Existence of TP(d, k, n)

by

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Abstract

Given a square grid of land, which has n rows and n columns. It is required to plant trees on the land so that there are k trees in every row and column and there is at most 1 tree in any small square part of the land with d rows and d columns. What should be the values of n, k and d? How to plant the trees?

The objective of this dissertation is to analyze the problem and come up with the answers to the questions proposed above. The dissertation consists of three main parts. The first part provides necessary conditions and the second part provides sufficient conditions regarding the values of n, k and d. The final part of the dissertation delivers the method to plant the trees on the land under given constraints.

Acknowledgments

First of all, I would like to express my sincere gratitude and appreciation to my advisor Dr. Dean G. Hoffman for providing me continuous support and being patient with me throughout my PhD study. His help, patience and endless support guided me during the research and writing process. He has shown me, by his example, what a good advisor, teacher and person should be.

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Last but not least, I would like to express my deepest gratitude to my mother Zohra and father Sabir, who have always believed in me. I would also like to thank my wife Ulviyya (together with our three musketeers, Zafar, Salim and Salih) for her love, support and patience that has enabled me to successfully complete this challenging process.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this chapter we discuss necessary and sufficient conditions for the existence of TP(d, k, n) and assume $k \geq 1$, $n > d \geq 2$ and $m > d$.

We start by introducing the following definitions.

Definition 1.1. *A grid is an $m \times n$ array with rows labeled $1, 2, ..., m$ and columns labeled $1, 2, ..., n$. An $n-$grid is a grid where $m = n$.*

Definition 1.2. *A subgrid $S(s, t, u, v)$ of $m \times n$ grid, where $1 \leq s \leq t \leq m$, $1 \leq u \leq v \leq n$, is the intersection of rows s through t, and columns u through v. Let $t - s + 1 = p$ and $v - u + 1 = r$, then we call $S(s, t, u, v)$ as block $B[p, r]$.*

Definition 1.3. *Let $k, d, n \in \mathbb{Z}^+$, where $1 \leq k$, $2 \leq d < n$. Then a $(d, k, n)-$tree planting (denoted as TP(d, k, n)) is a planting of exactly k trees in each row and column of $n-$grid such that there is at most one tree in any square block $B[d, d]$.*

1.1 Necessary condition for TP(d, k, n)

Theorem 1.4. *If TP(d, k, n) exists, then $\left\lfloor \dfrac{n}{d^2} \right\rfloor \geq k$.*

Proof. Let us take a look at subgrid $S(1, d, 1, n)$ of TP(d, k, n) as in Figure 1.1. Since every row contains k trees, then there are kd trees in $S(1, d, 1, n)$ in total.

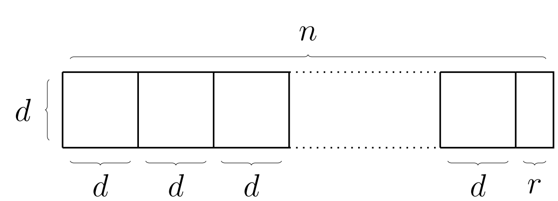


Figure 1.1: Subgrid $S(1, d, 1, n)$ of TP(d, k, n)

On the other hand, consider the following $\lceil \frac{n}{d} \rceil$ blocks in $S(1, d, 1, n)$, which we call *special blocks*:

(i) x number of square blocks $B[d, d]$, defined as $S(1, d, (i-1)d + 1, id)$, where $1 \leq i \leq x$ and $x = \lfloor \frac{n}{d} \rfloor$.

(ii) A block $B[d, n - xd]$, defined as $S(1, d, xd + 1, n)$, if $d \nmid n$

Since the number of special blocks in $S(1, d, 1, n)$ is $\lceil \frac{n}{d} \rceil$ and there is at most one tree in any $B[d, d]$, then we get the following inequality:

$$kd \leq \left\lceil \frac{n}{d} \right\rceil$$

Obviously, then:

$$k \leq \left\lfloor \frac{\lceil \frac{n}{d} \rceil}{d} \right\rfloor$$

Let $n = q_1 d - r_1$ for some $q_1, r_1 \in \mathbb{Z}$, $0 \leq r_1 < d$, then $\lceil \frac{n}{d} \rceil = q_1$. And let $q_1 = q_2 d + r_2$ for some $q_2, r_2 \in \mathbb{Z}$, $0 \leq r_2 < d$, then $\lfloor \frac{q_1}{d} \rfloor = q_2$.

Since $n = (q_2 d + r_2)d - r_1$, therefore

$$\left\lfloor \frac{n}{d^2} \right\rfloor = q_2 + \left\lfloor \frac{r_2 d - r_1}{d^2} \right\rfloor = \left\lfloor \frac{\lceil \frac{n}{d} \rceil}{d} \right\rfloor + \left\lfloor \frac{r_2 d - r_1}{d^2} \right\rfloor \tag{1.1}$$

Meanwhile, following equation is true.

$$\left\lfloor \frac{r_2 d - r_1}{d^2} \right\rfloor = \begin{cases} 0 & \text{if } r_1 \leq r_2 d \\ -1 & \text{if } r_1 \neq 0, r_2 = 0 \end{cases} \tag{1.2}$$

Combining the equations 1.1 and 1.2 we get the following result:

$$k \leq \left\lfloor \frac{\lceil \frac{n}{d} \rceil}{d} \right\rfloor = \begin{cases} \left\lfloor \dfrac{n}{d^2} \right\rfloor + 1 & \text{if } d \nmid n \text{ and } d \mid \lceil \frac{n}{d} \rceil \qquad (i) \\ \\ \left\lfloor \dfrac{n}{d^2} \right\rfloor & \text{otherwise} \qquad (ii) \end{cases} \tag{1.3}$$

Now we need to show that case (i) does not hold with equality for $\mathrm{TP}(d, k, n)$; in other words $k \neq \lfloor \frac{n}{d^2} \rfloor + 1$.

Suppose that $k = \lfloor \frac{n}{d^2} \rfloor + 1 = \lfloor \lceil \frac{n}{d} \rceil / d \rfloor$. Since $d \nmid n$ and $d \mid \lceil \frac{n}{d} \rceil$ is the case, thus $k = \lceil \frac{n}{d} \rceil / d$. Therefore $kd = \lceil \frac{n}{d} \rceil$, which means there is a tree in every one of the $\lceil \frac{n}{d} \rceil$ special blocks in $S(1, d, 1, n)$.

Obviously, using this fact about $S(1, d, 1, n)$ and a symmetry of $n-$grid, we obtain the result as in Figure 1.2.

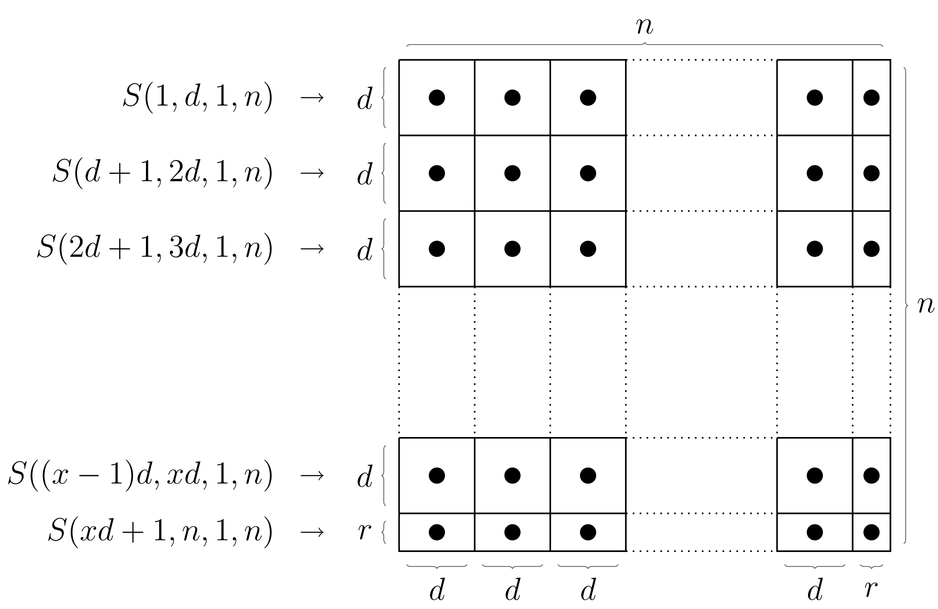


Figure 1.2: $n-$grid

Now, let us take a look at $S(1, n, n-d+1, n)$ in Figure 1.3. To be more precise, let's observe $S(1, d, n-d+1, n)$. Since every special block contains a tree and there is at most one tree in any $B[d, d]$, then a tree in $S(1, d, xd+1, n)$ is the only tree in $S(1, d, n-d+1, n)$

and $S(1, d, n-d+1, xd)$ contains no tree (recall that $x = \left\lfloor \frac{n}{d} \right\rfloor$). This can be generalized for the rest of $S(1, n, n-d+1, n)$. That is $S(1, n, n-d+1, xd)$ contains no tree, i.e., there is no tree in the columns $n-d+1$ trough xd, which contradicts the definition of $\mathrm{TP}(d, k, n)$. Therefore, $k \neq \left\lfloor \frac{n}{d^2} \right\rfloor + 1$ and $k \leq \left\lfloor \frac{n}{d^2} \right\rfloor$.

□

In the propositions and theorems, which are introduced in the later chapters, we consider $k = \left\lfloor \frac{n}{d^2} \right\rfloor$. By the Marriage Theorem[1], if the $\mathrm{TP}(d, k, n)$ exists for $k = \left\lfloor \frac{n}{d^2} \right\rfloor$, then it exists for the values of $k < \left\lfloor \frac{n}{d^2} \right\rfloor$

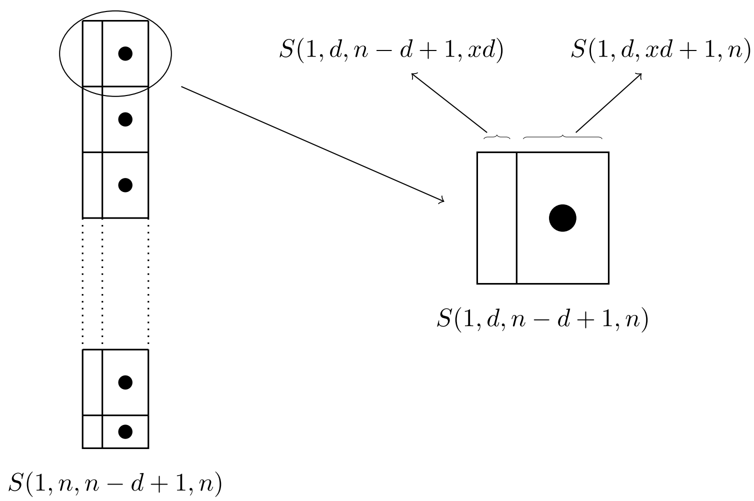


Figure 1.3: $S(1, n, n-d+1, n)$

1.2 Sufficient conditions for $\mathrm{TP}(d, k, n)$

In this section we find possible values of n in terms of d and k, where $\mathrm{TP}(d, k, n)$ exists. To prove our claims, we will mainly show in which cells of $n-$grid trees should be planted. We start with the following lemmas which will be used in the proof of proposition 1.6.

Lemma 1.5. *Let z and i be integers, where $1 \leq i \leq d$, $1 \leq z \leq d$ and $P_z = \left\{ j \mid \left\lfloor \frac{jd+i-1}{kd} \right\rfloor = z-1 \right\}$. Then,*

(i) If $(z-1)k \leq j \leq zk-1$, then $0 \leq \left\lfloor \frac{jd+i-1}{kd} \right\rfloor \leq d-1$.

(ii) If $\left\lfloor \frac{jd+i-1}{kd} \right\rfloor = z-1$, then $(z-1)k \leq j \leq zk-1$.

(iii) $|P_z| = k$.

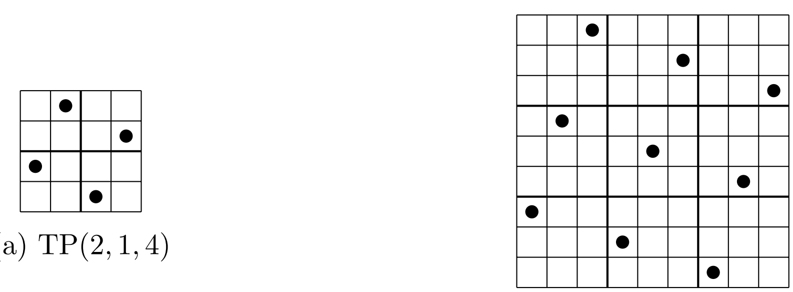
Proof. Under the given conditions on j and i, one can easily show that $(z-1)kd \leq jd+i-1 \leq zkd-1$. Consequently, $z-1 \leq \frac{jd+i-1}{kd} \leq z - \frac{1}{kd}$, that is, $\left\lfloor \frac{jd+i-1}{kd} \right\rfloor = z-1$. Since $z \in \{1, 2, ..., d\}$, we obtain $0 \leq \left\lfloor \frac{jd+i-1}{kd} \right\rfloor \leq d-1$. This proves (i).

If $\left\lfloor \frac{jd+i-1}{kd} \right\rfloor = z-1$, then $z-1 \leq \frac{jd+i-1}{kd} < z$. Thus, $(z-1)k - 1 + \frac{1}{d} \leq j < zk$ is obtained. Since j is an integer, preceding inequality can be written as $(z-1)k \leq j \leq zk-1$ and it proves (ii).

Finally, the number of j's satisfying $\left\lfloor \frac{jd+i-1}{kd} \right\rfloor = z-1$ for a given z, can be determined by using (ii), that is, $zk-1-(z-1)k+1 = k$. Therefore, $|P_z| = k$. $\square$

Proposition 1.6. *TP(d, k, kd^2) can be attained by planting trees in the cells* $\left(jd+i, d(ki+l-k) - \left\lfloor \frac{jd+i-1}{kd} \right\rfloor\right)$, *where* $1 \leq i \leq d$, $0 \leq j \leq dk-1$ *and* $1 \leq l \leq k$.

Proof. Let $k=1$, then TP$(d, 1, d^2)$. Therefore, the formula above for the cells containing trees reduces to $(jd+i, id-j)$, where $1 \leq i \leq d$ and $0 \leq j \leq d-1$. One can easily show that is true.



(a) TP$(2, 1, 4)$

(b) TP$(3, 1, 9)$

Figure 1.4: Two examples of TP$(d, 1, d^2)$

Let $k \geq 2$. Then, we need to show the following:

(i) There are k trees in each row. Obviously, to determine the number of trees in a row, one needs to keep $jd+i$ constant, and look at the number of possible values of

$d(ki+l-k)-\lfloor\frac{jd+i-1}{kd}\rfloor$. Since i and j must be fixed, and $1\leq l\leq k$, there are k columns containing a tree in a fixed row.

(ii) There are k trees in each column. We use similar idea to prove it. Obviously, $d(ki+l-k)-\lfloor\frac{jd+i-1}{kd}\rfloor$ needs to be constant, that is, l, i and $\lfloor\frac{jd+i-1}{kd}\rfloor$ must be fixed. As can be seen, $\lfloor\frac{jd+i-1}{kd}\rfloor$ is constant for the k values of j in the set P_z from Lemma 1.5. Therefore the number of all possible values of $jd+i$, i.e the number of rows containing a tree in a fixed column is k.

(iii) There is at most one tree in any B$[d,d]$. Suppose there are trees in cells (r_1,c_1) and (r_2,c_2). Then $r_1=j_1d+i_1$, $c_1=d(ki_1+l_1-k)-\lfloor\frac{j_1d+i_1-1}{kd}\rfloor$, $r_2=j_2d+i_2$ and $c_2=d(ki_2+l_2-k)-\lfloor\frac{j_2d+i_2-1}{kd}\rfloor$.

Let $R=|r_1-r_2|=|d(j_1-j_2)+(i_1-i_2)|$ and
$C=|c_1-c_2|=|d[k(i_1-i_2)+(l_1-l_2)]-A|$, where
$A=\lfloor\frac{j_1d+i_1-1}{kd}\rfloor-\lfloor\frac{j_2d+i_2-1}{kd}\rfloor$. We need to show either $R\geq d$ or $C\geq d$ (unless of course $R=0=C$, in which case the two cells are equal).

(a) Trivial case: $i_1=i_2$, $j_1=j_2$ and $l_1=l_2$, then $R=C=0$, i.e these two cells are the same cell.

(b) $i_1=i_2$, $j_1=j_2$ and $l_1\neq l_2$. We obtain $R=0$ and $C=|d(l_1-l_2)|>d$.

(c) $l_1=l_2$, $j_1=j_2$ and $i_1\neq i_2$. Since $j_1=j_2$, it is easy to verify that $\lfloor\frac{j_1d+i_1-1}{kd}\rfloor=\lfloor\frac{j_2d+i_2-1}{kd}\rfloor$ for $1\leq i_1,i_2\leq d$ by Lemma 1.5. Therefore, $A=0$. For this reason, we obtain $R=|i_1-i_2|$ and $C=|dk(i_1-i_2)|>d$.

(d) $i_1=i_2$, $l_1=l_2$, and $j_1\neq j_2$. We obtain $C=|A|$ and
$R=|d(j_1-j_2)|>d$.

(e) $i_1=i_2$, $l_1\neq l_2$ and $j_1\neq j_2$. We obtain $C=|d(l_1-l_2)-A|$ and
$R=|d(j_1-j_2)|>d$.

(f) $j_1 = j_2$, $i_1 \neq i_2$ and $l_1 \neq l_2$. As we verified above, $A = 0$ when $j_1 = j_2$. Thus, we obtain $R = |i_1 - i_2|$ and $C = |d[k(i_1 - i_2) + (l_1 - l_2)]| > d$.

(g) $l_1 = l_2$, $i_1 \neq i_2$ and $j_1 \neq j_2$. We obtain $C = |dk(i_1 - i_2) - A|$ and $R = |d(j_1 - j_2) + (i_1 - i_2)|$. We may assume $i_1 > i_2$.

(i) Let $j_1 > j_2$. Then $R > d$.

(ii) Let $j_1 < j_2$. Obviously, $1 \leq i_1 - i_2 \leq d - 1$ and $1 - d \leq A \leq 0$. Hence, we obtain $dk \leq dk(i_1 - i_2) - A \leq dk(d-1) + d - 1$. Therefore, $dk \leq C \leq dk(d-1) + d - 1$.

(h) $i_1 \neq i_2$, $j_1 \neq j_2$ and $l_1 \neq l_2$. We obtain
$C = |d[k(i_1 - i_2) + (l_1 - l_2)] - A|$ and $R = |d(j_1 - j_2) + (i_1 - i_2)|$. We may assume $i_1 > i_2$, then:

(i) Let $j_1 > j_2$ and $l_1 \neq l_2$. Then $R > d$.

(ii) Let $j_1 < j_2$. Obviously, $1 \leq i_1 - i_2 \leq d - 1$, $1 - d \leq A \leq 0$.

(1) If $l_1 > l_2$, then $1 \leq l_1 - l_2 \leq k - 1$. Thus, we obtain $d(k+1) \leq d[k(i_1 - i_2) + (l_1 - l_2)] - A \leq d^2k - 1$. Therefore, $d(k+1) \leq C \leq d^2k - 1$.

(2) If $l_1 < l_2$, then $1 - k \leq l_1 - l_2 \leq -1$. Thus, we obtain $d \leq d[k(i_1 - i_2) + (l_1 - l_2)] - A \leq d^2k - dk - 1$. Therefore, $d \leq C \leq d^2k - dk - 1$.

□

Lemma 1.7. *Let x, y and z be integers, where $1 \leq x \leq dk-1$, $1 \leq y \leq m-1$, $1-k \leq z \leq -1$. Let $A \in \{-1, 0\}$. Also, let $R = mx - y$ and $C = dky + dz + A$. Then, if $R < d$, then $x = 1$, $m - d < y \leq m - 1$ and $C \geq d$. (Hence either $R \geq d$ or $C \geq d$).*

Proof. The proof is as following:

1. If instead $x \geq 2$, (i.e. $2 \leq x \leq dk - 1$), then we obtain $m + 1 \leq R \leq m(dk-1) - 1$, due to the given condition $1 - m \leq -y \leq -1$. As a result, $R \geq m + 1 > d$. Thus $x = 1$.

2. Also, if instead $y \leq m - d$ (i.e. $1 \leq y \leq m - d$), then $d \leq R \leq m(dk - 1) - 1$, since $1 \leq x \leq dk - 1$. Thus $m - d < y \leq m - 1$.

3. To prove $C \geq d$, we apply direct proof using $x = 1$ and $m - d < y \leq m - 1$ when $R < d$. Because, $dk(m - d) < dky \leq dk(m - 1)$, $d(1 - k) \leq dz \leq -d$, we get $d[k(m-d-1)+1]-1 < C \leq dk(m-1)-d$. That is, $C > d[k(m-d-1)+1]-1 \geq d-1$, since $m > d$. Then, $C \geq d$.

□

Proposition 1.8. *TP(d, k, mdk) can be attained by planting trees in the cells* $\left(jm + i, d(ki + l) - dk + \left\lfloor \frac{jm+i-1}{mk} \right\rfloor + 1\right)$, *where* $1 \leq i \leq m$, $0 \leq j \leq dk-1$ *and* $0 \leq l \leq k-1$.

Proof. Let $k = 1$. Then, the formula above for the cells containing trees reduces to $(jm + i, id + j - d + 1)$, where $1 \leq i \leq m$ and $0 \leq j \leq d - 1$. One can easily show that is true.

Let $k \geq 2$. Then, we need to show the following:

(i) There are k trees in each row. Obviously, to determine the number of trees in a row, one needs to keep $jm + i$ constant, and look at the number of possible values of $d(ki + l) - dk + \left\lfloor \frac{jm+i-1}{mk} \right\rfloor + 1$. Since i and j must be fixed, and $0 \leq l \leq k - 1$, there are k columns containing a tree in a fixed row.

(ii) There are k trees in each column. We use similar idea to prove it. Obviously, $d(ki + l) - dk + \left\lfloor \frac{jm+i-1}{mk} \right\rfloor + 1$ needs to be constant, that is, l, i and $\left\lfloor \frac{jm+i-1}{mk} \right\rfloor$ must be fixed. As can be seen, $\left\lfloor \frac{jm+i-1}{mk} \right\rfloor$ is constant for the k values of j in the set P_z from Lemma 1.5. Therefore the number of all possible values of $jm+i$, i.e the number of rows containing a tree in a fixed column is k.

(iii) There is at most one tree in any B$[d, d]$. Suppose there are trees in cells (r_1, c_1) and (r_2, c_2). Then $r_1 = j_1 m + i_1$,
$c_1 = d(ki_1 + l_1) - dk + \left\lfloor \frac{j_1 m+i_1-1}{mk} \right\rfloor + 1$, $r_2 = j_2 m + i_2$ and
$c_2 = d(ki_2 + l_2) - dk + \left\lfloor \frac{j_2 m+i_2-1}{mk} \right\rfloor + 1$.

Let $R = |r_1 - r_2| = |m(j_1 - j_2) + (i_1 - i_2)|$ and

$C = |c_1 - c_2| = |d[k(i_1 - i_2) + (l_1 - l_2)] + A|$, where

$A = \left\lfloor \frac{j_1 m + i_1 - 1}{mk} \right\rfloor - \left\lfloor \frac{j_2 m + i_2 - 1}{mk} \right\rfloor$. We need to show either $R \geq d$ or $C \geq d$ (unless of course $R = 0 = C$, in which case the two cells are equal).

(a) Trivial case: $i_1 = i_2$, $j_1 = j_2$ and $l_1 = l_2$, then $R = C = 0$, i.e these two cells are the same cell.

(b) $i_1 = i_2$, $j_1 = j_2$ and $l_1 \neq l_2$. We obtain $R = 0$ and $C = |d(l_1 - l_2)| > d$.

(c) $l_1 = l_2$, $j_1 = j_2$ and $i_1 \neq i_2$. Since $j_1 = j_2$, it is easy to verify that $\left\lfloor \frac{j_1 m + i_1 - 1}{mk} \right\rfloor = \left\lfloor \frac{j_2 m + i_2 - 1}{mk} \right\rfloor$ for $1 \leq i_1, i_2 \leq d$ by Lemma 1.5. Therefore, $A = 0$. For this reason, we obtain $R = |i_1 - i_2|$ and $C = |dk(i_1 - i_2)| > d$.

(d) $i_1 = i_2$, $l_1 = l_2$, and $j_1 \neq j_2$. We obtain $C = |A|$ and $R = |m(j_1 - j_2)| > d$.

(e) $i_1 = i_2$, $l_1 \neq l_2$ and $j_1 \neq j_2$. We obtain $C = |d(l_1 - l_2) + A|$ and $R = |m(j_1 - j_2)| > d$.

(f) $j_1 = j_2$, $i_1 \neq i_2$ and $l_1 \neq l_2$. As we verified above, $A = 0$ when $j_1 = j_2$. Thus, we obtain $R = |i_1 - i_2|$ and $C = |d[k(i_1 - i_2) + (l_1 - l_2)]| > d$.

(g) $l_1 = l_2$, $i_1 \neq i_2$ and $j_1 \neq j_2$. We obtain $C = |dk(i_1 - i_2) + A|$ and $R = |m(j_1 - j_2) + (i_1 - i_2)|$. We may assume $i_1 > i_2$.

   (i) Let $j_1 > j_2$. Then $R > d$.

   (ii) Let $j_1 < j_2$. Obviously, $1 \leq i_1 - i_2 \leq m - 1$ and $1 - d \leq A \leq 0$. Hence, we obtain $dk - d + 1 \leq dk(i_1 - i_2) + A \leq dk(m - 1)$, i.e $dk - d + 1 \leq C \leq dk(m - 1)$. In other words, $C \geq dk - d + 1 > d$.

(h) $i_1 \neq i_2$, $j_1 \neq j_2$ and $l_1 \neq l_2$. We obtain $C = |d[k(i_1 - i_2) + (l_1 - l_2)] + A|$ and $R = |m(j_1 - j_2) + (i_1 - i_2)|$. We may assume $i_1 > i_2$, then:

(i) Let $j_1 > j_2$ and $l_1 \neq l_2$. Then $R > d$.

(ii) Let $j_1 < j_2$. Obviously, $1 \leq i_1 - i_2 \leq m-1$, $1-d \leq A \leq 0$.

(1) If $l_1 > l_2$, then $1 \leq l_1 - l_2 \leq k-1$. Thus, we obtain $dk+1 \leq d[k(i_1 - i_2) + (l_1 - l_2)] + A \leq d(km-1)$. Therefore, $dk+1 \leq C \leq d(km-1)$.

(2) If $l_1 < l_2$, then $R = m(j_2 - j_1) - (i_1 - i_2)$ and $C = dk(i_1 - i_2) + d(l_1 - l_2) + A$. Let $y = i_1 - i_2$, $x = j_2 - j_1$, $z = l_1 - l_2$. Then $R = mx - y$ and $C = dky + dz + A$. One can easily show that, $A \in \{-1, 0\}$ when $j_2 - j_1 = 1$. Therefore, by Lemma 1.7 we obtain either $R \geq d$ or $C \geq d$.

Proving all the possible eight cases concludes our proof. $\square$

Now, let's see some examples of TP(d,k,mdk) in the Figures 1.5 and 1.6 and observe the pattern that occurs in each example.

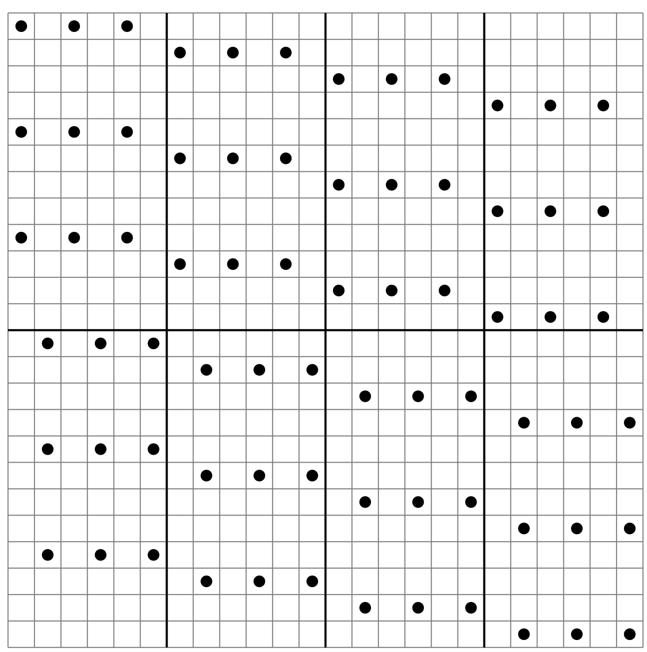


Figure 1.5: TP$(2,3,24)$

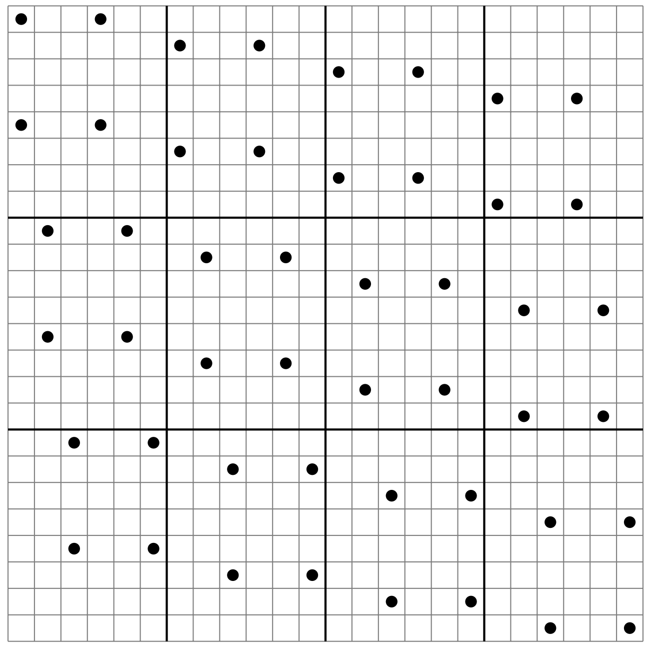


Figure 1.6: TP$(3,2,24)$

But how does one construct any TP(d,k,mdk). We provide two construction methods here. First one is, simply create $mdk \times mdk$ grid first and then fill in the trees in the cells defined in Propostion 1.8.

Second approach to construct TP(d,k,mdk) is following the simple construction trick which is obtained by observing the pattern of the trees which is discussed below in details. It is important to mention that these two methods give the same construction.

Now, let's dive into more details of the second method. It is easy to see the that there is a clear pattern in the previous two examples TP$(2,3,24)$ and TP$(3,2,24)$.

First, let's try to explain the pattern in TP$(2,3,24)$ (see Figure 1.5).

Instead of the whole grid, let us consider 9 trees located in the subgrid $S(1,12,1,6)$ (see Figure 1.7).

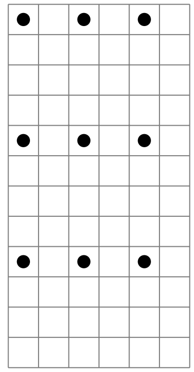


Figure 1.7: $S(1, 12, 1, 6)$ of $\mathrm{TP}(2, 3, 24)$

One can easily observe that $S(1, 12, 1, 6)$ consists of 9 copies of the same 4×2 block which contains a tree in the top-left cell as shown in Figure 1.8.

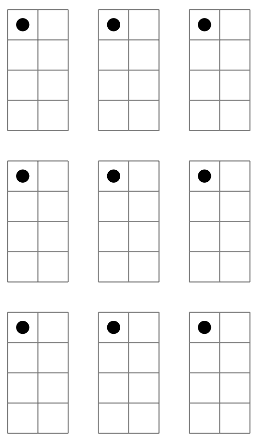


Figure 1.8: The $S(1, 12, 1, 6)$ consists of nine 4×2 blocks

In the next step, we take a look at the trees located in the subgrid $S(1, 12, 1, 24)$ described in the Figure 1.9 below.

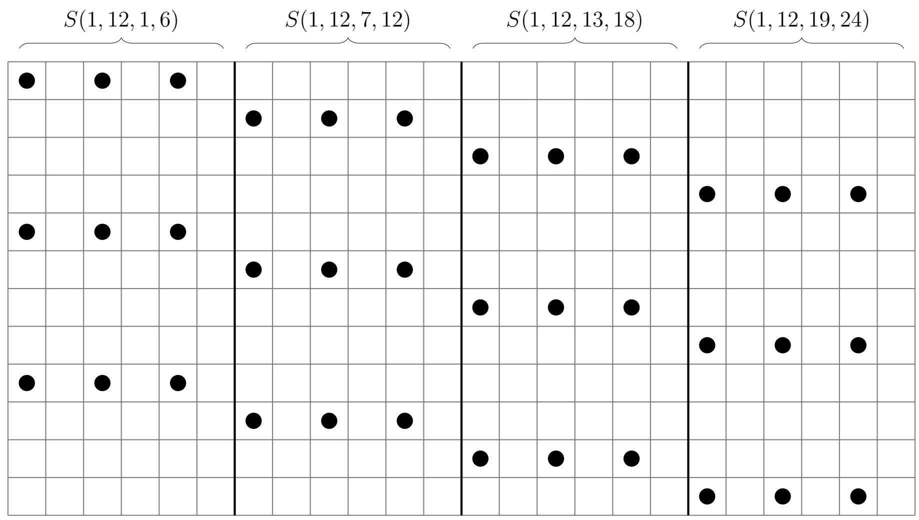


Figure 1.9: $S(1, 12, 1, 24)$ of TP$(2, 3, 24)$

Let us consider 9 trees located in $S(1, 12, 1, 6)$. It is easy to see that top 3 trees in $S(1, 12, 7, 12)$ are just 1 row below than the top 3 trees in $S(1, 12, 1, 6)$. Also, middle 3 trees in $S(1, 12, 7, 12)$ are 1 row below than the middle 3 trees in $S(1, 12, 1, 6)$ and bottom 3 trees in $S(1, 12, 7, 12)$ are 1 row below than the bottom 3 trees in $S(1, 12, 1, 6)$.

We can also observe that there is similar pattern between the 9 trees in $S(1, 12, 7, 12)$ and $S(1, 12, 12, 18)$. This is true for the 9 trees in $S(1, 12, 12, 18)$ and $S(1, 12, 19, 24)$.

Then, we can say that, using 9 trees in $S(1, 12, 1, 6)$, we can complete the rest of the trees in the remaining three subgrids. So, in order to locate 9 trees in $S(1, 12, 7, 12)$ just need to shift the 9 trees one row downward. As we shift it one row downward again we will obtain the locations of the trees in $S(1, 12, 13, 18)$ and shifting 9 trees one more row downward we obtain the locations for the last subgrid $S(1, 12, 19, 24)$.

We analized the pattern of the trees in the top half subgrid $S(1, 12, 1, 24)$ of TP$(2, 3, 24)$ so far. Finally, we observe the trees in the other half of TP$(2, 3, 24)$ which is the subgrid $S(13, 24, 1, 24)$. Then it is very obvious that 36 trees in $S(13, 24, 1, 24)$ can be obtained by simply shifting all the 36 trees in $S(1, 12, 1, 24)$ one column to right.

We can observe the similar pattern also in TP$(3, 2, 24)$ (see figure 1.6) as well. .

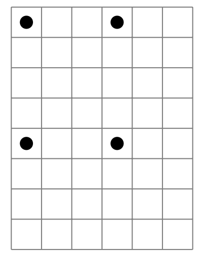


Figure 1.10: $S(1, 8, 1, 6)$ of TP$(3, 2, 24)$

It is also easy to see that the subgrid $S(1, 8, 1, 6)$ consists of 4 copies of the same 4×3 block containign a tree in the top-left cell as shown in Figure 1.11

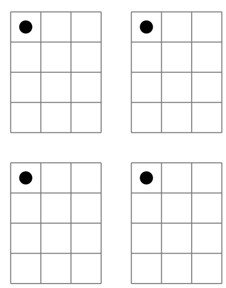


Figure 1.11: The $S(1, 8, 1, 6)$ consists of four 4×3 blocks

Similarly, we see that $S(1, 8, 1, 24)$ consists of 4 subgrids and 4 trees in each subgrid is obtained by shifting 4 trees (in the previous subgrid) one row downward. In other words, shifting 4 trees in the subgrid $S(1, 8, 1, 6)$ one row downward gives us the locations of 4 trees in the $S(1, 8, 7, 12)$. Using similar shifting we can locate 4 trees in each of the remaining subgrids. As a result, we obtain the locations of 16 trees in $S(1, 8, 1, 24)$ as in Figure 1.12

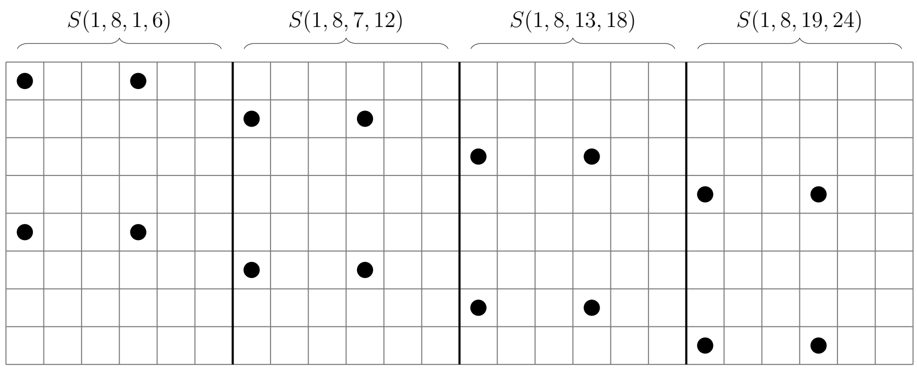


Figure 1.12: $S(1,8,1,24)$ of TP$(3,2,24)$

Finally, 16 trees in the subgrid $S(9,16,1,24)$ can be obtained by shifting 16 trees in the subgrid $S(1,8,1,24)$ one column to right. Similarly, 16 trees in the subgrid $S(17,24,1,24)$ can be obtained by shifting 16 trees in the pevious subgrid $S(9,16,1,24)$ one column to right. We can see that TP$(3,2,24)$ consists of these 3 subgrids as shown in Figure 1.13

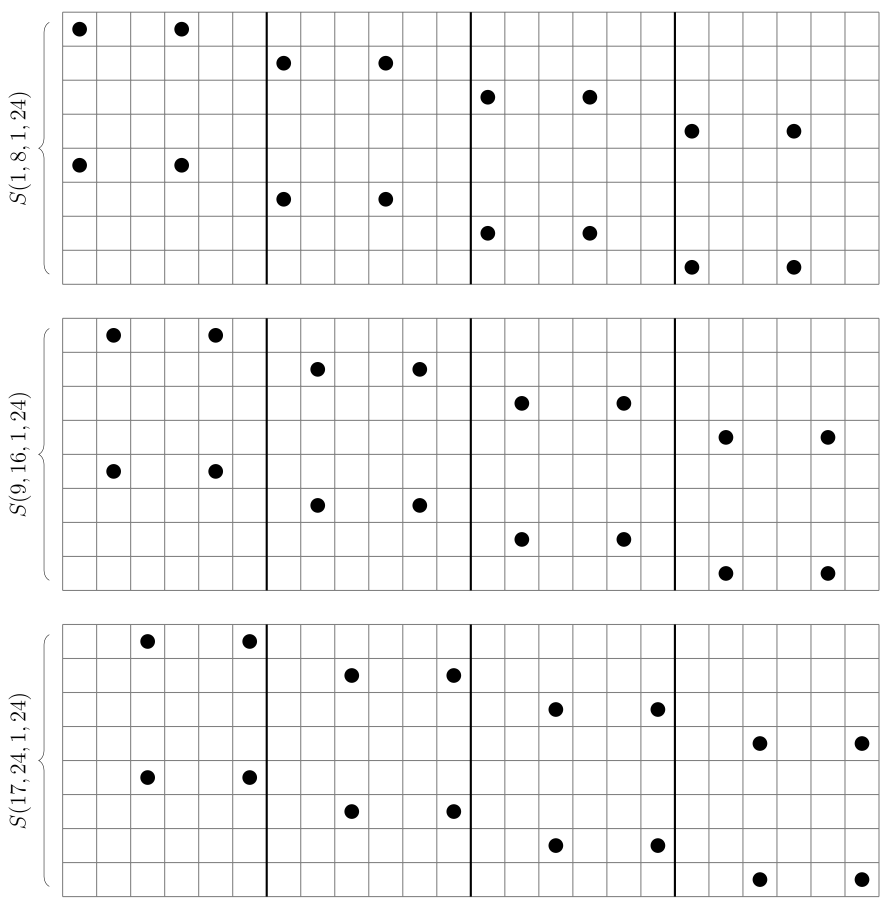


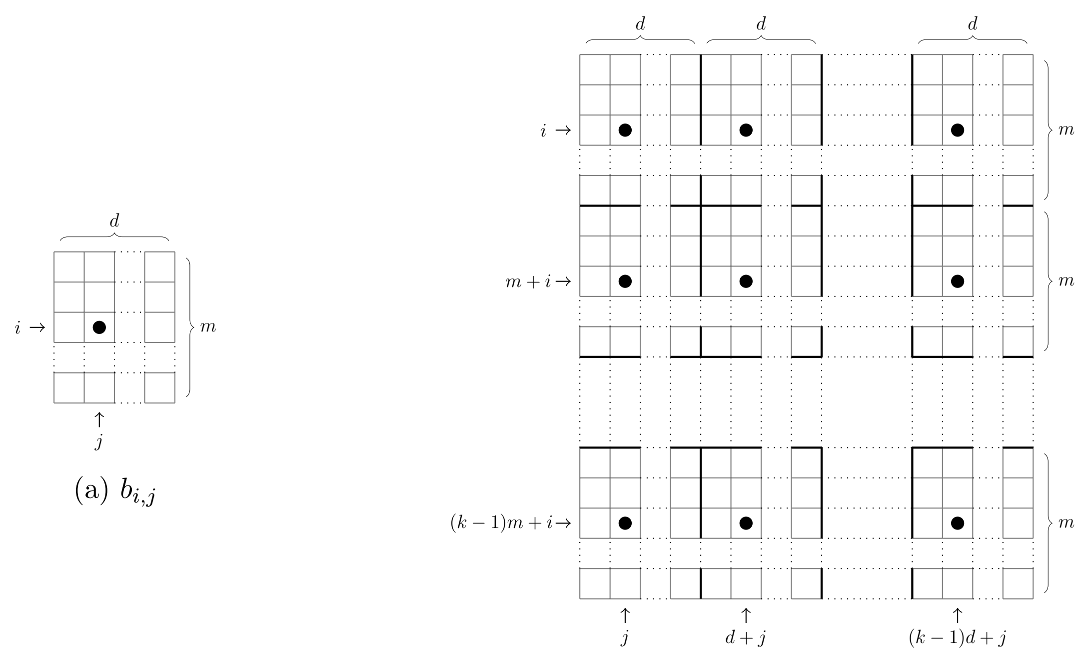
Figure 1.13: TP$(3,2,24)$ divided into $S(1,8,1,24)$, $S(9,16,1,24)$ and $S(17,24,1,24)$

1.3 Procedure to construct TP(d, k, mdk)

Now, after observing the similar pattern for the two examples, we can generalize the construction method for any TP(d, k, mdk). Before we introduce the method, let us introduce several definitions.

Definition 1.9. *A base block is a block $B[m, d]$ containing a tree in the cell (i, j). We denote it by $b_{i,j}$. Also, we define a Big block that is a block $B[mk, dk]$ consisting of k^2 copies of $b_{i,j}$ and denote it by $B_{i,j}$.*

Definition 1.10. *A Big block is a block $B[mk, dk]$ consisting of k^2 copies of $b_{i,j}$. We denote it by $B_{i,j}$.*



(a) $b_{i,j}$

(b) $B_{i,j}$

Figure 1.14

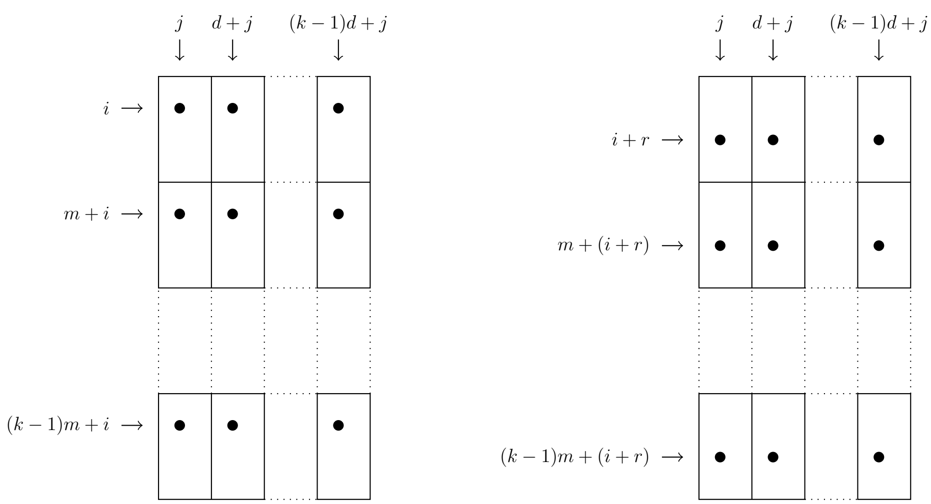
Lemma 1.11. *Let $1 \leq i \leq m$, $1 \leq j \leq d$, $r \leq m - i$ and $q \leq d - j$. Also let, $B_{i,j}$ be a big block. Then,*

(i) Shifting all the k^2 trees of $B_{i,j}$ by r rows downward results in the big block $B_{i+r,j}$.

(ii) Shifting all the k^2 trees of $B_{i,j}$ by q columns to the right results in the big block $B_{i,j+q}$.

(iii) Shifting all the k^2 trees of $B_{i,j}$ by r rows downward and q columns to the right results in the big block $B_{i+r,j+q}$.

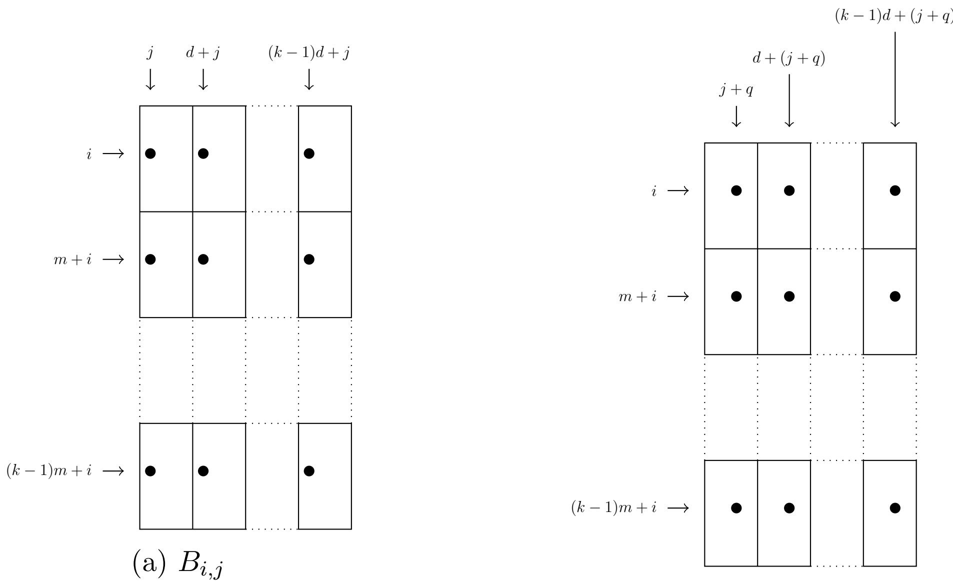
Proof. (i) Shifting all the trees of $B_{i,j}$ (Figure 1.15(a)) by r rows downward would change only row locations of the trees. Locations of the trees after shifting would be as in the Figure 1.15(b). Obviously, the resulting block is big block consisting of k^2 copies of $b_{i+r,j}$. Hence, according to the definition of a big block, the resulting big block in the Figure 1.15(b) is $B_{i+r,j}$.



(a) $B_{i,j}$ (b) $B_{i+r,j}$

Figure 1.15

(ii) Similarly, shifting all the trees of $B_{i,j}$ (Figure 1.16(a)) by q columns to the right would change only column locations of the trees. Locations of the trees after shifting would be as in the Figure 1.16(b). It is easy to see that, the resulting block is big block consisting of k^2 copies of $b_{i,j+q}$. Hence, according to the definition of a big block, the resulting big block in the Figure 1.16(b) is $B_{i,j+q}$.



(a) $B_{i,j}$

(b) $B_{i,j+q}$

Figure 1.16

(iii) It is obvious that shifting all the trees of $B_{i,j}$ by r rows downward results in the big block $B_{i+r,j}$ and shifting all the trees of $B_{i+r,j}$ by q columns to the right produces $B_{i+r,j+q}$. Therefore, shifting all the trees of $B_{i,j}$ by r rows downward and q columns to the right results in the big block $B_{i+r,j+q}$.

□

Let us now, demonstrate the following method, which is step by step procedure to construct $\mathrm{TP}(d, k, mdk)$:

Step 1: Construct the base block $b_{1,1}$.

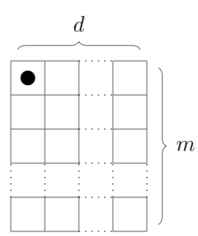


Figure 1.17: Base block $b_{1,1}$

Step 2: Construct the big block $B_{1,1}$ which consists of k^2 copies of $b_{1,1}$. This gives us the subgrid $S(1, mk, 1, dk)$ of $\mathrm{TP}(d, k, mdk)$.

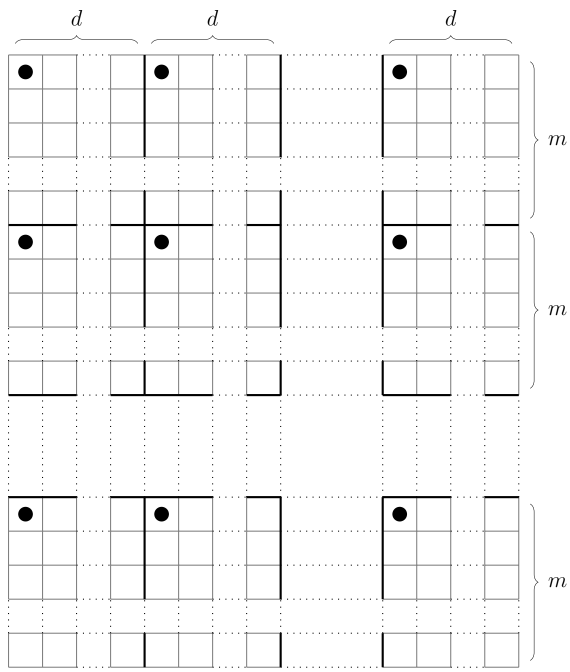


Figure 1.18: $B_{1,1}$

Step 3: Construct the subgrid $S(1, mk, 1, mdk)$ of $\mathrm{TP}(d, k, mdk)$, which is the block $B[mk, mdk]$ consisting of m big blocks $B_{i,1}$, where $1 \leq i \leq m$, as shown in the figure below. It is obvious that $B_{1,1}$ is the block obtained in Step 2 and according to the Lemma 1.11, the block $B_{i+1,1}$ is obtained by shifting k^2 trees of $B_{1,1}$ by i rows downward, for every i with $1 \leq i \leq m-1$. As a result, the block $B[mk, mdk]$ contains k trees in its all rows.

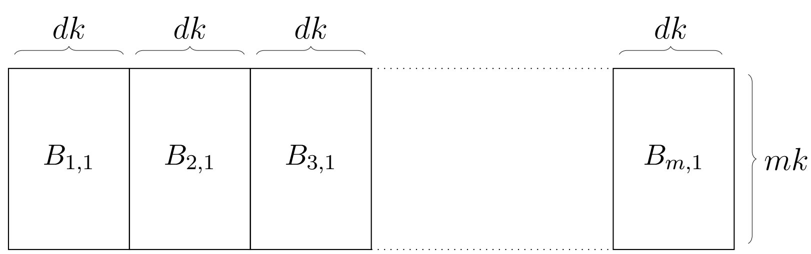


Figure 1.19: The subgrid $S(1, mk, 1, mdk)$ of $\mathrm{TP}(d, k, mdk)$

Step 4: For every p, where $1 \leq p \leq d-1$, construct the subgrid $S(pmk+1,(p+1)mk,1,mdk)$ of $TP(d,k,mdk)$, by shifting all the mk^2 trees in the subgrid $S(1,mk,1,mdk)$ by p columns to the right. Finally, stack all the subgrids on top of eachother. The result is TP(d,k,mdk).

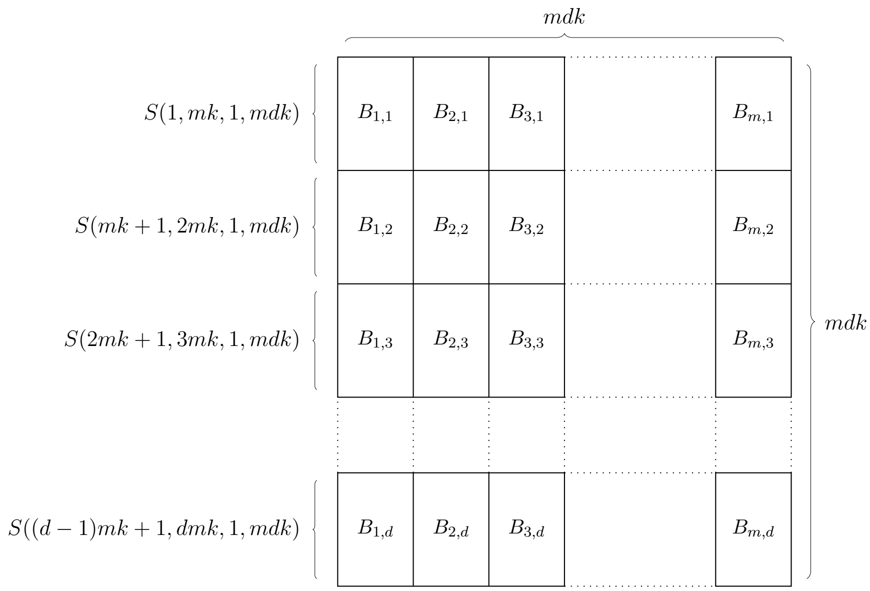


Figure 1.20: TP(d,k,mdk)

In order to see how the construction works, let's construct TP$(3,3,45)$ using the step-by-step process described above. For TP$(3,3,45)$, we obtain the values $d=3$, $k=3$ and $m=5$.

Step 1: Create the base block $b_{1,1}$. According to a definiton of base block, $b_{1,1}$ has 5 rows and 3 columns.

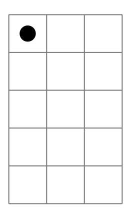


Figure 1.21: Base block $b_{1,1}$

Step 2: Create the big block $B_{1,1}$ which consists of 9 copies of $b_{1,1}$.

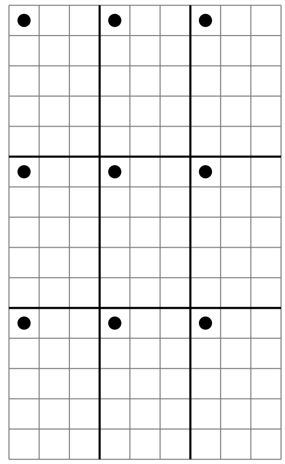


Figure 1.22: $B_{1,1}$

Step 3: Construct the subgrid $S(1, 15, 1, 45)$ of $\mathrm{TP}(3, 3, 45)$, which is a block $B[15, 45]$ consisting of 5 blocks of $B_{i,1}$, where $1 \leq i \leq 5$. It is easy to see that, the block $B_{i+1,1}$ is the same as the block $B_{1,1}$ except the 9 trees are shifted i rows downward, for every integer i with $1 \leq i \leq 4$.

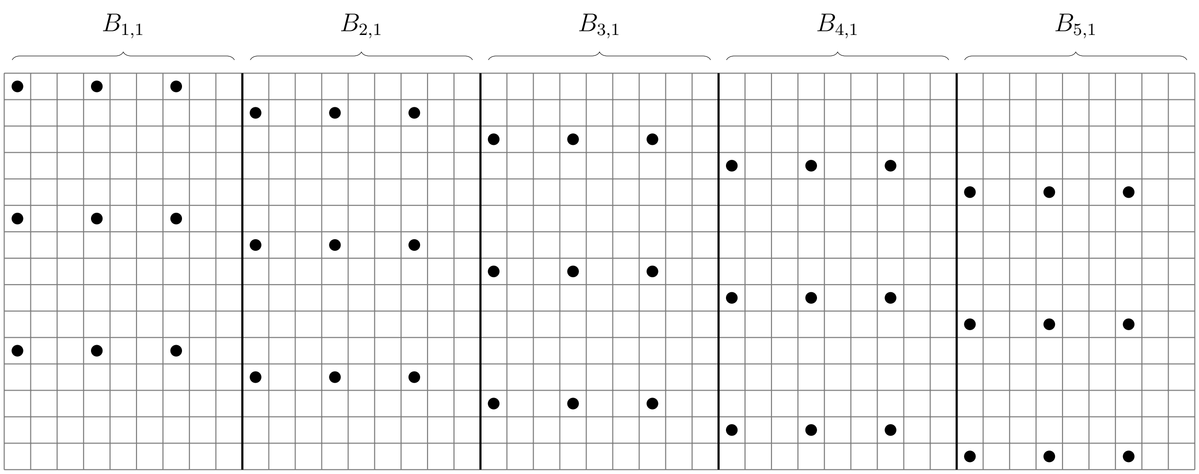


Figure 1.23: $S(1, 15, 1, 45)$

Step 4: Construct the subgrid $S(16, 30, 1, 45)$, by shifting all the 45 trees in the subgrid $S(1, 15, 1, 45)$ by one column to the right and $S(31, 45, 1, 45)$, by shifting all the 45 trees in the subgrid $S(1, 15, 1, 45)$ by two columns to the right. Finally, stack all the subgrids on top of eachother. The result is $\mathrm{TP}(3, 3, 45)$.

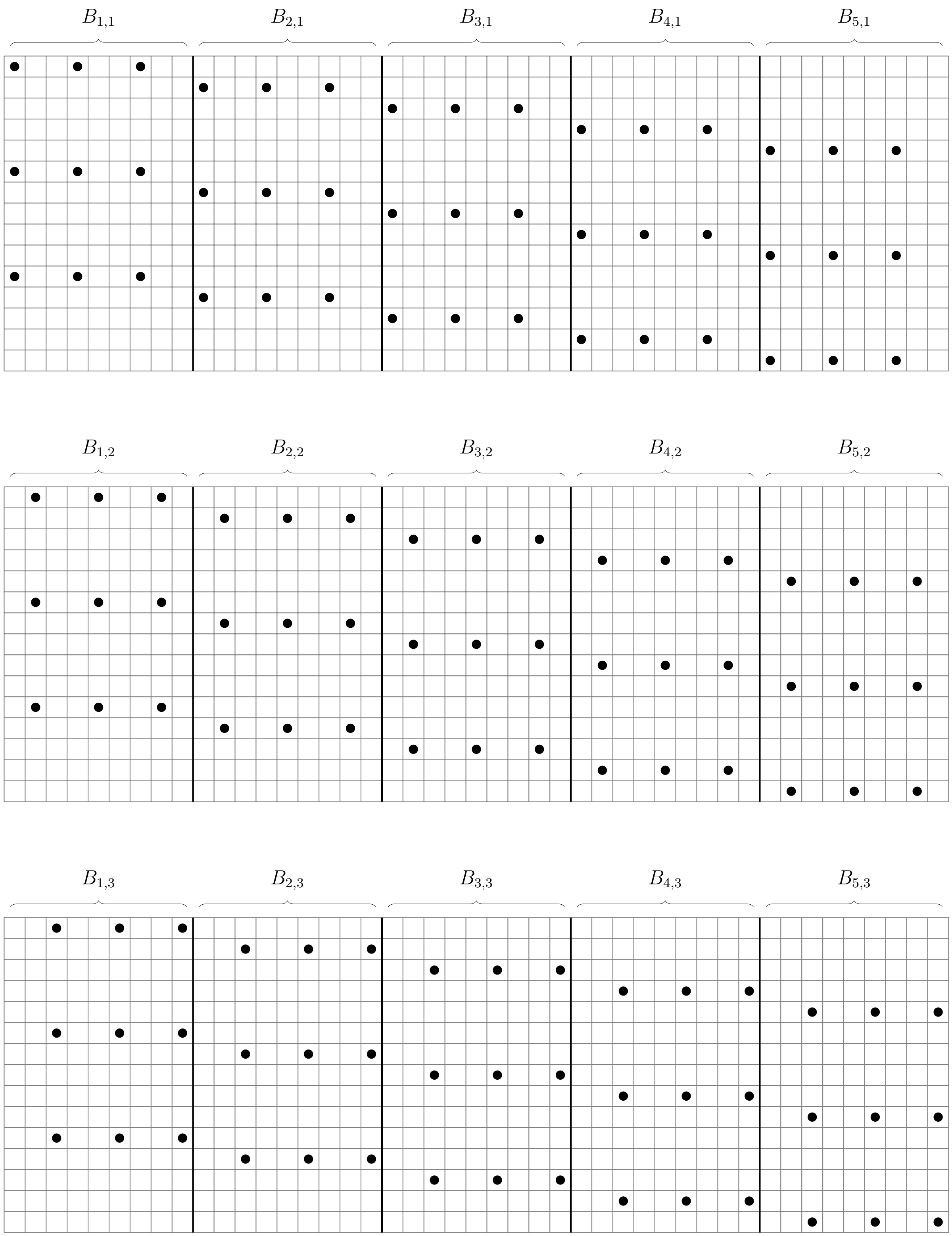


Figure 1.24: $S(1,15,1,45)$, $S(16,30,1,45)$, $S(31,45,1,45)$

Finally, stack all three subgrids in the Figure 1.24 on top of eachother. Finally, the result is TP$(3, 3, 45)$.

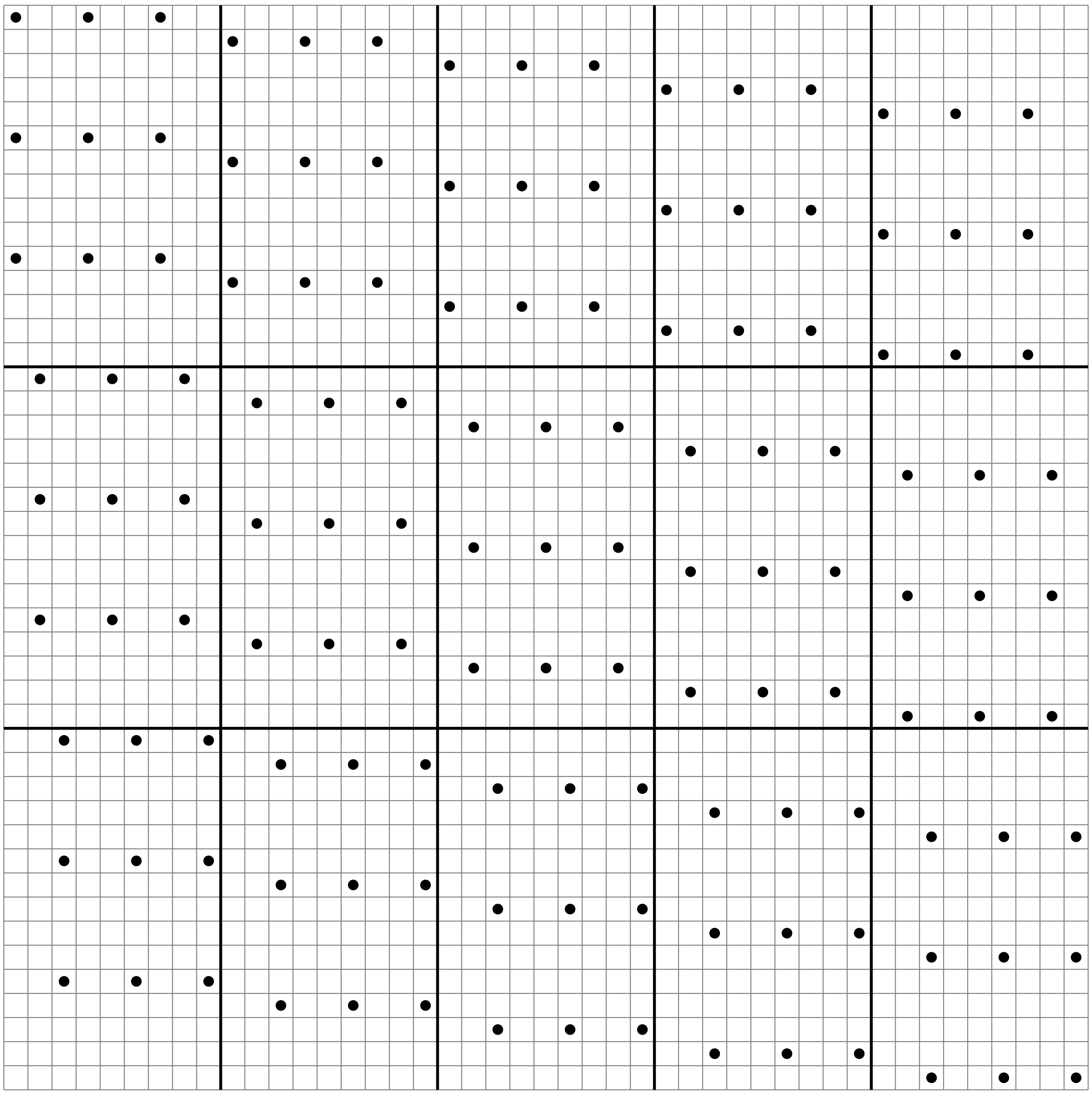


Figure 1.25: TP$(3, 3, 45)$

Chapter 2

Existence of TP$(d, k, mdk + i)$ and Its Construction

In this final chapter, we introduce our main result for the sufficient condition of the existence of TP(d, k, n). In order to make the proof of the theorem easy to follow, we start by stating lemmas which will be very useful. In this chapter we assume $k \geq 1$, $n > d \geq 2$ and $m > d$.

Before stating lemmas, we need to first introduce few definitions.

Definition 2.1. *A column block is a block $B[mdk, dk]$, which is the vertical stack of the big blocks $B_{i,i}$ of $TP(d, k, mdk)$, where $1 \leq i \leq d$, and it is denoted by B_c.*

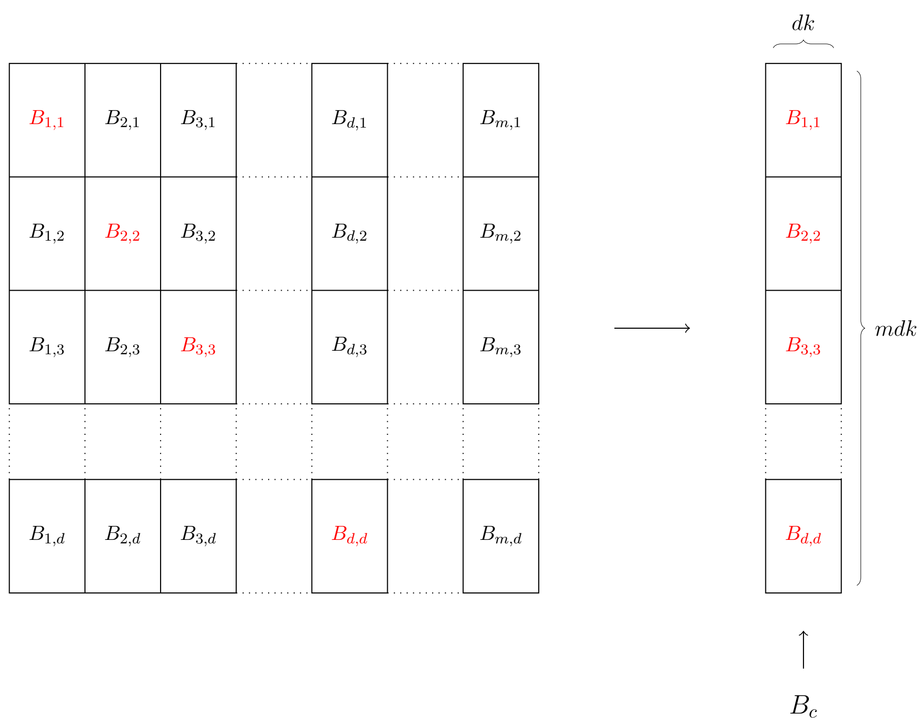


Figure 2.1: B_c obtained from TP(d, k, mdk)

Lemma 2.2. *Let B_c be a column block. Then,*

(i) There are k trees in each column of B_c.

(ii) There is at most one tree in any $B[d, d]$ of B_c.

Proof. (i) We know from the previous chapter that every big block contains k^2 trees which are located in k columns of the block. We also know that, column locations of k^2 trees in $B_{1,1}$ are 1, $d+1$,...,$(k-1)d+1$ (See Figure 1.14(b)). According to the Lemma 1.11, for every $B_{i+1,i+1}$, where $1 \leq i \leq d-1$, column locations of k^2 trees are i columns to the right of the column locations of the trees of $B_{1,1}$. Hence, k^2 trees in each of the d blocks $B_{1,1}$, $B_{2,2}$,...,$B_{d,d}$ are located in distinct k columns. In other words, there are k trees in each dk columns of the B_c.

(ii) Since $B_{i,i}$ is the subgrid of $\mathrm{TP}(d, k, mdk)$ and there is at most one tree in any $B[d, d]$ of $\mathrm{TP}(d, k, mdk)$, there is at most one tree in every $B[d, d]$ of $B_{i,i}$. Therefore, we only need to show that the bottom k trees of $B_{i-1,i-1}$ is at a row distance of at least d from the top k trees of $B_{i,i}$. Let's take a look at two consecutive blocks $B_{i-1,i-1}$ and $B_{i,i}$ of B_c and determine the distance between the row locations of the bottom k trees of $B_{i-1,i-1}$ and top k trees of $B_{i,i}$ as in Figure 2.2. According to the Figure 1.14(b), row locations of the bottom k trees of $B_{i-1,i-1}$ is $(k-1)m + (i-1)$ and top k trees of $B_{i,i}$ is i. The distance between the row locations would be $r_1 + r_2 = mk - [(k-1)m + (i-1)] + i = m + 1 > d$, since $m > d$ in $\mathrm{TP}(d, k, mdk)$.

□

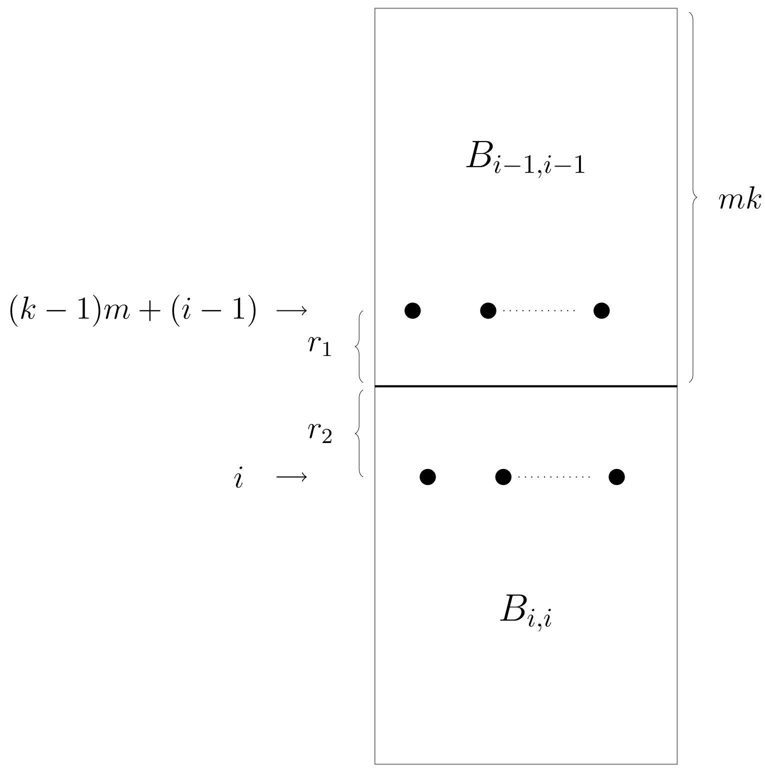


Figure 2.2

Definition 2.3. *A row block is a block $B[mk, ddk]$, which is the horizontal stack of the big blocks $B_{i,i}$ of $TP(d, k, mdk)$, where $1 \leq i \leq d$, and it is denoted by B_r.*

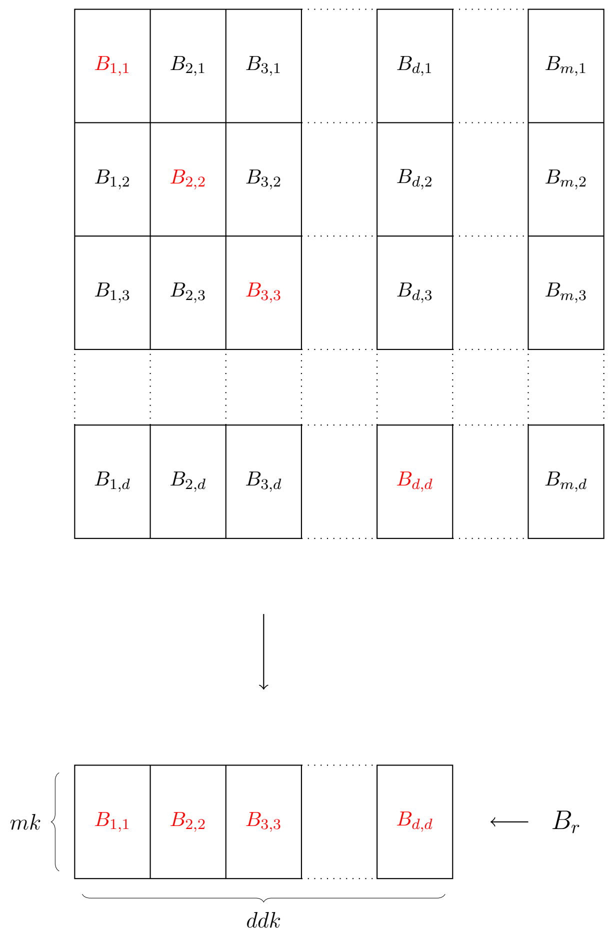


Figure 2.3: B_r obtained from $\mathrm{TP}(d, k, mdk)$

Let us look at the row locations of the trees in B_r. It is easy to see in the Figure 2.4 that trees are located between $(t-1)m+1$ and $(t-1)m+d$, where $1 \le t \le k$. In other words, dk rows (out of mk rows) contain trees and remaining $(m-d)k$ rows don't (see shaded region in Figure 2.5).

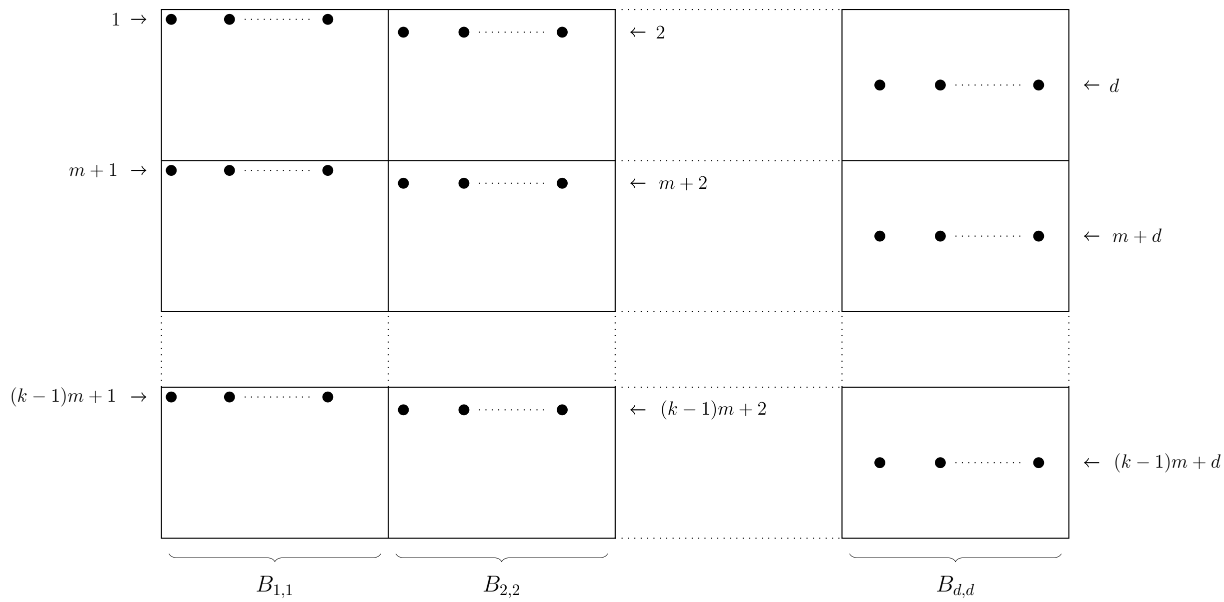


Figure 2.4: Row locations of the trees of the B_r

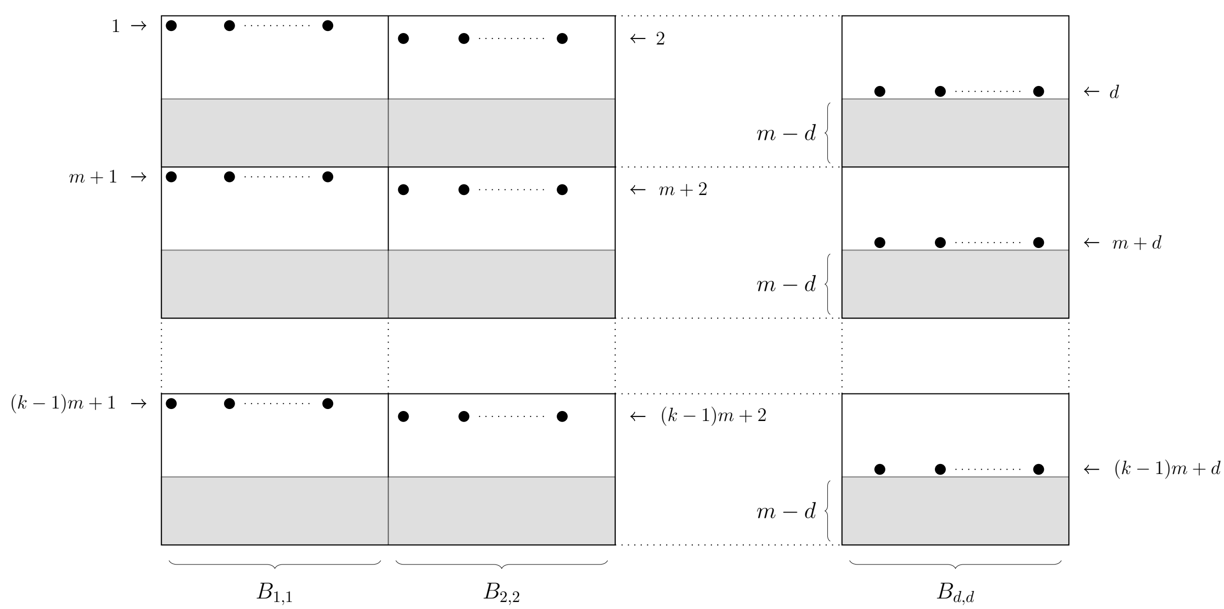


Figure 2.5: Empty (shaded) rows in B_r

Definition 2.4. *A reduced row block is a block $B[dk, ddk]$, which is obtained by dropping $(m-d)k$ empty rows from B_r and we denote it by $\hat{B}_r$.*

Note that, since B_r contains dk^2 trees, $\hat{B}_r$ also contains dk^2 trees.

Definition 2.5. *Reduced big block is a block $B[dk, dk]$ obtained by removing $(m-d)k$ rows of $B_{i,i}$ located between $(t-1)m+d+1$ and tm, where $1 \leq t \leq k$, $1 \leq i \leq d$ and we denote it by $\hat{B}_{i,i}$.*

It is obvious that $\hat{B}_r$ is a horizontal stack of the reduced big blocks $\hat{B}_{1,1}$, $\hat{B}_{2,2}$,...,$\hat{B}_{d,d}$ as in the Firgure 2.6.

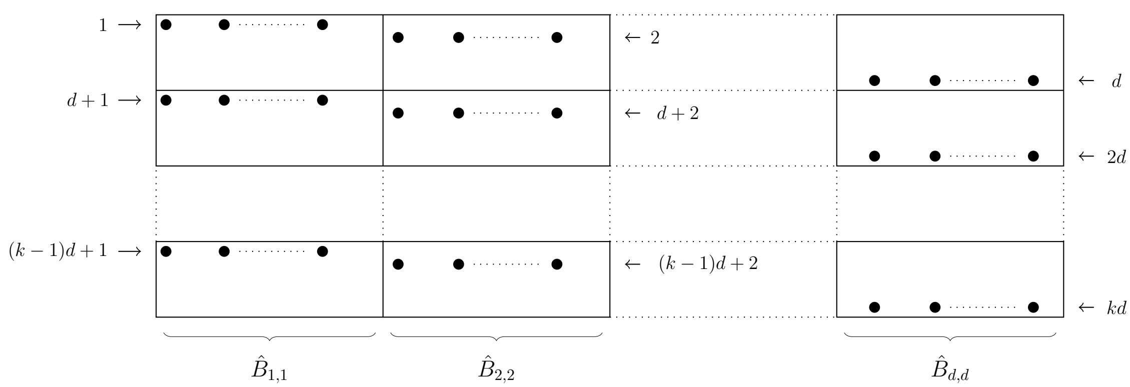


Figure 2.6: $\hat{B}_r$

Lemma 2.6. *Let $\hat{B}_r$ be a reduced row block $B[dk, ddk]$ obtained by removing empty rows of row block B_r. Then,*

(i) There are k trees in every row of $\hat{B}_r$.

(ii) There is at most one tree in any $B[d, d]$ of $\hat{B}_r$.

Proof. We will prove this lemma in a similar way to the proof we used for Lemma 2.2.

(i) As we mentioned earlier, a big block $B_{i,i}$ contains k^2 trees which are located in k rows of this block. Since $\hat{B}_{i,i}$ is obtained by removing the rows of $B_{i,i}$ and the removed rows contain no tree, $\hat{B}_{i,i}$ also contains k^2 trees which are located in k rows.

We also know that, the row locations of k^2 trees in $\hat{B}_{1,1}$ are 1, $d+1$,...,$(k-1)d+1$ (See Figure 2.6). According to the Lemma 1.11, for every $B_{i+1,i+1}$, where $1 \leq i \leq d-1$, row locations of k^2 trees are i rows below the row locations of the trees of $B_{1,1}$. Obviously, this is also correct for $\hat{B}_{i+1,i+1}$, where $1 \leq i \leq d-1$, as can be seen in the Figure 2.6. Therefore, k^2 trees in each of the d reduced big blocks $\hat{B}_{1,1}$, $\hat{B}_{2,2}$,...,$\hat{B}_{d,d}$ are located in the distinct k rows, i.e., there are k trees in each dk rows of the $\hat{B}_r$.

(ii) We first start to prove that there is at most one tree in any $B[d,d]$ within $\hat{B}_{i,i}$, where $1 \leq i \leq d$. We already know that there is at most one tree in a $B_{i,i}$, since it is a subgrid of $TP(d,k,mdk)$. Since the column locations of k^2 trees of $\hat{B}_{i,i}$ are the same as the column locations of $B_{i,i}$, we need to only check whether row distance between the trees in $\hat{B}_{i,i}$ is at least d. According to the Figure 1.14(b) the row distance between any two trees in $B_{i,i}$ is at least m. Since $\hat{B}_{i,i}$ is obtained by removing $m-d$ rows of $B_{i,i}$, the row distance between any two trees in the $\hat{B}_{i,i}$ is at least $m-(m-d)=d$.

Next, let's take a look at two consecutive blocks $\hat{B}_{i-1,i-1}$ and $\hat{B}_{i,i}$ of $\hat{B}_r$ and determine the distance between the column locations of the rightmost k trees of $\hat{B}_{i-1,i-1}$ and leftmost k trees of $\hat{B}_{i,i}$ as in Figure 2.7. According to the Figure 1.14(b), row locations of the rightmost k trees of $\hat{B}_{i-1,i-1}$ is $(k-1)d+(i-1)$ and leftmost k trees of $\hat{B}_{i,i}$ is i. The distance between the column locations would be $r_1+r_2 = dk-[(k-1)d+(i-1)]+i = d+1 > d$.

$\square$

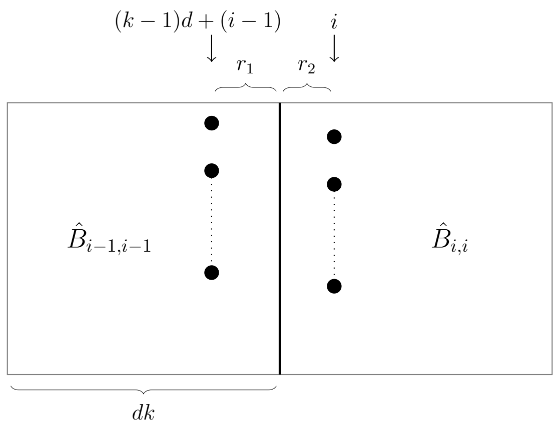


Figure 2.7

2.1 The Main Result

B_c and $\hat{B}_r$ will be used to state two more lemmas, which will play an important role to prove our main result.

Lemma 2.7. *Suppose H is a block $B[mdk, 2dk]$ which is the horizontal stack of the subgrid $S(1, mdk, (m-1)dk+1, mdk)$ of $TP(d, k, mdk)$ and column block B_c. Then, there is at most 1 tree in any $B[d, d]$ of the H.*

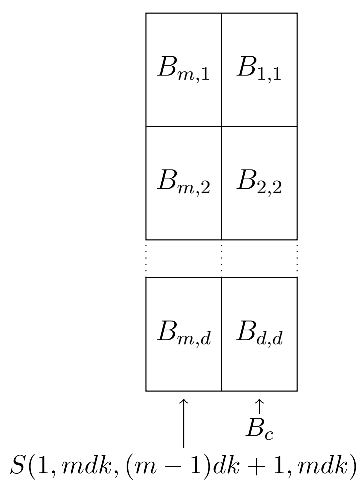


Figure 2.8: The block H

Proof. We already know that there is at most 1 tree in any $B[d, d]$ of $S(1, mdk, (m-1)dk+1, mdk)$, since it is true for $TP(d, k, mdk)$. The statement is true also for B_c due to the Lemma 2.2. Therefore, it is enough to show that the (row or column) distances between the particular trees, in the subgrid of H(see Figure 2.9), are at least d. Note that, as we defined earlier $B_{p,q}$ is big block with mk rows and dk columns.

1. Column distance between z and x is $dk - [(k-1)d + i] + i = d$.

2. Column distance between z and y is $dk - [(k-1)d + i] + i + 1 = d + 1$.

3. Column distance between w and y is $dk - [(k-1)d + i + 1] + i + 1 = d$.

4. Row distance between w and x is $mk - [(k-1)m + i] + m = 2m - i$ Since $1 \leq i \leq d-1$, it is not hard to find that $2m - i \geq 2m - d + 1 > m + 1 > d$.

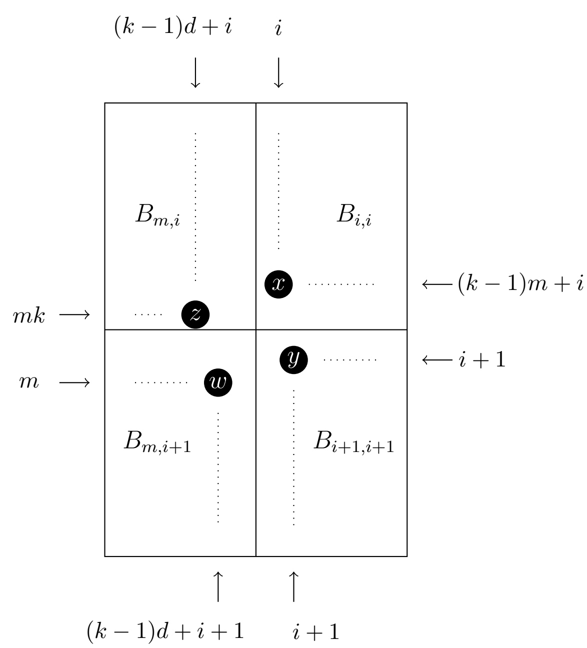


Figure 2.9: Subgrid of the block H, where $1 \leq i \leq d-1$

$\square$

Next, we state the following lemma which is similar to the one above. Hence its proof will also be similar.

Lemma 2.8. *Suppose V is a block $B[mk+dk, ddk]$ which is the vertical stack of the subgrid $S((d-1)mk+1, mdk, 1, ddk)$ of $TP(d,k,mdk)$ and reduced row block $\hat{B}_r$. Then, there is at most 1 tree in any $B[d,d]$ of the V.*

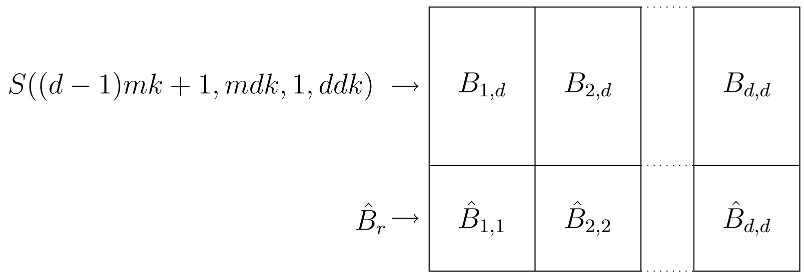


Figure 2.10: The block V

Proof. There is at most 1 tree in any $B[d,d]$ of $S((d-1)mk+1, mdk, 1, ddk)$, since it is true for $TP(d,k,mdk)$. The statement is true also for $\hat{B}_r$ due to the Lemma 2.6. Hence, it is enough to prove that the (row or column) distances between the particular trees, in the subgrid of V(see Figure 2.11), are at least d. Note that, $B_{i,d}$ is big block with mk rows and dk columns and $\hat{B}_{i,i}$ is reduced big block with dk rows and dk columns.

1. Row distance between z and x is $mk - [(k-1)m+i]+i = m > d$.

2. Row distance between z and y is $mk - [(k-1)m+i]+i+1 = m+1 > d$.

3. Row distance between w and y is $mk - [(k-1)m+i+1]+i+1 = m > d$.

4. Column distance between w and x is $dk-[(k-1)d+i]+d = 2d-i$ Since $1 \leq i \leq d-1$, it is easy to see that $2d-i \geq d+1$.

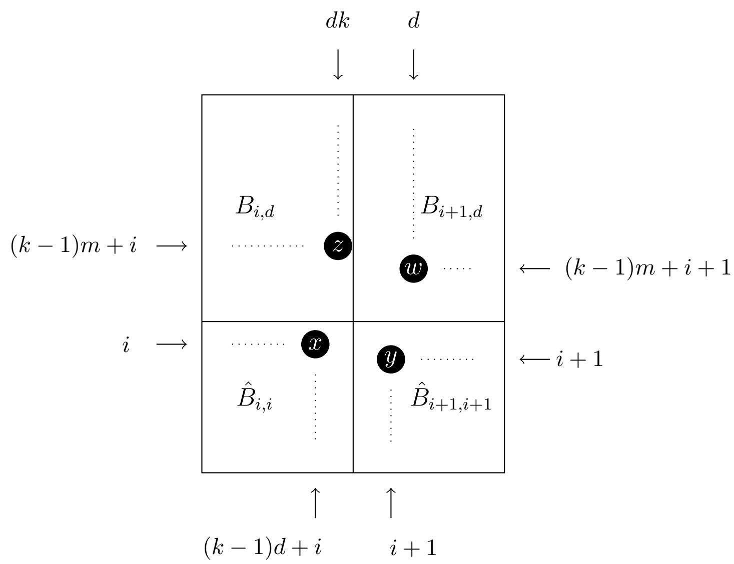


Figure 2.11: Subgrid of the block V, where $1 \le i \le d-1$

$\square$

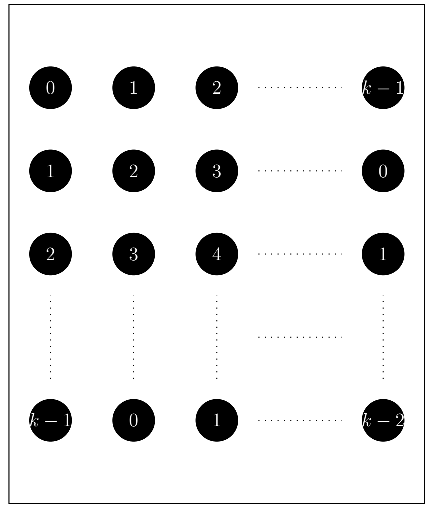
Before we jump to our main result, there are two more definitions left.

Consider a big block $B_{i,i}$, where $1 \le i \le d$. As we know, there are k trees in k rows and k columns of $B_{i,i}$, in total, there are k^2 trees in $B_{i,i}$.

Let's discuss a method to remove trees from $B_{i,i}$ so that there are t fewer trees (i.e., $k-t$ trees) in those particular k rows and k columns, where $1 \le t < k$. We propose the following method. Let each tree in $B_{i,i}$ correspond to a number in the addition table of $\mathbb{Z}_k$. As a result, k^2 trees of $B_{i,i}$ would be numbered from 0 to $k-1$, where each number would appear k times, as in the Figure 2.12.

$+$	0	1	2	$\dots$	$k-1$
0	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\boldsymbol{k-1}$
1	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\dots$	$\mathbf{0}$
2	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\dots$	$\mathbf{1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$\boldsymbol{k-1}$	$\mathbf{0}$	$\mathbf{1}$	$\dots$	$\boldsymbol{k-2}$

(a) Addition table of $\mathbb{Z}_k$



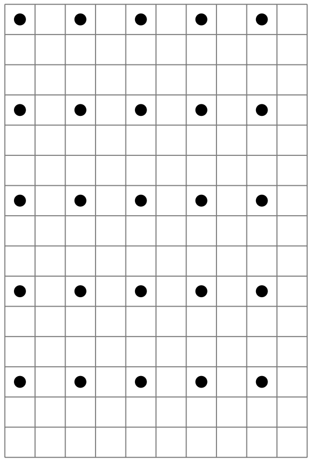
(b) k^2 trees of $B_{i,i}$

Figure 2.12

Definition 2.9. *Let $1 \leq i \leq d$ and $1 \leq t < k$. Given a big block $B_{i,i}$ of TP(d, k, mdk). In order to remove trees in $B_{i,i}$, to get $k - t$ trees in k columns and k rows, remove the trees corresponding to the numbers $0, 1, ..., t - 1$ in the addition table of $\mathbb{Z}_k$. We denote the resulting block by $B_{i,i}^{k-t}$.*

It is important to note that if $t = 0$, then $B_{i,i}^{k-t} = B_{i,i}^{k}$. That is, no tree is removed from $B_{i,i}$. Therefore, for $t = 0$, the block $B_{i,i}^{k-t}$ simply is $B_{i,i}$.

For the sake of clarity of the definition above, let's take an example. Consider the big block $B_{1,1}$ of TP$(2, 5, 30)$ and addition table of $\mathbb{Z}_5$ as in the figure below. $B_{1,1}$ contains 5 trees in 5 columns and 5 rows.



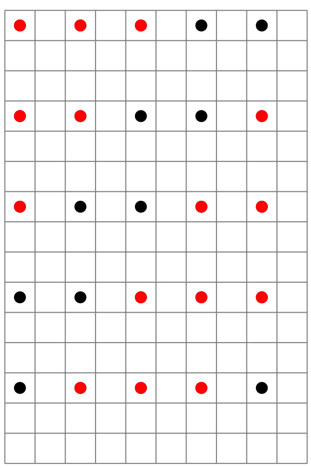
(a) $B_{1,1}$ of TP$(2, 5, 30)$

+	0	1	2	3	4
0	**0**	**1**	**2**	**3**	**4**
1	**1**	**2**	**3**	**4**	**0**
2	**2**	**3**	**4**	**0**	**1**
3	**3**	**4**	**0**	**1**	**2**
4	**4**	**0**	**1**	**2**	**3**

(b) Addition table of $\mathbb{Z}_5$

Figure 2.13

Suppose, we want to remove 3 trees from each row and column of $B_{1,1}$. Then, we must remove all the trees (red colored trees in the Figure 2.14) corresponding to the numbers 0,1 and 2.



(a) $B_{1,1}$ of TP$(2, 5, 30)$

+	0	1	2	3	4
0	**0**	**1**	**2**	**3**	**4**
1	**1**	**2**	**3**	**4**	**0**
2	**2**	**3**	**4**	**0**	**1**
3	**3**	**4**	**0**	**1**	**2**
4	**4**	**0**	**1**	**2**	**3**

(b) Addition table of $\mathbb{Z}_5$

Figure 2.14

As a result, we would get $B_{1,1}^2$, as in the figure below. There are 2 trees in 5 columns and 5 rows.

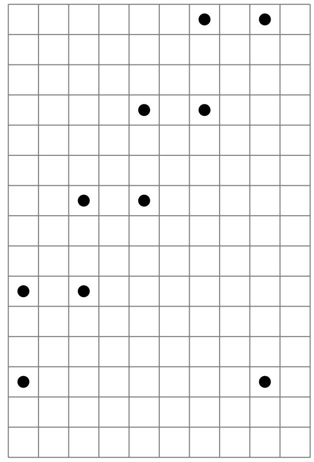


Figure 2.15: $B^2_{1,1}$

Definition 2.10. *Let $1 \leq i < dk$. Construct a new square block by the following step-by-step process:*

1. *First, construct $TP(d,k,mdk)$, column block B_c and $\hat{B}_r$.*

2. *Next, suppose $t = \lceil \frac{i}{d} \rceil$ and $p = i - d(t-1)$. Modify $TP(d,k,mdk)$ in the following way. For every q, where $1 \leq q \leq p$, replace big blocks $B_{q,q}$ of $TP(d,k,mdk)$ by $B^{k-t}_{q,q}$ and for every s, where $p+1 \leq s \leq d$ is true, replace big blocks $B_{s,s}$ of $TP(d,k,mdk)$ by $B^{k-t+1}_{s,s}$. Note that, s might not be satisfied if $d \mid i$. In that case, the only replacement occurs for the big blocks $B_{q,q}$.*

3. *Finally, attach i leftmost columns of column block B_c to the right, uppermost i rows of reduced row block $\hat{B}_r$ to the bottom and empty block $B[i,(m-d)dk+i]$ to the bottom right corner of the modified $TP(d,k,mdk)$ obtained in the previous step. The result is block $B[mdk+i, mdk+i]$. We denote it by $Z(i,d,k,mdk)$.*

Lemma 2.11. *Let $1 \leq i < dk$, $t = \lceil \frac{i}{d} \rceil$ and $p = i - d(t-1)$. Then,*

(i) the leftmost i columns of B_c contains kt trees (t trees in the k rows) from every big block $B_{q,q}$, where $1 \leq q \leq p$ and $k(t-1)$ trees ($t-1$ trees in the k rows) from every big block $B_{s,s}$, where $p+1 \leq s \leq d$ is true.

(ii) the uppermost i rows of $\hat{B}_r$ contains kt trees (t trees in the k columns) from every reduced big block $\hat{B}_{q,q}$, where $1 \leq q \leq p$ and $k(t-1)$ trees ($t-1$ trees in the k columns) from every reduced big block $\hat{B}_{s,s}$, where $p+1 \leq s \leq d$ is true.

Proof. (i) Obviously, the leftmost $d(t-1)$ columns of B_c contain k trees in every column. Since B_c consists of d big blocks, those $d(t-1)$ columns contain $k(t-1)$ trees ($t-1$ trees in the k rows) from every big block. Also, the remaining $p = i - d(t-1)$ columns contain k trees in every column. Hence, the leftmost i columns of B_c contains kt trees from every big block $B_{q,q}$, where $1 \leq q \leq p$ and $k(t-1)$ trees from every big block $B_{s,s}$, where $p+1 \leq s \leq d$ is true.

(ii) Similarly, the uppermost $d(t-1)$ rows of $\hat{B}_r$ contain k trees in every row. Since $\hat{B}_r$ consists of d reduced big blocks, those $d(t-1)$ rows contain $k(t-1)$ trees ($t-1$ trees in the k columns) from every reduced big block. Also, the remaining p rows contain k trees in every row. Therefore, the uppermost i rows of $\hat{B}_r$ contains kt trees from every reduced big block $\hat{B}_{q,q}$, where $1 \leq q \leq p$ and $k(t-1)$ trees from every big block $\hat{B}_{s,s}$, where $p+1 \leq s \leq d$ is true.

$\square$

Consider the example Z$(4, 3, 2, 24)$. That is, $i = 4$, $d = 3$, $k = 2$ and $m = 4$.

1. Construct TP$(3, 2, 24)$.

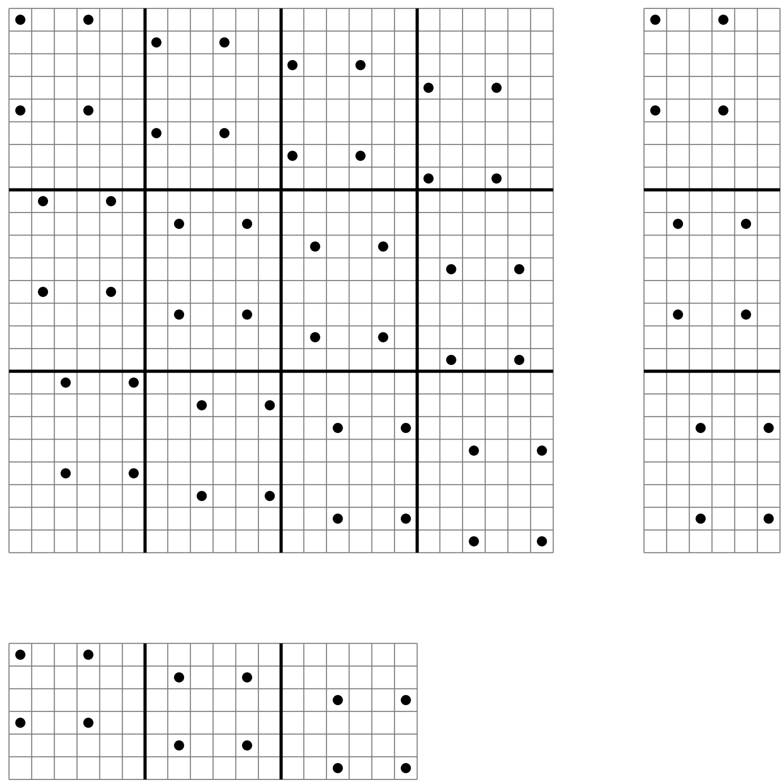


Figure 2.16: TP$(3, 2, 24)$(left), B_c(right) and $\hat{B}_r$(bottom)

2. $t = \lceil \frac{i}{d} \rfloor = \lceil \frac{4}{3} \rfloor = 2$ and $p = i - d(t-1)d = 4 - 3(2-1) = 1$, $k - t = 2 - 2 = 0$ and $k - t + 1 = 1$. We modify $B_{1,1}$, $B_{2,2}$ and $B_{3,3}$ of TP$(3, 2, 24)$ by $B^0_{1,1}$, $B^1_{2,2}$ and $B^1_{3,3}$, respectively.

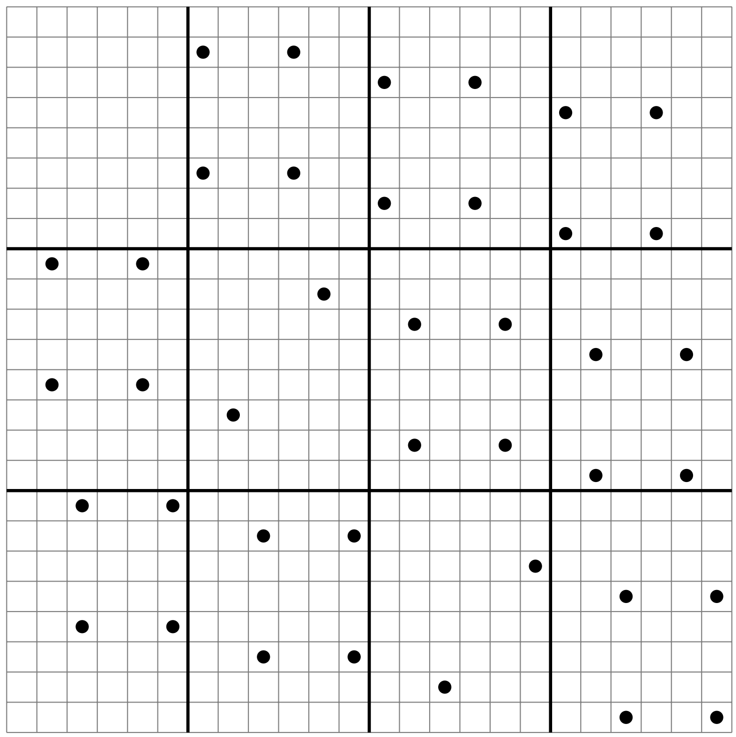


Figure 2.17: Modified TP$(3, 2, 24)$

3. Attach 4 leftmost columns of column block B_c to the right, uppermost 4 rows of reduced row block $\hat{B}_r$ to the bottom and empty block $B[4, 10]$ to the bottom right corner of the modified TP$(3, 2, 24)$ (see Figure 2.18). The result is Z$(4, 3, 2, 24)$ (see Figure 2.19).

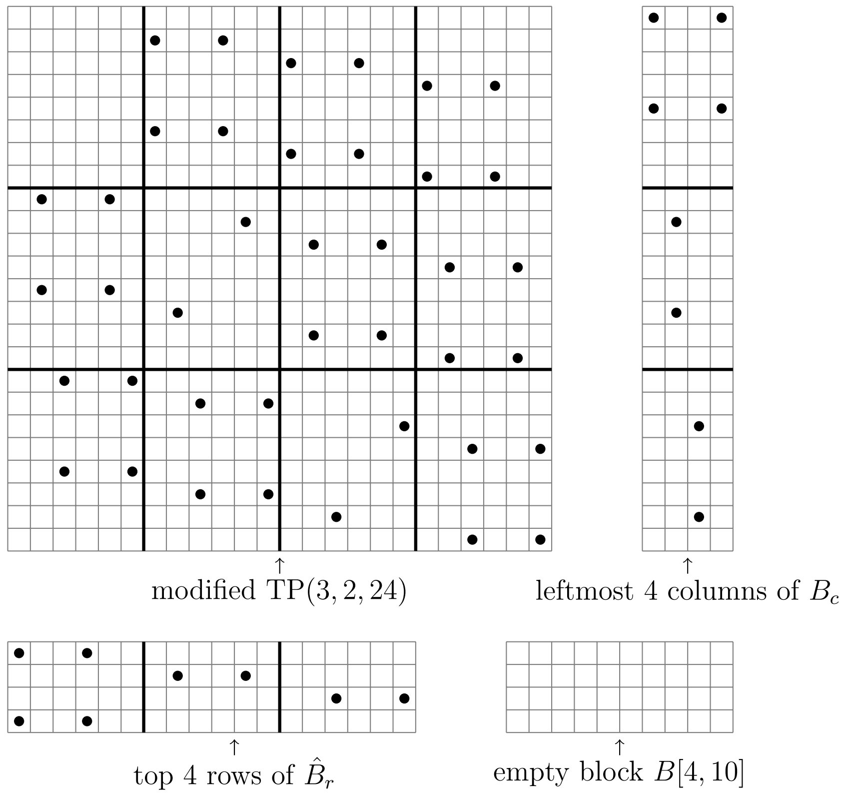


Figure 2.18

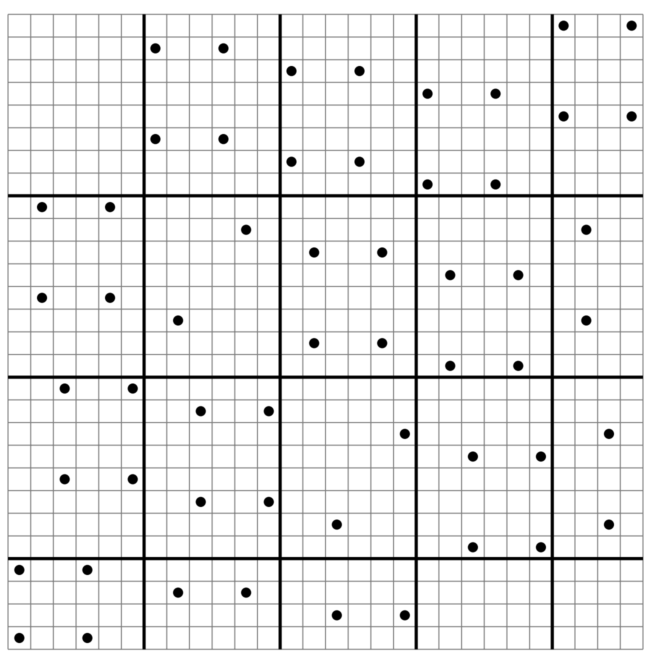


Figure 2.19: $\mathrm{Z}(4, 3, 2, 24)$

Finally, after providing all the preliminary definitions and lemmas, we state our main result as theorem below.

Theorem 2.12. *Let $1 \leq i < dk$. Then block $Z(i, d, k, mdk)$ is $TP(d, k, mdk + i)$. In other words, there are k trees in every row and column of Z and there is at most one tree in any $B[d, d]$.*

Proof. First, we start by proving there is at most one tree in any $B[d, d]$ of $\mathrm{Z}(i, d, k, mdk)$, i.e., row or column distance between the trees is at least d. We know that (row or column) distance between any two trees in the $\mathrm{TP}(d, k, mdk)$, B_c and $\hat{B}_r$ is at least d. It is also true between the rightmost trees of $\mathrm{TP}(d, k, mdk)$ and the leftmost trees of B_c and between the bottommost trees of $\mathrm{TP}(d, k, mdk)$ and the uppermost trees of $\hat{B}_r$ due to the lemmas 2.7 and 2.8. Also, since there is at most 1 tree in any $B[d, d]$ of $\mathrm{TP}(d, k, mdk)$, it would also be true for modified $\mathrm{TP}(d, k, mdk)$ obtained in second step of the Definition 2.10. Therefore, row or column distance between any two trees in $\mathrm{Z}(i, d, k, mdk)$ is at least d.

Now, we prove there are k trees in every row and column of $\mathrm{Z}(i, d, k, mdk)$. First, we prove there are k trees in every column of $\mathrm{Z}(i, d, k, mdk)$. Let us examine the columns of three disjoint subgrids that composes $\mathrm{Z}(i, d, k, mdk)$.

(a) $S(1, mdk + i, mdk + 1, mdk + i)$ of $\mathrm{Z}(i, d, k, mdk)$.

Obviously, $S(1, mdk + i, mdk + 1, mdk + i)$ is composed of $S(1, mdk, mdk + 1, mdk + i)$, which is the leftmost i columns of B_c and $S(mdk + 1, mdk + i, mdk + 1, mdk + i)$, which is empty block $B[i, i]$. We already know that, there are k trees in every column of B_c, i.e., there are k trees in the leftmost i columns of B_c. Hence, there are k trees in every column of $S(1, mdk + i, mdk + 1, mdk + i)$.

(b) $S(1, mdk + i, ddk + 1, mdk)$ of $\mathrm{Z}(i, d, k, mdk)$.

The subgrid $S(1, mdk + i, ddk + 1, mdk)$ is composition of $S(1, mdk, ddk + 1, mdk)$ of $\mathrm{TP}(d, k, mdk)$ and $S(mdk+1, mdk+i, ddk+1, mdk)$, which is empty block $B[i, (m-d)dk]$.

From the Definition 2.10, one can easily see that $S(1, mdk, ddk+1, mdk)$ of $\mathrm{TP}(d, k, mdk)$ stays untouched, i.e., no tree is removed in this particular subgrid. Therefore, there are k trees in the subgrid $S(1, mdk + i, ddk + 1, mdk)$.

(c) $S(1, mdk + i, 1, ddk)$ of $\mathrm{Z}(i, d, k, mdk)$.

The subgrid $S(1, mdk + i, 1, ddk)$ is composed of the subgrids $S(1, mdk + i, (j-1)dk + 1, jdk)$, where $1 \leq j \leq d$. Also, each subgrid $S(1, mdk+i, (j-1)dk+1, jdk)$ consists of the subgrid $S(1, mdk, (j-1)dk+1, jdk)$, which is the subgrid $S(1, mdk, (j-1)dk+1, jdk)$ of $\mathrm{TP}(d, k, mdk)$ except $B_{j,j}$ is replaced by $B_{j,j}^{k-t}$ and $S(mdk+1, mdk+i, (j-1)dk+1, jdk)$, which is the uppermost i rows of $\hat{B}_{j,j}$. Consider following cases:

(i) If $1 \leq j \leq p$, then $S(1, mdk, (j-1)dk + 1, jdk)$ contains k trees in every column except the columns of $B_{j,j}^{k-t}$, which contains $k - t$ trees in its particular columns. Also, according to the Lemma 2.11, the uppermost i rows of $\hat{B}_{j,j}$, contains t trees in its k particular columns, which are the same columns of $B_{j,j}^{k-t}$ that contains $k-t$ trees. Therefore, those particular k columns contain $k - t + t = k$ trees.

(ii) If $p + 1 \leq j \leq d$ is true, then $S(1, mdk, (j-1)dk + 1, jdk)$ contains k trees in every column except the columns of $B_{j,j}^{k-t+1}$, which contains $k - t + 1$ trees in its particular columns. Also, according to the Lemma 2.11, the uppermost i rows of $\hat{B}_{j,j}$, contains $t-1$ trees in its k particular columns, which are the same columns of $B_{j,j}^{k-t+1}$ that contains $k-t+1$ trees. Therefore, those particular k columns contain $k - t + 1 + t - 1 = k$ trees.

Hence there are k trees in every column of $\mathrm{Z}(i, d, k, mdk)$.

Let's examine the rows of two disjoint subgrids that composes $\mathrm{Z}(i, d, k, mdk)$ to prove there are k trees in every row of $\mathrm{Z}(i, d, k, mdk)$.

(a) $S(1, mdk, 1, mdk + i)$ of $\mathrm{Z}(i, d, k, mdk)$.

The subgrid $S(1, mdk, 1, mdk + i)$ is composed of the subgrids $S((j-1)mk+1, jmk, 1, mdk+i)$, where $1 \leq j \leq d$. Also, each subgrid $S((j-1)mk + 1, jmk, 1, mdk + i)$ consists of the subgrid $S((j-1)mk+1, jmk, 1, mdk)$, which is the subgrid $S((j-1)mk+1, jmk, 1, mdk)$ of $\mathrm{TP}(d, k, mdk)$ except $B_{j,j}$ is replaced by $B_{j,j}^{k-t}$ and $S((j-1)mk + 1, jmk, mdk + 1, mdk + i)$, which is the leftmost i columns of $B_{j,j}$. Consider following cases:

(i) If $1 \leq j \leq p$, then $S((j-1)mk + 1, jmk, 1, mdk)$ contains k trees in every row except the rows of $B_{j,j}^{k-t}$, which contains $k - t$ trees in its particular rows. Also, according to the Lemma 2.11, the leftmost i columns of $B_{j,j}$, contains t trees in its k particular rows, which are the same rows of $B_{j,j}^{k-t}$ that contains $k - t$ trees. Therefore, those particular k rows contain $k - t + t = k$ trees.

(ii) If $p + 1 \leq j \leq d$ is true, then $S((j-1)mk + 1, jmk, 1, mdk)$ contains k trees in every row except the rows of $B_{j,j}^{k-t+1}$, which contains $k-t+1$ trees in its particular rows. Also, according to the Lemma 2.11, the leftmost i columns of $B_{j,j}$, contains $t-1$ trees in its k particular rows, which are the same rows of $B_{j,j}^{k-t+1}$ that contains $k - t + 1$ trees. Therefore, those particular k rows contain $k - t + 1 + t - 1 = k$ trees.

(b) $S(mdk + 1, mdk + i, 1, mdk + i)$ of $\mathrm{Z}(i, d, k, mdk)$.

It is easy to see that, $S(mdk+1, mdk+i, 1, mdk+i)$ is composed of $S(mdk+1, mdk + i, 1, ddk)$, which is the uppermost i rows of $\hat{B}_r$ and $S(mdk+1, mdk+i, ddk+1, mdk+i)$, which is the empty block $B[i, (m-d)dk + i]$. We already know that, there are k trees in every row of $\hat{B}_r$, i.e., there are k trees in the uppermost i rows of $\hat{B}_r$. Hence, there are k trees in every row of $S(mdk + 1, mdk + i, 1, mdk + i)$.

Hence there are k trees in every row of $\mathrm{Z}(i, d, k, mdk)$.

Since there are k trees in every row and column and there is at most 1 tree in any $B[d, d]$ of $\mathrm{Z}(i, d, k, mdk)$, it is true that $\mathrm{Z}(i, d, k, mdk)$ is $\mathrm{TP}(d, k, mdk + i)$. □

Corollary 2.13. *$TP(d,k,mdk+i)$ exists for any i, where $1 \leq i < dk$.*

Proof. In the Definition 2.10, we see that $\mathrm{Z}(i,d,k,mdk)$ can be constructed for any i, where $1 \leq i < dk$. Also, in the Theorem 2.12, it is shown that, $\mathrm{Z}(i,d,k,mdk)$ is $\mathrm{TP}(d,k,mdk+i)$. In other words, $\mathrm{TP}(d,k,mdk+i)$ exists because $\mathrm{Z}(i,d,k,mdk)$ exists. $\square$

2.2 Conclusion

Let $k,d,n \in \mathbb{Z}^+$, where $1 \leq k$, $2 \leq d < n$. Then necessary condition for the existence of $\mathrm{TP}(d,k,n)$ is $\left\lfloor \frac{n}{d^2} \right\rfloor \geq k$.

On the other hand, the necessary condition is sufficient when $m > d$, $0 \leq i < dk$ and $n \in \{kd^2, mdk+i\}$. However, for the values of n, where $kd^2 < n < kd(d+1)$, the problem is open.

Bibliography

[1] J.A.Bondy and U.S.R.Murty, Graph Theory, Second Edition, Springer, 2008, Page 425-426