by showing there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v such that $c_1 = 2t$ and $c_i = v-1$, for $2 \leq i \leq 2t+1$.

Let $v = 2t+1+4tx$ for some integer $x \geq 0$. We assume $v > 1$, so $t \geq 1$. Let $b(v)+1 \leq 2t$. Define a set of vertices $V_0 = \{v_1, \ldots, v_{2t}\}$ and $\mathcal{G}_0 = K[V_0]$. Color all the edges of a $b(v)$-regular subgraph $\mathcal{H}_1$ of $\mathcal{G}_0$ with color $2t+1$. Note by Lemma 5.2 such a subgraph exists since $b(v) \leq 2t-1$ and $(b(v))(2t)$ is even. Thus for $1 \leq j \leq 2t$, $d_{\mathcal{H}_2(2t+1)}(v_j) = b(v)$. Let $k = 2t-1-b(v) = 2t-2x-2$ (by Lemma 3.2), then $\mathcal{H}_2 = \mathcal{G}_0 \backslash E(\mathcal{H}_1)$ is a k-regular graph with tk edges. Since $\frac{k}{2} = t-x-1$ is an integer and $2t > k$, by Lemma 5.3 there exists an equalized proper $2t$-edge-coloring of $\mathcal{H}_2$ with colors $1, \ldots, 2t$; so there are $k/2$ edges of each color. Note then that for $1 \leq i \leq 2t$,

($\star$) color i appears on k vertices in $\mathcal{H}_2$

and all edges of $\mathcal{G}_0$ have been colored. Now form a bipartite graph $\mathcal{B}$ with vertex set $V = \{v_1, \ldots, v_{2t}\}$ and $C = \{c_1, \ldots, c_{2t}\}$. Join c_i to v_j if and only if color i is missing from (i.e., no edge of color i is incident with) v_j in $\mathcal{H}_2$. Note since the coloring of $\mathcal{H}_2$ was proper, $d_{\mathcal{B}}(v_j) = 2t - d_{\mathcal{H}_2}(v_j) = 2t - k$. Also, since color i appears on k vertices in $\mathcal{H}_2$ and $|\mathcal{H}_2| = 2t$, $d_{\mathcal{B}}(c_i) = 2t - k$. Therefore $\mathcal{B}$ is a regular bipartite graph and by Lemma 5.4 there exists a 1-factor $\mathcal{F}$ of $\mathcal{B}$. Without loss of generality, we assume $\mathcal{F} = \{(v_j, c_j) \mid 1 \leq j \leq 2t\}$. The role of $\mathcal{F}$ is to guarantee that for $1 \leq i \leq 2t$, color i does not appear on v_i in our final edge-coloring of K_v. Next form a new edge-colored graph $\mathcal{G}_0^+$ from $\mathcal{G}_0$ as follows. Add a single vertex, called α, joining α to v_j with $v - 2t$ edges for $1 \leq j \leq 2t$ and adding $\binom{v-2t}{2}$ loops at α. We now turn to coloring the added edges and loops. For $1 \leq i, j \leq 2t$ with $i \neq j$, color $\big(b(v) - d_{H_2(i)}(v_j)\big) \in \{b(v), b(v) - 1\}$ edges between α and v_j with color i, so now $d_{\mathcal{G}_0^+(i)}(v_j) = b(v)$. Since $\mathcal{H}_2$ is k-regular and the coloring of $\mathcal{H}_2$ was proper, for each $v_j \in V_0$, this colors each of the $v - 2t$ edges between α and v_j since,

$$
\begin{aligned}
k(b(v)-1) + (2t-k-1)b(v) &= k(2x+1-1) + (2t-k-1)(2x+1) \\
&= 2kx + 4tx - 2kx - k + (2t - 2x - 2) + 1 \\
&= 4tx + 1 \\
&= v - 2t.
\end{aligned}
$$

Somewhat coincidentally α is currently incident with $k(b(v)-1) + (2t-k-1)b(v)$ edges colored i (using ($\star$)), which was just shown to be $v - 2t$. Now color $\frac{1}{2}(v-2t)(b(v)-1)$ loops of color $1, \dots, 2t$ at α. Notice this colors all the loops incident with α since

$$\begin{aligned} 2t\left(\frac{1}{2}(v-2t)(b(v)-1)\right) &= (v-2t)(4tx)/2 \\ &= (v-2t)(v-2t-1)/2 \\ &= \binom{v-2t}{2}. \end{aligned}$$

Now we intend to apply Theorem 3.1 to achieve the construction of a $(2t+1, 2t)$-equitable edge coloring of K_v. Let $H = \mathcal{G}_0^+$, which is a $(2t+1)$-edge-colored graph, and define $\eta(\alpha) = v - 2t$ and $\eta(v_i) = 1$ for $1 \leq i \leq 2t$. Thus by Theorem 3.1, a new graph G can be formed from H by detaching α into $v - 2t$ new vertices $\alpha_1, \dots, \alpha_{v-2t}$, so $|V(G)| = v$. The conditions G satisfies stated in Theorem 3.1 are presented in turn to show that $G = K_v$ and that it has a $(2t+1, 2t)$-equitable edge-coloring:

(i) For $1 \leq j \leq 2t$, since $\eta(v_j) = 1$, $d_G(v_j) \approx d_{\mathcal{G}_0^+}(v_j) = (2t-1) + (v-2t) = v-1$ as it should be in K_v. For $1 \leq k \leq v-2t$, the same is true for each α_k since by the construction of $\mathcal{G}_0^+$ and $\eta(\alpha) = v - 2t$, thus

$$\begin{aligned} d_G(\alpha_k) &\approx d_{\mathcal{G}_0^+}(\alpha)/(v-2t) \\ &= \frac{2\left(2t\left(\frac{1}{2}(v-2t)(b(v)-1)\right)\right) + 2t(v-2t)}{v-2t} \\ &= 2t(b(v)-1) + 2t \\ &= 4tx + 2t \\ &= v-1. \end{aligned}$$

(ii) For $1 \leq i \leq 2t+1$ and $1 \leq j \leq 2t$ with $i \neq j$, $d_{\mathcal{G}_0^+(i)}(v_j) = b(v)$, $d_{\mathcal{G}_0^+(j)}(v_j) = 0$, and $\eta(v_j) = 1$, so this condition simply states that $d_{G(i)}(v_j) = b(v)$ and $d_{G(j)}(v_j) = 0$ as well. Note $d_{\mathcal{G}_0^+(2t+1)}(\alpha) = 0$, so $d_{G(2t+1)}(\alpha_k) = 0$ for all $1 \leq k \leq v-2t$. Finally, for

$1 \leq i \leq 2t$ and $1 \leq k \leq v-2t$,

$$\begin{aligned} d_{G(i)}(\alpha_k) &\approx d_{\mathcal{G}_0^+(2t+1)}(\alpha)/\eta(\alpha) \\ &= \frac{(v-2t) + \left(2\left(\frac{1}{2}(v-2t)(b(v)-1)\right)\right)}{v-2t} \\ &= b(v) \end{aligned}$$

as required by Lemma 3.2.

(iii) For $1 \leq j \leq 2t$, $\eta(v_j) = 1 < 2$, so for all $v_j \in V_0$ this condition does not apply. Note $\eta(\alpha) = \binom{v-2t}{2} \geq 2$, so this condition tells us for $1 \leq i < i' \leq v-2t$ there is exactly one edge between each α_i and $\alpha_{i'}$ in G since,

$$\begin{aligned} m_G(\alpha_i, \alpha_{i'}) &\approx l_{\mathcal{G}_0^+}(\alpha)/\binom{\eta(\alpha)}{2} \\ &= \binom{v-2t}{2}/\binom{v-2t}{2} \\ &= 1. \end{aligned}$$

(iv) For $1 \leq i, i' \leq 2t$, $m_G(v_i, v_{i'}) \approx m_{\mathcal{G}_0^+}(v_i, v_{i'})/(\eta(v_i)\eta(v_{i'}) = 1$ so this condition ensures there is exactly one edge between any v_i and $v_{i'}$ in V_0. For $1 \leq i \leq 2t$ and $1 \leq i' \leq v-2t$, there is exactly one edge between each v_i and $\alpha_{i'}$ in G since,

$$\begin{aligned} m_{K_v}(v_i, \alpha_{i'}) &\approx m_{\mathcal{G}_0^+}(v_i, \alpha_i)/\left(\eta(v_i)\eta(\alpha_{i'})\right) \\ &= (v-2t)/(1(v-2t)) \\ &= 1. \end{aligned}$$

Therefore $G = K_v$ since by (iii) and (iv) G is simple and by (i) each of the v vertices has degree $v-1$. By (ii) for $1 \leq i \leq 2t+1$ and $1 \leq j \leq 2t$ with $i \neq j$, $d_{G(i)}(v_j) = b(v)$ and $d_{G(j)}(v_j) = 0$ and for $1 \leq i \leq 2t$ and $1 \leq k \leq v-2t$, $d_{G(i)}(\alpha_k) = b(v)$ and $d_{G(2t+1)}(\alpha_k) = 0$, so we have achieved a $(2t+1, 2t)$-equitable edge-coloring of K_v. Note color $2t+1$ only appears at each vertex in V_0 and for $2 \leq i \leq 2t+1$, color i appears at all but one vertex. It follows that, after renaming the colors $1, 2, \ldots, 2t+1$ with $2t+1, 2t, \ldots, 1$ respectively, $c_1 = 2t \geq b(v)+1$ by assumption and $c_i = v-1$ for $2 \leq i \leq 2t+1$. □

Theorem 5.1. *Let $v \equiv 2t+1 \pmod{4t}$ with $v > 1$. Then,*

1. $\psi_1'(K_2, K_v) = \max\{b(v)+1, 2t\}$,
2. *if* $b(v)+1 \leq 2t$, *then* $\overline{\psi_2'}(K_2, K_v) = v-1$,
3. *if* $b(v)+1 \geq 2t$, *then* $\overline{\psi_2'}(K_2, K_v) = \left\lfloor v - \frac{x+1}{t} \right\rfloor_e$, *and*
4. *for* $3 \leq i \leq 2t+1$, $\overline{\psi_i'}(K_2, K_v) = v-1$.

Proof. Let $v \equiv 2t+1 \pmod{4t}$ with $v > 1$. Property 1 holds by Propositions 5.2 and 5.3, property 2 holds by Proposition 5.3, property 3 holds by Proposition 5.1, and property 4 holds by Propositions 5.2 and 5.3. □

Corollary 5.1. *Let $v' \equiv 4t+2 \pmod{8t}$ with $v > 2$. Then*

(1) $\psi_1'(C_4, K_{v'} - F) = \max\{2(b'(v')+1), 4t\}$,

(3) if $b'(v')+1 \leq 2t$, *then* $\overline{\psi_2'}(C_4, K_{v'} - F) = v'-2$,

(3) if $b'(v')+1 \geq 2t$, *then* $\overline{\psi_2'}(C_4, K_{v'} - F) = \left\lfloor v' - \frac{2x+2}{t} \right\rfloor_e$, *and*

(4) for $3 \leq i \leq 2t+1$, $\overline{\psi_i'}(C_4, K_{v'} - F) = v'-2$.

Proof. Let $v = v'/2$. By Theorem 5.1, there exists a $(2t+1, 2t)$-equitable edge-coloring E of $G = K_v$ with $c_1(E) = \max\{b(v)+1, 2t\}$, $c_i(E) = v-1$ for $3 \leq i \leq 2t+1$, and $c_2(E) = v-1$ when $b(v)+1 \leq 2t$. There also exists a $(2t+1, 2t)$-equitable edge-coloring H of $G = K_v$ when $b(v)+1 \geq 2t$ with $|c_1(H) - c_{2t+1}(H)| \in \{0, 2\}$. Thus by Lemma 3.1, there exists a $(2t+1, 2t)$-equitable C_4-coloring E' of $G \times 2 = K_{v'} - F$ with $c_1(E') = 2\max\{b(v)+1, 2t\} = \max\{2(b'(v')+1), 4t\}$ by Lemma 3.2. The coloring E' also has the property that $c_i(E') = 2(v-1) = v'-2$ for $3 \leq i \leq 2t+1$ and $c_2(E') = 2(v-1) = v'-2$ when $b'(v') + 1 \leq 2t$ since $b(v) = b'(v')$. Also by Lemma 3.1, there exists a $(2t+1, 2t)$-equitable C_4-coloring H' of $G \times 2 = K_{v'} - F$ with $|c_1(H') - c_{2t+1}(H')| = 2|c_1(H) - c_{2t+1}(H)| \in \{0, 4\}$. Thus by Lemma 3.4 and 3.11 it follows that $\psi'_1(C_4, K_{v'} - F) = \max\{2(b'(v')+1), 4t\}$. Note as well, by Lemma 3.3, $\overline{\psi'_i}(C_4, K_{v'} - F) = v'-2$ for $3 \leq i \leq 2t+1$ and $\overline{\psi'_2}(C_4, K_{v'} - F) = v'-2$ if $b'(v') + 1 \leq 2t$. Finally, by Lemma 3.9, $\overline{\psi'_2}(C_4, K_{v'} - F) = \lfloor v' - \frac{2x+2}{t} \rfloor_e$ if $b'(v') + 1 \geq 2t$. $\square$

Thus in regards to equitable edge-colorings of K_v and equitable C_4-colorings of $K_{v'} - F$, we have established the other two extreme values for each; the smallest values that the smallest element of the color vector can attain and the largest value that the largest value of the color vector can attain. We have also in fact settled the largest any element of the color vector can be.

Chapter 6

Alternate Proofs via Direct Construction

Below is an alternate proof of Propositions 5.2 and 5.3 via a direct construction, displaying the diversity of proof techniques in the study of graph decompositions.

Proof. Assume $b(v)+1 \geq 2t$. Partition the vertices into $2t$ sets labeled $G_1, \ldots, G_{2t}$ such that for $1 \leq j \leq 2t-1$, $|G_j| = b(v)$ and $|G_{2t}| = b(v)+1$. Note by Lemma 3.2, the number of vertices in these sets is well-defined since,

$$\begin{aligned}(2t-1)b(v)+b(v)+1 &= (2t-1)(2x+1)+2x+2 \\ &= 4tx+2t+1 \\ &= v.\end{aligned}$$

Let $V(G_{2t}) = \{u_1, \ldots, u_{2t}, v_{2t+1} \ldots, v_{b(v)+1}\}$. Color all edges in $K[G_{2t}]$ with color $2t+1$. For $1 \leq i \leq 2t$, $1 \leq j \leq 2t-1$, and $k \equiv i+j \pmod{2t}$, color all edges between G_j and u_k with color i. Note for $1 \leq i,k \leq 2t$ such that $i \neq k$, color i appears at u_k and color k does not; so there are exactly $2t$ colors appearing at u_k. Furthermore, since $|G_j| = b(v)$ for $1 \leq j \leq 2t-1$, each color appearing at u_k does so on $b(v)$ edges. Now for $2t+1 \leq l \leq b(v)$ and $1 \leq i \leq 2t-1$, color the edges joining v_l to the vertices in G_i with color i; so there are $b(v)$ edges of color i incident with v_l and there are exactly $2t$ colors appearing at v_l. This completes the coloring of all edges incident with vertices in G_{2t}. Note, for $1 \leq i \leq 2t$ and $1 \leq j \leq 2t-1$ with $i \neq j$, each $u \in G_j$ is incident with one edge colored i and $1+(b(v)+1-2t)$ edges colored j.

To color the remaining edges, first for $1 \leq j \leq 2t-1$ color all edges in $K[G_j]$ with color $2t$. Second, let $\{F_1, \ldots, F_{2t-1}\}$ be a near 1-factorization of K_{2t-1} on the vertex set $\{i \mid 1 \leq i \leq 2t-1\}$ such that $i \notin V(F_i)$. For $1 \leq j < k \leq 2t-1$ color a $(b(v)+1-2t)$-factor of the bipartite graph formed by the edges between G_j and G_k with i if and only if $\{j,k\}$ is in F_i; this is possible by Lemma 5.4. Note all vertices in $G_1, \ldots, G_{2t-1}$ are now incident with $b(v)+2-2t$ edges of color i for $1 \leq i \leq 2t-1$, so each still needs to be incident with $2t-2$ more edges colored i. Finally, note the remaining uncolored edges form a $(2t-2)(2t-1)$-regular subgraph on the vertex set $\cup_{i=1}^{2t-1} G_i$, thus it has a 2-factorization by Petersen's Theorem (a well known corollary of Lemma 5.4); so it clearly has a $(2t-2)$-factorization. Hence, for $1 \leq i \leq 2t-1$, color all the edges in the i^{th} $(2t-2)$-factor with color i.

Thus a $(2t+1, 2t)$-equitable edge-coloring of K_v has been formed. Note that color $2t+1$ is only used on edges in $K[G_{2t}]$, color $2t$ appears at all vertices except those in $\{u_{2t}, v_{2t+1}, \ldots, v_{b(v)+1}\}$, and for $3 \leq i \leq 2t-1$ color i only misses u_i. Therefore, after renaming the colors $1, 2, \ldots, 2t+1$ with $2t+1, 2t, \ldots, 1$ respectively, $c_1 = 2t$, $c_2 = v-(b(v)+1-(2t-1))$, and $c_i = v-1$ for $3 \leq i \leq 2t+1$.

Therefore, in each case, a $(2t+1, 2t)$-equitable edge-coloring has been produced in which $c_1 = \max\{b(v)+1, 2t\}$, $c_i = v-1$ for $3 \leq i \leq 2t+1$ and $c_2 = v-1$ when $b(v) = 1 \leq 2t$. So, by Lemma 3.3 and 3.10 it follows that $\psi'_1(K_2, K_v) = \max\{b(v)+1, 2t\}$, and since v is odd, by Lemma 3.3 it follows that $\overline{\psi'_i}(K_2, K_v) = v-1$ for $3 \leq i \leq 2t+1$ and $\overline{\psi'_2}(K_2, K_v) = v-1$ when $b(v)+1 \leq 2t$. □

If only $\overline{\psi'_{2t+1}}(K_2, K_{v'/2})$ or $\overline{\psi'_{2t+1}}(C_4, K_{v'} - F)$ are of interest, they can be obtained much more easily via the simple direct construction shown in the proof of Theorem 6.1 where except for the smallest value of v and v', only one color class achieves the maximum value of $v-1$ and $v'-2$ respectively.

Theorem 6.1. *Let* $v \equiv 2t+1 \pmod{4t}$. *Then*

(1) $\overline{\psi'_{2t+1}}(K_2, K_v) = v-1$ *and*

(2) $\overline{\psi'_{2t+1}}(C_4, K_{2v} - F) = 2v-2$.

Proof. Let $v = 4tx + 2t + 1$ for some integer $x \geq 0$. By Lemma 3.3 $\overline{\psi'_{2t+1}}(K_2, K_v) \leq v-1$ so it simply needs to be shown that there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v

such that $c_{2t+1} = v - 1$. Partition the vertices with one vertex labeled P_1 and 2t groups $P_2, \ldots, P_{2t+1}$, each with $1+2x$ vertices. Let $\{F_1, \ldots, F_{2t+1}\}$ be a near 1-factorization of K_{2t+1} on the vertex set $\{1, 2, \ldots, 2t+1\}$. For $1 \leq i < j \leq 2t+1$ color the edges joining vertices in P_i to vertices in P_j with k if and only if $\{i, j\}$ is in F_k. Note for $1 \leq i, k \leq 2t+1$ such that $i \neq k$, color k appears at all vertices of group P_i. Now for $3 \leq i \leq 2t+1$, when group P_i is joined with color k to group P_1, color $K[P_i]$ with color k; there are $b(v) = 2x+1$ edges of color $k \neq i$ at each vertex in P_i. Note as well that $|P_k| = 2x+1$, so by construction we have $b(v)$ edges of color k incident with P_1. Therefore this creates a $(2t+1, 2t)$-equitable edge-coloring of K_v. Note as well by construction that color 1 appears at every vertex except P_1, so $c_{2t+1} = v - 1$. So by Lemma 3.3 $\overline{\psi'_{2t+1}}(K_2, K_v) = v - 1$. With $G = K_v$, by Lemma 3.1, there exists a $(2t+1, 2t)$-equitable C_4-coloring of $G \times 2 = K_{v'} - F$ with the property that $c_{2t+1} = 2(v-1) = v' - 2$. So by Lemma 3.4, it may be concluded that $\overline{\psi'_{2t+1}}(C_4, K_{2v} - F) = 2v - 2$. □

Chapter 7

Further Areas of Interest in Block Colorings of Graph Decompositions

Note by Theorem 2.2 if $v \equiv 0 \pmod 4$, there exists an (s,s)-equitable C_4-coloring of $K_v - F$ if and only if $v - 2 \geq 2s$. By Theorem 4.1 we know if $v \equiv 4t+2 \pmod{8t}$, there exists a $(2t+1, 2t)$-equitable C_4-coloring of $K_v - F$. Therefore,

- if $v \equiv 0 \pmod 4$ and $v - 2 \geq 2s$, then $\chi'_p(v) = p$,
- if $v \equiv 0 \pmod 4$ and $v - 2 < 2s$, then $\chi'_p(v) > p$, and
- if $v \equiv 4t+2 \pmod{8t}$, then $\chi'_{2t}(v) = 2t+1$.

These results leave some natural questions to be addressed:

(1) If $v \equiv 0 \pmod 4$ and $v - 2 < 2s$, what is $\chi'_p(v)$?

  - We already know $\chi'_p(v) > p$ by Theorem 2.2.

(2) If $v \equiv 2 \pmod 4$ and $v \not\equiv 4t+2 \pmod{8t}$, what is $\chi'_p(v)$?

  - Naturally, $\chi'_p(v) \geq p$, so we have our starting point and
  - for $p \in \{2,3,4\}$, in this case we know from Theorem 2.2, $\chi'_p(v) = p$ if and only if $v \geq 6$, $v \geq 8$, and $v \geq 10$ respectively.

(3) If $v \equiv 4t+2 \pmod{8t}$, for p odd, what is $\chi'_p(v)$?

  - Again naturally, $\chi'_p(v) \geq p$, so we have our starting point and
  - we know from Theorem 2.2, $\chi'_3(v) = 3$ if and only if $v \geq 8$.

Note for there to be a C_4-decomposition of $K_v - F$, it must be that $v \equiv 0 \pmod 2$ since the degree of each vertex must be even. So once these three questions are complete, we will know the value of $\chi'_s(v)$ for any v in regards to C_4-decompositions of $K_v - F$, that is the smallest value of s for a fixed value of p for which there exists an (s,p)-equitable block-coloring of some C_4-decomposition of $K_v - F$.

From Corollaries 4.3 and 5.1, for $v' \equiv 4t + 2 \pmod{8t}$ and $v' > 2$ we know:

- $\psi'_1(C_4, K_{v'} - F) = \max\{2(b'(v') + 1), 4t\}$,
- $\psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$,
- $\overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$,
- if $b'(v') + 1 \leq 2t$, then $\overline{\psi'_2}(C_4, K_{v'} - F) = v' - 2$,
- if $b'(v') + 1 \geq 2t$, then $\overline{\psi'_2}(C_4, K_{v'} - F) = \left\lfloor v' - \frac{2x+2}{t} \right\rfloor_e$, and
- for $3 \leq i \leq 2t + 1$, $\overline{\psi'_i}(C_4, K_{v'} - F) = v' - 2$.

These results introduce one immediate question to be first addressed.

(4) For $v \equiv 4t + 2 \pmod{8t}$, what is $\psi'_i(C_4, K_{v'} - F)$ for $2 \leq i \leq 2t$?

We have begun to work on answering question (4), but the work is proving more difficult than expected. There are many more restriction on the smallest the intermediate values of the color vector can be as compared to c_1 and c_{2t+1}.

Note in our work we have restricted ourselves to considering $\psi'_i(C_4, K_{v'} - F)$ and $\overline{\psi'_i}(C_4, K_{v'} - F)$ in the case where $s = \chi'_p(v)$, that is using as few colors as possible for a particular value of p. Thus for $v \equiv 0 \pmod 4$ and $v - 2 \geq 2s$, we are considering the size of the color classes for (p,p)-equitable block-colorings. Hence we are using p colors and want p colors appearing at each vertex. Necessarily then, every color class will contain every vertex. Therefore for $v \equiv 0 \pmod 4$, $v - 2 \geq 2s$, and $1 \leq i \leq s$,

$$\psi'_i(C_4, K_{v'} - F) = \overline{\psi'_i}(C_4, K_{v'} - F) = v. \tag{7.1}$$

By answering questions (1) and (3), if $\chi'_p(v) = p$, then we have $\overline{\psi'_i}(C_4, K_{v'} - F) = \psi'_i(C_4, K_{v'} - F) = v$ as well. The more interesting question then stems from question (2) where we already know $\chi'_p(v) > p$. Thus the following questions remain:

(5) If $v \equiv 0 \pmod 4$ and $v - 2 < 2s$, what are $\overline{\psi'_i}(C_4, K_{v'} - F)$ and $\psi'_i(C_4, K_{v'} - F)$ for $1 \leq i \leq s$?

- We already know $\chi'_p(v) > p$, so this will be an interesting question to resolve.

(6) If $v \equiv 2 \pmod 4$ and $v \not\equiv 4t+2 \pmod{8t}$, what are $\overline{\psi'_i}(C_4, K_{v'} - F)$ and $\psi'_i(C_4, K_{v'} - F)$ for $1 \leq i \leq s$?

- If in settling question (2), we find $\chi'_p(v) = p$, this question is automatically settled and equation (7.1) applies here.

(7) If $v \equiv 4t + 2 \pmod{8t}$, for p odd, what are $\overline{\psi'_i}(C_4, K_{v'} - F)$ and $\psi'_i(C_4, K_{v'} - F)$ for $1 \leq i \leq s$?

- Again, if in settling question (3), we find $\chi'_p(v) = p$, this question is automatically settled and equation (7.1) applies here as well.

In answering these questions, we'll again be addressing the analogous questions for edge-colorings of K_v. Having just introduced the idea of the color vector in our work, the final question to be considered would be, if $s > \chi'_p(v)$,

(8) what are $\overline{\psi'_i}(C_4, K_{v'} - F)$ and $\psi'_i(C_4, K_{v'} - F)$ for $1 \leq i \leq s$?

Thus the field of mathematics is ever changing and growing, leading us to continually ask more questions and expand our base of knowledge.

Chapter 8

Tiling Generalized Petersen Graphs

8.1 Introduction

The *Generalized Petersen Graph* $P(n, k)$ is defined for all $n \geq 3$ with $1 \leq k \leq (n-1)/2$ to be the graph on the vertex set $\mathbb{Z}_n$ x $\mathbb{Z}_2$ with,

$$E(P(n,k)) = \{\{(i,0),(i+1,0)\},\{(i,0),(i,1)\},\{(i,1),(i+k,1)\} \mid i \in \mathbb{Z}_n\}$$

reducing the sums modulo n. A *path* on j vertices of length $j-1$ is denoted by P_j with, $V = \{v_0, v_1, v_2, ..., v_{j-1}\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in \mathbb{Z}_{j-1}\}$. We define an H-*tiling* of a graph G to be a partition Π of the vertices of G into sets such that for all $\pi \in \Pi$, $G[\pi] = H$. In particular, we settle the existence of tilings of $P(n, k)$ with P_j for $j \in \{2, 4\}$ (see Theorem 8.1). We also propose necessary and sufficient conditions for the existence of a P_6-tiling of $P(n, k)$ in Conjecture 8.1 with a nearly complete proof and detailed plans for our future work with forming a P_8-tiling of $P(n, k)$.

The inspiration for our work with tiling generalized Petersen graphs with paths of various lengths came from a talk given by Jerrold Griggs and Kevin Milans on the Tilings of Hypercubes. They explored necessary and sufficient conditions with n sufficiently large for a graph G to tile the n-dimensional hypercube Q_n.

8.2 History

The study of tilings in mathematics has become a wide range of research with many variations in the questions explored. One area of interest is with tiling $m \times n$ checkerboards with *polyominoes*, which are connected figures formed from congruent squares placed side by side so that each square shares at least one side with another square. The most commonly known is the domino 2×1 tile, with further information in [13]. We can connect checkerboard tiling to graph theory by considering each square to be a vertex and having two vertices adjacent if their respective squares are adjacent. The shortcoming in this connection is when we consider tiling with 3 squares in an L-shape, also called a tromino (see [1] and [7]), which is viewed as different from tiling with 3 tiles stacked side by side, referred to as a 3×1 tile. In graph theory both would be considered a P_3–tiling. T–tetrominoes, which are formed by 4 congruent squares arranged in the shape of a T, can be thought of as stars of degree 3 (see [29]). In all of these tilings though, the partitions of vertices do not form an induced subgraph as is the case in our work.

Work has also been done in tiling nonorientable surfaces with Steiner Triple Systems (see [30]) where the triples $\{i, j, k\}$ of disjoint Steiner Triple Systems are thought of as black and white triangles with vertices i, j, and k joined together by their sides to tile some closed surface. In addition, others have looked at the minimum degree threshold for a bipartite graph to tile another bipartite graph (see [6]). Again, the difference in this specific work, versus our own, is that their partitioning of the vertices to form a particular subgraph does not induce that graph, versus our graphs are induced.

8.3 Results

We introduce some notation to ease our work. We refer to the set of vertices $V_0 = \{(i,0) \mid i \in \mathbb{Z}_n\}$ and their induced edges as the *outer vertices* and *outer cycle*, the set of vertices $V_1 = \{(i,1) \mid i \in \mathbb{Z}_n\}$ and their induced edges as the *inner vertices* and *inner graph*, denoted by $I(n,k)$, and the edges between vertices in V_0 and vertices in V_1 as the *spoke edges*. Let $\mathcal{D}$ be a set of induced vertex disjoint paths with span $V(G)$. A tiling naturally follows from such a partition as $\Pi = \{\{V(D)\} \mid D \in \mathcal{D}\}$. We say that D induces the tiling Π. Throughout the following proof it will also be useful to denote $\{s + i | s \in S\}$ by simply $S + i$.

Theorem 8.1. *For $j \in \{2,4\}$ and integers n and k with $n \geq 3$ and $1 \leq k \leq (n-1)/2$, there exists a P_j-tiling of $P(n,k)$ if and only if j divides $2n$.*

Proof. Let n and k be integers with $n \geq 3$ and $1 \leq k \leq (n-1)/2$. Since each of the $2n$ vertices in $P(n,k)$ must appear in exactly one path in any P_j-tiling, Π, of $P(n,k)$, $|\Pi| = 2n/j$, so j must divide $2n$ and the condition is necessary. To prove sufficiency we consider each value of $j \in \{2,4\}$ in turn.

Let $j = 2$. A P_2-tiling of $P(n,k)$ is simply formed by the spoke edges with,

$$\Pi_1 = \{\{(i,0),(i,1)\} \mid i \in \mathbb{Z}_n\}.$$

Let $j = 4$. Since we are assuming $n \geq 3$ and $j|2n$, we have $n \geq 4$. We consider two cases in turn.

Let $k = 1$ and

$$\Pi_2 = \{\pi_i = [(2i,0),(2i+1,0),(2i+1,1),(2i+2,1)] \mid i \in Z_{n/2}\}.$$

Note that $2i$ and $2i+1$ are different modulo 2 and cover all values $1 \leq j \leq n$, so each vertex on the outer cycle is in exactly one $\pi_k \in \Pi_2$. Similarly $2i+1$ and $2i+2$ are different modulo 2 and cover all values $1 \leq j \leq n$, so each vertex in the inner graph is in exactly one $\pi_i \in \Pi_2$ as well. Since $n \geq 4$, by the definition of $E(P(n,1))$ and π_i, clearly $G[\pi_i] = P_4$ for all $i \in \mathbb{Z}_{n/2}$.

Let $k > 1$ and

$$\Pi_3 = \{\pi_i = [(2i,1),(2i,0),(2i+1,0),(2i+1,1)] \mid i \in \mathbb{Z}_{n/2}\}.$$

Note that $2i$ and $2i+1$ are different modulo 2 and cover all values $1 \leq j \leq n$, so each vertex of $P(n,k)$ is in exactly one $\pi_i \in \Pi_3$. Since $k > 1$ and $n \geq 4$, by the definition of $E(P(n,k))$ and π_i, $G[\pi_i] = P_4$ for all $i \in \mathbb{Z}_{n/2}$.

□

Conjecture 8.1. *For integers n and k with $n \geq 3$ and $1 \leq k \leq (n-1)/2$, there exists a P_6-tiling of $P(n,k)$ if and only if 6 divides $2n$ and $n \geq 6$.*

The following proves the necessity of the conjecture, and lays out a plan for proving the sufficiency, settling nearly all cases in the process. Subcases that are not completely proved are indicated with a * (in fact there are only six subcases that remain to be proved).

Let n and k be integers with $n \geq 3$ and $1 \leq k \leq (n-1)/2$. Since each of the $2n$ vertices in $P(n,k)$ must appear in exactly one path in any P_6-tiling Π of $P(n,k)$, $|\Pi| = 2n/6$, so 6 must divide $2n$. If $n = 3$, then since $1 \leq k < n/2$, we have $k = 1$. Note though that $P(3,1)$ contains no induced subgraph isomorphic to P_6 as $|V(P(3,1)| = 6$; therefore $n \geq 6$. Thus the two conditions are necessary. To prove sufficiency, we consider six cases in turn. Note since $n \geq 6$ and $6|2n$, we have that $n \equiv 0 \pmod 3$.

Case 1: Suppose that $k = 1$ and let

$$\Pi_4 = \{\pi_i = [(3i,0),(3i+1,0),(3i+2,0),(3i+2,1),(3i+3,1),(3i+4,1)] \mid i \in Z_{n/3}\}.$$

Note that $3i$, $3i+1$, and $3i+2$ are all different modulo 3 and consist of every value $1 \leq j \leq n$, so each vertex on the outer cycle is in exactly one $\pi_i \in \Pi_4$. Also, $3i+2$, $3i+3$, and $3i+4$ are all different modulo 3 and consist of every value $1 \leq j \leq n$, so each vertex on the inner graph is in exactly one $\pi_i \in \Pi_4$. Since $n \geq 6$, by the definition of $E(P(n,k))$ and π_i, $G[\pi_i] = P_6$ for all $i \in \mathbb{Z}_{n/2}$. Therefore, Π_4 is a P_6-tiling of $P(n,k)$.

Case 2: Suppose that $k \equiv 1 \pmod 3$ and $k \neq 1$, then let

$$\Pi_5 = \{\pi_i = [(3i,1),(3i,0),(3i+1,0),(3i+1,1),(3i+1+k,1),(3i+1+k,0)] \mid i \in Z_{n/3}\}.$$

Since $k \neq 1$ and $n \geq 6$, $G[\pi_i] \cong P_6$ for each $\pi_i \in \Pi_5$. Note as $k \equiv 1 \pmod 3$, $3i$, $3i+1$, and $3i+1+k$ are all different modulo 3 and consist of every value $1 \leq j \leq n$.

Therefore each vertex of $P(n,k)$ is contained in exactly one elements of Π_5. Hence, Π_5 is a P_6-tiling of $P(n,k)$.

Case 3: Suppose that $k = 2$. We then consider three subcases in turn.

Case 3.1: Suppose that $n = 6$, then

$$D_1 = \{[(0,1),(0,0),(1,0),(1,1),(3,1),(3,0)],[(2,0),(2,1),(4,1),(4,0),(5,0),(5,1)]\}$$

clearly induces a P_6-tiling Π_6 of $P(6,2)$.

Case 3.2: Suppose that $n = 9$, then

$$D_2 = \{[(0,0),(1,0),(2,0),(3,0),(4,0),(5,0)],[(6,0),(7,0),(7,1),(5,1),(3,1),(1,1)],$$

$$[(8,0),(8,1),(6,1),(4,1),(2,1),(0,1)]\},$$

clearly induces a P_6 tiling Π_7 of $P(9,2)$.

*Case 3.3: Suppose that $n \geq 12$ and recall that $n \equiv 0 \pmod 3$, so we let $n = 3x$.

If x is even, then we propose that

$$\Pi_8 = \bigcup_{i \in \mathbb{Z}_{x/2}} (\Pi_6 + (6i, 0))$$

is a P_6-tiling of $P(n,2)$, but this remains to be sufficiently proved.

If x is odd, we propose that

$$\Pi_9 = \Pi_7 \cup \left(\bigcup_{i=1}^{(x-3)/2} (\Pi_6 + (6i+3, 0)) \right)$$

is a P_6-tiling of $P(n,2)$ which also remains to be proved.

Case 4: Suppose that $k \equiv 2 \pmod 3$ and $k \neq 2$, so we let

$$\Pi_{10} = \{\pi_i = [(3i+1,1),(3i+1,0),(3i,0),(3i,1),(3i+k,1),(3i+k,0)] \mid i \in Z_{n/3}\}.$$

Since $k \neq 2$, for each integer $1 \leq i \leq n/3$, π_i induces a P_6. As $k \equiv 2 \pmod 3$, $3i$, $3i+1$, and $3i+k$ are all different modulo 3 and consist of every value $1 \leq j \leq n$, so each vertex of $P(n,k)$ is contained in exactly one copy of P_6 induced by the elements of Π_{10}. Therefore, Π_{10} is a P_6-tiling of $P(n,k)$.

Case 5: Suppose that $k = 3$. We then again consider three subcases in turn.

Case 5.1: Suppose that $n = 6$, then

$$D_1 = \{[(1,1),(1,0),(0,0),(0,1),(3,1),(3,0)],[(2,0),(2,1),(5,1),(5,0),(4,0),(4,1)]\}$$

clearly induces a P_6-tiling Π_{11} of $P(6,3)$.

Case 5.2: Suppose that $n = 9$, then

$$D_2 = \{[(1,1),(1,0),(0,0),(0,1),(3,1),(3,0)],[(2,0),(2,1),(5,1),(5,0),(6,0),(6,1)],$$

$$[(4,0),(4,1),(7,1),(7,0),(8,0),(8,1)]\}$$

clearly induces a P_6-tiling Π_{12} of $P(9,3)$.

*Case 5.3: Suppose that $n \geq 9$. We again then let $n = 3x$.

If x is even, we propose that

$$\Pi_{13} = \bigcup_{i=1}^{(x/2)-1} (\Pi_{12} + (6i,0))$$

is a P_6-tiling of $P(n,3)$, which remains to be proved.

If x is odd, we propose that

$$\Pi_{14} = \Pi_{12} \cup (\bigcup_{i=1}^{(x-3)/2} (\Pi_{11} + (6i+3, 0)))$$

is a P_6-tiling of $P(n, 3)$, which remains to be proved.

Case 6: Suppose that $k \equiv 0 \pmod 3$ and $k \neq 3$. We focus on the cycles formed in the inner graph. Note by a simple number theory argument, the inner graph will have $\gcd(n, k)$ disjoint cycles, each of length $l(I(n, k)) = n/\gcd(n, k)$. Therefore we consider the following subcases to complete this result.

*Case 6.1 Suppose that $l(I(n, k)) \equiv 0 \pmod 2$.

*Case 6.2 Suppose that $l(I(n, k)) = 3$.

*Case 6.3 Suppose that $l(I(n, k)) \equiv 1 \pmod 2$, $l(I(n, k)) \neq 3$ and $\gcd(n, k) = 3$.

*Case 6.4 Suppose that $l(I(n, k)) \equiv 1 \pmod 2$, $l(I(n, k)) \neq 3$ and $\gcd(n, k) > 3$.

We have tiling schemes formed for Case 6, but simply lack the means to describe them using the current methods and require new visualization techniques. Therefore the proof of six subcases (Cases 3.3, 5.3, and 6.1–6.4) remain to complete the proof of this result. As tiling $P(n, k)$ with P_6 is nearly complete, we have also begun work on tiling $P(n, k)$ with P_8, breaking the proof down in a similar way, based on the modular value of k. We have thus resolved tiling the generalized Petersen graph with paths on j vertices for $j \in \{2, 4\}$ and will soon complete the result for $j \in \{6, 8\}$ as well.

Chapter 9

Final Comments

As can be seen from Chapters 3 through 6, extensive work has been completed in regards to equitable block-colorings of C_4-decompositions of $K_v - F$ and edge-coloring of K_v. A means of studying the structure within such colorings has been thoroughly developed, resolving many extremes in various regards. Further areas of interest which will next be explored in this field have also been presented in Chapter 7. As can be seen from Chapter 8, work with tiling $P(n, k)$ with P_j has just begun, with significant results already found and plans for future work.

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