Essays on Economic Growth, Population Growth, and Patent Policy

by

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Abstract

Economic growth is defined as an increase in the production of goods and services over a specific period of time and is calculated as the percentage change of real gross domestic product (GDP) from one year to another. Sustained economic growth rate helps a country to raise living standards. Economies with poor growth usually suffer from issues like higher rates of poverty, lower life expectancy, and higher infant mortality rates. There are three factors that influence economic growth - physical capital growth, human capital growth, and technological progress. In the early growth theories, the source of technological change was not explained and instead assumed it as a result of chance.

The modern growth theory overcomes this shortcoming. They believe that the technological advances were not by chance but driven by the firms in hope of profit. Technological advancement leads to accumulation of knowledge capital. As knowledge is nonrival and nonexcludable it is subject to increasing returns. Innovation leads to the introduction of - new and better - techniques and products. Sustained innovative activity increases consumption and production that in turn leads to a higher standard of living and economic growth. The two dimensions of innovation considered in the literature were horizontal (variety expansion) and vertical (quality improvements).

The first generation models show a positive relationship between economic growth and population size. This implies that higher population size increases the number of researchers which in turn leads to an increase in the growth rate. But this result was not supported empirically. To eliminate this prediction the second and third generation models emerge. These papers show a positive relationship between economic growth and population growth ("weak-scale effect"). However, even this relationship was not supported empirically.

The recent empirical literature established a non-monotonic relationship between economic growth and population growth. Recent theoretical papers proposed several modifications to align the theory with the empirical finding. A common element of modification is the

introduction of human capital as a productive input in the R&D sector. This induces substitution between the quantity and quality of workers, which increases effective labor supply and enhances economic growth.

The first two chapters of this dissertation extend the latter line of research in two different ways. In the first chapter, we modify the canonical third generation R&D based model to incorporate non-linear - human capital spillover and dynastic altruism - subject to congestion in fertility rate. For strong spillover and congestion effect, economic growth first increases with population growth and then slows down.

The second chapter emphasizes the role of the assumed demographic structure - that incorporates life-cycle and bequest saving motives - is considered. The sign of the weak scale effect is positive only if parental bequest saving motive is strong.

The third chapter reveals the implications of the demographic structure for patent policy. There is a large literature on patent policy and economic growth, but it was entirely written for infinitely lived agents. These papers conclude that growth is maximized with complete patent protection. This research shows that finite lifetime has immediate implications for patent policy. We find that both shortening patent length (duration of patent protection) and weakening patent breadth (lowering price of patented machines) are growth enhancing but shortening patent length is more effective in spurring growth. This is because the former decreases investment in old patents.

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Contents

Chapter 1

Human-Capital Spillover, Dynastic Altruism, Population and R&D-based Growth

1.1 Introduction

A recent line of research proposed several modifications to modern growth theory, aimed to remove the counterfactual "weak scale-effect" that presented in second and third generation models of R&D-based growth: that is a positive effect of population growth on economic prosperity. [1] A common element in this literature is the introduction of human capital as productive input in the R&D sector. This modification induces substitution between the quantity and quality of workers, which enables an increase in overall effective labor supply and thereby enhanced economic growth, even with a constant or declining population of workers.

Within this line of research the assumed process of human capital formation is crucial to the relation between population growth and economic growth. More specifically, the potential diluting effect of population growth on the average human capital level was emphasized in the literature as hindering economic growth; See for example Dalgaard and Kreiner (2001), Strulik (2005), Bucci (2013), and Chu et al.(2013). In these papers, young agents are assumed to enter the labor force with zero human capital. However, other related studies abstracted from this diluting effect entirely, following Lucas(1988) exact formulation, assuming that newborns are inherited with the same level of human capital as their parents; See for example Tournemaine and Luangaram (2012) and Bucci (2015).

When discussing his own formulation of human capital accumulation, Lucas (1988) emphasized the plausibility of fractional transmission of human capital within dynasties, from parents to their offspring: "..One needs to assume ... that the initial level each new member begins with is proportional to (not equal to!) the level already attained by older members of the

[1] Jones (1999) summarizes the role of population in the first three generations of R&D-based growth models. Recent summaries of the empirical literature can be found in Strulik et al.(2013) and Bucci (2015).

family. This is simply one instance of a general fact that I will emphasize ... : that human capital accumulation is a social activity, involving groups of people in a way that has no counterpart in the accumulation of physical capital" (p.19).

The present work pursues and elaborates such intermediate approach regarding the diluting effect of newborns on human capital accumulation. First, we allow for fractional human capital spillover from parents to their offspring. That is, we consider the entire range between the two extreme cases presented in the aforementioned literature. This kind of spillover was widely considered in other strands of the literature on growth and human capital accumulation, without R&D-based innovations (See for example Becker et al. 1990, Galor and Tsiddon 1997, De-la Croix and Deopke 2004, and Fioroni 2010), as well as in recent R&D-driven growth models written in the Overlapping Generations framework; See for example Strulik et al.(2013) and Prettner (2014).

Second, we consider congestion effects in the transmission of human capital within dynasties. That is, the degree of human capital spillover from parents to their offspring is decreasing with the number of kids. The intuition that motivates this analysis is the following: parental human capital spillover is transmitted through direct interaction between parents and their offspring in the household, where parenting time is not a pure public good.

This is consistent with the essential notion of trade-off between the quantity and quality of children, as phrased by Hanushek (1992, p.86): "The trade-off between child quantity and quality enters essentially because parents' time and resources must be spread thinner with more children".

Our framework is an extended version of Young's (1998) model of two-sector R&D, that incorporates population growth and human-capital accumulation. The analysis yields a rich set of possible relations between population growth and economic growth, including non-monotonic ones, depending on the assumed type of spillover. The welfare analysis shows that the rates of human capital accumulation and technological progress in the decentralized economy may deviate from the efficient ones, in various ways.

Several theoretical papers have established ambiguous effect of population growth on technological progress, which depends on the strength of dynastic altruism toward future generations (Dalgaard and Kreiner 2001, Strulik 2005, Bucci 2013), the (potentially adverse) effect of specialization on the production complexity (Bucci 2015), and the effect of technological progress on the stock of human capital - appreciation vs. depreciation (Bucci 2008). In these studies however, unlike in the present work, the effect of population growth on technological progress depends on the values of model parameters, and are monotonic given the parameters set.

Our work is closely related to the recent contribution by Boikos et al.(2013), which studies the effect of fertility on human capital accumulation, in a model with no R&D-based innovation and endogenous fertility. In their theoretical analysis the effect of population growth on human capital accumulation is allowed to be positive, negative, and non-monotonic. Hence,they allow for "negative dilution" effect of population growth on human capital accumulation, under which population growth enhances per-capita human-capital accumulation.

The theoretical part of their work shows that the overall effect of fertility on per-capita human capital accumulation and income growth depends crucially on the sign of the dilution effect, which is left unspecified (see discussion on p.50 and footnote there). Similar approach was taken also by Boucekkine and Fabbri (2013) and Marsiglio (2014), who assume unspecified and quadratic diluting effect, respectively, in models of physical capital accumulation and endogenous fertility [2].

All these studies were able to establish a hump shape relation between fertility growth and economic growth, consistent with the empirical findings reported by Boikos et al.(2013),[3] and Kelley and Schmidt, (1995)[4] In comparison with these studies, our results are derived in a full fledged R&D-based growth model [5] based on a simple specification of the diluting effect, which has an intuitive economic interpretation. Namely, the diluting effect here is defined by

[2]Their focus however is on the implications of different types of dynastic altruism to optimal fertility rates.

[3]Growth in their model is driven by human capital accumulation, and the empirical analysis focuses, accordingly, on the relation between fertility rates and human capital accumulation, based on a panel data analysis for ninety-nine countries (both OECD and non-OECD), over the years 1960-2000.

[4]In an earlier study Boucekkine et al.(2002) derived such relation between fertility and economic growth in an overlapping generations model of vintage human capital

[5]In concluding their study Boikos et al.(2013, p.57) propose this as a desirable extension to their analysis.

fractional human-capital spillover from parent to their offspring, which is subject to congestion in the number of offspring.

That is, we assume population growth always dilutes per-capita human capital accumulation, but not necessarily in a linear fashion. Yet, we establish non-monotonic polynomial relation between population growth and economic growth, which varies - from U shape to Hump shape, depending on the congestion factor in parental human capital spillover.

In another recent relevant paper by Prettner (2014), human capital is formed through public education system where higher fertility rate decreases schooling-intensity - i.e. per-student public spending. Prettner (2014) shows that for economies with developed public education system - in terms of spending level and teachers' productivity - there is a non-monotonic relation between fertility rate and economic growth: for initially low (high) rates increase in fertility has negative (positive) effect on economic growth. For economies with under-developed public education system the effect of fertility on population growth is definite-positive. [6] Prettner's (2014) results, derived in the overlapping generations framework, are similar to ours in the special case of no congestion in parental human-capital spillover, which we derive for infinitely living agents and decentralized human capital accumulation.

Our specification of human capital formation could yield similar qualitative relations between population growth and economic growth in models where human capital accumulation is the sole growth engine, like the one studies by Boikos et al.(2013). [7] Nonetheless, in R&D-based growth models, like the one studied here, it is the rate of increase in over effective labor supply that affects innovation rate and economic growth [8], whereas in typical models with human capital accumulation only economic growth requires an increase in average (per-capita) human capital level, as in Boikos et al.(2013). Hence, different parameter values will characterize the alternative relations between population growth and economic growth in these two prototype models.

[6]In another work written in the OLG framework, Strulik et al.(2013) explain the non-monotonic relation between population and economic growth within an unified growth model that incorporates endogenous fertility along with transition from neoclassical technology to R&D-based growth.

[7]Indeed, our specification of human capital formation is a particular case, with appealing economic interpretation, of their general formulation

[8]This is due to the basic scale effect, after labor measured in the number of workers is modified to effective labor supply accounting for workers productivity (i.e. human capital).

In the penultimate section of the paper, we introduce non-linear dynastic altruism toward future generations to study the relationship between economic growth and population growth. Several works in this literature have emphasized the role of dynastic altruism toward future generations in determining the effect of population growth on economic prosperity; see, for example, Dalgaard and Kreiner (2001), Strulik (2005), Bucci (2008), and Bucci (2013).[9] These studies show that dynastic altruism stimulates saving and investment in human capital. This positive effect of altruism on saving is increasing with the fertility rate [10] and may overcome the negative diluting effect of population growth on human capital accumulation.[11] In these studies, parents' altruistic utility is linear in the fertility rate (for a given per child consumption level), and the effect of population growth on technological progress depends on the values of model parameters - i.e., it is monotonic given the parameter set.

The paper is organized as follows. Section 1.2 presents the detailed model. section 1.3 analyzes the dynamic equilibrium and the effect of population growth on technological progress. Section 1.4 presents welfare analysis for the model economy. Section 1.5 presents second modification - dynastic altruism, and Section 1.6 concludes this study.

1.2 The Model

We extend Young's (1998) two-sector R&D model by adding population growth and human capital accumulation. Time is discrete, and population grows at exogenous rate n ≥ 0. Population size in each period is denoted $L_t = L_0(1+n)^t$, where L_0 is normalized to one. In each period, each worker is endowed with one unit of time. To enhance exposition clarity, the analysis focuses first on exogenous human capital accumulation, and then human capital accumulation is endogenized through education choice.

[9]In Bucci (2008) and Bucci (2013), the total effect of population growth on economic prosperity also depends on the effect of technological progress on human capital accumulation and on the returns to specialization.

[10]We identify the population growth rate, which is exogenous throughout our analysis, with the fertility rate.

[11] If human capital is not purely non-rival, population growth works to decrease per capita human capital, as the human capital of newborns is lower than the average of existing workers.

1.2.1 Preferences

Consumer's lifetime utility is given by

$$U = \sum_{t=0}^{\infty} \rho^t \ln(c_t) \tag{1.1}$$

where $\rho \in (0,1)$ is the subjective discount factor, and c is the per-capita instantaneous utility from consuming N differentiated products, i.e."varieties", subject to a CES utility function

$$c_t = \left(\sum_{i=1}^{N_t} c_{i,t}^{\frac{1}{\varepsilon}} \right)^{\varepsilon} \tag{1a}$$

with $\varepsilon = \frac{s}{s-1}$, and s is the elasticity of substitution across all varieties. The consumption level of each variety c_i is defined as c_i =$q_i x_i$, where x_i and q_i, denote the consumed quantity and product quality, respectively. The assumed preferences imply the instantaneous demand for each variety

$$x_{i,t}^d = q_{i,t}^{s-1} \lambda p_{i,t}^{-s} \left(\sum_{i=1}^{N_t} c_{i,t+1}^{\frac{1}{\varepsilon}} \right)^{\varepsilon} \tag{1b}$$

where λ is the Lagrange multiplier from the instantaneous utility maximization (i.e. the shadow value of the given periodic spending level). The logarithmic specification in (1) implies the standard Euler condition for optimal consumption smoothing, written in terms of aggregate spending, denoted E

$$\frac{E_{t+1}}{E_t} = \rho(1 + r_{t+1}) \tag{1.2}$$

where $1 + r_{t+1}$ is the (gross) interest rate earned between periods t and t+1.

1.2.2 Production and Innovation

We will start by analyzing a model with exogenous rate of human capital accumulation, subject to the following aggregate growth rate

$$1 + g_h \equiv \frac{h_t}{h_{t-1}} = \frac{(1+\omega n)(1+\tilde{g_h})}{1+n}$$

where h_t is the per-capita human capital, and the parameter $\omega \in (0,1)$ measures parental human capital spillover. With constant population, i.e. n=0, per-capita human capital grows at the rate $\tilde{g_h}$. For positive population growth and $\omega = 1$ parental human capital spillover is complete, and thus population growth has no effect on the per-capita human capital level. For $\omega = 0$ population growth rate works as a full dilution factor over h_t. Our analysis focuses on the intermediate cases with fractional transmission of human capital from parents to their offspring. Furthermore, we consider nonlinear spillover, due to congestion in the number of offspring, that is,$\omega \equiv \omega(n)$ and $\omega'(n) < 0$. To enhance tractability, we focus on the following specification

$$\omega(n) = \omega_0 \exp(-\mu n) \tag{3a}$$

Where $\omega_0 \in (0,1)$, and $\mu \geq 0$ is the congestion factor. With $\mu = 0$ there is no congestion in human capital spillover. Notice that (3)-(3a) imply that population growth slows down the accumulation of per-capita human capital, that is $\frac{\delta g_h}{\delta n} < 0$, [12] hence we assume effective diluting. Aggregate human capital, denoted H, is defined as the product of population size and per-capita human capital

$$H_t = L_t h_t \tag{3b}$$

Effective labor supply is the sole input for production and innovation, and the wage rate is normalized to one. One unit of labor produces one unit of consumption good (regardless of its quality).

[12] More generally, $\frac{\partial g_h}{\partial n} < 0$ is negative as long as $\omega'(n) < 0$.

We follow Young's (1998) specification of the innovation cost function

$$f(q_{i,t+1}, \bar{q}_t) = \begin{cases} \exp(\phi \frac{q_{i,t+1}}{\bar{q}_t}) & \text{if } q_{i,t+1} > q_{i,t} \\ \exp(\phi) & \text{if } q_{i,t+1} < q_{i,t} \end{cases} \tag{4}$$

Innovation cost in sector i is increasing with the rate of improving its product quality, over the highest quality that was already developed in the economy - denoted $\bar{q}_t$. This specification implies vertical knowledge-spillovers, as the present cost of providing a certain quality today is decreasing in the highest quality level that was already developed in the past. As innovation is assumed to be certain, vertical innovation (i.e. quality improvements) implies that the effective lifetime of each product is one period. Hence, each firm maximizes the profit

$$\Pi_{i,t} = \frac{(p_{i,t+1} - 1)x^d_{i,t+1} L_{t+1}}{1 + r_{t+1}} - f(q_{i,t+1}, \bar{q}_t) \tag{5}$$

Maximizing (5) for price $p_{i,t+1}$ yields the standard optimal monopolistic price $p^* = \varepsilon, \forall t, i$. The first order for optimal quality choice is derived after plugging the optimal price and the demand function (1b) into (5)

$$\frac{1}{q^*_{i,t+1}} \frac{(\varepsilon - 1)(s - 1)(\lambda\varepsilon)^{-s} \left(\sum_{i=1}^{N_t} c_{i,t+1}^{\frac{1}{\varepsilon}} \right)^{\varepsilon} L_{t+1}}{1 + r_{t+1}} = \frac{\phi}{\bar{q}_t} f(q^*_{i,t+1}, \bar{q}_t) \tag{5a}$$

The asterisk superscript denotes optimally chosen values for the variables in the decentralized economy. Assuming free entry to the R&D sector implies that in equilibrium the profit in (5) equals zero. Combining this assumption with the optimality condition (5a) we obtain the equilibrium rate of quality improvement

$$\forall_i : 1 + g_q \equiv \frac{q^*_{t+1}}{\bar{q}_t} = \frac{s - 1}{\phi} \tag{5b}$$

We assume the cost parameter ϕ is low enough to guarantee $g_q > 0$ and to make vertical competition between successive product generations redundant, i.e. $p^* < 1 + g_q \Rightarrow \varepsilon < \frac{s-1}{\phi}$.

As the rate of quality improvement is time invariant, so is equilibrium innovation cost $f(q_{i,t+1}, \bar{q}_t) = e^{s-1}, \forall t, i$.Notice that under symmetric equilibrium demand for each variety is $x_t^d = \frac{E_t}{\varepsilon N_t}\ \forall i$, and thus the free entry condition can be also written as

$$\frac{(\varepsilon - 1)\frac{E_{t+1}}{\varepsilon N_{t+1}}}{f} = (1 + r_{t+1}) \tag{6}$$

1.3 Equilibrium and Growth Dynamics

1.3.1 Exogenous Human Capital Accumulation

Combining (2) and (6) we have,

$$E_t = \frac{fN_{t+1}}{(1 - \frac{1}{\varepsilon})\rho} \tag{7}$$

and plugging (7) back into (6) yields the interest rate for the assumed stationary equilibrium

$$\frac{1 + g_N}{\rho} = (1 + r_{t+1}) \tag{8}$$

where $1 + g_N \equiv \frac{N_{t+1}}{N_t}$. The aggregate resources-uses constraint for the economy is defined by the allocation of labor between production and R&D investment

$$H_t = \frac{E_t}{\varepsilon} + fN_{t+1} \tag{9}$$

Plugging (7) into (9) yields

$$H_t = \frac{fN_{t+1}}{(\varepsilon - 1)\rho} + fN_{t+1} \Rightarrow N_{t+1} = \frac{H_t}{f\left(\frac{1}{(\varepsilon-1)\rho} + 1\right)} \tag{10}$$

Hence, variety expansion rate equals to the exogenous growth rate of effective labor supply $(1 + g_N) = (1 + g_H)$, which, following (3)-(3a), implies:

$$(1 + g_N) = (1 + \widetilde{g}_h)\left[1 + \omega\left(n\right)n\right] \tag{10a}$$

Observe that under symmetric equilibrium, equation (1a) can be written as

$$C_t = \left(\sum_{i=1}^{N_t} (q_{i,t} x_{i,t})^{\frac{1}{\varepsilon}} \right)^{\varepsilon} = N_t^{\varepsilon} q_t x_t = N_t^{\varepsilon} q_t \frac{E_t}{\varepsilon N_t}$$

After plugging (7) into C_t, the above expression implies that in the stationary equilibrium per-capita consumption grows at a constant rate

$$1 + g_c \equiv \frac{c_t}{c_{t-1}} = \frac{L_{t-1} N_t^{\varepsilon-1} q_t N_{t+1}}{L_t N_{t-1}^{\varepsilon-1} q_{t-1} N_t} = \frac{(1+g_q)(1+g_N)^{\varepsilon}}{1+n} \tag{11}$$

Then we substitute (10a) and (3a) into (11) to rewrite

$$1 + g_c = \frac{(1+g_q)(1+\tilde{g_h})^{\varepsilon} \left[1 + \omega_0 \exp(-\mu n) n\right]^{\varepsilon}}{1+n} \tag{11a}$$

Equation (11a) reveals the two opposing effect induced by population growth on per-capita consumption growth. The positive effect is due to the increase in aggregate human capital supply, which accelerates variety expansion - according to equation (10). This positive effect is generated through the spillover parameter and is then amplified by the preference parameter ε, which is decreasing with the elasticity of substitution across varieties - s. With lower s gains from faster variety expansion, driven by faster human capital accumulation are higher. The negative effect of population growth on per-capita consumption growth, which presents in the denominator of (11a), is the regular pure dilution effect.

Differentiating (11a) for n shows that $\frac{\delta g_c}{\delta n}$ is positive (negative) if the following (reverse) inequality holds

$$\varepsilon (1+n)(1-\mu n) - n > \frac{\exp(\mu n)}{\omega_0} \tag{11b}$$

Proposition 1.1. *With exogenous human capital accumulation, for sufficiently high ω_0 and μ, the function $g_c(n)$ is hump shape. That is for sufficiently strong base spillover and congestion effect economic growth first accelerates with population growth rate and then slows down.*

Proof. For $\mu > 0$, the right hand side of (11b) is increasing with n. For $\varepsilon(1-\mu) < 1$, the left hand side of (11b) is monotonically decreasing with n, and hence, for sufficiently high n it is guaranteed that (11b) does not hold, that is $\frac{\partial g_c}{\partial n} < 0$. If $\omega_0 > \frac{1}{\varepsilon}$, condition (11b) holds for $n = 0$. Hence under these conditions, $\frac{\partial g_c}{\partial n}$ is positive (negative) under sufficiently low (high) population growth rates □

For $\varepsilon(1-\mu) < 1$ and $\omega_0 < \frac{1}{\varepsilon}$, the function $g_c(n)$ is monotonically decreasing. Under lower values of μ, for which $\varepsilon(1-\mu) > 1$, the left-hand side of (11b) is increasing with n up to $n = \frac{\varepsilon(1-\mu)-1}{2\varepsilon\mu}$, and then starts decreasing (for high n values). Then, $g_c(n)$ still follows a hump shape for $\omega_0 > \frac{1}{\varepsilon}$. However, for $\omega_0 < \frac{1}{\varepsilon}$, condition (11b) holds only for intermediate values of n, implying that $g_c(n)$ is first decreasing with n - to a local minimum, and then it is increasing to a local maximum from where it is monotonically decreasing. Hence, within this parameters set the shape of $g_c(n)$ follows a co-sine shape, which combines U shape with Hump shape. As the value of μ decreases the range of the U shape is expanding.

Proposition 1.2. *With exogenous human capital accumulation, for sufficiently low ω_0 and μ, the function $g_c(n)$ is U shape. That is for sufficiently weak base spillover and congestion effect economic growth first slows down with population growth rate and then accelerates.*

Proof. For the limit case $\mu = 0$ (11b) is modified to $\varepsilon + n(\varepsilon - 1) > \frac{1}{\omega_0}$. The right-hand side of this condition is increasing with n. For $\omega_0 > \frac{1}{\varepsilon}$ the latter inequality does (not) hold for sufficiently high (low) n, implying that $\frac{\partial g_c}{\partial n}$ is negative (positive) for low (high) values of n □

For $\mu = 0$ and $\omega_0 > \frac{1}{\varepsilon}$ we have $\forall n > 0 : \frac{\partial(1+g_c)}{\partial n} > 0$, that is $g_c(n)$ is monotonically increasing.

1.3.2 Endogenous Human Capital Accumulation

We turn now to incorporate endogenous human capital accumulation in the model, subject to the conventional specification

$$h_{t+1} = \frac{(1+\omega n)(\xi e_t + 1 - \delta)h_t}{1+n} \tag{1.3}$$

$$\Delta h_{t+1} \equiv h_{t+1} - h_t = \left[\frac{(1+\omega n)(\xi e_t + 1 - \delta)h_{t-1}}{1+n} - 1\right] h_t \tag{12}$$

where $e \in (0,1)$ is the time invested in human capital formation, δ is a depreciation rate, and ξ captures the productivity of the human capital formation technology[13]. Equation (12) implies that $1+g_h \equiv \frac{h_{t+1}}{h_t} = \frac{(1+\omega n)(\xi e_t + 1 - \delta)}{(1+n)}$, and following (3b) we obtain

$$1+g_H \equiv \frac{H_{t+1}}{H_t} = (1+g_h)(1+n) = (1+\omega n)(\xi e_t + 1 - \delta) \tag{13}$$

The return on investment in human capital should equal the return on R&D investment defined in (5)

$$1+r_{t+1} = \frac{(\xi e_{t-1} + 1 - \delta) h_t}{e_t h_t} \tag{14}$$

Plugging the interest rate (8) into (14) and rearranging yields

$$\forall t : e^* = \frac{1-\delta}{\frac{1+g_N}{\rho} - \xi} \tag{15}$$

and plugging (15) back into (13) yields

$$1+g_H = \frac{(1+\omega n)(1-\delta)}{1 - \frac{\xi\rho}{1+g_N}} \tag{16}$$

Modifying the resources-uses constraint (10) for the time invested in human capital formation yields

$$(1-e^*) H_t = \frac{f N_{t+1}}{(\varepsilon - 1)\rho} + f N_{t+1} \tag{17}$$

Plugging the interest rate (8) into (17) and rearranging yields

$$N_{t+1} = \frac{(1-e^*) H_t}{f\left[\frac{1}{(\varepsilon-1)\rho} + 1\right]} \tag{17a}$$

Equation (17a) shows that the aggregate human-capital stock and the varieties span share the same growth rate, as in Section 3. Imposing $(1+g_H) = (1+g_N)$ in (16) and simplifying we

[13]With constant population and no depreciation equation (13) falls back to Lucas' (1988) original formulation: $\triangle h_t = \xi (e_{t-1}) h_{t-1}$.

obtain

$$1 + g_N = (1 - \delta)(1 + \omega n) + \xi\rho \tag{18}$$

Hence, following (11)-(11a), per-capita consumption growth rate remains $1 + g_c = \frac{(1+g_q)(1+g_h)^{\varepsilon}(1+\omega n)^{\varepsilon}}{1+n}$ which can be written explicitly as [14]

$$1 + g_c = \frac{(1 + g_q)\left[(1 - \delta)(1 + \omega_0 \exp(-\mu n) \cdot n) + \xi\rho\right]^{\varepsilon}}{1 + n} \tag{19}$$

Equation (19) shows that the effect of population growth on per-capita consumption growth under endogenous human capital accumulation is very similar to the one presented in equation (11a), for exogenous rate of human capital accumulation. Nonetheless, here, the effect of population growth rate on per-capita consumption growth depends also on the technological parameters of human capital formation, and the time preference parameter.

Following (19), $\frac{\partial g_c}{\partial n}$ is positive (negative) if the following (reverse) inequality holds

$$\varepsilon > \frac{1 + \omega_0 \exp(-\mu n) \cdot n + \frac{\xi\rho}{(1-\delta)}}{\omega_0 \exp(-\mu n) \cdot (1 - \mu n)(1 + n)} \tag{19a}$$

Proposition 1.3. *With endogenous human capital accumulation, for $\mu > 0$ and $\varepsilon > \frac{1}{\omega_0}\left(1 + \frac{\xi\rho}{(1-\delta)}\right)$, the relation between population growth and per-capita consumption growth follows a hump shape.*

Proof. Condition (19a) does not hold for $n > \frac{1}{\mu}$, as the denominator turns negative, but it does hold for sufficiently low n if $\varepsilon > \frac{1}{\omega_0}\left(1 + \frac{\xi\rho}{(1-\delta)}\right)$. Hence, under these conditions $g_c(n)$ is non-monotonic and follows a hump shape □

Proposition 1.4. *With endogenous human capital accumulation, for sufficiently low congestion effect and $\varepsilon < \frac{1}{\omega}\left(1 + \frac{\xi\rho}{(1-\delta)}\right)$, the relation between g_c and n follows non-monotonic U shape.*

Proof. For the limit case $\mu = 0$, condition (19a) becomes $\varepsilon > \frac{1+\omega_0 n+\frac{\xi\rho}{(1-\delta)}}{\omega_0(1+n)} \Rightarrow \varepsilon + n(\varepsilon - 1) > \frac{1+\frac{\xi\rho}{(1-\delta)}}{\omega}$. The latter condition holds for sufficiently high n , but it does not hold for sufficiently low (yet non-negative) n if $\omega_0 < \frac{1+\frac{\xi\rho}{(1-\delta)}}{\varepsilon}$. Hence, under these conditions $\frac{\partial g_c}{\partial n}$ is negative (positive) for sufficiently low (high) values of n, implying that $g_c(n)$ is Ushaped □

[14]Following (3a),(13) ,(16) and (18).

Having $\varepsilon > \frac{1+\frac{\xi\rho}{(1-\delta)}}{\omega}$ implies that $\forall n > 0 : \frac{\partial g_c}{\partial n} > 0$,that is positive monotonic relation between population growth and economic growth.

The relation between g_c and n established in Proposition 4 is similar to the one presented in Prettner (2014) for an overlapping generations economy with public education system. In Prettner's work, as in the present study, high productivity in human capital formation (interpreted there as teachers' productivity and schooling efficiency) is needed to obtain such non-monotonic relation. In addition, his result also requires a high level of public spending on schooling, which is set exogenously, whereas Proposition 4 above is derived for decentralized investment in education, chosen by the households.

1.4 Welfare Analysis

We turn now to evaluate the welfare performance of our extended version of Young's model. In Young's (1998) original work, growth is driven solely by vertical (quality improving) innovation. There, the rate of quality improvements is slower than the social optimum (see p.59 there) because investors do not internalize the vertical knowledge spillover defined in the innovation process (4). In our extended framework there are additional knowledge spillover through the process of human capital accumulation, which determines the rate of variety expansion. These spillover are not internalized either in the decentralized economy, as agents are choosing investment level according to the private rate of return, according to equation (14). Therefore, the introduction of human capital accumulation in our model generates a second source of efficiency distortion.

The social planner that maximizes (1) along the balanced growth path is facing the following objective function

$$U = \frac{1}{1-\rho}\left(\ln c_0 + \frac{\rho \ln(1+g_c)}{1-\rho}\right) \tag{20}$$

The welfare function (20) should be maximized by allocating labor efficiency over production and the two investment activities - quality improvements and human capital formation. This maximization problem is still subject to the resources-uses constraint (17), and also to the implied explicit expression for $(1+g_c)$ in (19). Imposing these restrictions on (20) we obtain the constrained objective function [15]

$$U = \frac{1}{1-\rho}\left(\begin{array}{c} \ln\left[\frac{N_0^{\varepsilon} q_0[(1-e)h_0 - fN_0(1+\omega n)(\xi e+1-\delta)]}{\varepsilon N_0}\right] + \\ + \frac{\rho \ln\left[\frac{\frac{1}{\phi}\ln f((1+\omega n)(\xi e+1-\delta))^{\varepsilon}}{1+n}\right]}{1-\rho} \end{array}\right) \tag{20a}$$

After normalizing all initial values to unity, we derive the first order conditions with respect to investment in education and quality improvements

$$\frac{\partial U}{\partial e} : \frac{1+\xi f(1+\omega n)}{((1-e^{**}) - f(1+\omega n)(\xi e^{**}+1-\delta))} = \frac{\rho}{1-\rho}\left[\frac{\xi\varepsilon}{\xi e^{**}+1-\delta}\right] \tag{21}$$

$$\frac{\partial U}{\partial f} : \frac{(1+\omega n)(\xi e+1-\delta)}{((1-e) - f^{**}(1+\omega n)(\xi e+1-\delta))} = \frac{\rho}{1-\rho}\frac{1}{f^{**}\ln f^{**}} \tag{22}$$

The superscript with double asterisk denotes the solution values for the maximization of (20a). Combining conditions (21)-(22) yields the efficient investment in quality

$$\xi(1+\omega n) = \frac{1}{f^{**}\left[\varepsilon(\ln f^{**}) - 1\right]} \tag{23}$$

Note that the efficient investment in quality improvement decreases with the productivity of human capital formation ξ and the degree of human capital spillover ω, and increases with the elasticity across varieties, s. By comparison, the decentralized investment in quality implied by equation(5b) depends only on the preferences parameter s and it does not account for the human capital formation and spillover parameters. The first optimization condition can be written as

$$\left(\frac{1}{\rho}-1\right)\ln f^{**} + 1 = \frac{\xi(1-e^{**})}{\xi e^{**}+1-\delta} \Rightarrow e^{**} = \frac{\varepsilon\ln f - 1 - \frac{1-\delta}{\xi}\left[\left(\frac{1}{\rho}-1\right)(\ln f)+1\right]}{\left(\frac{1}{\rho}-1\right)\ln f + \varepsilon\ln f} \tag{24}$$

The efficient investment in education implies the following rate of human capital accumulation

$$1+g_H = \frac{(1+\omega n)(\varepsilon\ln f - 1)(\xi+1-\delta)}{\left(\frac{1}{\rho}-1\right)(\ln f) + \varepsilon\ln f} = \frac{1+\frac{1-\delta}{\xi}}{f\ln f\left[\left(\frac{1}{\rho}-1\right)+\varepsilon\right]} \tag{24a}$$

[15]Here, ω can be any of the specification of human capital spillovers considered in Section 3. Following the innovation function (4), the quality growth rate is given by $\frac{\ln f}{\phi}$.

Proposition 1.5. *The growth rates of human capital accumulation and products' quality improvements may deviate from the efficient one in various ways. Overall efficiency is achieved iff* $\frac{1+\frac{1-\delta}{\xi}}{f^{**}\ln f^{**}\left[\left(\frac{1}{\rho}-1\right)+\varepsilon\right]} = (1-\delta)(1+\omega n)+\xi\rho$

Proof. Comparing (23) with (5a) shows that the market will provide efficient rate of quality improvements only if, $\xi(1+\omega n) = \frac{1}{\exp^{s-1}(s-1)}$. Hence, generally, the rate of quality improvements in the decentralized economy can be higher or lower than the efficient one. Comparing (24a) with (16) implies that the rate of human capital accumulation in the market is efficient only if

$$\frac{1+\frac{1-\delta}{\xi}}{f^{**}\ln f^{**}\left[\frac{1}{\rho}-1+\varepsilon\right]} = (1-\delta)(1+\omega n)+\xi\rho$$

Clearly, this condition may hold only for a very specific set of parameter □

As explained above, in the present model there are externalities in both the vertical and horizontal dimensions of innovation, due to knowledge externalities in quality improvements and in the process of human capital which determines the rate of variety expansion.

Both types of spillover are not internalized in the decentralized economy. Hence, the deviation of the decentralized economy from the efficient performance depend not only on the overall level of positive externalities but also on their relative strength.

1.5 Dynastic Altruism

In this section, we introduce non-linear parental altruism in the number of offspring to establish a non-monotonic relationship between economic growth and population growth. We extend Young's (1998) two-sector R&D model by incorporating population growth, human capital accumulation, and dynastic altruism.

1.5.1 Preferences

The consumer's lifetime utility is given by

$$U = \sum_{t=0}^{\infty} \rho^t (1+\theta n)^t \ln(c_t) \tag{25}$$

where $\rho, \theta \in (0,1)$ are the time preference and degree of altruism, respectively. The current literature is focused on the linear specification of the altruism factor, implying that θ is scalar; See, for example, Strulik (2005), Bucci (2008, 2013).[16] Here, we let the degree of altruism per child depend on the number of offspring, that is, on $\theta \equiv \theta(n)$. Following Barro and Becker [2, 3], Becker et al.(1990) and Becker (1992), we assume $\theta(n) = \theta_0 n^{-\gamma}$; hence, $\theta\left(n\right) n = \theta_0 n^{1-\gamma}$, where $\gamma, \theta_0 \in (0,1)$. The assumption $\theta_0 < 1$ implies that parents' "selfish" utility from their own consumption has a higher weight than their altruistic utility from per-child consumption, which is in line with the latter references. To ensure that (1) has finite values, we also assume $\rho(1+\theta n) < 1$.

The modified Euler condition for optimal consumption smoothing smoothing

$$\frac{E_{t+1}}{E_t} = \rho(1+\theta n)(1+r_{t+1}) \tag{26}$$

where $(1+r_{t+1})$ is the (gross) interest rate earned between periods t and $t+1$.

1.5.2 Population Growth and Economic Prosperity

The expression for (stationary) per capita consumption growth rate

$$1+g_c \equiv \frac{c_t}{c_{t-1}} = \frac{L_{t-1}M_t^{\varepsilon-1}q_t M_{t+1}}{L_t M_{t-1}^{\varepsilon-1}q_{t-1}M_t} = \frac{\left(1+g_q\right)\left(1+g_M\right)^{\varepsilon}}{1+n} \tag{27}$$

Which can be also written as

$$1+g_c = \frac{\left(1+g_q\right)\left[(1-\delta)+\xi\rho(1+\theta\left(n\right)n)\right]^{\varepsilon}}{1+n} \tag{27a}$$

Plugging the explicit expression for $\theta\left(n\right)$ into (27a) and then differentiating for n shows that the sign of $\frac{\partial g_c}{\partial n}$ depends on the sign of $\frac{\varepsilon(1-\gamma)\theta_0 n^{-\gamma}(1+n)}{\frac{(1-\delta)}{\xi\rho}+1+\theta_0 n^{1-\gamma}} - 1$. The latter expression is positive (negative) if the following (reverse) inequality holds

$$\varepsilon\left(1-\gamma\right)n^{-\gamma} - \left[1-\varepsilon\left(1-\gamma\right)\right]n^{1-\gamma} > \frac{1}{\theta_0}\left(\frac{1-\delta}{\xi\rho}+1\right) \tag{28}$$

[16] At the two extremes, $\theta = 0$ or $\theta = 1$, preferences are of Millian or Benthamite type, respectively.

Proposition 1.6. *For a sufficiently large γ, the relation between g_c and n is* hump shaped.

Proof. If γ is large enough to ensure that $\varepsilon(1-\gamma) < 1$, the left side of (28) is decreasing with n: starting from plus infinity for $n \to 0$, and becoming negative for $n > \frac{\varepsilon(1-\gamma)}{1-\varepsilon(1-\gamma)}$. Hence, for $\varepsilon(1-\gamma) < 1$, the sign of $\frac{\partial g_c}{\partial n}$ is positive (negative) for low (high) fertility rates, and thus, $g_c(n)$ is hump shaped. □

Following (28), with $\varepsilon(1-\gamma) \leq 1$ (i.e., $\gamma \geq \frac{1}{s}$), the per capita consumption growth rate is maximized for $n = \frac{\varepsilon(1-\gamma)}{1-\varepsilon(1-\gamma)}$. As γ increases (decreases), the range of n for which $\frac{\partial g_c}{\partial n} > 0$ is shrinking (widening). The result presented in proposition 1 summarizes the total impact of the two contradictory effects of population growth on economic work presented in equation (27a): the numerator shows the positive effect of population growth on savings and investment in the presence of altruism, i.e., for any $\theta(n) > 0$. The denominator in (27) shows the standard diluting effect of population growth on human capital accumulation and, thereby, on economic growth. However, under the current specification, the positive effect of altruism on growth depends on the rate of population growth, as $\frac{\partial\theta(n)n}{\partial n} = \frac{\partial\theta_0 n^{1-\gamma}}{\partial n} = (1-\gamma)\theta_0 n^{-\gamma}$. Hence, for high (low) levels of n, the positive (negative) effect dominates the overall impact of population growth on economic growth. This result is similar to the one we presented in Diwakar and Sorek (2016), albeit through a different mechanism, which is congestion in dynastic spillovers of human capital.

1.6 Conclusion

In this work we have established a polynomial relation between population growth and economic growth, building on the notion of human-capital spillover from parents to their offspring. We have shown that the shape of the non-monotonic relation between population growth and economic growth can be altered and even inverted in the presence of congestion in human-capital spillover. Our findings contribute to the recent literature that is aimed to modify R&D-based model to remove the counterfactual definite positive effect of population growth on technological progress, and economic growth ("weak scale effect").

In particular, this work adds to the few recent studies that established non-monotonic relation between population growth and economic growth. We have shown that under sufficient congestion impact, the effect of population growth on economic growth may follow a hump shape, that is consistent with the empirical finding of Boikos et al.(2013) and Kelley and Schmidt (1995). Finally, we have shown that the rates of human capital accumulation and products' quality improvements in the decentralized economy may deviate in various ways from the welfare maximizers.

Subsequent research is called to explore the implications of endogenous fertility rates to the results derived in this work, including the potential for equilibria multiplicity that was pointed out but not fully explored by Boikos et al.(2013, p.49).

Chapter 2

Weak Scale Effects in

Overlapping Generations Economy

2.1 Introduction

The second and third generations of R&D-based growth models were criticized for presenting a positive relation between population growth and economic prosperity, i.e. a "weak scale effect", which does not fit the empirical findings of an ambiguous, possibly non-monotonic, relation between these variables. [1] This literature, however, has focused almost exclusively on the analysis of infinitely lived homogenous agents. We study the implications of this demographic structure for the presence of the weak scale effects, through a comparative analysis of the Overlapping Generations (OLG) model of finitely lived agents. [2]

The two canonical demographic structures of the macroeconomic workhorse models imply different incentives for saving. The infinitely lived agents are assumed to share their assets (patent ownership in the current context) with their offspring. They fully internalize this into their saving decisions as they maximize the per-capita or aggregate lifetime utility of their dynasty members. Therefore, in this framework savings involve bequests, but they lack a life-cycle saving motive as workers' labor supply does not change with age. [3]

[1]Jones (1999) provides a compact comparative summary of the theoretical literature. Strulik et al. (2013) and Boikos et al.(2013) summarize the empirical literature.

[2]Earlier literature already showed that the different demographic structures have immediate implications for tax policy, convergence patterns, and the feasibility of growth itself. Dalgaard and Jensen (2009, p.1639) summarize this literature. Sorek (2011), and Diwakar and Sorek (2016c) highlight the implications of the OLG demographic structure to patent policy.

[3]The infinitely living agents can be thought equivalently, and more realistically, as finitely living ones with strong altruism toward their offspring.

By contrast, in the OLG framework saving is aimed to smooth consumption over a finite lifetime, which spans from working years to retirement period, and there are no intergenerational bequests. Hence, in this framework saving is motivated purely by life-cycle considerations. Clearly, the exclusive presentation of each saving motive in its corresponding demographic structure is unrealistically extreme. [4]

Our analysis decomposes the implications of the two saving motives for the presence of weak scale effect. First, we show that in the absence of bequest saving-motive, the sign of the weak scale effect in the OLG economy depends solely on the degree of intertemporal elasticity of substitution (IES). Then, we show how when both saving motives are active, the sign of the weak effect depends also on the relative strength of parents' utility from bequest vs. utility from their own consumption during retirement.

Our results contribute to a recent line of modified R&D-based growth model with infinitely lived agents, which aimed at aligning the role of population growth in R&D-based growth theory with the empirical evidence. [5]

Unlike the present work, these modified models introduce human capital as a productive input in the R&D sector, thereby forming a tension between a positive effect of population growth on saving in the presence of dynastic altruism and its negative (diluting) effect on human capital accumulation.

The present study is also related to the work by Dalgaard and Jensen (2009), hereafter "DJ", on the effect of alternative saving motives on the presence of strong scale effects - that is the effect of population size on economic growth. Their work adds bequest saving-motive to an otherwise standard OLG model with capital externalities, that is an AK model.

However, our research question differs from DJ's, as we study the effect of alternative saving motives on the presence of weak scale effects and our modeling approach differs from DJ's

[4] The empirical literature has not yet reached an agreement regarding their relative importance in driving saving behavior; See De Nardi et al. (2015) for a recent survey.

[5] See for example Dalgaard and Kreiner (2001), Strulik (2005), Bucci (2008, 2015), Bucci and Raurich (2016), and Diwakar and Sorek (2016a,b). Two recent works study this topic within the OLG framework. Prettner (2014) shows that the sign of the weak scale effect depends on the characteristics of the public education sector. Strulik et al.(2013) developed a unified growth model that incorporates endogenous fertility and human capital accumulation, and transition from neoclassical technology to R&D-based growth.

as we incorporate a full-fledged textbook model of R&D-based growth within the OLG framework. Therefore, our results are not fully comparable with those of DJ's. Nevertheless, we reconfirm that the different saving motives implied by the alternative demographic structures are crucial in determining the role of population in R&D-based growth.

The paper proceeds in a straightforward manner. Section 2.2 presents the detailed model. Section 2.3 studies the weak-scale effects with life-cycle saving only. Section 2.4 introduces bequest motive for saving. Lastly, Section 2.5 concludes this study.

2.2 The Model

We take the variety-expansion model presented in the textbook of Barro and Sala-I-Martin (2004, Chapter 6), hereafter "BS", and accommodate it to the OLG framework: each consumer lives for two periods. In the first period, she supplies one unit of labor and in the second period she retires. Cohort (generation) size is increasing at an exogenous constant rate n, which is also the growth rate of the labor force and overall population.

2.2.1 Production and Innovation

The final good Y is produced by perfectly competitive firms with labor and differentiated inputs, to which we refer as "machines"

$$Y_t = AL_t^{1-\alpha} \int_0^{M_t} K_{i,t}^{\alpha}\, \mathrm{d}i \qquad ,0 < \alpha < 1 \tag{1}$$

where A is a productivity factor, L_t and $K_{i,t}$ are labor input and the utilization level of machine i in period t, respectively, and M_t measures the number of available machine varieties. The final good price is normalized to one. Machines are capital goods, and thus they are formed one period ahead of utilization, and we assume they fully depreciate after one period. Once invented, the new machine variety is eternally patented. Under symmetric equilibrium, utilization level for all machines is uniform, i.e. $K_{i,t} = K_t \; \forall$ i, and thus total output is

$$Y_t = AM_tK_t{}^{\alpha}L_t^{1-\alpha} \tag{1a}$$

The labor market is perfectly competitive, and therefore the equilibrium wage and aggregate labor income are $w_t = A(1-\alpha)M_tK_t{}^{\alpha}L_t^{-\alpha}$ and $w_tL_t = A(1-\alpha)M_tK_t{}^{\alpha}L_t^{1-\alpha}$, respectively. The profit for the final good producer is $\pi_{i,t} = AL_t^{1-\alpha}\int\limits_0^{M_t} K_{i,t}^{\alpha}\,\mathrm{d}i - \int\limits_{i=1}^{M_t} p_{i,t}K_{i,t}\,\mathrm{d}i - w_tL_t$, where $p_{i,t}$ is the price of input i. Profit maximization yields the demand for each machine: $K_{i,t}^d = A^{\frac{1}{1-\alpha}}L_t\left(\frac{\alpha}{p_{i,t}}\right)^{\frac{1}{1-\alpha}}$, for which the periodic producer-surplus from machine i, denoted PS_i, is [6] $PS_{i,t} = [p_{i,t} - (1+r)]K_{i,t}^d$. This surplus is maximized by the standard monopolistic price $p_{i,t} = \frac{1+r_t}{\alpha}\ \forall i,t$. [7] Plugging this price in $K_{i,t}^d$, and then back in (1a), we obtain per-worker output,y_t:

$$y_t \equiv \frac{Y_t}{L_t} = A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r_t}\right)^{\frac{\alpha}{1-\alpha}} M_t \tag{1b}$$

The innovation technology follows the specification of BS in the analysis of scale effects (see p.302 in sub-chapter 6.1.7 there): [8]

$$\eta_t = \eta A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r_t}\right)^{\frac{\alpha}{1-\alpha}} L_t \tag{2}$$

where eta_t the cost of innovating a new variety ($\eta > 0$). This innovation technology implies that variety expansion (i.e. productivity growth) in this model depends positively on the share of output devoted to R&D. As we assume machine-varieties are patented forever, patents are being traded inter-generationally - young buy patents from old. New and old varieties play equivalent roles in the production, as reflected in their symmetric presentation in (1). Therefore

[6] Total surplus is the given by per-unit surplus times demand, and per-unit surplus is the selling price minus the marginal cost of capital, that is $\delta + r$ (full depreciation is assumed here).

[7] BS abstract from the timing of investment, setting the cost of each machine (in terms of output units) to one and therefore having the optimal monopolistic price $p = \frac{1}{\alpha}$ (equations 6.9-6.10 on pp. 291-292 there). In their continuous time framework this abstraction has no effect on any of the results.

[8] Equation (2) implies that variety expansion rate, which defines productivity growth in this model, depends positively on the share of output devoted to R&D. This specification aligns with the empirical regularities summarized in that chapter, which were originally presented by Jones (1995). See chapter 6.1.7 in Barro and Sala-I-Martin (2004) for a detailed discussion.

the market value of old varieties equals the cost of inventing a new one - η_t. Hence the return on patent ownership - over old and new technologies is $1 + r_{t+1} = \frac{PS_{i,t+1}+\eta_{t+1}}{\eta_t}$. Plugging the explicit expressions for the surplus and the innovation cost, we obtain the stationary interest rate: [9]

$$1 + r = (1 + n)\left[\frac{\alpha(1-\alpha)}{\eta} + 1\right], \forall t \tag{3}$$

Hence, population growth works to increase the rate of return on capital, due increased demand for patented machines. Following (1b), per-capita output growth (which coincides with per-worker output growth), denoted g_y, is determined by the expansion rate of machine-varieties range, g_M:

$$1 + g_{y,t+1} \equiv \frac{y_{t+1}}{y_t} = 1 + g_{M,t+1} \tag{4}$$

2.2.2 Preferences

Lifetime utility, for an agent born in period t, is derived from consumption over two periods, and bequest:

$$u(c_{t,1}, c_{t,2}, b_t) = \frac{(c_{t,1})^{1-\theta}}{1-\theta} + \beta\left[\frac{(c_{t,2})^{1-\theta}}{1-\theta} + \kappa\frac{(\frac{b_t}{1+n})^{1-\theta}}{1-\theta}\right] \tag{5}$$

where $\beta \in (0, 1)$ is the subjective discount factor, $\frac{1}{\theta}$ is the IES, [10] c_1, c_2 denote consumption when young and old, respectively, and b_t is the total bequest left by a representative parent in period t (hence $\frac{b_t}{1+n}$ denotes per-child bequest). The parameter $\kappa \geq 0$ measures the weight placed on utility from the bequest. This specification of the bequest motive for saving, which resembles a "joy-of-giving" is similar to DJ and is common to the literature written in the OLG framework (see for example Strulik et al. 2013). It implies that parents care about the per-child

[9] Any non-stationary interest rate path should satisfy $1 + r_{t+1}^{\frac{1}{1-\alpha}} = (1+n)\frac{\alpha(1-\alpha)+\eta}{\eta}(1+r_t)^{\frac{\alpha}{1-\alpha}}$. Our results would hold if we assume that patents ownership is transferred from parents to offspring, like in the model with infinitely living agents. Then, however, the interest rate would be $1 + r = \frac{(1+n)\alpha(1-\alpha)}{\eta}$, which corresponds to the one presented in BS (adjusted for continuous time).

[10] The empirical literature suggests that the IES is lower than one; See Hall (1988), Beaudry and Wincoop (1996), Engelhardt and Kumar (2009).

bequest level, which is in line with the Millian type of parental preferences employed by BS. In the extended working-paper version of this study. [11] we explore also the Benthamite and "Beckerian" types of parental preferences

2.3 Life-Cycle Saving

In the absence of bequest saving motive, which is the case we analyze first, we have $\kappa = 0$ and lifetime utility boils down to the standard form:

$$U(c_{t,1}, c_{t,2}) = \frac{(c_{t,1})^{1-\theta}}{1-\theta} + \beta\frac{(c_{t,2})^{1-\theta}}{1-\theta} \tag{5a}$$

Under (5a) young agents allocate their labor income between consumption and saving, denoted s. The solution for the standard optimal saving problem is $s_t = \frac{w_t}{1+\beta^{\frac{-1}{\theta}}(1+r)^{1-\frac{1}{\theta}}}$. Hence, aggregate saving is $S_t = \frac{w_t L_t}{1+\beta^{\frac{-1}{\theta}}(1+r)^{1-\frac{1}{\theta}}}$, which after substituting the explicit expressions for w_t becomes

$$S_t = \frac{M_t(1-\alpha)L_t A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}}}{1+\beta^{\frac{-1}{\theta}}(1+r)^{1-\frac{1}{\theta}}} \tag{6}$$

The saving from labor income in (6) is allocated to three types of investment: buying patents over old varieties, inventing new varieties, and forming specialized machines. Hence aggregate investment in each period, I_t, satisfies

$$I_t = M_{t+1}\left[\eta_t + A^{\frac{1}{1-\alpha}} L_{t+1}\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right] \tag{7}$$

Notice that a higher population growth rate between period t and t + 1, has a direct positive effect on the demand for each machine variety - due to the increase in L. However, following (3), a higher population growth rate also increases the interest rate, which thereby increases machine prices and therefore decreases the demand for each machine variety. By equalizing (6) and (7), we impose the equilibrium condition $I_t = S_t$ to obtain the dynamic equation that

[11] Available at: http://cla.auburn.edu/econwp/Arc

governs the variety expansion rate:

$$1+g_y=\frac{(1-\alpha)L_tA^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}}}{\left[\eta_t+A^{\frac{1}{1-\alpha}}L_{t+1}\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right]\left[1+\beta^{\frac{-1}{\theta}}(1+r)^{1-\frac{1}{\theta}}\right]} \tag{8}$$

Plugging (2)and (3) in (8) yields

$$1+g_y=\frac{\left(\frac{\alpha(1-\alpha)}{\eta}+1\right)(1-\alpha)}{(\alpha+\eta)\left[1+\beta^{\frac{-1}{\theta}}\left[(1+n)\left(\frac{\alpha(1-\alpha)}{\eta}+1\right)\right]^{1-\frac{1}{\theta}}\right]} \tag{8a}$$

Proposition 2.1. *With no bequest motive, the sign of the weak scale effect depends on the IES* $\equiv \frac{1}{\theta}$*: for* $\frac{1}{\theta}<1(\frac{1}{\theta}>1)$ *there is negative (positive) weak scale effect i.e.* $\frac{\partial g_y}{\partial n}<0(\frac{\partial g_y}{\partial n}>0)$.

Proof. Proof is by inspection of equation (8a). □

The counterpart model with infinitely lived agents (presented in BS) yields no relation between population growth and economic growth regardless of the IES value. [12] In both models, population growth increases future demand for patented machines, thereby increasing the equilibrium interest rate. However, for the infinitely lived agents, population growth works also as a demographic discounting factor which discourages saving, and thus the two effects cancel out. In the OLG economy, population growth does not generate direct negative effects on saving and, due to the life-cycle structure of this framework, the effect of the increased interest rate on saving depends on the IES.

2.4 Bequests

In the presence of bequest saving motive, each young agent maximizes her lifetime utility (5), subject to the budget constraint: $w_t+\frac{b_{t-1}}{1+n}=c_{t,1}+\frac{c_{t,2}+b_t}{1+r}$. Applying this budget constraint to (5) we write the indirect utility function

$$u(s_t,w_t,b_{t-1},b_t,r)=\frac{(w_t+\frac{b_{t-1}}{1+n}-s_t)^{1-\theta}}{1-\theta}+\beta\left[\frac{[s_t(1+r)-b_t]^{1-\theta}}{1-\theta}+\kappa\frac{\left(\frac{b_t}{1+n}\right)^{1-\theta}}{1-\theta}\right] \tag{9}$$

[12] The growth equation for this model, defined by the regular Euler condition, are presented in the following section.

Differentiating (9) with respect to s and b we obtain the following first order conditions

$$s_t = \frac{w_t + \frac{b_{t-1}}{1+n}}{\frac{\beta^{\frac{-1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}{1+(1+n)^{\frac{\theta-1}{\theta}}\kappa^{\frac{1}{\theta}}}+1}$$

$$b_t = s_t \frac{1+r}{\frac{(1+n)^{\frac{1-\theta}{\theta}}}{\kappa^{\frac{1}{\theta}}}+1} \tag{10}$$

Optimal saving is still a fraction of the resources available to the young (worker), which now combine labor income and her inherited bequest. Hence, the operative bequest motive relaxes the former dependency of saving (and thereby investment and innovation rate) on labor income. Saving depends now not only on the interest rate and the IES, but also on the bequest motive parameter κ, through the expression $(1+n)^{\frac{\theta-1}{\theta}}\kappa^{\frac{1}{\theta}}$.

The effect of population growth rate on this expression (and thereby on saving) depends on the IES. Here, the population growth rate works as a depreciation rate that erodes the per-child bequest level. Hence, its effect is inverse to the effect of the interest rate. This effect has life-cycle saving properties due to the timing of parents utility from bequest-giving during the second period of life. The second factor has a positive effect on saving, due to the increased marginal utility from per-child bequest.

The optimal per-child bequest level is a certain fraction of capital income, $s_t(1+r)$. This fraction is a function of the population growth rate and the bequest motive. As explained above, the population growth rate erodes the per-child bequest level, and thus works like a decrease in the interest rate: as the utility from bequest takes place during retirement, the effect of lower return on the bequest per-child depends on the IES. The effect of the strength of bequest motive, κ, on per-child optimal bequest is positive.

The first condition in (10) implies that aggregate savings is given by

$$S_t = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}M_t L_t\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}} + B_{t-1}}{\frac{\beta^{\frac{-1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}{1+(1+n)^{\frac{\theta-1}{\theta}}\kappa^{\frac{1}{\theta}}}+1} \tag{11}$$

where $B_{t-1} = \frac{L_t b_{t-1}}{1+n}$, is aggregate bequests given to workers who were born in period t. Notice that for $\kappa = 0$ the aggregate saving level defined in (11) falls back to the one presented in (6). in

(6). The second condition in (10) implies that $B_{t-1} = S_{t-1}\frac{1+r}{(1+n)^{\frac{1-\theta}{\theta}}\kappa^{\frac{-1}{\theta}}+1}$, and the equilibrium condition $S_{t-1} = I_{t-1}$ requires

$$B_{t-1} = \frac{1+r}{(1+n)^{\frac{1-\theta}{\theta}}\kappa^{\frac{-1}{\theta}}+1}M_t\left(\eta_{t-1} + A^{\frac{1}{1-\alpha}}L_t\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}}\right) \tag{12}$$

Substituting (12), along with (3), back into (11) and equalizing to (7), i.e. setting $S_t = I_t$, we obtain

$$1+g_y = \frac{\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]\left[\frac{(1-\alpha)}{\alpha+\eta}(1+n)^{\frac{1-\theta}{\theta}} + \frac{(1+\eta)}{\alpha+\eta}\kappa^{\frac{1}{\theta}}\right]}{\beta^{\frac{-1}{\theta}}\left(\frac{\alpha(1-\alpha)}{\eta}+1\right)^{\frac{\theta-1}{\theta}} + \left[(1+n)^{\frac{1-\theta}{\theta}}\kappa^{\frac{1}{\theta}}\right]} \tag{13}$$

Proposition 2.2. *In the presence of bequest saving-motive, the sign of the weak scale effect,* $\frac{\partial g_y}{\partial n}$ *is positive (negative) for* $\theta > 1(\theta < 1)$ *and sufficiently strong (weak) bequest motive.*

Proof. Differentiating (13) for n reveals that, for $\theta > 1(\theta < 1)$, $\frac{\partial g_y}{\partial n}$ iff
$\beta^{-1}\left(\frac{1-\alpha}{\alpha+\eta}\right)^{\theta}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\theta-1} < \kappa\left(\beta^{-1}\left(\frac{1-\alpha}{\alpha+\eta}\right)^{\theta}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\theta-1} > \kappa\right)$.
For $\theta = 1$, we have $\frac{\partial g_y}{\partial n} = 0$ independent of κ. □

In the counterpart model of infinitely lived agents, presented in BS, households maximize per-capita utility of their dynasty members (following Millian preferences). Hence, aggregate consumption growth follows the standard Euler equation [13]: $\frac{\dot{C}}{C} = \frac{1}{\theta}(r-\beta)$, and per-capita consumption follows $\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\beta-n)$ where interest rate is given by [14] $r = n + \frac{\alpha(1-\alpha)}{\eta}$. Combining the two latter conditions yields the stationary growth rates for per-capita income: $g_{c,y} = \frac{1}{\theta}\left[\frac{\alpha(1-\alpha)}{\eta} - \beta\right]$. Hence, in the counterpart economy of infinitely lived homogeneous agents the IES plays no role in the presence (or sign) of the weak scale effect. Notice that, by Proposition 2, for $\theta = 1$ the weak scale effect is also muted in our model, for any κ.

In reference to the results obtained by DJ, it is worthwhile noting that under the technological parameters used in our model, they find a strong scale-effect will prevail for any $\theta \leq 1$.

[13] Equation (6.22) on p.295 there, in which the parameter ρ the time preference parameter (denoted here as β).
[14] Equation (6.35) on p. 302 there.

However, if $\theta < 1$ is sufficiently larger (smaller) than one, strong scale effect in their model will prevail only if κ is sufficiently large (small). [15]

2.5 Conclusion

This study highlights the implications of alternative demographic structures, and the saving motives they imply, to the presence of weak scale effect on R&D-based growth models. To this end, we have placed a basic variety-expansion textbook model (without human capital accumulation) in the overlapping-generations demographic framework, and showed how the interaction between the two alternative saving motives - life-cycle consumption smoothing and parental bequests - determine the sign of the weak scale effect. In particular, for the empirically valid degree of the IES, positive(negative) weak scale effect presents in the OLG economy only if parental-bequest saving motive is sufficiently strong (weak).

[15] See Theorem 1 and Corollary 2 (on pp. 1642 and 1643 respectively) there, for $\sigma = 1$ (by their notation), which is the elasticity of substitution between labor and capital in our model.

Chapter 3

Patents and Growth in OLG Economy

with Physical Capital

3.1 Introduction

There is a relatively large literature on the role of patent policy in modern growth theory and the implications of patent strength to R&D-based growth and welfare. The current literature, however, is almost exclusively written about models with infinitely lived agents. This paper utilizes an overlapping generations model to highlight some unique implications of finite lifetimes to patent policy.

In an economy of finitely lived agents, the limited longevity sets a barrier to growth by inducing integenerational trade in productive assets. This point was emphasized by Jones and Manuelli (1992) in a model of physical capital accumulation, and by Chou and Shy (1993) in an endogenous growth model of variety expansion with no physical capital. Both studies employed the canonical Overlapping Generations (OLG) model pioneered by Samuelson (1958) and Diamond (1965), where saving and investment are constrained by labor income. [1]

Jones and Manuelli (1992) showed that perpetual growth cannot prevail in the neoclassical OLG economy [2] due to the limited ability of the young to purchase capital held by the old. One of the remedies they consider to support sustained growth in such economy is direct income transfers from old to young. Chou and Shy (1993) emphasized that inter-generational trade in old patent slows down growth as investment in old patents crowds out innovative (R&D) investment in new varieties. They showed that due to this crowding-out effect, which is not present in infinitely-lived agent economy, shortening patent length enhances growth.

To the best of our knowledge, Sorek (2011) is the only other work to study the growth implications of patents in the OLG framework. However, this work focuses on the effect of

[1] More generally, in economies with finitely lived agents the accumulation of assets is limited by the agent's consumption horizon (longevity).

[2] In other words, the perpetual accumulation of physical capital per-capita.

patents' breadth and length on quality growth (i.e. vertical innovation), where differentiated consumption goods are only produced with labor (i.e. there is no physical capital as in Chou and Shy 1993). In Sorek's (2011) setup, the effect of patent policy on growth depends crucially on the elasticity of inter-temporal substitution, through the effect of the interest rate on life-cycle saving in the OLG model. This effect plays no role in the current analysis (proof is available upon request).

The present work studies an OLG economy that incorporates both variety expansion and physical capital accumulation, to highlight a unique mechanism through which loosening patents' strength spurs growth. Our analysis places the variety-expansion model proposed by Rivera-Batiz and Romer (1991),[3] into the canonical OLG demographic framework of Samuelson (1958) and Diomaond (1965). Previous works on Rivera-Batiz and Romerís (1991) model economy with infinitely lived agents concluded that growth is maximized with complete patent protection, that is,infinite patent length and complete patent breadth; See Kwan and Lai (2003), Cysne and Turchick (2012), and Zeng et al. (2014). [4],[5]

In order to isolate the main effect under study from the aforementioned crowding-out effect [6], we first show that under infinite patent length growth is maximized with incomplete patent breadth. The mechanism at work behind this result involves the trade-off between the static and dynamic effects faced by the patents policy maker. Weakening patent breadth protection works to lower the price of patented machines (by weakening sellers' market power), which in turn increases demand for machines. With more machines being utilized, output and labor income are higher, thus increasing aggregate saving and investment. This is the positive static effect of

[3]Barro and Sala-i-Martin (2004) and Aghion and Howitt (2008) adopted this framework as the textbook model of variety expansion; See chapters 6 and 8, respectively.

[4]These studies differ only in their modelling approach of patent policy. The first two model patent policy through constant imitation rate, which can be also interpreted as stochastic patent duration, as will be explained in Section 4. The last study models patent policy along two (more natural) dimensions: deterministic patent duration and price regulation which is, technically, equivalent to our modeling approach of patent breadth. All these works assume the differentiated inputs are intermediate goods that are formed are formed in the same period they are being used, whereas we consider the differentiated inputs as investment goods (i.e. physical capital) that are formed one period ahead of utilization. Nonetheless, for the infinitely lived agents this assumption does not effect the implications of patent breadth for growth.

[5]In another related work, Iwaisako and Futagami (2013) study the implications of patent policy for growth in a model of infinitely lived agents with physical capital. However, the role of physical capital is completely different than in the present analysis. They use homogenous (raw) physical capital, along with labor, as an input in the production of differentiated consumption goods - to which patent policy applies.

[6]The weakening of breadth protection over all patents evenly (as considered here), does not reduce the crowding out effect induced by intergenerational trade in old patents.

loosening patent breadth protection on growth. [7] However, higher demand for machines shifts investment away from patents and innovation toward physical capital. This is the negative dynamic effect of weakening patent breadth protection on growth.

We show that the growth maximizing patent breadth depends negatively on the depreciation rate of capital due to the effect of the latter on machines' price. The lower the depreciation rate, the lower the price of physical capital and, therefore, the higher is the demand for physical capital. With initial lower machine prices, there is less potential for growth enhancement through further price decrease induced by loosening patent protection.

The effect of patent policy on growth we are highlighting here is not present in the counterpart model of infinitely living agents, where saving is not bounded by labor income. With infinitely lived agents, the growth rate is determined by the standard Euler condition, [8] and thus the effect of patent protection strength on growth works solely through its positive impact on the returns to innovation and, thereby, the interest rate.

Next, we show that, for any positive depreciation rate on physical capital, shortening patent length is more effective in spurring growth than loosening patent breadth protection. Shortening patent length triggers the mechanism presented above while mitigating the crowding out effect as in Chou and Shy (1993).[9] Shortening patent length induces the same effect as loosening patent breadth protection by lowering the average price of machine varieties. The expiration of patent over a certain specialized machine results in competition among imitators of this specific variety, which brings its price down to marginal cost. Shorter patent length increases the fraction of competitive machine-industries, thus lowering average machines' price. Compared with Chou and Shy (1993) and Sorek (2011), who found that one-period patent length yields higher growth then infinite patents protection in OLG economy with no physical capital, we also find that one period patent length never maximizes growth in our model economy.

[7] Since the old are the patent owners, this effect of weakening patent breadth protection is similar to income transfers from the old to the young considered by Jones and Manuelli (1992). Similarly, Uhlig and Yanagawa (1996) showed that reliance on capital-income taxation can also enhance growth.

[8] The familiar Euler condition is given by $\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$, where c is per-capita consumption, θ is the intertemporal elasticity of substitution, ρ is the time preference parameter and r is the interest rate. See for example equations (3),(14) and (15), in Zeng et al. (2014).

[9] This crowding-out reduction could be also achieved by weakening patent breadth protection gradually along patents' lifetime. Either way the market value of an old patent will decrease, freeing investment resources for R&D activity.

Our welfare analysis shows that enhancing growth by loosening patent breadth protection is preferred by all generations if time preference and the degree of substitution across machine varieties are sufficiently low. This result concurs with Chou and Shy's (1993) analysis of the welfare implications of patent length in their OLG model economy. [10]

Finally, in the last section of the analysis, we present an implication of our main finding for patent policy and economic development. We show that when labor productivity increases relative to innovation cost, due to human capital accumulation, the growth maximizing patent strength corresponds to labor productivity. Hence, as the economy develops, the growth maximizing patent strength is increasing as well. This result provides a normative case for the documented positive correlation between the strength of intellectual property rights (IPR) and economic development worldwide (See Eicher and Newiak 2013, and Chu et al. 2014).

Chu et al. (2014) presented the first analysis of stage-dependent optimal IPR, based on a tradeoff between imitation from foreign direct investment (FDI) and reliance on domestic innovation. Our last result provides a complementary case for growth enhancing stage-dependent IPR policy for a closed economy (which is independent of the imitation motive). In an earlier analysis of the topic, Diwakar and Sorek (2016) provide evidence that major developing economies strongly restrict (physical) capital inflows.

The paper proceeds in a straightforward manner. Section 3.2 presents the model. Section 3.3 studies the implications of alternative patent policies to growth and welfare. Lastly, Section 3.4 concludes.

3.2 The Model

Our model uses the variety expansion model with lab-equipment innovation technology and differentiated capital goods proposed by Rivera-Batiz and Romer (1991) together with Diamond's (1965) canonical OLG demographic structure. Each period two overlapping generations of measure L; the "young" and the "old", are economically active. Each agent is endowed with one unit of labor to be supplied inelastically when young. Old agents retire and consume their saving.

[10] See Propositions 3-4 on page 310 there.

The benchmark model presented in this section assumes full patent protection - i.e. infinite patent duration and complete patent breadth protection, implying that in any period innovators can charge the unconstrained monopolistic price for their patented machines. We study the implications of incomplete patent protection in Section 3.3.

3.2.1 Production and Innovation

The final good Y is produced by perfectly competitive firms with labor and differentiated capital goods, to which we refer also as “specialized machines”.

$$Y_t = AL_t^{1-\alpha} \int_0^{M_t} K_{i,t}^{\alpha} \, \mathrm{d}i \tag{1}$$

where $0 < \alpha < 1$ A is a productivity factor, L is the constant labor supply, $K_{i,t}$ is the utilization level of machine-variety i in period t, respectively, and M_t measures the number of available machine varieties. [11] Machines are subject to the depreciation rate $\delta \in (0,1)$ per usage-period, and the price of the final good is normalized to one. Under symmetric equilibrium, utilization level for all machines is the same, i.e. $K_{i,t} = K_t \forall i$, and thus total output is

$$Y_t = AM_t {K_t}^{\alpha} L_t^{1-\alpha} \tag{1a}$$

The representative (perfectly-competitive) firm in the final-good production sector employs specialized machines at the rental price p_i and labor at the market wage w, in order to maximize the profit function

$$\pi_{i,t} = AL_t^{1-\alpha} \int_0^{M_t} K_{i,t}^{\alpha} \, \mathrm{d}i - \int_{i=1}^{M_t} p_{i,t} K_{i,t} \, \mathrm{d}i - w_t L_t \tag{2}$$

The labor market is perfectly competitive and the equilibrium wage and aggregate labor income are $w_t = A(1-\alpha) M_t {K_t}^{\alpha} L_t^{-\alpha}$ and $w_t L_t = A(1-\alpha) M_t {K_t}^{\alpha} L_t^{1-\alpha}$, respectively.ctively.

[11] The elasticity of substitution between different varieties is $\frac{1}{\alpha}$.

The profit maximization with respect to each machine variety yields the familiar demand function: $K^d_{i,t} = A^{\frac{1}{1-\alpha}} L_t \left(\frac{\alpha}{p_{i,t}}\right)^{\frac{1}{1-\alpha}}$. Assuming symmetric equilibrium prices and plugging the latter expression back into (1a) we obtain

$$Y_t = A^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{p_t}\right)^{\frac{\alpha}{1-\alpha}} M_t L \tag{2a}$$

Innovation technology follows lab-equipment specification, and the cost of a new blue print is η output units.

3.2.2 Preferences

Lifetime utility of the representative agent born in period t is derived from consumption (denoted by c) over two periods, based on the logarithmic instantaneous-utility specification [12]

$$U_t = \ln c_{1,t} + \rho \ln c_{2,t} \tag{3}$$

where ρ is the subjective discount factor. Young agents allocate their labor income between consumption and saving (denoted by s). The solution for the standard optimal saving problem is $s_t = \frac{w_t}{1+\rho^{-1}}$. Hence, aggregate saving is $S_t = \frac{w_t L}{1+\rho^{-1}}$, which after substituting the explicit expression for w_t becomes

$$S_t = \frac{(1-\alpha) A^{\frac{1}{1-\alpha}} M_t L \left(\frac{\alpha}{p}\right)^{\frac{\alpha}{1-\alpha}}}{1+\rho^{-1}} \tag{4}$$

3.2.3 Equilibrium and Growth

The patent owners of each machine variety borrow raw physical capital from savers/lenders at the rate $(\delta + r_t)$, where r_t is the net interest. They then transform each unit of raw capital into one specialized machine, at no cost, and the specialized machines are then rented to final output producers at the price p. Hence, given the demand for each machine as previously specified, the per-period surplus from each patented machine, denoted PS, is: $PS_{i,t} = [p_{i,t} - (\delta + r_t)] K^d_{i,t}$.

[12] It is well known that under the assumed demographic structure, the logarithmic instantaneous utility implies that the saving (and investment) level is independent of the interest rate. We have also considered the implications of the general CEIS preference form. The proof is available upon request.

This surplus is maximized by the standard monopolistic price $p_{i,t} = \frac{\delta + r_t}{\alpha}$. Under infinite patent duration, all new and old varieties are priced equally and, therefore, share the same utilization level. As long as innovation takes place, the market value of old varieties equals the cost of inventing a new one, η. The gross rate of return on investment in patents is given by $1 + r_t = \frac{PS+\eta}{\eta}$. Notice that the numerator in the interest expression contains η because each and every period all patents held by old agents are sold to the young agents.

Plugging the explicit term for the surplus into the interest rate expression yields an implicit expression for the equilibrium interest:

$$\forall t : 1 + r = \frac{[p_{i,t} - (\delta + r_t)]K^d_{i,t} + \eta}{\eta} = \frac{(\delta + r)^{\frac{-\alpha}{1-\alpha}}(\frac{1}{\alpha} - 1)A^{\frac{1}{1-\alpha}}L\alpha^{\frac{2}{1-\alpha}} + \eta}{\eta} \tag{5}$$

Equation (5) also defines the no-arbitrage condition that equalizes the net rate of return on investment in patents and investment in physical capital.

Lemma 3.1. *There exists a unique stationary interest rate, r^*, which solves (5).*

Proof. The left hand side of (2) is increasing linearly in r , from one (for r = 0) to infinity. The right hand side of (5) is decreasing in r from $\frac{\delta^{\frac{-\alpha}{1-\alpha}}(\frac{1}{\alpha}-1)A^{\frac{1}{1-\alpha}}L\alpha^{\frac{2}{1-\alpha}}+\eta}{\eta} > 1$ (for r=0) to one (for r $\to \infty$). Hence, by the Intermediate Value theorem, there exists a positive stationary interest rate r^* that solves (5). □

As the right hand side of (5) is decreasing with the depreciation rate, so does the equilibrium interest rate, that is $\frac{\partial r^*}{\partial \delta} < 0$. For the case $\delta = 0$, equation (5) yields an explicit solution for the stationary equilibrium rate:

$$r = \left[\left(\frac{1}{\alpha} - 1\right)\frac{L}{\eta}\right]^{1-\alpha} A\alpha^2, \qquad for\ \delta = 0, \forall t. \tag{5a}$$

Under the equilibrium interest rate, aggregate saving is allocated over investment in old and young patents and in physical capital, where the investment in physical capital is set to meet the demand for specialized machines

$$I_t = M_{t+1}\left[\eta + A^{\frac{1}{1-\alpha}}L\left(\frac{\alpha^2}{r+\delta}\right)^{\frac{1}{1-\alpha}}\right] \tag{6}$$

Equation (5) implies that machines prices are stationary: $\forall t, i\ :\ p_{i,t} = \frac{\delta+r^*}{\alpha}$. Hence, Following (2), the output growth rate, denoted $g_{Y,t+1} \equiv \frac{Y_{t+1}}{Y_t} - 1$, which coincides with per-capita output growth [13], is equal to the rate of machine-varieties expansion, i.e. $g_{Y,t+1} \equiv \frac{Y_{t+1}}{Y_t} - 1 = g_{M,t+1} \equiv \frac{M_{t+1}}{M_t} - 1$. Imposing the equilibrium condition S = I, we equalize (4) and (6), to derive the stationary rate of variety expansion which defines the output growth rate:

$$1 + g_y = \frac{\frac{1-\alpha}{1+\rho^{-1}} \left(\frac{\alpha^2}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}}{\hat{\eta} + \left(\frac{\alpha^2}{r+\delta}\right)^{\frac{1}{1-\alpha}}} \tag{7}$$

where $\hat{\eta} = \frac{\eta}{A^{\frac{1}{1-\alpha}} L}$. For sufficiently low $\hat{\eta}$ the growth rate defined in (7) is positive, and we will generally assume that this the case. To see that, notice that as $\hat{\eta}$ approaches zero, the right hand side in (7) approaches $\frac{1-\alpha}{1+\rho^{-1}} \frac{r+\delta}{\alpha^2}$. However, by (5), as $\hat{\eta}$ approaches zero, the interest rate r approaches infinity, and thus $\lim_{\hat{\eta} \to 0} \frac{1-\alpha}{1+\rho^{-1}} \frac{r+\delta}{\alpha^2}$ approaches infinity as well.

3.3 Patents

We now explore the implications of patent policy to growth and welfare. The growth implications of patent breadth protection under infinite patent length are studied first. We then demonstrate the greater effectiveness of finite patent length in spurring economic growth. Lastly, we examine welfare and the issue of stage-dependent patent policy.

3.3.1 Patent Breadth and Growth

We model patent breadth protection with the parameter λ, which limits the ability of patent holders to charge the unconstrained monopolistic price: $p(\lambda) = \lambda p^* = \frac{\delta+r_t}{\alpha}$ where $\lambda \in [\alpha, 1]$, and thus $p(\lambda) \in [\delta + r_t, \frac{\delta+r_t}{\alpha}]$. One can think of $p(\lambda)$ as the maximal price a patent holder can set and still deter competition by imitators. Weaker breadth protection lowers the cost of imitation, thereby imposing a lower deterrence price on patent holders. [14] When $\lambda = 1$, patent breadth protection is complete and patent holders can charge the unconstrained monopolistic

[13] As both total population and the labor force are constant.

[14] Similar modeling approach for patent breadth protection was used (among others) by Goh and Olivier (2002), Iwaisako and Futagami (2013), and Chu et al. (2016).

price $p = \frac{\delta + r_t}{\alpha}$. With zero protection $\lambda = \alpha$, patent holders lose their market power completely and sell at marginal cost. Note that as patent breadth protection is weakened, machines' price is reduced and demand for each machine-variety is increasing. Under this patent breadth policy, the equilibrium interest rate in equation (5) modifies to

$$\forall t : 1 + r = \frac{(\delta + r)^{\frac{-\alpha}{1-\alpha}} (\frac{\lambda}{\alpha} - 1) A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}} + \eta}{\eta} \tag{8}$$

$$For\ \delta = 0, \forall t : r = \left[\left(\frac{\lambda}{\alpha} - 1\right) \frac{L}{\eta}\right]^{1-\alpha} A \left(\frac{\alpha^2}{\lambda}\right) \tag{8a}$$

Lemma 3.2. *The stationary equilibrium interest rate r^* is increasing with patent breadth protection, i.e.* $\frac{\partial r^*}{\partial \lambda} > 0$

Proof. Differentiating the right hand side for λ yields a positive derivative for any $\lambda < 1$. Hence, the value of r^*, which solves (8), is increasing with patent breadth protection λ. □

Lemma 3.2 implies that loosening patent breadth protection decreases machines' price, $p(\lambda) = \frac{\lambda(\delta + r^*)}{\alpha}$, through capping the monopolistic markup and by decreasing the marginal cost (of capital) on which this mark up builds. Thus, loosening patent breadth protection increases the demand for each machine variety. This increase in demand for machines has positive effect on aggregate saving (4), for a given variety span:

$$S_t = \frac{(1-\alpha) A^{\frac{1}{1-\alpha}} M_t L \left(\frac{\alpha}{p(\lambda)}\right)^{\frac{\alpha}{1-\alpha}}}{1 + \rho^{-1}}$$

This is the positive static effect of loosening patent breadth protection on aggregate saving (for a given variety span M_t) and, thereby, innovation and growth. However, for a given level of saving, the increased demand for machines works to shift investment toward physical capital and away from patents. This is the dynamic negative effect of loosening patent breadth protection on innovation and growth. From equation (6) we have:

$$I_t = M_{t+1} \left[\eta + A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha}{p(\lambda)} \right)^{\frac{1}{1-\alpha}} \right]$$

Plugging $p(\lambda) = \frac{\lambda(\delta + r^*)}{\alpha}$ in the above saving and investment expressions and imposing the aggregate constraint S =I we obtain,

$$1 + g_y = \frac{1-\alpha}{1+\rho^{-1}} \frac{\psi^{\frac{\alpha}{1-\alpha}}}{\hat{\eta} + \psi^{\frac{1}{1-\alpha}}} \tag{9}$$

where $\hat{\eta} = \frac{\eta}{A^{\frac{1}{1-\alpha}} L}$, and $\psi \equiv \frac{\alpha^2}{\lambda(\delta + r^*)}$. Finally, we denote the growth maximizing policy by λ^{**}.

Proposition 3.1. *For any positive depreciation rate, the growth-maximizing patent breadth is positive but incomplete, and is decreasing with depreciation rate. That is* $\forall \delta > 0 : \alpha < \lambda^{**} < 1$ *and* $\frac{\partial \lambda^{**}}{\partial \delta} < 0$.

Proof. Differentiating (9) for reveals that the growth rate is increasing with ψ , if $\frac{\alpha}{1-\alpha}\hat{\eta} > \psi^{\frac{1}{1-\alpha}}$, that is $\frac{\alpha}{1-\alpha}\hat{\eta} > \left[\frac{\alpha^2}{\lambda(\delta + r^*)} \right]^{\frac{1}{1-\alpha}}$. Hence, under this condition the growth rate is decreasing in λ. Plugging the interest rate in (8a), i.e. for $\delta = 0$, in the latter condition yields equality for $\lambda = 1$, implying that this condition holds any $\delta > 0$ (evaluated at $\lambda = 1$). Hence, $\lambda^{**} < 1$. For $\lambda = \alpha$, equation (8) yields $r^* = 0$, $\forall \delta$. Then, setting $r^* = 0$ in (9) yields negative growth rate, $g_y < 0$, $\forall \delta$. Hence, $\alpha < \lambda^{**} < 1$. Because the interest rate is increasing with δ, the degree of patent breadth protection that maximizes growth, to satisfy $\left[\frac{\alpha^2}{\lambda^{**}(\delta + r^{**})} \right]^{\frac{1}{1-\alpha}} = \frac{\alpha}{1-\alpha}\hat{\eta}$, is decreasing with the depreciation rate, i.e. $\frac{\partial \lambda^{**}}{\partial \delta} < 0$. □

Proposition 3.2. *The maximal growth rate that can be achieved with incomplete patent breadth protection is unique:* $1 + g_y^{**} = \frac{\alpha^{\alpha}(1-\alpha)^{2-\alpha}}{(1+\rho^{-1})\hat{\eta}^{1-\alpha}}$

Proof. By setting $\psi = \psi^{**} \equiv \left(\frac{\alpha}{1-\alpha}\hat{\eta} \right)^{1-\alpha}$ in the growth equation (9). □

3.3.2 Patent Length and Growth

We turn to study the implications of patent length for growth, under complete patent breadth protection. We study stochastic patent length, assuming that each period a fraction

$1-\pi$ of the existing patents expire, where $\pi \in (0,1)$. [15],[16] However, all new patents are certain to grant patent for one period (which will expire with probability $1-\pi$ in the second period). This means that the actual lifetime of a patent, denoted T, is the value of $E(T) = 1+\frac{\pi}{1-\pi}$ for all new and old patented technologies. Under this specification, the stationary fraction of patented industries, μ, is

$$\mu = \frac{g}{1+g-\pi} \Rightarrow 1-\mu = \frac{1-\pi}{1+g-\pi} \tag{10}$$

Applying (10) to (2a) we write the modified output equation:

$$Y_t = A^{\frac{1}{1-\alpha}} LM_t \left[\frac{g}{1+g-\pi}\left(\frac{\alpha^2}{\delta+r_t}\right)^{\frac{\alpha}{1-\alpha}} + \frac{1-\pi}{1+g-\pi}\left(\frac{\alpha}{\delta+r_t}\right)^{\frac{\alpha}{1-\alpha}} \right] \tag{11}$$

Aggregate saving is still a constant fraction of total output: $S_t = \frac{(1-\alpha)}{1+\rho^{-1}} Y_t$, and the modified investment equation is

$$I_t = M_{t+1}\left[\frac{g}{1+g-\pi}\eta + \frac{g}{1+g-\pi} A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\delta+r_{t+1}}\right)^{\frac{1}{1-\alpha}} + \frac{1-\pi}{1+g-\pi} A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha}{\delta+r_{t+1}}\right)^{\frac{1}{1-\alpha}} \right] \tag{12}$$

Imposing $I_t = S_t$ yields the following implicit equation for the stationary-growth rate:

$$1+g = \frac{\frac{(1-\alpha)}{1+\rho^{-1}}\left(\frac{\alpha^2}{\delta+r}\right)^{\frac{\alpha}{1-\alpha}}\left(1+\frac{1-\pi}{g}\alpha^{\frac{-\alpha}{1-\alpha}}\right)}{\hat{\eta} + \left(\frac{\alpha^2}{\delta+r}\right)^{\frac{1}{1-\alpha}}\left(1+\frac{1-\pi}{g}\alpha^{\frac{-1}{1-\alpha}}\right)} \tag{13}$$

This equation has only one positive root, and for $\pi = 1$ it coincides with (7). The interest rate under the current patent policy is given by

$$\forall t : 1+r = \frac{(\delta+r)^{\frac{-\alpha}{1-\alpha}}(\frac{1}{\alpha}-1)\alpha^{\frac{2}{1-\alpha}} + \pi\hat{\eta}}{\hat{\eta}} \tag{14}$$

[15] This formulation has two practical advantages. First, it is a continuous policy instrument although time in this model is discrete. Second, it greatly enhances tractability.

[16] This modelling approach follows Helpman (1993), Kwan and Lai (2003), and Rubens and Turchick (2012). Their original interpretation was that a fraction π of the patented technologies are being imitated due to a lack of patent protection enforcement.

The stationary equilibrium interest rate that satisfies (14), r^*, is increasing with the stochastic patent length π, and for $\pi = 1$ it coincides with (8).

Remark 3.1. *Setting $\pi = (1-\delta)$ in (14) yields $\delta + r^* = \left(\frac{1-\alpha}{\alpha\hat{\eta}}\right)^{1-\alpha}\alpha^2$. Thus, by Proposition 2 we have: $\psi(\pi = 1-\delta, \lambda = 1) = \psi^{**}(\pi = 1, \lambda^{**}) \equiv \left(\frac{\alpha}{1-\alpha}\hat{\eta}\right)^{1-\alpha}$.*

Applying the implicit function theorem to (13) we obtain the following expression for $\frac{dg}{d\pi}$:

$$\frac{\frac{1-\alpha}{1+\rho^{-1}}\psi^{\frac{\alpha}{1-\alpha}}\left[\frac{\alpha}{1-\alpha}\psi^{-1}|\frac{\partial\psi}{\partial\pi}|\left(1+\frac{1-\pi}{g}\alpha^{\frac{-\alpha}{1-\alpha}}\right)-C\right]-(1+g)\psi^{\frac{1}{1-\alpha}}\left[\frac{1}{1-\alpha}\psi^{-1}|\frac{\partial\psi}{\partial\pi}|\left(1+\frac{1-\pi}{g}\alpha^{-\frac{1}{1-\alpha}}\right)-C\right]}{B+\left[\frac{(1-\alpha)}{1+\rho^{-1}}\psi^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{-\alpha}{1-\alpha}}(\frac{1-\pi}{g^2})\right]-(1+g)\left[\psi^{\frac{1}{1-\alpha}}\alpha^{\frac{-1}{1-\alpha}}(\frac{1-\pi}{g^2})\right]} \tag{15}$$

Where B is the denominator in the right hand side of (13) and $C = \frac{\alpha^{-\frac{1}{1-\alpha}}}{g}$. Based on the above remark and equation (15), we obtain the following proposition.

Proposition 3.3. *For any positive depreciation rate, finite patent length can yield higher growth than incomplete patent breadth protection.*

Proof. Substituting $\psi = \psi^{**}$ into (13) yields the growth rate obtained in Proposition 2. That is for $\pi = 1-\delta$ and $\lambda = 1$: $1 + g_y^{**} = \frac{1-\alpha}{1+\rho^{-1}}\frac{\alpha^\alpha(1-\alpha)^{2-\alpha}}{\hat{\eta}^{1-\alpha}}$. Then, substituting $\psi = \psi^{**}$ and $g = g_y^{**}$ into (15) reveals that, for any positive depreciation rate, both the numerator and denominator are negative, and for zero depreciation rate the numerator also equals zero (while the denominator remains negative). That is $\forall\delta > 0 : \frac{dg}{d\pi}|_{\pi=1-\delta} > 0$, and for $\delta = 0 : \frac{dg}{d\pi}|_{\pi=1-\delta} = 0$. Hence, for any positive depreciation rate, growth under finite patent length can be enhanced beyond the maximal rate defined in Proposition 2, by marginal increase in expected patent length. That is, $\pi^{**} < 1-\delta$, and thus $E(T^{**}) > 1 + \frac{1-\delta}{\delta}$. For zero depreciation physical capital growth is maximized with infinite patent length. □

3.3.3 Patents and Welfare

This section will briefly explore the welfare implications of loosening patent protection. To maintain tractability we focus on patent breadth protection. We follow Chou and Shy (1993) in comparing the lifetime utility of all living generations under alternative stationary patent policies. Substituting the explicit expressions for c_1 and c_2, implied by the saving equations in Subsection 3.2.2, into the lifetime utility function (3) yields the indirect lifetime utility of the representative consumer who was born in period t:

$$U_t = \ln\left[\frac{(1-\alpha)M_t A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{p}\right)^{\frac{\alpha}{1-\alpha}}}{1+\rho}\right] + \rho\ln\left[\frac{\rho(1-\alpha)M_t A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{p}\right)^{\frac{\alpha}{1-\alpha}}}{1+\rho}(1+r^*)\right] \qquad (16)$$

Equation (16) implies that $U_t = U_{t-1} + (1+\rho)\ln(1+g)$, and thus

$$U_t = U_0 + t(1+\rho)\ln(1+g) \qquad (16a)$$

where U_0 is given by evaluating (16) for M_0. By Propositions 1, for any positive depreciation rate, the stationary growth rate g is maximized with incomplete breadth protection. Hence, by (16), a sufficient condition for incomplete breadth protection to increase the lifetime utility of all generations who are born in the present and future periods, is having $\frac{\partial U_0}{\partial \lambda}|_{\lambda=1} < 0$. The derivative $\frac{\partial U_0}{\partial \lambda}$ depends has the following expression:

$$\left[\frac{\rho}{1+r} - \frac{\alpha}{1-\alpha}\frac{1+\rho}{(1+r^*)}\right]\frac{\partial r}{\partial \lambda} - \frac{\alpha}{1-\alpha}\frac{(1+\rho)}{\lambda} \qquad (16b)$$

However, Lemma 3.2 implies $\frac{\partial r^*}{\partial \lambda} > 0$. Hence, having a negative term in the brackets of (16b) is sufficient condition to assure $\frac{\partial U_0}{\partial \lambda} < 0$. This sufficient condition holds if $\frac{\rho}{1+\rho} < \frac{\alpha}{1-\alpha}$. [17] This result is summarized in our next proposition.

Proposition 3.4. *Incomplete patent breadth protection benefits all generations if time preference and the degree of substitution across varieties are sufficiently low, such that* $\frac{\rho}{1+\rho} < \frac{\alpha}{1-\alpha} \Leftrightarrow \frac{1}{\alpha} < \frac{1}{p} + 2$.

Proposition 4, which relies on comparison between two alternative stationary policies concurs with the results obtained in the welfare analysis of Chou and Shy (1993). However, the direct transitional impact of loosening patent breadth policy at a certain period will not yield

[17] Recall that the elasticity of substitution across varieties is $\frac{1}{\alpha}$

Pareto improvement even if the above proposition holds. At period zero, the amount of available machines is already pre-determined, and thus, decreasing their price can not increase their utilization level.

Hence, the positive effect on aggregate saving will not prevail, and only the negative effect on second-period consumption (due to lower interest rate) will be at work. Therefore, in this case transfers from the next young generation (to be born in period one) to the current young generation will be required to maintain Pareto improvement. However, the complete analysis of this issue falls beyond the scope of the current study.

3.3.4 Stage-Dependent Patent Policy

Proposition 2 implies that the growth maximizing patent policy depends on the value of $\hat{\eta} \equiv \frac{\eta}{A^{\frac{1}{1-\alpha}} L}$. This term can be interpreted as innovation-cost per effective labor supply, denote $H \equiv A^{\frac{1}{1-\alpha}} L$ [18]. However, the value of these parameters may be associated with the economy's development stage. Labor productivity is typically increasing along the course of economic development through the accumulation of human capital. Similarly, the literature on R&D driven growth has considered alternative endogenous dynamics of the innovation cost due to inter-temporal knowledge spillover. This works to decrease per-variety invention cost and the stepping on toes ("fishing out") effect (which works to increase invention cost as R&D efforts increase), and distance from the frontier. [19]

Proposition 3.2 suggests that the growth-maximizing patent protection is decreasing with $\hat{\eta}$. In this subsection, we attempt to formalize this result. Adding the time subscript to the relevant parameters, we re-write the output and growth equation

$$Y_t = M_t H_t \left[\frac{\alpha}{p_t(\lambda_t)} \right]^{\frac{\alpha}{1-\alpha}} \tag{17}$$

[18] If $A^{\frac{1}{1-\alpha}}$ is interpreted as labor augmented productivity factor we can write (1) as: $Y = MK^{\alpha} \left(A^{\frac{1}{1-\alpha}} L \right)^{1-\alpha}$

[19] See Jones (1995) and Segerstrom (1998). Jones (1999) provides a compact summary of the topic.

$$1+g_{M,t+1}=\frac{(1-\alpha)}{1+\rho^{-1}}\frac{H_t\left[\frac{\alpha}{p_t(\lambda_t)}\right]^{\frac{\alpha}{1-\alpha}}}{\eta+H_{t+1}\left[\frac{\alpha}{p_{t+1}(\lambda_{t+1})}\right]^{\frac{1}{1-\alpha}}} \tag{17a}$$

where $p_t(\lambda_t)=\frac{\lambda_t(\delta+r-t)}{\alpha}$, as before, and the interest rate follows the modified no-arbitrage condition

$$1+r^*_{t+1}=\frac{(\delta+r^*_{t+1})^{\frac{-\alpha}{1-\alpha}}(\frac{\lambda_{t+1}}{\alpha}-1)\lambda_{t+1}^{\frac{-1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}+\hat{\eta}_{t+1}}{\hat{\eta}_{t+1}} \tag{17b}$$

Equation (17) implies the following growth rate of per-capita output.

$$1+g_{y,t+1}=(1+g_{M,t+1})(1+g_{H,t+1})(1+g_{p(\lambda),t+1})^{\frac{\alpha}{1-\alpha}} \tag{18}$$

Combining equations (18) with (17a) yields

$$1+g_{y,t+1}=\frac{(1-\alpha)}{1+\rho^{-1}}\frac{\psi_{t+1}^{\frac{\alpha}{1-\alpha}}}{\hat{\eta}_{t+1}+\psi_{t+1}^{\frac{\alpha}{1-\alpha}}} \tag{18a}$$

Notice that the growth equation (18a) depends only on the patent policy expected to prevail in period t+1.

Proposition 3.5. *The growth-maximizing policy, $\psi_t^{**}\equiv\left(\frac{\alpha}{1-\alpha}\hat{\eta}_t\right)^{1-\alpha}$, implies that patent breadth is decreasing with $\hat{\eta}_{t+1}$.*

Proof. Proposition 3.1 implies that this growth rate is maximized with $\psi_{t+1}^{**}\equiv\left(\frac{\alpha}{1-\alpha}\hat{\eta}_{t+1}\right)^{1-\alpha}$. Rewriting the interest rate expression (17b) we obtain: $\left(\frac{r^*_{t+1}}{\delta+r^*_{t+1}}\frac{\alpha}{\lambda_{t+1}-\alpha}\hat{\eta}_{t+1}\right)^{1-\alpha}=\frac{\alpha^2}{\lambda_{t+1}(\delta+r^*_{t+1})}$. Hence, the growth-maximizing condition is satisfied with $\frac{r^*_{t+1}}{\delta+r^*_{t+1}}=\frac{\lambda_{t+1}-\alpha}{1-\alpha}$. Clearly, for $\delta=0$ growth is maximized by the stationary policy of complete breadth protection. However, for any $\delta>0$, as $\hat{\eta}_{t+1}$ is decreasing (increasing), the left hand side of the latter condition is also increasing. Then, in order to restore the equality patent breadth protection should also increase (decrease). Then, by strengthening patent breadth protection, the right hand side is increasing while the left hand side is increasing at a lower rate. □

3.4 Conclusion

This work proposes a contribution to the literature on patent policy and economic growth by exploring the implications of patent policy in an OLG framework with physical capital. We have highlighted a novel mechanism through which weakening patent protection can enhance growth, which is unique to the OLG demographic structure of finitely lived agents. This mechanism involves a trade-off between the effect of patent strength on aggregate saving and investment and the allocation of total investment between patent ownership and physical capital. This positive effect can be induced by either shortening patent length or loosening patent breadth protection. However, shortening patent length also mitigates the crowding out effect of trade in old patents on R&D investment. Hence, shortening patent length can be more effective at generating growth than loosening patent breadth protection. These effects are not present in similar models with infinitely lived agents. Consequently, growth in these models is maximized with eternal patent life and complete patent breadth protection.

Finally, we have also presented an important implication of the main mechanism under study to patent policy and economic development. A stage-dependent patent policy for which patent strength is increasing over the course of economic development may be growth maximizing. This result provides a normative case for the often observed positive correlation between patent strength and economic development around the world.

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