# The Complete Solution of the Intersection Problem for Maximum Packings of $K_n$ with Triples

by

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Abstract

In the late 1980's the intersection problem for maximum packings of $K_n$ with triples was solved by Hoffman, Lindner, and Quattrocchi. Their combined results showed that for any $n \equiv i \pmod 6$ such that $i \in \{0, 2, 4, 5\}$ the intersection spectrum is $I(n) = \{0, 1, \ldots, x\} \backslash \{x-1, x-2, x-3, x-5\}$ where $x$ is the size of a maximum packing. Each result was formed when all leaves are the same. However, in this thesis we show that if the leaves are not necessarily the same we can eliminate the exceptions $\{x-1, x-2, x-3, x-5\}$ of the given results. We show that the intersection spectrum for $n \equiv i \pmod 6$ such that $i \in \{4, 5\}$ is $I(n) = \{0, 1, \ldots, x\}$ where $x$ is the size of a maximum packing and $I(n) = \{0, 1, \ldots, x\} \backslash \{x-1\}$ for $n \equiv j \pmod 6$ such that $j \in \{0, 2\}$ and $n \neq 8$; $I(8) = \{0, 1, 2, 3, 4, 5, 8\}$.

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## Chapter 1

## Introduction

A *Steiner Triple System*, more simply a *triple system*, of order $n$ is a pair $(S, T)$, where $T$ is a set of edge-disjoint *triangles* (or *triples*) which partition the edge set of $K_n$ (the complete undirected graph on $n$ vertices) with vertex set $S$, denoted STS(n). It is well known that the spectrum for Steiner Triple Systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$, and that if $(S, T)$ is a triple system of order $n$ then $|T| = \frac{n(n-1)}{6}$ [4]. Define $I(n)$ and $J(n)$ as follows:

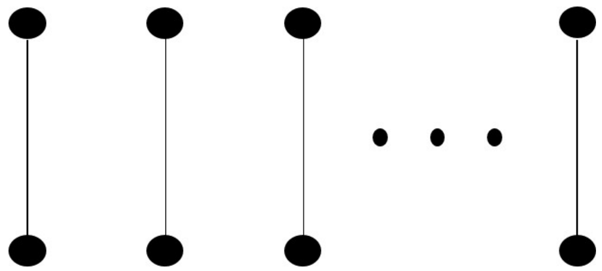
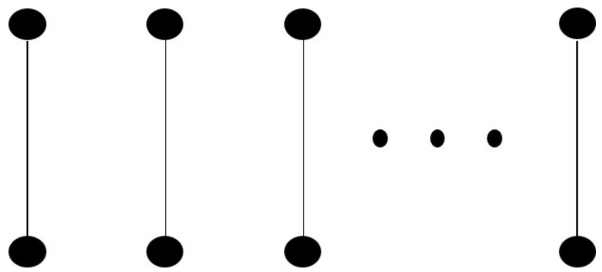
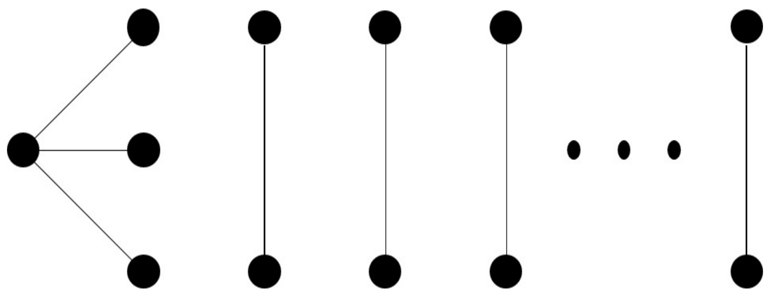
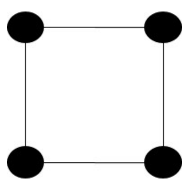
$$\begin{cases} I(n) & = \{0,\, 1,\, 2,\, \ldots,\, x = \frac{n(n-1)}{6}\} \backslash \{x-1,\, x-2,\, x-3,\, x-5\}, \text{ and} \\ J(n) & = \{k \mid \text{there exists a pair of triple systems of order } n \text{ having exactly } k \text{ triples} \\ & \quad \text{in common}\}. \end{cases}$$

Here, $J(n)$ is a pair of triple systems of order $n$ with the same leave having exactly $k$ triples in common. A natural question to ask is the following: *for which* $k \in \{0,\, 1,\, 2,\, \ldots,\, \frac{n(n-1)}{6}\}$ *does there exist a triple system of order* $n$ *having* $k$ *triples in common?* The following theorem gives a complete solution of the intersection problem for triple systems.

**Theorem 1.1** (C.C. Lindner, A. Rosa[6])**.** Let $n \equiv 1$ or $3 \pmod 6$. Then $J(n) = I(n)$, if $n \neq 9$, and $J(9) = I(9) \backslash \{5,\, 8\}$[5]. ■

Now when $n \not\equiv 1$ or $3 \pmod 6$ there does not exist a triple system and so the intersection problem for maximum packings of $K_n$ with triples is immediate. A *packing* of $K_n$ with triples is a pair $(S, P)$ where $P$ in a collection of edge disjoint triples of $K_n$ with vertex set $S$. If $P$ is as large as possible, then $(S, P)$ is said to be a *maximum packing* of $K_n$ with triples.

The set of unused edges is called the *leave*. The following easy to read table gives the leave for a maximum packing of $K_n$ with triples for each $n \equiv 0,\ 1,\ 2,\ 3,\ 4,\ 5 \pmod 6$.

| $n \equiv \pmod 6$ | leave |
|---|---|
| 0 | *1-factor*<br> |
| 1 | *Steiner Triple System* |
| 2 | *1-factor*<br> |
| 3 | *Steiner Triple System* |
| 4 | *tripole*<br> |
| 5 | *4-cycle*<br> |

*Theorem 1.1* gives a complete solution of the intersection problem for Steiner Triple Systems ($n \equiv 1$ or 3 (mod 6)). Subsequently, the intersection problem for maximum packings of $K_n$ with triples was completely solved for $n \equiv 0, 2, 4, 5$ when the leave is the SAME in the following two papers [3, 7]. (Best described with a table.)

| $n \equiv \pmod 6$ | Intersection spectrum |
|---|---|
| 0 or 2 | *D. Hoffman and C. C. Lindner*[3]<br>$I(6) = \{0, 4\}$, $I(8) = \{0, 2, 8\}$ and for all $n \geq 12$, $n \equiv 0$ or $2 \pmod 6$,<br>$I(n) = \{0, 1, 2, \ldots, \frac{n(n-2)}{6} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ |
| 4 | *G. Quattrocchi*[7]<br>$I(4) = \{1\}$ and for all $n \geq 10$, $n \equiv 4 \pmod 6$,<br>$I(n) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2} - \frac{(n+2)}{2})}{3} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ |
| 5 | *G. Quattrocchi*[7]<br>$I(5) = 2$ and for all $n \geq 11$, $n \equiv 5 \pmod 6$,<br>$I(n) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2} - 4)}{3} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ |

As mentioned, the leaves in the above table are always the same. The object of this thesis is the extension of the intersection problem for maximum packings of $K_n$ with triples when the leaves are isomorphic but not necessarily the same. In particular we remove most of the exceptions in the above table (best described in another table).

| $n \equiv \pmod 6$ | New Intersection Spectrum |
|---|---|
| 0 or 2 | $I(6) = \{0, 1, 2, 4\}$, $I(8) = \{0, 1, 2, 3, 4, 5, 8\}$ and for all $n \geq 12$, $n \equiv 0$ or $2 \pmod 6$, $I(n) = \{0, 1, 2, \ldots, \frac{n(n-2)}{6} = x\} \backslash \{x-1\}$ |
| 4 | $I(4) = \{0, 1\}$ and for all $n \geq 10$, $n \equiv 4 \pmod 6$, $I(n) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2} - \frac{(x+2)}{2})}{3}\}$ |
| 5 | $I(5) = \{0, 1, 2\}$ and for all $n \geq 11$, $n \equiv 5 \pmod 6$, $I(n) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2} - 4)}{3}\}$ |

To be specific we improve $I(6)$ from $\{0, 4\}$ to $\{0, 1, 2, 4\}$; $I(8)$ from $\{0, 2, 8\}$ to $\{0, 1, 2, 3, 4, 5, 8\}$; and remove the exceptions $\{x-2, x-3, x-5\}$ for all $n \equiv 0$ or $2 \pmod 6 \geq 12$. We improve $I(4)$ from $\{1\}$ to $\{0, 1\}$ and for all $n \equiv 4 \pmod 6 \geq 10$ remove the exceptions $\{x-1, x-2, x-3, x-5\}$; and finally improve $I(5)$ from $\{2\}$ to $\{0, 1, 2\}$ and for all $n \equiv 5 \pmod 6 \geq 11$ remove the exceptions $\{x-1, x-2, x-3, x-5\}$.

## Chapter 2

## Examples for $n \equiv 0$ or $2 \pmod 6$

As stated above, almost all exceptions from previous papers have been removed. However, in the equivalence classes $n \equiv 0$ or $2 \pmod 6$ it is impossible to obtain the intersection number $x - 1$ where $x$ is the number of triples in a maximum packing.

### 2.1 $n = 6$

In the following maximum packings, the triples are listed in the left column and the leave is listed in the right column.

$$S_1 = \left\{ \begin{array}{ccc|cc} \{1, & 3, & 5\} & 1, & 2 \\ \{1, & 4, & 6\} & 3, & 4 \\ \{2, & 3, & 6\} & 5, & 6 \\ \{2, & 4, & 5\} & & \end{array} \right. \qquad S_2 = \left\{ \begin{array}{ccc|cc} \{1, & 3, & 6\} & 1, & 2 \\ \{1, & 4, & 5\} & 3, & 4 \\ \{2, & 3, & 5\} & 5, & 6 \\ \{2, & 4, & 6\} & & \end{array} \right.$$

$$S_3 = \left\{ \begin{array}{ccc|cc} \{1, & 2, & 4\} & 1, & 3 \\ \{1, & 5, & 6\} & 2, & 5 \\ \{2, & 3, & 6\} & 4, & 6 \\ \{3, & 4, & 5\} & & \end{array} \right. \qquad S_4 = \left\{ \begin{array}{ccc|cc} \{1, & 3, & 4\} & 1, & 2 \\ \{1, & 5, & 6\} & 3, & 5 \\ \{2, & 3, & 6\} & 4, & 6 \\ \{2, & 4, & 5\} & & \end{array} \right.$$

The intersection numbers $\{0, 1, 2, 4\}$ can be found by intersecting the following maximum packings:

- **0**: $S_1 \cap S_2$
- **1**: $S_1 \cap S_3$
- **2**: $S_1 \cap S_4$
- **4**: $S_1 \cap S_1$

### 2.2 $n = 8$

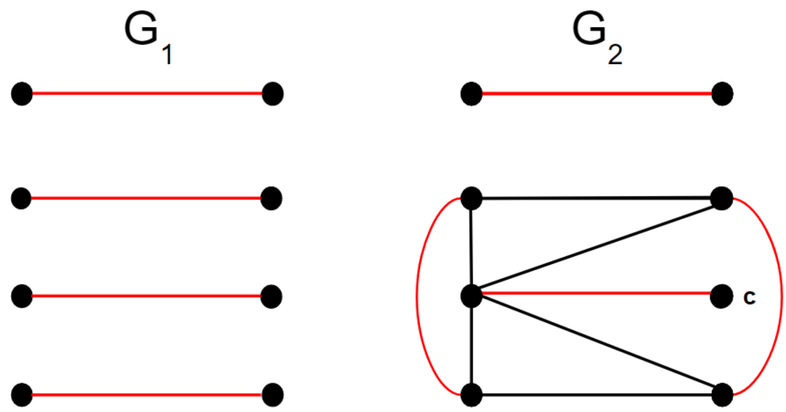
In the following maximum packings, the triples are listed in the left column and the leave is listed in the right column. We need intersection numbers $\{3, 5\}$; 6 is not possible.

$$S_1 = \left\{ \begin{array}{l|l} \{0,\ 1,\ 4\} & 0,\ 7 \\ \{0,\ 2,\ 5\} & 1,\ 6 \\ \{0,\ 3,\ 6\} & 2,\ 4 \\ \{1,\ 2,\ 3\} & 3,\ 5 \\ \{1,\ 5,\ 7\} & \\ \{2,\ 6,\ 7\} & \\ \{3,\ 4,\ 7\} & \\ \{4,\ 5,\ 6\} & \end{array} \right. \qquad S_2 = \left\{ \begin{array}{l|l} \{0,\ 1,\ 4\} & 0,\ 7 \\ \{0,\ 2,\ 5\} & 1,\ 5 \\ \{0,\ 3,\ 6\} & 2,\ 6 \\ \{1,\ 2,\ 3\} & 3,\ 4 \\ \{1,\ 6,\ 7\} & \\ \{2,\ 4,\ 7\} & \\ \{3,\ 5,\ 7\} & \\ \{4,\ 5,\ 6\} & \end{array} \right.$$

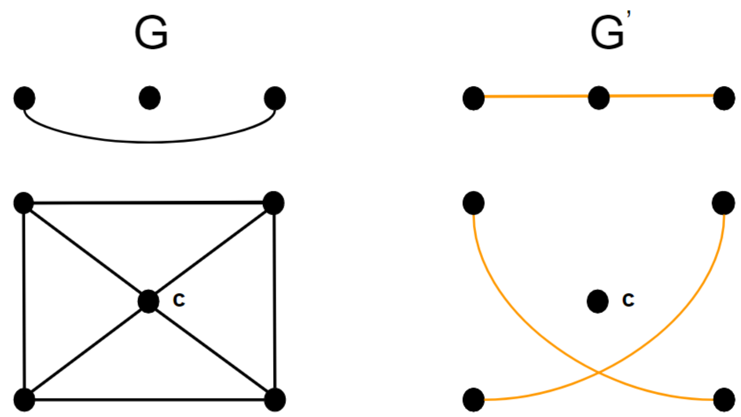
$$S_3 = \left\{ \begin{array}{l|l} \{0,\ 1,\ 2\} & 0,\ 7 \\ \{0,\ 3,\ 4\} & 1,\ 5 \\ \{0,\ 5,\ 6\} & 2,\ 3 \\ \{1,\ 3,\ 6\} & 4,\ 6 \\ \{1,\ 4,\ 7\} & \\ \{2,\ 4,\ 5\} & \\ \{2,\ 6,\ 7\} & \\ \{3,\ 5,\ 7\} & \end{array} \right. \qquad S_4 = \left\{ \begin{array}{l|l} \{0,\ 1,\ 2\} & 0,\ 7 \\ \{0,\ 3,\ 4\} & 1,\ 6 \\ \{0,\ 5,\ 6\} & 2,\ 4 \\ \{1,\ 4,\ 5\} & 3,\ 5 \\ \{1,\ 3,\ 7\} & \\ \{2,\ 3,\ 6\} & \\ \{2,\ 5,\ 7\} & \\ \{4,\ 6,\ 7\} & \end{array} \right.$$

So $S_1 \cap S_2$ gives intersection number 5 and $S_3 \cap S_4$ gives intersection number 3.

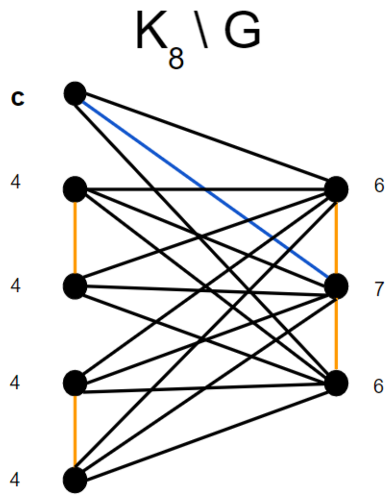
We now show that the intersection number 6 is impossible. Consider two maximum packings from $K_8$, namely $S'$ and $S''$. Assume in these maximum packings that they have an intersection number of 6, where $S'$ and $S''$ differ in two triples. The following is an illustration of the two aforementioned triples and the one-factors $G_1$ and $G_2$ of the maximum packing leaves, where $G_1 \subset S'$ and $G_2 \subset S''$.




The two triples that $S''$ contains but $S'$ does not are colored black and the one-factors of each are colored red. Along with the configuration $G_1$ in $S'$, we can also examine the configuration $G \subset S'$.




In $G$, there are two possible triples with three edges remaining, which would be included in the leave above. Without loss of generality, assume that these two triples are different than those found in $G_2$. If the decomposition of $K_8 \backslash G$ is six triples with only one edge remaining that completes the leave then the intersection number $x - 2$ is possible.




By looking at the graph of $K_8 \backslash G$ it is clear to see that this decomposition is impossible to form. The two vertices of odd degree have a blue edge to denote that it will need to be the last edge placed in the leave, shown in red in $G_1$. Once that edge is labeled as part of the leave, the remaining graph cannot be decomposed into triples. This contradicts our assumption that the two triples colored black are the only two triples that $S'$ and $S''$ do not have in common. Hence, the intersection number $x - 2$ is impossible for $K_8$.

### 2.3 $n = 12$

We begin by listing the triples in a cyclic triple system of order 13.

$$T_{13} = \left\{ \begin{array}{lll} \{\,0, & 1, & 4\} \\ \{\,1, & 2, & 5\} \\ \{\,2, & 3, & 6\} \\ \{\,3, & 4, & 7\} \\ \{\,4, & 5, & 8\} \\ \{\,5, & 6, & 9\} \\ \{\,6, & 7, & 10\} \\ \{\,7, & 8, & 11\} \\ \{\,8, & 9, & 12\} \\ \{\,9, & 10, & 0\} \\ \{\,10, & 11, & 1\} \\ \{\,11, & 12, & 2\} \\ \{\,12, & 0, & 3\} \end{array} \right. \left\{ \begin{array}{lll} \{\,0, & 2, & 7\} \\ \{\,1, & 3, & 8\} \\ \{\,2, & 4, & 9\} \\ \{\,3, & 5, & 10\} \\ \{\,4, & 6, & 11\} \\ \{\,5, & 7, & 12\} \\ \{\,6, & 8, & 0\} \\ \{\,7, & 9, & 1\} \\ \{\,8, & 10, & 2\} \\ \{\,9, & 11, & 3\} \\ \{\,10, & 12, & 4\} \\ \{\,11, & 0, & 5\} \\ \{\,12, & 1, & 6\} \end{array} \right.$$

First deleting vertex 5, we attain a maximum packing of $K_{12}$ as shown below. We need intersection numbers $\{15, 17, 18\}$.

| | | | | |
|---|---|---|---|---|
| 6 12 | 3 6 | 4 8 | 0 3 12 | 4 6 11 |
| 7 9 | 8 10 | 3 10 | 0 6 8 | 4 10 12 |
| 3 8 | 4 9 | 6 9 | 0 9 10 | 6 7 10 |
| 10 11 | 11 12 | 7 12 | 3 4 7 | 7 8 11 |
| 0 4 | 0 7 | 0 11 | 3 9 11 | 8 9 12 |


To find intersection number {18}, intersect the table above with the table below, obtained by simply swapping the orange edges in column two with the orange factors in the leave.

| | | | | |
|---|---|---|---|---|
| 6 12 | 3 6 | 4 8 | 0 3 12 | 4 6 11 |
| 7 9 | 8 10 | 3 10 | 0 6 8 | 4 10 12 |
| 3 8 | 4 9 | 6 9 | 0 9 10 | 6 7 10 |
| 10 11 | 7 12 | 11 12 | 3 4 7 | 7 8 11 |
| 0 4 | 0 11 | 0 7 | 3 9 11 | 8 9 12 |


To find intersection number {17}, intersect the top table with the table below, obtained by simply swapping the blue edges in column two with the blue factors in the leave.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 12 | 4 | 8 | 3 | 6 | 0 | 3 | 12 | 4 | 6 | 11 |
| 7 | 9 | 3 | 10 | 8 | 10 | 0 | 6 | 8 | 4 | 10 | 12 |
| 3 | 8 | 6 | 9 | 4 | 9 | 0 | 9 | 10 | 6 | 7 | 10 |
| 10 | 11 | 11 | 12 | 7 | 12 | 3 | 4 | 7 | 7 | 8 | 11 |
| 0 | 4 | 0 | 7 | 0 | 11 | 3 | 9 | 11 | 8 | 9 | 12 |

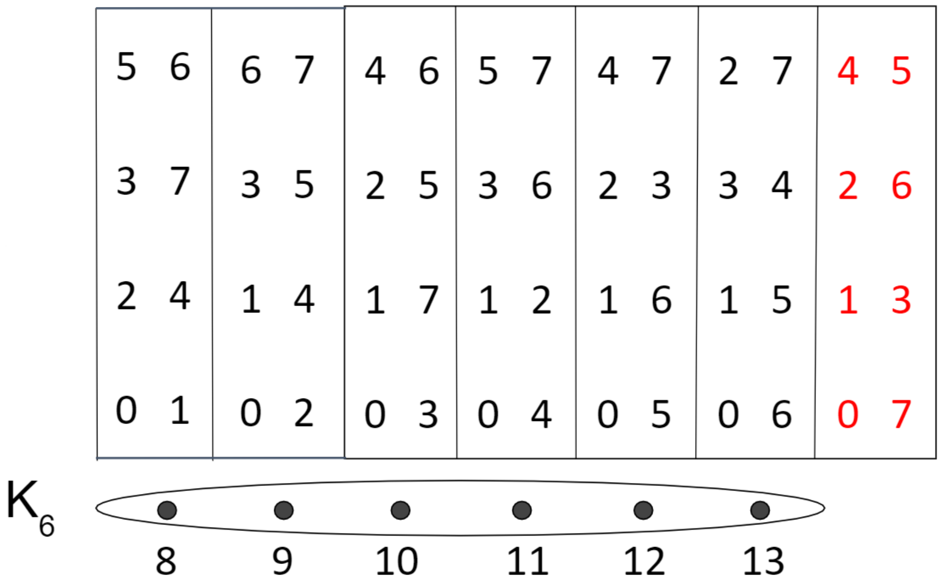

To find intersection number $\{15\}$, intersect the first table with the table below, obtained by simply swapping the orange and blue edges in column two with the orange and blue edges in the leave.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 12 | 4 | 8 | 3 | 6 | 0 | 3 | 12 | 4 | 6 | 11 |
| 7 | 9 | 3 | 10 | 8 | 10 | 0 | 6 | 8 | 4 | 10 | 12 |
| 3 | 8 | 6 | 9 | 4 | 9 | 0 | 9 | 10 | 6 | 7 | 10 |
| 10 | 11 | 7 | 12 | 11 | 12 | 3 | 4 | 7 | 7 | 8 | 11 |
| 0 | 4 | 0 | 11 | 0 | 7 | 3 | 9 | 11 | 8 | 9 | 12 |

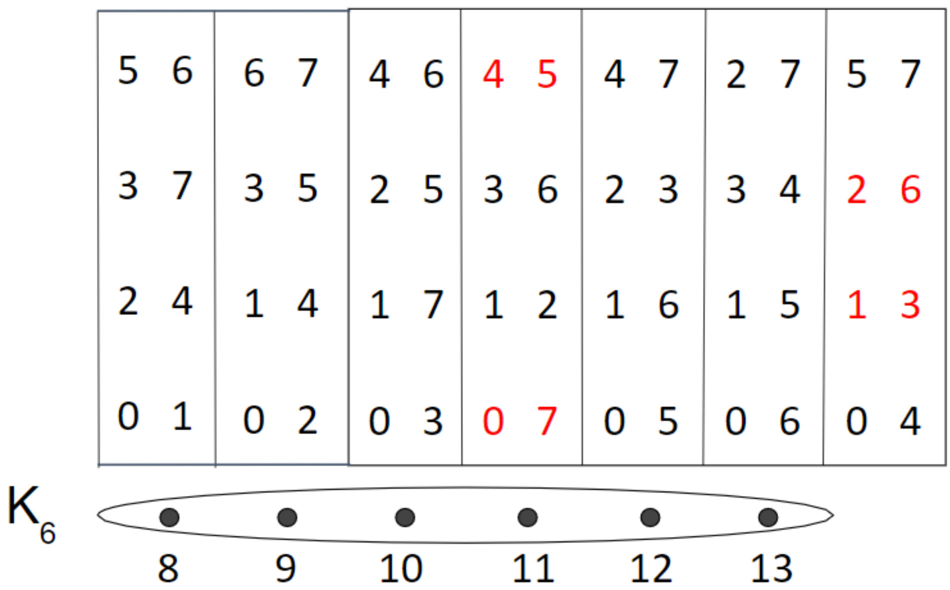

This gives intersection numbers $\{15, 17, 18\}$ for $K_{12}$.

## 2.4 $n = 14$

The intersection numbers needed for $K_{14}$ are $\{23, 25, 26\}$. Below is a one-factorization of $K_{14}$ with a $K_6$ in the oval, where the leave is in red.



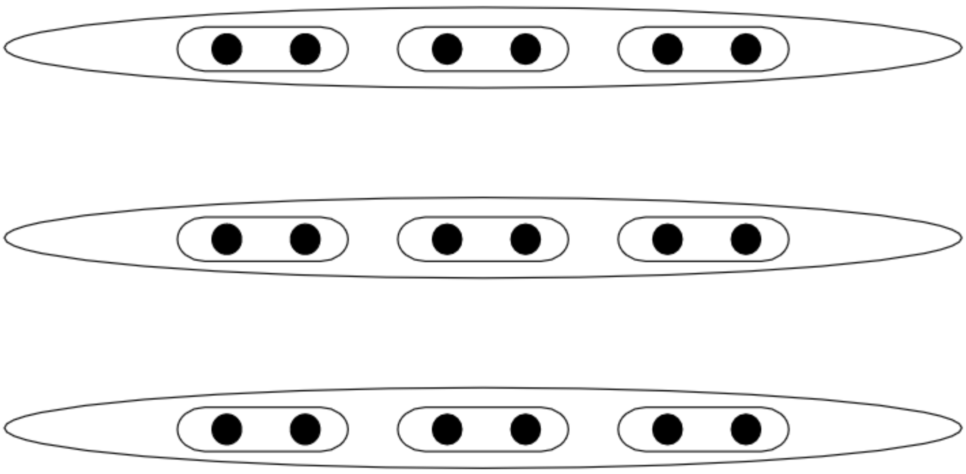

Because we know that the intersection numbers for $K_6$ are $\{0, 1, 2, 4\}$ we have intersection numbers $\{25, 26\}$ immediately.



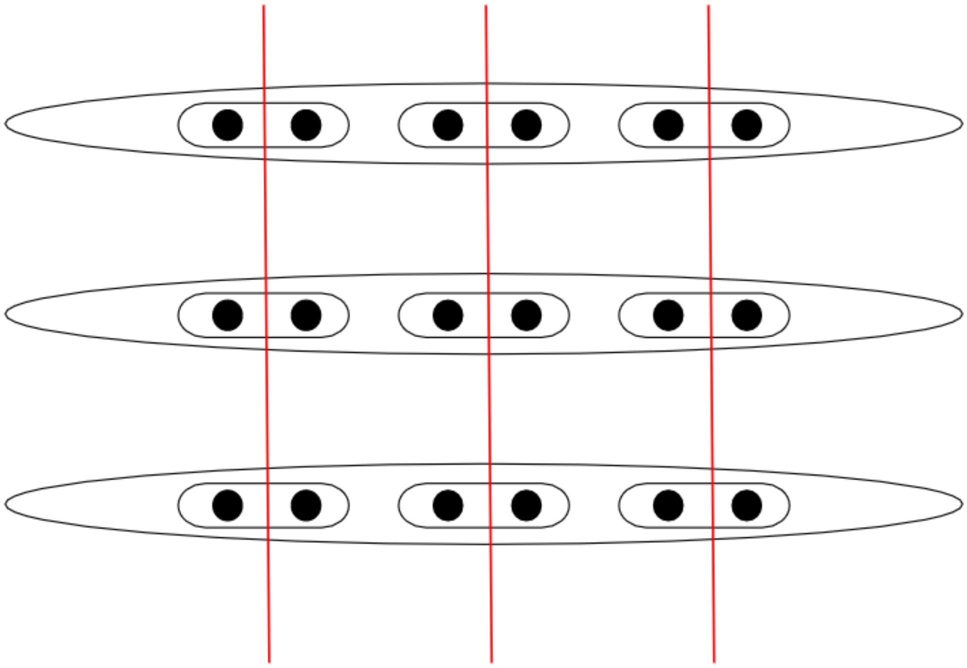
To find intersection number 23, take the intersection of the two one-factorizations above. Here, we switch edges $\{4,7\}$ and $\{0,5\}$ with $\{0,7\}$ and $\{4,5\}$. This reduces the number of triples by two. Now, if we take the intersection of the $K_6$ graphs to be 1 then this reduces the intersection number by another three triples. Thus, we have reduced the intersection number by 5 giving us the intersection number 23.

### 2.5 $n = 18$

In this case, we need find only intersection numbers $\{43, 45, 46\}$. We begin by constructing a $GDD(18, 2, 3)$ where each level of six points will have 0, 1, 2, or 4 triples.



To find the triples that cut across each level we simply place a $K_{3,3,3}$, as shown in red, into the design.



Each of the four $K_{3,3,3}$ will yield 9 triples. By intersecting two group divisible designs, $GDD(18, 2, 3)$, where the intersection of each of the four $K_{3,3,3}$ is equal to 36, we can use the intersections of two $K_6$ to find the intersection numbers $\{43, 45, 46\}$. For convenience, let $R_1$ be the first row of six vertices, $R_2$ be the second, and $R_3$ be the third. The intersection number 43 is found by $R_1 \cap R_1 = 4$, $R_2 \cap R_2 = 2$, and $R_3 \cap R_3 = 1$ summed together with the original 36 triples. The intersection number 45 is found by $R_1 \cap R_1 = 4$, $R_2 \cap R_2 = 4$, and $R_3 \cap R_3 = 1$ summed together with the original 36 triples. The intersection number 46 is found by $R_1 \cap R_1 = 4$, $R_2 \cap R_2 = 4$, and $R_3 \cap R_3 = 2$ summed together with the original 36 triples. Hence, we have found the intersection numbers $\{43, 45, 46\}$.

### 2.6 $n = 20$

In this case, we need find only intersection numbers $\{55, 57, 58\}$. We begin by constructing the following one-factorization:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 9 | 10 11 | 12 13 | 11 12 | 9 13 | 7 11 | 8 12 | 0 2 5 | 7 9 12 |
| 7 13 | 6 7 | 9 10 | 6 13 | 6 12 | 6 10 | 5 13 | 1 3 6 | 8 10 13 |
| 5 6 | 5 12 | 7 8 | 4 8 | 5 11 | 5 9 | 4 11 | 2 4 7 | 9 11 0 |
| 4 12 | 3 13 | 4 5 | 3 10 | 4 10 | 3 4 | 3 9 | 3 5 8 | 10 12 1 |
| 3 11 | 2 8 | 2 3 | 2 9 | 3 7 | 2 12 | 2 6 | 4 6 9 | 11 13 2 |
| 2 10 | 1 9 | 1 11 | 1 5 | 1 2 | 1 8 | 1 7 | 5 7 10 | 12 0 3 |
| 0 1 | 0 4 | 0 6 | 0 7 | 0 8 | 0 13 | 0 10 | 6 8 11 | 13 1 4 |

14 15 16 17 18 19

To find the intersection numbers $\{57, 58\}$ we simply take two one-factorizations to be the same and take maximum packings of the two $K_6$ graphs below the one-factorizations to have 1 or 2 triples in common. To find the intersection number $\{55\}$ we find the intersection between the one factorization above and the one-factorization below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 9 | 4 11 | 12 13 | 11 12 | 9 13 | 7 11 | 8 12 | 0 2 5 | 7 9 12 |
| 7 13 | 6 7 | 9 10 | 6 13 | 6 12 | 6 10 | 5 13 | 1 3 6 | 8 10 13 |
| 5 6 | 5 12 | 7 8 | 4 8 | 5 11 | 5 9 | 10 11 | 2 4 7 | 9 11 0 |
| 4 12 | 3 13 | 4 5 | 3 10 | 4 10 | 3 4 | 3 9 | 3 5 8 | 10 12 1 |
| 3 11 | 2 8 | 2 3 | 2 9 | 3 7 | 2 12 | 2 6 | 4 6 9 | 11 13 2 |
| 2 10 | 1 9 | 1 11 | 1 5 | 1 2 | 1 8 | 1 7 | 5 7 10 | 12 0 3 |
| 0 1 | 0 10 | 0 6 | 0 7 | 0 8 | 0 13 | 0 4 | 6 8 11 | 13 1 4 |

14 15 16 17 18 19

By taking the intersection of two maximum packings of the $K_6$ graphs below, with the one-factorizations to have 1 triple in common, we obtain maximum packings of $K_{20}$ with $14 + 40 + 1 = 55$ triples in common.

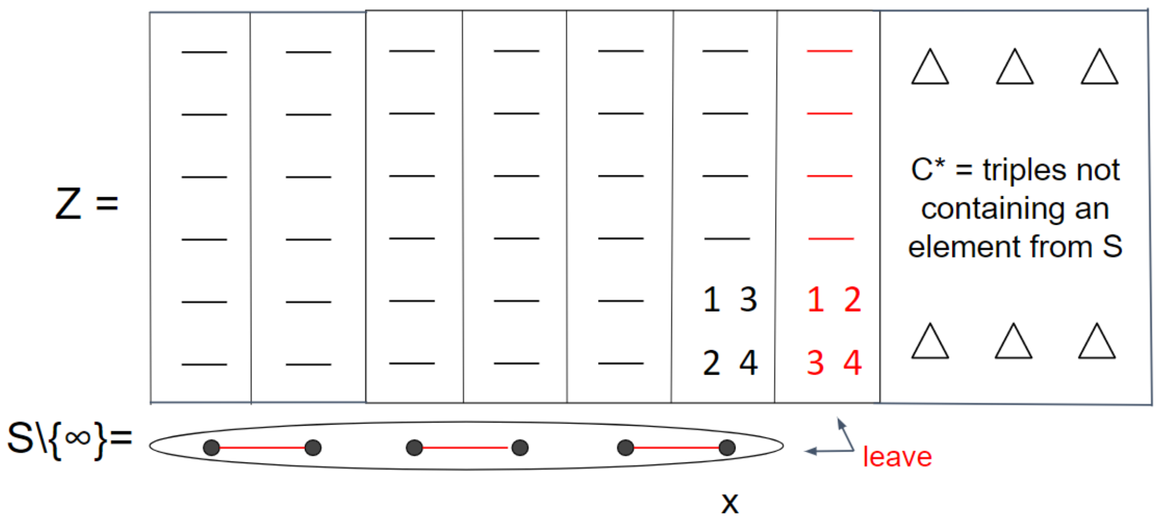
## Chapter 3

## General Construction: $n \equiv 0$ or $2 \pmod 6$

In view of the examples in *Chapter 2*, we need only look at $n \equiv 0$ or $2 \pmod 6$, $n \geq 24$ and construct the intersection numbers $\{x-2,\, x-3,\, x-5\}$, where $x$ is the number of triples in a maximum packing of order $n \equiv 0$ or $2 \pmod 6$. Let $(S, T)$ be a triple system of order 7 (any triple system of order 7 will do), there is only one up to isomorphism. Now construct a partial triple system $(X, P)$ of order 11 as follows.

$$\begin{cases} X &= S \cup \{1,\, 2,\, 3,\, 4\}, \text{ and} \\ P &= T \cup \{\{\infty,\, 1,\, 2\},\, \{\infty,\, 3,\, 4\},\, \{x,\, 1,\, 3\},\, \{x,\, 2,\, 4\}\}, \text{ where } \infty \text{ and } x \text{ belong to } S \end{cases}$$

Then $(X,\, P)$ is a partial triple system of order 11. Embed $(X,\, P)$ in a complete triple system $(Y,\, C)$ of order $n \equiv 1$ or $3 \pmod 6 \geq 25$ [1]. Let $C^* = C \backslash \{\text{all triples containing } \infty\} \geq 24$. Then $(Y \backslash \{\infty\},\, C^*, L)$ is a maximum packing of order $\equiv 0$ or $2 \pmod 6$; where the leave $W = \{\{x,\, y\} \,|\, \{\infty,\, x,\, y\} \in (Y,\, C)\}$.



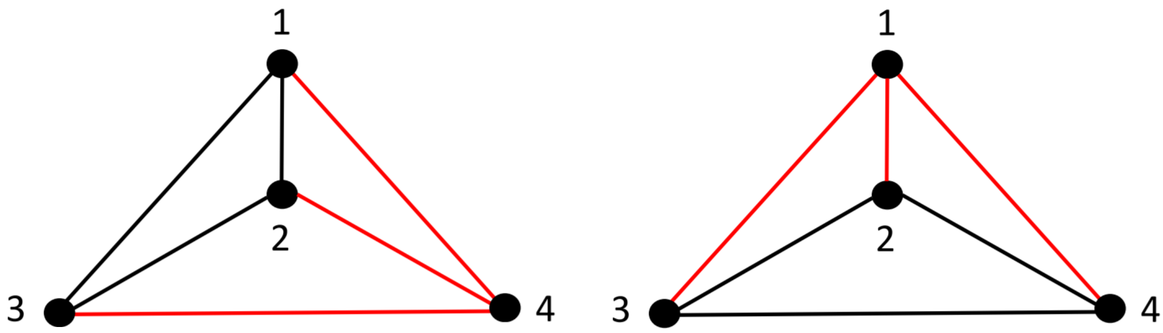
Since $I(6) = \{0,\, 1,\, 2,\, 4\}$ if we interchange $\{\{1,3\},\, \{2,4\}\}$ with $\{\{1,2\},\, \{3,4\}\}$ and take the $S\backslash\{\infty\}$s to have one triple in common, we reduce the intersection number by five. If we take $Z$ twice and use a pair of maximum packings of order 6, since $I(6) = \{0,\, 1,\, 2,\, 4\}$ we obtain a pair of maximum packing of order $n$ intersecting in $x-2$ and $x-3$ triples. It follows that $I(n) = \{0,\, 1,\, 2,\, \ldots,\, \frac{n(n-2)}{6} = x\}\backslash\{x-1\}$. ■

Chapter 4

$n \equiv 4 \pmod 6$

## 4.1 $n = 4$

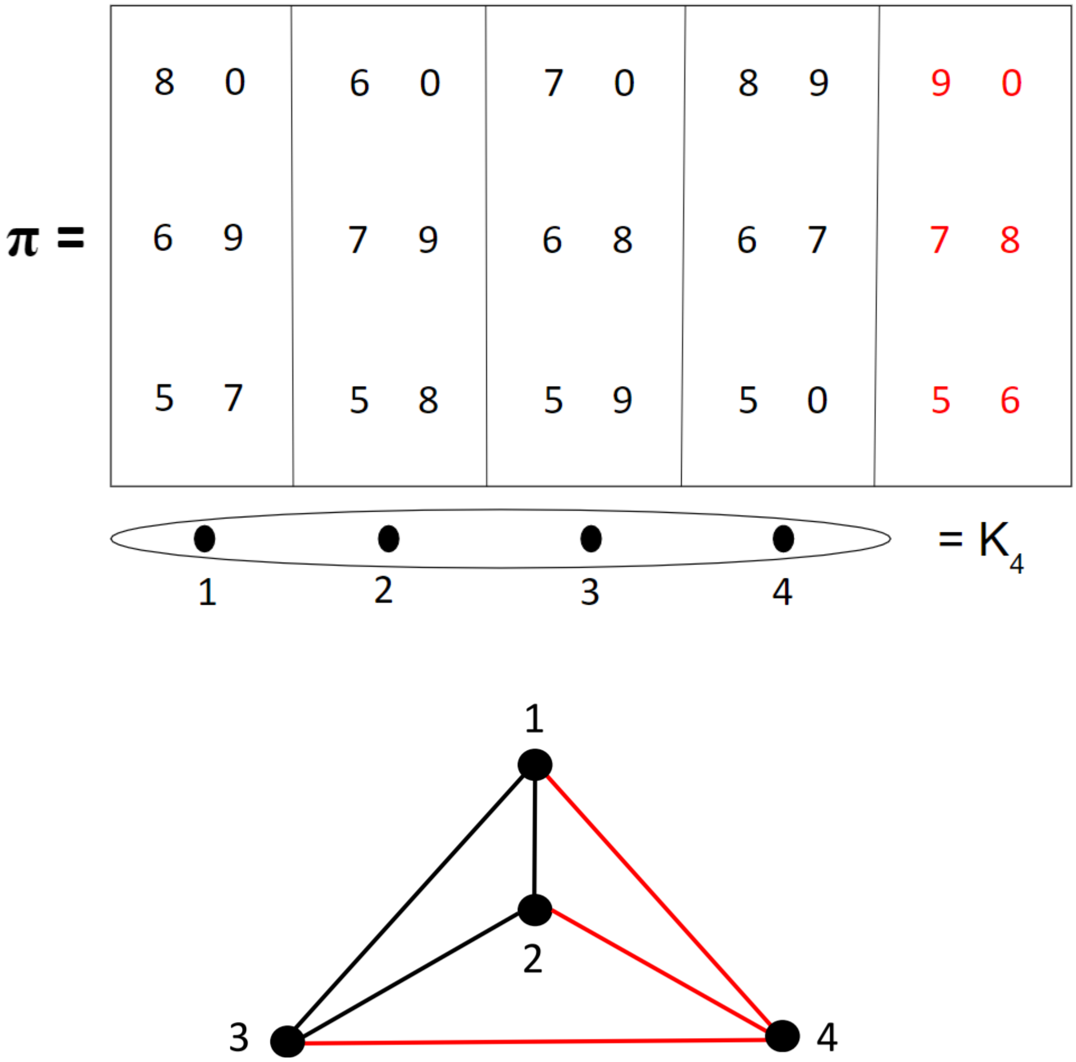
Since the leave of $K_4$ is a tripole, a maximum packing on $K_4$ consists of one triple and one tripole. Hence, the intersection number of two maximum packings on $K_4$ is either 0 or 1.




## 4.2 $n = 10$

We need intersection numbers 8, 10, 11, and 12.

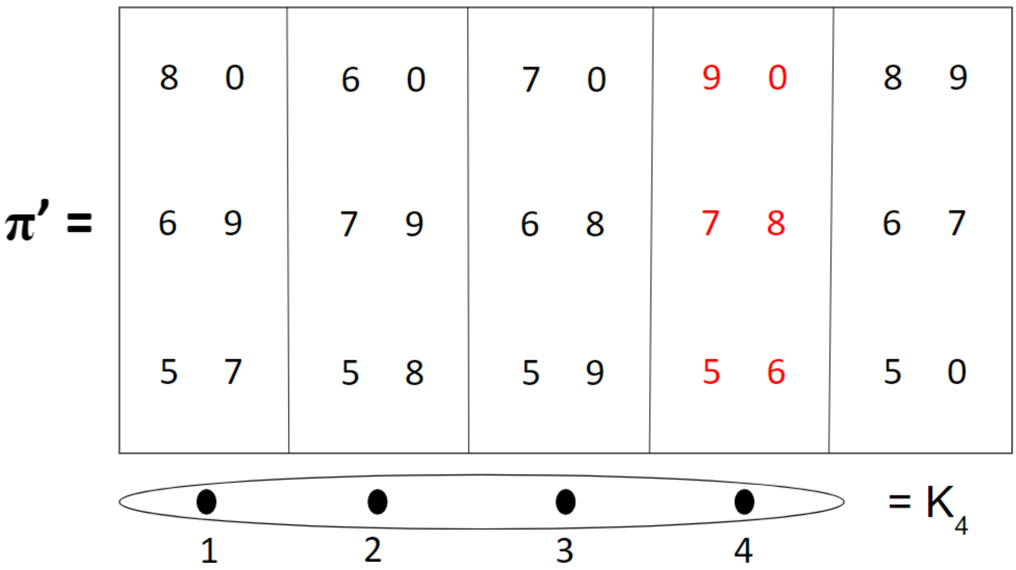
Here is a maximum packing $\prod$ of order 10 where the leave is represented in red font with the accompanying $K_4$.



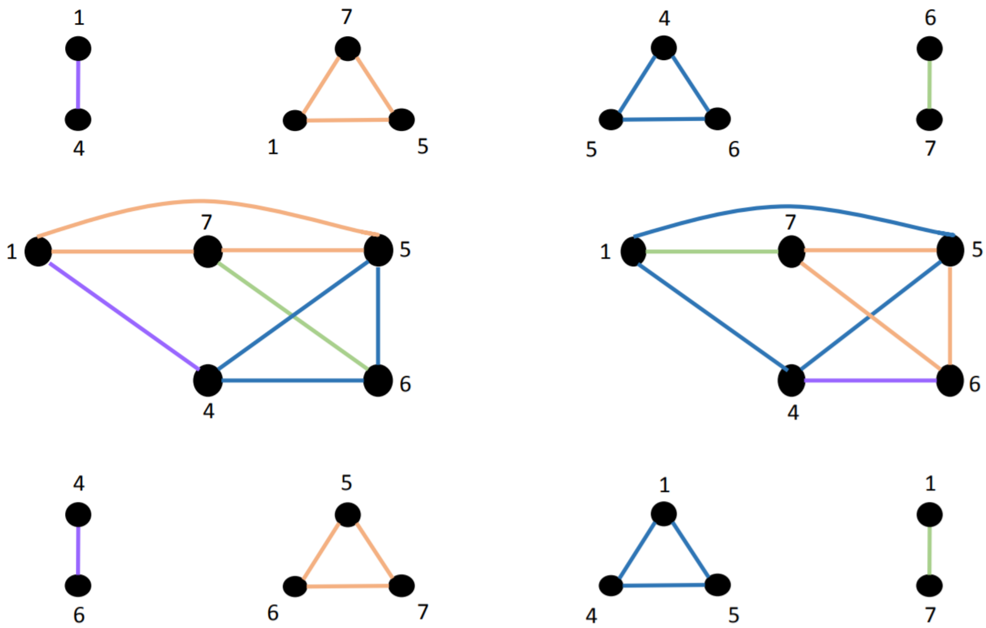

To find intersection number 12 replace the triple $\{1, 2, 3\}$ in the $K_4$ with $\{2, 3, 4\}$. So we have reduced the triples by one giving us an intersection number of 12.

To find the intersection number 11, replace the triples {{1, 5, 7}, {4, 6, 7}} and edges {{1, 4}, {5, 6}} in $\prod$ with triples {{0, 5, 7}, {1, 4, 7}} and edges{{1, 5}, {4, 6}}. The resulting maximum packing has two less triples than $\prod$. The leave is {{$4^*$, 2, 3, 6}, {0, 9}, {1, 5}, {7, 8}} where $4^*$ is the root in the tripole.

To find intersection number 10, we take the intersection of $\prod$ with the accompanying $K_4$ and $\prod'$ below, using the same $K_4$.




To find intersection number 8 we begin with $\prod'$ and use the below construction to replace triples {1, 5, 7} and {4, 5, 6} with {5, 6, 7} and {1, 4, 5}. Let's call this new construction $\prod''$. Then we take the intersection of $\prod''$ and $\prod$ but instead of choosing the triple $\{1, 2, 3\}$ in the $K_4$ of the $\prod'$ we will choose $\{2, 3, 4\}$.



More simply, $\prod \cap \prod' = 10$ and $\prod \cap \prod'' = 8$ with triples from $\prod$, $\prod'$, $\prod''$ listed below.

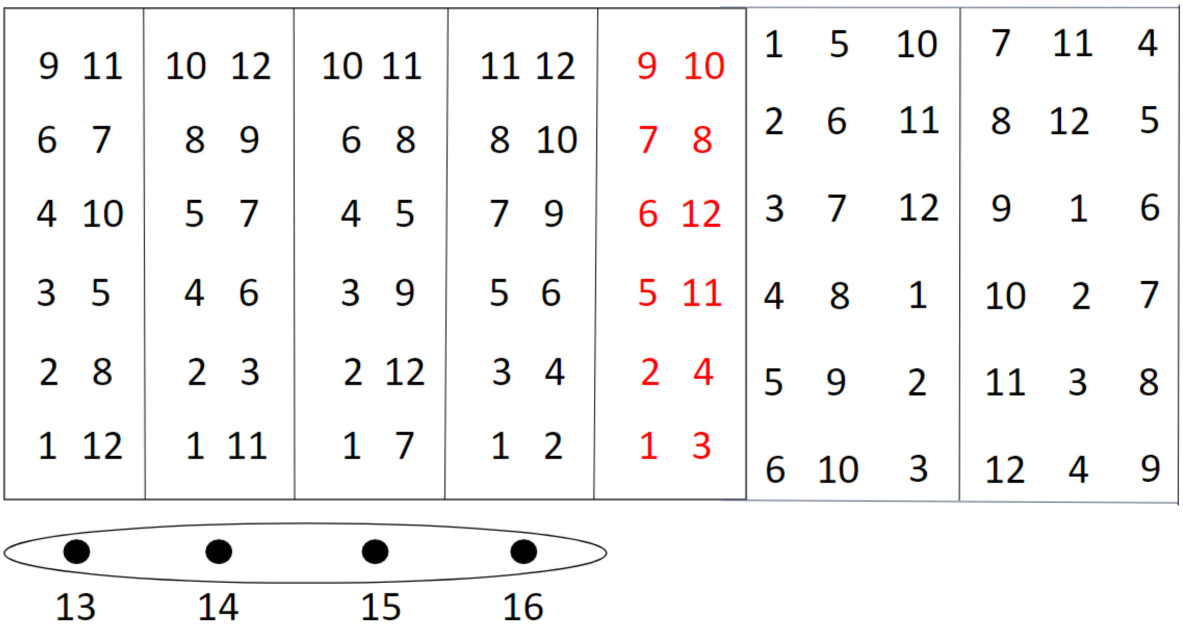
$$\prod = \begin{cases} \{\,1, & 5, & 7\} \\ \{\,1, & 6, & 9\} \\ \{\,1, & 0, & 8\} \\ \{\,2, & 5, & 8\} \\ \{\,2, & 7, & 9\} \\ \{\,2, & 0, & 6\} \\ \{\,3, & 5, & 9\} \\ \{\,3, & 6, & 8\} \\ \{\,3, & 0, & 7\} \\ \{\,4, & 0, & 5\} \\ \{\,4, & 6, & 7\} \\ \{\,4, & 8, & 9\} \end{cases} \quad \prod' = \begin{cases} \{\,1, & 5, & 7\} \\ \{\,1, & 6, & 9\} \\ \{\,1, & 0, & 8\} \\ \{\,2, & 5, & 8\} \\ \{\,2, & 7, & 9\} \\ \{\,2, & 0, & 6\} \\ \{\,3, & 5, & 9\} \\ \{\,3, & 6, & 8\} \\ \{\,3, & 0, & 7\} \\ \{\,4, & 5, & 6\} \\ \{\,4, & 7, & 8\} \\ \{\,4, & 0, & 9\} \end{cases} \quad \prod'' = \begin{cases} \{\,1, & 4, & 5\} \\ \{\,1, & 6, & 9\} \\ \{\,1, & 0, & 8\} \\ \{\,2, & 5, & 8\} \\ \{\,2, & 7, & 9\} \\ \{\,2, & 0, & 6\} \\ \{\,3, & 5, & 9\} \\ \{\,3, & 6, & 8\} \\ \{\,3, & 0, & 7\} \\ \{\,5, & 6, & 7\} \\ \{\,4, & 7, & 8\} \\ \{\,4, & 0, & 9\} \end{cases}$$

### 4.3 $n = 16$

We need intersection numbers 32, 34, 35, and 36.

Here is a one factorization of a $K_{16}$ where the leave is represented in red font with an accompanying $K_4$. As seen in the example of $K_{10}$, the $K_4$ yields one or zero triples in common with the original.

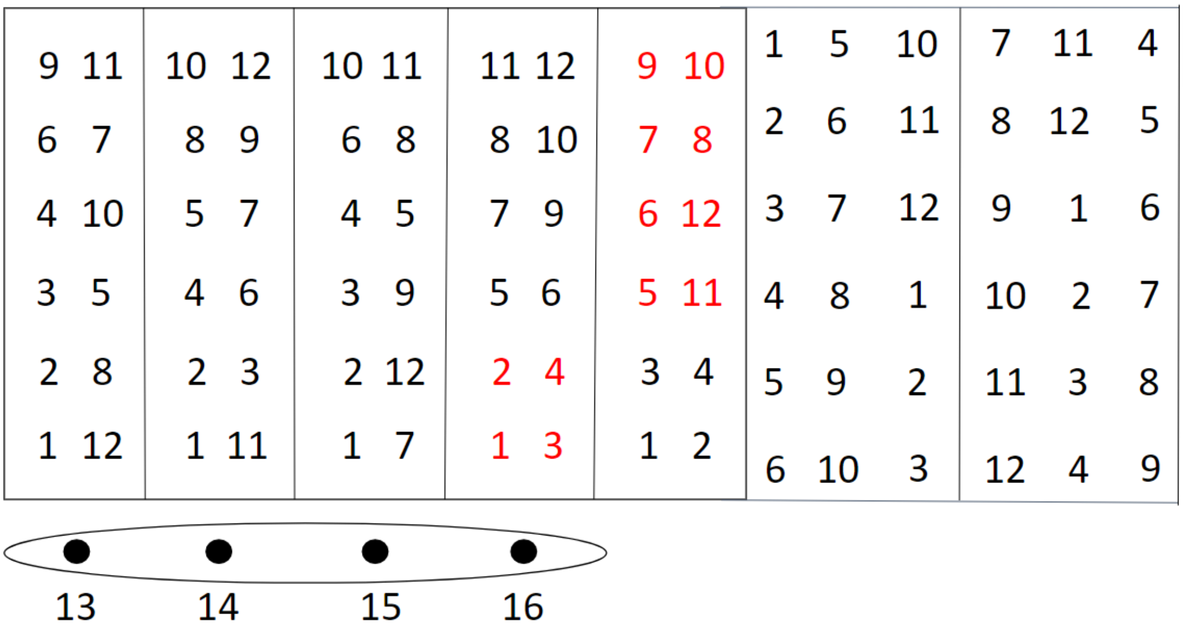
$\pi_1 =$




To find the intersection number 36, simply take two 1-factorizations to be the same and choose the triple from two $K_4$s to be different.

To find the intersection numbers 34 and 35, find the intersection between the 1-factorization $\prod_1$ above and the 1-factorization $\prod_2$ below. By taking the two $K_4$ to have either 0 or 1 triple in common we find the intersection numbers 34 and 35, respectively.

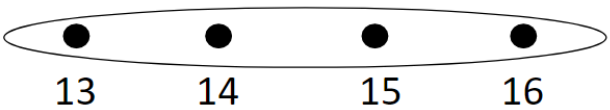
$\pi_2 =$



To find the intersection number 32, find the intersection between the 1-factorization $\prod_1$ above and the following 1-factorization $\prod_3$, where the intersection between the two $K_4$s is zero.

$\pi_3 =$

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 11 | 10 12 | 10 11 | 9 10 | 11 12 | 1 5 10 | 7 11 4 |
| 6 7 | 8 9 | 6 8 | 7 8 | 8 10 | 2 6 11 | 8 12 5 |
| 4 10 | 5 7 | 4 5 | 6 12 | 7 9 | 3 7 12 | 9 1 6 |
| 3 5 | 4 6 | 3 9 | 5 11 | 5 6 | 4 8 1 | 10 2 7 |
| 2 8 | 2 3 | 2 12 | 3 4 | 2 4 | 5 9 2 | 11 3 8 |
| 1 12 | 1 11 | 1 7 | 1 2 | 1 3 | 6 10 3 | 12 4 9 |




These 1-factorizations have shown all intersection numbers are possible for $K_{16}$.

Chapter 5

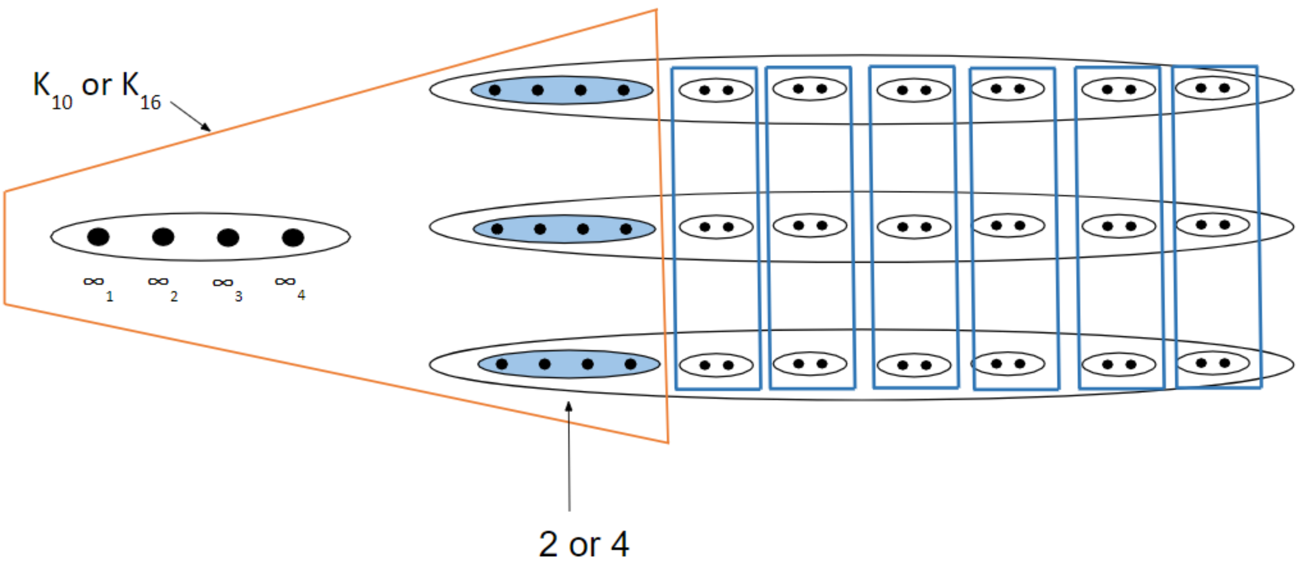
General Construction: $n \equiv 4 \pmod 6$

With the three examples in *Chapter 4* in hand we can proceed to the main construction showing that $I(6n+4) = J^*(6n+4) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2} + \frac{(n+2)}{2})}{3}\}$ for all $n$.

**<u>The $6n+4$ Construction:</u>** Let $6n+4 \geq 22$ and let $(X, G, B)$ be a $GDD(2n, 2, 3)$ or $GDD(2n, \{2, 4^*\}, 3)$, where $\{2, 4^*\}$ means there is exactly one group of size 4 and the rest have size 2. Set $S = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1, 2, 3\})$ and define a maximum packing, $P$ of $K_{6n+4}$ as follows:

1. Place an example of order 10 or 16 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2, 3\})$ where $g$ is a block of size 2 if all blocks have size 2; or 4 if $g$ in the unique block of size 4.

2. For all other blocks (which necessarily have size 2) place an example of order 10 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2, 3\})$ minus the edges between $\{\infty_1, \infty_2, \infty_3, \infty_4\}$.



3. For each triple $\{a, b, c\} \in B$ decompose $K_{3,3,3}$ into 9 triples with parts $a \times \{1, 2, 3\}$, $b \times \{1, 2, 3\}$ and $c \times \{1, 2, 3\}$.

Then $(S, P)$ is a maximum packing of $K_{6n+4}$ with triples with leave a 4–cycle. Now take two copies of $(S, P)$. We need construct only the intersection numbers $x-1$, $x-2$, $x-3$, and $x-5$ since Quattrocchi has taken care of everything else. But this is easily done by defining a pair of maximum packings of order 10 or 16 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2, 3\})$ intersecting in $x-1$, $x-2$, $x-3$, or $x-5$ triples, where $x = 13$ or 37 as the case may be. This completes the proof. We have the following theorem:

**Theorem 5.1.** $I(6n+4) = J^*(6n+4)$ for all $6n+4$. ■

Chapter 6

# $n \equiv 5 \pmod 6$

In everything that follows $J^*(n) = \{0, 1, 2, \ldots, \left(\binom{n}{2} - 4\right)/3\}$. We will need examples for $n = 5$, 11, and 17. In each case we will show that $I(n) = J^*(n)$, thereby removing the exceptions in Quatrocchi's constructions for $n \equiv 5 \pmod 6$.

## 6.1 $n = 5$

Define three maximum packings of order 5 $(X,\, P_1)$, $(X,\, P_2)$, and $(X,\, P_3)$ as follows:

$$\begin{cases} 1.\ X = \{1,\, 2,\, 3,\, 4,\, 5\},\ P_1 = \{\{1,\, 2,\, 3\}, \{1,\, 4,\, 5\}\} \text{ with leave } (2,\, 4,\, 3,\, 5); \\ 2.\ X = \{1,\, 2,\, 3,\, 4,\, 5\},\ P_2 = \{\{1,\, 4,\, 5\}, \{2,\, 3,\, 4\}\} \text{ with leave } (1,\, 2,\, 5,\, 3); \\ 3.\ X = \{1,\, 2,\, 3,\, 4,\, 5\},\ P_3 = \{\{1,\, 2,\, 4\}, \{1,\, 5,\, 3\}\} \text{ with leave } (2,\, 3,\, 4,\, 5). \end{cases}$$

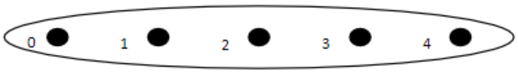
Then $|P_1 \cap P_3| = 0$, $|P_1 \cap P_2| = 1$, and $|P_1 \cap P_1| = 2$. It follows that $I(5) = J^*(5) = \{0,\, 1,\, 2\}$.

## 6.2 $n = 11$

Let $(X, F)$ be a 1-factorization of $K_6$ with vertex set $X$ and $(Y, P_1)$ and $(Y, P_2)$ any two maximum packings of $K_5$ with triples in *Example 6.1.* Define a pair of maximum packings $C_1$ and $C_2$ of $K_{11}$ with triples with vertex set $X \cup Y$ as follows:

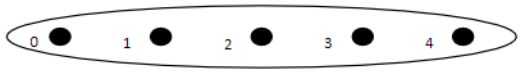
$F =$

| $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|---|
| 8, 9 | 9, 10 | 8, 10 | 7, 8 | 7, 9 |
| 7, 10 | 6, 8 | 6, 7 | 6, 9 | 6, 10 |
| 5, 6 | 5, 7 | 5, 9 | 5, 10 | 5, 8 |

$P_1 =$ 

$F =$

| $F_0$ | $F_1$ | $F_4$ | $F_2$ | $F_3$ |
|---|---|---|---|---|
| 8, 9 | 9, 10 | 7, 9 | 8, 10 | 7, 8 |
| 7, 10 | 6, 8 | 6, 10 | 6, 7 | 6, 9 |
| 5, 6 | 5, 7 | 5, 8 | 5, 9 | 5, 10 |

$P_2 =$ 

1. $\{i,\, x,\, y\} \in C_1$ and $C_2$ for each $i \in \{0,\, 1,\, 2,\, 3,\, 4\}$ and $\{x,\, y\} \in F_i$,
2. $P_1 \subseteq C_1$ and $P_2 \subseteq C_2$. The leave in each case are the leaves in $P_1$ and $P_2$.

By permuting the columns of $F$ and using the examples in *6.1* independently we obtain the intersection numbers $0, 1, 2, \ldots, 9, 10, 11, 15, 16, 17$. So it remains to obtain the intersection numbers 12, 13, 14. Let $Z_1$ and $Z_2$ be the following two mutually balanced configurations consisting of a 4–cycle and 3–triples.

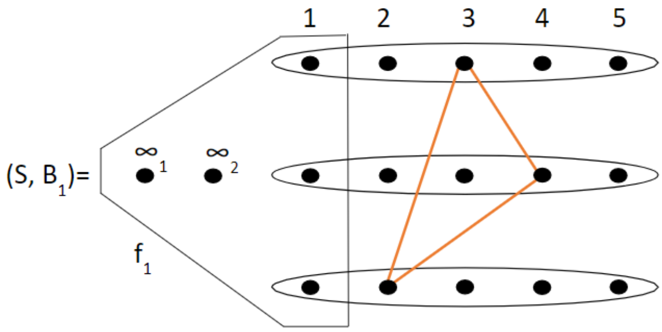
$$Z_1 = \left\{ \begin{array}{llll} (1, & 2, & 3, & 4) \\ & \{2, & 8, & 10\} \\ & \{3, & 5, & 10\} \\ & \{4, & 5, & 8\} \end{array} \right. \qquad Z_2 = \left\{ \begin{array}{llll} (1, & 4, & 8, & 2) \\ & \{3, & 4, & 5\} \\ & \{2, & 3, & 10\} \\ & \{5, & 8, & 10\} \end{array} \right.$$

None of the triples in $Z_1$ and $Z_2$ belong to $P_1$ or $P_2$. So removing $Z_1$ from $C_1$ and replacing it with $Z_2$ reduces the number of type (1) triples by 3. Taking $P_1$ and $P_2$ to have 0, 1, or 2 triples in common gives intersection numbers 12, 13, and 14.

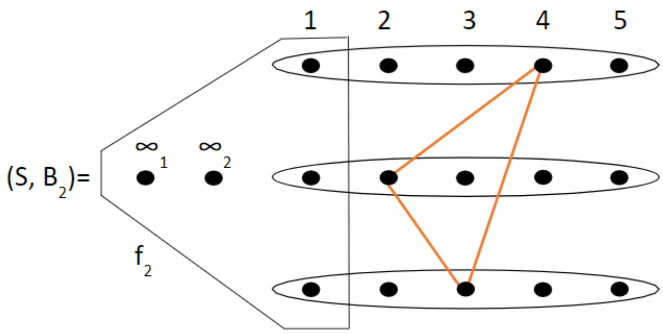
### 6.3 $n = 17$

Let $\mathcal{Q} = \{1, 2, 3, 4, 5\}$ and let $(\mathcal{Q}, \circ_1)$ and $(\mathcal{Q}, \circ_2)$ be two quasigroups such that $1 \circ_1 1 = 1 \circ_2 1 = 1$. Set $\mathcal{S} = \{\infty_1, \infty_2\} \cup (\{1, 2, 3, 4, 5\} \times \{1, 2, 3\})$ and define PBDs

$(\mathcal{S},\, B_1)$ and $(\mathcal{S},\, B_2)$ of order 17 as follows:




| $O_1$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | | | | | |
| 3 | | | | 2 | |
| 4 | | | | | |
| 5 | | | | | |




| $O_2$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | 3 | | | |
| 5 | | | | | |

1. $f_1 = f_2 = \{\infty_1,\, \infty_2,\, 11,\, 12,\, 13\} \in B_1 \cap B_2$. We can define copies of *Example 6.1* independently on $f_1$ and $f_2$ so that $|\, f_1 \cap f_2 \,| \in \{0,\, 1,\, 2\}$.

2. For each $i,\ j \in \{1, 2, 3, 4, 5\}$, let $\{i1,\ j2,\ (i \circ_1 j, 3)\} \in B_1$ and $\{i2,\ j2,\ (i \circ_2 j, 3)\} \in B_2$. (Note that $\{11,\ 12,\ 13\} \in B_1 \cap B_2$.) Since the intersection numbers for quasigroups of order 5 are $\{0,\ 1,\ 2,\ \ldots,\ 25\} \backslash \{24,\ 23,\ 22,\ 20\}$[2] and since in each of the quasigroups $(\mathcal{Q},\ \circ_1)$ and $(\mathcal{Q},\ \circ_2)$ $1 \circ_1 1 = 1 \circ_2 1 = 1$ and the triple $\{11,\ 21,\ 31\} \in f_1 \cap f_2$ the type (2) intersection numbers are $\{0,\ 1,\ 2,\ \ldots,\ 17,\ 18,\ 20,\ 24\}$.

3. For each $i \in \{1, 2, 3\}$ set $X(i) = \{\infty_1,\ \infty_2\} \cup \{\{1,\ 2,\ 3,\ 4,\ 5\} \times \{i\}\}$ and define a triple system $(X(i), T(i))$ where $\{\infty_1,\ \infty_2,\ 1i\} \in T(i)$. Since the intersection numbers for triple systems of order 7 are 0, 1, 3, 7; the intersection numbers for $T(i) \backslash \{\infty_1,\ \infty_2,\ 1i\}$ and $T(j) \backslash \{\infty_1,\ \infty_2,\ 1j\}$ for each $i$ and $j$ are 0, 2, and 6.

The intersection numbers in (1), (2), and (3) are independent of each other and so the intersection numbers for $(S,\ B_1)$ and $(S,\ B_2)$ consists of $x + y + z$, where $x = \mid f_1 \cap f_2 \mid \in \{0,\ 1,\ 2\}$, $y \in \{0,\ 1,\ 2,\ \ldots,\ 17,\ 18,\ 20,\ 24\}$, and $z \in \{0,\ 2,\ 6\} + \{0,\ 2,\ 6\} + \{0,\ 2,\ 6\}$. A straightforward computation shows that $x + y + z \in \{0,\ 1,\ 2,\ \ldots,\ 44\} \backslash \{41\}$. So all that remains is to show that $41 \in J^*(17) = \{0,\ 1,\ 2,\ \ldots,\ 44\}$ (no exceptions). Take $(S,\ B_1)$ and $(S,\ B_2)$ to be the same. Define $T(1)$ in $B_1$ to be

$$T(1) = \begin{cases} \infty_1 & \infty_2 & 11 \\ 11 & 21 & 31 \\ 11 & 41 & 51 \\ \infty_1 & 21 & 51 \\ \infty_1 & 31 & 41 \\ \infty_2 & 21 & 41 \\ \infty_2 & 31 & 51 \end{cases}$$

We can assume in $f_1$ that the leave is the $4-$cycle $(\infty_1,\ \infty_2,\ 11,\ 12)$. Then the configuration

$$Z_1 = \begin{cases} (\infty_1, & \infty_2, \quad 11, \quad 12) \\ & \{\ 21, \quad 31, \quad 11\ \} \\ & \{\ 31, \quad 41, \quad \infty_1\ \} \\ & \{\ 21, \quad 41, \quad \infty_2\ \} \end{cases}$$

belongs to $B_1$. If we replace $Z_1$ in $B_1$ with

$$Z_2 = \begin{cases} (\infty_1, & 12, \quad 11, \quad 31) \\ & \{\ \infty_1, \quad \infty_2, \quad 41\ \} \\ & \{\ \infty_2, \quad 11, \quad 21\ \} \\ & \{\ 21, \quad 31, \quad 41\ \} \end{cases}$$

we reduce the intersection number between $B_1$ and $B_2$ from 44 to 41.

Chapter 7

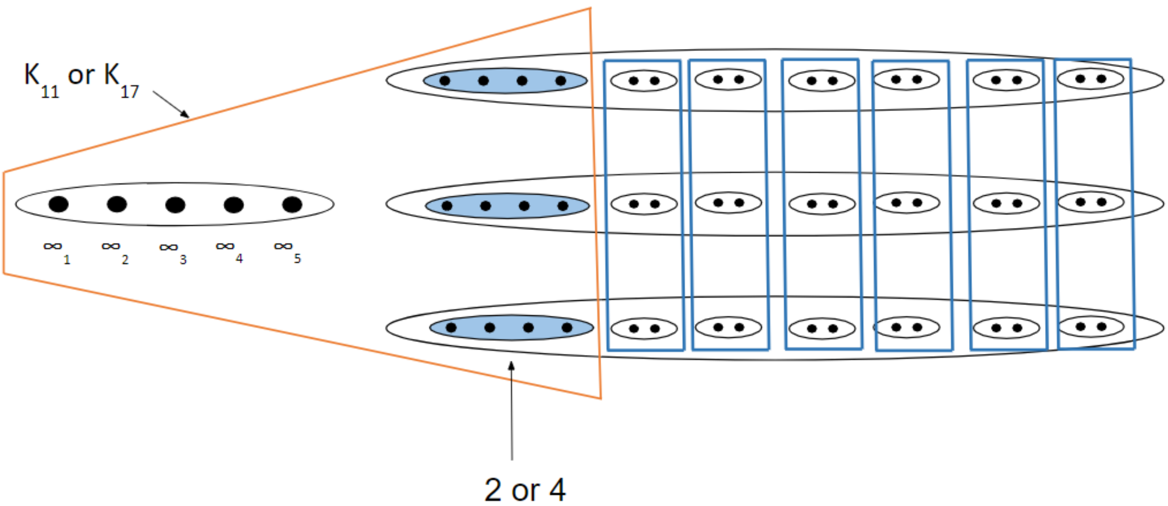
General Construction: $n \equiv 5 \pmod 6$

With the three examples in *Section 6* in hand we can proceed to the main construction showing that $I(6n+5) = J^*(6n+5) = \{0,\ 1,\ 2,\ \ldots,\ \frac{(\binom{n}{2}-4)}{3}\}$ for all $n$.

**<u>The $6n+5$ Construction:</u>** Let $6n+5 \geq 23$ and let $(X,\ G,\ B)$ be a $GDD(2n,\ 2,\ 3)$ or $GDD(2n,\ \{2,\ 4^*\},\ 3)$, where $\{2,\ 4^*\}$ means there is exactly one group of size 4 and the rest have size 2. Set $S = \{\infty_1,\ \infty_2,\ \infty_3,\ \infty_4,\ \infty_5\} \cup (X \times \{1,\ 2,\ 3\})$ and define a maximum packing, $P$ of $K_{6n+5}$ as follows:

1. Place an example of order 11 or 17 on $\{\infty_1,\ \infty_2,\ \infty_3,\ \infty_4,\ \infty_5\} \cup (g \times \{1,\ 2,\ 3\})$ where $g$ is a block of size 2 or 4 as the case may be.

2. For all other blocks (which necessarily have size 2) place a copy of *Example 6.2* or *6.3* on $\{\infty_1,\ \infty_2,\ \infty_3,\ \infty_4,\ \infty_5\} \cup (g \times \{1,\ 2,\ 3\})$ minus the block $\{\infty_1,\ \infty_2,\ \infty_3,\ \infty_4,\ \infty_5\}$ of size 5.




3. For each triple $\{a,\ b,\ c\} \in B$ decompose $K_{3,3,3}$ into 9 triples with parts $a \times \{1,\ 2,\ 3\}$, $b \times \{1,\ 2,\ 3\}$ and $c \times \{1,\ 2,\ 3\}$.

Then $(S, P)$ is a maximum packing of $K_{6n+5}$ with triples with leave a 4–cycle. Now take two copies of $(S, P)$. We need construct only the intersection numbers $x-1$, $x-2$, $x-3$, and $x-5$ since Quattrocchi has taken care of everything else. But this is easily done by defining a pair of maximum packings of order 11 or 17 on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (g \times \{1, 2, 3\})$ intersecting in $x-1$, $x-2$, $x-3$, or $x-5$ triples, where $x = 17$ or $44$ as the case may be. This completes the proof. We have the following theorem:

**Theorem 7.1.** $I(6n+5) = J^*(6n+5)$ for all $6n+5$. ■

Chapter 8

Concluding Remarks

We summarize the results in this thesis with the following table (a reprint from Chapter 1).

**Theorem 8.1.** The following table gives a complete solution of the intersection problem for maximum packings of $K_n$ with triples for $n \equiv 0, 2, 4, 5$:

| $n \equiv \pmod 6$ | New Intersection Spectrum |
|---|---|
| 0 or 2 | $I(6) = \{0, 1, 2, 4\}$, $I(8) = \{0, 1, 2, 3, 4, 5, 8\}$ and for all $n \equiv 0$ or $2 \geq 12$, $I(n) = \{0, 1, 2, \dots, \frac{n(n-2)}{6} = x\} \backslash \{x-1\}$ |
| 4 | $I(4) = \{0, 1\}$ and for all $n \equiv 4 \pmod 6 \geq 10$, $I(n) = \{0, 1, 2, \dots, \frac{(\binom{n}{2} - \frac{(x+2)}{2})}{3}\}$ |
| 5 | $I(5) = \{0, 1, 2\}$ and for all $n \equiv \pmod 6 \geq 11$, $I(n) = \{0, 1, 2, \dots, \frac{(\binom{n}{2}-4)}{3}\}$ |

■

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