A Property of GO-Topologies on the Reals

by

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Abstract

In 1981 Peter de Caux proved that finite powers of the Sorgenfrey line are hereditarily D-spaces. In this paper we build on de Caux's technique to show that any subspace of a finite power of the reals with a generalized ordered topology is a finite union of D-spaces and therefore a transitively D-space. We also note that finite powers of Sorgenfrey Suslin lines are D-spaces.

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Chapter 1

D-Spaces

A *neighborhood assignment* on a topological space X is a function φ from X to its topology such that $x \in \varphi(x)$ for each $x \in X$. We say X is a *D-space* if for every neighborhood assignment φ on X there exists a closed discrete subset D of X such that $\varphi[D]$ covers X. A neighborhood assignment is *transitive* if $\varphi(y) \subset \varphi(x)$ whenever $y \in \varphi(x)$, so a *transitively D-space* is one in which we can find such a closed discrete subset for every transitive neighborhood assignment. Certainly any D-space is a transitively D-space, but Dauvergne showed recently [1] that a transitively D-space need not be a D-space.

For terms or notation left undefined, see [3].

1.1 About closed discrete sets

Let A be a subset of a space X. To be precise we say *A is closed* if every point not in A has a neighborhood missing A, *A is discrete* if every point in A has a neighborhood missing every other point of A, and *x is a limit point of A* if every neighborhood of x meets A at a point other than x. Some ways of identifying closed discrete sets follow directly from our definitions.

Lemma 1.1. *The following are equivalent for a subset D of a space X.*

(a) *D is closed discrete in X.*

(b) *D has no limit points in X.*

(c) *There is a closed subset H of X containing D such that no point of H is a limit point of D.*

(d) *There is a closed subspace H of X containing D such that D is closed discrete in H.*

(e) *There is a closed subset H of X containing D such that every $x \in H$ has a neighborhood U with $U \cap D \subset \{x\}$.*

(f) *Every $x \in X$ has a neighborhood U with $U \cap D \subset \{x\}$.*

We will frequently need to build closed discrete sets by combining smaller closed discrete sets. The next few lemmas give some conditions under which we may do so.

Lemma 1.2. *A finite union of closed discrete subsets of a space X is closed discrete.*

Recall that a collection of subsets of a space is *discrete* if every point of the space has a neighborhood meeting at most one member of the collection.

Lemma 1.3. *Let $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ be a discrete collection of closed subsets of a space X. If $D_\alpha \subset F_\alpha$ is closed discrete in F_α for each $\alpha < \kappa$, then $D = \bigcup_{\alpha<\kappa} D_\alpha$ is closed discrete in X.*

Proof. Let $F = \bigcup \mathcal{F}$ and fix $x \in F$; $x \in F_\alpha$ for some $\alpha < \kappa$. There is a neighborhood U of x open-in-F missing F_β for $\beta \neq \alpha$, and a neighborhood V of x open-in-F with $V \cap D_\alpha \subset \{x\}$. It follows that $(U \cap V) \cap D \subset \{x\}$, so D is closed discrete in F. But F is closed in X [3, Lemma 135], so D is closed discrete in X. ■

Lemma 1.4. *Let $D = \bigcup_{\alpha<\kappa} D_\alpha$ be a union of closed discrete subsets of a space X. If every point of X has a neighborhood meeting only finitely many of the D_α's, then D is closed discrete.*

Proof. Fix $x \in X$, and let U_x be a neighborhood of x such that $F = \{\alpha < \kappa : U_x \cap D_\alpha \neq \varnothing\}$ is finite. For each $\alpha \in F$, let U_α be a neighborhood of x with $U_\alpha \cap D_\alpha \subset \{x\}$. Then $U = U_x \cap \bigcap_{\alpha \in F} U_\alpha$ is a neighborhood of x such that $U \cap D \subset \{x\}$. ■

Lemma 1.5. *Let $\{D_\alpha : \alpha < \kappa\}$ be a pairwise-disjoint collection of closed discrete subsets of a first-countable space X. If every $\{x_\alpha : \alpha < \kappa\} \subset X$ with $x_\alpha \in D_\alpha$ is closed discrete in X, then $D = \bigcup_{\alpha<\kappa} D_\alpha$ is closed discrete in X.*

Proof. If $\kappa < \omega$ the result is immediate, so we assume $\kappa \geqslant \omega$. Suppose D is not closed discrete. Fix a limit point x of D and let $\{B_n : n < \omega\}$ be a decreasing local base at x. For each $n < \omega$, let A_n consist of those $\alpha < \kappa$ for which B_n meets D_α at a point other than x. Observe that for each $n < \omega$ we have $|A_n| \geqslant \omega$ and $A_n \supset A_{n+1}$.

Set $\alpha_0 = \min A_0$, and let x_0 be a point of $B_0 \cap D_{\alpha_0}$ other than x. Suppose for some $n > 0$ that α_m and x_m have been defined for every $m < n$. Take $\alpha_n = \min A_n \setminus \{\alpha_m : m < n\}$, and let x_n be a point of $B_n \cap D_{\alpha_n}$ other than x. In this way, define α_n and x_n for each $n < \omega$.

By construction each B_n contains infinitely many of the x_n's, so x is a limit point of $\{x_n : n < \omega\}$. It follows that there is a subset $\{x_\alpha : \alpha < \kappa\}$ of X with $x_\alpha \in D_\alpha$ which is not closed discrete. ■

1.2 Unions of (transitively) D-spaces

In general we cannot say much about unions of D-spaces; in [6] Soukup and Szeptycki gave a consistent example of a space which is the union of two D-spaces but not itself a D-space. However, Lemma 1.3 implies the following.

Lemma 1.6. *Let $\{Y_\alpha : \alpha < \kappa\}$ be a discrete collection of closed subsets of a space X. If each Y_α is a D-space, then $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ is a D-space.*

The conditions in Lemma 1.6 can be relaxed a bit; the following is proved in [4].

Lemma 1.7. *Let $\{Y_\alpha : \alpha < \kappa\}$ be a collection of subsets of a space X such that $\bigcup_{\alpha<\beta} Y_\alpha$ is closed in X for each $\beta \leqslant \kappa$. If each Y_α is a D-space, then $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ is a D-space.*

Transitively D-spaces are a bit nicer. First, observe that being a (transitively) D-space is weakly hereditary.

Lemma 1.8. *Any closed subspace of a (transitively) D-space is a (transitively) D-space.*

Proof. Let φ be a (transitive) neighborhood assignment on a closed subset Y of a (transitively) D-space X. For each $x \in X$, define $\psi(x) = X \setminus Y$ when $x \notin Y$, and $\psi(x) = \varphi(x) \cup X \setminus Y$ when $x \in Y$; certainly ψ is a (transitive) neighborhood assignment on X. If $D \subset X$ is closed discrete and $\psi[D]$ covers X, then $D \cap Y$ is closed discrete in Y and $\varphi[D \cap Y]$ covers Y. ■

Lemma 1.9. *A finite union of transitively D-spaces is a transitively D-space.*

Proof. Let $X = Y \cup Z$, where Y and Z are transitively D-spaces, and fix a transitive neighborhood assignment φ on X. For each $y \in Y$ define $\psi(y) = \varphi(y) \cap Y$; then ψ is a transitive neighborhood assignment on Y, so let D_Y be closed discrete in Y such that $\psi[D_Y]$ is a cover of Y.

Put $A = \overline{D}_Y \setminus D_Y$. Since no point of Y can be a limit point of D_Y, A is closed in X and contained in Z. This means A is a transitively D-space; let D_A be closed discrete in A such that $\varphi[D_A]$ covers A.

Set $D'_Y = D_Y \setminus \bigcup \varphi[D_A]$, and observe $\bigcup \varphi[D_Y] \subset \bigcup \varphi[D'_Y \cup D_A]$ by transitivity. If D'_Y is closed discrete in X, then $D'_Y \cup D_A$ is closed discrete in X such that $\varphi[D'_Y \cup D_A]$ covers $A \cup Y$. To that end, first note that any limit point of D'_Y must belong to $\overline{D}_Y$. Since D_Y is discrete in Y, any limit point of D'_Y must in fact belong to $\overline{D}_Y \setminus D_Y = A$. But $A \subset \bigcup \varphi[D_A]$ and $\bigcup \varphi[D_A]$ misses D'_Y by construction, so D'_Y has no limit points.

Let $K = Z \setminus \bigcup \varphi[D'_Y \cup D_A]$; K is closed in X and contained in Z. Let D_K be closed discrete in K (hence, in X) such that $\varphi[K]$ covers K. Then $D'_Y \cup D_A \cup D_K$ is closed discrete in X and $\varphi[D'_Y \cup D_A \cup D_K]$ covers X. ■

1.3 Two easy examples

Recall that a *network* for a space X is a collection $\mathcal{N}$ of subsets of X such that, whenever U is open and contains x, there is some $N \in \mathcal{N}$ with $x \in N \subset U$. The following is straightforward.

Lemma 1.10. *Let $\mathcal{N}$ be a network for a space X. If for every $\psi\colon X \to \mathcal{N}$ with $x \in \psi(x)$ there is a closed discrete subset D of X such that $\psi[D]$ covers X, then X is a D-space.*

Proof. Let φ be a neighborhood assignment on X. For each $x \in X$ pick $\psi(x) \in \mathcal{N}$ such that $x \in \psi(x) \subset \varphi(x)$. If D is a subset of X such that $\psi[D]$ covers X, then $\varphi[D]$ covers X. ■

This allows us to devise a particularly nice base for our space and then investigate only those neighborhood assignments whose image is in the base. For first-countable spaces, we can make use of the following.

Theorem 1.11. *Let φ be a neighborhood assignment on a space X. Suppose there exists a function $j\colon X \to \omega$ such that, whenever $F \subset X$ is closed, there exists $D \subset F$ closed discrete in X such that $\varphi[D]$ covers $\{x \in F : j(x) = \min j[F]\}$. Then there exists a closed discrete subset D of X such that $\varphi[D]$ covers X.*

Proof. Set $F_0 = X$, $j_0 = \min j[F_0]$ and $G_0 = \{x \in F_0 : j(x) = j_0\}$. Let $D_0 \subset F_0$ be closed discrete in X such that $\varphi[D_0]$ covers G_0. If $\varphi[D_0]$ covers F_0 we are done; otherwise, note $\varphi[D_0]$ covers those x with $j(x) \leqslant j_0$ and proceed.

Suppose for some $\beta > 0$ that F_α, j_α, G_α, and D_α have been defined for each $\alpha < \beta$. If $\varphi[\bigcup_{\alpha<\beta} D_\alpha]$ covers F_0, take $\kappa = \beta$ and the process terminates. Otherwise put $F_\beta = F_0 \setminus \bigcup_{\alpha<\beta} \varphi[D_\alpha]$, $j_\beta = \min j[F_\beta]$, and $G_\beta = \{x \in F_\beta : j(x) = j_\beta\}$. Let $D_\beta \subset F_\beta$ be closed discrete in X such that $\varphi[D_\beta]$ covers G_β. Observe $j_0 < \cdots < j_\beta$ and $\varphi[\bigcup_{\alpha\leqslant\beta} D_\alpha]$ covers those x with $j(x) \leqslant j_\beta$. Continue with the induction.

This process must terminate; for some $\kappa \leqslant \omega$ we get a collection $\{D_\alpha : \alpha < \kappa\}$ of closed discrete subsets of X such that $\varphi[\bigcup_{\alpha<\kappa} D_\alpha]$ covers X. Fix $x \in X$. There exists $\alpha < \kappa$ and $y \in D_\alpha$ such that $x \in \varphi(y)$. By construction $\varphi(y)$ misses D_β whenever $\alpha < \beta < \kappa$, so $\bigcup_{\alpha<\kappa} D_\alpha$ is closed discrete in X by Lemma 1.4. ■

We provide two examples of using Theorem 1.11. Recall that the continuous image of a separable metric space is called a *cosmic space*.

Theorem 1.12. *Cosmic spaces are hereditarily D-spaces.*

Proof. Fix a continuous surjection f from a separable metric space Y onto a space X. If $\mathcal{B} = \{B_k : k < \omega\}$ is a base for Y, then $\mathcal{N} = \{N_k : k < \omega\}$ is a network for X, where $N_k = f[B_k]$. If Z is a subspace of X then $\{Z \cap N_k : k < \omega\}$ is a countable network for Z, so it suffices to show X is a D-space. Fix a neighborhood assignment φ on X, and for each $x \in X$ let $j(x)$ be minimal so that $x \in N_{j(x)} \subset \varphi(x)$.

Let $F \subset X$ be closed, set $j^* = \min j[F]$, and take $G = \{x \in F : j(x) = j^*\}$. Observe that $x \in N_{j^*} \subset \varphi(x)$ for each $x \in G$, so $G \subset N_{j^*}$. For each $x \in X$, $\{x\}$ is a closed discrete subset of X with $\varphi[\{x\}]$ a cover of G. By Theorem 1.11, X is a D-space. ■

Theorem 1.13. *Metrizable spaces are hereditarily D-spaces.*

Proof. Fix a metric space $\langle X, d\rangle$; every subspace of X is metrizable, so it suffices to show X is a D-space. Fix a neighborhood assignment φ on X, and for each $x \in X$ define $j(x)$ to be the least nonnegative integer such that $B(x, 2^{-j(x)})$ lies within $\varphi(x)$.

Let $F \subset X$ be closed, set $j^* = \min j[F]$, and take $G = \{x \in F : j(x) = j^*\}$. Well-order G and set $x_0 = \min G$. If x_α has been defined for each $\alpha < \beta$, take $x_\beta = \min G \setminus \bigcup_{\alpha<\beta} \varphi(x_\alpha)$. Continue in this way until the $\varphi(x_\alpha)$'s cover G. The set D consisting of the x_α's is closed discrete, since for each $x \in X$ the ball $B(x, 4^{-j^*})$ contains at most one point of D. By Theorem 1.11, X is a D-space. ■

Chapter 2

GO-Topologies

Let $\langle X, \leqslant\rangle$ be a linearly ordered set. Denote by $\mathcal{T}_\leqslant$ the order topology on X; that is, the topology generated by the subbase $\{X\} \cup \{(-\infty, x) : x \in X\} \cup \{(x, \infty) : x \in X\}$. A topology $\mathcal{T}$ on X is a *generalized ordered topology* (*GO-topology*) if $\mathcal{T}_\leqslant \subset \mathcal{T}$ and $\mathcal{T}$ has a base of $\leqslant$-convex sets. Note that $\mathcal{T}_\leqslant$ is itself a GO-topology on X.

In [8] van Douwen and Pfeffer showed that every finite power of the Sorgenfrey line is a D-space. Peter de Caux proved in [2] that finite powers of the Sorgenfrey line are in fact hereditarily D-spaces. In this paper we build on de Caux's technique to show that any subspace of a finite power of the reals with a GO-topology is a finite union of D-spaces. This will imply that finite powers of the reals with a GO-topology are hereditarily transitively D-spaces.

2.1 GO-partitions

We generate GO-topologies using *GO-partitions*, functions from X to the set of symbols $\{\ell, e, r, i\}$. From our definitions we immediately get the following.

Lemma 2.1. *If τ is a GO-partition of a linearly ordered set $\langle X, \leqslant\rangle$, then the topology $\mathcal{T}$ generated by the subbase $\mathcal{T}_\leqslant \cup \{(-\infty, x] : \tau(x) = \ell\} \cup \{[x, \infty) : \tau(x) = r\} \cup \{\{x\} : \tau(x) = i\}$ is a GO-topology on X.*

We will let $\mathcal{T}_\tau$ denote the GO-topology generated in Lemma 2.1, and X_τ will denote the *generalized ordered space* (*GO-space*) $\langle X, \mathcal{T}_\tau\rangle$. For any linearly ordered set, the GO-partitions ε and λ are those which assign to every point the values e and ℓ, respectively.

If the linearly ordered set is $\mathbb{R}$, for example, then $\mathbb{R}_\varepsilon$ is the set of reals with the usual metric topology and $\mathbb{R}_\lambda$ is the Sorgenfrey line. The next result says that we can examine the GO-topologies on X by considering the GO-partitions of X.

Lemma 2.2. *If $\mathcal{T}$ is a GO-topology on the linearly ordered set X, then there exists a GO-partition τ such that $\mathcal{T} = \mathcal{T}_\tau$.*

Proof. For each $x \in X$ define $\tau(x)$ to be i if x is isolated in $\mathcal{T}$, ℓ if x is nonisolated and has a neighborhood $(a_x, x]$ in $\mathcal{T}$, r if x is nonisolated and has a neighborhood $[x, b_x)$ in $\mathcal{T}$, and e otherwise. Observe that τ is well-defined and a GO-partition on X.

First note that a point x is isolated in $\mathcal{T}$ if and only if it is isolated in $\mathcal{T}_\tau$. Suppose $\tau(x) = \ell$ and $y < x$. Then $(y, x] = (y, \infty) \cap (-\infty, x]$ is open in $\mathcal{T}_\tau$. Moreover $(y, x] = (y, x) \cup (a_x, x]$ or $(y, x] = (a_x, x] \cap (y, \infty)$, both of which are open in $\mathcal{T}$. Similarly, if $\tau(x) = r$ then $[x, y)$ is open in $\mathcal{T}$ and $\mathcal{T}_\tau$ for each $y > x$. Finally both $\mathcal{T}$ and $\mathcal{T}_\tau$ contain $\mathcal{T}_\leqslant$, so $\mathcal{T} = \mathcal{T}_\tau$. ■

Let $\mathrm{GP}(X)$ denote the set of all GO-partitions on X, and let $\mathrm{GP}'(X)$ denote the set of all GO-partitions on X whose image does not include i. We will see that, for the property we are interested in, we may do most of our work in $\mathrm{GP}'(\mathbb{R})$ and then easily extend the result to all of $\mathrm{GP}(\mathbb{R})$.

2.2 Paracompactness

Recall that a space X is *paracompact* if every open cover of X has a locally finite open refinement, where a collection of subsets of X is *locally finite* whenever each point of X has a neighborhood meeting only finitely many members of the collection, and a collection $\mathcal{U}$ of subsets of X is a *refinement* of a collection $\mathcal{V}$ of subsets of X if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subset V$. In the class of GO-spaces, being a D-space is equivalent to being paracompact.

Lemma 2.3 [5, Theorem 4.10]**.** *Let $\langle X, \leqslant\rangle$ be a linearly ordered set. If X_ε is hereditarily paracompact then X_τ is hereditarily paracompact for each $\tau \in \mathrm{GP}(X)$.*

Lemma 2.4 [7, Theorem 1.2]**.** *Let $\langle X, \leqslant \rangle$ be a linearly ordered set. For $\tau \in \mathrm{GP}(X)$, X_τ is paracompact if and only if X_τ is a D-space.*

Since a subspace of a GO-space is itself a GO-space, we conclude the following.

Theorem 2.5. *Let $\langle X, \leqslant \rangle$ be a linearly ordered set. If X_ε is hereditarily paracompact then X_τ is hereditarily a D-space for each $\tau \in \mathrm{GP}(X)$.*

In particular, $\mathbb{R}_\tau$ is hereditarily a D-space for each $\tau \in \mathrm{GP}(\mathbb{R})$.

Chapter 3

Every Subspace of $\mathbb{R}_\tau^N$ is a Finite Union of D-Spaces

Fix $\tau \in \mathrm{GP}'(\mathbb{R})$. For $x \in \mathbb{R}$ and $m < \omega$ let $Q_\mathbb{R}(m)$ be the set of integer multiples of 2^{-m}, $a_m(x)$ the greatest point of $Q_\mathbb{R}(m)$ less than x, and $b_m(x)$ the least point of $Q_\mathbb{R}(m)$ greater than x. Define $B_\tau(x,m)$ to be $(a_m(x), x]$ if $\tau(x) = \ell$, $(a_m(x), b_m(x))$ if $\tau(x) = e$, and $[x, b_m(x))$ if $\tau(x) = r$; certainly the $B_\tau(x,m)$'s form a base for $\mathbb{R}_\tau$. Let $B^\circ_\tau(x,m)$ denote the interior of $B_\tau(x,m)$ in $\mathbb{R}_\varepsilon$.

3.1 Single-type subspaces

Fix $1 < N < \omega$. Let X be a subspace of $\mathbb{R}_\tau^N$ such that $\tau(x_n) = \tau(y_n)$ for all $\vec{x}, \vec{y} \in X$ and $n < N$; a subspace with this property is called a *single-type subspace*. For two points $\vec{x}$ and $\vec{y}$ of X, say $\vec{x} \lhd \vec{y}$ if $x_n < y_n$ when $\tau(y_n) = \ell$ and $y_n < x_n$ when $\tau(y_n) = r$.

For $\vec{x} \in X$ and $m < \omega$ let $Q(\vec{x}, m) = \{n < N : x_n \in Q_\mathbb{R}(m)\}$; let $Q(m)$ consist of those $\vec{x}$ with $Q(\vec{x}, m)$ nonempty. If $\vec{x} \in Q(m)$ for some m, let $q(\vec{x})$ be the least such m. Set $B(\vec{x}, m) = X \cap \prod_{n<N} B_\tau(x_n, m)$ and $B^\circ(\vec{x}, m) = X \cap \prod_{n<N} B^\circ_\tau(x_n, m)$. The $B(\vec{x}, m)$'s form a base for X and have the following properties.

Lemma 3.1. *Fix $\vec{x} \in X$, $m < \omega$, $\vec{y} \in B^\circ(\vec{x}, m)$, and $\vec{z} \in B(\vec{x}, m)$ such that $\vec{y} \lhd \vec{z}$.*

(a) *$B(\vec{y}, m) \subset B^\circ(\vec{x}, m)$.*

(b) *If $\{n \in Q(\vec{x}, m) : \tau(x_n) = e\} \subset Q(\vec{z}, m)$, then $\vec{y} \in B^\circ(\vec{z}, m)$.*

(c) *If $\vec{x} \in Q(m) \setminus Q(0)$, then $\vec{y} \in B^\circ(\vec{z}, q(\vec{x}) - 1)$.*

Proof.

(a) Certainly $B_\tau(y_n, m) \subset B^\circ_\tau(x_n, m)$ for each $n < N$.

(b) Fix $n < N$. If $\tau(x_n) = \ell$ then $a_m(z_n) = a_m(x) < y_n < z_n \leqslant x_n$, so $y_n \in B^\circ_\tau(z_n, m)$. Similarly $y_n \in B^\circ_\tau(z_n, m)$ if $\tau(x_n) = r$.

Now suppose $\tau(x_n) = e$. If $n \notin Q(\vec{x}, m)$ then $n \notin Q(\vec{z}, m)$, so $a_m(x_n) = a_m(z_n)$ and $b_m(x_n) = b_m(z_n)$. If $n \in Q(\vec{x}, m)$ and $n \in Q(\vec{z}, m)$ then $x_n = z_n$, so $a_m(x_n) = a_m(z_n)$ and $b_m(x_n) = b_m(z_n)$. In either case $y_n \in B_\tau(z_n, m)$.

If $n \in Q(\vec{x}, m)$ but $n \notin Q(\vec{z}, m)$ it is possible, for example, that $y_n < x_n = a_m(z_n) < z_n$, in which case $y_n \notin B^\circ_\tau(z_n, m)$.

(c) Fix $n < N$ and set $m' = q(\vec{x}) - 1$. Note that $0 \leqslant m' < m$, $a_{m'}(z_n) \leqslant a_m(x_n)$, and $b_m(x_n) \leqslant b_{m'}(z_n)$. ■

Let φ be a neighborhood assignment on X. We may assume that for each $\vec{x} \in X$ there exists $m < \omega$ such that $\varphi(\vec{x}) = B(\vec{x}, m)$; let $j(\vec{x})$ be the least such m. Take $\varphi^\circ(\vec{x})$ to mean $B^\circ(\vec{x}, j(\vec{x}))$. The following result is immediate.

Lemma 3.2. *Fix $\vec{x} \in X$, $\vec{y} \in \varphi^\circ(\vec{x})$, and $\vec{z} \in \varphi(\vec{x})$ with $\vec{y} \triangleleft \vec{z}$.*

(a) *If $j(\vec{x}) \leqslant j(\vec{y})$, then $\varphi(\vec{y}) \subset \varphi^\circ(\vec{x})$.*

(b) *If $j(\vec{z}) \leqslant j(\vec{x})$ and $\{n \in Q(\vec{x}, j(\vec{x})) : \tau(x_n) = e\} \subset Q(\vec{z}, j(\vec{x}))$, then $\vec{y} \in \varphi^\circ(\vec{z})$.*

(c) *If $\vec{x} \in Q(j(\vec{x}))$ and $j(\vec{z}) < q(\vec{x})$, then $\vec{y} \in \varphi^\circ(\vec{z})$.*

3.2 Extending de Caux's lemmas

Suppose for each $1 \leqslant M < N$ that every single-type subspace of $\mathbb{R}^M_\tau$ is a D-space. We extend the three lemmas in [2]. If D is a subset of X and we refer to the *assigned neighborhoods* of D, we mean the collection $\varphi[D]$.

Let $\mathcal{A}$ be a collection of subsets of X such that $A \in \mathcal{A}$ if and only if $A \subset X$ is closed and nonempty, $\varphi^\circ(\vec{x}) \cap A = \varnothing$ for each $\vec{x} \in A$, and for each $\vec{x} \in A$ there exists $\vec{y} \in A \setminus \{\vec{x}\}$ with $\vec{x} \in \varphi(\vec{y})$. For $\vec{x}, \vec{y} \in X$, let $\delta(\vec{x}, \vec{y}) = \{n < N : \tau(x_n) \in \{\ell, r\} \wedge x_n = y_n\}$.

Fix $A \in \mathcal{A}$, and for each $\vec{x} \in A$ define $T(\vec{x}) = \{\vec{y} \in A \setminus \{\vec{x}\} : \vec{x} \in \varphi(\vec{y})\}$ and $t(\vec{x}) = \min\{|\delta(\vec{x},\vec{y})| : \vec{y} \in T(\vec{x})\}$. Let $T'(\vec{x})$ be a maximal subset of $T(\vec{x})$ such that $|\delta(\vec{x},\vec{y})| = t(\vec{x})$ for all $\vec{y} \in T'(\vec{x})$, and $\delta(\vec{x},\vec{y}) \neq \delta(\vec{x},\vec{z})$ whenever $\vec{y}$ and $\vec{z}$ are two points of $T'(\vec{x})$. Set $O(\vec{x}) = \varphi(\vec{x}) \cap \bigcap \varphi[T'(\vec{x})]$ and $\delta(\vec{x}) = \{\delta(\vec{x},\vec{y}) : \vec{y} \in T'(\vec{x})\}$.

Lemma 3.3. *If $\vec{x} \neq \vec{y} \in A$, $\vec{y} \in O(\vec{x})$, and $\vec{z} \in T'(\vec{x})$, then $\delta(\vec{y},\vec{z}) \subset \delta(\vec{x},\vec{z})$ and $t(\vec{y}) \leqslant |\delta(\vec{y},\vec{z})| \leqslant |\delta(\vec{x},\vec{z})| = t(\vec{x})$. Hence, if $t(\vec{y}) = t(\vec{x})$ then $\delta(\vec{y},\vec{z}) = \delta(\vec{x},\vec{z})$.*

Proof. Suppose $n \in \delta(\vec{y},\vec{z}) \setminus \delta(\vec{x},\vec{z})$. If $\tau(x_n) = \ell$ then $x_n < z_n = y_n$, and if $\tau(x_n) = r$ then $y_n = z_n < x_n$. In either case $\vec{y} \notin \varphi(\vec{x})$, a contradiction. ■

Lemma 3.4. *If $K = \{\vec{x} \in A : t(\vec{x}) = \max t[A]\}$, then K contains a closed discrete subset of X whose assigned neighborhoods cover K.*

Proof. Set $t^* = \max t[A]$. If $\vec{x} \in A \setminus K$ and $\vec{y} \in O(\vec{x}) \cap A$ then $t(\vec{y}) \leqslant t(\vec{x}) < t^*$, so K is closed. For $\vec{x},\vec{y} \in K$, say $\vec{x} \prec \vec{y}$ if and only if $\delta(\vec{x}) \subsetneq \delta(\vec{y})$. Set $K_0 = K$.

Let $\vec{x}_0$ be $\prec$-minimal in K_0, and put $L_0 = \{\vec{x} \in K_0 : \delta(\vec{x}) = \delta(\vec{x}_0)\}$. Suppose $\vec{x} \in K_0 \setminus L_0$ and $\vec{y} \in O(\vec{x}) \cap K_0$. Note $\delta(\vec{y},\vec{z}) = \delta(\vec{x},\vec{z})$ for each $\vec{z} \in T'(\vec{x})$, so $\delta(\vec{x}) \subset \delta(\vec{y})$. It follows that $\vec{y} \notin L_0$, so L_0 is closed in X. Let Δ be the unique member of $\delta[L_0]$, and form a paritition $\mathcal{L}$ of L_0 so that two points $\vec{x}$ and $\vec{y}$ of L_0 belong to the same member of $\mathcal{L}$ if and only if $x_n = y_n$ for each $n \in \bigcup \Delta$. Suppose $\vec{x} \in L_0$ and $\vec{y} \in O(\vec{x}) \cap L_0$. For each $n \in \bigcup \Delta$ there exists $\vec{z} \in T'(\vec{x})$ with $n \in \delta(\vec{x},\vec{z})$; we have $x_n = y_n$ since $\delta(\vec{x},\vec{z}) = \delta(\vec{y},\vec{z})$. It follows that $\mathcal{L}$ is a discrete collection of closed sets, each of which is a D-space by the inductive hypothesis. Then $L_0 = \bigcup \mathcal{L}$ is a D-space, so let $D_0 \subset L_0$ be closed discrete in X such that $\varphi[D_0]$ covers L_0.

If $\varphi[D_0]$ does not cover K_0, put $K_1 = K_0 \setminus \bigcup \varphi[D_0]$. Let $\vec{x}_1$ be $\prec$-minimal in K_1, and put $L_1 = \{\vec{x} \in K_1 : \delta(\vec{x}) = \delta(\vec{x}_1)\}$. By a similar argument to that for L_0, let $D_1 \subset L_1$ be closed discrete in X such that $\varphi[D_1]$ covers L_1.

If $\varphi[D_1]$ does not cover K_1, put $K_2 = K_1 \setminus \bigcup \varphi[D_1]$. Let $\vec{x}_2$ be $\prec$-minimal in K_2, and ...

This process must end since $\delta[K]$ is finite, and the union of the D_n's is a closed discrete subset of X whose assigned neighborhoods cover K. ■

Lemma 3.5. *Let $H \subset X$ be closed such that $\varphi^\circ(\vec{x}) \cap H = \varnothing$ for all $\vec{x} \in H$. Then there exists $A \in \mathcal{A}$ and a closed discrete subset D of X such that $A \subset H$, $D \subset H \setminus A$, and $\varphi[D]$ covers $H \setminus A$.*

Proof. For any subset H' of H, let $D(H') = \{\vec{x} \in H' : \vec{x} \notin \bigcup \varphi[H' \setminus \{\vec{x}\}]\}$. Certainly $D(H')$ is closed discrete in H'.

Set $H_0 = H$. Suppose for some $\beta > 0$ that H_α has been defined for each $\alpha < \beta$ so that $H_\alpha = H_0 \setminus \bigcup \varphi[\bigcup_{\xi<\alpha} D(H_\xi)]$. Put $H_\beta = H_0 \setminus \bigcup \varphi[\bigcup_{\alpha<\beta} D(H_\alpha)]$. If $D(H_\beta) \neq \varnothing$ proceed to the next step; otherwise, set $\kappa = \beta$ and the construction terminates.

This must terminate. We claim $\{D(H_\alpha) : \alpha < \kappa\}$ is a discrete collection. Certainly $X \setminus H_0$ is open and misses each $D(H_\alpha)$, and if $\vec{x} \in H_\kappa$ and $\vec{y} \in \varphi(\vec{x})$ then $\vec{y} \notin D(H_\alpha)$ for each $\alpha < \kappa$. It remains to be seen that each point of $H_0 \setminus H_\kappa \subset \bigcup \varphi[\bigcup_{\alpha<\kappa} D(H_\alpha)]$ has a neighborhood meeting at most one of the $D(H_\alpha)$'s. To that end fix $\alpha < \kappa$ and $\vec{x} \in D(H_\alpha)$; observe $\vec{y} \in \varphi(\vec{x})$ implies $\vec{y} \notin D(H_\xi)$ for each $\xi < \alpha$, and $\alpha < \xi < \kappa$ implies $D(H_\xi) \subset H_\xi \subset H_0 \setminus \varphi(\vec{x})$ by construction.

We have $H_\kappa \in \mathcal{A}$, and $\bigcup_{\alpha<\kappa} D(H_\alpha)$ is a closed discrete subset of X contained in $H \setminus H_\kappa$ whose assigned neighborhoods cover $H \setminus H_\kappa$. ■

Lemma 3.6. *If $H \subset X$ is closed such that $\varphi^\circ(\vec{x}) \cap H = \varnothing$ for all $\vec{x} \in H$, then H contains a closed discrete subset of X whose assigned neighborhoods cover H.*

Proof. Set $H_0 = H$. By Lemma 3.5 there exists $A_0 \in \mathcal{A}$ such that $A_0 \subset H_0$ and $H_0 \setminus A_0$ contains a closed discrete subset E_0 of X whose assigned neighborhoods cover $H_0 \setminus A_0$. If $A_0 = \varnothing$ put $D_0 = \varnothing$ and the construction terminates. Otherwise put $t_0 = \max t[A_0]$, and let $K_0 = \{\vec{x} \in A_0 : t(\vec{x}) = t_0\}$. By Lemma 3.4, let $D_0 \subset K_0$ be closed discrete in X such that $\varphi[D_0]$ covers K_0.

If the assigned neighborhoods of D_0 do not cover A_0, let $H_1 = A_0 \setminus \bigcup \varphi[D_0]$. There exists $A_1 \in \mathcal{A}$ and E_1 closed discrete in X such that $A_1 \subset H_1$, $E_1 \subset H_1 \setminus A_1$, and the assigned neighborhoods of E_1 cover $H_1 \setminus A_1$. If $A_1 = \varnothing$ put $D_1 = \varnothing$ and the construction terminates. Otherwise put $t_1 = \max t[A_1]$ and take $K_1 = \{\vec{x} \in A_1 : t(\vec{x}) = t_1\}$; let $D_1 \subset K_1$ be closed discrete in X such that $\varphi[D_1]$ covers K_1.

If the assigned neighboorhoods of D_1 do not cover A_1, let $H_2 = A_1 \setminus \bigcup \varphi[D_1]$. There exists ...

This process ends since the t_n's are strictly decreasing. The union of the D_n's and E_n's is a closed discrete subset of X whose assigned neighborhoods cover H. ■

Lemma 3.7. *Let $G \subset X$ such that $j(\vec{x}) = j(\vec{y})$ for all $\vec{x}, \vec{y} \in G$ and $\overline{G} \subset \bigcup \varphi^\circ[G]$. Then there is a countable $D \subset G$ closed discrete in X such that $\bigcup \varphi^\circ[D] = \bigcup \varphi^\circ[G]$.*

Proof. Since finite powers of $\mathbb{R}_\varepsilon$ are hereditarily Lindelöf, there is a countable subset C of G with $\bigcup \varphi^\circ[C] = \bigcup \varphi^\circ[G]$. Let C be well-ordered with order type ω and put $\vec{x}_0 = \min C$. Inductively define $\vec{x}_n = \min\{\vec{x} \in C : \varphi^\circ(\vec{x}) \not\subset \bigcup_{m<n} \varphi^\circ(\vec{x}_m)\}$ for each $0 < n < \omega$, and put $D = \{\vec{x}_n : n < \omega\}$. Certainly $\bigcup \varphi^\circ[D] = \bigcup \varphi^\circ[C]$.

Let $\vec{x} \in \overline{G}$; there exists $m < \omega$ such that $\vec{x} \in \varphi^\circ(\vec{x}_m)$. Observe that $n \neq m$ and $\vec{x}_n \in \varphi^\circ(\vec{x}_m)$ implies $\varphi^\circ(\vec{x}_n) \subset \varphi(\vec{x}_n) \subset \varphi^\circ(\vec{x}_m)$, since $j(\vec{x}_n) = j(\vec{x}_m)$. It follows that $\varphi^\circ(\vec{x}_m)$ is a neighborhood of $\vec{x}$ which does not contain $\vec{x}_n$ whenever $m < n$, so $\vec{x}$ is not a limit point of D. ■

Lemma 3.8. *If $F \subset X$ is closed and $G = \{\vec{x} \in F : j(\vec{x}) = \min j[F]\}$, then F contains a closed discrete subset of X whose assigned neighborhoods cover G.*

Proof. Set $j^* = \min j[F]$. Fix $n < N$ and let $K = \{\vec{x} \in \overline{G} : n \in Q(\vec{x}, j^*)\}$; note K is closed. Form a partition $\mathcal{K}$ of K so that two points $\vec{x}$ and $\vec{y}$ of K belong to the same member of $\mathcal{K}$ if and only if $x_n = y_n$. Observe $\mathcal{K}$ is a discrete collection of closed sets whose every member is a D-space, so K is a D-space. This is true for each $n < N$, so let $E \subset \overline{G} \cap Q(j^*)$ be a closed discrete subset of X whose assigned neighborhoods cover $\overline{G} \cap Q(j^*)$.

Set $G_0 = G \setminus \bigcup \varphi[E]$. If $\overline{G_0}$ is covered by $\varphi^\circ[G_0]$, we are done by Lemma 3.7; otherwise, let $H_0 = \overline{G_0} \setminus \bigcup \varphi^\circ[G_0]$ and note that H_0 is closed and misses $Q(j^*)$. Suppose there exist $\vec{x}, \vec{y} \in H_0$ with $\vec{y} \in \varphi^\circ(\vec{x})$. Then $\vec{x} \in \overline{G_0} \setminus G_0$, so there exists $\vec{z} \in G_0$ such that $\vec{z} \in \varphi(\vec{x})$ and $\vec{y} \lhd \vec{z}$. By construction, $j(\vec{z}) = j^* \leqslant j(\vec{x})$ and $\vec{x} \in \overline{G} \setminus Q(j^*)$. If $\vec{x} \notin Q(j(\vec{x}))$ then $Q(\vec{x}, j(\vec{x})) = \varnothing$, so $\vec{y} \in \varphi^\circ(\vec{z})$ by Lemma 3.2(b). If $\vec{x} \in Q(j(\vec{x}))$ then

$j(\vec{z}) = j^* < q(\vec{x})$, so $\vec{y} \in \varphi^\circ(\vec{z})$ by Lemma 3.2(c). In either case we get $\vec{y} \notin H_0$, a contradiction. It follows that $\varphi^\circ(\vec{x}) \cap H_0 = \varnothing$ for all $\vec{x} \in H_0$. By Lemma 3.6, let $D_0 \subset H_0$ be a closed discrete subset of X whose assigned neighborhoods cover H_0. Note $\bigcup \varphi[D_0]$ meets G_0 since $H_0 \subset \overline{G}_0$; if the assigned neighborhoods of D_0 cover G_0 we are done, otherwise continue.

Suppose for some $\beta > 0$ that for each $\alpha < \beta$ we have defined $G_\alpha \subset G_0$, $H_\alpha \subset \overline{G}_\alpha$, and D_α such that $\varphi[D_\alpha]$ covers H_α. If the assigned neighborhoods of $\bigcup\limits_{\alpha<\beta} D_\alpha$ cover G_0 the construction terminates; put $\kappa = \beta$, $G_\beta = \varnothing$, and $D_\beta = \varnothing$. Otherwise put $G_\beta = G_0 \setminus \bigcup \varphi[\bigcup\limits_{\alpha<\beta} D_\alpha]$ and do as follows.

(a) If $\overline{G}_\beta \subset \bigcup \varphi^\circ[G_\beta]$, use Lemma 3.7 to get $D_\beta \subset G_\beta$ closed discrete in X with $\varphi^\circ[D_\beta]$ a cover of $\overline{G}_\beta$. The construction terminates; take $\kappa = \beta$.

(b) If $\overline{G}_\beta \not\subset \bigcup \varphi^\circ[G_\beta]$, define $H_\beta = \overline{G}_\beta \setminus \bigcup \varphi^\circ[G_\beta]$ and note that H_β is closed and misses $Q(j^*)$. A similar argument to that for H_0 shows $\varphi^\circ(\vec{x}) \cap H_\beta = \varnothing$ for all $\vec{x} \in H_\beta$. By Lemma 3.6, let $D_\beta \subset H_\beta$ be a closed discrete subset of X whose assigned neighborhoods cover H_β. Note that $\bigcup \varphi[D_\beta]$ meets G_β, and proceed to the next step.

This process must terminate; we get collections $\{G_\alpha : \alpha < \kappa\}$, $\{H_\alpha : \alpha < \kappa\}$, and $\{D_\alpha : \alpha < \kappa\}$ defined as above for some κ, and $D_\kappa \subset G_\kappa \subset G_0$ is a closed discrete subset of X whose assigned neighborhoods cover G_κ.

Let $Y = \{\vec{y}_\alpha : \alpha < \kappa\}$ so that $\vec{y}_\alpha \in D_\alpha$ for each $\alpha < \kappa$. Fix $\alpha < \beta < \kappa$. Note $\vec{y}_\alpha \in D_\alpha \subset H_\alpha = \overline{G}_\alpha \setminus \bigcup \varphi^\circ[G_\alpha]$, so $\vec{y}_\alpha \notin \bigcup \varphi^\circ[G_\alpha]$. Recall $G_\beta = G_0 \setminus \bigcup \varphi[\bigcup\limits_{\xi<\beta} D_\xi]$ and $G_\alpha = G_0 \setminus \bigcup \varphi[\bigcup\limits_{\xi<\alpha} D_\xi]$, so $G_\beta \subset G_\alpha$ and $\overline{G}_\beta \cap \bigcup \varphi[D_\alpha] = \varnothing$. By construction $H_\alpha = \overline{G}_\alpha \setminus \bigcup \varphi^\circ[G_\alpha] \subset \bigcup \varphi[D_\alpha]$, so $\overline{G}_\beta \cap H_\alpha = \varnothing$. But $\overline{G}_\beta \subset \overline{G}_\alpha$, so $\vec{y}_\beta \in D_\beta \subset H_\beta \subset \overline{G}_\beta \subset \overline{G}_\alpha \setminus H_\alpha = \bigcup \varphi^\circ[G_\alpha]$. This also shows $D_\alpha \cap D_\beta = \varnothing$.

Finally, suppose $\vec{x}$ is a limit point of Y; there exists $\alpha < \beta < \kappa$ such that $\vec{y}_\alpha, \vec{y}_\beta \in B(\vec{x}, j^*)$ and $\vec{y}_\alpha \triangleleft \vec{y}_\beta$. Since $\vec{x} \notin Q(j^*)$ we have $\vec{y}_\alpha \in B(\vec{y}_\beta, j^*)$, and $\vec{y}_\beta$ but not $\vec{y}_\alpha$ belongs to $\bigcup \varphi^\circ[G_\alpha]$. Take $\vec{z} \in G_\alpha$ such that $\vec{y}_\beta$ but not $\vec{y}_\alpha$ belongs to $\varphi^\circ(\vec{z})$. Note $j(\vec{z}) = j^*$, so $B(\vec{y}_\beta, j^*) \subset B^\circ(\vec{z}, j^*) = \varphi^\circ(\vec{z})$ by Lemma 3.1. This implies $\vec{y}_\alpha \in \varphi^\circ(\vec{z})$, a contradiction.

Since Y is closed discrete in X, we have that $D = \bigcup_{\alpha<\kappa} D_\alpha$ is a closed discrete subset of X whose assigned neighborhoods cover $G_0 \setminus G_\kappa$.

Then $D \cup D_\kappa \cup E \subset F$ is a closed discrete subset of X whose assigned neighborhoods cover G. ■

3.3 The result

We have that X is a D-space by Theorem 1.11. We have shown that every single-type subspace of $\mathbb{R}_\tau^N$ is a D-space if every single-type subspace of $\mathbb{R}_\tau^M$ is a D-space for $1 \leqslant M < N$. But $\mathbb{R}_\tau$ is hereditarily a D-space, so the following is true.

Theorem 3.9. *If $\tau \in \mathrm{GP}'(\mathbb{R})$ and $N \geqslant 1$, then every single-type subspace of $\mathbb{R}_\tau^N$ is a D-space.*

Extending the previous theorem to every GO-partition in $\mathrm{GP}(\mathbb{R})$ is straightforward. Let $\tau \in \mathrm{GP}(\mathbb{R})$. We know $\mathbb{R}_\tau$ is hereditarily a D-space, so fix $N > 1$ and let X be a single-type subspace of $\mathbb{R}_\tau^N$. Pick any $\vec{x} \in X$ and set $i^* = \{n < N : \tau(x_n) = i\}$. Form a partition $\mathcal{X}$ of X so that two points $\vec{x}$ and $\vec{y}$ of X belong to the same member of $\mathcal{X}$ if and only if $x_n = y_n$ for each $n \in i^*$. Then $X = \bigcup \mathcal{X}$ is a D-space, as $\mathcal{X}$ is a discrete collection of closed subsets of X whose every member is a D-space.

We have that every single-type subspace of a finite power of $\mathbb{R}$ with a GO-topology is a D-space. Since any subspace is a finite union of single-type subspaces, the following is proved.

Theorem 3.10. *Every subspace of a finite power of $\mathbb{R}$ with a GO-topology is a finite union of D-spaces.*

Corollary 3.11. *Every finite power of $\mathbb{R}$ with a GO-topology is hereditarily a transitively D-space.*

Chapter 4

A Note on Sorgenfrey Suslin Lines

We can generalize the argument used in [8] to show that finite powers of the Sorgenfrey line are D-spaces. A space X is called a *generalized left-separated space* (*GLS-space*) if there is a reflexive binary relation $\preccurlyeq$ on X, called a *GLS-relation*, such that every nonempty closed subset of X has a $\preccurlyeq$-minimal element and $\{y \in X : x \preccurlyeq y\}$ is open for each $x \in X$. We need the following results.

Lemma 4.1 [8, Theorem 2]**.** *Every GLS-space is a D-space.*

Lemma 4.2 [8, Lemma 2.1]**.** *Let $\preccurlyeq$ be a reflexive and transitive binary relation of a space X such that for every nonempty $\preccurlyeq$-chain K in X there is an $m \in \overline{K}$ with $m \preccurlyeq x$ for all $x \in K$. Then each non-empty closed subset of X has a $\preccurlyeq$-minimal element.*

Lemma 4.3 [7, Theorem 1.1(g)]**.** *Suppose X is a paracompact GO-space and $C \subset X$ is closed. There are closed discrete sets S and T which are, respectively, well-ordered and reverse well-ordered by the given ordering on X, have $S \cup T \subset C$, and have the property that if $x \in C$ then some points $s \in S$ and $t \in T$ have $s \leqslant x \leqslant t$.*

Let $\langle X, \leqslant \rangle$ be a linearly ordered set, and fix $N \geqslant 1$.

Lemma 4.4. *Suppose every nonempty subset of X_λ which is bounded above has a least upper bound. Then $(-\infty, x]^N$ is a D-space for each $x \in X_\lambda$.*

Proof. Fix $x \in X$ and put $Y = (-\infty, x]^N$. Define a reflexive and transitive binary relation $\preccurlyeq$ on Y so that $\vec{x} \preccurlyeq \vec{y}$ if and only if $y_n \leqslant x_n$ for all $n < N$. Certainly $\{\vec{y} \in Y : \vec{x} \preccurlyeq \vec{y}\} = \prod_{n<N} (-\infty, x_n]$ is open in Y for each $\vec{x} \in Y$.

Let K be a nonempty $\preccurlyeq$-chain in Y, and put $K_n = \{x_n : \vec{x} \in K\}$ for each $n < N$. Note that each K_n is bounded above in X_λ, so we may define $z_n = \sup K_n$. Put $\vec{z} = \langle z_0, \ldots, z_{N-1} \rangle$; observe that $\vec{z} \in Y$ and $\vec{z} \preccurlyeq \vec{x}$ for all $\vec{x} \in K$.

Consider a basic open neighborhood $U = \prod_{n<N} (z'_n, z_n]$ of $\vec{z}$ in Y. For each $n < N$, there exists $\vec{x}_n \in K$ such that $z'_n < \vec{x}_n(n) \leqslant z_n$. Since K is a $\preccurlyeq$-chain, we may define $\vec{x} = \max\{\vec{x}_0, \ldots, \vec{x}_{N-1}\}$. Then $\vec{x} \in U$, so $\vec{z} \in \overline{K}$. It follows that $\preccurlyeq$ is a GLS-relation on Y. ■

Theorem 4.5. *Suppose every nonempty subset of X_λ which is bounded above has a least upper bound, and suppose there is a sequence $\langle x_n \rangle_{n<\omega}$ such that for every $x \in X_\lambda$ there exists $n < \omega$ such that $x \leqslant x_n$. Then X_λ^N is a D-space.*

Proof. Each $(-\infty, x_n]^N$ is closed in X_λ^N and a D-space. ■

We can now give some sufficient conditions for finite powers of Sorgenfrey topologies to be D-spaces. Recall that a space is *connected* if it is not the union of two proper nonempty open subsets.

Theorem 4.6. *Let $\langle X, \leqslant \rangle$ be a linearly ordered set. If X_ε is connected and paracompact, then every finite power of X_λ is a D-space.*

Proof. Connectedness gives that every nonempty bounded-above subset has a least upper bound [3, Theorem 183], and paracompactness gives a cofinal sequence $\langle x_n \rangle_{n<\omega}$. Then $X_\lambda^N = \bigcup_{n<\omega} (-\infty, x_n]^N$, with each $(-\infty, x_n]^N$ closed and a D-space. ■

Recall that a space satisfies the *countable chain condition* (or, is *ccc*) if every pairwise-disjoint collection of open subsets of the space is countable. Now suppose that X_ε is a Suslin line; that is, X_ε is *ccc* and connected, has no first or last point, and is not homeomorphic to $\mathbb{R}_\varepsilon$. Since a GO-space is *ccc* if and only if it is hereditarily Lindelöf [5, Proposition 2.10] and regular Lindelöf spaces are paracompact [3, Theorem 126], every finite power of X_λ is a D-space.

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