The Frictional Impact of two Link Chain with a Flat

by

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Abstract

Mechanical impact is an important phenomena in mechanical systems such as robotics, multibody systems, spacecraft vehicles, ground vehicles, etc. It is a complex phenomenon that occurs when two or more bodies collide. The studies on the impact focus on reaction forces created during the collision and the dynamic response of the objects. Impact happens in such a small period and large forces are generated.

In this study, the impact of two link kinematic chain with a flat is studied. The links in the chain are connected with a revolute joint and the motion of the chain is planar. Only the first link of the chain is impacting and the end point of the second link is free to move. First, computer simulations using the Jackson Green model and a smooth coulomb friction coefficient are conducted. Simulation results are compared with the experimental data. The motion of the kinematic chain has been recorded with a high speed camera at 6,000 fps. Generated videos have been analyzed with an image processing algorithm. Pre-impact and post-impact velocities of each link are calculated from the video. To gather better understanding of impact of the kinematic chain, the kinetic energies of each link and total kinetic energy of the chain is analyzed.

Experimental results, and computer simulations showed a connection between the first link initial impact orientation, and the second link kinetic energy. The kinetic energy of the second link increases after the impact for some impact configurations.

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Chapter 1

Introduction

Mechanical impact is apparently one of the oldest phenomena in mechanical engineering. It is a crucial subject of a wide range of engineering applications such as engineering design, robotics, biomechanics, tribology, machining, etc. Mostly, the studies on the impact focus on reaction forces that created during the collision and the dynamic response of the objects to that. A large number of studies have been conducted to understand what occurs during the impact since the last centuries. Impact happens in such a small period and large forces are generated. Due to the complexity of the collision event, there is no simple solution for such a problem.

Probably the first scientist who study the collision problem was Galileo who is widely known as the father of the modern science. Galileo inspected falling objects. Wallis considered the plastic deformation is occurring during impact for the first time. Wren and Huygens investigated the elastic impact based on the conservation of momentum. In 1687, Sir Isaac Newton defined the coefficient of restitution which is the most used parameters in the impact mechanics, is the coefficient of restitution the ratio of the velocity after impact and the velocity before rebound in his work Principia Mathematica [1].

In 1724, Maclaurin approached the collision with an elastic element between two different rigid bodies and divided it into two phases, compression and restitution [2]. The compression phase starts at the moment when bodies are touched each other and start being deformed. It ends when the normal velocity of the contact point becomes zero, and at this specific moment, the contact forces between bodies and surface deformation reaches their maximum values like it seen in Fig. 1.1. The restitution phase starts from that moment, bodies start regaining their shapes and lasts until surfaces are completely separated. Also, Poisson worked on the impact

problem and defined kinetic coefficient of restitution as the ratio between impulse of compression and restitution [3].



Figure 1.1: Force-time diagram of an impact

In the 19th century, Routh [4] developed a graphical method to treat collision problems with friction based on Poisson's hypothesis [3]. Whittaker [5] developed a model using Newton's hypothesis [1] that can be used to analyze rigid body collisions. The model assumes there is an instantaneous, point contact during rigid body collision. However, for certain problems and initial conditions, the outcome of the methods based on Newton's hypothesis [1] may violate conservation of energy principle [6, 7, 8]. The energetic coefficient of restitution, which is the square root of the ratio of kinetic energy released during the restitution to the kinetic energy absorbed during the compression is introduced by Stronge [9].

During collision of the kinematic chains, multiple impact may appear. An accurate collision model needed in order to successfully design an impacting system. The model is needed to include a number of hypothesis due to nature of impact [10]. Even though, laws used in order to describe mechanical collision, the nature of happening includes inherent imprecisions. Impulsive motions and dynamic response of corresponding system are directly linked to the impact. Also, an interesting set of events are observed: shock, a rapid momentum distribution, indentation, elastic wave propagation, and energy dissipation trough noise and heat.

The coefficient of restitution depends on the collision velocity, material properties and surface topography. According to Lu and Kuo [11] materials with higher ratio of Young's

modulus to yield stress have smaller coefficient of restitution. As impacting velocity increase, the sensitivity of coefficient of restitution decreases.



Figure 1.2: Kinetic energy-time diagram of an impact

A Sample kinetic energy - time response of a sphere is presented in Fig. 1.2. The compression and restitution phases as well as the impact duration, the maximum contact force, and minimum kinetic energy are shown. Contact force increases, kinetic energy decreases as the sphere continues to deform and reaches their peak point when the maximum deformation occurs.



Figure 1.3: The illustration of an impact process

Figure 1.3 represents a typical deformation-time behavior of the impact where the first half of the curve represents compression phase and the second half represent restitution phase. In the case of perfectly elastic impact, where there is no permanent deformation, the cure is symmetric at the point of maximum penetration. The symmetry of the curve disappears for the case of elastic-plastic impact deformation and the second half of the cure vanishes for fully plastic impact. One of known primary cause of energy loss during the impact is permanent deformation [12]. The classical theory of the impact is presented in details in [13].

The study of collision is mainly separated into two main groups, rigid body collisions and direct solution of the equation of motion [14, 15, 16, 17, 18, 19, 20, 21, 22]. The classical rigid-body impact theory treats the collision as discrete event. Solution is based on impulse-momentum balance relations and it does not consider any resulting deformations and the associated transient stress wave propagation. When we are dealing with a perfectly elastic body collision between smooth surfaces, post-impact velocities are calculated using the impulse momentum and conservation of energy equations. If permanent deformations or indentation produced during the impact, the restitution relationship is needed to use in place of energy conservation relation. Whittenburg [23], Haug et al. [24], and Lankarani and Nikravesh [25] studied frictionless and direct central impact. However, Whittaker [26], Keller [6], and Han and Gilmore [27] take into account the friction. There are many friction force model can be found in the literature, the Classical Coulomb friction model [28] is used by researchers due to its simplicity.

Numerical simulation of impact is mathematically complex. Friction force is always in opposite direction to sliding velocity. This causes several numerical issues in terms of computation. Because of friction force being discontinuous at zero velocity, The classic Coulomb model causes stiff equations of motion. Moreover the exact value of zero velocity is never found by numerical computations. To overcome any discontinuity issue adopting smoothening functions can be useful .

Impact can be studied as a continuous phenomenon and the dynamic analysis involves before, during, and after the impact. In this approach, deformations of colliding bodies and generated forces are mathematically modeled during collision event, and dynamic equations of motion are numerically solved with contact forces. This approach describes the real behavior of the system more realistically, but also adds much more complexity to the problem. It is clear that the contact force during collision is the most important factor in such a numerical and analytical solution as the integration of the equations of motion highly depends on the generated contact force. Therefore, using an accurate contact model is of a uttermost importance.

1.1 Contact Mechanics

Contact mechanics is one of the most important topics of tribology, from macro-scale to nano-scale. It has critical importance in bearing applications, rough surface contact [29, 30, 31, 32, 33, 34, 35, 36, 37], gear designs, tires on roads, biomechanical system designs, nano-lubrication and impact mechanics [38, 39, 40, 41, 42].

There are various models in the literature for the contact behavior of elastic-plastic objects of various geometries yet, one of the pioneering model is published by Hertz [43]. The Hertzian theory based on elastic half-space theory, parabolic contact profiles, frictionless surface, and isotropic homogeneous solids. The theory assumes the contact between nonconforming bodies is a single line or a point. As the body start to compress, a contact load starts to develop, and small deformations occurs on surface in the vicinity of initial contact point. The Hertz theory allows predicting stress distribution in the contact zone, values of the contact size, compression and maximum pressure. A large amount of information for the contact problem can be found Johnson [12]. The contact problem with interesting events including wear, roughness, viscoelasticity, and stick-slip phenomena presented in Galin [44] and Popov [45].

The field of contact mechanic includes various models for the elastic-plastic objects with various geometries. A brief description and literature review of sphere contact models are presented below in subsections.

1.1.1 Sphere Contact Models

The contact of a hemisphere with a massive flat surface has been studied by many researchers over many years. Elastic contact of sphere can be given by Hooke's law F = kx, where k is stiffness af the ball, x is deformation of the ball, and F is the contact force. The Hertzian law of the form $F = kx^{\frac{3}{2}}$ gives good agreement for low speed collision of elastic sphere but it has limited applicability for most of the cases. Most of the impact cases bodies undergo permanent deformations. Therefore, Hertzian theory itself is not sufficient and more advanced approaches to model impact behavior of the sphere are needed. Scientists are came up with two main approaches: indentation and flattening. Indentation approach assumes the hemisphere is rigid and all deformations occurs on the flat surface. On the other hand, flattening approach assumes the opposite of the indentation models.

1.1.1.1 Indentation Contact of a Sphere

Indentation models refers to contact of a rigid sphere with a massive deformable flat surface. Fig illustrates the identation type of contact. Where δ is the deformation of the flat, and R is the radius of the sphere.



Figure 1.4: Identation of a rigid sphere into deformable flat

The indentation of a flat by a hemisphere has been studied by many researchers for over a century [46, 47, 12, 48, 49, 50, 51, 52, 53, 54, 55, 56]. In the reality however, there is no ideally rigid body and therefore a criteria is needed to define in order to define contact as indentation. Tabor showed in his study that if the yield strength of the sphere is more than 2.5 times the

yield strength of the flat, the deformations on the sphere will be elastic for a normal hardness test, and the effect of the elastic deformation of sphere is neglectable [55]. Ghaednia et al. [47] also verified that if the yield strength of the sphere is 1.7 times larger than the yield strength of the flat, the contact can be described as indentation.

The study of indentation contact models has been used for the Brinell [57] and Meyer [58] hardness tests. In 1900, Brinell [57] developed a a hardness test in order to determine material properties of the test subject. Later in 1908, Meyer [58] designed a different hardness tests and presented the Meyer hardness. Also, Tabor [55] introduced the simple theory of static and dynamic hardness in 1948. Tabors theory is probably one of the most used definition for many mechanical indentation applications since presented [59].

1.1.1.2 Flattening Contact of a Sphere

Flattening models refers to contact of a deformable sphere with a massive rigid flat surface. Figure 1.5 illustrates the contact of deformable sphere with a rigid surface. Where δ is the deformation of the flat with respect to its free geometry.



Figure 1.5: Flatenning of a deformable sphere on rigid flat

Researchers studied flattening models since early 1960, In 1968, Johnson conducted experimental works over the fully plastic contact between spheres and cylinders. However, the study of elastic-plastic contact is done ten year later.

Li et al.,[60] modified Thorntons model [61, 62], with detailed pressure distribution based on FEA modeling. Wu et al., presented a new model for the restitution phase of the impact of an elastic-plastic sphere with a rigid flat [63]. Kogut and Etsion [64] presented a new empirical formulation for elastic-plastic contact of a hemisphere and a rigid flat. They considered the material is elastic-perfectly plastic and the contact is divided into elastic and elastic-plastic phases during the compression. The elastic phase follows the Hertzian theory and was defined as $\delta/\delta_c \leq 1$, where δ and δ_c are the deformation and the critical deformation respectively. The elastic-plastic phase is also divided into three sub-phases. Later Wang [65] improved the formulation of Kogut and Etsion [64] specially for the restitution period. Etsion et al., [66] developed a new improved empirical formulation for the restitution phase

In 2005, Jackson and Green [67] developed a new contact model for an elastic-plastic half sphere and a rigid flat. Hardness is not considered to be constant and it changes during the contact. Later, Brake [53] developed a new analytical model and divided the contact into four phases: fully elastic, elasto-plastic, fully plastic, and restitution phases. In [68] Ghaednia presented improved Jackson and Green Formula.

1.1.2 Contact Models in the Study of Impact

Contact models have been used in many impact related studies. Stronge defined a new energetic coefficient of restitution [9] and used Johnsons [12] elastic-fully plastic model. Also, Stronge separated the impact into three different phases: the fully elastic phase, the fully plastic phase, and the restitution phase [99, 101, 102, 104, 100]. The elastic phase and the restitution phase follows the Hertzian contact theory [43], The fully plastic phase follows Johnsons expression [12]. Thornton [61] provided a numerical model and studied the impact of a sphere with a wall. He also worked on elastic-perfectly plastic impact and developed an analytical solution to a simplified model [62]. Even though, The restitution phase follows the Hertzian

theory [43] in his study. Thornton assumes the radius of curvature of the restitution stage is larger than the radius of curvature of the compression stage.

1.2 Scope of the Study

Researchers in the field of contact mechanics and dynamics are also show interest to multibody mechanical systems collision behaviors. In general, multibody mechanical systems addresses mechanisms, open or closed kinematic chains, machines. It has given less attention to impulsive motion in constrained multibody systems aside to largely focused problem of the two-body collision.

Hurmuzlu et al, [69] studied rigid body collision of a planar kinematic chain with rough and flat surfaces. They developed a methodology to solve impacting problem of planar kinematic chain with one and stationary on the surface and the other end is resting. Difficulties among velocity reveals, normal impulses, and velocities at contact points are addressed. Marghitu et al. [70] Studied three-dimensional rigid body collisions. Ahmed et. al, [71] presented a methodology for frictional impact of open link multi-body mechanical systems based on Poissons hypothesis for coefficient of friction and Rouths graphical method. The author presented an algorithm to analyze multibody collisions. Marghitu et al.[70] Inspected simultaneous impact and one end resting impact of a two-link planar chain by using Jackson and Green method [67] and gathered good agreement between simulations and experiments.

In this thesis, frictional impact of two link kinematic chain with a flat is mathematically modeled and experimentally validated. Pre-impact and post-impact kinetic energy allocation of the link 1, link 2, and the whole chain is investigated.

Chapter 2

Mathematical Background

2.1 Dynamics of the System

The dynamics of multibody systems is based on the classical mechanics. History of multibody dynamic can be found in [72, 73]. With the improvement on computers capacity computational dynamics has made rapid progress in the 20th century. The first step in the modeling of a multibody system is assigning the appropriate relations between links. Dynamics of a system can be modeled by using Newton Euler method, Euler Lagrange method, Gibbs Appel method, or Kanes equation. Selection of method used to model the system dynamics depends on the application and complexity of the application.

2.1.1 Dynamics of two Link Chain Impacting on a Flat

In this section, we analyze dynamics of free two link open kinematic chain. Figure 2.1 shows schematic of the links. The dynamic of the given two link chain during the impact is developed by using Lagrangian equations.



Figure 2.1: Two link open kinematic chain over rigid flat

In Fig. 2.1 $x_0O_0y_0$ reference frame illustrates is global coordinate system, and $x_iO_iy_i$ illustrates i^{th} link local coordinates system. The center of the mas of i^{th} link is C_i , and length of the i^{th} link is given by L_i notation. R illustrades radious of the tip point of the link, Tillustrades the tip point of the link. The system has four degree of freedom and generalized active coordinates are given by q_i where q_1 is x position of the tip point of the impacting link, q_2 is the y position of the tip point of the impacting link, q_3 is the rotation angle θ of the impacting link, and q_4 is the rotation angle β of the follower link. The gravitational force in i^{th} link is G_i , the normal force during the impact is F_n , friction force during the impact is F_f . Contact force, friction force are presented in later sections.

$$q_1 = x, \ q_2 = y, \ q_3 = \theta, \ q_4 = \beta,$$
 (2.1)

The position vector of the contact point T is

$$\mathbf{r}_T = x \, \mathbf{i} + y \, \mathbf{j}. \tag{2.2}$$

The position vector of the center of the mass of the impacting link is

$$\mathbf{r}_{C_1} = [x\mathbf{l} + y\mathbf{j} + R\mathbf{j} + \frac{L_1}{2}[\cos\theta\mathbf{l} - \sin\theta\mathbf{j}].$$
(2.3)

The position vector of the center of the mass of link 2 is

$$\mathbf{r}_{C_2} = [x\mathbf{i} + y\mathbf{j}] + R\mathbf{j} + L_1[\cos\theta\mathbf{i} - \sin\theta\mathbf{j}] + \frac{L_2}{2}[\cos\beta\mathbf{i} - \sin\beta\mathbf{j}].$$
(2.4)

Angular velocity and acceleration of the impacting link is

$$\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}} \, \mathbf{k}, \quad \boldsymbol{\alpha}_1 = \ddot{\boldsymbol{\theta}} \, \mathbf{k}, \tag{2.5}$$

where ω_1 is the angular velocity of the impacting link, and α_1 is the angular acceleration of the impacting link.

Angular velocity and acceleration of the follower link is

$$\boldsymbol{\omega}_2 = \dot{\beta} \, \mathbf{k}, \quad \boldsymbol{\alpha}_2 = \ddot{\beta} \, \mathbf{k}, \tag{2.6}$$

where ω_2 is the angular velocity of the impacting link, and α_2 is the angular acceleration of the impacting link.

The velocity of the center of the mass of the links, and the tip point of the impacting link is calculated by taking first derivatives of the position vectors

$$\mathbf{v}_T = \frac{d\mathbf{r}_T}{dt}, \quad \mathbf{v}_{C_1} = \frac{d\mathbf{r}_{C_1}}{dt}, \quad \mathbf{v}_{C_2} = \frac{d\mathbf{r}_{C_2}}{dt}.$$
(2.7)

The equations for acceleration of the center of the mass of the links, and the tip point of the impacting link can developed by taking second derivatives of the position vectors

$$\mathbf{a}_{T} = \frac{d^{2}\mathbf{r}_{T}}{dt^{2}}, \quad \mathbf{a}_{C_{1}} = \frac{d^{2}\mathbf{r}_{C_{1}}}{dt^{2}}, \quad \mathbf{a}_{C_{2}} = \frac{d^{2}\mathbf{r}_{C_{2}}}{dt^{2}}.$$
(2.8)

2.1.1.1 Lagrange Equations of the Motion

The equations of motion for the given link in section 2.1.1 during the impact is solved using Lagrange's method. One of the advantages of the Lagrangian method is the operator does not required to deal with joint forces. The lagrange equation of motion for four degree of freedom is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1, 2, 3, 4$$
(2.9)

Where T is total kinetic energy of the system, and Q is the generalized forces. The kinetic energy of the first link T_1 is

$$T_{1} = \frac{1}{2}m_{1}\mathbf{v}_{C_{1}} \cdot \mathbf{v}_{C_{1}} + \frac{1}{2}I_{C_{1}}\boldsymbol{\omega}_{1} \cdot \boldsymbol{\omega}_{1}.$$
 (2.10)

The kinetic energy of the second link T_2 is

$$T_{2} = \frac{1}{2}m_{2}\mathbf{v}_{C_{2}} \cdot \mathbf{v}_{C_{2}} + \frac{1}{2}I_{C_{2}}\boldsymbol{\omega}_{2} \cdot \boldsymbol{\omega}_{2}.$$
 (2.11)

The total kinetic energy is

$$T = \sum_{n=1}^{2} T_n.$$
 (2.12)

The generalized active force Q_i for the chain is

$$Q_i = \frac{\partial \mathbf{r}_T}{\partial q_i} \cdot \left[\mathbf{F}_n + \mathbf{F}_f\right] + \frac{\partial \mathbf{r}_{C_1}}{\partial q_i} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{r}_{C_2}}{\partial q_i} \cdot \mathbf{G}_2, \qquad (2.13)$$

where \mathbf{F}_n represents contact force, \mathbf{F}_f represents friction force, and \mathbf{G}_i represents i^{th} link mass. By using Equations 2.9, 2.10, 2.11, 2.12, and 2.13 the equations of motion of the system can be derived and solved for the given two link, four degree of freedom open kinematic chain. The equations of motions are automatically created by using MATLAB symbolic toolbox, therefore final equations are not presented here.

2.2 Contact Forces During the Impact

The modeling of elastic-plastic hemispheres contacting with a rigid flat surface is an important problem in contact mechanics in both micro and macro surfaces. The mathematical procedure used for spherical impact is illustrated below.

2.2.1 Elastic Contact Force

The first state of the impact is the elastic compression phase. The elastic compression starts with the moment when bodies touched each other. The elastic contact phase of the impact employs Hertzian contact theory. A closed form analytical solution for elastic contact of two spheres are provided by Hertz [43] in 1882. The Hertzian contact theory Hertz' contact theory provides a very good result for collisions between hard bodies where the contact region is relatively small in compassion with the size of contacting bodies.

Surfaces are infinitely large half-spaces.

Pressure profile is parabolic (which assumes that the shape of the bodies in contact can also be approximated well with parabolic shapes, e.g., sphere, ellipse or a cylinder)

Contacting bodies are isotropic homogeneous solids

At the begging of the elastic contact, the normal contact force P_e is zero, and the elastic indentation δ is zero. The pressure distribution equation that has contact region given by Hertz is shown below

$$p = p_0 [1 - \left(\frac{r}{a}\right)^2]^{1/2}, \quad r \le a$$
 (2.14)

where p_0 illustrates the maximum pressure in the contact area, r is the radial distance from the center of the contact, and a is the radius of the contact area. Equation 2.14 yields to displacement field as in shown below.

$$u_z = \frac{1 - \nu^2}{E} \frac{\pi p_0}{4a} (2a^2 - r^2).$$
(2.15)

Using the boundary conditions given in [12] and Eq. 2.14 and 2.15 following expressions are found.

$$a = \left(\frac{3PR}{4E'}\right)^{1/3},\tag{2.16}$$

$$\delta = \left(\frac{9P^2}{16RE'^2}\right),\tag{2.17}$$

$$p_0 = \frac{3P}{2\pi a^2},$$
 (2.18)

where the material modulus is E', and R are given by

$$E'^{-1} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2},$$
(2.19)

$$R^{-1} = \frac{1}{R_1} + \frac{1}{R_2}.$$
(2.20)

Note that the radios of an flat is infinitive and in the case of contact between a sphere and a flat the second part of the expression in Eq. 2.20 vanishes. Until the maximum Von-Mises stress reaches the yield strength S_y , the deformation beneath the contact surface is assumed to be elastic. The critical deformation δ_c expression is given by

$$\delta_c = \left(\frac{\pi C S_y}{2E'}\right)^{1/2} R,\tag{2.21}$$

where the coefficient C is

$$C = 1.295 e^{0.736\nu}.$$
 (2.22)

Finally, the contact force expression for elastic contact is obtained by reordering Eq. 2.21

$$P_e(\delta) = \frac{4}{3} E' \sqrt{R} \delta^{3/2}, \quad 0 \le \delta \le 1.9 \delta_c$$
(2.23)

Hertzian contact theory [43] provides good results, but in most applications the contacting material reaches their yield stress. when the identation δ reaches 1.9 δ_c , and elastic phase ends.

2.2.2 Elastic-Plastic Contact Force

The elastic-plastic contact occurs in the second phase of the impact when the contact force is greater than the critical contact force where maximum von misses stress reaches the yield strength. Even though several models considers elastic-plastic deformation occurs at initiation of yield when deformation reaches critical deformation, δ_c . In this thesis, Jackson-Green contact model is followed [68].

Contact force for elastic-plastic state is presented following formulation.

$$P_{ep} = P_c \left[e^{-0.17(\delta/\delta_c)^{5/12}} \left(\frac{\delta}{\delta_c}\right)^{1.5} + \frac{4H_G}{CS_y} \left(1 - e^{-\frac{1}{78}(\delta/\delta_c)^{5/9}} \right) \left(\frac{\delta}{\delta_c}\right)^{1.1} \right], \quad (2.24)$$

where P_c constant is given by

$$P_c = \frac{4}{3} \left(\frac{R}{E'}\right)^2 \left(\frac{\pi C S_y}{2}\right)^3,\tag{2.25}$$

where $\frac{H_G}{S_y}$ is described as

$$\frac{H_G}{S_y} = 2.84 - 0.92 \left[1 - \cos(\pi \frac{r}{R}) \right],$$
(2.26)

where r is shown below

$$r = \sqrt{R\delta_c \left(\frac{\delta}{\delta_c 1.9}\right)^B},\tag{2.27}$$

where B constant is

$$B = 0.11 \, e^{23 \, S_y/E'}.\tag{2.28}$$

2.2.3 Restitution Force

Restitution phase of the contact starts when the normal relative motion of the contacting bodies becomes zero. At the initial point of the restitution phase, contact force and deformation, δ reaches their maximum values, contrariwise kinetic energy gains its lowest. Restitution phase is assumed to be elastic and can be modeled as a Hertizan contact force shown below.

$$P_r = \frac{4}{3} E' \sqrt{R_{res}} \left(\delta - \delta_r\right)^{3/2},$$
(2.29)

where δ_r illustrates permanent plastic deformation and shown as below.

$$\delta_r = 1.02\delta_m \left[1 - \left(\frac{\delta_m / \delta_c + 5.9}{6.9}\right)^{-0.54} \right],$$
(2.30)

where R_{res} is permanent radius of hemisphere. δ_m and P_m are maximum penetration and maximum contact force generated during elastic-plastic phase respectively.

$$R_{res} = \frac{1}{\left(\delta_m - \delta_r\right)^3} \left(\frac{3P_m}{4E'}\right)^2.$$
(2.31)

2.3 Friction Force

Friction is an inherent phenomenon which is the outcome of the interface between surfaces. As simplest definition, friction is the tangential reaction force that resisting the relative motion of the contacting surfaces. Leonardo Da Vinci (1452-1519) is the first known scientist who work on friction. He studied all types of friction were the focused subjects and differences between sliding and rolling friction were determined. Leonardo stated the two basic laws of friction 200 years before Newton defined what force is.

the areas in contact have no effect on friction.

if the load of an object is doubled, its friction will also be doubled

French physicist Amontons (1663-1705) conducted an experimental work on dry friction and confirmed Leonardos statements on friction. The main outcomes of his experiments were that frictional force is independent of the areas in contact and directly proportional to the normal load. The most systematic research on friction was done by Coulomb (1736-1806). Coulomb published the work referring to Amontons. As the result of Amontons and Couloms's studies well known Amontons-Coulombs law of friction are stated

magnitude of friction force is independent of contact area magnitude of friction force is proportional to magnitude of normal force magnitude of kinetic friction is independent of speed of slippage

The classical Coulomb model which is an elementary formulation of friction expressed as

$$F = \begin{cases} N\mu_k sgn(v) & if \quad v > 0, \\ F_{app} & if \quad v = 0 \quad and \quad F_{app} \le N\mu_s, \end{cases}$$
(2.32)

where v is the relative velocity between the contacting bodies, μ_s is the static friction coefficient, μ_k is the dynamic friction coefficient, N is the normal force, and F_{app} is the frictional reaction at rest.

Even though given law of friction are pretty simple and clear, true nature of the effective friction is extremely complex phenomenon, depends of many parameters such as mechanical properties of material, the contact surface roughens, temperature, humidity, etc. It contains elastic and plastic deformations of the surface layers of the contacting bodies, chemical reactions, transfer of particles from one body to the other, micro-fractures and the restoration of the continuity of materials. Therefore the classic Coulomb model is not always accurate.

Coulomb also discovered that the static frictional force increases with the amount of time an object is at rest. Physical reasons for the time dependence is very diverse. For metals the real contact area in the micro-contacts increases with time. At higher temperatures, it increase faster. Since rolling is a continuous process of reestablishment of new contact between bodies, for lower rolling speed, the contact time is larger and a larger friction can be expected.

During the sliding of contacting surfaces, micro asperities come into contact. Seeing that the contact time of micro asperities hinge on sliding speed, friction force is also dependent on sliding speed. For simplicity, the coefficient of kinetic friction is assumed to be independent of sliding speed, but this approximation may cause poor results on applications where sliding speed vary much.

In many engineering applications, the success of dynamic simulation in predicting experimental results are sensitive of accurate friction modeling. In dynamic system simulation researcher is needed to define the most accurate way to include friction in a numerical or analytical model. The selection of friction models is relay on the operational conditions during application.

Numerical simulation of impact is mathematically complex with multiple events, elastic loading, elastic-plastic loading, and unloading. Friction force is always in opposite direction to sliding velocity. This causes several numerical issues in terms of computation. Because of friction force being discontinuous at zero velocity, The classic Coulomb model causes stiff equations of motion. Moreover the exact value of zero velocity is never found by numerical computations [74].

In this study smooth coulomb friction model given in Eq. 2.33 is used in order to avoid any computation burden it may caused by the friction force discontinuity.

$$\mu = \mu_k \tanh v_{Tx} / v_d, \tag{2.33}$$

where the term v_{Tx} is relative tangential velocity of the contacting surfaces, and v_d is velocity tolerance which is chosen 0.01 [74]. On the other hand, the joint friction between link 1 and link 2 is neglected in this study.

Chapter 3

Experimental Studies

In tis section, details of the experimental studies are presented. The motion of the dynamic system was recorded with a high speed camera for various conditions. Recordings are later analyzed using image processing techniques to gather position and velocity data. Details of the experimental setup and image processing are explained following sections.

3.1 Experimental Setup

A special experimental setup is built in order to perform consistent and repeatable experiments. The kinematic chain is held by electromagnetic solenoids. Three electromagnetic solenoids keep the links with the desired orientation and height. Two solenoids clutch each links at center of the mass and the third solenoid holds the joint that attaches kinematic links. A release button controlled by the operator is installed to deactivate solenoids and to release kinematic chain. A schematic right view illustration of the experimental setup is given in Fig. 3.1, and front view illustration is given in Fig. 3.2



Figure 3.1: Experimental setup right view



Figure 3.2: Experimental setup front view

For image processing, the quality of the recording is vital. The video must be fairly lightened. Any reflection or bright object may cause noise or even miscalculations. In this aspect, exposure triangle, a well known to phenomena in photography is introduced [75]. The exposure triangle consist of there key parameter acts on quantity of light sustained by camera : the ISO, aperture and shutter speed. ISO is a camera setting to measure cameras ability to capture light. Digital cameras convert the light into electrical signals for processing. ISO sensitivity is raised by amplifying the signal. As the ISO sensitivity increases at a camera photos will grow brighter. Increasing ISO value puts extra pressure on the sensor and higher values may cause degradation of the image quality.

Aperture is a value of how open or closed the lens. As the hole of lens get wider, more light penetrate trough to the sensor, but wider apertures may lead to loose depth of field. In order to shoot fairly focused, and sharp images an appropriate aperture setting is needed to be employed.

Exposure speed stands for the duration that the sensor exposed of light. As the frequency of a video camera increases, exposure speed decreases proportionally. In order to capture decently bright video, two 1000 W projectors have been used to provide essential light. The camera and light sources are shown in Fig. 3.3



Figure 3.3: Experimental setup (a) HotShot high speed camera (b) 1000W light source

3.2 Image Processing Technique

Digital video to analyze the moving objects have been widely utilized. Experiments including video analysis helps to improve the understanding of dynamic system response. Analyzing the motion of the impacting object is always been a big challenge due to remarkably small contact time. The capacity of standard video cameras, and other measuring techniques are insufficient to capture data during the impact. High accuracy devices or methods are needed to be used in order to capture the motion of the objects during the impact.

Image processing techniques have been widely adopted to analyze impacting objects. Many researchers employed the technique for impact related studies. Marghitu et al., [76], Kharaz, Gorham, and Salman [77, 78, 79], Cermik et al.,[80], Ghaednia et al., [81, 82, 83, 84, 85], Pffeier [86], Kardel et al., [87] used high speed cameras to detect the motion before and after the impact, and the coefficient of restitution. Stoianovici and Hurmuzlu [88] used high-speed camera at 1,000 frames per second (fps) to detect velocities of the rod before and after impact. The frequency of the high-speed camera was insufficient to record enough data to fairly analyze motion of the rode during the impact, but velocities of the rod before and after the impact are detected. Impact time is also measured by using an electrical circuit [88].

For the recording experiments, a HotShot high-speed camera is used. The camera is settled to 6,000 fps after several pre-experiments. It has to be noted that the resolution of the camera remarkably decrease as the frame per second rates increase. Lover resolution videos causes significant noise and inaccurate measurements. Each frame has a resolution of 512 x 418 pixel where each pixel is an 8 bit number for settled recording frequency.

To track motion, three white markers on black background are applied to the each links. On link 1, The first marker applied to the tip point of the link, the second marker applied to the mid point of the center of the mass and the tip point of the link, the third marker is applied to the center of the mass of the link. On link 2, The first marker applied to the center point of the link, and other two markers are applied to an equal distance of the center of the link on different directions. The distance between each markers is equal and a = 25.4 mm. This positioning provide advantage of gathering data for both tip point and center motion data. A schematic drawing of the marker placements of the kinematic chain is shown in Fig. 3.4.



Figure 3.4: Marker placement on links

A MATLAB code is created to calculate the position and velocity of the links. Since the recording host dynamic of two link chain, the video is cropped into two pieces and each links motion is analyzed separately.

Some modifications on each frame of the recording is needed in order to remove undesired edges and clearly detect markers. An integer from 0 to 255 called gray value has been assigned to each pixel. The value 0 refers to pure black, and 255 refers to pure white. In order to find the edges of each marker, a number for critical gray value has been defined. The pixels with a larger gray value have been set to pure white and the pixels with a smaller gray value have been set to pure white and the pixels with a smaller gray value have been set to pure white and the pixels with a smaller gray value have been set to pure white and the pixels with a smaller gray value have been set to pure black. A picture of gray modificated frame is shown in Fig. 3.5

Later, counter of the markers are found by using an edge detection algorithm. The center of markers are found by calculating average of each counter pixel position. A linear regression for center positions of the markers are done to find orientation of the link. The distance between the first and the third marker is measured for each frames. The average distance between the markers is used in order to define a scale factor to convert measurement from pixel to meter. In Fig. 3.6 edges and centers of the markers are shown.



Figure 3.5: Contrast modification



Figure 3.6: Calculation the position of markers center

As the impact happens in a significantly short time, we were not able to gather enough data point to see motion of the rod during the impact. Contact speed and rebound speed of the
links are measured precisely for pre-impact and post-impact then, compared to the theory at later section.

Chapter 4

Results and Conclusion

Experimental and simulation results are presented in this section. The kinematic chain is dropped onto massive flat under various initial speed and different orientation of impacting link.



Figure 4.1: Two link open kinematic chain

A chain of two-link interconnected rigid links 1 and 2 is shown in Fig. 4.1. The end T of the chain collides with the massive flat surface. The free end of the second link is parallel to horizontal axis. The material properties, the Initial velocity and the orientation of the links, frictional relation between the surface and the chain may lead to different outcomes. The impulsive forces and friction force are generated at the collision point T. Smooth Coulomb friction model is applied to the simulations.

The length of the nylon links is L1 = L2 = 0.1524 m. The radius of the ends of the link is R = 0.00762 m. The mass density of the material of the links is $\rho_L = 1400 \text{ kg/m}^2$. The Youngs elastic modulus of the link is $E = 9 (10^9)$ Pa. The Poissons ratio of the link is $\nu = 0.3$. The yield strength of the link is $S_{Y_L} = 140 (10^6)$ Pa. The mass density of the massive steel flat is $\rho_L = 7800 \text{ kg/m}^2$. The Youngs elastic modulus of the massive flat is $E = 200 (10^9)$ Pa. The yield strength of the massive flat is $S_{Y_f} = 1.12 (10^9)$ Pa.

A series of incline friction experiments are conducted to calculate the coefficients of friction of the nylon link on steel flat. First, the angle of the steel flat is gradually increased until the instant moment the link starts to sliding. The critical angle is determined and static coefficient of friction, μ_s , is calculated. Later, another set of experiments are conducted at higher angles in order to measure kinematic friction coefficient μ_k . The motion of the link is recorded with the high speed camera. The average average kinetic coefficient of friction is calculated $\mu_k = 0.18$.

The ordinary nonlinear equations of motion are found by using the MATLAB Symbolic Toolbox. In order to conduct numerical integrations of the equations of motion the MATLAB function ode45 (simultaneously fourth- and fifth-order Runge-Kutta method with a variable time step) is used. The computer simulation of the impact is needed to define end of the contact force conditions, elastic compression, elastic-plastic compression, and restitution. Different sets of the MATLAB event location function is used in order to detect the end of the three phases of the impact period, and to define new initial conditions for the following contact law.

An experimental setup is built in order to study the planar collisions of two-link open kinematic chain with a massive flat. The collision experiments performed by dropping link onto massive flat with desired initial conditions. The setup provides ability to test different links with different geometries. The pre-colliding velocities and orientations of the links are easily adjustable.

Experiments are recorded with a high-speed video camera and later analyzed by using a MATLAB code. To obtain kinematic data, white circular markers are attached on black links. The experimental setup is placed into a black box to enable the markers detection and noise cancellation. For the impacting link, markers are placed at the impacting tip of the link, centroid of the link and the mid-point between other two marker. For follower link, markers are placed to the centroid of the link and the other two is placed to an equal distance of the centroid in different directions.

More than a hundred experiments are performed and analyzed for total nine different set of experimental condition. Only there of the experiments that fulfill desired initial conditions in the best way are considered. Obtained results are compared with the theory. The generalized energetic coefficient of restitution [89] given in Eq. 4.1 which is square root of the ratio of the post-impact kinetic energy, E1, and pre-impact kinetic energy, E2, of the constrained motion is calculated in order to achieve better understanding of the dynamic response of the system. Results of each simulations and experiments are presented in following sections with figure illustrations.

$$e = \sqrt{\frac{E1}{E2}} \tag{4.1}$$

4.1 The Experiment and Simulation Results for Test Case 1

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.2$ m/s, the orientation angle of the link 1 is $\theta = 50^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.2 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 621$ N,



Figure 4.2: The normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.3 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.4 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.5 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.3 is calculated $v_{Ty} = -1.044$ m/s from the simulation. The experimental result is $v_{Ty} = -0.99$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.4 is calculated $v_{Tx} = -1.249$ m/s from the simulation. The experimental result is $v_{Tx} = -1.245$ m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s. The angular output velocity of link 1 shown in Fig. 4.5 m/s.



Figure 4.3: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.4: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.5: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.6 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.7 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.8 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.6 is calculated $v_{C_{2y}} = 1.005 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.98 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.7 is calculated $v_{C_{2x}} = 0.614 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.614 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.6 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.8 is calculated $\omega_2 = 6.348 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 6.49 \text{ rad/s}$.



Figure 4.6: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.7: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.2 \text{ m/s}$, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.8: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.9 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0257 J for each link, and total pre-impact kinetic energy of the system is 0.0513 J. The post-impact kinetic energy of the link 1 is 0.0137 J from the simulation, and 0.0142 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.7308 from simulations, and 0.77 from experimental data. The post-impact kinetic energy ratio of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.0264 J from the simulation, and 0.0260 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 1.0143 from the simulation, and 1.0067 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 1.0143 from the simulation, and 1.0067 from the experimental data. The sequence of the system is e = 0.8842 from the simulation, and e = 0.8964 from the experimental data.



Figure 4.9: Kinetic energy variation of the links for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy increases slightly.

4.2 The Experiment and Simulation Results for Test Case 2

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.2$ m/s, the orientation angle of the link 1 is $\theta = 55^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.10 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 671$ N,



Figure 4.10: The normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.11 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.12 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.13 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.11 is calculated $v_{Ty} = -1.034$ m/s from the simulation. The experimental result is $v_{Ty} = -1.04$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.12 is calculated $v_{Tx} = -1.191$ m/s from the simulation. The experimental result is $v_{Tx} = -1.1$ m/s. The angular output velocity of link 1 shown in

Fig. 4.13 is calculated $\omega_1 = 15.955 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 15 \text{ rad/s}$.



Figure 4.11: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.12: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.13: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.14 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.15 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.16 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.14 is calculated $v_{C_{2y}} = 0.924 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.9 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.14 is calculated $v_{C_{2y}} = 0.924 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.9 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.15 is calculated $v_{C_{2x}} = 0.622 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.58 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.16 is calculated $\omega_2 = 8.982 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 9.1 \text{ rad/s}$.



Figure 4.14: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.2 \text{ m/s}$, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.15: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.2 \text{ m/s}$, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.16: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.17 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0257 J for each link, and total pre-impact kinetic energy of the system is 0.0513 J. The post-impact kinetic energy of the link 1 is 0.0132 J from the simulation, and 0.0135 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.7163 from simulations, and 0.7224 from experimental data. The post-impact kinetic energy of the link 2 is calculated 0.0255 J from the simulation, and 0.0239 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9964 from the simulation, and 0.9650 from the experimental data. The generalized energetic coefficient of restitution of the system is e = 0.8679 from the simulation, and e = 0.8534 from the experimental data.



Figure 4.17: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.3 The Experiment and Simulation Results for Test Case 3

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.5$ m/s, the orientation angle of the link 1 is $\theta = 60^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.18 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 715$ N,



Figure 4.18: The normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.19 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.20 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.5 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.19 is calculated $v_{Ty} = -1.025$ m/s from the simulation. The experimental result is $v_{Ty} = -1.01$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.20 is calculated $v_{Tx} = -1.054$ m/s from the simulation. The experimental result is $v_{Tx} = -0.98$ m/s. The angular output velocity of link 1 shown

in Fig. 4.21 is calculated $\omega_1 = 12.552 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 13.02 \text{ rad/s}$.



Figure 4.19: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.20: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.21: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.22 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.23 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.24 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.22 is calculated $v_{C_{2y}} = 0.835 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.82 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.22 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.23 m/s. The angular output velocity of link 2 shown in Fig. 4.24 is calculated $\omega_2 = 11.839 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 12.10 \text{ rad/s}$.



Figure 4.22: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}, v_{Ty} = 1.2 \text{ m/s}, \theta = 60^{\circ}, \beta = 0^{\circ}$



Figure 4.23: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.2 \text{ m/s}$, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.24: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.25 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0257 J for each link, and total pre-impact kinetic energy of the system is 0.0513 J. The post-impact kinetic energy of the link 1 is 0.0125 J from the simulation, and 0.0128 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.7072 from simulations, and 0.6981 from experimental data. The post-impact kinetic energy at the simulation, and 0.0239 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9831 from the simulation, and 0.0239 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9831 from the simulation, and 0.9654 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9831 from the simulation, and 0.9654 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9831 from the simulation, and 0.9654 from the experimental data.



Figure 4.25: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.4 The Experiment and Simulation Results for Test Case 4

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.5$ m/s, the orientation angle of the link 1 is $\theta = 50^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.26 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 798$ N,



Figure 4.26: The normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.27 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.28 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.29 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.27 is calculated $v_{Ty} = -1.263$ m/s from the simulation. The experimental result is $v_{Ty} = -1.22$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.28 is calculated $v_{Tx} = -1.533$ m/s from the simulation. The experimental result is $v_{Tx} = -1.47$ m/s. The angular output velocity of link 1 shown

in Fig. 4.29 is calculated $\omega_1 = 19.64 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 20.5 \text{ rad/s}$.



Figure 4.27: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.28: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.29: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.30 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.31 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.32 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.30 is calculated $v_{C_{2y}} = 1.259 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.22 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.30 is calculated $v_{C_{2y}} = 1.259 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.22 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.31 is calculated $v_{C_{2x}} = 0.756 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.72 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.32 is calculated $\omega_2 = 7.812 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 8.1 \text{ rad/s}$.



Figure 4.30: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.5 \text{ m/s}$, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.31: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.5 \text{ m/s}$, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.32: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.33 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0401 J for each link, and total pre-impact kinetic energy of the system is 0.0802 J. The post-impact kinetic energy of the link 1 is 0.0205 J from the simulation, and 0.0218 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.7155 from simulations, and 0.7375 from experimental data. The post-impact kinetic energy of the link 2 is calculated 0.0410 J from the simulation, and 0.0408 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 1.0113 from the simulation, and 1.0089 from the experimental data. The sequence root of the system is e = 0.8761 from the simulation, and e = 0.8839 from the experimental data.



Figure 4.33: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy increases slightly.

4.5 The Experiment and Simulation Results for Test Case 5

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.5$ m/s, the orientation angle of the link 1 is $\theta = 55^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.34 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 857$ N,



Figure 4.34: The normal contact force variation at point T for initial condition conditions, $v_{Tx} = 0 \text{ m/s}, v_{Ty} = 1.5 \text{ m/s}, \theta = 55^{\circ}, \beta = 0^{\circ}$

Figure 4.35 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.36 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.37 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.35 is calculated $v_{Ty} = -1.251$ m/s from the simulation. The experimental result is $v_{Ty} = -1.22$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.36 is calculated $v_{Tx} = -1.466$ m/s from the simulation. The experimental result is $v_{Tx} = -1.42$ m/s. The angular output velocity of link 1 shown

in Fig. 4.37 is calculated $\omega_1 = 17.903 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 18.2 \text{ rad/s}$.



Figure 4.35: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.36: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.37: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.38 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.39 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.40 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.38 is calculated $v_{C_{2y}} = 1.159 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.13 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.38 is calculated $v_{C_{2y}} = 1.159 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.13 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.39 is calculated $v_{C_{2x}} = 0.756 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.72 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.40 is calculated $\omega_2 = 11.05 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 11.5 \text{ rad/s}$.



Figure 4.38: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}, v_{Ty} = 1.5 \text{ m/s}, \theta = 55^{\circ}, \beta = 0^{\circ}$



Figure 4.39: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.5 \text{ m/s}$, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.40: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.41 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0401 J for each link, and total pre-impact kinetic energy of the system is 0.0802 J. The post-impact kinetic energy of the link 1 is 0.0195 J from the simulation, and 0.0203 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.6983 from simulations, and 0.7119 from experimental data. The post-impact kinetic energy ratio of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.0395 J from the simulation, and 0.0375 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9927 from the simulation, and 0.9675 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9927 from the simulation, and 0.9675 from the experimental data. The sequence root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9927 from the simulation, and 0.9675 from the experimental data.



Figure 4.41: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.6 The Experiment and Simulation Results for Test Case 6

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.5$ m/s, the orientation angle of the link 1 is $\theta = 60^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.42 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 918$ N,



Figure 4.42: the normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.43 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.44 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.45 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.43 is calculated $v_{Ty} = -1.238$ m/s from the simulation. The experimental result is $v_{Ty} = -1.20$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.44 is calculated $v_{Tx} = -1.292$ m/s from the simulation. The experimental result is $v_{Tx} = -1.24$ m/s. The angular output velocity of link 1 shown

in Fig. 4.45 is calculated $\omega_1 = 15.446 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 15.9 \text{ rad/s}$.



Figure 4.43: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.44: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$


Figure 4.45: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.46 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.47 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.48 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.46 is calculated $v_{C_{2y}} = 1.050 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.97 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.46 is calculated $v_{C_{2y}} = 1.050 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 0.97 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.47 is calculated $v_{C_{2x}} = 0.747 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.69 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.48 is calculated $\omega_2 = 14.561 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 14.91 \text{ rad/s}$.



Figure 4.46: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.5 \text{ m/s}$, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.47: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.5 \text{ m/s}$, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.48: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.49 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0401 J for each link, and total pre-impact kinetic energy of the system is 0.0802 J. The post-impact kinetic energy of the link 1 is 0.0189 J from the simulation, and 0.0191 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.6862 from simulations, and 0.6900 from experimental data. The post-impact kinetic energy of the link 2 is calculated 0.0384 J from the simulation, and 0.0364 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9786 from the simulation, and 0.9529 from the experimental data. The simulation, and 0.9529 from the experimental data. The simulation of the system is e = 0.8453 from the simulation, and e = 0.8321 from the experimental data.



Figure 4.49: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.5$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.7 The Experiment and Simulation Results for Test Case 7

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.8$ m/s, the orientation angle of the link 1 is $\theta = 50^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.50 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 978$ N,



Figure 4.50: the normal contact force variation at point T for initial condition, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.51 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.52 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.53 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.51 is calculated $v_{Ty} = -1.472$ m/s from the simulation. The experimental result is $v_{Ty} = -1.42$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.52 is calculated $v_{Tx} = -1.817$ m/s from the simulation. The experimental result is $v_{Tx} = -1.72$ m/s. The angular output velocity of link 1 shown

in Fig. 4.53 is calculated $\omega_1 = 23.261 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 24.1 \text{ rad/s}$.



Figure 4.51: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.52: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.53: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.54 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.55 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.56 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.54 is calculated $v_{C_{2y}} = 1.515 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.44 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.54 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.55 m/s from the simulation. The experimental result is $v_{C_{2x}} = 0.895 \text{ m/s}$ from the simulation. The simulation. The angular output velocity of link 2 shown in Fig. 4.56 is calculated $\omega_2 = 9.246 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 9.91 \text{ rad/s}$.



Figure 4.54: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.8 \text{ m/s}$, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.55: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.8 \text{ m/s}$, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$



Figure 4.56: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

Figure 4.57 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0577 J for each link, and total pre-impact kinetic energy of the system is 0.1154 J. The post-impact kinetic energy of the link 1 is 0.0285 J from the simulation, and 0.0299 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.7025 from simulations, and 0.7139 from experimental data. The post-impact kinetic energy of the link 2 is calculated 0.0587 J from the simulation, and 0.0572 J from the simulation and 0.0587 J from the simulation and 0.0572 J from the simulation and 0.0572 J from the simulation and 0.0587 J from the simulation and 0.0572 J from the simulation is 0.8087 from the simulation, and 0.8686 from the experimental data.



Figure 4.57: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 50^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy increases slightly in experiments, but increases in simulations.

4.8 The Experiment and Simulation Results for Test Case 8

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.8$ m/s, the orientation angle of the link 1 is $\theta = 55^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.58 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 1050$ N,



Figure 4.58: The normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.59 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.60 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.61 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.59 is calculated $v_{Ty} = -1.457$ m/s from the simulation. The experimental result is $v_{Ty} = -1.41$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.60 is calculated $v_{Tx} = -1.735$ m/s from the simulation. The experimental result is $v_{Tx} = -1.64$ m/s. The angular output velocity of link 1 shown

in Fig. 4.61 is calculated $\omega_1 = 21.196 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 22.03 \text{ rad/s}$.



Figure 4.59: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.60: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.61: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.62 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.63 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.64 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.62 is calculated $v_{C_{2y}} = 1.396 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.3 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.62 is calculated $v_{C_{2y}} = 1.396 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.3 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.63 is calculated $v_{C_{2x}} = 0.906 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.83 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.64 is calculated $\omega_2 = 13.077 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 13.71 \text{ rad/s}$.



Figure 4.62: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.8 \text{ m/s}$, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.63: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.8 \text{ m/s}$, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$



Figure 4.64: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

Figure 4.65 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0577 J for each link, and total pre-impact kinetic energy of the system is 0.1154 J. The post-impact kinetic energy of the link 1 is 0.0269 J from the simulation, and 0.0273 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.6831 from simulations, and 0.6876 from experimental data. The post-impact kinetic energy ratio of the link 2 is calculated 0.0565 J from the simulation, and 0.0542 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9896 from the simulation, and 0.9690 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9896 from the simulation, and 0.9690 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9896 from the simulation, and 0.9690 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9896 from the simulation, and 0.9690 from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9896 from the simulation, and 0.9690 from the experimental data.



Figure 4.65: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 55^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.9 The Experiment and Simulation Results for Test Case 9

The simulation results and the experimental results for given initial conditions are explained. The initial tangential velocity of the tip point of link is $v_{Tx} = 0$ m/s, the initial normal velocity of the tip point of link 1 is $v_{Ty} = 1.8$ m/s, the orientation angle of the link 1 is $\theta = 60^{\circ}$ with respect to the horizontal plane and the link 2 is to parallel to the horizontal axis ($\beta = 0$).

Figure 4.66 shows the variation of the normal contact force during the impact. Maximum normal contact force is calculated $F_N = 1123$ N,



Figure 4.66: the normal contact force variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.67 depicts normal velocity of the tip of link 1 simulation, and experimental result. Fig. 4.68 depicts tangential velocity of the link 1 simulation, and experimental results. Fig. 4.53 depicts angular velocity of the link 1 simulation, and experimental results. The normal output velocity of the tip point of link 1 shown in Fig. 4.67 is calculated $v_{Ty} = -1.441$ m/s from the simulation. The experimental result is $v_{Ty} = -1.40$ m/s. The tangential output velocity of the tip point of link 1 shown in Fig. 4.68 is calculated $v_{Tx} = -1.53$ m/s from the simulation. The experimental result is $v_{Tx} = -1.45$ m/s. The angular output velocity of link 1 shown

in Fig. 4.69 is calculated $\omega_1 = 18.28 \text{ rad/s}$ from the simulation. The experimental result is $\omega_1 = 19.11 \text{ rad/s}$.



Figure 4.67: Normal velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.68: Tangential velocity variation at point T for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.69: Angular velocity of link 1 variation for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.70 depicts normal velocity of the center of link 2 simulation, and experimental result. Fig. 4.71 depicts tangential velocity of the center of link 2 simulation, and experimental results. Fig. 4.72 depicts angular velocity of the link 2 simulation, and experimental results. The normal output velocity of center of link 2 shown in Fig. 4.70 is calculated $v_{C_{2y}} = 1.268 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2y}} = 1.19 \text{ m/s}$. The tangential output velocity of center of link 2 shown in Fig. 4.70 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.71 m/s. The tangential output velocity of center of link 2 shown in Fig. 4.71 is calculated $v_{C_{2x}} = 0.878 \text{ m/s}$ from the simulation. The experimental result is $v_{C_{2x}} = 0.81 \text{ m/s}$. The angular output velocity of link 2 shown in Fig. 4.72 is calculated $\omega_2 = 17.23 \text{ rad/s}$ from the simulation. The experimental result is $\omega_2 = 18.13 \text{ rad/s}$.



Figure 4.70: Normal velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}, v_{Ty} = 1.8 \text{ m/s}, \theta = 60^{\circ}, \beta = 0^{\circ}$



Figure 4.71: Tangential velocity variation at center of the link 2 for initial conditions, $v_{Tx} = 0 \text{ m/s}$, $v_{Ty} = 1.2 \text{ m/s}$, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$



Figure 4.72: Angular velocity variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

Figure 4.73 depicts the kinetic energy variations of the link 1, link 2, and the total kinetic energy of the system. The notation T_1 in the figure illustrates kinetic energy of the link 1, the notation T_2 in the figure illustrates kinetic energy of the link 2, the notation T in the figure illustrates the total kinetic energy of the kinematic chain. The pre-impact kinetic energy is 0.0577 J for each link, and total pre-impact kinetic energy of the system is 0.1154 J. The post-impact kinetic energy of the link 1 is 0.0258 J from the simulation, and 0.0254 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of the link 1 is 0.6683 from simulations, and 0.6628 from experimental data. The post-impact kinetic energy of the link 2 is calculated 0.0549 J from the simulation, and 0.0544 J from the experimental data. The square root of the pre-impact and post-impact kinetic energy ratio of link 2 is calculated 0.9748 from the simulation, and 0.9690 from the experimental data. The generalized energetic coefficient of restitution of the system is e = 0.8359 from the simulation, and e = 0.8303 from the experimental data.



Figure 4.73: Kinetic energy variation of the link 2 for initial conditions, $v_{Tx} = 0$ m/s, $v_{Ty} = 1.8$ m/s, $\theta = 60^{\circ}$, $\beta = 0^{\circ}$

The experiments and simulations are in good agreement. In this test case major decrease in link 1, and whole system kinetic energy is observed. However link 2 kinetic energy decreases slightly.

4.10 Impact Angle, θ , and Link 2 Kinetic Energy Relation

The experiments and the simulations displays a relation between link 1 initial impact angle and link 2 kinetic energy. In this purpose a series of simulation are conducted for various link 1 initial impact angle θ . The other 3 initials are set to $v_{Tx} = 0$ m/s, $v_{Ty} = 1.2$ m/s, $\beta = 0^{\circ}$. The simulation results are shown in Fig. 4.74, where dotted line indicates pre-impact kinetic energy. The simulation results reveals existence of critical angle which divides link 2 into energy loose and energy gain zone.



Figure 4.74: Link 2 kinetic energy response for Link 1 impact angle for initial conditions, $v_{Tx} = 0 \text{ m/s}, v_{Ty} = 1.2 \text{ m/s}, \beta = 0^{\circ}$

In Fig. 4.75 square root ratios of the link 2 post-impact and pre-impact kinetic energies for various initial normal velocity and initial link 1 impact angle obtained by computer simulations are shown. The term $T_2(f)$ illustrates post-impact kinetic energy and the term $T_2(0)$ illustrates pre-impact kinetic energy.



Figure 4.75: Link 2 kinetic energy ratio for Link 1 impact angle for initial conditions, $v_{Tx} = 0 \text{ m/s}, \beta = 0^{\circ}$

4.11 Conclusion

In this study, Impact of two link kinematic chain with a flat is studied. Only the first link of the chain is impacting and the end point of the second link is free to move. First, computer simulations using the Jackson Green formula [67] and smooth Coulomb friction conducted for three different initial velocities with three different orientation. Later, a special setup is built and more than a hundred experiments are conducted. The motion of the kinematic chain has been recorded with a high speed camera at 6,000 fps. Generated videos have been analyzed with a image processing algorithm. Pre-impact and post-impact velocities of each link is calculated from the video. Since, two link kinematic chain has four degree of freedom in two dimensional space most of the experiments are mismatched with desired initials and experiments are repeated until gathering at least three adequate data set. Simulation results and experimental results are in good agreement with less than 10% error for each case.

To gather better understanding on dynamic impact response of the kinematic chain, kinetic energies of the links calculated. Square root of the ratio of post-impact and pre-impact kinetic energies are calculated. Inherently, Link 1 kinetic energy and total kinetic energy of the kinematic chain decreases after elasto-plastic impact, however increase on link 2 kinetic energy is observed in same cases. Experiments and simulation results illustrates that the impact has very limited affect on link two for given horizontal orientation of the link two. The maximum chance of the kinetic energy of link 2 is calculated less then 5%, however the maximum chance of kinetic energy of link 1 is 55%. On the other hand, Results with 50° orientation of link 1 shows a slight increase of link 2 kinetic energy, but 55°, 60° orientation of link 1 showed decrease on link two kinetic energy. The results reveals existence of a critical angle which divides the impact into energy loose and energy gain zone for link two.

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