Dynamics of Chemotaxis Models in Heterogeneous Environments

by

Tahir Bachar Issa

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 03, 2019

Keywords: chemotaxis, persistence, coexistence, extinction, optimal rectangles, stability of entire solutions

Approved by

Wenxian Shen, Chair, Professor of Mathematics and Statistics
Ash Abebe, Professor of Mathematics and Statistics
Georg Hetzer, Professor of Mathematics and Statistics
Dmitry Glotov, Professor of Mathematics and Statistics

Abstract

Chemotaxis describes the oriented movements of biological cells or organisms in response to chemical gradients in their environments and is crucial for many aspects of behaviour such as the location of food sources, avoidance of predators and attracting mates, slime mold aggregation, tumor angiogenesis, and primitive streak formation. Chemotaxis is also crucial in macroscopic process such as population dynamics and gravitational collapse. In 1970, Keller and Segel introduced a celebrated mathematical model to describe chemotaxis. Since then a tremendous effort has been dedicated to understand the classical chemotaxis model and its various variants. But there are still a lot of open interesting problems in the understanding of chemotaxis models. In particular, to the best of our knowledge, there has been no study of chemotaxis models in heterogeneous environments. This dissertation aims to study the dynamics of chemotaxis models of both one and two species in bounded heterogeneous environments.

Regarding chemotaxis models of one species in heterogeneous environments, we first investigate and prove the local existence and uniqueness of classical solutions. Next under some natural conditions on the parameters, we prove the boundedness of classical solutions and the existence of positive entire solutions. Finally, under some further conditions on the parameters, we establish the uniqueness and stability of positive entire solutions. Our results on the existence, uniqueness and stability of positive entire solutions are new and original. Important new techniques have been established to prove those results.

Concerning chemotaxis models of two species in heterogeneous environments, we first find various conditions on the parameters which guarantee the global existence and boundedness of classical solutions. Next, we find further conditions on the parameters which establish the persistence of the two species. Furthermore, under the same set of conditions for the persistence of the two species, we prove the existence of coexistence states. We then prove the extinction phenomena in the sense that one of the species dies out asymptotically and the other reaches its carrying capacity as time goes to infinity. Finally, we study the asymptotic dynamics of two species competition systems with/without chemotaxis in heterogeneous media and find conditions on the parameters for the uniqueness and stability of positive coexistence states of such systems. The persistence in general two species chemotaxis systems is studied for the first time. Several important techniques are developed to study the persistence and coexistence of the two species chemotaxis systems. Many existing results on the persistence, coexistence, and extinction on two species competition systems without chemotaxis are recovered. The established results on the asymptotic dynamics of two species competition systems are new even for the two species competition systems without chemotaxis but with space dependent coefficients.

Acknowledgments

First and foremost, I would like to praise and thank the Almigty God for giving me the strength, guidance and patience to complete this dissertation. Next, I would like to sincerely express my deepest appreciation and thank to Professor Wenxian Shen for her excellent guidance into research, patience, attentiveness, care and generosity. This work would not have come to an end without her support and guidance. I would like also to sincerely thank Professor Ash Abebe, Professor Dmitry Glotov, Professor Georg Hetzer and Professor Paul Schmidt for their continuous support and guidance throughout my graduate studies here at Auburn University and for accepting to be on my PhD committee member. I would like also to thank the outside reader Professor Mark Dougherty for reading the draft, for his valuable comments and for his support and valuable advice throughout my stay here at Auburn. I would like also to thank Professor Geraldo De-souza and Professor Ulrich F. Albrecht for their valuable advice and support throughout my stay here at Auburn.

Last but not least, I would like to express my gratitude to all my parents for their support, love and prayers. Specially to my beloved mom Cherifa Ahmat, to my dad Mahamat Tahir Bachar, to my late brother Youssouf Mahamat Tahir Bachar, to my late uncle Mahamat Sougui Ahmat and to all brothers and sisters. I would like also to express my gratitude to my uncle Professor Ali Souleyman Dabye for his guide, advices and support throughout all my studies. Finally, to all my family members and friends, I say thank you for being there for me all the time and God bless all of you.

Table of Contents

Chapter 1

Introduction

Chemotaxis, the oriented movements of mobile species toward the increasing or decreasing concentration of a signaling chemical substance, has a crucial role in a wide range of biological phenomena such as immune system response, embryo development, tumor growth, population dynamics, gravitational collapse, etc. (see [28, 41, 50]). At the beginning of the 1970s, Keller and Segel proposed a celebrated mathematical model, referred to as the classical Keller-Segel model, to describe the aggregation process of Dictyostelium discoideum, a soil-living amoebae [33, 34]. It is well known that in homogeneous environments, finite-time blow-up of some classical solutions may occur in the classical Keller-Segel model and its variants in space dimension $n \geq 2$ (see [9, 19, 32, 66] for a one species chemotaxis model and [4] two species chemotaxis models). However, it is also known that logistic sources of Lotka-Volterra type may preclude such blow-up phenomenon (see [30, 51, 59] for one species and [44, 60] for two species) and that, at least numerically, chemotaxis with logistic sources may exhibit quite a rich variety of colorful dynamical features, up to periodic and even chaotic solution behavior [38, 49]. For a broad survey on the progress of various chemotaxis models in homogeneous environments and a rich selection of references, we refer the reader to the survey papers [5, 24, 25]. To the best of our knowledge, there has been no study on the dynamics of chemotaxis models in heterogeneous environments previous to our study. In reality, the underlying environments of many biological systems are subject to various spatial and temporal variations. It is of both biological and mathematical importance to study chemotaxis models in heterogeneous environments.

This dissertation focuses on the study of the dynamics of the following chemotaxis models in bounded heterogeneous environments,

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)\int_\Omega u\Big), \quad x\in\Omega \\ \tau w_t = d_2\Delta w + ku - \lambda w, \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big), \quad x\in\Omega \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + v\Big(b_0(t,x) - b_1(t,x)u - b_2(t,x)v\Big), \quad x\in\Omega \\ \tau w_t = d_3\Delta w + ku + lv - \lambda w, \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary.

System (1.1) is referred to as one species chemotaxis model and arises in mathematical biology as a model for the spatio-temporal evolution of the population of a species which proliferates according to a Lotka-Volterra-type kinetics, and in which individuals are moreover able to move according to both random diffusion and chemotaxis toward or away a signal produced by themselves. In this setting, $u = u(x,t)$ represents the population density of the species, $w = w(x,t)$ denotes the concentration of the chemical, $\chi > 0$ describes the chemotaxis sensitivity, $d_i > 0, i = 1,2,$ describe the diffusion rate of u and w respectively, $\tau \geq 0$ describes the diffusion speed of the chemical substance, a_0 and a_1 describe respectively the intrinsic growth rate and the self limitation effect of the species u, $a_2\int_\Omega u$ describes the influence of the total mass of the species in the growth of the population. System (1.1) with $a_i \equiv 0$ ($i = 0,1,2$) reduces to the classical Keller-Segel model.

A quite rich dynamical features in system (1.1) with constant coefficients have been observed, including spatial pattern formation and spatio-temporal chaos, at least numerically (see [38, 49]). For example, it is proved that blow-up never happens in one dimension [47] and that chemotactic-cross diffusion has a very strong destabilizing action in space dimension $n \geq 2$ in the sense that finite-time blow-up of some classical solutions may occur (see [32, 66]). It is also known that logistic sources of Lotka-Volterra type preclude such blow-up phenomenon in certain sense (see [30, 59]) and that, at least numerically, chemotaxis may exhibit quite a rich variety of colorful dynamical features, up to periodic and even chaotic solution behavior [38, 49]. But there is little study of the dynamics of (1.1) with time and space dependent coefficients, including the case that $\tau = 0$.

One of the main objectives of this dissertation is to study the fundamental dynamical aspects in system (1.1) with $\tau = 0$ such as

- Global existence of nonnegative classical solutions.

- Existence and stability of positive entire solutions.

We obtained many important results on the dynamical aspects of system (1.1) in [30]. For example, we proved the following and others in [30] for (1.1) with $a_2(t,x) = 0$, $d_i = 1$, $\tau = 0$, and $k = l = 1$.

(i) Assume that $\inf_{t\in\mathbb{R}, x\in\bar{\Omega}} a_1(t,x) > \frac{\chi(n-2)}{n}$. Then for any $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, system (1.1) has a unique global classical solution $(u(x,t;t_0,u_0), w(x,t;t_0,u_0))$ which satisfies that $\lim_{t\to t_0} \|u(\cdot,t;t_0,u_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0$.

(ii) Assume that $\inf_{t\in\mathbb{R}, x\in\bar{\Omega}} a_1(t,x) > \chi$. Then there is a positive bounded entire solution $(u,w) = (u^*(x,t), w^*(x,t))$ of (1.1). Moreover, if there is $T > 0$ such that $a_i(t+T,x) = a_i(t,x)$ for $i = 0,1$, then (1.1) has a positive periodic solution $(u,v) = (u^*(x,t), w^*(x,t))$ with period T; and if $a_i(t,x) \equiv a_i(x)$ for $i = 0,1$, then (1.1) has a positive steady state solution $(u,w) = (u^*(x), w^*(x))$.

The result (i) provides some parameter region for the global existence of nonnegative classical solutions. In addition to the difficulties related to chemotaxis, for example the lack of comparison principle, time and space dependence in (1.1) introduces several other new difficulties. The existence and nonlinear stability of positive entire solutions of (1.1) with time and space dependent coefficients are much more difficult to study than the case with constant coefficients. The result (ii) provides for some parameter region of global existence, the existence of positive entire solutions. Several new techniques have been developed in [30, Lemma 5.1, Lemma 5.2, and Lemma 5.3] to obtain result (ii).

System (1.2) also arises in mathematical biology as a model for the spatio-temporal evolution of the populations of two species which proliferate and compete according to a Lotka-Volterra-type kinetics, and in which individuals are moreover able to move according to both random diffusion and chemotaxis toward a signal produced by themselves. In the setting of (1.2), $u = u(x,t)$ and $v = v(x,t)$ represent the population densities of two species; $w = w(x,t)$ denotes the concentration of the chemical; $\chi_i > 0, i = 1,2$, describe the chemotaxis sensitivities of u and v respectively; $d_i > 0, i = 1,2,3$, describe the diffusion rate of u, v and w respectively; $\tau \geq 0$ describes the diffusion speed of the chemical substance; a_0 and a_1 (resp. b_0 and b_2) describe respectively the intrinsic growth rate and the self limitation effect of the species u (resp. of the species v), and $b_1 \in \mathbb{R}$ (resp. $a_2 \in \mathbb{R}$) describes the local effect of the species u (resp. of the species v) on the species v (resp. on the species u).

Among interesting dynamical issues in (1.2) are persistence, coexistence, exclusion, and nonlinear stability of positive entire solutions. Several authors have studied these issues for system (1.2) with constant coefficients [29, 44, 60]. There is little study of these important issues for (1.2) with time and space dependent coefficients even in the case of $\tau = 0$.

The second main objective of this dissertation is to study the following dynamical issues of system (1.2) with $\tau = 0$

- Uniform persistence and coexistence.
- Existence of positive entire solutions.
- Competitive exclusion of one of the two species.
- Uniqueness and stability of coexistence states.

Among others, we proved the following.

(iii) (Persistence) Assume

$$a_{1,\inf} > \frac{k\chi_1}{d_3}, \quad a_{2,\inf} \geq \frac{l\chi_1}{d_3}, \quad b_{1,\inf} \geq \frac{k\chi_2}{d_3}, \quad b_{2,\inf} > \frac{l\chi_2}{d_3},$$

$$a_{0,\inf} > a_{2,\sup}\bar{A}_2 \quad \text{and} \quad b_{0,\inf} > b_{1,\sup}\bar{A}_1,$$

where

$$\bar{A}_1 = \frac{a_{0,\sup}}{a_{1,\inf} - \frac{k\chi_1}{d_3}}, \quad \bar{A}_2 = \frac{b_{0,\sup}}{b_{2,\inf} - \frac{l\chi_2}{d_3}}.$$

Then there are $\underline{A}_1 > 0$ and $\underline{A}_2 > 0$ such that for any $\epsilon > 0$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$, and $u_0, v_0 \not\equiv 0$, there exists t_{ϵ,u_0,v_0} such that the unique global classical solution $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ of system (1.2) with $(u(x,t_0;t_0,u_0,v_0), v(x,t_0;t_0,u_0,v_0)) = (u_0(x), v_0(x))$ in certain sense satisfies

$$\underline{A}_1 \leq u(x,t;t_0,u_0,v_0) \leq \bar{A}_1 + \epsilon, \;\; \underline{A}_2 \leq v(x,t;t_0,u_0,v_0) \leq \bar{A}_2 + \epsilon \tag{1.3}$$

for all $x \in \bar{\Omega}$, $t \geq t_0 + t_{\epsilon,u_0,v_0}$, and $t_0 \in \mathbb{R}$.

(iv) (Coexistence) Under the same assumption of (iii), there is a coexistence state $(u^{**}(x,t)$, $v^{**}(x,t)$, $w^{**}(x,t))$ of system (1.2) (i.e. $(u^{**}(x,t)$, $v^{**}(x,t)$, $w^{**}(x,t))$ is a solution of (1.2) defined for all $t \in \mathbb{R}$ and $\inf_{t\in\mathbb{R},x\in\Omega} u^{**}(x,t) > 0$ and $\inf_{t\in\mathbb{R},x\in\bar{\Omega}} v^{**}(x,t) > 0$). Moreover, if there is $T > 0$ such that $a_i(t+T,x) = a_i(t,x)$, $b_i(t+T,x) = b_i(t,x)$ for $i = 0,1,2$, then system (1.2) has a T-periodic coexistence state $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$; and if $a_i(t,x) \equiv a_i(x)$, $b_i(t,x) \equiv b_i(x)$ for $i = 0,1,2$, then system (1.2) has a steady state coexistence state $(u^{**}(t,x), v^{**}(t,x), w^{**}(t,x)) \equiv (u^{**}(x), v^{**}(x), w^{**}(x))$.

(v) (Competitive exclusion) Assume that

$$a_{1,\inf} > \frac{k\chi_1}{d_3}, \quad a_{2,\inf} \geq \frac{l\chi_1}{d_3}, \quad b_{1,\inf} \geq \frac{k\chi_2}{d_3}, \quad b_{2,\inf} > 2\frac{\chi_2}{d_3}l,$$

$$a_{2,\inf}\big(b_{0,\inf}(b_{2,\inf} - l\frac{\chi_2}{d_3}) - b_{0,\sup}\frac{\chi_2}{d_3}l\big) \geq a_{0,\sup}\big((b_{2,\inf} - l\frac{\chi_2}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - (l\frac{\chi_2}{d_3})^2\big),$$

and

$$
\begin{aligned}
&\big(a_{1,\inf} - \frac{\chi_1 k}{d_3}\big)\Big(b_{0,\inf}(b_{2,\inf} - \frac{l\chi_2}{d_3}) - b_{0,\sup}\frac{l\chi_2}{d_3}\Big) \\
&> \Big[\Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big)(b_{2,\inf} - \frac{\chi_2 l}{d_3}) + \frac{l\chi_2}{d_3}\Big(b_{1,\inf} - k\frac{\chi_2}{d_3}\Big)_-\Big]a_{0,\sup}.
\end{aligned}
$$

Then for every $t_0 \in \mathbb{R}$ and nonnegative initial functions $u_0, v_0 \in C^0(\overline{\Omega})$, $u_0 \geq 0$, $v_0 \geq 0$, with $\|v_0\|_\infty > 0$, the unique bounded and globally defined classical solution $(u(\cdot,\cdot;t_0,u_0,v_0), v(\cdot,\cdot;t_0,u_0;v_0), w(\cdot,\cdot;t_0,u_0,v_0))$ of system (1.2) with $(u(x,t_0;t_0,u_0,v_0), v(x,t_0;t_0,u_0,v_0)) = (u_0(x), v_0(x))$ in certain sense satisfies

$$
\lim_{t\to\infty} \|u(\cdot,t+t_0;t_0,u_0;v_0)\|_\infty = 0,
$$

$$
\alpha \leq \liminf_{t\to\infty}(\min_{x\in\bar{\Omega}} v(x,t)) \leq \limsup_{t\to\infty}(\max_{x\in\bar{\Omega}} v(x,t)) \leq \beta,
$$

for some $0 < \alpha < \beta < \infty$.

Global asymptotic stability and uniqueness of coexistence states are obtained for system (1.2) when the coefficients are constants and satisfy certain weak competition condition in [6], [55], [60]. In such cases, the persistence follows from the asymptotic stability and uniqueness of coexistence states. The persistence in two species chemotaxis systems without assuming the asymptotic stability of coexistence states is studied for the first time in [31], even when the coefficients are constants. Several new nontrivial techniques have been developed in [31, Lemma 3.1 to 3.5] to prove the persistence result (iii).

The rest of the dissertation is organized as follows. In Chapter 2, we will study the dynamics of system (1.1) with $\tau = 0$, the so called parabolic-elliptic chemotaxis model. We first state basic assumptions, definitions, notations and main results. Next, we study respectively local existence and global existence of classical solutions. Finally, we study the existence, uniqueness and stability of positive entire solutions. Chapter 3 is dedicated to the study of the dynamical aspects of system (1.2) with $\tau = 0$, the so called parabolic-parabolic-elliptic chemotaxis model. We first study the global existence of classical solutions. Next, we study persistence of solutions, existence of coexistent states and the exclusion phenomenon. We then study the existence of optimal rectangles. Finally, we study the uniqueness and stability of coexistence states. Chapter 4 is dedicated to concluding remarks and future works.

Chapter 2

Dynamics in Chemotaxis Models of One Species on Bounded Heterogeneous Environments

2.1 Introduction

In this chapter, we study the dynamics of system (1.1) with $\tau = 0$ and $d_1 = d_2 = k = \lambda = 1$, that is the following chemotaxis system of parabolic-elliptic type with both local and nonlocal heterogeneous logistic source,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a_0(t,x) - a_1(t,x)u - a_2(t,x)\int_\Omega u), \quad x \in \Omega \\ 0 = \Delta v + u - v, \quad x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded subset of $\mathbb{R}^n$ with smooth boundary, $u(x,t)$ and $v(x,t)$ represent the population densities of living organisms and some chemoattractant substance, respectively, $\chi > 0$ is the chemotactic sensitivity, a_0, a_1 are nonnegative bounded functions and a_2 is a bounded real valued function.

System (2.1) with constant coefficients was introduced recently in [44] by Negreanu and Tello. As mentioned in [44], the logistic growth describes the competition of the individuals of the species for the resources of the environment and the cooperation to survive. The coefficient a_0 induces an exponential growth for low density populations and the term $a_1 u$ describes a local competition of the species. At the time that the population grows, the competitive effect of the local term $a_1 u$ becomes more influential. The non-local term $a_2 \int_\Omega u$ describes the influence of the total mass of the species in the growth of the population. If $a_2 > 0$, we have a competitive term which limits such growth and when $a_2 < 0$ the individuals cooperate globally to survive. In the last case, the individuals compete locally but cooperate globally and the effects of $a_1 u$

and $a_2 \int_\Omega u$ balance the system. Note that $(u, v) \equiv (0, 0)$ is always a solution of system (2.1), which will be called the *trivial solution* of system (2.1). Due to biological reasons, we are only interested in nonnegative solutions, in particular, nonnegative and nontrivial solutions, of system (2.1).

In the case that chemotaxis and nonlocal competition are absent (i.e. $\chi = 0$ and $a_2 \equiv 0$) in system (2.1), the population density $u(x, t)$ of the living organisms satisfies the following scalar reaction diffusion equation,

$$\begin{cases} u_t = \Delta u + u(a_0(t, x) - a_1(t, x)u), \quad x \in \Omega \\ \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.2}$$

Equation (2.2) is called Fisher or KPP type equation in literature because of the pioneering works by Fisher ([12]) and Kolmogorov, Petrowsky, Piscunov ([36]) in the special case $a_0(t, x) = a_1(t, x) = 1$, and has been extensively studied (see [7], [21], [46], [54], [68], etc.). The dynamics of (2.2) is quite well understood. For example, if $a_0(t, x) \equiv a_0(t)$ and $a_1(t, x) \equiv a_1(t)$, it is proved in [46] that system (2.2) has a unique bounded entire solution, that is positive, does not approach the zero-solution in the past and in the future and attracts all positive solutions. If $a_0(t, x)$ and $a_1(t, x)$ are positive and almost periodic in t, it is proved in [54] that (2.2) has a unique globally stable time almost periodic positive solution.

In the case of constant coefficients with $a_0 > 0$ and $a_1 - |\Omega|(a_2)_- > 0$, it is clear that $(u, v) \equiv (\frac{a_0}{a_1 + a_2|\Omega|}, \frac{a_0}{a_1 + a_2|\Omega|})$ is the unique nontrivial spatially and temporally homogeneous steady state solution of system (2.1), where $|\Omega|$ is the Lebesgue measure of Ω. It is proved in [44] that the condition $a_1 > 2\chi + |a_2|$ ensures the global stability of the homogeneous steady state (see [59] when $a_2 = 0$) and that, if furthermore $a_2 = 0$, the assumption $a_1 > \frac{n-2}{n}\chi$ ensures the global existence of a unique bounded classical solution $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ with given nonnegative initial function $u_0 \in C^{0,\alpha}(\bar{\Omega})$ (i.e. $u(x, t_0; t_0, u_0) = u_0(x) \geq 0$) (see [59]). It should be pointed that, when $n \geq 3$ and $a_1 \leq \frac{n-2}{n}\chi$ ($a_2 = 0$), it remains open whether for any given nonnegative initial function $u_0 \in C^{0,\alpha}(\bar{\Omega})$, system (2.1) possesses a global classical solution $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ with $u(x, t_0; t_0, u_0) = u_0(x)$, or whether finite-time blow-up occurs for some initial data. We mention the works [39], [66], [67] along this direction. It is shown in [39], [67] that in presence of suitably weak logistic dampening (that is, small a_1) certain transient growth phenomena do occur for some initial data. It is shown in [66] that replacing $a_1 u$ by $a_1 u^\kappa$ with suitable $\kappa < 1$ (for instance, $\kappa = 1/2$) and replacing $u - v$ by $u - \frac{1}{|\Omega|}\int_\Omega u(x)dx$, then finite-time blow-up is possible.

However, as far as $\chi > 0$ and $a_0(t,x), a_i(t,x), a_2(t,x)$ are not constants, there is little study of system (2.1). The objective in this chapter is to investigate thoroughly the asymptotic dynamics of system (2.1). To this end, we first study the local and global existence of classical solutions of system (2.1) with given nonnegative initial functions, next study the existence of entire positive solutions, and then investigate the uniqueness and stability of positive entire solutions and the asymptotic behavior of positive solutions of system (2.1).

2.2 Notations, Assumptions, Definitions and Main results

2.2.1 Notations, assumptions and definitions

Throughout this chapter, we assume that a_i $(i = 0, 1, 2)$ satisfy the following standard assumption.

(H1) *$a_0(t,x)$, $a_1(t,x)$ and $a_2(t,x)$ are Hölder continuous in $t \in \mathbb{R}$ with exponent $\nu > 0$ uniformly with respect to $x \in \bar{\Omega}$, continuous in $x \in \bar{\Omega}$ uniformly with respect to $t \in \mathbb{R}$, and there are nonnegative constants α_i, A_i $(i = 0, 1, 2)$ with $\alpha_1 + \alpha_2 > 0$ such that*

$$\begin{cases} 0 < \alpha_0 \le a_0(t,x) \le A_0 \\ 0 \le \alpha_1 \le a_1(t,x) \le A_1 \\ 0 \le \alpha_2 \le |a_2(t,x)| \le A_2. \end{cases}$$

We put

$$a_{i,\inf} = \inf_{t\in\mathbb{R}, x\in\bar{\Omega}} a_i(t,x), \quad a_{i,\sup} = \sup_{t\in\mathbb{R}, x\in\bar{\Omega}} a_i(t,x), \tag{2.3}$$

$$a_{i,\inf}(t) = \inf_{x\in\bar{\Omega}} a_i(t,x), \quad a_{i,\sup}(t) = \sup_{x\in\bar{\Omega}} a_i(t,x), \tag{2.4}$$

unless specified otherwise. For convenience, we introduce the following assumptions.

(H2) *$a_1(t,x)$, $a_2(t,x)$, and χ satisfy*

$$\inf_{t\in\mathbb{R}} \Big\{ a_{1,\inf}(t) - |\Omega| \Big(a_{2,\inf}(t) \Big)_- \Big\} > \chi, \tag{2.5}$$

where $|\Omega|$ is the Lebesgue measure of Ω.

(H2)$'$ *$a_1(t,x)$, $a_2(t,x)$, and χ satisfy $\inf_{t\in\mathbb{R}} \big\{ a_{1,\inf}(t) - |\Omega| \big(a_{2,\inf}(t) \big)_- \big\} > 0$ and if $n \geq 3$, $a_{1,\inf} > \frac{\chi(n-2)}{n}$.*

For given $1 \leq p < \infty$, let $X = L^p(\Omega)$ and $A = -\Delta + I$ with

$$D(A) = \Big\{ u \in W^{2,p}(\Omega) \,|\, \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \Big\}.$$

It is well known that A is a sectorial operator in X (see, for example, [17, Example 1.6]) and thus generates an analytic semigroup $\big(e^{-At}\big)_{t\geq 0}$ in X (see, for example, [17, Theorem 1.3.4]). Moreover $0 \in \rho(A)$ and

$$\|e^{-At}u\|_X \leq e^{-t}\|u\|_X \quad \text{for} \quad t \geq 0 \,\text{and}\, u \in X.$$

Because Δ is a dissipative operator and $range(I - \Delta) = X$, so it generates a strongly continuous semigroup of contraction on X.

Let $X^\alpha = D(A^\alpha)$ be equipped with the graph norm $\|u\|_\alpha := \|u\|_{X^\alpha} = \|A^\alpha u\|_p$ (see, for example, [17, Definition 1.4.7]).

Throughout this chapter, A and X^α are defined as in the above. For given $-\infty \leq t_1 < t_2 \leq \infty$ and $0 \leq \delta < 1$, $C^\delta((t_1,t_2), X^\alpha)$ is the space of all locally Hölder continuous functions from (t_1,t_2) to X^α with exponent δ.

A vector valued function $(u(x,t), v(x,t))$ is called a *classical solution* of system (2.1) on $\Omega \times (t_1,t_2)$ $(-\infty \leq t_1 < t_2 \leq \infty)$ if $(u,v) \in C(\bar{\Omega} \times (t_1,t_2)) \cap C^{2,1}(\bar{\Omega} \times (t_1,t_2))$ and satisfies system (2.1) for $t \in (t_1,t_2)$ in the classical sense. A classical solution $(u(x,t), v(x,t))$ of system (2.1) on $\Omega \times (t_1,t_2)$ is called *nonnegative* if $u(x,t) \geq 0$ and $v(x,t) \geq 0$ for $(x,t) \in \bar{\Omega} \times (t_1,t_2)$, and is called *positive* if $\inf_{(x,t)\in\bar{\Omega}\times(t_1,t_2)} u(x,t) > 0$ and $\inf_{(x,t)\in\bar{\Omega}\times(t_1,t_2)} v(x,t) > 0$.

$(u(x,t),v(x,t))$ is called an *entire classical solution* of system (2.1) if it is a classical solution of system (2.1) on $(-\infty,\infty)$. For a given $t_0 \in \mathbb{R}$ and a given function $u_0(\cdot)$ on Ω, it is said that system (2.1) has a *classical solution with initial condition* $u(x,t_0) = u_0(x)$ if system (2.1) has a classical solution, denoted by $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$, on (t_0,T) for some $T > t_0$ satisfying that $\lim_{t\to t_0+} u(\cdot,t;t_0,u_0) = u_0(\cdot)$ in certain sense. A classical solution of system (2.1) with initial condition $u(x,t_0) = u_0(x)$ exists globally if system (2.1) has a classical solution $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ with $u(x,t_0;t_0,u_0) = u_0(x)$ on (t_0,∞).

2.2.2 Main results

First of all, we have the following local existence theorem.

Theorem 2.1. *Suppose that $p > 1$ and $1/2 < \alpha < 1$ are such that $X^\alpha \subset C^1(\bar{\Omega})$.*

(1) For any $t_0 \in \mathbb{R}$ and $u_0 \in X^\alpha$ with $u_0 \geq 0$, there exists $T_{\max} \in (0,\infty]$ such that system (2.1) *has a unique non-negative classical solution $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ on $(t_0, t_0+T_{\max})$ satisfying that $\lim_{t\to t_0} \|u(\cdot,t;t_0,u_0) - u_0(\cdot)\|_{X^\alpha} = 0$, and*

$$u(\cdot,\cdot;t_0,u_0) \in C([t_0,t_0+T_{\max}), X^\alpha) \cap C^\delta((t_0,t_0+T_{\max}), X^\alpha) \tag{2.6}$$

for some $0 < \delta < 1$. Moreover if $T_{\max} < \infty$, then

$$\limsup_{t\nearrow T_{\max}} \|u(\cdot,t+t_0;t_0,u_0)\|_{X^\alpha} = \limsup_{t\nearrow T_{\max}} \|u(\cdot,t+t_0;t_0,u_0)\|_{C^0(\bar{\Omega})} = \infty. \tag{2.7}$$

(2) For any given $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, there exists $T_{\max} \in (0,\infty]$ such that system (2.1) *has a unique non-negative classical solution $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ on $(t_0,t_0+T_{\max})$ satisfying that $\lim_{t\to t_0} \|u(\cdot,t;t_0,u_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0$, and*

$$u(\cdot,\cdot;t_0,u_0) \in C((t_0,t_0+T_{\max}), X^\alpha) \cap C^\delta((t_0,t_0+T_{\max}), X^\alpha) \tag{2.8}$$

for some $0 < \delta < 1$. Moreover if $T_{\max} < \infty$, then

$$\limsup_{t\nearrow T_{\max}} \|u(\cdot,t+t_0;t_0,u_0)\|_{C^0(\bar{\Omega})} = \infty. \tag{2.9}$$

Remark 2.1. *(1) Since $X^\alpha \subset C^1(\bar{\Omega}) \subset C^0(\bar{\Omega})$, the existence of a local classical solution in Theorem 2.1(1) is guaranteed by Theorem 2.1(2). However* $\lim_{t\to t_0} u(\cdot,\cdot;t_0,u_0) = u_0(\cdot)$ *in the X^α-norm in Theorem 2.1(1) is not included in Theorem 2.1 (2).*

(2) Theorem 2.1(2) is consistent (one species version) with [55, Lemma 2.1].

(3) Semigroup theory and fixed point theorems together with regularity and a prior estimates for elliptic and parabolic equations are among basic tools used in literature to prove the local existence of classical solutions of chemotaxis models with various given initial functions. For the self-completeness, we will give a proof of Theorem 2.1(1) by using semigroup theory and give a proof of Theorem 2.1(2) based on the combination of fixed point theorems and semigroup theory.

We next consider the global existence of classical solutions of system (2.1) with given initial functions and the following is our main result on the global existence of positive classical solutions to system (2.1).

Theorem 2.2. *(1) Assume that* **(H2)** *holds. Then for any $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, system* (2.1) *has a unique global classical solution* $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ *which satisfies that* $\lim_{t\to t_0} \|u(\cdot,t;t_0,u_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0$ *and* (2.8), (2.9). *Moreover, we have*

$$
\begin{aligned}
0 \leq v(x,t;t_0,u_0) &\leq \max_{x\in\bar{\Omega}} u(x,t;t_0,u_0) \\
&\leq \max\Big\{\sup u_0(x), \frac{a_{0,\sup}}{\inf_{t\geq t_0}\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_- - \chi\}}\Big\}
\end{aligned}
\tag{2.10}
$$

for all $(x,t) \in \bar{\Omega}\times[t_0,\infty)$.

(2) Assume that **(H2)**$'$ *holds. Then for any $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, system* (2.1) *has a unique global classical solution* $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ *which satisfies that* $\lim_{t\to t_0} \|u(\cdot,t;t_0,u_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0$ *and* (2.8), (2.9). *Moreover,*

$$
\|u(\cdot,t;t_0,u_0)\|_{C^0(\bar{\Omega})} + \|v(\cdot,t;t_0,u_0)\|_{C^0(\bar{\Omega})} \leq C
$$

for all $t \geq t_0$, where $C = C(\|u_0\|_{C^0(\bar{\Omega})})$, i.e, C depends only on $\|u_0\|_{C^0(\bar{\Omega})}$, and

$$0 \leq \int_\Omega u(x,t;t_0,u_0)dx \leq \max\Big\{\int_\Omega u_0(x), \frac{|\Omega|a_{0,\sup}}{\inf_{t\in\mathbb{R}}\big\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\big\}}\Big\} \, \forall t \geq t_0.$$

Remark 2.2. *(1) When $a_2(t,x) \geq 0$, **(H2)**$'$ becomes $a_{1,\inf} > \max\{\frac{\chi(n-2)}{n}, 0\}$. In particular, if $a_2(t,x) = 0$, Theorem 2.2 is consistent with the result by Tello and Winkler in [59].*

*(2) When the coefficients are constant, the condition **(H2)** becomes $a_1 - |\Omega|(a_2)_- > \chi$ which is consistent with the result of global existence by Negreanu and Tello in [44].*

*(3) **(H2)** implies **(H2)**$'$. Therefore the global existence of bounded classical solutions of system* (2.1) *in Theorem 2.2(1) follows from Theorem 2.2(2). However the explicit bound given by* (2.10) *is not included in Theorem 2.2(2). Note that the explicit bound* (2.10) *will be used in the proof of the existence of periodic solutions (resp. steady state solutions) when the coefficients $a_i(t,x)$ are periodic (resp. when $a_i(t,x) = a_i(x)$) (see Theorem 2.3).*

(4) In general, assuming that $\inf_{t\in\mathbb{R}}\big\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\big\} > 0$, it remains open whether for any given $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$, system (2.1) *has a global classical solution $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$. This is open even in the case that $a_i(t,x) \equiv a_i$ for $i = 0,1$ and $a_2(t,x) = 0$.*

We now state our main result on the existence of positive bounded entire solutions of system (2.1).

Theorem 2.3. *Suppose that **(H2)** holds. Then there is a positive bounded entire classical solution $(u,v) = (u^*(x,t), v^*(x,t))$ of system* (2.1)*. Moreover, the following hold.*

(1) If there is $T > 0$ such that $a_i(t+T,x) = a_i(t,x)$ for $i = 0,1,2$, then system (2.1) *has a positive periodic solution $(u,v) = (u^*(x,t), v^*(x,t))$ with period T.*

(2) If $a_i(t,x) \equiv a_i(t)$ for $i = 0,1,2$, then system (2.1) *has a unique positive spatially homogeneous entire solution $(u,v) = (u^*(t), v^*(t))$ with $v^*(t) = u^*(t)$, and if $a_i(t)$ $(i = 0,1,2)$ are periodic or almost periodic, so is $(u^*(t), v^*(t))$.*

(3) If $a_i(t,x) \equiv a_i(x)$ for $i = 0, 1, 2$, then system (2.1) *has a positive steady state solution $(u, v) = (u^*(x), v^*(x))$.*

Remark 2.3. *(1) When the coefficients are only time dependent, i.e, $a_i(t, x) = a_i(t)$ for $i = 0, 1, 2$, every positive entire solution $u(t)$ of the ODE*

$$u_t = u[a_0(t) - (a_1(t) + |\Omega| a_2(t))u]$$

is a positive entire solution of the first equation of system (2.1) *and then $(u(t), v(t))$ with $v(t) = u(t)$ is an entire positive solution of system* (2.1)*. Thus system* (2.1) *has an entire solution under the weaker assumption $\inf_{t\in\mathbb{R}}\{a_1(t) - |\Omega|(a_2(t))_-\} > 0$ (see Lemma 2.5). In general, due to the lack of comparison principle for system* (2.1)*, it is fairly nontrivial to prove the existence of positive entire solutions.*

(2) It should be mentioned that there may be lots of positive entire solutions (see [38], [59]).

(3) The existence of positive bounded entire classical of system (2.1) *also holds under the weaker assumption* **(H2)**$'$ *(see Remarks 2.6 and 2.7). However under* **(H2)**$'$*, it reminds open whether there are periodic solutions of system* (2.1) *when the coefficients $a_i(t, x)$ are periodic (resp. steady state solutions of system* (2.1) *when $a_i(t, x) \equiv a_i(x)$).*

Finally we state the main results on the stability and uniqueness of positive entire solutions and asymptotic behavior of positive solutions of system (2.1).

Theorem 2.4. *(1) If $a_i(t, x) \equiv a_i(t)$ for $i = 0, 1, 2$ and*

$$\inf_t \{a_1(t) - |\Omega|\, |a_2(t)|\,\} > 2\chi, \tag{2.11}$$

then for any $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$ and $u_0 \not\equiv 0$, the unique global classical solution $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ of system (2.1) *satisfies*

$$\lim_{t\to\infty} \left(\|u(\cdot, t; t_0, u_0) - u^*(t)\|_{C^0(\bar{\Omega})} + \|v(\cdot, t; t_0, u_0) - u^*(t)\|_{C^0(\bar{\Omega})} \right) = 0, \tag{2.12}$$

where $u^*(t)$ is the unique spatially homogeneous positive entire solution of system (2.1).

(2) Suppose that

$$\inf_{t\in\mathbb{R}}\Big\{a_{1,\inf}(t)-|\Omega|\Big(a_{2,\inf}(t)\Big)_-\Big\}>\Big\{\chi+\frac{a_{0,\sup}}{a_{0,\inf}}\Big(\chi+|\Omega|\sup_{t\in\mathbb{R}}\Big(a_{2,\sup}(t)\Big)_+\Big)\Big\} \tag{2.13}$$

and

$$\limsup_{t-s\to\infty}\frac{1}{t-s}\int_s^t (L_2(\tau)-L_1(\tau))d\tau<0, \tag{2.14}$$

where

$$L_1(t)=2r_2\big(a_{1,\inf}(t)+|\Omega|(a_{2,\inf}(t))_+\big), \tag{2.15}$$

$$L_2(t)=a_{0,\sup}(t)+\frac{\chi}{2}(r_1-r_2)+\frac{(\chi r_1)^2}{2}+|\Omega|r_1\big(2(a_{2,\inf}(t))_-+\big(a_{2,\sup}(t)\big)_+\big), \tag{2.16}$$

and

$$r_1=\frac{\sup_{t\in\mathbb{R}}\{a_{1,\sup}(t)-|\Omega|(a_{2,\sup}(t))_- -\chi\}a_{0,\sup}-a_{0,\inf}\big(\chi+|\Omega|\inf_t(a_{2,\inf}(t))_+\big)}{h(\chi)}, \tag{2.17}$$

$$r_2=\frac{\inf_{t\in\mathbb{R}}\{a_{1,\inf}(t)-|\Omega|(a_{2,\inf}(t))_- -\chi\}a_{0,\inf}-a_{0,\sup}\big(\chi+|\Omega|\sup_t(a_{2,\sup}(t))_+\big)}{h(\chi)}, \tag{2.18}$$

$$\begin{aligned}h(\chi)=&\inf_{t\in\mathbb{R}}\{a_{1,\inf}(t)-|\Omega|(a_{2,\inf}(t))_- -\chi\}\sup_{t\in\mathbb{R}}\{a_{1,\sup}(t)-|\Omega|(a_{2,\sup}(t))_- -\chi\}\\ &-\big(\chi+|\Omega|\inf_{t\in\mathbb{R}}(a_{2,\inf}(t))_+\big)\big(\chi+|\Omega|\sup_{t\in\mathbb{R}}(a_{2,\sup}(t))_+\big).\end{aligned}$$

Then system (2.1) *has a unique positive entire solution* $(u^*(x,t),v^*(x,t))$*, and, for any* $t_0\in\mathbb{R}$ *and* $u_0\in C^0(\bar{\Omega})$ *with* $u_0\geq 0$ *and* $u_0\not\equiv 0$*, the global classical solution* $(u(x,t;t_0,u_0)$*,* $v(x,t;t_0,u_0))$ *of system* (2.1) *satisfies*

$$\lim_{t\to\infty}\Big(\|u(\cdot,t;t_0,u_0)-u^*(\cdot,t)\|_{C^0(\bar{\Omega})}+\|v(\cdot,t;t_0,u_0)-v^*(\cdot,t)\|_{C^0(\bar{\Omega})}\Big)=0. \tag{2.19}$$

If, in addition, $a_i(t,x)\equiv a_i(x)$ *(resp.* $a_i(t+T,x)=a_i(t,x)$*,* $a_i(t,x)$ *is almost periodic in* t *uniformly with respect to* x*) for* $i=0,1,2$*, then system* (2.1) *has a unique positive* *steady state solution* $(u^*(x), v^*(x))$ *(resp. system* (2.1) *has a unique time periodic positive solution* $(u^*(x,t), v^*(x,t))$ *with period* T*, system* (2.1) *has a unique time almost periodic positive solution* $(u^*(x,t), v^*(x,t))$.

Theorem 2.5. *Suppose that* (2.13) *holds and* r_1 *and* r_2 *are as in Theorem 2.4(2). Then*

(1) For any $t_0 \in \mathbb{R}$, $u_0 \in C^0(\bar{\Omega})$ *with* $u_0 \geq 0$ *and* $u_0 \not\equiv 0$, *and* $\epsilon > 0$, *there exists* t_ϵ *such that*

$$r_2 - \epsilon \leq u(x,t;t_0,u_0) \leq r_1 + \epsilon, \ \ r_2 - \epsilon \leq v(x,t;t_0,u_0) \leq r_1 + \epsilon$$

for all $x \in \bar{\Omega}$ *and* $t \geq t_0 + t_\epsilon$.

(2) Moreover if the coefficients a_i *are periodic in* t *with period* $T > 0$ *(resp.* a_i *are almost periodic in* t*), then there are* T*-periodic functions* $m(t)$ *and* $M(t)$ *(resp. almost periodic functions* $m(t)$ *and* $M(t)$*) with*

$$r_2 \leq \inf_{t\in\mathbb{R}} m(t) \leq m(t) \leq M(t) \leq \sup_{t\in\mathbb{R}} M(t) \leq r_1$$

such that for any $t_0 \in \mathbb{R}$, $u_0 \in C(\bar{\Omega})$ *with* $u_0 \geq 0$ *and* $u_0 \not\equiv 0$, *and* $\epsilon > 0$, *there is* $t_\epsilon > 0$ *such that*

$$m(t) - \epsilon \leq u(x,t;t_0,u_0) \leq M(t) + \epsilon, \quad m(t) - \epsilon \leq v(x,t;t_0,u_0) \leq M(t) + \epsilon,$$

for all $x \in \bar{\Omega}$, $t \geq t_0 + t_\epsilon$.

Remark 2.4. *(1) When* a_i $(i = 0,1,2)$ *are constants, the condition* (2.11) *becomes*

$$a_1 - |\Omega|\,|a_2| > 2\chi. \tag{2.20}$$

Theorem 2.4(1) is then an extension of [44, Theorem 0.1] by Negreanu and Tello. When the nonlocal term is zero, the result in Theorem 2.4(1) is consistent with the result by Tello and Winkler in [59].

(2) In Theorem 2.4(2), when a_i $(i = 0,1,2)$ *are constants , we have* $r_1 = r_2 = \frac{a_0}{a_1+|\Omega|a_2}$ *and*

$$L_1(t) = \frac{2a_0(a_1 + |\Omega|(a_2)_+)}{a_1 + |\Omega|a_2}, \quad L_2(t) = a_0 + \frac{\chi^2 a_0^2}{2(a_1 + |\Omega|a_2)^2} + \frac{a_0|\Omega|[2(a_2)_- + (a_2)_+]}{a_1 + |\Omega|a_2}.$$

Hence the condition (2.13) *becomes* (2.20) *and the condition* (2.14) *becomes*

$$\frac{\chi^2 a_0}{2(a_1 + |\Omega| a_2)} < a_1 - |\Omega|(a_2)_-.$$

Furthermore when $\chi = 0$, *the condition* (2.14) *becomes*

$$a_1 - |\Omega|(a_2)_- > 0.$$

(3) In Theorem 2.4(2), if $a_i(t+T,x) = a_i(t,x)$ $(i = 0,1,2)$, *then* (2.14) *becomes*

$$\int_0^T \big(L_2(t) - L_1(t)\big) dt < 0.$$

(4) It is seen from Theorem 2.3 that (2.5) *ensures the existence of positive entire solutions of system* (2.1). *In the case that* $a_i(t,x) \equiv a_i(t)$ $(i = 0,1,2)$, *the condition* (2.11) *ensures the stability and uniqueness of positive entire solutions of system* (2.1). *In the general case, Theorem 2.5 provides some positive attracting set for positive solutions of system* (2.1) *under the condition* (2.13). *It remains open whether in the general case, the condition* (2.13) *also ensures the stability and uniqueness of positive entire solutions of system*(2.1).

(5) The reader is referred to Definition 2.3 for the definition of almost periodic functions.

The rest of the chapter is organized as follows. In section 2.3, we collect some important results from literature that will be used in the proofs of our main results. In section 2.4, we study the local existence of classical solutions of (2.1) with given initial functions and prove Theorem 2.1. In section 2.5, we investigate the global existence of classical solutions of (2.1) with given initial functions and prove Theorem 2.2. We consider the existence of positive entire solutions of (2.1) and prove Theorem 2.3 in section 2.6. Finally, in section 2.7, we study the asymptotic behavior of global positive solutions and prove Theorems 2.4 and 2.5.

2.3 Preliminaries

In this section, we recall some standard definitions and lemmas from semigroup theory. We also present some known results on non-autonomous logistic equations and Lotka-Volterra competition systems.

2.3.1 Semigroup theory

In this subsection, we recall some standard definitions and lemmas from semigroup theory. The reader is referred to [17], [48] for the details.

Recall that for given $1 \le p < \infty$, $A = -\Delta + I$ with

$$D(A) = \big\{ u \in W^{2,p}(\Omega) \,|\, \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \big\}$$

and $X^\alpha = D(A^\alpha)$ equipped with the graph norm $\|x\|_\alpha = \|A^\alpha x\|_p$. Note that $X^0 = L^p(\Omega)$.

Lemma 2.1. *(See [17, Theorem 1.6.1]) Let $1 \le p < \infty$. For any $0 \le \alpha \le 1$, we have*

$$X^\alpha \subset C^\nu(\bar{\Omega}) \text{ when } 0 \le \nu < 2\alpha - \frac{n}{p},$$

where the inclusion is continuous. In particular when $\frac{n}{2p} < \alpha \le 1$, we get $X^\alpha \subset C^0(\bar{\Omega})$.

Lemma 2.2. *(See [27, Lemma 2.1]) Let $\beta \ge 0$ and $p \in (1, \infty)$. Then for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for any $w \in C_0^\infty(\Omega)$ we have*

$$\|A^\beta e^{-tA} \nabla \cdot w\|_{L^p(\Omega)} \le C(\epsilon) t^{-\beta - \frac{1}{2} - \epsilon} e^{-\mu t} \|w\|_{L^p(\Omega)} \quad \text{for all } t > 0 \text{ and some } \mu > 0. \tag{2.21}$$

Accordingly, for all $t > 0$ the operator $A^\beta e^{-tA}\nabla\cdot$ admits a unique extension to all of $L^p(\Omega)$ which is again denoted by $A^\beta e^{-tA}\nabla\cdot$ and satisfies (2.21) *for all $w \in L^p(\Omega)$.*

Consider

$$\begin{cases} u_t + Au = F(t, u),\ t > t_0 \\ u(t_0) = u_0. \end{cases} \tag{2.22}$$

We assume that F maps some open set U of $\mathbb{R} \times X^\alpha$ into X^0 for some $0 \le \alpha < 1$, and F is locally Hölder continuous in t and locally Lipschitz continuous in u for $(t, u) \in U$.

Definition 2.1 (Mild solution)**.** *For given $u_0 \in X^\alpha$. A continuous function $u : [t_0, t_1) \to X^0$ is called a* mild solution *of* (2.22) *on $t_0 < t < t_1$ if $u(t) \in X^\alpha$ for $t \in [t_0, t_1)$ and the following integral equation holds on $t_0 < t < t_1$,*

$$u(t) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)}F(s, u(s))ds. \tag{2.23}$$

Definition 2.2 (Strong solution)**.** *(see [17, Definition 3.3.1]) A* strong solution *of the Cauchy problem* (2.22) *on (t_0, t_1) is a continuous function $u : [t_0, t_1) \to X^0$ such that $u(t_0) = u_0$, $u(t) \in D(A)$ for $t \in (t_0, t_1)$, $\frac{du}{dt}$ exists for $t \in (t_0, t_1)$, $(t_0, t_1) \ni t \to F(t, u(t)) \in X^0$ is locally Hölder continuous, and $\int_{t_0}^{t_0+\sigma} \|F(t, u(t))\|dt < \infty$ for some $\sigma > 0$, and the differential equation $u_t + Au = F(t, u)$ is satisfied on (t_0, t_1).*

Lemma 2.3 (Existence of mild/strong solutions)**.** *(1) For any $(t_0, u_0) \in U$ there exists $T_{\max} = T_{\max}(t_0, u_0) > 0$ such that* (2.22) *has a unique strong solution $u(t; t_0, u_0)$ on $(t_0, t_0 + T_{\max})$ with initial value $u(t_0; t_0, u_0) = u_0$. Moreover, $u(\cdot; t_0, u_0) \in C([t_0, t_0+T_{\max}), X^\alpha)$ and if $T_{\max} < \infty$, then*

$$\limsup_{t \nearrow T_{\max}} \|u(t + t_0; t_0, u_0)\|_{X^\alpha} = \infty.$$

(2) For given $(t_0, u_0) \in U$, if $u(t)$ is a strong solution of (2.22) *on (t_0, t_1), then u satisfy the integral equation* (2.23)*. Conversely, if $u(t)$ is continuous function from (t_0, t_1) into X^α, $\int_{t_0}^{t_0+\sigma} \|F(t, u(t))\|dt < \infty$ for some $\sigma > 0$, and if the integral equation* (2.23) *holds for $t_0 < t < t_1$, then $u(t)$ is a strong solution of the differential equation* (2.22) *on (t_0, t_1). Furthermore,*

$$u \in C^\delta((t_0, t_1), X^\alpha) \text{ for all } \delta \text{ such that } 0 < \delta < 1 - \alpha$$

Proof. (1) It follows from [17, Theorem 3.3.3] and [17, Theorem 3.3.4].

(2) The equivalence part follows from [17, Lemma 3.3.2] and $u \in C^{\delta}((t_0, t_1); X^{\alpha})$ follows from the proof of [17, Lemma 3.3.2]. □

2.3.2 Nonautonomous logistic equations and Lotka-Volterra competition systems

In this subsection, we first recall the definition of almost periodic functions and some basic properties of almost periodic functions. We then review some known results for nonautonomous logistic equations and Lotka-Volterra competition systems.

Definition 2.3. *(1) A continuous function $f : \mathbb{R} \to \mathbb{C}$ is Bohr almost periodic if for any $\epsilon > 0$, the set of ϵ-periods $\{\tau \,|\, |f(t+\tau) - f(t)| < \epsilon\}$ is relatively dense in $\mathbb{R}$, i.e, there exists an $l = l(\epsilon)$ such that every interval of the form $[t, t+l]$ intersects the set of ϵ-periods.*

(2) Let $g(t,x)$ be a continuous function of $(t,x) \in \mathbb{R} \times \bar{\Omega}$. g is said to be almost periodic in t uniformly with respect to $x \in \bar{\Omega}$ *if g is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{\Omega}$, and for each $x \in \bar{\Omega}$, $g(t,x)$ is almost periodic in t.*

Lemma 2.4. *Let $g(t,x)$ be a continuous function of $(t,x) \in \mathbb{R} \times \bar{\Omega}$. g is almost periodic in t uniformly with respect to $x \in \bar{\Omega}$ if and only if g is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{\Omega}$, and for any sequences $\{\beta_n'\}$, $\{\gamma_n'\} \subset \mathbb{R}$, there are subsequences $\{\beta_n\} \subset \{\beta_n'\}$, $\{\gamma_n\} \subset \{\gamma_n'\}$ such that*

$$\lim_{n\to\infty} \lim_{m\to\infty} g(t + \beta_n + \gamma_m, x) = \lim_{n\to\infty} g(t + \beta_n + \gamma_n, x) \quad \forall\ (t,x) \in \mathbb{R} \times \bar{\Omega}.$$

Proof. See [11, Theorems 1.17 and 2.10]. □

Consider the following nonautonomous logistic equation

$$\frac{du}{dt} = u(a(t) - b(t)u), \tag{2.24}$$

where $a(t)$ and $b(t)$ are continuous functions. For given $u_0 \in \mathbb{R}$, let $u(t; t_0, u_0)$ be the solution of (2.24) with $u(t_0; t_0, u_0) = u_0$.

Lemma 2.5. *(see [46],Theorems 2.1, 3.1 and 4.1) Suppose that $a(t)$ and $b(t)$ are continuous and satisfy that* $0 < \inf_{t\in\mathbb{R}} a(t) \le \sup_{t\in\mathbb{R}} a(t) < \infty, \quad 0 < \inf_{t\in\mathbb{R}} b(t) \le \sup_{t\in\mathbb{R}} b(t) < \infty.$ *Then*

(1) The non-autonomous equation (2.24) *has exactly one bounded entire solution $u^*(t)$ that is positive and satisfies*

$$\frac{\inf_{t\in\mathbb{R}} a(t)}{\sup_{t\in\mathbb{R}} b(t)} \le u^*(t) \le \frac{\sup_{t\in\mathbb{R}} a(t)}{\inf_{t\in\mathbb{R}} b(t)} \quad \forall\, t \in \mathbb{R}.$$

(2) $u^*(\cdot)$ is an attractor for all positive solutions of (2.24)*, that is, for any $u_0 > 0$ and $t_0 \in \mathbb{R}$,*

$$\lim_{t\to\infty} \|u(t+t_0; t_0, u_0) - u^*(t+t_0)\| = 0.$$

(3) If furthermore $a(t)$ and $b(t)$ are periodic with period T (resp. almost periodic), $u^*(t)$ is also periodic with period T (resp. almost periodic).

Consider now the following nonautonomous Lotka-Volterra competition systems

$$\begin{cases} \frac{du}{dt} = u(a_1(t) - b_1(t)u - c_1(t)v) \\ \frac{dv}{dt} = v(a_2(t) - b_2(t)u - c_2(t)v), \end{cases} \tag{2.25}$$

where $a_i(t)$, $b_i(t)$, and $c_i(t)$ $(i = 1, 2)$ are continuous and bounded above and below by positive constants.

Given a function $f(t)$, which is bounded above and below by positive constants, we let

$$f^L = \inf_{t\in\mathbb{R}} f(t) \quad \text{and} \quad f^M = \sup_{t\in\mathbb{R}} f(t).$$

Lemma 2.6. *Suppose that $a_1^L > \frac{c_1^M a_2^M}{c_2^L}$ and $a_2^L > \frac{a_1^M b_2^M}{b_1^L}$.*

(1) Suppose that $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ are two solutions of the system (2.25) *with $u_k(t_0) > 0$, $v_k(t_0) > 0$ $(k = 1, 2)$. Then $u_1(t) - u_2(t) \to 0$ and $v_1(t) - v_2(t) \to 0$ as $t \to \infty$.*

(2) For any $t_0 \in \mathbb{R}$, there exists a solution $(u_0(t), v_0(t))$ of system (2.25) *for $t \ge t_0$ such that*

$$0 < \frac{a_1^L c_2^L - c_1^M a_2^M}{b_1^M c_2^L - c_1^M b_2^L} \le u_0(t) \le \frac{a_1^M c_2^M - c_1^L a_2^L}{b_1^L c_2^M - c_1^L b_2^M} \quad \forall\, t \ge t_0,$$

$$0 < \frac{b_1^L a_2^L - a_1^M b_2^M}{b_1^L c_2^M - c_1^L b_2^M} \le v_0(t) \le \frac{b_1^M a_2^M - a_1^L b_2^L}{b_1^M c_2^L - c_1^M b_2^L} \quad \forall\, t \ge t_0.$$

(3) If moreover the coefficients are positive and T-periodic, then there exist exactly one T-periodic solution of the system (2.25) *with positive components, which attracts all solutions that begin in the open first quadrant.*

(4) If moreover the coefficients are positive and almost periodic, then there exist exactly one almost periodic solution of the system (2.25) *with positive components, which attracts all solutions that begin in the open first quadrant.*

Proof. (1), (2), (3) follow from [1, Theorems 1 and 2], and (4) follows from [22, Theorem C]. □

2.4 Local Existence and Uniqueness of Classical Solutions

In this section, we study the local existence and uniqueness of classical solutions of system (2.1) with given initial functions and give main steps of the proof of Theorem 2.1 (the details of the proof of Theorem 2.1 are given in our paper [30, proof of Theorem 2.1]).

First, observe that $C^0(\bar{\Omega}) \subset L^p(\Omega)$ for any $1 \le p < \infty$. Throughout this section, unless specified otherwise, $p > 1$ and $\alpha \in (1/2, 1)$ are such that $X^\alpha \subset C^1(\bar{\Omega})$, where $X^\alpha = D(A^\alpha)$ with the graph norm $\|u\|_\alpha = \|A^\alpha u\|_{L^p(\Omega)}$ and $A = I - \Delta$ with domain $D(A) = \{u \in W^{2,p}(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Note that $A : D(A) \to X^0 (= L^p(\Omega))$ is a linear, bounded bijection, and $A^{-1} \colon X^0 \to X^\alpha$ is compact.

Next, we note that if $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ is a classical solution of system (2.1) satisfying the properties in Theorem 2.1 (1) or (2), then $v(\cdot,t;t_0,u_0) = A^{-1}u(\cdot,t;t_0,u_0)$ and $u(x,t;t_0,u_0)$ is a classical solution of

$$\begin{cases} u_t = (\Delta - 1)u + f(t,x,u), \quad x \in \Omega \\ \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega \end{cases} \tag{2.26}$$

with $u(x, t_0; t_0, u_0) = u_0(x)$, where

$$f(t, x, u) = -\chi \nabla u \cdot \nabla A^{-1} u + \chi u(u - A^{-1} u) + u\Big(1 + a_0(t, x) - a_1(t, x) u - a_2(t, x) \int_\Omega u\Big).$$

Conversely, if $u_0 \in X^\alpha$ (resp. $u_0 \in C^0(\bar{\Omega})$) and $u(x, t; t_0, u_0)$ is a classical solution of (2.26) satisfying the properties in Theorem 2.1 (1) (resp. (2)), then $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ is a classical solution of system (2.1) satisfying the properties in Theorem 2.1 (1) (resp. (2)), where $v(\cdot, t; t_0, u_0) = A^{-1} u(\cdot, t; t_0, u_0)$.

We now give main steps of the proof of Theorem 2.1 . In the rest of this section, C denotes a constant independent of the initial conditions and the solutions under consideration, unless otherwise specified.

Proof of Theorem 2.1. (1) We use the semigroup approach to prove (1) and divide the proof into four steps.

Step 1. (Existence of strong solution). In this step, we prove the existence of a unique strong solution $u(\cdot, t; t_0, u_0)$ of (2.26) in X^α with $u(\cdot, t_0; t_0, u_0) = u_0$ and satisfying (2.6) and (2.7). In order to do so, we write (2.26) as

$$u_t + Au = F(t, u), \tag{2.27}$$

where $F(t, u) = -\chi \nabla u \cdot \nabla A^{-1} u + \chi u(u - A^{-1} u) + u\Big(1 + a_0(t, \cdot) - a_1(t, \cdot) u - a_2(t, \cdot) \int_\Omega u\Big)$.

It is not difficult to prove that $F : \mathbb{R} \times X^\alpha \to X^0$ is locally Hölder continuous in t and locally Lipschitz continuous in u. Then by Lemma 2.3, (2.27) has a strong solution $u(\cdot, \cdot; t_0, u_0) \in C([t_0, t_0 + T_{\max}), X^\alpha)$. Moreover, $u \in C^\delta((t_0, t_0 + T_{\max}), X^\alpha) \cap C^1((t_0, t_0 + T_{\max}), X_0)$ for any δ satisfying $0 < \delta < 1 - \alpha$. Hence (2.6) holds. Moreover, $u(x, t) := u(x, t; t_0, u_0)$ is a mild solution of (2.27) given by

$$
\begin{aligned}
u(\cdot,t) =& e^{-A(t-t_0)}u_0 - \chi\int_{t_0}^t e^{-A(t-s)}\nabla u(\cdot,s)\cdot\nabla A^{-1}u(\cdot,s)ds \\
&+\chi\int_{t_0}^t e^{-A(t-s)}u(\cdot,s)\big(u(\cdot,s)-A^{-1}u(\cdot,s)\big)ds \\
&+\int_{t_0}^t e^{-A(t-s)}u(\cdot,s)\Big(1+a_0(s,\cdot)-a_1(s,\cdot)u(\cdot,s)-a_2(s,\cdot)\int_\Omega u(\cdot,s)\Big)ds.
\end{aligned}
$$

Furthermore if $T_{\max}<\infty$, then (2.7) holds.

Step 2. (Regularity). In this step, we prove that $u(x,t) := u(x,t;t_0,u_0)$ obtained in (i) is a classical solution of (2.26) on $(t_0, t_0+T_{\max})$ and then $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ is a classical solution of system (2.1) on $(t_0, t_0+T_{\max})$ satisfying the properties in Theorem 2.1(1), where $v(\cdot,t;t_0,u_0) = A^{-1}u(\cdot,t;t_0,u_0)$.

Fix $t_0 < t_1 < T < t_0+T_{\max}$ and consider the problem

$$
\begin{cases}
\tilde{u}_t(x,t) = (\Delta-1)\tilde{u}(x,t) + g(x,t), & x\in\Omega, \quad t\in(t_1,T) \\
\tilde{u}(x,t_1) = u(x,t_1), & x\in\Omega \\
\frac{d\tilde{u}}{dn} = 0, \quad x\in\partial\Omega,
\end{cases} \tag{2.28}
$$

where

$$
\begin{aligned}
g(x,t) =& -\chi\nabla A^{-1}u(x,t)\cdot\nabla u(x,t) \\
&+\Big(1+a_0(x,t)-\chi A^{-1}u(x,t)+\chi u(x,t)-a_1(x,t)u(x,t)-a_2(x,t)\int_\Omega u(\cdot,t)\Big)u(x,t).
\end{aligned}
$$

By Lemma 2.1, $t\to g(\cdot,t)\in C^\theta(\bar\Omega)$ is Hölder continuous in $t\in(t_0,t_0+T_{\max})$ for some $\theta\in(0,1)$. Then by [2, Theorem 15.1, Corollary 15.3], (2.28) has a unique classical solution $\tilde{u}\in C^{2,1}(\bar\Omega\times(t_1,T))\cap C^0(\bar\Omega\times[t_1,T))$. Moreover, by Lemma 2.3,

$$\tilde{u}(\cdot,t) =$$

$$e^{-A(t-t_1)}u(\cdot,t_1) - \chi\int_{t_1}^{t} e^{-A(t-s)}\Big(\nabla u(\cdot,s)\cdot\nabla A^{-1}u(\cdot,s) - u(\cdot,s)\big(u(\cdot,s) - A^{-1}u(\cdot,s)\big)\Big)ds$$
$$+\int_{t_1}^{t} e^{-A(t-s)}u(\cdot,s)\Big(1 + a_0(s,\cdot) - a_1(s,\cdot)u(\cdot,s) - a_2(s,\cdot)\int_\Omega u(\cdot,s)\Big)ds.$$

Thus $\tilde{u}(x,t) = u(x,t)$ for $t \in [t_1,T)$ and $u \in C^{2,1}(\bar{\Omega}\times(t_1,T)) \cap C^0(\bar{\Omega}\times[t_1,T))$. Letting $t_1 \to t_0$ and $T \to T_{\max}$, we have $u \in C^{2,1}(\bar{\Omega}\times(t_0,t_0+T_{\max})) \cap C^0(\bar{\Omega}\times[t_0,t_0+T_{\max}))$.

Let $v(\cdot,t;t_0,u_0) = A^{-1}u(\cdot,t;t_0,u_0)$. We then have that $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ is a classical solution of system (2.1) on $(t_0,t_0+T_{\max})$ satisfying the properties in Theorem 2.1.

Step 3. (Uniqueness). In this step, we prove the uniqueness of classical solutions of system (2.1) satisfying the properties in Theorem 2.1(1).

Suppose that $(u_1(x,t), v_1(x,t))$ and $(u_2(x,t), v_2(x,t))$ are two classical solutions of system (2.1) on $(t_0,t_0+T_{\max})$ satisfying the properties in Theorem 2.1. First, set $u = u_1 - u_2$ and $v = v_1 - v_2$. Then (u,v) satisfies

$$\begin{cases} u_t = \Delta u - \chi\nabla(u\cdot\nabla v_1) - \chi\nabla(u_2\cdot\nabla v) \\ \qquad +u\left(a_0(t,x) - a_1(t)(u_1+u_2) - a_2(t,x)\int_\Omega u_1\right) - a_2(t,x)\left(\int_\Omega u\right)u_2, & x\in\Omega, t>t_0 \\ \Delta u + u - v = 0, & x\in\Omega, t>t_0 \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \\ u(x,t_0) = 0, & \text{on } x\in\Omega. \end{cases}$$

Next, fix t_1, T such that $t_0 < t_1 < T < t_0 + T_{\max}$. It is clear that, for $t \in [t_1, t_0+T_{\max})$,

$$u(\cdot,t) =$$
$$e^{-A(t-t_1)}\big(u_1(\cdot,t_1) - u_2(\cdot,t_1)\big) - \chi\int_{t_1}^{t} e^{-A(t-s)}\nabla\big[u(\cdot,s)\cdot\nabla v_1(\cdot,s) + u_2(\cdot,s)\cdot\nabla v(\cdot,s)\big]ds$$
$$+\int_{t_1}^{t} e^{-A(t-s)}u(\cdot,s)\Big(1 + a_0(s,\cdot) - a_1(s,\cdot)(u_1(\cdot,s)+u_1(\cdot,s)) - a_2(s,\cdot)\int_\Omega u_1(\cdot,s)\Big)ds$$
$$-\int_{t_1}^{t} e^{-A(t-s)}a_2(s,\cdot)\big(\int_\Omega u(\cdot,s)\big)u_2(\cdot,s)ds. \tag{2.29}$$

Then using the generalized Gronwall's inequality (see [17, page 6]), we get $u(\cdot, t) = 0$ for $t \in [t_0, T]$. Letting $T \to t_0 + T_{\max}$, we get $u(\cdot, t) = 0$ for $t \in [t_0, t_0 + T_{\max})$. Since $v(\cdot, t) = A^{-1}u(\cdot, t)$, $v(\cdot, t) = 0$ for $t \in [t_0, t_0 + T_{\max})$. Therefore $(u_1(x, t), v_1(x, t)) = (u_2(x, t), v_2(x, t))$ for $(x, t) \in \bar{\Omega} \times [t_0, t_0 + T_{\max})$.

Step 4. (Nonnegativity). In this last step, we prove the nonnegativity of the classical solutions. Since $u(x, t; t_0, u_0)$ is classical solution of (2.26), by maximum principle for parabolic equations, we have that $u(x, t; t_0, u_0)$ is nonnegative (see [13, Theorem 7 on page 41]). And now, since $u(x, t; t_0, u_0)$ is nonnegative, by maximum principle for elliptic equations, $v(x, t; t_0, u_0)$ is nonnegative (see [13, Theorem 18 on page 53]).

(2) We prove (2) by Banach Fixed Point Theorem and some arguments in (1) and divide the proof into three steps. To this end, we first introduce the notion of generalized mild solution of (2.27). A function $u \in C^0([t_0, t_0 + T), C^0(\bar{\Omega}))$ is called a *generalized mild solution* of (2.27) with $u(t_0) = u_0$ if

$$
\begin{aligned}
u(t) =& e^{-A(t-t_0)}u_0 - \chi \int_{t_0}^{t} e^{-A(t-s)} \nabla \cdot (u(s)\nabla A^{-1}u(s))ds \\
&+ \int_{t_0}^{t} e^{-A(t-s)}u(s)\Big(1 + a_0(s, \cdot) - a_1(s, \cdot)u(s) - a_2(s, \cdot)\int_\Omega u(s)\Big)ds
\end{aligned}
$$

for $t \in [t_0, t_0 + T)$.

Step 1. (Existence of generalized mild solution). In this step, we prove the existence of a unique generalized mild solution $u(\cdot, t; t_0, u_0)$ of (2.27).

In order to do so, fix $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$. For given $T > 0$ and $R > \|u_0\|_{C^0(\bar{\Omega})}$, let

$$
\mathcal{X}_T = C^0([t_0, t_0 + T], C^0(\bar{\Omega}))
$$

with the supremum norm $\|u\|_{\mathcal{X}_T} = \max_{t_0 \le t \le t_0 + T} \|u(t)\|_{C^0(\bar{\Omega})}$, and let

$$
\mathcal{S}_{T,R} = \left\{ u \in \mathcal{X}_T \mid \|u\|_{\mathcal{X}_T} \le R \right\}.
$$

Note that $\mathcal{S}_{T,R}$ is a closed subset of the Banach space $\mathcal{X}_T$.

First, it is not difficult to prove that, for given $u \in \mathcal{S}_{T,R}$ and $t \in [t_0, t_0+T]$, $(Gu)(t)$ is well defined, where

$$\begin{aligned}(Gu)(t) =& e^{-A(t-t_0)}u_0 - \chi \int_{t_0}^t e^{-A(t-s)}\nabla\cdot(u(s)\nabla A^{-1}u(s))ds \\ &+ \int_{t_0}^t e^{-A(t-s)}u(s)\Big(1+a_0(s,\cdot)-a_1(s,\cdot)u(s)-a_2(s,\cdot)\int_\Omega u(s)\Big)ds\end{aligned}$$

and the integrals are taken in $C^0(\bar{\Omega})$. Furthermore G is a contraction for $0 < T \ll 1$. Then, By Banach fixed point Theorem, G has a unique fixed point $u \in \mathcal{S}_{T,R}$. That means $u \in C^0([t_0, t_0+T], C^0(\bar{\Omega}))$ and

$$\begin{aligned}u(t) =& e^{-A(t-t_0)}u_0 - \chi \int_{t_0}^t e^{-A(t-s)}\nabla\cdot(u(s)\nabla A^{-1}u(s))ds \\ &+ \int_{t_0}^t e^{-A(t-s)}u(s)\Big(1+a_0(s,\cdot)-a_1(s,\cdot)u(s)-a_2(s,\cdot)\int_\Omega u(s)\Big)ds.\end{aligned}$$

Hence $u(\cdot, t; t_0, u_0) := u(t)(x)$ is a generalized mild solution of (2.27). The generalized mild solution $u(\cdot, t; t_0, u_0)$ may be prolonged by standard method into a maximal interval $[t_0, t_0 + T_{\max})$ such that if $T_{\max} < \infty$ then $\limsup_{t\nearrow T_{\max}} \|u(\cdot, t+t_0; t_0, u_0)\|_{C^0(\bar{\Omega})} = \infty$.

Step 2. (Regularity). In this step, we prove that $u(t) = u(\cdot, t; t_0, u_0)$ is a classical solution of (2.26) satisfying the properties in Theorem 2.1(2), where $u(\cdot, t; t_0, u_0))$ is obtained in Step 1. Then $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ with $v(\cdot, t; t_0, u_0) = A^{-1}u(\cdot, t; t_0, u_0)$ is a classical solution of (2.1) satisfying the properties in Theorem 2.1(2).

First, for any $0 \le \beta < \frac{1}{2}$ and σ such that $\beta + \sigma < \frac{1}{2}$, by the arguments in Step 1, $u(t)$ is locally Hölder continuous from $(t_0, t_0 + T_{\max})$ to X^β with exponent σ.

Next, fix $\frac{1}{2} < \alpha < 1$. We define the map $B(t) : (t_0, t_0 + T_{\max}) \to \mathcal{L}(X^\alpha, L^p(\Omega))$ by

$$B(t)\tilde{u} = -\chi\nabla A^{-1}u(t)\cdot\nabla\tilde{u} + \Big(a_0(t,\cdot) - \chi A^{-1}u(t) + \chi u(t) - a_1(t,\cdot)u(t) - a_2(t,\cdot)\int_\Omega u(t)\Big)\tilde{u}.$$

It is not difficult to prove that B is well defined and is Hölder continuous in t.

Finally, fix any $t_1 \in (t_0, t_0 + T_{\max})$. By [17, Theorem 7.1.3], we have that

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + B(t)\tilde{u}, \quad t \in (t_1, t_0 + T_{\max}) \\ \tilde{u}(t_1) = u(t_1) \end{cases} \tag{2.30}$$

has a unique strong solution $\tilde{u}$ which satisfy $\tilde{u}(t) \in X^\gamma$ for any $\gamma < 1$ and $t_1 < t < t_0 + T_{\max}$. By Lemma 2.3(2), $\tilde{u}$ is given by the formula

$$\begin{aligned} \tilde{u}(t) =& e^{-A(t-t_1)}u(t_1) - \chi \int_{t_1}^t e^{-A(t-s)} \nabla \cdot (\tilde{u}(s)\nabla A^{-1}u(s))ds \\ &+ \int_{t_1}^t e^{-A(t-s)}\tilde{u}(s)\Big(1 + a_0(s,\cdot) - a_1(s,\cdot)u(s) - a_2(s,\cdot)\int_\Omega u(s)\Big)ds. \end{aligned}$$

Fix $t_0 < t_1 < t_2 < t_0 + T_{\max}$. We have by Lemma 2.2 with $\beta < \frac{1}{2}$ and $\epsilon \in (0, \frac{1}{2} - \beta)$ that

$$\begin{aligned} &\|\tilde{u}(t) - u(t)\|_{C^0(\bar{\Omega})} \\ &\le C\int_{t_1}^t (t-s)^{-\frac{1}{2}-\beta-\epsilon}\|\tilde{u}(s) - u(s)\|_{C^0(\bar{\Omega})}ds + C\int_{t_1}^t (t-s)^{-\beta}\|\tilde{u}(s) - u(s)\|_{C^0(\bar{\Omega})}ds \\ &\le C\int_{t_1}^t (t-s)^{-\frac{1}{2}-\beta-\epsilon}\|\tilde{u}(s) - u(s)\|_{C^0(\bar{\Omega})}ds \end{aligned}$$

for $t_1 \le t \le t_2$ and some $C = C(\sup_{t_1 \le t \le t_2} \|u(t)\|_{C^0(\bar{\Omega})})$. Then by generalized Gronwall's inequality (see [17, page 6]), we get $\tilde{u}(t) = u(t)$ in $C^0(\bar{\Omega})$ on $[t_1, t_2]$. Letting $t_1 \to t_0$ and $t_2 \to t_0 + T_{\max}$, we have $\tilde{u}(t) = u(t) \in X^\gamma$ for any $0 \le \gamma < 1$ and $t \in (t_0, t_0 + T_{\max})$. It then follows from Theorem 2.1(1) that $u(x, t; 0, u_0) := u(t)(x)$ is a classical solution of (2.26) satisfying the properties in Theorem 2.1(2).

Step 3. (Nonnegativity and uniqueness) By the similar arguments as in Steps 3 and 4 in the proof of Theorem 2.1(1), we have that $(u(x, t; t_0, u_0), v(x, t; t_0, u_0))$ is the unique non-negative classical solution of system (2.1) satisfying Theorem 2.1(2), where $v(\cdot, t; t_0, u_0) = A^{-1}u(\cdot, t; t_0, u_0)$. □

Remark 2.5. *Let* $\{t_n\} \subset \mathbb{R}$. *Suppose that* $\lim_{n\to\infty} a_i(t + t_n, x) = \hat{a}_i(t, x)$ *locally uniformly in* $(t, x) \in \mathbb{R} \times \bar{\Omega}$. *Then* $\hat{a}_i(t, x)$ $(i = 0, 1, 2)$ *also satisfy the hypothesis (H1) in the introduction.*

Hence for any $t_0 \in \mathbb{R}$ and $u_0 \in X^\alpha$ or $u_0 \in C^0(\bar\Omega)$, system (2.1) *with $a_i(t,x)$ being replaced by $\hat a_i(t,x)$ $(i=0,1,2)$ has also a unique solution $(\hat u(x,t;t_0,u_0), \hat v(x,t;t_0,u_0))$ satisfying the properties in Theorem 2.1(1) or (2).*

The following corollary follows directly from Theorem 2.1 and its proof.

Corollary 2.1. *(1) Let $t_0 \in \mathbb{R}$ and $u_0 \in X^\alpha$ or $C^0(\bar\Omega)$ be given and let $(u(x,t;t_0,u_0)$, $v(x,t;t_0,u_0))$ be the unique solution of system* (2.1) *with initial condition $u(\cdot,t_0;t_0,u_0) = u_0(\cdot)$ in Theorem 2.1(1) or (2). For any $t_0 < t_1 < t_2 < t_0 + T_{\max}$, there holds*

$$(u(x,t_2;t_0,u_0), v(x,t_2;t_0,u_0)) = (u(x,t_2;t_1,u(\cdot,t_1;t_0,u_0)), v(x,t_2;t_1,u(\cdot,t_1;t_0,u_0)).$$

(2) Let $(u(x,t;t_0,u_0)$, $v(x,t;t_0,u_0))$ be the unique solution of system (2.1) *with initial condition $u(\cdot,t_0;t_0,u_0) = u_0(\cdot) \in X$ in Theorem 2.1(1) or (2), where $X = X^\alpha$ or $C^0(\bar\Omega)$. Then $\mathbb{R} \times X \ni (t_0,u_0) \mapsto (u(\cdot,t;t_0,u_0), v(\cdot,t;t_0,u_0)) \in X \times X$ is continuous locally uniformly with respect to $t \in (t_0, t_0+T_{\max})$.*

(3) Let $\{t_n\} \subset \mathbb{R}$. Suppose that $\lim_{n\to\infty} a_i(t+t_n,x) = \hat a_i(t,x)$ locally uniformly in $(t,x) \in \mathbb{R} \times \bar\Omega$. For given $t_0 \in \mathbb{R}$ and $u_0 \in X^\alpha$ or $C^0(\bar\Omega)$, let $(u_n(x,t;t_0,u_0), v_n(x,t;t_0,u_0))$ be the solution of system (2.1) *with $a_i(t,x)$ being replaced by $a_i(t+t_n,x)$ $(i=0,1,2)$ and with initial condition $u_n(\cdot,t_0;t_0,u_0) = u_0(\cdot)$ and $(\hat u(x,t;t_0,u_0), \hat v(x,t,;t_0,u_0))$ be the solution of system* (2.1) *on $(t_0, t_0+\hat T_{\max})$ with $a_i(t,x)$ being replaced by $\hat a_i(t,x)$ $(i = 0,1,2)$ and with initial condition $\hat u(\cdot,t_0;t_0,u_0) = u_0(\cdot)$. Then for any $t \in (t_0, t_0+\hat T_{\max})$,*

$$\lim_{n\to\infty} (u_n(\cdot,t;t_0), v_n(\cdot,t;t_0,u_0)) = (\hat u(\cdot,t;t_0,u_0), \hat v(\cdot,t;t_0,u_0)) \quad \text{in} \quad C^0(\bar\Omega).$$

2.5 Global Existence and Uniform Boundedness of Classical Solutions

In this section, we investigate the global existence and the uniform boundedness of classical solutions of system (2.1) with given initial functions and prove Theorem 2.2. We first prove two important lemmas.

Consider the following Lotka-Volterra Competition system of ordinary differential equations,

$$\begin{cases} \overline{u}' = \chi\overline{u}(\overline{u}-\underline{u}) + \overline{u}\left[a_{0,\sup}(t) - a_{1,\inf}(t)\overline{u} - |\Omega|\Big(a_{2,\inf}(t)\Big)_{+}\underline{u} + |\Omega|\Big(a_{2,\inf}(t)\Big)_{-}\overline{u}\right] \\ \underline{u}' = \chi\underline{u}(\underline{u}-\overline{u}) + \underline{u}\left[a_{0,\inf}(t) - a_{1,\sup}(t)\underline{u} - |\Omega|\Big(a_{2,\sup}(t)\Big)_{+}\overline{u} + |\Omega|\Big(a_{2,\sup}(t)\Big)_{-}\underline{u}\right]. \end{cases} \tag{2.31}$$

For given $u_0 \in C^0(\bar{\Omega})$ with $u_0(x) \geq 0$ and $t_0 \in \mathbb{R}$, let $\overline{u}_0 = \max_{x\in\bar{\Omega}} u_0(x)$, $\underline{u}_0 = \min_{x\in\bar{\Omega}} u_0(x)$ and

$$(\overline{u}(t), \underline{u}(t)) = (\overline{u}(t; t_0, \overline{u}_0, \underline{u}_0), \underline{u}(t; t_0, \overline{u}_0, \underline{u}_0)) \tag{2.32}$$

be the solution of (2.31) with $(\overline{u}(t_0; t_0, \overline{u}_0, \underline{u}_0), \underline{u}(t_0; t_0, \overline{u}_0, \underline{u}_0)) = (\overline{u}_0, \underline{u}_0)$.

Lemma 2.7. *Suppose* $\inf_{t\geq t_0}\left\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_{-}\right\} > \chi$. *Then* $(\bar{u}(t), \underline{u}(t))$ *exists for all* $t > t_0$ *and*

$$0 \leq \underline{u}(t) \leq \overline{u}(t) \quad \forall t \geq t_0. \tag{2.33}$$

Moreover, $0 \leq \overline{u}(t) \leq \max\left\{\overline{u}_0, \frac{a_{0,\sup}}{\inf_{t\geq t_0}\left\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_{-} - \chi\right\}}\right\}$.

Proof. First, note that

$$\inf_{t\geq t_0}\left\{a_{1,\sup}(t) - |\Omega|\Big(a_{2,\sup}(t)\Big)_{-}\right\} \geq \inf_{t\in\mathbb{R}}\left\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_{-}\right\} > \chi.$$

The existence of $(\bar{u}(t), \underline{u}(t))$ for all $t > t_0$ is then clear. For any $\epsilon > 0$, let $\overline{u}_0^\epsilon = \overline{u}_0 + \epsilon$ and $a_{0,\sup}^\epsilon(t) = a_{0,\sup}^\epsilon(t) + \epsilon$. Let

$$(\overline{u}^\epsilon(t), \underline{u}^\epsilon(t)) = (\overline{u}^\epsilon(t; t_0, \overline{u}_0^\epsilon, \underline{u}_0), \underline{u}^\epsilon(t; t_0, \overline{u}_0^\epsilon, \underline{u}_0)),$$

where $(\overline{u}^\epsilon(t; t_0, \overline{u}_0^\epsilon, \underline{u}_0), \underline{u}^\epsilon(t; t_0, \overline{u}_0^\epsilon, \underline{u}_0))$ is the solution of (2.31) with $a_{0,\sup}(t)$ being replaced by $a_{0,\sup}^\epsilon(t)$ and $(\overline{u}^\epsilon(t_0; t_0, \overline{u}_0^\epsilon, \underline{u}_0), \underline{u}^\epsilon(t_0; t_0, \overline{u}_0^\epsilon, \underline{u}_0)) = (\overline{u}_0^\epsilon, \underline{u}_0)$. We claim that $0 \leq \underline{u}^\epsilon(t) \leq \overline{u}^\epsilon(t)$ for all $t \geq t_0$. Suppose by contradiction that this claim does not hold. Then since $0 \leq \underline{u}_0 < \overline{u}_0^\epsilon$, there exist $\bar{t} \in (t_0, \infty)$ such that

$$\underline{u}^\epsilon(t) < \overline{u}^\epsilon(t), \ \forall t \in [t_0, \bar{t}) \ \text{ and } \ \underline{u}(\bar{t}) = \overline{u}^\epsilon(\bar{t}).$$

Thus $(\overline{u}^\epsilon - \underline{u}^\epsilon)'(\overline{t}) \le 0$. Note that $\overline{u}^\epsilon(t) > 0$ for $t \ge t_0$. Using (2.31) at $t = \overline{t}$, we get

$$\begin{aligned}(\overline{u}^\epsilon - \underline{u})'(\overline{t}) =& \overline{u}^\epsilon(\overline{t})\Big[a^\epsilon_{0,\sup}(\overline{t}) - a_{0,\inf}(\overline{t}) \\ & + \{a_{1,\sup}(\overline{t}) - a_{1,\inf}(\overline{t}) + |\Omega|(a_{2,\sup}(\overline{t}) - a_{2,\inf}(\overline{t}))\}\overline{u}^\epsilon(\overline{t})\Big].\end{aligned}$$

It then follows that $(\overline{u}^\epsilon - \underline{u})'(\overline{t}) \ge 0$, which implies that $(\overline{u}^\epsilon - \underline{u})'(\overline{t}) = 0$ and then

$$0 = a^\epsilon_{0,\sup}(\overline{t}) - a_{0,\inf}(\overline{t}) + \{a_{1,\sup}(\overline{t}) - a_{1,\inf}(\overline{t}) + |\Omega|(a_{2,\sup}(\overline{t}) - a_{2,\inf}(\overline{t}))\}\overline{u}^\epsilon(\overline{t}) > 0,$$

which is a contradiction. Thus the claim holds. Letting $\epsilon \to 0$ and using continuity of solutions of (2.31) with respect to initial data and coefficients, (2.33) follows.

Furthermore, we have

$$\begin{aligned}\overline{u}' &= \chi\overline{u}(\overline{u} - \underline{u}) + \overline{u}\left[a_{0,\sup}(t) - a_{1,\inf}(t)\overline{u} - |\Omega|\Big(a_{2,\inf}(t)\Big)_+\underline{u} + |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}\right] \\ &\le \overline{u}\Big[a_{0,\sup}(t) - \Big\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_- - \chi\Big\}\overline{u}\Big].\end{aligned}$$

Thus if $\inf_{t\ge t_0}\Big\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\Big\} > \chi$, by comparison principle, we have

$$0 < \overline{u}(t) \le \max\Big\{\overline{u}_0, \frac{a_{0,\sup}}{\inf_{t\ge t_0}\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_- - \chi\}}\Big\}.$$

□

Lemma 2.8. *Suppose* $\inf_{t\in\mathbb{R}}\Big\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\Big\} > 0$. *Then*

$$0 \le \int_\Omega u(t) \le \max\Big\{\int_\Omega u_0(x), \frac{|\Omega|a_{0,\sup}}{\inf_{t\in\mathbb{R}}\{a_{1,\inf}(t) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\}}\Big\} := M_0(\|u_0\|_{L^1}, a_i, |\Omega|)$$

for all $t \in [t_0, t_0 + T_{\max})$, *where* $u(t) = u(\cdot, t; t_0, u_0)$ *and* $u_0 \in C^0(\bar{\Omega})$ *with* $u_0(x) \ge 0$.

Proof. By integrating the first equation of system (2.1) over Ω, we get for any $t \in [t_0, t_0+T_{\max})$ that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(t) &= \int_\Omega u(t)\Big\{a_0(t,x) - a_1(t,x)u(t) - a_2(t,x)\int_\Omega u(t)\Big\} \\
&\le \int_\Omega u(t)\Big\{a_{0,\sup} - a_{1,\inf}(t)u(t) - \big(a_{2,\inf}(t)\big)_+\int_\Omega u(t) + \big(a_{2,\inf}(t)\big)_-\int_\Omega u(t)\Big\} \\
&\le \int_\Omega u(t)\Big\{a_{0,\sup} - \frac{1}{|\Omega|}\Big[a_{1,\inf}(t) - |\Omega|\big(a_{2,\inf}(t)\big)_-\Big]\int_\Omega u(t)\Big\}
\end{aligned}$$

Thus if $\inf_{t\in\mathbb{R}}\Big\{a_{1,\inf}(t) - |\Omega|\big(a_{2,\inf}(t)\big)_-\Big\} > 0$, we get by comparison principle for ODEs that

$$0 \le \int_\Omega u(t) \le \max\Big\{\int_\Omega u_0(x), \frac{|\Omega| a_{0,\sup}}{\inf_{t\in\mathbb{R}}\big\{a_{1,\inf}(t) - |\Omega|\big(a_{2,\inf}(t)\big)_-\big\}}\Big\}.$$

□

We now prove Theorem 2.2.

Proof of Theorem 2.2. (1) Let $(\bar{u}(t), \underline{u}(t))$ be as in (2.32). It suffices to prove that $0 \le \underline{u}(t) \le u(x,t;t_0,u_0) \le \overline{u}(t)$ for all $t_0 \le t < t_0 + T_{\max}$ and $x \in \bar{\Omega}$.

Observe that for any $\epsilon > 0$, there exists $t_0 < t_\epsilon < t_0 + T_{\max}$ such that

$$\underline{u}(t) - 2\epsilon < u(x,t;t_0,u_0) < \overline{u}(t) + 2\epsilon, \quad \text{for all } (x,t) \in \Omega \times [t_0, t_\epsilon).$$

Let

$$T_\epsilon = \sup\{t_\epsilon \in (t_0, t_0+T_{\max}) \,|\, \underline{u}(t) - 2\epsilon < u(x,t;t_0,u_0) < \overline{u}(t) + 2\epsilon \quad \forall\ (x,t) \in \Omega \times [t_0, t_\epsilon)\}.$$

It then suffices to prove that $T_\epsilon = t_0 + T_{\max}$.

Assume by contradiction that $T_\epsilon < t_0 + T_{\max}$. Then there is $x_0 \in \bar{\Omega}$ such that

$$\text{either } \ u(x_0, T_\epsilon; t_0, u_0) = \underline{u}(T_\epsilon) - 2\epsilon \ \text{ or } \ u(x_0, T_\epsilon; t_0, u_0) = \overline{u}(T_\epsilon) + 2\epsilon.$$

Let $\overline{U}(x,t) = u(x,t;t_0,u_0) - \overline{u}(t)$ and $\underline{U}(x,t) = u(x,t;t_0,u_0) - \underline{u}(t)$.

Note that for $t \in (t_0, t_0 + T_{\max})$, $\overline{U}$ satisfies

$$\overline{U}_t - \Delta\overline{U} \le -\chi\nabla\overline{U}\cdot\nabla v + \overline{U}\left[a_{0,\sup}(t) - \Big(a_{1,\inf}(t)-\chi\Big)(u+\overline{u}) - \chi\underline{u}\right]$$
$$- \chi u(v-\underline{u}) - a_{2,\inf}(t)(\int_\Omega u)u + |\Omega|\Big(a_{2,\inf}(t)\Big)_+\underline{u}\overline{u} - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}^2.$$

By $-a_{2,\inf}(t)(\int_\Omega u)u = -\Big(a_{2,\inf}(t)\Big)_+(\int_\Omega u)u + \Big(a_{2,\inf}(t)\Big)_-(\int_\Omega u)u$, we get for $t \in (t_0, t_0 + T_{\max})$ that

$$\begin{aligned}
&\overline{U}_t - \Delta\overline{U} \le \\
&-\chi\nabla\overline{U}\cdot\nabla v + \overline{U}\left[a_{0,\sup}(t) - \Big(a_{1,\inf}(t)-\chi\Big)(u+\overline{u}) - \chi\underline{u}\right] \\
&-\chi u(v-\underline{u}) - \Big(a_{2,\inf}(t)\Big)_+\Big((\int_\Omega u)u - |\Omega|\underline{u}\overline{u}\Big) + \Big(a_{2,\inf}(t)\Big)_-(\int_\Omega u)u - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}^2 \\
&\le -\chi\nabla\overline{U}\cdot\nabla v + \overline{U}\left[a_{0,\sup}(t) - \Big(a_{1,\inf}(t)-\chi\Big)(u+\overline{u}) - \chi\underline{u}\right] \\
&-\chi u(v-\underline{u}) - \Big(a_{2,\inf}(t)\Big)_+\Big((\int_\Omega u)u - |\Omega|\underline{u}\overline{u}\Big) \\
&+\Big(a_{2,\inf}(t)\Big)_-\Big(u\int_\Omega(u-\overline{u})\Big) - |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}(\overline{u}-u) \\
&\le -\chi\nabla\overline{U}\cdot\nabla v + \overline{U}\left[a_{0,\sup}(t) - \Big(a_{1,\inf}(t)-\chi\Big)(u+\overline{u}) - \chi\underline{u} + |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}\right] \\
&-\chi u(v-\underline{u}) - \Big(a_{2,\inf}(t)\Big)_+\Big((\int_\Omega u)u - |\Omega|\underline{u}\overline{u}\Big) + \Big(a_{2,\inf}(t)\Big)_-(\int_\Omega\overline{U})u. \qquad (2.34)
\end{aligned}$$

We claim that $\int_\Omega \overline{U}_+^2(x,t)dx$ is weakly differentiable in t and moreover

$$\frac{d}{dt}\int_\Omega \overline{U}_+^2(x,t)dx = 2\int_\Omega \overline{U}_+(x,t)\bar{U}_t(x,t)dx \quad \text{for} \quad a.e.\, t\in(t_0,t_0+T_{\max}), \qquad (2.35)$$

and

$$\int_\Omega \overline{U}_+^2(x,t)dx = \int_\Omega \overline{U}_+^2(x,t_0)dx + \int_{t_0}^t\Big(\frac{d}{dt}\int_\Omega \overline{U}_+^2(x,\tau)dx\Big)d\tau \quad \forall\, t\in(t_0,t_0+T_{\max}). \quad (2.36)$$

In order to prove the claim we define for $r > 0$,

$$F_r(z) = \begin{cases} (z^2+r)^{\frac{1}{2}} - r, & \text{if } z > 0; \\ 0, & \text{if } z \le 0. \end{cases}$$

Then $F_r \in C^1(\mathbb{R})$,

$$F_r^{'}(z) = \begin{cases} z(z^2+r)^{-\frac{1}{2}}, & \text{if } z > 0; \\ 0, & \text{if } z \leq 0. \end{cases}$$

Note that $|{F_r}^{'}| \leq 1$ and that we have the following pointwise convergence,

$$\overline{U}_+(x,t) = \lim_{r \to 0} F_r(\overline{U}(x,t)).$$

This implies that

$$\int_\Omega \overline{U}_+^2(x,t)dx = \lim_{r\to 0} \int_\Omega F_r^2(\overline{U}(x,t))dx \quad \forall\ t \in (t_0, t_0 + T_{\max}). \tag{2.37}$$

Note also that $\int_\Omega F_r^2(\overline{U}(x,t))dx$ is differentiable in t and

$$\frac{d}{dt}\int_\Omega F_r^2(\overline{U}(x,t))dx = 2\int_\Omega \big((\overline{U}_+^2(x,t)+r)^{\frac{1}{2}} - r\big)\overline{U}_+(x,t)(\overline{U}_+^2(x,t)+r)^{-\frac{1}{2}}\bar{U}_t(x,t)dx. \tag{2.38}$$

By (2.38), for any $\delta > 0$, there is $M_\delta > 0$ such that for any $r > 0$

$$\Big|\int_\Omega F_r^2(\overline{U}(x,t_1))dx - \int_\Omega F_r^2(\overline{U}(x,t_2))dx\Big| \leq M_\delta|t_1 - t_2| \quad \forall\ t_1, t_2 \in [t_0+\delta, t_0+T_{\max}-\delta]. \tag{2.39}$$

Then by (2.37) and (2.39), we have

$$\Big|\int_\Omega \overline{U}_+^2(x,t_1)dx - \int_\Omega \overline{U}_+^2(x,t_2)dx\Big| \leq M_\delta|t_1 - t_2| \quad \forall\ t_1, t_2 \in [t_0+\delta, t_0+T_{\max}-\delta]. \tag{2.40}$$

Let $\phi \in C_c^\infty((t_0, t_0 + T_{\max}))$. We have by integration by part that

$$\int_{t_0}^{T_{\max}} \frac{d}{dt}\left(\int_\Omega F_r(\overline{U}(x,t))^2 dx\right)\phi(t)dt = -\int_{t_0}^{T_{\max}}\left(\int_\Omega F_r(\overline{U}(x,t))^2dx\right)\phi_t(t)dt. \tag{2.41}$$

By Lebesgue Dominated Theorem we get from (2.37) that

$$\lim_{r\to 0}\left(-\int_{t_0}^{T_{\max}}\left(\int_\Omega F_r(\overline{U}(x,t))^2dx\right)\phi_t(t)dt\right) = -\int_{t_0}^{T_{\max}}\int_\Omega (\overline{U}_+(x,t))^2dx\phi_t(t)dt,$$

and from (2.38) that

$$\lim_{r\to 0}\int_{t_0}^{T_{\max}}\frac{d}{dt}\left(\int_\Omega F_r(\overline{U}(x,t))^2dx\right)\phi(t)dt = 2\int_{t_0}^{T_{\max}}\int_\Omega \overline{U}_+(x,t)\bar{U}_t(x,t)dx\phi(t)dt.$$

Thus it follows from equations (2.41) that

$$\int_{t_0}^{T_{\max}}\int_\Omega (\overline{U}_+(x,t))^2dx\phi_t(t)dt = -2\int_{t_0}^{T_{\max}}\int_\Omega \overline{U}_+(x,t)\bar{U}_t(x,t)dx\phi(t)dt.$$

This implies that $\int_\Omega \overline{U}_+^2(x,t)dx$ is weakly differentiable and (2.35) holds. By (2.35), (2.40), and the Fundamental Theorem of Calculus for Lebesgue Integrals, we have for any $t, t_1 \in (t_0, t_0+T_{\max})$ that

$$\int_\Omega \overline{U}_+^2(x,t)dx = \int_\Omega \overline{U}_+^2(x,t_1)dx + \int_{t_1}^{t}\Big(\frac{d}{dt}\int_\Omega \overline{U}_+^2(x,\tau)dx\Big)d\tau. \tag{2.42}$$

Letting $t_1 \to t_0$, (2.36) follows.

By (2.35), multiplying (2.34) by $\overline{U}_+$ and integrating with respect to x over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_+)^2 + \int_\Omega|\nabla(\overline{U}_+)|^2\\
&\le \int_\Omega(\overline{U}_+)^2\left[a_{0,\sup}(t) + \chi\frac{1}{2}u - \frac{1}{2}\chi v - \Big(a_{1,\inf}(t)-\chi\Big)(u+\overline{u}) - \chi\underline{u} + |\Omega|\Big(a_{2,\inf}(t)\Big)_-\overline{u}\right]\\
&\quad -\chi\int_\Omega(\overline{U}_+)u(v-\underline{u}) - \Big(a_{2,\inf}(t)\Big)_+\int_\Omega(\overline{U}_+)\left[(\int_\Omega u)u - |\Omega|\underline{u}\overline{u}\right]\\
&\quad +\Big(a_{2,\inf}(t)\Big)_-\int_\Omega \overline{U}_+(u\int_\Omega\overline{U})
\end{aligned}$$

for a.e. $t\in(t_0,t_0+T_{\max})$. Note that

$$\Big(a_{2,\inf}(t)\Big)_-\int_\Omega\overline{U}_+(u\int_\Omega\overline{U}) \le \Big(a_{2,\inf}(t)\Big)_-\int_\Omega\overline{U}_+u(\int_\Omega\overline{U}_+) \le |\Omega|\Big(a_{2,\inf}(t)\Big)_-(\overline{u}+2\epsilon)\int_\Omega\overline{U}_+^2$$

and

$$
\begin{aligned}
&-\Big(a_{2,\inf}(t)\Big)_+\int_\Omega \overline{U}_+\Big((\int_\Omega u)u-|\Omega|\underline{u}\overline{u}\Big)\\
=&-\Big(a_{2,\inf}(t)\Big)_+\int_\Omega \overline{U}_+^2(\int_\Omega u)-\Big(a_{2,\inf}(t)\Big)_+\int_\Omega \overline{U}_+\overline{u}\int_\Omega \underline{U}\\
\le&-\Big(a_{2,\inf}(t)\Big)_+\int_\Omega \overline{U}_+\overline{u}\int_\Omega \underline{U}\le \Big(a_{2,\inf}(t)\Big)_+\overline{u}\int_\Omega \overline{U}_+\int_\Omega \underline{U}_-\\
\le&|\Omega|\Big(a_{2,\inf}(t)\Big)_+\frac{\overline{u}}{2}\Big(\int_\Omega \overline{U}_+^2+\int_\Omega \underline{U}_-^2\Big).
\end{aligned}
$$

Moreover by using the second equation of system (2.1), we get

$$
\int_\Omega |\nabla(v-\underline{u})_-|^2+\int_\Omega (v-\underline{u})_-^2=-\int_\Omega(\underline{U})(v-\underline{u})_-\le\int_\Omega(\underline{U})_-(v-\underline{u})_-.
$$

Thus by Young's inequality, we have $\int_\Omega(v-\underline{u})_-^2\le\int_\Omega(\underline{U}_-)^2$. Therefore

$$
-\chi\int_\Omega(\overline{U}_+)u(v-\underline{u})\le\frac{\chi(\overline{u}+2\epsilon)}{2}\Big(\int_\Omega(\overline{U}_+)^2+\int_\Omega(\underline{U}_-)^2\Big).
$$

Combining all these inequalities, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_+)^2+\int_\Omega|\nabla(\overline{U}_+)|^2\\
\le&\int_\Omega(\overline{U}_+)^2\left[a_{0,\sup}(t)+\chi\frac{1}{2}u+2|\Omega|\Big(a_{2,\inf}(t)\Big)_-(\overline{u}+\epsilon)+|\Omega|\Big(a_{2,\inf}(t)\Big)_+\frac{\overline{u}}{2}+\frac{\chi(\overline{u}+2\epsilon)}{2}\right]\\
&+\left[\frac{\chi(\overline{u}+2\epsilon)}{2}+|\Omega|\Big(a_{2,\inf}(t)\Big)_+\frac{\overline{u}}{2}\right]\int_\Omega(\underline{U}_-)^2\\
\le&\int_\Omega(\overline{U}_+)^2\left[a_{0,\sup}(t)+\Big(2|\Omega|\big(a_{2,\inf}(t)\big)_-+\chi\Big)(\overline{u}+2\epsilon)+|\Omega|\Big(a_{2,\inf}(t)\Big)_+\frac{\overline{u}}{2}\right]\\
&+\left[\frac{\chi(\overline{u}+2\epsilon)}{2}+|\Omega|\Big(a_{2,\inf}(t)\Big)_+\frac{\overline{u}}{2}\right]\int_\Omega(\underline{U}_-)^2\quad\text{for } a.e.\ t\in(t_0,T_\epsilon].
\end{aligned}
\tag{2.43}
$$

Similarly, we have that $\int_\Omega \underline{U}_-^2(x,t)dx$ is weakly differentiable in t and moreover

$$
\frac{d}{dt}\int_\Omega \underline{U}_-^2(x,t)dx=2\int_\Omega \underline{U}_-(x,t)\underline{U}_t(x,t)dx\quad\text{for}\quad a.e.\,t\in(t_0,t_0+T_{\max}),\tag{2.44}
$$

and

$$
\int_\Omega \underline{U}_-^2(x,t)dx=\int_\Omega \underline{U}_-^2(x,t_0)dx+\int_{t_0}^t\Big(\frac{d}{dt}\int_\Omega \underline{U}_-^2(x,\tau)dx\Big)d\tau\quad\forall\ t\in(t_0,t_0+T_{\max}).\tag{2.45}
$$

Also we have

$$\underline{U}_t - \Delta \underline{U} \geq -\chi \nabla \overline{U} \cdot \nabla v + \overline{U}\left[a_{0,\inf}(t) - \Big(a_{1,\sup}(t) - \chi\Big)(u + \underline{u}) - \chi \overline{u} + |\Omega|\Big(a_{2,\sup}(t)\Big)_{-}\underline{u}\right]$$
$$- \chi u(v - \overline{u}) - \Big(a_{2,\sup}(t)\Big)_{+}\Big((\int_\Omega u)u - |\Omega|\underline{u}\overline{u}\Big) + \Big(a_{2,\sup}(t)\Big)_{-}(\int_\Omega \underline{U})u.$$

By multiplying the above inequality by $-\underline{U}_-$ and integrating with respect to x over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\underline{U}_-^2) + \int_\Omega |\nabla(\underline{U}_-)|^2 \\
&\leq \int_\Omega (\underline{U}_-)^2\left[a_{0,\inf}(t) + (2|\Omega|\Big(a_{2,\sup}(t)\Big)_{-} + \chi)(\overline{u} + 2\epsilon) + |\Omega|\Big(a_{2,\sup}(t)\Big)_{+}\frac{u}{2}\right] \\
&\quad + \left[\frac{\chi(\overline{u} + 2\epsilon)}{2} + |\Omega|\Big(a_{2,\sup}(t)\Big)_{+}\frac{u}{2}\right]\int_\Omega (\bar{U}_+)^2 \quad \text{for} \quad a.e.\ t \in (t_0, T_\epsilon].
\end{aligned} \tag{2.46}$$

By (2.36), (2.43), (2.45), and (2.46), we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big(\overline{U}_+^2(x,t) + \underline{U}_-^2(x,t)\Big)dx \\
&\leq \frac{1}{2}\int_\Omega \Big(\overline{U}_+^2(x,t_0) + \underline{U}_-^2(x,t_0)\Big)dx \\
&\quad + \int_{t_0}^t \int_\Omega \Big(\overline{U}_+^2(x,\tau) + \underline{U}_-^2(x,\tau)\Big)\left[a_{0,\sup}(\tau) + \Big(2|\Omega|\big(a_{2,\inf}(\tau)\big)_{-} + \chi\Big)(\overline{u} + 2\epsilon)\right]d\tau \\
&\quad + \int_{t_0}^t \int_\Omega \Big(\overline{U}_+^2(x,\tau) + \underline{U}_-^2(x,\tau)\Big)\left[|\Omega|\Big(a_{2,\sup}(\tau)\Big)_{+}\frac{\overline{u}}{2}\right]d\tau \\
&\quad + \int_{t_0}^t \left[\frac{\chi(\overline{u} + 2\epsilon)}{2} + |\Omega|\Big(a_{2,\sup}(\tau)\Big)_{+}\frac{\overline{u}}{2}\right]\int_\Omega \Big(\overline{U}_+^2(x,\tau) + \underline{U}_-^2(x,\tau)\Big)d\tau \quad \forall t \in (t_0, T_\epsilon].
\end{aligned}$$

This together with $\overline{U}_+(\cdot,t_0) = \underline{U}_-(\cdot,t_0) = 0$ and Gronwall's inequality implies $\overline{U}_+(x,t) = \underline{U}_-(x,t) = 0$ for $(x,t) \in \Omega \times [t_0, T_\epsilon]$. Therefore,

$$\underline{u}(t) \leq u(x,t;t_0,u_0) \leq \overline{u}(t) \quad (x,t) \in \overline{\Omega} \times [t_0, T_\epsilon].$$

This is a contradiction. Therefore, $T_\epsilon = t_0 + T_{\max}$. We then have $T_{\max} = \infty$ and (2.10) holds.

(2) We divide the proof in three steps. Note that the statements in these steps have already been establish in the case of constant coefficients and $a_2 = 0$, by Tello and Winkler in [59,

Lemma 2.2, 2.3 and 2.4]. For simplicity in notation, we denote $(u(\cdot,t;t_0,u_0),v(\cdot,t;t_0,u_0))$ by $(u(t),v(t))$.

Step 1. In this step, we prove that for any $\gamma \in \left(1, \frac{\chi}{\left(\chi-a_{1,\inf}\right)_+}\right)$, there is $C = C(\gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|)$ such that

$$\int_\Omega u^\gamma(t) \le C \quad \forall\ t \in [t_0, t_0+T_{\max}), \tag{2.47}$$

and

$$\int_{t_0}^T \int_\Omega u^{\gamma+1}(t) + \int_{t_0}^T \int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \le \widetilde{C}(T+1) \quad \forall T \in (t_0, t_0+T_{\max}). \tag{2.48}$$

By multiplying the first equation of system (2.1) by $u^{\gamma-1}(t)$ and integrating with respect to x over Ω, we have for $t \in (t_0, t_0+T_{\max})$ that

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) + \frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 =& (\gamma-1)\chi\int_\Omega u^{\gamma-1}(t)\nabla u(t)\cdot\nabla v(t)\\ &+\int_\Omega u^\gamma(t)\Big[a_0(t,\cdot)-a_1(t,\cdot)u(t)-a_2(t,\cdot)\int_\Omega u(t)\Big].\end{aligned}$$

By multiplying the second equation of system (2.1) by $u^\gamma(\cdot)$ and integrating over Ω, we get

$$(\gamma-1)\chi\int_\Omega u^{\gamma-1}(t)\nabla u(t)\cdot\nabla v(t) = -\frac{\chi(\gamma-1)}{\gamma}\int_\Omega v(t)u^\gamma(t) + \frac{\chi(\gamma-1)}{\gamma}\int_\Omega u^{\gamma+1}(t).$$

Thus we have for $t \in (t_0, t_0+T_{\max})$ that

$$\begin{aligned}&\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) + \frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2\\ &= -\frac{\chi(\gamma-1)}{\gamma}\int_\Omega vu^\gamma(t) + \int_\Omega u^\gamma(t)\Big[a_0(t,\cdot)-a_1(t,\cdot)u(t)-a_2(t,\cdot)\int_\Omega u(t)\Big]\\ &+\frac{\chi(\gamma-1)}{\gamma}\int_\Omega u^{\gamma+1}(t).\end{aligned}$$

By Lemma 2.8, we have $\Big(a_{2,\inf}(t)\Big)_-\int_\Omega u(t) \le A_2M_0$. Therefore

$$
\begin{aligned}
&\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) + \frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \\
&\le \frac{\chi(\gamma-1)}{\gamma}\int_\Omega u^{\gamma+1}(t) + \int_\Omega u^\gamma(t)\left[a_0(t,\cdot) - a_{1,\inf}u(t) + \Big(a_{2,\inf}(t)\Big)_-\int_\Omega u(t)\right] \\
&\le \frac{\chi(\gamma-1)}{\gamma}\int_\Omega u^{\gamma+1}(t) + \int_\Omega u^\gamma(t)\left[A_0 + A_2M_0 - a_{1,\inf}u(t)\right] \\
&\le -\left[a_{1,\inf} - \frac{\chi(\gamma-1)}{\gamma}\right]\int_\Omega u^{\gamma+1}(t) + (A_0 + A_2M_0)\int_\Omega u^\gamma(t). \qquad (2.49)
\end{aligned}
$$

Note that $\mu := a_{1,\inf} - \frac{\chi(\gamma-1)}{\gamma} > 0$. By Young's inequality, we have

$$
(A_0 + A_2M_0)\int_\Omega u^\gamma(t) \le \frac{1}{2}\mu\int_\Omega u^{\gamma+1}(t) + C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|).
$$

Thus

$$
\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) + \frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \le -\frac{\mu}{2}\int_\Omega u^{\gamma+1}(t) + C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|). \qquad (2.50)
$$

This together with Hölder's inequality implies that

$$
\frac{d}{dt}\int_\Omega u^\gamma(t) \le -\frac{\mu\gamma}{|\Omega|^{\frac{1}{\gamma}}}\left(\int_\Omega u^\gamma(t)\right)^{\frac{\gamma+1}{\gamma}} + C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|).
$$

It then follows that

$$
\int_\Omega u^\gamma(t) \le \max\Big\{\int_\Omega u_0^\gamma, \Big(\frac{C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|)}{\mu}\Big)^{\frac{\gamma}{\gamma+1}}\Big\} \quad \forall t \in [t_0, t_0 + T_{\max}).
$$

Now by integrating (2.50) on (t_0, T), we get

$$
\int_{t_0}^T\int_\Omega u^{\gamma+1}(t) + \int_{t_0}^T\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \le \widetilde{C}(T+1).
$$

(2.47) and (2.48) then follow.

Step 2. In this step, we prove that for any $\gamma > 1$, there is $C = C(\gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, a_i, |\Omega|)$ such that

$$
\int_\Omega u^\gamma(t) \le C \quad \forall t \in [t_0, t_0 + T_{\max}). \qquad (2.51)
$$

Since $a_{1,\inf} > \frac{\chi(n-2)}{n}$, we get $\frac{n}{2} < \frac{\chi}{(\chi - a_{1,\inf})_+}$. Choose $\gamma_0 \in (\frac{n}{2}, \frac{\chi}{(\chi - a_{1,\inf})_+})$, then by (2.47), we have

$$\int_\Omega u^{\gamma_0}(t) \le C = C(\gamma_0, \|u_0\|_{L^{\gamma_0}}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|).$$

Let $\gamma > 1$. If $1 < \gamma \le \gamma_0$, the result follows by the continuous inclusion $L^{\gamma_0}(\Omega) \subset L^\gamma(\Omega)$. Suppose $\gamma > \gamma_0$. Let $\tilde{\mu} = 2|(a_{1,\inf} - \frac{\chi(\gamma-1)}{\gamma})| + 1 > 0$. By Young's inequality we get

$$(A_0 + A_2 M_0) \int_\Omega u^\gamma(t) \le \frac{\tilde{\mu}}{2} \int_\Omega u^{\gamma+1}(t) + C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|).$$

This together with (2.49) implies that

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) + \frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \le \tilde{\mu}\int_\Omega u^{\gamma+1}(t) + C(\gamma, A_0, A_2, a_1, \|u_0\|_{L^1}, |\Omega|).$$

Note that

$$\int_\Omega u^{\gamma_0}(t) = \|u^{\frac{\gamma}{2}}(t)\|_{L^{\frac{2\gamma_0}{\gamma}}}^{\frac{2\gamma_0}{\gamma}} \le C \quad \text{and} \quad \int_\Omega u^{\gamma+1}(t) = \|u^{\frac{\gamma}{2}}(t)\|_{L^{\frac{2(\gamma+1)}{\gamma}}}^{\frac{2(\gamma+1)}{\gamma}}.$$

By Gagliardo-Nirenberg inequality, there exists C_0 depending on the domain Ω and γ such that

$$\begin{aligned}\int_\Omega u^{\gamma+1}(t) = \|u^{\frac{\gamma}{2}}(t)\|_{L^{\frac{2(\gamma+1)}{\gamma}}}^{\frac{2(\gamma+1)}{\gamma}} &\le C_0\|\nabla u^{\frac{\gamma}{2}}(t)\|_{L^2}^{\frac{2(\gamma+1)a}{\gamma}}\|u^{\frac{\gamma}{2}}(t)\|_{L^{\frac{2\gamma_0}{\gamma}}}^{\frac{2(\gamma+1)(1-a)}{\gamma}} + C_0\|u^{\frac{\gamma}{2}}(t)\|_{L^{\frac{2\gamma_0}{\gamma}}}^{\frac{2\gamma_0}{\gamma}} \\ &\le C(\gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|)\Big(\|\nabla u^{\frac{\gamma}{2}}(t)\|_{L^2}^{\frac{2(\gamma+1)a}{\gamma}} + 1\Big),\end{aligned}$$

where $a = \frac{\frac{n\gamma}{2\gamma_0} - \frac{n\gamma}{2(\gamma+1)}}{1 + \frac{n}{2}(\frac{\gamma}{\gamma_0} - 1)}$. Since $\frac{n}{2} < \gamma_0 < \gamma$, we have $0 < a < 1$ and $2\frac{(\gamma+1)}{\gamma}a - 2 = -\frac{2 - \frac{n}{\gamma_0}}{1 + \frac{n}{2}(\frac{\gamma}{\gamma_0} - 1)} < 0$. By applying Young's Inequality, we get for any $\epsilon > 0$

$$\begin{aligned}&C(\gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|)\|\nabla u^{\frac{\gamma}{2}}(t)\|_{L^2}^{\frac{2(\gamma+1)a}{\gamma}} \\ &\le \epsilon\|\nabla u^{\frac{\gamma}{2}}(t)\|_{L^2}^2 + C(\epsilon, \gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|).\end{aligned}$$

Therefore

$$\int_\Omega u^{\gamma+1}(t) \le \epsilon\|\nabla u^{\frac{\gamma}{2}}(t)\|_{L^2}^2 + C(\epsilon, \gamma, \|u_0\|_{L^\gamma}, \|u_0\|_{L^1}, A_0, A_2, a_1, |\Omega|)$$

and then

$$-\frac{4(\gamma-1)}{\gamma^2}\int_\Omega |\nabla u^{\frac{\gamma}{2}}(t)|^2 \le -\frac{4(\gamma-1)}{\epsilon\gamma^2}\int_\Omega u^{\gamma+1}(t)+\frac{4(\gamma-1)}{\epsilon\gamma^2}C(\epsilon,\gamma,\|u_0\|_{L^\gamma},\|u_0\|_{L^1},a_i,|\Omega|).$$

It then follows that

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) \le -\Big(\frac{4(\gamma-1)}{\epsilon\gamma^2}-\tilde{\mu}\Big)\int_\Omega u^{\gamma+1}(t)+C(\epsilon,\gamma,\|u_0\|_{L^\gamma},\|u_0\|_{L^1},A_0,A_2,a_1,|\Omega|).$$

By choosing $\epsilon=\frac{4(\gamma-1)}{\gamma^2(1+\tilde{\mu})}$, we get

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma(t) \le -\int_\Omega u^{\gamma+1}(t)+C(\gamma,\|u_0\|_{L^\gamma}) \le -\frac{1}{|\Omega|^{\frac{1}{\gamma}}}\Big(\int_\Omega u^\gamma(t)\Big)^{\frac{\gamma+1}{\gamma}}+C(\gamma,\|u_0\|_{L^\gamma}).$$

This implies that

$$\int_\Omega u^\gamma(t) \le C(\gamma,\|u_0\|_{L^\gamma},\|u_0\|_{L^1},A_0,A_2,a_1,|\Omega|) \quad \forall t\in[t_0,t_0+T_{\max}).$$

(2.51) then follows.

Step 3. In this sept, we prove that there is $C=C(\|u_0\|_{L^\infty})$ such that

$$\|u(t)\|_{C^0(\bar{\Omega})}+\|v(t)\|_{C^0(\bar{\Omega})} \le C \quad \forall t\in[t_0,t_0+T_{\max}). \tag{2.52}$$

By the variation of constant formula, we have

$$u(t)=$$
$$e^{-A(t-t_0)}u_0-\chi\int_{t_0}^t e^{-(t-s)A}\nabla(u(s)\cdot\nabla v(s))ds$$
$$+\int_{t_0}^t e^{-A(t-s)}u(s)\Big[\underbrace{1+a_0(s,\cdot)-a_1(s,\cdot)u(s)-(a_2(s,\cdot))_+\int_\Omega u(s)+(a_2(s,\cdot))_-\int_\Omega u(s)}_{I_0(\cdot,s)}\Big]ds,$$

Note that $u(s)I_0(\cdot,s) \le u(s)[\underbrace{1 + A_2M_0 + a_0(\cdot,s) - a_1(s,\cdot)u(s)}_{I_1(\cdot,s)}]$ and by parabolic comparison principle, we get $\int_{t_0}^t e^{-A(t-s)}u(s)I_0(\cdot,s)ds \le \int_{t_0}^t e^{-A(t-s)}u(s)I_1(\cdot,s)ds$. Therefore

$$u(t) \le u_1(t) + u_2(t) + u_3(t),$$

where

$$u_1(t) = e^{-A(t-t_0)}u_0, \quad u_2(t) = -\chi \int_{t_0}^t e^{-(t-s)A}\nabla(u(s)\cdot\nabla v(s))ds$$

and

$$u_3(t,x) = \int_{t_0}^t e^{-A(t-s)}u(s)\left[1 + A_2M_0 + a_0(\cdot,s) - a_1(s,\cdot)u(s)\right]ds.$$

Note that there are $c_0, c_1 > 0$ such that $(1 + A_2M_0 + a_0(t,x))r - a_1(t,x)r^2 \le c_0 - c_1r^2$ for all $t \in \mathbb{R}$, $x \in \Omega$, and $r \ge 0$. We then have that

$$\|u_1(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \quad \forall\, t \in [t_0, t_0 + T_{\max})$$

and

$$u_3(t) \le C\int_{t_0}^t e^{-A(t-s)}ds \le C\int_{t_0}^t e^{-(t-s)} \le C \quad \forall\, t \in [t_0, t_0 + T_{\max}).$$

Choose $p > n$ and $\alpha \in (\frac{n}{2p}, \frac{1}{2})$. Then $X^\alpha \subset L^\infty(\Omega)$ and the inclusion is continuous (see [17] exercise 10, page 40.) Choose $\epsilon \in (0, \frac{1}{2} - \alpha)$, then we have

$$\begin{aligned}
\|u_2(t)\|_{L^\infty(\Omega)} &\le C\|A^\alpha u_2(t)\|_{L^p(\Omega)} \le C\chi\int_{t_0}^t \|A^\alpha e^{-(t-s)A}\nabla(u(s)\cdot\nabla v(s))\|_{L^p(\Omega)}ds \\
&\le C\int_{t_0}^t (t-s)^{-\alpha-\frac{1}{2}-\epsilon}e^{-\mu(t-s)}\|u(s)\cdot\nabla v(s)\|\|_{L^p(\Omega)}ds \\
&\le C\int_{t_0}^t (t-s)^{-\alpha-\frac{1}{2}-\epsilon}e^{-\mu(t-s)}\|u(s)\|_{L^{p_1}(\Omega)}\|\nabla v(s)\|_{L^{p_2}(\Omega)}ds
\end{aligned}$$

for $t \in [t_0, t_0+T_{\max})$, where $p_1 > p$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Note that $\|\nabla v(s)\|_{L^{p_2}(\Omega)} \le C\|u(s)\|_{L^{p_2}(\Omega)}$. By (2.51), we get

$$\|u_2(t)\|_{L^\infty(\Omega)} \le C(\|u_0\|_{L^\infty(\Omega)})\int_{t_0}^\infty (t-s)^{-\alpha-\frac{1}{2}-\epsilon}e^{-\mu(t-s)}ds < \infty.$$

Therefore

$$\|v(t)\|_\infty \le \|u(t)\|_{L^\infty(\Omega)} \le C(\|u_0\|_{L^\infty(\Omega)}) \quad \forall\, t \in [t_0, t_0 + T_{\max}).$$

(2.52) then follows. Theorem 2.2(2) is thus proved. □

Remark 2.6. *Assume* **(H2)**$'$. *It follows from the proof of Theorem 2.2.(2) that for any* $M \ge \frac{a_{0,\sup}}{\inf_{t\in\mathbb{R}}\left\{a_{1,\inf}(t)-|\Omega|\left(a_{2,\inf}(t)\right)_-\right\}}$*, there is a positive constant* $C = C(M)$ *depending only on* M *such that for any* $u_0 \in C^0(\bar{\Omega})$ *with* $u_0 \ge 0$ *and* $\|u_0\|_{C^0(\bar{\Omega})} \le M$, $0 \le u(\cdot, t; t_0; u_0) \le C$.

2.6 Existence of Positive Entire Solutions

In this section, we explore the existence of positive entire solutions of system (2.1) in the general case; the existence of time almost periodic, time periodic, and time independent positive solutions of system (2.1) in the case that the coefficients of system (2.1) are time almost periodic, time periodic, and time independent, respectively; and prove Theorem 2.3.

We first prove three lemmas. Throughout this section, we assume that **(H2)** holds and we let

$$M = \frac{a_{0,\sup}}{\inf_{t\in\mathbb{R}}\left\{a_{1,\inf}(t) - |\Omega|\left(a_{2,\inf}(t)\right)_-\right\} - \chi}. \tag{2.53}$$

Let $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ be the solution of system (2.1) with $u(x,t_0;t_0,u_0) = u_0(x)$ ($u_0 \in C^0(\bar{\Omega})$). By Corollary 2.1, for any $t_2 > t_1 > t_0$,

$$u(x,t_2;t_0,u_0) = u(x,t_2;t_1,u(\cdot,t_1;t_0,u_0)).$$

By Theorem 2.2, the global existence of system (2.1) holds, and for any $0 \le u_0(\cdot) \le M$,

$$0 \le u(\cdot,t;t_0,u_0) \le M \quad \text{for} \quad t \ge t_0. \tag{2.54}$$

Lemma 2.9. *Fix a* $T > 0$. *For any* $\epsilon > 0$, *there is* $\delta = \delta(T) > 0$ *such that for any given* $u_0(\cdot) \ge 0$ *with* $\sup u_0 < \delta$ *and any* $t_0 \in \mathbb{R}$, $u(x, t+t_0; t_0, u_0) < \epsilon$ *for* $0 \le t \le T$.

Proof. It follows from the continuity with respect to initial conditions. □

Fix a $T > 0$. Fix ϵ_0 such that $\epsilon_0 < \frac{a_{0,\inf}}{\chi + |\Omega| \cdot |a_{2,\sup}|}$. Let $\delta_0 = \delta$ be as in Lemma 2.9 with $\epsilon = \epsilon_0$. By Lemma 2.9, for given $0 \le u_0(x) < \delta_0$, $u(x, t + t_0; t_0, u_0) < \epsilon_0$ for $0 \le t \le T$. This implies that $v(x, t + t_0; t_0, u_0) = V(u(\cdot, t + t_0; t_0, u_0)) := A^{-1}u(\cdot, t + t_0; t_0, u_0) < \epsilon_0$ for $0 \le t \le T$.

Lemma 2.10. *For any* $t_0 \in \mathbb{R}$ *and any* $0 < u_0(x) < \min\{\delta_0, \frac{a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|)}{a_{1,\sup}}\}$ *for* $x \in \Omega$, $u(x, t + t_0; t_0, u_0) > \inf u_0$ *for* $0 < t \le T$ *and* $x \in \Omega$.

Proof. By Lemma 2.9, $V(u(\cdot, t + t_0; t_0, u_0)) < \epsilon_0$ for $0 \le t \le T$. Hence

$$
\begin{aligned}
u_t &= \Delta u - \chi \nabla u \cdot \nabla V(u) - \chi u(V(u) - u) + u(a_0(t,x) - a_1(t,x)u - a_2(t,x) \int_\Omega u) \\
&\ge \Delta u - \chi \nabla u \cdot \nabla V(u) + u(a_0(t,x) - \epsilon_0 \chi - a_1(t,x)u - a_2(t,x) \int_\Omega u) \\
&\ge \Delta u - \chi \nabla u \cdot \nabla V(u) + u(a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|) - a_{1,\sup} u).
\end{aligned}
$$

Then by comparison principle, we have

$$
u(x, t + t_0; t_0, u_0) \ge u(t; \inf u_0) \quad 0 \le t \le T
$$

where $u(t; \inf u_0)$ is the solution of the ODE

$$
\dot{u} = u(a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|) - a_{1,\sup} u). \tag{2.55}
$$

with $u(0; \inf u_0) = \inf u_0$. Note that $u(t; \inf u_0)$ increases as t increases. The lemma then follows. □

Lemma 2.11. *There is* $\delta^* = \delta^*(T) > 0$ *such that for any* $0 < \delta \le \delta^*$, $t_0 \in \mathbb{R}$, *and* $u_0(\cdot)$ *with* $\delta \le \inf u_0 \le \sup u_0 \le M$, $u(x, t_0 + T; t_0, u_0) \ge \delta$ *for* $x \in \Omega$.

Proof. We prove the lemma by contradiction. Assume that the lemma does not hold. Then there are $\delta_n \to 0$, $t_n \in \mathbb{R}$, and $u_n(\cdot)$ with $\delta_n \le \inf u_n \le M$ such that $\inf u(\cdot, t_n + T; t_n, u_n) < \delta_n$. Without loss of generality, we assume that $\delta_n < \min\{\delta_0, \frac{a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|)}{a_{1,\sup}}\}$. By Lemma

2.10, we must have $\sup u_n \ge \min\{\delta_0, \frac{a_{0,\inf}-\epsilon_0(\chi+|\Omega|\cdot|a_{2,\sup}|)}{a_{1,\sup}}\}$. Let

$$\Omega_n = \Big\{x \in \Omega \,|\, u_n(x) \ge \frac{1}{2}\min\{\delta_0, \frac{a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|)}{a_{1,\sup}}\}\Big\}.$$

Without loss of generality, we may assume that $m_0 = \lim_{n\to\infty} |\Omega_n|$ exists, where $|\Omega_n|$ is the Lebesgue measure of Ω_n. Assume that $m_0 = 0$. Then there is $\tilde{u}_n \in C^0(\bar{\Omega})$ such that

$$\delta_n \le \tilde{u}_n(x) \le \frac{1}{2}\min\{\delta_0, \frac{a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|)}{a_{1,\sup}}\}$$

and

$$\lim_{n\to\infty} \|u_n - \tilde{u}_n\|_{L^p(\Omega)} = 0 \quad \forall\, 1 \le p < \infty.$$

This implies that

$$\lim_{n\to\infty} \|u(\cdot,t;t_n,u_n) - u(\cdot,t;t_n,\tilde{u}_n)\|_{L^p(\Omega)} = 0$$

uniformly in $t \in [t_n, t_n + T]$ for all $1 \le p < \infty$. Indeed, let $G(\cdot)$ be as in the proof of Theorem 1.1(1). Then $G(u(\cdot,t;t_n,u_n))(t) = u(\cdot,t;t_n,u_n)$, $G(u(\cdot,t;t_n,\tilde{u}_n))(t) = u(\cdot,t;t_n,\tilde{u}_n)$. Let

$$\hat{G}(u_n)(t) = G(u(\cdot,t;t_n,u_n))(t), \quad \hat{G}(\tilde{u}_n)(t) = G(u(\cdot,t;t_n,\tilde{u}_n))(t),$$

$$w_n(\cdot,t) = G(u(\cdot,t;t_n,u_n))(t) - G(u(\cdot,t;t_n,\tilde{u}_n))(t)$$

and

$$W_n(\cdot,t) = V(G(u(\cdot,t;t_n,u_n))(t)) - V(G(u(\cdot,t;t_n,\tilde{u}_n))(t)).$$

Then

$$\begin{aligned}
&w_n(\cdot,t) = \\
&e^{-A(t-t_n)}\big(u_n - \tilde{u}_n\big) - \chi\int_{t_n}^t e^{-A(t-s)}\nabla\big[w_n(\cdot,s)\cdot\nabla V(\hat{G}(u_n)(s) + \hat{G}(\tilde{u}_n)(s)\cdot\nabla W_n(\cdot,s)\big]ds \\
&+ \int_{t_n}^t e^{-A(t-s)} w_n(\cdot,s)\Big(1 + a_0(s,\cdot) - a_1(s,\cdot)(\hat{G}(u_n) + \hat{G}(\tilde{u}_n))(s) - a_2(s,\cdot)\int_\Omega \hat{G}(u_n)(s)\Big)ds \\
&- \int_{t_n}^t e^{-A(t-s)} a_2(s,\cdot)\big(\int_\Omega w_n(\cdot,s)\big)\hat{G}(\tilde{u}_n)(s)ds. \qquad (2.56)
\end{aligned}$$

Now, fix $1 < p < \infty$. By regularity and a priori estimates for elliptic equations, [17, Theorem 1.4.3], Lemma 2.2, and (2.56), for any $\epsilon \in (0, \frac{1}{2})$, we have

$$
\begin{aligned}
&\|w_n(\cdot,t)\|_{L^p(\Omega)} \\
&\le \|u_n - \tilde{u}_n\|_{L^p(\Omega)} + C\chi \max_{t_n \le s \le t_n+T} \|\nabla V(\hat{G}(u_n)(s))\|_{C^0(\bar{\Omega})} \int_{t_n}^t (t-s)^{-\epsilon-\frac{1}{2}} \|w_n(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ C\chi \max_{t_n \le s \le t_n+T} \|\hat{G}(\tilde{u}_n)(s)\|_{C^0(\bar{\Omega})} \int_{t_n}^t (t-s)^{-\epsilon-\frac{1}{2}} \|w_n(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ C \int_{t_n}^t \{1 + A_0 + A_1[\max_{t_n \le s \le t_n+T} (\|\hat{G}(u_n)(s)\|_{C^0(\bar{\Omega})} + \|\hat{G}(\tilde{u}_n)(s)\|_{C^0(\bar{\Omega})})]\} \|w_n(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ C \int_{t_n}^t A_2 |\Omega| \max_{t_n \le s \le t_n+T} \|\hat{G}(u_n)(s)\|_{C^0(\bar{\Omega})} \|w_n(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ C \int_{t_n}^t A_2 \|\hat{G}(\tilde{u}_n)(s)\|_{C^0(\bar{\Omega})} \|w_n(\cdot,s)\|_{L^p(\Omega)} ds. \qquad (2.57)
\end{aligned}
$$

Therefore there exists a positive constant C_0 independent of t and n such that

$$
\|w_n(\cdot,t+t_n)\|_{L^p(\Omega)} \le \|u_n - \tilde{u}_n\|_{L^p(\Omega)} + C_0 \int_0^t (t-s)^{-\epsilon-\frac{1}{2}} \|w_n(\cdot,s+t_n)\|_{L^p(\Omega)} ds \qquad \forall t \in [0,T]. \tag{2.58}
$$

By (2.58) and the generalized Gronwall's inequality (see [17, page 6]), we get

$$
\lim_{n\to\infty} \|u(\cdot,t;t_n,u_n) - u(\cdot,t;t_n,\tilde{u}_n)\|_{L^p(\Omega)} = 0,
$$

uniformly in $t \in [t_n, t_n+T]$ for all $1 \le p < \infty$. Therefore,

$$
\lim_{n\to\infty} \|V(u(\cdot,t;t_n,u_n)) - V(u(\cdot,t;t_n;\tilde{u}_n))\|_{C^1(\bar{\Omega})} = 0
$$

uniformly in $t \in [t_n, t_n+T]$. Note that

$$
V(u(\cdot,t;t_n,\tilde{u}_n))(x) \le \epsilon_0
$$

for all $t \in [t_n, t_n+T]$ and $x \in \Omega$. It then follows that

$$
V(u(\cdot,t;t_n,u_n))(x) \le 2\epsilon_0
$$

for all $t \in [t_n, t_n+T]$, $x \in \Omega$, and $n \gg 1$. Then by the arguments of Lemma 2.10, $\inf u(\cdot, t_n + T; t_n, u_n) \ge \delta_n$, which is a contradiction. Therefore, $m_0 \ne 0$.

By $m_0 \ne 0$ and comparison principle for parabolic equations, without loss of generality, we may assume that

$$\liminf_{n\to\infty} \|e^{-At}u_n\|_{C^0(\bar\Omega)} > 0 \quad \forall\, t \in [0,T].$$

This together with the arguments in the proof of Theorem 1.1(2) implies that there is $T_0 > 0$ and $\delta_\infty > 0$ such that

$$\sup u(\cdot, t_n + T_0; t_n, u_n) \ge \delta_\infty$$

for all $n \gg 1$. By a priori estimates for parabolic equations, without loss of generality, we may assume that

$$u(\cdot, t_n + T_0; t_n, u_n) \to u_0^*, \quad u(\cdot, t_n + T; t_n, u_n) \to u^*$$

as $n \to \infty$. By (H1), without loss of generality, we may also assume that

$$a_i(t + t_n, \cdot) \to a_i^*(t,x)$$

as $n \to \infty$ locally uniformly in $(t,x) \in \mathbb{R} \times \bar\Omega$. Then by Corollary 2.1,

$$u^*(x) = u^*(x, T; T_0, u_0^*) \quad \text{and} \quad \inf u^* = 0,$$

where $(u^*(x,t;T_0,u_0^*), v^*(x,t;T_0,u_0^*))$ with $v^*(\cdot,t;T_0,u_0^*) = A^{-1}u^*(\cdot,t;T_0,u_0^*)$ is the solution of (2.1) with $a_i(t,x)$ being replaced by $a^*(t,x)$. By comparison principle, we must have $u_0^* \equiv 0$. But

$$\sup u_0^* \ge \delta_\infty.$$

This is a contradiction. $\square$

Proof of Theorem 2.3. We first prove the existence of positive entire solutions of (2.1) in the general case.

Let $\delta^* > 0$ be given by Lemma 2.11 with $T = 1$. Choose $u_0 \in C^0(\bar\Omega)$ such that $\delta^* \le u_0(x) \le M$. By Lemma 2.11 and (2.54),

$$\delta^* \le u(x, t_0 + n; t_0, u_0) \le M \quad \forall\, x \in \bar{\Omega},\ t_0 \in \mathbb{R},\, n \in \mathbb{N}. \tag{2.59}$$

Set $t_n = -n$ and define $u_n(x) = u(x, 0; t_n, u_0)$. Choose $\tilde{t}$ such that $-2 < \tilde{t} < -1$. Then there is $\tilde{M} > 0$ such that for each $n \ge 3$, we have

$$\|u_n\|_\alpha = \|u(\cdot, 0; t_n, u_0)\|_\alpha = \|u(\cdot, 0; \tilde{t}, u(\cdot, \tilde{t}; t_n, u_0))\|_\alpha \le \tilde{M}.$$

Therefore by Arzela-Ascoli Theorem, there exist n_k, $u_0^* \in C^0(\bar{\Omega})$ such that u_{n_k} converges to u_0^* in $C^0(\bar{\Omega})$ as $n_k \to \infty$. Then by Corollary 2.1, we have

$$u(\cdot, t; t_{n_k}, u_0) = u(\cdot, t; 0, u(\cdot, 0; t_{n_k}, u_0)) = u(\cdot, t; 0, u_{n_k}) \to u(\cdot, t; 0, u_0^*)$$

in $C^0(\bar{\Omega})$ as $n \to \infty$ for $t \ge 0$. Moreover, by (2.54) and Lemma 2.11,

$$\delta^* \le u(x, n; 0, u_0^*) \le M \quad \forall\, x \in \bar{\Omega},\ n \in \mathbb{N}. \tag{2.60}$$

We need to prove that $u(\cdot, t; 0, u_0^*)$ has backward extension. To see that, fix $m \in \mathbb{N}$. Then $u(\cdot, t; t_n, u_0)$ is defined for $t > -m$ and $n > m$. Observe that

$$u_n(\cdot) = u(\cdot, 0; t_n, u_0) = u(\cdot, 0; -m, u(\cdot, -m; t_n, u_0)).$$

Without loss of generality, we may assume that $u(\cdot, -m; t_{n_k}, u_0) \to u_m^*(\cdot)$ in $C^0(\bar{\Omega})$. Then

$$u(\cdot, t; t_{n_k}, u_0) = u(\cdot, t; -m, u(\cdot, -m; t_{n_k}, u_0)) \to u(\cdot, t; -m, u_m^*)$$

for $t > -m$ and $u(\cdot, t; 0, u_0^*) = u(\cdot, t; -m, u_m^*)$ for $t \ge 0$. This implies that $u^*(x, t; 0, u_0^*)$ has a backward extension up to $t = -m$. Let $m \to \infty$, we have that $u^*(x, t)$ has a backward extension on $(-\infty, 0)$.

Let $u^*(x, t) = u^*(x, t; 0, u_0^*)$ and $v^*(x, t) = A^{-1}u^*(\cdot, t)$, Then $(v^*(x, t), u^*(x, t))$ is an entire nonnegative solution of system (2.1). Moreover,

$$\delta^* \le u^*(x, n) \le M \quad \forall\, x \in \bar{\Omega},\ n \in \mathbb{Z}. \tag{2.61}$$

This implies that

$$0 < \inf_{x\in\bar{\Omega},t\in\mathbb{R}} u^*(x,t) \le M, \quad 0 < \inf_{x\in\bar{\Omega},t\in\mathbb{R}} v^*(x,t) \le M.$$

Therefore, $(v^*(x,t), u^*(x,t))$ is an entire positive bounded solution of system (2.1).

Next, we prove (1), (2), and (3).

(1) Assume that $a_i(t+T,x) = a_i(t,x)$ for $i = 0,1,2$. Let $\delta^* = \delta^*(T) > 0$ be given by Lemma 2.11 and set

$$E(T) = \{u_0 \in C^0(\bar{\Omega}) : \delta^* \le u_0 \le M\}. \tag{2.62}$$

Note that $E(T)$ is nonempty, closed, convex and bounded subset of $C^0(\bar{\Omega})$. Define the map $\mathcal{T}(T) : E(T) \to C^0(\bar{\Omega})$ by $\mathcal{T}(T)u_0 = u(\cdot, T; 0, u_0)$. Note that $\mathcal{T}(T)$ is well defined and continuous by continuity with respect to initial conditions.

Let $u_0 \in E(T)$. Then by Theorem 2.2, we have $0 < u(\cdot, T; 0, u_0) \le M$ and by Lemma 2.11, we have $u(\cdot, T; 0, u_0) \ge \delta^*$. Thus $u(\cdot, T; 0, u_0) \in E(T)$ and $\mathcal{T}(T)E(T) \subset E(T)$.

Let $\frac{n}{2p} < \alpha < \frac{1}{2}$, and $\epsilon \in (0, \frac{1}{2} - \alpha)$. By the similar arguments as those in the proof of local existence, we have that

$$\|u(\cdot, T; 0, u_0)\|_\alpha \le CMT^{-\alpha} + CM^2T^{\frac{1}{2}-\alpha-\epsilon} + CM[1 + A_0 + k_1(A_1 + |\Omega|A_2)]T^{1-\alpha}.$$

Now choose ν such that $0 \le \nu < 2\alpha - \frac{n}{p}$, then $X^\alpha \subset C^\nu(\bar{\Omega})$, where the inclusion is continuous. Thus by Arzela-Ascoli Theorem, $\mathcal{T}(T)E(T)$ is precompact. Therefore by Schauder fixed point theorem, there exists $u^T \in E(T)$ such that $\mathcal{T}(T)u^T = u^T$, i.e $u(\cdot, T; 0, u^T) = u^T(\cdot)$. Since $u(\cdot, t+T; 0, u^T) = u(\cdot, t; T, u(\cdot, T; 0, u^T)) = u(\cdot, t; 0, u^T)$, $u(\cdot, t; 0, u^T)$ is periodic with period T. Now from the facts that $u(., t; 0, u^T)$ is periodic with period T and the uniqueness of solutions of

$$\begin{cases} -\Delta v + v = u(x,t;0,u^T) & x \in \Omega \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

we get $v(\cdot, t; 0, u^T) = A^{-1}u(\cdot, t; 0, u^T)$ is periodic with period T. Then $(u(\cdot, t; 0, u^T), v(\cdot, t; 0, u^T))$ is a positive periodic solution of system (2.1).

(2) Assume that $a_i(t,x) \equiv a_i(t)$. Note that in this case, every solution of the ODE

$$u_t = u(a_0(t) - (a_1(t)u + |\Omega| a_2(t))u)$$

is a solution of the first equation of the system (2.1) with Neumann boundary. (2) then follows from Lemma 2.5.

(3) Assume that $a_i(t,x) \equiv a_i(x)$ $(i = 0, 1, 2)$. In this case, each $\tau > 0$ is a period for a_i. By (2), there exist $u^\tau \in E(\tau)$ such that $(u(\cdot, t; 0, u^\tau), A^{-1}(u(\cdot, t; 0, u^\tau)))$ is a positive periodic solution of system (2.1) with period τ. Note that there is $\tilde{M} > 0$ such that for each $\tau > 0$ and $u_0 \in E(\tau)$, $\|u(\cdot, t; 0, u_0)\|_\alpha \le \tilde{M}$ for each $1 \le t \le 2$. Let $\tau_n = \frac{1}{n}$, then there exists $u_n \in E(\tau_n)$ such that $u(\cdot, t; 0, u_n)$ is periodic with period τ_n and

$$\|u_n\|_\alpha = \|u(\cdot, \tau_n; 0, u_n)\|_\alpha = \|u(\cdot, N\tau_n; 0, u_n)\|_\alpha \le \tilde{M}, \tag{2.63}$$

where N is such that $1 \le N\tau_n \le 2$.

We claim that there is $\delta > 0$ such that

$$\|u_n(\cdot)\|_{C^0(\bar{\Omega})} \ge \delta \quad \forall\, n \ge 1. \tag{2.64}$$

Suppose by contradiction that this does not hold. Then there exists n_k such that $\|u_{n_k}\|_{C^0(\bar{\Omega})} < \frac{1}{n_k}$ for every $k \ge 1$. Let k_0 such that $\frac{1}{n_k} < \epsilon_0$ for all $k \ge k_0$. By the proof of Lemma 2.10 we get that $u(\cdot, t; 0, u_{n_k}) \ge u(t; \inf u_{n_k})$ for all $t > 0$ and $k \ge k_0$, where $u(t; \inf u_{n_k})$ is the solution of (2.55) with $u(0; \inf u_{n_k}) = \inf u_{n_k}$. Let $\delta_* = \frac{a_{0,\inf} - \epsilon_0(\chi + |\Omega| \cdot |a_{2,\sup}|)}{2a_{1,\sup}}$ and choose k large enough such that $\frac{1}{n_k} < \delta_*$. There is $t_0 > 0$ such that $u(t; \inf u_{n_k}) > \delta_*$ for all $t \ge t_0$. Then we have

$$u_{n_k}(x) = u(\cdot, m\tau_{n_k}; 0, u_{n_k}) \ge u(m\tau_{n_k}; \inf u_{n_k}) > \delta^*$$

for all $m \in \mathbb{N}$ satisfying that $m\tau_{n_k} > t_0$. This is a contradiction. Therefore, (2.64) holds.

By (2.63) and Arzela-Ascoli theorem, there exist n_k, $u^* \in C^0(\bar{\Omega})$ such that u_{n_k} converges to u^* in $C^0(\bar{\Omega})$. By (2.64), $\|u^*(\cdot)\|_{C^0(\bar{\Omega})} \ge \frac{\delta}{2}$. We claim that $(u(\cdot, t; 0, u^*), v(\cdot, t; 0, u^*))$ with $v(\cdot, t; 0, u^*) = A^{-1}u(\cdot, t; 0, u^*)$ is a steady state solution of system (2.1), that is,

$$u(\cdot,t;0,u^*) = u^*(\cdot) \quad \text{for all } t \geq 0. \tag{2.65}$$

In fact, let $\epsilon > 0$ be fix and let $t > 0$. Note that

$$[n_k t]\tau_{n_k} = \frac{[n_k t]}{n_k} \leq t \leq \frac{[n_k t]+1}{n_k} = ([n_k t]+1)\tau_{n_k}.$$

By Corollary 2.1, we can choose k large enough such that

$$|u(x,t;0,u^*) - u(x,t;0,u_{n_k})| < \epsilon, |u_{n_k}(x) - u^*(x)| < \epsilon, |u(x,\frac{[n_k t]}{n_k};0,u_{n_k}) - u(x,t;0,u_{n_k})| < \epsilon$$

for all $x \in \bar{\Omega}$. We then have

$$\begin{aligned} |u(x,t;0,u^*) - u^*| \leq& |u(x,t;0,u^*) - u(x,t;0,u_{n_k})| + |u(x,t;0,u_{n_k}) - u(x,[n_k t]\tau_{n_k};0,u_{n_k})| \\ &+ |u_{n_k}(x) - u^*(x)| < 3\epsilon \quad \forall\, x \in \bar{\Omega}. \end{aligned}$$

Letting $\epsilon \to 0$, (2.65) follows. □

Remark 2.7. *It follows from the proof of the existence of positive entire solutions in Theorem 2.3 and Remark 2.6 that the existence of positive entire solutions also holds under the weaker condition* **(H2)**$'$.

2.7 Asymptotic Stability of Positive Entire Solutions

In this section, we investigate the stability and uniqueness of positive entire solutions of system (2.1), the asymptotic behavior of global positive solutions of system (2.1), and prove Theorems 2.4 and 2.5. We first prove Theorem 2.5.

Proof of Theorem 2.5. Suppose that (2.13) holds. For given $u_0 \in C^0(\bar{\Omega})$ with $u_0(x) \geq 0$, $u_0(\cdot) \not\equiv 0$, and $t_0 \in \mathbb{R}$, let $(u(\cdot,t;t_0,u_0), v(\cdot,t;t_0,u_0))$ be the solution of system (2.1) satisfying the properties in Theorem 2.1(2). By Theorem 2.2, $(u(\cdot,t;t_0,u_0)$, $v(\cdot,t;t_0,u_0))$ exists for all $t > t_0$. Note that $u(x,t;t_0,u_0) > 0$ for all $x \in \bar{\Omega}$ and $t > t_0$. Without loss of generality, we may assume that $u_0(x) > 0$ for all $x \in \bar{\Omega}$.

Let $(\bar{u}(t), \underline{u}(t))$ be as in (2.32). By Lemma 2.7 and the proof of Theorem 2.2,

$$\underline{u}(t) \le u(x,t;t_0,u_0) \le \bar{u}(t) \quad \forall\, x \in \bar{\Omega},\ t \ge t_0. \tag{2.66}$$

Let r_1 and r_2 be as in (2.17) and (2.18), respectively.

(1) By Lemma 2.6(1) and (2), for any $\epsilon > 0$, there is $t_\epsilon > 0$ such that

$$r_2 - \epsilon \le \underline{u}(t) \le \bar{u}(t) \le r_1 + \epsilon \quad \text{for} \quad t \ge t_0 + t_\epsilon. \tag{2.67}$$

(1) then follows from (2.66) and (2.67).

(2) We first consider the case that $a_i(t,x)$ $(i = 0,1,2)$ are periodic in t with period T. By Lemma 2.6(1), (2), and (3), there are periodic functions $m(t)$ and $M(t)$ with period T such that

$$r_2 \le m(t) \le M(t) \le r_1 \quad \forall\, t \in \mathbb{R}$$

and for any $\epsilon > 0$, there is $t_\epsilon > 0$ such that

$$m(t) - \epsilon \le \underline{u}(t) \le \bar{u}(t) \le M(t) + \epsilon \quad \forall\, t \ge t_0 + t_\epsilon. \tag{2.68}$$

In this case, (2) then follows from (2.66) and (2.68).

Next, we consider the cases that $a_i(t,x)$ $(i = 0,1,2)$ are almost periodic in t. By Lemma 2.6(1), (2), and (4), there are almost periodic functions $m(t)$ and $M(t)$ such that

$$r_2 \le m(t) \le M(t) \le r_1 \quad \forall\, t \in \mathbb{R}$$

and for any $\epsilon > 0$, there is $t_\epsilon > 0$ such that (2.68) holds. (2) then follows from (2.66) and (2.68). □

We now prove Theorem 2.4

Proof of Theorem 2.4. (1) Suppose that $a_i(t,x) \equiv a_i(t)$ for $i = 0,1,2$ and

$$\inf_{t \in \mathbb{R}} \left\{ a_1(t) - |\Omega|\, |a_2(t)| \right\} > 2\chi. \tag{2.69}$$

For given $u_0 \in C^0(\bar{\Omega})$ with $u_0(x) \ge 0$, $u_0(\cdot) \ne 0$, and $t_0 \in \mathbb{R}$, let $(u(\cdot,t;t_0,u_0), v(\cdot,t;t_0,u_0))$ be the solution of system (2.1) satisfying the properties in Theorem 2.1(2). Again, by Theorem

2.2, $(u(\cdot,t;t_0,u_0),\ v(\cdot,t;t_0,u_0))$ exists for all $t > t_0$ and without loss of generality, we may assume that $u_0(x) > 0$ for all $x \in \bar{\Omega}$.

Let $(\bar{u}(t), \underline{u}(t))$ be as in (2.32). Let $(u^*(t), v^*(t))$ be the unique entire positive spatially homogeneous solution of system (2.1) in Theorem 2.3(2). By Lemma 2.7 and the proof of Theorem 2.2,

$$\underline{u}(t) \le u(x,t;t_0,u_0) \le \bar{u}(t) \quad \forall\ x \in \bar{\Omega},\ t \ge t_0. \tag{2.70}$$

By Lemma 2.6(1), for any $\epsilon > 0$, there is $t_\epsilon > 0$ such that

$$\underline{u}(t) - \epsilon \le u^*(t) \le \bar{u}(t) + \epsilon \quad \forall\ t \ge t_0 + t_\epsilon. \tag{2.71}$$

By (2.70) and (2.71), it suffices to show $0 \le \ln \frac{\overline{u}(t)}{\underline{u}(t)} \longrightarrow 0$ as $t \to \infty$. Assume that $t > t_0$. By dividing the first equation of (2.31) by $\overline{u}$, and the second by $\underline{u}$, we get

$$\begin{cases} \frac{\overline{u}'}{\overline{u}} = [a_0(t) - (a_1(t) - |\Omega|(a_2(t))_- - \chi)\overline{u} - (|\Omega|(a_2(t))_+ + \chi)\underline{u}] \\ \frac{\underline{u}'}{\underline{u}} = [a_0(t) - (a_1(t) - |\Omega|(a_2(t))_- - \chi)\underline{u} - (|\Omega|(a_2(t))_+ + \chi)\overline{u}] \end{cases}$$

This together with (2.69) implies that

$$\frac{d}{dt}\Big(\ln\frac{\overline{u}}{\underline{u}}\Big) = \frac{\overline{u}'}{\overline{u}} - \frac{\underline{u}'}{\underline{u}} = -\left(a_1(t) - |\Omega|\,|a_2(t)| - 2\chi\right)(\overline{u} - \underline{u}) \le 0.$$

Thus by integrating over (t_0, t), we get

$$0 \le \ln\frac{\overline{u}}{\underline{u}} \le \ln\frac{\overline{u}_0}{\underline{u}_0}, \quad \text{and then} \quad \frac{\overline{u}(t)}{\underline{u}(t)} \le \frac{\overline{u}_0}{\underline{u}_0}.$$

We have by mean value theorem that

$$\overline{u} - \underline{u} = e^{\ln\overline{u}} - e^{\ln\underline{u}} = e^{\ln\hat{u}}\Big(\ln\frac{\overline{u}}{\underline{u}}\Big) = \hat{u}\Big(\ln\frac{\overline{u}}{\underline{u}}\Big),$$

where $\underline{u} \le \hat{u} \le \overline{u}$. Therefore

$$\frac{d}{dt}\Big(\ln\frac{\overline{u}}{\underline{u}}\Big) \le -\Big(a_1(t) - |\Omega|\,|a_2(t)| - 2\chi\Big)\Big(\inf_{t\ge t_0}\overline{u}(t)\frac{\underline{u}_0}{\overline{u}_0}\Big)\Big(\ln\frac{\overline{u}}{\underline{u}}\Big).$$

By letting $\epsilon_0 = \inf_{t\in\mathbb{R}}\{a_1(t) - |\Omega|\,|a_2(t)| - 2\chi\}\left(\inf_{t\geq t_0}\overline{u}(t)\frac{\underline{u}_0}{\overline{u}_0}\right)$, we have $\epsilon_0 > 0$ and

$$0 \leq \ln\frac{\overline{u}}{\underline{u}} \leq \ln\frac{\overline{u}_0}{\underline{u}_0}e^{-\epsilon_0 t} \to 0 \quad \text{as} \quad t\to\infty.$$

(2) Let $L_1(t)$ and $L_2(t)$ be as in (2.15) and (2.16), respectively. By (2.14),

$$\mu = \limsup_{t-s\to\infty}\frac{1}{t-s}\int_s^t (L_1(\tau) - L_2(\tau))d\tau < 0.$$

Fix $0 < \epsilon < -\mu$. Let r_1 and r_2 be as in (2.17) and (2.18), respectively. By (2.13), Theorem 2.5(1), and definition of μ, for any $\epsilon > 0$, there exists $T_\epsilon > 0$ such that

$$r_2 - \epsilon \leq u(\cdot, t_0+t;t_0;u_0) \leq r_1+\epsilon,\;\; r_2-\epsilon \leq u^*(x,t) \leq r_1+\epsilon \;\;\forall x\in\bar{\Omega},\;\; t\geq t_0+T_\epsilon,$$

and

$$\int_{t_0}^{t_0+t}(L_1(s) - L_2(s))ds \leq (\mu+\epsilon)t,\;\forall\; t_0\in\mathbb{R},\;\; t\geq t_0+T_\epsilon.$$

We first prove that for any entire positive solution $(u^*(x,t), v^*(x,t))$ of system (2.1), (2.19) holds. To simplify the notation, for given $t_0 \in \mathbb{R}$ and $u_0 \in C^0(\bar{\Omega})$ with $u_0(x) \geq 0$ and $u_0(\cdot) \neq 0$, set $u(t) = u(\cdot, t;t_0;u_0)$ and $u^*(t) = u^*(\cdot, t)$. Let $w(t) = u(t) - u^*(t)$. Then w satisfy the equation

$$\begin{aligned} w_t =&\Delta w - \chi\nabla(w\cdot\nabla A^{-1}u) - \chi\nabla(u^*\cdot\nabla A^{-1}w) + w\left(a_0(t,x) - a_1(t,x)(u+u^*)\right.\\ &\left. - a_2(t,x)\int_\Omega u\right) - a_2(t,x)\Big(\int_\Omega w\Big)u^* \end{aligned} \tag{2.72}$$

for $t > t_0$. By the similar arguments for (2.35), we have that $\int_\Omega w_+^2$ is weakly differentiable and moreover

$$\frac{d}{dt}\int_\Omega w_+^2 = 2\int_\Omega w_+ w_t \quad \forall\; a.e.\, t > t_0.$$

Next, by multiplying (2.72) by w_+ and integrating it over Ω, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w_+^2 + \int_\Omega |\nabla w_+|^2 =& \chi\int_\Omega w_+\nabla w_+\cdot\nabla A^{-1}u + \chi\int_\Omega u^*\nabla w_+\cdot\nabla A^{-1}w\\ &+\int_\Omega w_+^2\big(a_0(t,x) - a_1(t,x)(u+u^*) - a_2(t,x)\int_\Omega u\big) - \Big(\int_\Omega w\Big)\int_\Omega a_2(t,x)u^*w_+ \end{aligned}$$

for a.e $t > t_0$. Integrating by part and using the equation of $A^{-1}u$, we get for a.e $t > t_0 + T_\epsilon$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega w_+^2 + \int_\Omega |\nabla w_+|^2 &= \frac{\chi}{2}\int_\Omega w_+^2(u - A^{-1}u) + \chi\int_\Omega u^*\nabla w_+\cdot\nabla A^{-1}w \\
&\quad + \int_\Omega w_+^2\big(a_0(t,x) - a_1(t,x)(u+u^*) - a_2(t,x)\int_\Omega u\big) \\
&\quad - \Big(\int_\Omega w\Big)\int_\Omega a_2(t,x)u^*w_+ \\
&\le \frac{\chi}{2}\int_\Omega w_+^2(u - A^{-1}u) + \chi\int_\Omega u^*\nabla w_+\cdot\nabla A^{-1}w \\
&\quad + \int_\Omega w_+^2\big(a_{0,\sup} - a_{1,\inf}(t)(u+u^*) - (a_{2,\inf}(t))_+\int_\Omega u\big) \\
&\quad + \int_\Omega w_+^2\big((a_{2,\inf}(t))_-\int_\Omega u\big) \\
&\quad - \Big(\int_\Omega w\Big)\int_\Omega a_2(t,x)u^*w_+.
\end{aligned}$$

We have by Young's inequality that

$$\chi\int_\Omega u^*\nabla w_+\cdot\nabla A^{-1}w \le \int_\Omega |\nabla w_+|^2 + \frac{(\chi(r_1+\epsilon))^2}{4}\int_\Omega |\nabla A^{-1}w|^2.$$

Using the equation of $A^{-1}u$, we get

$$\int_\Omega |\nabla A^{-1}w|^2 \le \int_\Omega w_+^2 + \int_\Omega w_-^2$$

for $t > t_0$. Also we have for $t > t_0 + T_\epsilon$ that

$$\begin{aligned}
&-\int_\Omega w\int_\Omega a_2(t,x)u^*w_+ \\
&\le \int_\Omega w_-\int_\Omega \big(a_{2,\sup}(t)\big)_+u^*w_+ - (a_{2,\inf}(t))_+\int_\Omega w_+\int_\Omega u^*w_+ + (a_{2,\inf}(t))_-\int_\Omega w_+\int_\Omega u^*w_+ \\
&\le [(r_1+\epsilon)(a_{2,\inf}(t))_- - (r_2-\epsilon)(a_{2,\inf}(t))_+]\Big(\int_\Omega w_+\Big)^2 + (r_1+\epsilon)\big(a_{2,\sup}(t)\big)_+\Big(\int_\Omega w_-\Big)\Big(\int_\Omega w_+\Big).
\end{aligned}$$

By combining all these inequalities we have for a.e $t > t_0 + T_\epsilon$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w_+^2 \le& \Big(a_{0,\sup}(t) + \frac{\chi}{2}\big((r_1+\epsilon)-(r_2-\epsilon)\big) + \frac{(\chi(r_1+\epsilon))^2}{4}\Big)\int_\Omega w_+^2 \\ &- \Big((r_2-\epsilon)(2a_{1,\inf}(t) + |\Omega|(a_{2,\inf}(t))_+)\Big)\int_\Omega w_+^2 \\ &+ 2|\Omega|(r_1+\epsilon)(a_{2,\inf}(t))_-\int_\Omega w_+^2 + \big(\frac{(\chi(r_1+\epsilon))^2}{4}\big)\int_\Omega w_-^2 \\ &- (r_2-\epsilon)(a_{2,\inf}(t))_+\big(\int_\Omega w_+\big)^2 + (r_1+\epsilon)\big(a_{2,\sup}(t)\big)_+\big(\int_\Omega w_-\big)\big(\int_\Omega w_+\big). \quad (2.73)\end{aligned}$$

Similarly we have that $\int_\Omega w_-^2$ is weakly differentiable with $\frac{d}{dt}\int_\Omega w_-^2 = -2\int_\Omega w_- w_t$, and for a.e $t > t_0 + T_\epsilon$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w_-^2 \le& \Big(a_{0,\sup}(t) + \frac{\chi}{2}\big((r_1+\epsilon)-(r_2-\epsilon)\big) + \frac{(\chi(r_1+\epsilon))^2}{4}\Big)\int_\Omega w_-^2 \\ &- \Big((r_2-\epsilon)(2a_{1,\inf}(t) + |\Omega|\big(a_{2,\inf}(t)\big)_+)\Big)\int_\Omega w_-^2 \\ &+ 2|\Omega|(r_1+\epsilon)(a_{2,\inf}(t))_-\int_\Omega w_-^2 + \big(\frac{(\chi(r_1+\epsilon))^2}{4}\big)\int_\Omega w_+^2 \\ &- (r_2-\epsilon)(a_{2,\inf}(t))_+\big(\int_\Omega w_-\big)^2 + (r_1+\epsilon)\big(a_{2,\sup}(t)\big)_+\big(\int_\Omega w_-\big)\big(\int_\Omega w_+\big). \quad (2.74)\end{aligned}$$

Note that

$$\begin{aligned}&- (r_2-\epsilon)(a_{2,\inf}(t))_+\Big(\big(\int_\Omega w_+\big)^2 + \big(\int_\Omega w_-\big)^2\Big) + 2(r_1+\epsilon)\big(a_{2,\sup}(t)\big)_+\big(\int_\Omega w_-\big)\big(\int_\Omega w_+\big) \\ &\le 2\Big((r_1+\epsilon)\big(a_{2,\sup}(t)\big)_+ - (r_2-\epsilon)(a_{2,\inf}(t))_+\Big)\big(\int_\Omega w_-\big)\big(\int_\Omega w_+\big) \\ &\le |\Omega|\Big[\epsilon\Big(\big(a_{2,\sup}(t)\big)_+ + (a_{2,\inf}(t))_+\Big) + \Big(r_1\big(a_{2,\sup}(t)\big)_+ - r_2(a_{2,\inf}(t))_+\Big)\Big]\Big(\int_\Omega w_-^2 + \int_\Omega w_+^2\Big).\end{aligned}$$

Set

$$K(t,\epsilon) = \chi\epsilon + \chi^2\frac{\epsilon}{2}(2r_1+\epsilon) + 2|\Omega|\epsilon(a_{2,\inf})_- + |\Omega|\epsilon(a_{2,\sup}(t) + (a_{2,\inf}(t))_+) + \epsilon(2a_{1,\inf}(t) + |\Omega|a_{2,\inf}(t)).$$

Adding (2.73) and (2.74), we then have

$$\frac{d}{dt}\int_\Omega (w_+^2 + w_-^2)(t) \le 2\Big\{L_2(t) - L_1(t) + K(t,\epsilon)\Big\}\int_\Omega (w_+^2 + w_-^2)$$

for a.e $t > t_0 + T_\epsilon$. Therefore by the continuity with respect to time of both sides of this last inequality, we get

$$\frac{d}{dt}\int_\Omega (w_+^2 + w_-^2)(t) \le 2\Big\{L_2(t) - L_1(t) + K(t,\epsilon)\Big\}\int_\Omega (w_+^2 + w_-^2)$$

for $t > t_0 + T_\epsilon$. Then by Gronwall's inequality,

$$\int_\Omega (w_+^2(t)+w_-^2(t)) \le \int_\Omega (w_+^2(t_0+T_\epsilon)+w_-^2(t_0+T_\epsilon))e^{2\int_{t_0}^t (L_1(s)-L_2(s)+K(s,\epsilon))ds} \quad \text{for all } t > t_0 + T_\epsilon.$$

Note that $0 \le \sup_{t\in\mathbb{R}} |K(t,\epsilon)| \to 0$ as $\epsilon \to 0$ and choose $\epsilon_0 \ll 1$ ($\epsilon_0 < -\mu$) such that

$$0 \le \sup_{t\in\mathbb{R}} |K(t,\epsilon)| < \frac{-\mu-\epsilon_0}{2}.$$

By $\int_{t_0}^t (L_1(s) - L_2(s))ds \le (\mu + \epsilon_0)(t - t_0)$ for $t \ge t_0 + T_{\epsilon_0}$, we have

$$\begin{aligned}\int_\Omega (w_+^2(t) + w_-^2(t)) &\le (\int_\Omega w_+^2(t_0 + T_{\epsilon_0}) + w_-^2(t_0 + T_{\epsilon_0}))e^{2(\mu+\epsilon_0)(t-t_0)}e^{2(\frac{-\mu-\epsilon_0}{2})(t-t_0)} \\ &\le (\int_\Omega w_+^2(t_0 + T_{\epsilon_0}) + w_-^2(t_0 + T_{\epsilon_0}))e^{(\mu+\epsilon_0)(t-t_0)} \quad \forall\ t > t_0 + T_{\epsilon_0}.\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \|u(\cdot, t + t_0; t_0, u_0) - u^*(\cdot, t + t_0)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|w(t + t_0)\|^2_{L^2(\Omega)} = 0 \tag{2.75}$$

uniformly in $t_0 \in \mathbb{R}$.

We claim that (2.19) holds. Suppose by contradiction that there is $t_0 \in \mathbb{R}$ such that

$$u(\cdot, t; t_0, u_0) \nrightarrow u^*(\cdot, t)$$

in $C^0(\bar{\Omega})$ as $t \to \infty$. Then there exists $\epsilon_0 > 0$ and a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$\|u(\cdot, t_n; t_0, u_0) - u^*(\cdot, t_n)\|_{C^0(\bar{\Omega})} > \epsilon_0.$$

Since $u(\cdot, t_n; t_0, u_0), u^*(\cdot, t_n) \in C^0(\bar{\Omega})$ are uniformly bounded and equicontinuous, there exists up to subsequence $u^1, u_*^1 \in C^0(\bar{\Omega})$ such that $u(\cdot, t_n; t_0, u_0)$, $u^*(\cdot, t_n)$ converges respectively to u^1, u_*^1 in $C^0(\bar{\Omega})$. Therefore by dominated convergence theorem, $u(\cdot, t_n; t_0, u_0) \to u_1$ and $u^*(\cdot, t_n) \to u_*^1$ in $L^2(\Omega)$ as $t \to \infty$. This implies that

$$\lim_{t_n \to \infty} \|u(\cdot, t_n; t_0, u_0) - u^*(\cdot, t_n)\|_{L^2(\Omega)} = 0.$$

Hence we have that $u^1 = u_*^1$. But also from $\|u(\cdot, t_n; t_0, u_0) - u^*(\cdot, t_n)\|_{C^0(\bar{\Omega})} > \epsilon_0$, we get as $n \to \infty$, $\|u^1 - u_1^*\|_{C^0(\bar{\Omega})} \geq \epsilon_0$, which is a contradiction. Hence (2.19) holds.

Next, we prove that system (2.1) has a unique entire positive solution. Suppose that $(u_1^*(x,t), v_1^*(x,t))$ and $(u_2^*(x,t), v_2^*(x,t))$ are two entire positive solutions of system (2.1). We claim that $(u_1^*(x,t), v_1^*(x,t)) \equiv (u_2^*(x,t), v_2^*(x,t))$ for any $t \in \mathbb{R}$. Indeed, fix any $t \in \mathbb{R}$, by the arguments in the proof of (2.75),

$$\|u_1^*(\cdot,t) - u_2^*(\cdot,t)\|_{L^2(\Omega)} = \|u(\cdot,t;t_0,u_1^*(\cdot,t_0)) - u(\cdot,t;t_0,u_2^*(\cdot,t_0))\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t_0 \to -\infty.$$

This together with the continuity of $u_i^*(x,t)$ $(i = 1, 2)$ implies that $u_1^*(x,t) \equiv u_2^*(x,t)$ and then $v_1^*(x,t) \equiv v_2^*(x,t)$. Hence system (2.1) has a unique entire positive solution.

Assume now that $a_i(t,x) \equiv a_i(x)$ $(i = 0, 1, 2)$. By Theorem 2.3(3) and the uniqueness of entire positive solutions of system (2.1), (2.1) has a unique positive steady state solution.

Assume that $a_i(t+T,x) = a_i(t,x)$ $(i = 0, 1, 2)$. By Theorem 2.3(1) and the uniqueness of entire positive solutions of (2.1), (2.1) has a unique positive periodic solution with period T.

Finally assume that $a_i(t,x)$ $(i = 0, 1, 2)$ are almost periodic in t uniformly with respect to $x \in \bar{\Omega}$. Let $(u^*(x,t), v^*(x,t))$ be the unique positive solution of system (2.1). We claim that $(u^*(x,t), v^*(x,t))$ is almost periodic in t. Indeed, for any sequences $\{\beta_n'\}$, $\{\gamma_n'\} \subset \mathbb{R}$, by the almost periodicity of $a_i(t,x)$ in t, there are subsequences $\{\beta_n\} \subset \{\beta_n'\}$ and $\{\gamma_n\} \subset \{\gamma_n'\}$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} a_i(t + \beta_n + \gamma_m, x) = \lim_{n\to\infty} a_i(t + \beta_n + \gamma_n, x)$$

uniformly in $t \in \mathbb{R}$ and $x \in \bar{\Omega}$ for $i = 0, 1, 2$. Let

$$\hat{a}_i(t,x)=\lim_{n\to\infty}a_i(t+\beta_n,x),\quad \check{a}_i(t,x)=\lim_{m\to\infty}\hat{a}_i(t+\gamma_m,x),\quad \tilde{a}_i(t,x)=\lim_{n\to\infty}a_i(t+\beta_n+\gamma_n,x)$$

for $i=0,1,2$. Observe that $\hat{a}_i$ $(i=0,1,2)$, $\check{a}_i$ $(i=0,1,2)$, and $\tilde{a}_i$ $(i=0,1,2)$ also satisfy the hypothesis (H1) in the introduction, and $\check{a}_i=\tilde{a}_i$ for $i=0,1,2$.

Without loss of generality, we may assume that $\lim_{n\to\infty}(u^*(\cdot,t+\beta_n),v^*(\cdot,t+\beta_n))$ exists in $C^0(\bar{\Omega})$. Let

$$(\hat{u}^*(x,t),\hat{v}^*(x,t))=\lim_{n\to\infty}(u^*(\cdot,t+\beta_n),v^*(\cdot,t+\beta_n)).$$

Then $(\hat{u}^*(x,t),\hat{v}^*(x,t))$ is an entire positive solution of system (2.1) with $a_i(t,x)$ being replaced by $\hat{a}_i(t,x)$ $(i=0,1,2)$.

We may also assume that $\lim_{n\to\infty}(\hat{u}^*(\cdot,t+\beta_n),\hat{v}^*(\cdot,t+\beta_n))$ exists in $C^0(\bar{\Omega})$. Let

$$(\check{u}^*(x,t),\check{v}^*(x,t))=\lim_{n\to\infty}(\hat{u}^*(\cdot,t+\beta_n),\hat{v}^*(\cdot,t+\beta_n)).$$

Then $(\check{u}^*(x,t),\check{v}^*(x,t))$ is an entire positive solution of system (2.1) with $a_i(t,x)$ being replaced by $\check{a}_i(t,x)$ $(i=0,1,2)$.

Furthermore, we may assume that $\lim_{n\to\infty}(\hat{u}^*(\cdot,t+\beta_n+\gamma_n),\hat{v}^*(\cdot,t+\beta_n+\gamma_n))$ exists in $C^0(\bar{\Omega})$. Let

$$(\tilde{u}^*(x,t),\tilde{v}^*(x,t))=\lim_{n\to\infty}(\hat{u}^*(\cdot,t+\beta_n+\gamma_n),\hat{v}^*(\cdot,t+\beta_n+\gamma_n)).$$

Then $(\tilde{u}^*(x,t),\tilde{v}^*(x,t))$ is an entire positive solution of system (2.1) with $a_i(t,x)$ being replaced by $\tilde{a}_i(t,x)$ $(i=0,1,2)$. By the uniqueness of entire positive solutions of system (2.1) with $a_i(t,x)$ being replaced by $\tilde{a}_i(t,x)$ $(i=0,1,2)$, we have that

$$(\tilde{u}^*(x,t),\tilde{v}^*(x,t))=(\check{u}^*(x,t),\check{v}^*(x,t))\quad \forall\ x\in\bar{\Omega},\ t\in\mathbb{R}.$$

It then follows from $\check{a}_i=\tilde{a}_i$ for $i=0,1,2$ that

$$\lim_{m\to\infty}\lim_{n\to\infty}(u^*(x,t+\beta_n+\gamma_m),v^*(x,t+\beta_n+\gamma_m))=\lim_{n\to\infty}(u^*(x,t+\beta_n+\gamma_n),v^*(x,\beta_n+\gamma_n))$$

and hence $(u^*(x,t),v^*(x,t))$ is almost periodic in t. The theorem is thus proved. $\square$

Chapter 3

Persistence, Coexistence and Extinction in Two Species Chemotaxis Models on Bounded Heterogeneous Environments

3.1 Introduction

In this chapter, we study system (1.2) with $\tau = 0$, which reduces to the following two species parabolic-parabolic-elliptic chemotaxis system with heterogeneous Lotka-Volterra type competition terms,

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big), \quad x\in\Omega \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + v\Big(b_0(t,x) - b_1(t,x)u - b_2(t,x)v\Big), \quad x\in\Omega \\ 0 = d_3\Delta w + ku + lv - \lambda w, \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary, d_i ($i = 1, 2, 3$) are positive constants, $\chi_1, \chi_2, k, l, \lambda$ are nonnegative constants, and $a_i(t, x)$ and $b_i(t, x)$ ($i = 0, 1, 2$) are positive bounded smooth functions.

Note that, in the absence of chemotaxis, that is, $\chi_1 = \chi_2 = 0$, the dynamics of (3.1) is determined by the first two equations, that is, the following two species competition system,

$$\begin{cases} u_t = d_1\Delta u + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big), \quad x\in\Omega \\ v_t = d_2\Delta v + v\Big(b_0(t,x) - b_1(t,x)u - b_2(t,x)v\Big), \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x\in\partial\Omega. \end{cases} \tag{3.2}$$

Among interesting dynamical issues in (3.1) and (3.2) are persistence, coexistence, and extinction. These dynamical issues for (3.2) have been extensively studied (see [1], [15], [22], [23], etc.). Several authors have studied these issues for system (3.1) with constant coefficients [6, 29, 44, 55, 60]. For example in [29], the authors considered a more general competitive-cooperative chemotaxis system with nonlocal terms logistic sources and proved both the phenomena of coexistence and of exclusion for parameters in some natural range. However, there is little study of these important issues for (3.1) with time and space dependent coefficients. The objective of this chapter is to investigate the persistence, coexistence, and extinction dynamics of (3.2). In particular, we identify the circumstances under which persistence or extinction occurs, and in the case that persistence occurs, we study the existence, uniquenss and stability of coexistence states.

In order to do so, we first study the global existence of classical solutions of (3.1) with any given nonnegative initial functions. Note that for any given $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, system (3.2) has a unique bounded global classical solution

$$(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0))$$

with $(u(x,t_0;t_0,u_0,v_0), v(x,t_0;t_0,u_0,v_0)) = (u_0(x), v_0(x))$. However, it is not known whether for any given $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, (3.1) has a unique bounded global classical solution $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ with $(u(x,t_0;t_0,u_0,v_0), v(x,t_0;t_0,u_0,v_0)) = (u_0(x), v_0(x))$.

3.2 Notations, Assumptions, Definitions and Main results

3.2.1 Notations, assumptions and definitions

For a given function $f_i(t,x)$ defined on $\mathbb{R} \times \bar{\Omega}$ we put

$$f_{i,\inf} = \inf_{t\in\mathbb{R}, x\in\bar{\Omega}} f_i(t,x), \quad f_{i,\sup} = \sup_{t\in\mathbb{R}, x\in\bar{\Omega}} f_i(t,x),$$

$$f_{i,\inf}(t) = \inf_{x\in\bar{\Omega}} f_i(t,x), \quad f_{i,\sup}(t) = \sup_{x\in\bar{\Omega}} f_i(t,x),$$

unless specified otherwise.

We also introduce the following assumptions for our global existence results.

(H3) *$a_i(t,x)$, $b_i(t,x)$, χ_i and d_3, k and l satisfy*

$$a_{1,\inf} > \frac{k\chi_1}{d_3}, \quad a_{2,\inf} \geq \frac{l\chi_1}{d_3}, \quad b_{1,\inf} \geq \frac{k\chi_2}{d_3}, \quad \textit{and} \quad b_{2,\inf} > \frac{l\chi_2}{d_3}.$$

(H4) *$a_i(t,x)$, $b_i(t,x)$, χ_i and d_3, k and l satisfy*

$$a_{1,\inf} > \frac{k\chi_1}{d_3}, \quad b_{2,\inf} > \frac{l\chi_2}{d_3}, \quad \text{and} \quad \big(a_{1,\inf} - \frac{k\chi_1}{d_3}\big)\big(b_{2,\inf} - \frac{l\chi_2}{d_3}\big) > \frac{k\chi_2}{d_3}\frac{l\chi_1}{d_3}.$$

(H5) *$a_i(t,x)$, $b_i(t,x)$, χ_i and d_3, k and l satisfy*

$$a_{1,inf} > \max\{0, \frac{\chi_1 k(n-2)}{d_3 n}\}, \qquad a_{2,\inf} > \max\{0, \frac{\chi_1 l(n-2)}{d_3 n}\},$$

and

$$b_{1,\inf} > \max\{0, \frac{\chi_2 k(n-2)}{d_3 n}\}, \qquad b_{2,\inf} > \max\{0, \frac{\chi_2 l(n-2)}{d_3 n}\}.$$

For our results on persistence and coexistence, we further introduce the following assumptions.

(H6) *$a_i(t,x)$, $b_i(t,x)$, χ_i and d_3, k and l satisfy (H3) and*

$$a_{0,\inf} > a_{2,\sup}\bar{A}_2 \quad \textit{and} \quad b_{0,\inf} > b_{1,\sup}\bar{A}_1,$$

where

$$\bar{A}_1 = \frac{a_{0,\sup}}{a_{1,\inf} - \frac{k\chi_1}{d_3}}, \quad \bar{A}_2 = \frac{b_{0,\sup}}{b_{2,\inf} - \frac{l\chi_2}{d_3}}.$$

(H7) *$a_i(t,x)$, $b_i(t,x)$, χ_i and d_3, k and l satisfy (H4) and*

$$a_{0,inf} > \big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+\bar{B}_2 + \frac{\chi_1 l}{d_3}\bar{B}_2 \quad \textit{and} \quad b_{0,inf} > \big(b_{1,\sup} - \frac{\chi_2 k}{d_3}\big)_+\bar{B}_1 + \frac{\chi_2 k}{d_3}\bar{B}_1,$$

where

$$\bar{B}_1 = \frac{a_{0,\sup}(b_{2,\inf} - \frac{l\chi_2}{d_3}) + \frac{l\chi_1}{d_3}b_{0,\sup}}{(a_{1,\inf} - \frac{k\chi_1}{d_3})(b_{2,\inf} - \frac{l\chi_2}{d_3}) - \frac{lk\chi_1\chi_2}{d_3^2}} \tag{3.3}$$

and

$$\bar{B}_2 = \frac{b_{0,\sup}(a_{1,\inf} - \frac{k\chi_1}{d_3}) + \frac{k\chi_2}{d_3}a_{0,\sup}}{(a_{1,\inf} - \frac{k\chi_1}{d_3})(b_{2,\inf} - \frac{l\chi_2}{d_3}) - \frac{lk\chi_1\chi_2}{d_3^2}}, \tag{3.4}$$

and $(\cdots)_+$ *represents the positive part of the expression inside the brackets.*

Note that both (H6) and (H7) imply

$$a_{0,\inf}b_{2,\inf} > a_{2,\sup}b_{0,\sup}, \quad a_{1,\inf}b_{0,\inf} > a_{0,\sup}b_{1,\sup}. \tag{3.5}$$

Finally for our results on the stability and uniqueness of coexistence states in (3.1), we introduce the following assumptions.

(H8) *Assume (H3) and*

$$a_{0,\inf} > a_{2,\sup}\bar{A}_2 + k\frac{\chi_1}{d_3}\bar{A}_1, \quad b_{0,\inf} > b_{1,\sup}\bar{A}_1 + l\frac{\chi_2}{d_3}\bar{A}_2. \tag{3.6}$$

(H9) *Assume (H4) and*

$$a_{0,\inf} > (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{B}_2 + k\frac{\chi_1}{d_3}\bar{B}_1, \quad b_{0,\inf} > (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{B}_1 + l\frac{\chi_2}{d_3}\bar{B}_2. \tag{3.7}$$

(H10) $a_i(t,x) \equiv a_i(t)$ *and* $b_i(t,x) \equiv b_i(t)$ *(*$i = 0,1,2$*) satisfy* (3.5) *and*

$$\inf_t \{a_1(t) - b_1(t)\} > 2\frac{\chi_1}{d_3}(k+l), \quad \inf_t \{b_2(t) - a_2(t)\} > 2\frac{\chi_2}{d_3}(k+l). \tag{3.8}$$

Remark 3.1. *(1) (H8) implies (H6) and (H9) implies (H7).*

(2) When $\chi_1 = \chi_2 = 0$*, (H8) and (H9) are the same, and both* (3.6) *and* (3.7) *become* (3.5).

A solution $(u(x,t), v(x,t), w(x,t))$ of (3.1) defined for all $t \in \mathbb{R}$ is called an *entire solution*. A *coexistence state* of (3.1) is a positive entire solution $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$ with

$$\inf_{t\in\mathbb{R}, x\in\bar{\Omega}} u^{**}(x,t) > 0, \qquad \inf_{t\in\mathbb{R}, x\in\bar{\Omega}} v^{**}(x,t) > 0.$$

We say that *persistence* occurs in (3.1) if there is $\eta > 0$ such that for any $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 > 0$ and $v_0 > 0$, there is $\tau(u_0, v_0) > 0$ such that

$$u(x,t;t_0,u_0,v_0) \geq \eta, \quad v(x,t;t_0,u_0,v_0) \geq \eta \quad \forall\ x \in \bar{\Omega},\ t \geq t_0 + \tau(u_0,v_0),\ t_0 \in \mathbb{R}.$$

We say that *extinction of one species* or *competitive exclusion* occurs in (3.1) if for any $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 > 0$ and $v_0 > 0$, there holds

$$\lim_{t\to\infty} \|v(\cdot, t+t_0; t_0, u_0, v_0)\|_\infty = 0$$

or for any $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 > 0$ and $v_0 > 0$, there holds

$$\lim_{t\to\infty} \|u(\cdot, t+t_0; t_0, u_0, v_0)\|_\infty = 0.$$

3.2.2 Main results

Our results on global existence and boundedness of nonnegative classical solutions of (3.1) are stated in the following theorem.

Theorem 3.1. *(Global Existence)*

(1) Assume that (H3) holds. Then for any $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, (3.1) *has a unique bounded global classical solution $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ which satisfies that*

$$\lim_{t\to t_0^+} \left(\|u(\cdot,t;t_0,u_0,v_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} + \|v(\cdot,t;t_0,u_0,v_0) - v_0(\cdot)\|_{C^0(\bar{\Omega})} \right) = 0. \quad (3.9)$$

Moreover, for any $\epsilon > 0$*, there is* $T(u_0, v_0, \epsilon) \geq 0$ *such that*

$$0 \leq u(x,t;t_0,u_0,v_0) \leq \bar{A}_1 + \epsilon \quad \text{and} \quad 0 \leq v(x,t;t_0,u_0,v_0) \leq \bar{A}_2 + \epsilon$$

for all $t \geq t_0 + T(u_0, v_0, \epsilon)$*. If* $u_0 \leq \bar{A}_1 + \epsilon$*,* $v_0 \leq \bar{A}_2 + \epsilon$*, then* $T(u_0, v_0, \epsilon)$ *can be chosen to be zero.*

(2) Assume that (H4) holds. Then for any $t_0 \in \mathbb{R}$ *and* $u_0, v_0 \in C^0(\bar{\Omega})$ *with* $u_0 \geq 0$ *and* $v_0 \geq 0$*,* (3.1) *has a unique bounded global classical solution* $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ *which satisfies* (3.9)*. Moreover, for any* $\epsilon > 0$*, there is* $T(u_0, v_0, \epsilon) > 0$ *such that*

$$0 \leq u(x,t;t_0,u_0,v_0) \leq \bar{B}_1 + \epsilon \quad \text{and} \quad 0 \leq v(x,t;t_0,u_0,v_0) \leq \bar{B}_2 + \epsilon$$

for all $t \geq t_0 + T(u_0, v_0, \epsilon)$*. If* $u_0 \leq \bar{B}_1 + \epsilon$*,* $v_0 \leq \bar{B}_2 + \epsilon$*,* $T(u_0, v_0, \epsilon)$ *can be chosen to be zero.*

(3) Assume (H5) holds. Then for any $t_0 \in \mathbb{R}$ *and nonnegative functions* $u_0, v_0 \in C^0(\bar{\Omega})$*, system* (3.1) *has a unique bounded global classical solution* $(u(x,t;t_0,u_0,v_0)$*,* $v(x,t;t_0,u_0,v_0)$*,* $w(x,t;t_0,u_0,v_0))$ *which satisfies* (3.9)*. Moreover,*

$$0 \leq \int_\Omega u(x,t;t_0,u_0,v_0)dx \leq \max\left\{\int_\Omega u_0, \frac{a_{0,\sup}}{a_{1,\inf}}\right\}$$

and

$$0 \leq \int_\Omega v(x,t;t_0,u_0,v_0) \leq \max\left\{\int_\Omega v_0, \frac{b_{0,\sup}}{b_{2,\inf}}\right\}$$

for all $t \geq t_0$*.*

Remark 3.2. *(1) Under the assumption (H3), $(\bar{A}_1, \bar{A}_2)$ is the unique positive equilibrium of the following decoupled system,*

$$\begin{cases} u_t = u\big(a_{0,\sup} - (a_{1,\inf} - \frac{k\chi_1}{d_3})u\big) \\ v_t = v\big(b_{0,\sup} - (b_{2,\inf} - \frac{l\chi_2}{d_3})v\big). \end{cases}$$

Under the assumption (H4), $(\bar{B}_1, \bar{B}_2)$ is the unique positive equilibrium of the following cooperative system,

$$\begin{cases} u_t = u\big(a_{0,\sup} - (a_{1,\inf} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\big) \\ v_t = v\big(b_{0,\sup} - (b_{2,inf} - l\frac{\chi_2}{d_3})v + k\frac{\chi_2}{d_3}u\big). \end{cases}$$

(2) Conditions (H3), (H4) and (H5) are natural in the sense that when no chemotaxis is present, i.e., $\chi_1 = \chi_2 = 0$, conditions (H3) and (H4) become the trivial conditions $a_{1,\inf} > 0$ and $b_{2,\inf} > 0$ while (H5) becomes $a_{1,\inf} > 0$, $a_{2,\inf} > 0$, $b_{1,\inf} > 0$, and $b_{2,\inf} > 0$.

(3) By (H5), finite time blow up cannot happen when $n = 1$ or $n = 2$. In general, it remains open whether for any $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ exists for all $t \geq t_0$.

(4) It is proved in [30] that, under the assumption (H3), (H4), or (H5), there are semitrivial entire solutions $(u^*(x,t), 0, w_u^*(x,t))$ and $(0, v^*(x,t), w_v^*(x,t))$ of (3.1) with

$$\inf_{t\in\mathbb{R}, x\in\bar{\Omega}} u^*(x,t) > 0, \qquad \inf_{t\in\mathbb{R}, x\in\bar{\Omega}} v^*(x,t) > 0.$$

In the absence of chemotaxis (i.e. $\chi_1 = \chi_2 = 0$), such semitrivial solutions are unique.

(5) The condition of global existence and boundedness of classical solutions in [29, Theorem 1.1(1)] implies (H4). Therefore Theorem 3.1(2) is an improvement of the global existence result in [29, Theorem 1.1(1)]. Notice also that when $d_3 = l = 1, a_1 = \mu_1, b_2 = \mu_2,$ (H4)

coincide with the boundedness condition in [55, Lemma 2.2]. Thus (H4) is a generation of the global existence condition in [55].

We have the following theorem on the persistence in (3.1).

Theorem 3.2 (Persistence). *(1) Assume (H6). Then there are $\underline{A}_1 > 0$ and $\underline{A}_2 > 0$ such that for any $\epsilon > 0$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$, and $u_0, v_0 \not\equiv 0$, there exists t_{ϵ,u_0,v_0} such that*

$$\underline{A}_1 \leq u(x,t;t_0,u_0,v_0) \leq \bar{A}_1 + \epsilon, \ \ \underline{A}_2 \leq v(x,t;t_0,u_0,v_0) \leq \bar{A}_2 + \epsilon \tag{3.10}$$

for all $x \in \bar{\Omega}$, $t \geq t_0 + t_{\epsilon,u_0,v_0}$, and $t_0 \in \mathbb{R}$.

(2) Assume (H7). Then there are $\underline{B}_1 > 0$ and $\underline{B}_2 > 0$ such that for any $\epsilon > 0$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$, and $u_0, v_0 \not\equiv 0$, there exists t_{ϵ,u_0,v_0} such (3.10) *holds with $\underline{A}_1$, $\bar{A}_1$, $\underline{A}_2$, and $\bar{A}_2$ being replaced by $\underline{B}_1$, $\bar{B}_1$, $\underline{B}_2$, and $\bar{B}_2$, respectively.*

Remark 3.3. *(1) It should be pointed out that in [6], [55], [60], global asymptotic stability and uniqueness of coexistence states are obtained for* (3.1) *when the coefficients are constants and satisfy certain weak competition condition (see also [44] when the system involves nonlocal terms). In such cases, the persistence follows from the asymptotic stability and uniqueness of coexistence states. The persistence in two species chemotaxis systems without assuming the asymptotic stability of coexistence states is studied for the first time, even when the coefficients are constants. It should be also pointed out that the authors of [56] studied the persistence of a parabolic-parabolic chemotaxis system with logistic source. The persistence in* (3.1) *implies the persistence of mass, that is, if persistence occurs in* (3.1)*, then for any $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 > 0$ and $v_0 > 0$, there is $m(u_0, v_0) > 0$ such that*

$$\int_\Omega u(x,t;t_0,u_0,v_0)dx \geq m(u_0,v_0), \ \int_\Omega v(x,t;t_0,u_0,v_0)dx \geq m(u_0,v_0), \forall t \geq t_0, \ t_0 \in \mathbb{R}.$$

We will study persistence in fully parabolic two species competition system with chemotaxis somewhere else.

(2) *It is well known that, in the absence of chemotaxis (i.e., $\chi_1 = \chi_2 = 0$), the instability of the unique semitrivial solutions* $(u^*, 0)$ *and* $(0, v^*)$ *of* (3.2) *implies that the persistence occurs in* (3.2). *Note that both (H6) and (H7) imply* (3.5)*, which implies that the semitrivial solutions* $(u^*, 0)$ *and* $(0, v^*)$ *of* (3.2) *are unstable. When $\chi_1 = \chi_2 = 0$, the conditions (H6) and (H7) coincide and become* (3.5)*, and*

$$\bar{A}_1 = \bar{B}_1 = \frac{a_{0,\sup}}{a_{1,\inf}}, \quad \bar{A}_2 = \bar{B}_2 = \frac{b_{0,\sup}}{b_{2,\inf}}.$$

Hence theorem 3.2 recovers the uniform persistence result of (3.2) *in [23, Theorem E(1)].*

(3) *The conditions (H6) and (H7) are sufficient conditions for semi-trivial positive entire solutions of* (3.1) *to be unstable. In fact, assume (H6) or (H7) and suppose that* $(u^*, 0, w_u^*)$ *is a semi-trivial solution of* (3.1). *Then we have the following linearized equation of* (3.1) *at* $(u^*, 0, w_u^*)$,

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u^* \nabla w) - \chi_1 \nabla \cdot (u \nabla w_u^*) \\ \qquad\qquad + \Big(a_0(t,x) - 2a_1(t,x) u^*) u - a_2(t,x) u^* v \Big), \quad x \in \Omega \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w_u^*) + \Big(b_0(t,x) - b_1(t,x) u^* \Big) v, \quad x \in \Omega \\ 0 = d_3 \Delta w + ku + lv - \lambda w, \quad x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega. \end{cases}$$

Note that the second equation in the above system is independent of u and w. Assume (H6). Then

$$u^* \le \bar{A}_1, \quad w_u^* \le \frac{k}{\lambda} \bar{A}_1$$

and

$$
\begin{aligned}
v_t &= d_2\Delta v - \chi_2\nabla v\cdot\nabla w_u^* - \chi_2 v\Delta w_u^* + \Big(b_0(t,x) - b_1(t,x)u^*\Big)v\\
&= d_2\Delta v - \chi_2\nabla v\cdot\nabla w_u^* + \Big(b_0(t,x) - (b_1(t,x) - \frac{\chi_2 k}{d_3})u^* - \chi_2\frac{\lambda w_u^*}{d_3}\Big)v\\
&\geq d_2\Delta v - \chi_2\nabla v\cdot\nabla w_u^* + \Big(b_{0,\inf} - (b_{1,\sup} - \frac{\chi_2 k}{d_3})\bar{A}_1 - \chi_2\frac{\lambda\frac{k}{\lambda}\bar{A}_1}{d_3}\Big)v\\
&= d_2\Delta v - \chi_2\nabla v\cdot\nabla w_u^* + \Big(b_{0,\inf} - b_{1,\sup}\bar{A}_1\Big)v.
\end{aligned}
$$

This together with $b_{0,\inf} > b_{1,\sup}\bar{A}_1$ *implies that* $(u^*, 0, w_u^*)$ *is linearly unstable. Other cases can be proved similarly. The proof that (H6) or (H7) implies persistence* (3.1) *is very nontrivial. To prove Theorem 3.2, we first prove five nontrivial lemmas (i.e. Lemmas 3.4 to 3.8), some of which also play an important role in the study of coexistence.*

(4) Consider the following one species parabolic-elliptic chemotaxis model,

$$
\begin{cases}
u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u\Big), \quad x\in\Omega\\
0 = d_3\Delta w + ku - \lambda w, \quad x\in\Omega\\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega
\end{cases}
\tag{3.11}
$$

and assume that

$$
a_{1,\inf} > \frac{k\chi_1}{d_3}. \tag{3.12}
$$

By the arguments of Theorem 3.2, we have the following persistence for (3.11)*, which is new. There is* $\underline{A}_1$ *such that for any* $\epsilon > 0$, $t_0\in\mathbb{R}$, $u_0\in C^0(\bar{\Omega})$ *with* $u_0\geq 0$, *and* $u_0\not\equiv 0$, *there exists* t_{ϵ,u_0} *such that*

$$
\underline{A}_1 \leq u(x,t;t_0,u_0) \leq \bar{A}_1 + \epsilon
$$

for all $x\in\bar{\Omega}$ *and* $t\geq t_0 + t_{\epsilon,u_0}$, *where* $(u(x,t;t_0,u_0), w(x,t;t_0,u_0))$ *is the global solution of* (3.11) *with* $u(x,t_0;t_0,u_0) = u_0(x)$ *(see Corollary 3.2).*

The next theorem is about the existence of coexistence states of (3.1).

Theorem 3.3 (Coexistence). *(1) Assume (H6). Then there is a coexistence state* $(u^{**}(x,t),$ $v^{**}(x,t),$ $w^{**}(x,t))$ *of* (3.1). *Moreover, the following holds.*

(i) If there is $T > 0$ *such that* $a_i(t+T,x) = a_i(t,x)$, $b_i(t+T,x) = b_i(t,x)$ *for* $i = 0,1,2$, *then* (3.1) *has a* T*-periodic coexistence state* $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$, *that is,*

$$(u^{**}(x,t+T), v^{**}(x,t+T), w^{**}(x,t+T)) = (u^{**}(x,t), v^{**}(x,t), w^{**}(x,t)).$$

(ii) If $a_i(t,x) \equiv a_i(x)$, $b_i(t,x) \equiv b_i(x)$ *for* $i = 0,1,2$, *then* (3.1) *has a steady state coexistence state*

$$(u^{**}(t,x), v^{**}(t,x), w^{**}(t,x)) \equiv (u^{**}(x), v^{**}(x), w^{**}(x)).$$

(iii) If $a_i(t,x) \equiv a_i(t)$, $b_i(t,x) = b_i(t)$ *for* $i = 0,1,2$, *then* (3.1) *has a spatially homogeneous coexistence state*

$$(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t)) \equiv (u^{**}(t), v^{**}(t), w^{**}(t))$$

with $w^{**}(t) = ku^{**}(t) + lv^{**}(t)$, *and if* $a_i(t)$, $b_i(t)$ $(i = 0,1,2)$ *are periodic or almost periodic, so is* $(u^{**}(t), v^{**}(t), w^{**}(t))$.

(2) Assume (H7). Then there is a coexistence state $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$ *of* (3.1) *which satisfies (i)-(iii) of (1).*

Remark 3.4. *(1) By Theorem 3.2, (H6) or (H7) implies the persistence in* (3.1). *It is known that persistence in* (3.2) *implies the existence of a coexistence state. In the spatially homogeneous case, persistence in* (3.1) *also implies the existence of a coexistence state* *by the fact that the solutions of the following systems of ODEs are solutions of* (3.1)*,*

$$\begin{cases} u_t = u\big(a_0(t) - a_1(t)u - a_2(t)v\big) \\ v_t = v\big(b_0(t) - b_1(t)u - b_2(t)v\big) \\ 0 = ku + lv - \lambda w. \end{cases}$$

In general, it is very nontrivial to prove that persistence in (3.1) *implies the existence of a coexistence state.*

(2) As it is mentioned in Remark 1.2(1), when $\chi_1 = \chi_2 = 0$, the conditions (H6) and (H7) coincide and become (3.5)*. Hence theorem 3.3 recovers the coexistence result for* (3.2) *in [23, Theorem E(1)].*

We now state our result about the extinction of one of the species.

Theorem 3.4. *Assume that (H3) or (H4), and suppose furthermore that*

$$b_{2,\inf} > 2\frac{\chi_2}{d_3}l, \quad a_{2,\inf} \geq \frac{\chi_1}{d_3}l, \tag{3.13}$$

$$a_{2,\inf}\big(b_{0,\inf}(b_{2,\inf} - l\frac{\chi_2}{d_3}) - b_{0,\sup}\frac{\chi_2}{d_3}l\big) \geq a_{0,\sup}\big((b_{2,\inf} - l\frac{\chi_2}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - (l\frac{\chi_2}{d_3})^2\big), \tag{3.14}$$

and

$$\begin{aligned}&\big(a_{1,\inf} - \frac{\chi_1 k}{d_3}\big)\Big(b_{0,\inf}(b_{2,\inf} - \frac{l\chi_2}{d_3}) - b_{0,\sup}\frac{l\chi_2}{d_3}\Big) \\ &> \Big[\Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big)(b_{2,\inf} - \frac{\chi_2 l}{d_3}) + \frac{l\chi_2}{d_3}\Big(b_{1,\inf} - k\frac{\chi_2}{d_3}\Big)_-\Big]a_{0,\sup}.\end{aligned} \tag{3.15}$$

Then for every $t_0 \in \mathbb{R}$ and nonnegative initial functions $u_0, v_0 \in C^0(\overline{\Omega})$, $u_0 \geq 0$, $v_0 \geq 0$, with $\|v_0\|_\infty > 0$, the unique bounded and globally defined classical solution

$$(u(\cdot,\cdot;t_0,u_0,v_0), v(\cdot,\cdot;t_0,u_0;v_0), w(\cdot,\cdot;t_0,u_0,v_0))$$

of (3.1) *satisfies*

$$\lim_{t\to\infty} \|u(\cdot, t+t_0; t_0, u_0; v_0)\|_\infty = 0, \tag{3.16}$$

$$\alpha \le \liminf_{t\to\infty}(\min_{x\in\bar\Omega} v(x,t)) \le \limsup_{t\to\infty}(\max_{x\in\bar\Omega} v(x,t)) \le \beta, \tag{3.17}$$

$$l\alpha \le \lambda \liminf_{t\to\infty}(\min_{x\in\bar\Omega} w(x,t)) \le \lambda \limsup_{t\to\infty}(\max_{x\in\bar\Omega} w(x,t)) \le l\beta, \quad \forall x \in \bar\Omega \quad t \ge t_0, \tag{3.18}$$

where

$$\beta = \frac{b_{0,\sup}(b_{2,\sup} - l\frac{\chi_2}{d_3}) - l\frac{\chi_2}{d_3} b_{0,\inf}}{(b_{2,\inf} - l\frac{\chi_2}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - (l\frac{\chi_2}{d_3})^2},$$

and

$$\alpha = \frac{b_{0,\inf} - l\frac{\chi_2}{d_3}\beta}{b_{2,\sup} - l\frac{\chi_2}{d_3}} > 0.$$

Furthermore, if there is a unique positive entire solution $(v^*(x,t;\tilde b_0, \tilde b_2), w^*(x,t;\tilde b_0,\tilde b_2))$ *of*

$$\begin{cases} v_t = d_2\Delta v - \chi_2 \nabla\cdot(v\nabla w) + v\Big(\tilde b_0(t,x) - \tilde b_2(t,x) v\Big), & x \in \Omega \\ 0 = d_3\Delta w + lv - \lambda w, \quad x \in \Omega \\ \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \end{cases} \tag{3.19}$$

for any $(\tilde b_0, \tilde b_2) \in H(b_0, b_2)$*, where*

$$H(b_0,b_2) =$$
$$\{(c_0(\cdot,\cdot), c_2(\cdot,\cdot)) \,|\, \exists\, t_n \to \infty \text{ such that}$$
$$\lim_{n\to\infty}(b_0(t+t_n,x), b_2(t+t_n,x)) = (c_0(t,x), c_2(t,x)) \text{ locally uniformly in } (t,x) \in \mathbb{R}\times\mathbb{R}^N\},$$

then

$$\lim_{t\to\infty} \|v(\cdot, t+t_0; t_0, u_0, v_0) - v^*(\cdot, t+t_0; b_0, b_2)\|_\infty = 0. \tag{3.20}$$

Remark 3.5. *(1)* (3.14) *and* (3.15) *imply*

$$\frac{a_{0,\sup}}{b_{0,\inf}} \le \frac{a_{2,\inf}}{b_{2,\sup}}, \quad \frac{a_{0,\sup}}{b_{0,\inf}} < \frac{a_{1,\inf}}{b_{1,\sup}}. \tag{3.21}$$

To see this, we first note that (3.15) *implies that*

$$\big(a_{1,\inf}-\frac{\chi_1 k}{d_3}\big)b_{0,\inf}(b_{2,\inf}-\frac{l\chi_2}{d_3})>\Big[\Big(\Big(b_{1,\sup}-k\frac{\chi_2}{d_3}\Big)_+ +k\frac{\chi_2}{d_3}\Big)(b_{2,\inf}-\frac{\chi_2 l}{d_3})\Big]a_{0,\sup}$$
$$\geq b_{1,\sup}(b_{2,\inf}-\frac{\chi_2 l}{d_3})a_{0,\sup}.$$

Thus since $b_{2,\inf}-\frac{\chi_2 l}{d_3}>0$, *we get*

$$\big(a_{1,\inf}-\frac{\chi_1 k}{d_3}\big)b_{0,\inf}>b_{1,\sup}a_{0,\sup},$$

which implies the second inequality in (3.21)*. Second, note that* (3.14) *implies that*

$$a_{2,\inf}\big(b_{0,\inf}(b_{2,\inf}-l\frac{\chi_2}{d_3})-b_{0,\sup}\frac{\chi_2}{d_3}l\big)\geq a_{0,\sup}\big((b_{2,\inf}-l\frac{\chi_2}{d_3})b_{2,\sup}-l\frac{\chi_2}{d_3}b_{2,\inf}\big)$$
$$\geq a_{0,\sup}(b_{2,\inf}-2l\frac{\chi_2}{d_3})b_{2,\sup}.$$

This together with the fact that $a_{2,\inf}\big(b_{0,\inf}(b_{2,\inf}-l\frac{\chi_2}{d_3})-b_{0,\sup}\frac{\chi_2}{d_3}l\big)\leq a_{2,\inf}b_{0,\inf}(b_{2,\inf}-2l\frac{\chi_2}{d_3})$ *implies that*

$$a_{2,\inf}b_{0,\inf}(b_{2,\inf}-2l\frac{\chi_2}{d_3})\geq a_{0,\sup}(b_{2,\inf}-2l\frac{\chi_2}{d_3})b_{2,\sup},$$

which combines with $b_{2,\inf}-2l\frac{\chi_2}{d_3}>0$ *implies the first inequality in* (3.21).

(2) When $\chi_1=\chi_2=0$, (3.13) *becomes*

$$b_{2,\inf}>0,\quad a_{2,\inf}>0,\quad a_{1,\inf}>0;$$

(3.14) *and* (3.15) *become*

$$\frac{a_{0,\sup}}{b_{0,\inf}}\leq\frac{a_{2,\inf}}{b_{2,\sup}}\quad\text{and}\quad\frac{a_{0,\sup}}{b_{0,\inf}}<\frac{a_{1,\inf}}{b_{1,\sup}},$$

respectively. Therefore, the extinction results for (3.2) *in [23] are recovered.*

(3) When the coefficients are constants, Theorem 3.4 coincide with the exclusion Theorem in [29, Theorem 1.4]. Thus Theorem 3.4 give a natural extension to the phenomenon of exclusion in heterogeneous media.

(4) The reader is referred to [30] for the existence and uniqueness of positive entire solutions of (3.19).

Now, we state our result about optimal attracting rectangles for (3.1) under the assumption (H8) (resp., (H9)).

Theorem 3.5 (Optimal attracting rectangle)**.** *For given $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$, let* $\overline{u}_0 = \max_{x\in\bar{\Omega}} u_0(x)$, $\underline{u}_0 = \min_{x\in\bar{\Omega}} u_0(x)$, $\overline{v}_0 = \max_{x\in\bar{\Omega}} v_0(x)$, $\underline{v}_0 = \min_{x\in\bar{\Omega}} v_0(x)$.

(1) Assume (H8) and that the following system has a unique solution $(\bar{r}_1, \bar{r}_2, \underline{r}_1, \underline{r}_2)$

$$\begin{cases} (a_{1,\inf} - k\frac{\chi_1}{d_3})\bar{r}_1 = a_{0,\sup} - a_{2,\inf}\underline{r}_2 - k\frac{\chi_1}{d_3}\underline{r}_1 \\ (b_{2,\inf} - l\frac{\chi_2}{d_3})\bar{r}_2 = b_{0,\sup} - b_{1,\inf}\underline{r}_1 - k\frac{\chi_1}{d_3}\underline{r}_2 \\ (a_{1,\sup} - k\frac{\chi_1}{d_3})\underline{r}_1 = a_{0,\inf} - a_{2,\sup}\bar{r}_2 - k\frac{\chi_1}{d_3}\bar{r}_1 \\ (b_{2,\sup} - l\frac{\chi_2}{d_3})\underline{r}_2 = b_{0,\inf} - b_{1,\sup}\bar{r}_1 - l\frac{\chi_2}{d_3}\bar{r}_2. \end{cases} \tag{3.22}$$

Then $0 < \underline{r}_1 \leq \bar{r}_1$, $0 < \underline{r}_2 \leq \bar{r}_2$, and for any $\epsilon > 0$, $t_0 \in \mathbb{R}$, and $u_0, v_0 \in C^0(\bar{\Omega})$ with $\inf u_0 > 0$, $\inf v_0 > 0$, *there exists $t_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ such that*

$$\begin{cases} 0 < \underline{r}_1 - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{r}_1 + \epsilon \\ 0 < \underline{r}_2 - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{r}_2 + \epsilon, \end{cases} \tag{3.23}$$

for all $x \in \bar{\Omega}$ and $t \geq t_0 + t_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. Furthermore

$$\underline{r}_1 \leq u_0 \leq \bar{r}_1 \text{ and } \underline{r}_2 \leq v_0 \leq \bar{r}_2 \tag{3.24}$$

implies

$$\underline{r}_1 \leq u(x,t;t_0,u_0,v_0) \leq \bar{r}_1 \text{ and } \underline{r}_2 \leq v(x,t;t_0,u_0,v_0) \leq \bar{r}_2 \ \forall t \geq t_0. \tag{3.25}$$

(2) *Assume (H9) and that there is a unique solution* $(\bar{s}_1, \bar{s}_1, \underline{s}_1, \underline{s}_2)$ *of the following system,*

$$\begin{cases} \bar{s}_1 = \frac{\Big(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2 - k\frac{\chi_1}{d_3}\underline{s}_1\Big)(b_{2,\inf} - l\frac{\chi_2}{d_3})}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{l\chi_1}{d_3}\Big(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1 - l\frac{\chi_2}{d_3}\underline{s}_2\Big)}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\ \bar{s}_2 = \frac{\Big(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1 - l\frac{\chi_2}{d_3}\underline{s}_2\Big)(a_{1,\inf} - k\frac{\chi_1}{d_3})}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{k\chi_2}{d_3}\Big(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2 - k\frac{\chi_1}{d_3}\underline{s}_1\Big)}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\ \underline{s}_1 = \frac{\Big(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{s}_2 - k\frac{\chi_1}{d_3}\bar{s}_1\Big)(b_{2,\sup} - l\frac{\chi_2}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{l\chi_1}{d_3}\Big(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{s}_1 - l\frac{\chi_2}{d_3}\bar{s}_2\Big)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\ \underline{s}_2 = \frac{\Big(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{s}_1 - l\frac{\chi_2}{d_3}\bar{s}_2\Big)(a_{1,\sup} - k\frac{\chi_1}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{k\chi_2}{d_3}\Big(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{s}_2 - k\frac{\chi_1}{d_3}\bar{s}_1\Big)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}}. \end{cases}$$

Then $0 < \underline{s}_1 \le \bar{s}_1$, $0 < \underline{s}_2 \le \bar{s}_2$, *and for any* $\epsilon > 0$, $t_0 \in \mathbb{R}$, *and* $u_0, v_0 \in C^0(\bar{\Omega})$ *with* $\inf u_0 > 0$, $\inf v_0 > 0$, *there exists* $t_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$, *such that* (3.23)-(3.25) *hold with* $\bar{r}_1, \bar{r}_2, \underline{r}_1$, *and* $\underline{r}_2$ *being replaced by* $\bar{s}_1, \bar{s}_2, \underline{s}_1$, *and* $\underline{s}_2$, *respectively.*

Remark 3.6. *(1) Under the assumptions in Theorem 3.5(1),* $(\bar{r}_1, \bar{r}_2)$ *is the unique positive equilibrium of the system,*

$$\begin{cases} u_t = u\Big(a_{0,\sup} - \big(a_{1,\inf} - k\frac{\chi_1}{d_3}\big)u - a_{2,\inf}\underline{r}_2 - k\frac{\chi_1}{d_3}\underline{r}_1\Big) \\ v_t = v\Big(b_{0,\sup} - \big(b_{2,\inf} - l\frac{\chi_2}{d_3}\big)v - b_{1,\inf}\underline{r}_1 - l\frac{\chi_2}{d_3}\underline{r}_2\Big), \end{cases}$$

hence,

$$\bar{r}_1 < \bar{A}_1, \quad \bar{r}_2 < \bar{A}_2,$$

and $(\underline{r}_1, \underline{r}_2)$ *is the unique positive equilibrium of the system,*

$$\begin{cases} u_t = u\Big(a_{0,\inf} - \big(a_{1,\sup} - k\frac{\chi_1}{d_3}\big)u - a_{2,\sup}\bar{r}_2 - k\frac{\chi_1}{d_3}\bar{r}_1\Big) \\ v_t = v\Big(b_{0,\inf} - \big(b_{2,\sup} - l\frac{\chi_2}{d_3}\big)v - b_{1,\sup}\bar{r}_1 - l\frac{\chi_2}{d_3}\bar{r}_2\Big). \end{cases}$$

(2) Under the assumptions in Theorem 3.5(2), $(\bar{s}_1, \bar{s}_2)$ *is the unique positive equilibrium of the system,*

$$\begin{cases} u_t = u\Big(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2 - k\frac{\chi_1}{d_3}\underline{s}_1 - (a_{1,\inf} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\Big) \\ v_t = v\Big(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1 - l\frac{\chi_2}{d_3}\underline{s}_2 - (b_{2,\inf} - l\frac{\chi_2}{d_3})v + k\frac{\chi_2}{d_3}u\Big), \end{cases}$$

hence,

$$\bar{s}_1 < \bar{B}_1, \quad \bar{s}_2 < \bar{B}_2,$$

and $(\underline{s}_1, \underline{s}_2)$ *is the unique positive equilibrium of the system,*

$$\begin{cases} u_t = u\Big(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{s}_2 - k\frac{\chi_1}{d_3}\bar{s}_1 - (a_{1,\sup} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\Big) \\ v_t = v\Big(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{s}_1 - l\frac{\chi_2}{d_3}\bar{s}_2 - (b_{2,\sup} - l\frac{\chi_2}{d_3})v + k\frac{\chi_2}{d_3}u\Big). \end{cases}$$

(3) When $\chi_1 = \chi_2 = 0$,

$$\underline{r}_1 = \underline{s}_1 = \frac{a_{0,\inf}b_{2,\inf} - a_{2,\sup}b_{0,\sup}}{a_{1,\sup}b_{2,\inf} - a_{2,\sup}b_{1,\inf}}, \quad \bar{r}_1 = \bar{s}_1 = \frac{a_{0,\sup}b_{2,\sup} - a_{2,\inf}b_{0,\inf}}{a_{1,\inf}b_{2,\sup} - a_{2,\inf}b_{1,\sup}},$$

$$\underline{r}_2 = \underline{s}_2 = \frac{a_{1,\inf}b_{0,\inf} - a_{0,\sup}b_{1,\sup}}{a_{1,\inf}b_{2,\sup} - a_{2,\inf}b_{1,\sup}}, \quad \bar{r}_2 = \bar{s}_2 = \frac{a_{1,\sup}b_{0,\sup} - a_{0,\inf}b_{1,\inf}}{a_{1,\sup}b_{2,\inf} - a_{2,\sup}b_{1,\inf}}.$$

Thus Theorem 3.5 recovers the result on ultimate bounds of solutions of (3.2) *in [1]. Note that this result can be proven directly by using the competitive comparison principle. Note also that, in this case,* $(\bar{r}_1, \underline{r}_2)$ *is the unique coexistence state of*

$$\begin{cases} u_t = u(a_{0,\sup} - a_{1,\inf}u - a_{2,\inf}v) \\ v_t = v(b_{0,\inf} - b_{1,\sup}u - b_{2,\sup}v) \end{cases}$$

and $(\underline{r}_1, \bar{r}_2)$ *is the unique coexistence state of*

$$\begin{cases} u_t = u(a_{0,\inf} - a_{1,\sup}u - a_{2,\sup}v) \\ v_t = v(b_{0,\sup} - b_{1,\inf}u - b_{2,\inf}v). \end{cases}$$

(4) When the coefficients are constants, i.e $a_i(t,x) = a_i$ *and* $b_i(t,x) = b_i$ $(i = 0, 1, 2)$*, we have*

$$\underline{r}_1 = \bar{r}_1 = \underline{s}_1 = \bar{s}_1 = \frac{a_0b_2 - a_2b_0}{b_2a_1 - b_1a_2},$$

and

$$\underline{r}_2 = \bar{r}_2 = \underline{s}_2 = \bar{s}_2 = \frac{b_0 a_1 - b_1 a_0}{b_2 a_1 - b_1 a_2}.$$

Thus Theorem 3.5 implies the uniqueness and stability of coexistence states and we recover the results on asymptotic stability and uniqueness of the constant positive steady states in [29, Theorem 1.3] and [6]. Moreover, we get the optimal attracting rectangles $[\underline{r}_1 - \epsilon, \bar{r}_1 + \epsilon] \times [\underline{r}_2 - \epsilon, \bar{r}_2 + \epsilon]$ *($\epsilon > 0$) for* (3.1).

Finally, we state our result on the uniqueness and stability of coexistence states of (3.1).

Theorem 3.6 (Stability and uniqueness of coexistence states)**.**

(1) Assume (H8). Furthermore, assume that

$$\limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t \max\{Q_1(\tau) - q_1(\tau), Q_2(\tau) - q_2(\tau)\}d\tau < 0, \tag{3.26}$$

where

$$q_1(t) = 2a_{1,\inf}(t)\underline{r}_1 + a_{2,\inf}(t)\underline{r}_2 + \frac{\chi_1\left(k\underline{r}_1 + l\underline{r}_2\right)}{2d_3},$$

$$Q_1(t) = a_{0,\sup}(t) + \frac{\chi_1}{2d_3}\big(k\bar{r}_1 + l\bar{r}_2\big) + \frac{k^2}{4\lambda d_3}\Big(\frac{\chi_1^2\bar{r}_1^2}{d_1} + \frac{\chi_2^2\bar{r}_2^2}{d_2}\Big) + \frac{a_{2,\sup}(t)\bar{r}_1 + b_{1,\sup}(t)\bar{r}_2}{2},$$

$$q_2(t) = 2b_{2,\inf}(t)\underline{r}_2 + b_{1,\inf}(t)\underline{r}_1 + \frac{\chi_2\left(k\underline{r}_1 + l\underline{r}_2\right)}{2d_3},$$

and

$$Q_2(t) = b_{0,\sup}(t) + \frac{\chi_2}{2d_3}\big(k\bar{r}_1 + l\bar{r}_2\big) + \frac{l^2}{4\lambda d_3}\Big(\frac{\chi_1^2\bar{r}_1^2}{d_1} + \frac{\chi_2^2\bar{r}_2^2}{d_2}\Big) + \frac{a_{2,\sup}(t)\bar{r}_1 + b_{1,\sup}(t)\bar{r}_2}{2}.$$

Then (3.1) *has a unique coexistence state* $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$*, and, for any* $t_0 \in \mathbb{R}$ *and* $u_0, v_0 \in C^0(\bar{\Omega})$ *with* $u_0, v_0 \geq 0$ *and* $u_0, v_0 \not\equiv 0$*, the global classical solution* $(u(x,t;t_0,u_0,v_0)$*,* $v(x,t;t_0,u_0,v_0)$*,* $w(x,t;t_0,u_0,v_0))$ *of* (3.1) *satisfies*

$$\lim_{t\to\infty}\Big(\|u(\cdot,t;t_0,u_0,v_0) - u^{**}(\cdot,t)\|_{C^0(\bar{\Omega})} + \|v(\cdot,t;t_0,u_0,v_0) - v^{**}(\cdot,t)\|_{C^0(\bar{\Omega})}\Big) = 0, \tag{3.27}$$

and

$$\lim_{t\to\infty} \|w(\cdot,t;t_0,u_0,v_0) - w^{**}(\cdot,t)\|_{C^0(\bar\Omega)} = 0. \tag{3.28}$$

(2) Assume (H9). Furthermore, assume that (3.26) *holds with* $\bar r_1$*,* $\bar r_2$*,* $\underline{r}_1$*, and* $\underline{r}_2$ *being replaced by* $\bar s_1$*,* $\bar s_2$*,* $\underline{s}_1$*, and* $\underline{s}_2$*, respectively, where* $\underline{s}_i$ *and* $\bar s_i$ $(i = 1, 2)$ *are as in Theorem 3.5(2). Then the conclusion in (1) also holds.*

(3) Assume (H10). Then (3.1) *has a unique spatially homogeneous coexistence state* $(u^{**}(t), v^{**}(t), w^{**}(t))$*, and for any* $t_0 \in \mathbb{R}$ *and* $u_0, v_0 \in C^0(\bar\Omega)$ *with* $u_0, v_0 \geq 0$ *and* $u_0, v_0 \not\equiv 0$*, the unique global classical solution* $(u(x,t;t_0,u_0,v_0), v(x,t;t_0,u_0,v_0), w(x,t;t_0,u_0,v_0))$ *of* (3.1) *satisfies*

$$\lim_{t\to\infty} \big(\|u(\cdot,t;t_0,u_0,v_0) - u^{**}(t)\|_{C^0(\bar\Omega)} + \|v(\cdot,t;t_0,u_0,v_0) - v^{**}(t)\|_{C^0(\bar\Omega)} \big) = 0, \tag{3.29}$$

$$\lim_{t\to\infty} \|w(\cdot,t;t_0,u_0,v_0) - ku^{**}(t) - lv^{**}(t)\|_{C^0(\bar\Omega)} = 0. \tag{3.30}$$

Remark 3.7. *(1) Assume (H10).* (3.5) *implies that*

$$\begin{cases} u_t = u(a_0(t) - a_1(t)u - a_2(t)v) \\ v_t = v(b_0(t) - b_1(t)u - b_2(t)v) \end{cases}$$

has a positive entire solution $(u^{**}(t), v^{**}(t))$ *which is globally stable (see Lemma 3.9). Thus* $(u^{**}(t), v^{**}(t), w^{**}(t))$ *with* $w^{**}(t) = \frac{ku^{**}(t)+lv^{**}(t)}{\lambda}$*, is a positive entire solution of* (3.1) *in the case of space homogeneous coefficients, i.e,* $a_i(t,x) = a_i(t)$ *and* $b_i(t,x) = b_i(t)$*. The uniqueness results is new even for the case* $\chi_1 = \chi_2 = 0$ *with general time dependence. When the coefficients are periodic, Alvarez and Lazer proved in [3] the uniqueness of the entire solution* $(u^{**}(t), v^{**}(t))$ *only under the assumption* (3.5)*. It remains open whether such uniqueness result holds even in the case of* $\chi_1 = \chi_2 = 0$ *with general time dependence under only the assumption* (3.5)

(2) (3.8) *implies (H4). It is the analogue of the condition* $a_{1,\inf} > \frac{2\chi_1 k}{d_3}$ *for the global stability of the unique spatially homogeneous positive entire solution of the following one species*

chemotaxis model,

$$\begin{cases} u_t = d_1\Delta u - \chi_1 \nabla\cdot(u\nabla w) + u\Big(a_0(t) - a_1(t)u\Big), \quad x\in\Omega \\ 0 = d_3\Delta w + ku - \lambda w, \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases}$$

(see [29, Theorem 1.7]).

(3) When $\chi_1 = \chi_2 = 0$, (3.26) *becomes*

$$\begin{cases} \overline{\lim}_{t-s\to\infty}\frac{1}{t-s}\int_s^t\Big\{a_{0,\sup}(\tau) + \frac{a_{2,\sup}(\tau)}{2}\bar{r}_1 - 2a_{1,\inf}(\tau)\underline{r}_1 + \frac{b_{1,\sup}(\tau)}{2}\bar{r}_2 - a_{2,\inf}(\tau)\underline{r}_2\Big\}d\tau < 0 \\ \overline{\lim}_{t-s\to\infty}\frac{1}{t-s}\int_s^t\Big\{b_{0,\sup}(\tau) + \frac{b_{1,\sup}(\tau)}{2}\bar{r}_2 - 2b_{2,\inf}(\tau)\underline{r}_2 + \frac{a_{2,\sup}(\tau)}{2}\bar{r}_1 - b_{1,\inf}(\tau)\underline{r}_1\Big\}d\tau < 0. \end{cases}$$

If furthermore the coefficients are time homogeneous i.e $a_i(t,x) = a_i(x)$ *and* $b_i(t,x) = b_i(x)$, *then* (3.26) *becomes*

$$\begin{cases} a_{0,\sup} + \frac{a_{2,\sup}}{2}\bar{r}_1 + \frac{b_{1,\sup}}{2}\bar{r}_2 < 2a_{1,\inf}\underline{r}_1 + a_{2,\inf}\underline{r}_2 \\ b_{0,\sup} + \frac{b_{1,\sup}}{2}\bar{r}_2 + \frac{a_{2,\sup}}{2}\bar{r}_1 < 2b_{2,\inf}\underline{r}_2 + b_{1,\inf}\underline{r}_1. \end{cases} \tag{3.31}$$

We have the following corollary for the uniqueness and stability of coexistence states of (3.2), which is new in the general space dependence case.

Corollary 3.1. *Consider* (3.2). *Assume that* $\frac{a_{0,\sup}}{a_{0,\inf}} < 2\frac{a_{1,\inf}}{a_{1,\sup}}$ *and* $\frac{b_{0,\sup}}{b_{0,\inf}} < 2\frac{b_{2,\inf}}{b_{2,\sup}}$. *Then* (3.2) *has a unique stable coexistence state provided that the competition coefficients* a_2 *and* b_1 *are such small so that* (3.5) *and the following hold,*

$$\begin{cases} a_{2,\sup}\big(\frac{\bar{r}_1}{2} + \frac{2a_{1,\inf}b_{0,\sup} - a_{0,\sup}b_{1,\inf}}{a_{1,\sup}b_{2,\inf} - a_{2,\sup}b_{1,\inf}}\big) + \frac{b_{1,\sup}}{2}\bar{r}_2 - a_{2,\inf}\underline{r}_2 < b_{2,\inf}\frac{2a_{1,\inf}a_{0,\inf} - a_{0,\sup}a_{1,\sup}}{a_{1,\sup}b_{2,\inf} - a_{2,\sup}b_{1,\inf}} \\ \\ b_{1,\sup}\big(\frac{\bar{r}_2}{2} + \frac{2b_{2,\inf}a_{0,\sup} - b_{0,\sup}a_{2,\inf}}{b_{2,\sup}a_{1,\inf} - b_{1,\sup}a_{2,\inf}}\big) + \frac{a_{2,\sup}}{2}\bar{r}_1 - b_{1,\inf}\underline{r}_1 < a_{1,\inf}\frac{2b_{2,\inf}b_{0,\inf} - b_{0,\sup}b_{2,\sup}}{b_{2,\sup}a_{1,\inf} - b_{1,\sup}a_{2,\inf}}. \end{cases}$$

The rest of the chapter is organized as follows. In section 3.3, we study the global existence of classical solutions and prove Theorem 3.1. Section 3.4 is devoted to the study of the persistence and boundedness of classical solutions. It is here that we present the proof of Theorem 3.2. In section 3.5, we study the existence of coexistence states and prove Theorem 3.3. Next, in section 3.6, we study the phenomenon of exclusion and prove Theorem 3.4. Finally in section 3.7, we study existence of optimal rectangles and stability and uniqueness of coexistence states and prove Theorem 3.5 and Theorem 3.6.

3.3 Global Existence of Bounded Classical Solutions

In this section, we study the existence of bounded classical solutions of system (3.1) and prove Theorem 3.1. We start with the following important result on the local existence of classical solutions of system (3.1) with nonnegative initial functions in $C^0(\bar{\Omega})$.

Lemma 3.1. *For any given $t_0 \in \mathbb{R}$, $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, there exists $T_{\max}(t_0, u_0, v_0) \in (0, \infty]$ such that* (3.1) *has a unique nonnegative classical solution $(u(x,t;t_0,u_0,v_0)$, $v(x,t;t_0,u_0,v_0)$, $w(x,t;t_0,u_0,v_0))$ on $(t_0, t_0 + T_{\max}(t_0,u_0,v_0))$ satisfying that*

$$\lim_{t \nearrow t_0} \|u(\cdot,t;t_0,u_0,v_0) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0, \quad \lim_{t \nearrow t_0} \|v(\cdot,t;t_0,u_0,v_0) - v_0(\cdot)\|_{C^0(\bar{\Omega})} = 0,$$

and moreover if $T_{\max}(t_0,u_0,v_0) < \infty$, then

$$\limsup_{t \nearrow T_{\max}(t_0,u_0,v_0)} \Big(\|u(\cdot,t_0+t;t_0,u_0,v_0)\|_{C^0(\bar{\Omega})} + \|v(\cdot,t_0+t;t_0,u_0,v_0)\|_{C^0(\bar{\Omega})} \Big) = \infty. \quad (3.32)$$

Proof. If follows from the similar arguments as those in [55, Lemma 2.1]. □

Next, we consider the following system of ODEs induced from system (3.1),

$$\begin{cases} \overline{u}' = \frac{\chi_1}{d_3}\overline{u}\big(k\overline{u}+l\overline{v}-k\underline{u}-l\underline{v}\big)+\overline{u}\big[a_{0,\sup}(t)-a_{1,\inf}(t)\overline{u}-a_{2,\inf}(t)\underline{v}\big] \\ \underline{u}' = \frac{\chi_1}{d_3}\underline{u}\big(k\underline{u}+l\underline{v}-k\overline{u}-l\overline{v}\big)+\underline{u}\big[a_{0,\inf}(t)-a_{1,\sup}(t)\underline{u}-a_{2,\sup}(t)\overline{v}\big] \\ \overline{v}' = \frac{\chi_2}{d_3}\overline{v}\big(k\overline{u}+l\overline{v}-k\underline{u}-l\underline{v}\big)+\overline{v}\big[b_{0,\sup}(t)-b_{2,\inf}(t)\overline{v}-b_{1,\inf}(t)\underline{u}\big] \\ \underline{v}' = \frac{\chi_2}{d_3}\underline{v}\big(k\underline{u}+l\underline{v}-k\overline{u}-l\overline{v}\big)+\underline{v}\big[b_{0,\inf}(t)-b_{2,\sup}(t)\underline{v}-b_{1,\sup}(t)\overline{u}\big]. \end{cases} \tag{3.33}$$

For convenience, we let

$$\begin{aligned}&(\overline{u}(t),\underline{u}(t),\overline{v}(t),\underline{v}(t))\\&=\left(\overline{u}\left(t;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\underline{u}\left(t;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\overline{v}\left(t;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\underline{v}\left(t;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right)\right)\end{aligned}$$

be the solution of (3.33) with initial condition

$$\begin{aligned}&\left(\overline{u}\left(t_0;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\underline{u}\left(t_0;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\overline{v}\left(t_0;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right),\underline{v}\left(t_0;t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right)\right)\\&=(\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0)\in\mathbb{R}^4_+.\end{aligned}\tag{3.34}$$

Then for given $t_0 \in \mathbb{R}$ and $(\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0)\in\mathbb{R}^4_+$, there exists $T_{\max}\left(t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right)>0$ such that (3.33) has a unique classical solution $(\overline{u}(t),\underline{u}(t),\overline{v}(t),\underline{v}(t))$ on $(t_0,t_0+T_{\max}\left(t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right))$ satisfying (3.34). Moreover if $T_{\max}\left(t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right)<\infty$, then

$$\limsup_{t\nearrow T_{\max}(t_0,\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0)}\left(|\overline{u}(t_0+t)|+|\underline{u}(t_0+t)|+|\overline{v}(t_0+t)|+|\underline{v}(t_0+t)|\right)=\infty.$$

We now state and prove the following important lemma which provides sufficient conditions for the boundedness of classical solutions of system (3.33).

Lemma 3.2. *let* $(\overline{u}(t),\underline{u}(t),\overline{v}(t),\underline{v}(t))$ *be the solution of* (3.33) *which satisfies* (3.34)*. Then*

(i) $0\le\underline{u}_0\le\overline{u}_0$ *and* $0\le\underline{v}_0\le\overline{v}_0$ *imply* $0\le\underline{u}(t)\le\overline{u}(t)$ *and* $0\le\underline{v}(t)\le\overline{v}(t)$ *for all* $t\in[t_0,t_0+T_{\max}\left(\overline{u}_0,\underline{u}_0,\overline{v}_0,\underline{v}_0\right))$.

(ii) If (H4) holds, then $T_{\max}(t_0, \overline{u}_0, \underline{u}_0, \overline{v}_0, \underline{v}_0) = \infty$ *and*

$$\limsup_{t\to\infty} \overline{u}(t) \le \bar{B}_1, \quad \limsup_{t\to\infty} \overline{v}(t) \le \bar{B}_2,$$

where $\bar{B}_1$ *and* $\bar{B}_2$ *are as in* (3.3) *and* (3.4)*, respectively.*

Proof. (i) Let $\epsilon > 0$ and $(\overline{u}_\epsilon(t), \underline{u}_\epsilon(t), \overline{v}_\epsilon(t), \underline{v}_\epsilon(t))$ be solution of (3.33) with $a_{0,\sup}(t)$ and $b_{0,\sup}(t)$ being replaced by $a_{0,\sup}(t) + \epsilon$ and $b_{0,\sup}(t) + \epsilon$, respectively, and satisfying (3.34) with $\overline{u}_0$, $\overline{v}_0$ being replaced respectively by $\overline{u}_0^\epsilon = \overline{u}_0 + \epsilon$ and $\overline{v}_0^\epsilon = \overline{v}_0 + \epsilon$. We claim first that (i) holds for $(\overline{u}_\epsilon(t), \underline{u}_\epsilon(t), \overline{v}_\epsilon(t), \underline{v}_\epsilon(t))$. Suppose by contradiction that our claim does not hold. Then there exists $\bar{t} \in (t_0, t_0 + T_{\max}(t_0, \overline{u}_0^\epsilon, \underline{u}_0, \overline{v}_0^\epsilon, \underline{v}_0))$ such that

$$0 \le \underline{u}_\epsilon(t) < \overline{u}_\epsilon(t), \quad 0 \le \underline{v}_\epsilon(t) < \overline{v}_\epsilon(t)\,, \forall t \in [t_0, \bar{t}) \tag{3.35}$$

and

$$\text{either} \quad \underline{u}_\epsilon(\bar{t}) = \overline{u}_\epsilon(\bar{t}) \quad \text{or} \quad \underline{v}_\epsilon(\bar{t}) = \overline{v}_\epsilon(\bar{t}).$$

Without loss of generality, assume that $\underline{u}_\epsilon(\bar{t}) = \overline{u}_\epsilon(\bar{t})$. Then on one hand (3.35) implies that

$$\left(\overline{u}_\epsilon - \underline{u}_\epsilon\right)'(\bar{t}) \le 0,$$

and on the other hand the difference between the first and the second equations of (3.33) gives

$$\begin{aligned} \left(\overline{u}_\epsilon - \underline{u}_\epsilon\right)'(\bar{t}) = \overline{u}_\epsilon(\bar{t}) \left\{ a_{0,\sup}(\bar{t}) + \epsilon - a_{0,\inf}(\bar{t}) + (a_{1,\sup}(\bar{t}) - a_{1,\inf}(\bar{t}))\overline{u}_\epsilon(\bar{t}) + 2l\frac{\chi_1}{d_3}\left(\overline{v}_\epsilon - \underline{v}_\epsilon\right)(\bar{t}) \right\} \\ + \overline{u}_\epsilon(\bar{t}) \left\{ a_{2,\sup}(\bar{t})\overline{v}_\epsilon(\bar{t}) - a_{2,\inf}(\bar{t})\underline{v}_\epsilon(\bar{t}) \right\} > 0, \end{aligned}$$

which is a contradiction. Thus (i) holds for $\left(\overline{u}_\epsilon(t), \underline{u}_\epsilon(t), \overline{v}_\epsilon(t), \underline{v}_\epsilon(t)\right)$. Letting $\epsilon \to 0$, we have that (i) holds for $(\overline{u}(t), \underline{u}(t), \overline{v}(t), \underline{v}(t))$.

(ii) First from the first and third equations of (3.33) we get

$$\begin{cases} \overline{u}' \le \overline{u}\Big[a_{0,\sup} - \Big(a_{1,\inf} - k\frac{\chi_1}{d_3}\Big)\overline{u} + l\frac{\chi_1}{d_3}\overline{v}\Big] \\ \overline{v}' \le \overline{v}\Big[b_{0,\sup} - \Big(b_{2,\inf} - l\frac{\chi_2}{d_3}\Big)\overline{v} + k\frac{\chi_2}{d_3}\overline{u}\Big]. \end{cases}$$

Thus the result follows from comparison principle for cooperative systems and the fact that $(\bar{B}_1, \bar{B}_2)$ is a uniformly asymptotically stable solution for the following system of ODEs,

$$\begin{cases} u' = u\Big\{a_{0,\sup} - (a_{1,\inf} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\Big\} \\ v' = v\Big\{b_{0,\sup} - \Big(b_{2,inf} - l\frac{\chi_2}{d_3}\Big)v + k\frac{\chi_2}{d_3}u\Big\}. \end{cases}$$

□

Next, we state and prove the following lemma used in some of our proofs.

Lemma 3.3. *[29, Proof Theorem 1.1(1)] Assume (H4). Given $t_0 \in \mathbb{R}$, $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \ge 0$, let $\overline{u}_0 = \max_{x\in\bar{\Omega}} u_0(x)$, $\underline{u}_0 = \min_{x\in\bar{\Omega}} u_0(x)$, $\overline{v}_0 = \max_{x\in\bar{\Omega}} v_0(x)$, $\underline{v}_0 = \min_{x\in\bar{\Omega}} v_0(x)$ and let $(\overline{u}(t), \underline{u}(t), \overline{v}(t), \underline{v}(t))$ be solution of* (3.33) *satisfying initial condition* (3.34)*. Then if $(u(x,t), v(x,t), w(x,t))$ is the solution of equation* (3.1) *with initials $u(\cdot, t_0) = u_0$ and $v(\cdot, t_0) = v_0$, we have*

$$0 \le \underline{u}(t) \le u(x,t) \le \overline{u}(t) \quad \textit{and} \quad 0 \le \underline{v}(t) \le v(x,t) \le \overline{v}(t)\,, \forall x \in \bar{\Omega} \;\; t \ge t_0.$$

Proof. By the similar arguments as those in [29, Theorem 1.1(1)], under the condition (H4), we have

$$0 \le \underline{u}(t) \le u(x,t) \le \overline{u}(t) \quad \text{and} \quad 0 \le \underline{v}(t) \le v(x,t) \le \overline{v}(t)\,, \forall x \in \bar{\Omega} \;\; t \ge (t_0, t_0 + T_{\max}).$$

By (H2) and Lemma 3.2, we get $T_{\max} = \infty$. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \ge 0$ and $v_0 \ge 0$.

(1) From the first equation of system (3.1), we have that for $t \in (t_0, t_0 + T_{\max}(t_0, u_0, v_0))$,

$$
\begin{aligned}
u_t =& d_1 \Delta u - \chi_1 \nabla u \cdot \nabla w + u \left\{ a_0(t,x) - \left(a_1(t,x) - k\frac{\chi_1}{d_3} \right) u - \left(a_2(t,x) - l\frac{\chi_1}{d_3} \right) v - \frac{\chi_1}{d_3} \lambda w \right\} \\
\leq& d_1 \Delta u - \chi_1 \nabla u \cdot \nabla w + u \left\{ a_{0,\sup} - \left(a_{1,\inf} - k\frac{\chi_1}{d_3} k \right) u - \left(a_{2,\inf} - l\frac{\chi_1}{d_3} \right) v - \frac{\chi_1}{d_3} \lambda w \right\}.
\end{aligned}
$$

This together with (H3) gives for $t \in (t_0, t_0 + T_{\max}(t_0, u_0, v_0))$,

$$
u_t \leq d_1 \Delta u - \chi_1 \nabla u \cdot \nabla w + u \left\{ a_{0,\sup} - \left(a_{1,\inf} - k\frac{\chi_1}{d_3} k \right) u \right\}. \tag{3.36}
$$

Therefore by comparison principle for parabolic equations, we get

$$
0 \leq u(x,t;t_0,u_0,v_0) \leq \max \left\{ \|u_0\|_\infty, \frac{a_{0,\sup}}{a_{1,\inf} - k\frac{\chi_1}{d_3}} \right\} \quad \forall\ t \in [t_0, t_0 + T_{\max}(t_0, u_0, v_0)). \tag{3.37}
$$

Similarly, the second equation of system (3.1) gives

$$
0 \leq v(x,t;t_0,u_0,v_0) \leq \max \left\{ \|v_0\|_\infty, \frac{b_{0,\sup}}{b_{2,\inf} - l\frac{\chi_2}{d_3}} \right\} \quad \forall\ t \in [t_0, t_0 + T_{\max}(t_0, u_0, v_0)).
$$

This together with (3.32) and (3.37) implies that $T_{\max}(t_0, u_0, v_0) = \infty$.

Moreover, by (3.36) and comparison principle for parabolic equations again, for any $\epsilon > 0$, there is $T_1(u_0, v_0, \epsilon) \geq 0$ such that

$$
0 \leq u(x,t;t_0,u_0,v_0) \leq \frac{a_{0,\sup}}{a_{1,\inf} - k\frac{\chi_1}{d_3}} + \epsilon \quad \forall\ x \in \bar{\Omega},\ t \geq t_0 + T_1(u_0, v_0, \epsilon),
$$

and $T_1(u_0, v_0, \epsilon)$ can be chosen to be zero if $u_0 \leq \bar{A}_1 + \epsilon$. Similarly, for any $\epsilon > 0$, there is $T_2(u_0, v_0, \epsilon) \geq 0$ such that

$$
0 \leq v(x,t;t_0,u_0,v_0) \leq \frac{b_{0,\sup}}{b_{2,\inf} - l\frac{\chi_2}{d_3}} + \epsilon \quad \forall\ x \in \bar{\Omega},\ t \geq t_0 + T_2(u_0, v_0, \epsilon),
$$

and $T_2(u_0, v_0, \epsilon)$ can be chosen to be zero if $v_0 \leq \bar{A}_2 + \epsilon$. (1) thus follows with $T(u_0, v_0, \epsilon) = \max\{T_1(u_0, v_0, \epsilon), T_2(u_0, v_0, \epsilon)\}$.

(2) Let $\overline{u}_0 = \max_{x\in\bar\Omega} u_0(x)$, $\underline{u}_0 = \min_{x\in\bar\Omega} u_0(x)$, $\overline{v}_0 = \max_{x\in\bar\Omega} v_0(x)$, $\underline{v}_0 = \min_{x\in\bar\Omega} v_0(x)$ and let $(\overline{u}(t), \underline{u}(t), \overline{v}(t), \underline{v}(t))$ be solution of (3.33) satisfying initial condition (3.34). By the similar arguments as those in [29, Theorem 1.1(1)], under the condition (H4), we have

$$0 \le \underline{u}(t) \le u(x,t) \le \overline{u}(t) \quad \text{and} \quad 0 \le \underline{v}(t) \le v(x,t) \le \overline{v}(t)\,, \forall x \in \bar\Omega \;\; t \in (t_0, t_0 + T_{\max}).$$

This together with Lemma 3.2 implies Theorem 3.1 (2).

(3) It follows from the similar arguments as those in [29, Theorem 1.1(2)]. □

3.4 Persistence

In this section, we study the persistence in (3.1) and prove Theorem 3.2.

Fix $T > 0$. We first prove five Lemmas.

Lemma 3.4. *(1) Assume (H3). For any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for any $0 \le u_0 \le \bar A_1 + \epsilon$, $0 \le v_0 \le \bar A_2 + \epsilon$, the following hold.*

(i) If $0 \le u_0 \le \delta$, then $u(x, t; t_0, u_0, v_0) \le \epsilon$ for $t \in [t_0, t_0 + T]$ and $x \in \bar\Omega$.

(ii) If $0 \le v_0 \le \delta$, then $v(x, t; t_0, u_0, v_0) \le \epsilon$ for $t \in [t_0, t_0 + T]$ and $x \in \bar\Omega$.

(2) Assume (H4). For any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for any $0 \le u_0 \le \bar B_1 + \epsilon$, $0 \le v_0 \le \bar B_2 + \epsilon$, the following hold.

(i) If $0 \le u_0 \le \delta$, then $u(x, t; t_0, u_0, v_0) \le \epsilon$ for $t \in [t_0, t_0 + T]$ and $x \in \bar\Omega$.

(ii) If $0 \le v_0 \le \delta$, then $v(x, t; t_0, u_0, v_0) \le \epsilon$ for $t \in [t_0, t_0 + T]$ and $x \in \bar\Omega$.

Proof. (1)(i) By Theorem 3.1(1),

$$0 \le u(x, t + t_0; t_0, u_0, v_0) \le \bar A_1 + \epsilon, \;\; 0 \le v(x, t + t_0; t_0, u_0, v_0) \le \bar A_2 + \epsilon \quad \forall\, t \ge 0, \quad x \in \bar\Omega.$$

Assume (H3). Then

$$
\begin{aligned}
u_t &= d_1\Delta u - \chi_1 \nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})v - \frac{\chi_1\lambda}{d_3}w\Big)\\
&\le d_1\Delta u - \chi_1\nabla u\cdot\nabla w + a_{0,\sup}u.
\end{aligned}
$$

Hence, by comparison principle for parabolic equations, we have

$$
u(x,t;t_0,u_0) \le e^{a_{0,\sup}(t-t_0)}\|u_0\| \quad \forall\ t\ge t_0.
$$

(1)(i) thus follows with $\delta = \epsilon e^{-a_{0,\sup}T}$ for any given $\epsilon > 0$.

(1)(ii) It can be proved by the similar arguments as in (1)(i).

(2)(i) By Theorem 3.1(2),

$$
u(x,t+t_0;t_0,u_0,v_0) \le \bar B_1 + \epsilon,\ \ v(x,t+t_0;t_0,u_0,v_0)\le \bar B_2 + \epsilon \quad \forall\ t\ge 0, \quad x\in\bar\Omega.
$$

Assume (H4). Then

$$
\begin{aligned}
u_t &= d_1\Delta u - \chi_1 \nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})v - \frac{\chi_1\lambda}{d_3}w\Big)\\
&\le d_1\Delta u - \chi_1\nabla u\cdot\nabla w + \big(a_{0,\sup} + \frac{\chi_1 l}{d_3}(\bar B_2+\epsilon)\big)u.
\end{aligned}
$$

By comparison principle for parabolic equations, we have

$$
u(x,t;t_0,u_0) \le e^{\big(a_{0,\sup}+\frac{\chi_1 l}{d_3}(\bar B_2+\epsilon)\big)(t-t_0)}\|u_0\| \quad \forall\ t\ge t_0.
$$

(2)(i) thus follows with $\delta = \epsilon e^{-\big(a_{0,\sup}+\frac{\chi_1 l}{d_3}(\bar B_2+\epsilon)\big)T}$ for any given $\epsilon > 0$.

(2)(ii) It can be proved by the similar arguments as in (2)(i). □

Remark 3.8. *Consider* (3.11) *and assume* (3.12)*. By the arguments of Lemma 3.4, we have that, for any* $\epsilon > 0$*, there is* $\delta = \delta(\epsilon) > 0$ *such that for any* $0 \le u_0 \le \bar{A}_1 + \epsilon$*, if* $0 \le u_0 \le \delta$*, then* $u(x,t;t_0,u_0) \le \epsilon$ *for* $t \in [t_0, t_0+T]$ *and* $x \in \bar{\Omega}$*.*

Lemma 3.5. *(1) Assume (H6). Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(1) holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*,*

$$a_{0,\inf} > a_{2,\sup}(\bar{A}_2 + \epsilon_0) + \frac{\chi_1 k}{d_3}\epsilon_0, \quad b_{0,\inf} > b_{1,\sup}(\bar{A}_1 + \epsilon_0) + \frac{\chi_2 l}{d_3}\epsilon_0,$$

and

$$\delta_0 < \min\Big\{\frac{a_{0,\inf} - a_{2,\sup}(\bar{A}_2 + \epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0}{a_{1,\sup} - \frac{\chi_1 k}{d_3}}, \frac{b_{0,\inf} - b_{1,\sup}(\bar{A}_1 + \epsilon_0) - \frac{\chi_2 l}{d_3}\epsilon_0}{b_{2,\sup} - \frac{\chi_2 l}{d_3}}\Big\}.$$

For given $0 \le u_0 \le \bar{A}_1 + \epsilon_0$*,* $0 \le v_0 \le \bar{A}_2 + \epsilon_0$*, the following hold.*

(i) If $0 < u_0 < \delta_0$*, then* $u(x, t+t_0; t_0, u_0, v_0) > \inf u_0(x) \quad \forall\ 0 < t \le T$.

(ii) If $0 < v_0 < \delta_0$*, then* $v(x, t+t_0; t_0, u_0, v_0) > \inf v_0(x) \quad \forall\ 0 < t \le T$.

(2) Assume (H7). Let ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(2) holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*,*

$$a_{0,\inf} > \big[\big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+ + \frac{\chi_1 l}{d_3}\big](\bar{B}_2 + \epsilon_0) + \frac{\chi_1 k}{d_3}\epsilon_0,$$

$$b_{0,\inf} > \big[\big(b_{1,\sup} - \frac{\chi_2 k}{d_3}\big)_+ + \frac{\chi_2 k}{d_3}\big](\bar{B}_1 + \epsilon_0) + \frac{\chi_2 l}{d_3}\epsilon_0,$$

and

$$\delta_0 < \min\Big\{\frac{a_{0,\inf} - \big[\big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+ + \frac{\chi_1 l}{d_3}\big](\bar{B}_2 + \epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0}{a_{1,\sup} - \frac{\chi_1 k}{d_3}},$$
$$\frac{b_{0,\inf} - \big[\big(b_{1,\sup} - \frac{\chi_2 k}{d_3}\big)_+ + \frac{\chi_2 k}{d_3}\big](\bar{B}_1 + \epsilon_0) - \frac{\chi_2 l}{d_3}\epsilon_0}{b_{2,\sup} - \frac{\chi_2 l}{d_3}}\Big\}.$$

For given $0 \le u_0 \le \bar{B}_1 + \epsilon_0$*,* $0 \le v_0 \le \bar{B}_2 + \epsilon_0$*, the following hold.*

(i) If $0 < u_0 < \delta_0$*, then* $u(x, t+t_0; t_0, u_0, v_0) > \inf u_0(x) \quad \forall\ 0 < t \le T$.

(ii) If $0 < v_0 < \delta_0$, *then* $v(x,t+t_0;t_0,u_0,v_0) > \inf v_0(x) \quad \forall\ 0 < t \le T$.

Proof. (1)(i) Without loss of generality, assume $\inf_{x\in\Omega} u_0(x) > 0$. By Theorem 3.1 (1),

$$u(x,t+t_0;t_0,u_0,v_0) \le \bar{A}_1 + \epsilon_0,\ \ v(x,t+t_0;t_0,u_0,v_0) \le \bar{A}_2 + \epsilon_0 \quad \forall\ t \ge 0, \quad x \in \bar{\Omega}.$$

This together with Lemma 3.4 (1) implies that

$$\begin{aligned}
u_t &= d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})v - \frac{\chi_1\lambda}{d_3}w\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w\\
&\quad + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})(\bar{A}_2+\epsilon_0) - \frac{\chi_1\lambda}{d_3}\big(\frac{k}{\lambda}\epsilon_0 + \frac{l}{\lambda}(\bar{A}_2+\epsilon_0)\big)\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - a_2(t,x)(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_1(t,x) - \frac{\chi_1 k}{d_3})u\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_{0,\inf} - a_{2,\sup}(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})u\Big)
\end{aligned}$$

for $0 < t \le T$. Let $\tilde{u}(t)$ be the solution of

$$\tilde{u}_t = \tilde{u}\Big(a_{0,\inf} - a_{2,\sup}(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})\tilde{u}\Big)$$

with $\tilde{u}(t_0) = \inf_{x\in\bar{\Omega}} u_0(x)$. We have $\tilde{u}(t)$ is monotonically increasing in $t \ge t_0$ and

$$\lim_{t\to\infty}\tilde{u}(t) = \frac{a_{0,\inf} - a_{2,\sup}(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0}{(a_{1,\sup} - \frac{\chi_1 k}{d_3})}.$$

By comparison principle for parabolic equations, we have

$$u(x,t+t_0;t_0,u_0) \ge \tilde{u}(t+t_0) > \inf_{x\in\bar{\Omega}} u_0(x) \quad \forall\ 0 < t \le T.$$

(1)(ii) It can be proved by the similar arguments as those in (1)(i).

(2)(i) Again, without loss of generality, assume $\inf_{x\in\Omega} u_0(x) > 0$. By Theorem 3.1 (2),

$$u(x,t+t_0;t_0,u_0,v_0) \le \bar{B}_1 + \epsilon_0, \;\; v(x,t+t_0;t_0,u_0,v_0) \le \bar{B}_2 + \epsilon_0 \quad \forall\, t \ge 0, \quad x \in \bar{\Omega}.$$

This together with Lemma 3.4 (2) implies that

$$\begin{aligned}
u_t &= d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})v - \frac{\chi_1\lambda}{d_3}w\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w +\\
&\quad u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - \big(a_2(t,x) - \frac{\chi_1 l}{d_3}\big)_+(\bar{B}_2+\epsilon_0) - \frac{\chi_1\lambda}{d_3}\big(\frac{k}{\lambda}\epsilon_0 + \frac{l}{\lambda}(\bar{B}_2+\epsilon_0)\big)\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w\\
&\quad + u\Big(a_{0,\inf} - \big[\big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+ + \frac{\chi_1 l}{d_3}\big](\bar{B}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})u\Big)
\end{aligned}$$

for $0 < t \le T$. Let $\tilde{u}(t)$ be the solution of

$$\tilde{u}_t = \tilde{u}\Big(a_{0,\inf} - \big[\big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+ + \frac{\chi_1 l}{d_3}\big](\bar{B}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})\tilde{u}\Big)$$

with $\tilde{u}(t_0) = \inf_{x\in\bar{\Omega}} u_0(x)$. We have $\tilde{u}(t)$ is monotonically increasing in $t \ge t_0$ and

$$\lim_{t\to\infty}\tilde{u}(t) = \frac{a_{0,\inf} - \big[\big(a_{2,\sup} - \frac{\chi_1 l}{d_3}\big)_+ + \frac{\chi_1 l}{d_3}\big](\bar{B}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0}{(a_{1,\sup} - \frac{\chi_1 k}{d_3})}.$$

By comparison principle for parabolic equations, we have

$$u(x,t+t_0;t_0,u_0) \ge \tilde{u}(t+t_0) > \inf_{x\in\bar{\Omega}} u_0(x) \quad \forall\, 0 < t \le T.$$

(2)(ii) It can be proved by the similar arguments as those in (2)(i). □

Remark 3.9. *Consider* (3.11) *and assume* (3.12). *By the arguments of Lemma 3.5, the following holds. Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Remark 3.8 holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$,

and

$$a_{0,\inf} > \frac{\chi_1 k}{d_3}\epsilon_0 \quad \text{and} \quad \delta_0 < \frac{a_{0,\inf} - \frac{\chi_1 k}{d_3}\epsilon_0}{a_{1,\sup} - \frac{\chi_1 k}{d_3}}.$$

For given $0 \le u_0 \le \bar{A}_1 + \epsilon_0$*, if* $0 < u_0 < \delta_0$*, then* $u(x, t+t_0; t_0, u_0) > \inf u_0(x) \quad \forall\ 0 < t \le T$.

Lemma 3.6. *(1) Assume (H3). Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(1) holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. There are* $\underline{A}_1^1 > 0$ *and* $\underline{A}_2^1 > 0$ *such that for any* $t_0 \in \mathbb{R}$ *and* $0 < u_0 < \bar{A}_1 + \epsilon_0$ *and* $0 < v_0 < \bar{A}_2 + \epsilon_0$*, the following hold.*

(i) For any $t \ge T$*, if* $\sup_{x\in\bar{\Omega}} u(x, t+t_0; t_0, u_0, v_0) \ge \delta_0$*,* $\inf_{x\in\bar{\Omega}} u(x, t+t_0; t_0, u_0, v_0) \ge \underline{A}_1^1$.

(ii) For any $t \ge T$*, if* $\sup_{x\in\bar{\Omega}} v(x, t+t_0; t_0, u_0, v_0) \ge \delta_0$*,* $\inf_{x\in\bar{\Omega}} v(x, t+t_0; t_0, u_0, v_0) \ge \underline{A}_2^1$.

(2) Assume (H4). Let ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(2) holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. There are* $\underline{B}_1^1 > 0$ *and* $\underline{B}_2^1 > 0$ *such that for any* $t_0 \in \mathbb{R}$ *and* $0 < u_0 < \bar{B}_1 + \epsilon_0$ *and* $0 < v_0 < \bar{B}_2 + \epsilon_0$*, the following hold.*

(i) For any $t \ge T$*, if* $\sup_{x\in\bar{\Omega}} u(x, t+t_0; t_0, u_0, v_0) \ge \delta_0$*,* $\inf_{x\in\bar{\Omega}} u(x, t+t_0; t_0, u_0, v_0) \ge \underline{B}_1^1$.

(ii) For any $t \ge T$*, if* $\sup_{x\in\bar{\Omega}} v(x, t+t_0; t_0, u_0, v_0) \ge \delta_0$*,* $\inf_{x\in\bar{\Omega}} v(x, t+t_0; t_0, u_0, v_0) \ge \underline{B}_2^1$.

Proof. (1)(i) Assume that (1)(i) does not hold. Then there are $t_{0n} \in \mathbb{R}$, $t_n \ge T$, and u_n, v_n with $0 < u_n < \bar{A}_1 + \epsilon_0$ and $0 < v_n < \bar{A}_2 + \epsilon_0$ such that

$$\sup_{x\in\bar{\Omega}} u(x, t_n + t_{0n}; t_{0n}, u_n, v_n) \ge \delta_0, \quad \lim_{n\to\infty} \inf_{x\in\bar{\Omega}} u(x, t_n + t_{0n}; t_{0n}, u_n, v_n) = 0.$$

By Theorem 3.1(1),

$$0 < u(x, t+t_0; t_0, u_0, v_0) < \bar{A}_1 + \epsilon_0, \quad 0 < v(x, t+t_0; t_0, u_0, v_0) < \bar{A}_2 + \epsilon_0 \quad \forall\ t > 0,\ x \in \bar{\Omega}.$$

Without loss of generality, we may assume that

$$\lim_{n\to\infty} a_i(x, t+t_n+t_{0n}) = \tilde{a}_i(x,t), \quad \lim_{n\to\infty} b_i(x, t+t_n+t_{0n}) = \tilde{b}_i(x,t)$$

and

$$\lim_{n\to\infty} u(x, t+t_n+t_{0n}; t_{0n}, u_n, v_n) = \tilde{u}(x,t), \quad \lim_{n\to\infty} v(x, t+t_n+t_{0n}; t_{0n}, u_n, v_n) = \tilde{v}(x,t)$$

uniformly in $x \in \bar{\Omega}$ and t in bounded closed sets of $(-T, \infty)$. Note that

$$\begin{aligned} & u(x, t+t_n+t_{0n}; t_{0n}, u_n, v_n) \\ & = u(x, t+t_n+t_{0n}; t_n+t_{0n}, u(\cdot, t_n+t_{0n}; t_{0n}, u_n, v_n), v(\cdot, t_n+t_{0n}; t_{0n}, u_n, v_n)), \end{aligned}$$

and

$$\begin{aligned} & v(x, t+t_n+t_{0n}; t_{0n}, u_n, v_n) \\ & = v(x, t+t_n+t_{0n}; t_n+t_{0n}, u(\cdot, t_n+t_{0n}; t_{0n}, u_n, v_n), v(\cdot, t_n+t_{0n}; t_{0n}, u_n, v_n)). \end{aligned}$$

Therefore

$$\tilde{u}(x,t) = \tilde{u}(x,t;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)), \quad \tilde{v}(x,t) = \tilde{v}(x,t;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)),$$

where $(\tilde{u}(x,t;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)), \tilde{v}(x,t;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)), \tilde{w}(x,t;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)))$ is the solution of (3.1) on $(-T,\infty)$ with a_i being replaced by $\tilde{a}_i$ and b_i being replaced by $\tilde{b}_i$, and

$$\big(\tilde{u}(x,0;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0)), \tilde{v}(x,0;0,\tilde{u}(\cdot,0),\tilde{v}(\cdot,0))\big) = \big(\tilde{u}(x,0), \tilde{v}(x,0)\big).$$

Moreover $\tilde{u}(x,-T/2) \geq 0$, $\tilde{v}(x,-T/2) \geq 0$ for $x \in \bar{\Omega}$, with $\sup_{x\in\bar{\Omega}} \tilde{u}(x,0) \geq \delta_0$ and $\inf_{x\in\bar{\Omega}} \tilde{u}(x,0) = 0$, which is a contradiction by comparison principle for parabolic equations. Hence (1)(i) holds.

(1)(ii), (2)(i), (2)(ii) can be proved by the similar arguments as those in (1)(i). □

Remark 3.10. *Consider* (3.11) *and assume* (3.12)*. By the arguments of Lemma 3.6, the following holds. Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Remark 3.8 holds with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. There is* $\underline{A}_1^1 > 0$ *such that for any* $t_0 \in \mathbb{R}$ *and* $0 < u_0 < \bar{A}_1 + \epsilon_0$*, for any* $t \geq T$*, if* $\sup_{x\in\bar{\Omega}} u(x, t + t_0; t_0, u_0) \geq \delta_0$*, then* $\inf_{x\in\bar{\Omega}} u(x, t + t_0; t_0, u_0) \geq \underline{A}_1^1$*.*

Lemma 3.7. *(1) Assume (H6). Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(1) and Lemma 3.5(1) hold with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. There are* $\underline{A}_1^2 > 0$ *and* $\underline{A}_2^2 > 0$ *such that for any* $t_0 \in \mathbb{R}$ *and* $0 < u_0 < \bar{A}_1 + \epsilon_0$ *and* $0 < v_0 < \bar{A}_2 + \epsilon_0$*, the following holds.*

(i) For any $\underline{A}_1 \leq \underline{A}_1^2$*, if* $\inf_{x\in\bar{\Omega}} u_0(x) \geq \underline{A}_1$*, then* $\inf_{x\in\bar{\Omega}} u(x, T + t_0; t_0, u_0, v_0) \geq \underline{A}_1$*.*

(ii) For any $\underline{A}_2 \leq \underline{A}_2^2$*, if* $\inf_{x\in\bar{\Omega}} v_0(x) \geq \underline{A}_2$*, then* $\inf_{x\in\bar{\Omega}} v(x, T + t_0; t_0, u_0, v_0) \geq \underline{A}_2$*.*

(2) Assume (H7). Let ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(2) and Lemma 3.5(2) hold with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. There are* $\underline{B}_1^2 > 0$ *and* $\underline{B}_2^2 > 0$ *such that for any* $t_0 \in \mathbb{R}$ *and* $0 < u_0 < \bar{B}_1 + \epsilon_0$ *and* $0 < v_0 < \bar{B}_2 + \epsilon_0$*, the following holds.*

(i) For any $\underline{B}_1 \leq \underline{B}_1^2$*, if* $\inf_{x\in\bar{\Omega}} u_0(x) \geq \underline{B}_1$*, then* $\inf_{x\in\bar{\Omega}} u(x, T + t_0; t_0, u_0, v_0) \geq \underline{B}_1$*.*

(ii) For any $\underline{B}_2 \leq \underline{B}_2^2$*, if* $\inf_{x\in\bar{\Omega}} v_0(x) \geq \underline{B}_2$*, then* $\inf_{x\in\bar{\Omega}} v(x, T + t_0; t_0, u_0, v_0) \geq \underline{B}_2$*.*

Proof. (1)(i) We prove it using properly modified similar arguments of [30, Lemma 5.3].

Assume that (1)(i) does not hold. Then there are $\underline{A}_{1,n} \to 0$, $0 < u_n < \bar{A}_1 + \epsilon_0$, $0 < v_n < \bar{A}_2 + \epsilon_0$, $t_n \in \mathbb{R}$, and $x_n \in \Omega$ such that

$$u_n(x) \geq \underline{A}_{1,n} \quad \forall\, x \in \bar{\Omega} \quad \text{and} \quad u(x_n, T + t_n; t_n, u_n, v_n) < \underline{A}_{1,n}.$$

Let

$$\Omega_n = \{x \in \Omega \,|\, u_n(x) \geq \frac{\delta_0}{2}\}.$$

Without loss of generality, we may assume that $\lim_{n\to\infty} |\Omega_n|$ exists. Let

$$m_0 = \lim_{n\to\infty} |\Omega_n|.$$

Assume that $m_0 = 0$. Then there is $\tilde{u}_n \in C^0(\bar{\Omega})$ such that

$$\underline{\mathbf{A}}_{1,n} \le \tilde{u}_n(x) \le \frac{\delta_0}{2} \quad \text{and} \quad \lim_{n\to\infty} \|u_n - \tilde{u}_n\|_{L^p(\Omega)} = 0 \quad \forall\ 1 \le p < \infty.$$

This implies that

$$\lim_{n\to\infty} \|\phi_n^1(\cdot,t)\|_{L^p(\Omega)} + \lim_{n\to\infty} \|\phi_n^2(\cdot,t)\|_{L^p(\Omega)} = 0$$

uniformly in $t \in [t_n, t_n + T]$ for all $1 \le p < \infty$, where $\phi_n^1(\cdot,t) = u(\cdot,t;t_n,u_n,v_n) - u(\cdot,t;t_n,\tilde{u}_n,v_n)$ and $\phi_n^2(\cdot,t) = v(\cdot,t;t_n,u_n,v_n) - v(\cdot,t;t_n,\tilde{u}_n,v_n)$. Indeed, let

$$G_n^1(\cdot,t) = u(\cdot,t;t_n,u_n,v_n),\ G_n^2(\cdot,t) = v(\cdot,t;t_n,u_n,v_n),\ W_n(\cdot,t) = w(\cdot,t;t_n,u_n,v_n),$$

$$\tilde{G}_n^1(\cdot,t) = u(\cdot,t;t_n,\tilde{u}_n,v_n),\ \tilde{G}_n^2(\cdot,t) = v(\cdot,t;t_n,\tilde{u}_n,v_n),\ \tilde{W}_n(\cdot,t) = w(\cdot,t;t_n,\tilde{u}_n,v_n),$$

and

$$\hat{W}_n(\cdot,t)(\cdot,t) = w(\cdot,t;t_n,u_n,v_n) - w(\cdot,t;t_n,\tilde{u}_n,v_n).$$

Then

$$\begin{aligned}
&\phi_n^1(\cdot,t)\\
&= e^{-A(t-t_n)}\big(u_n - \tilde{u}_n\big) - \chi_1 \int_{t_n}^t e^{-A(t-s)}\nabla\cdot\big[\phi_n^1(\cdot,s)\nabla W_n(\cdot,s) + \tilde{G}_n^1(\cdot,s)\nabla\hat{W}_n(\cdot,s)\big]ds\\
&+ \int_{t_n}^t e^{-A(t-s)}\phi_n^1(\cdot,s)\Big(1 + a_0(s,\cdot) - a_1(s,\cdot)(G_n^1(\cdot,s) + \tilde{G}_n^1(\cdot,s)) - a_2(s,\cdot)G_n^2(\cdot,s)\Big)ds\\
&- \int_{t_n}^t e^{-A(t-s)}a_2(s,\cdot)\big(\tilde{G}_n^1(\cdot,t)\big)\phi_n^2(\cdot,s)ds, \qquad (3.38)
\end{aligned}$$

and

$$
\begin{aligned}
&\phi_n^2(\cdot,t) = \\
&-\chi_2 \int_{t_n}^t e^{-A(t-s)} \nabla\cdot \big[\phi_n^2(\cdot,s)\nabla W_n(\cdot,s) + \tilde{G}_n^2(\cdot,s)\nabla \hat{W}_n(\cdot,s)\big] ds \\
&+ \int_{t_n}^t e^{-A(t-s)} \phi_n^2(\cdot,s)\Big(1 + b_0(s,\cdot) - b_2(s,\cdot)(G_n^2(\cdot,s) + \tilde{G}_n^2(\cdot,s)) - b_1(s,\cdot)G_n^1(\cdot,s)\Big) ds \\
&- \int_{t_n}^t e^{-A(t-s)} b_1(s,\cdot)\big(\tilde{G}_n^2(\cdot,t)\big)\phi_n^1(\cdot,s) ds,
\end{aligned} \tag{3.39}
$$

where $A = -\Delta + I$ with $D(A) = \Big\{u \in W^{2,p}(\Omega) \,|\, \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\Big\}$ (it is known that A is a sectorial operator in $X = L^p(\Omega)$). Now, fix $1 < p < \infty$. By regularity and a priori estimates for elliptic equations, [17, Theorem 1.4.3], [30, Lemma 2.2], (3.38), and (3.39), for any $\epsilon \in (0, \frac{1}{2})$, there is $C = C(\epsilon) > 0$ such that

$$
\begin{aligned}
&\|\phi_n^1(\cdot,t)\|_{L^p(\Omega)} \\
&\le \|u_n - \tilde{u}_n\|_{L^p(\Omega)} + C\chi_1 \max_{t_n \le s \le t_n + T} \|\nabla W_n(\cdot,s))\|_{C^0(\bar{\Omega})} \int_{t_n}^t (t-s)^{-\epsilon-\frac{1}{2}} \|\phi_n^1(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ C\chi \max_{t_n \le s \le t_n + T} \|\hat{W}_n(\cdot,s)\|_{C^0(\bar{\Omega})} \int_{t_n}^t (t-s)^{-\epsilon-\frac{1}{2}} (\|\phi_n^1(\cdot,s)\|_{L^p(\Omega)} + \|\phi_n^2(\cdot,s)\|_{L^p(\Omega)}) ds \\
&+ C\int_{t_n}^t \{1 + a_{0,sup} + a_{1,\sup}[\max_{t_n \le s \le t_n + T}(\|G^1(\cdot,s)\|_{C^0(\bar{\Omega})} + \|\tilde{G}^1(\cdot,s)\|_{C^0(\bar{\Omega})})]\} \|\phi_n^1(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ Ca_{2,\sup} \max_{t_n \le s \le t_n + T} \|G^2(\cdot,s)\|_{C^0(\bar{\Omega})} \int_{t_n}^t \|\phi_n^1(\cdot,s)\|_{L^p(\Omega)} ds \\
&+ Ca_{2,\sup} \max_{t_n \le s \le t_n + T} \|\tilde{G}^1(\cdot,s)\|_{C^0(\bar{\Omega})} \int_{t_n}^t \|\phi_n^2(\cdot,s)\|_{L^p(\Omega)} ds.
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\phi_n^2(\cdot,t)\|_{L^p(\Omega)}\\
&\le C\chi_2 \max_{t_n\le s\le t_n+T}\|\nabla W_n(\cdot,s))\|_{C^0(\bar\Omega)}\int_{t_n}^t (t-s)^{-\epsilon-\frac12}\|\phi_n^2(\cdot,s)\|_{L^p(\Omega)}ds\\
&+C\chi \max_{t_n\le s\le t_n+T}\|\hat W_n(\cdot,s)\|_{C^0(\bar\Omega)}\int_{t_n}^t (t-s)^{-\epsilon-\frac12}(\|\phi_n^1(\cdot,s)\|_{L^p(\Omega)}+\|\phi_n^2(\cdot,s)\|_{L^p(\Omega)})ds\\
&+C\int_{t_n}^t\{1+b_{0,sup}+b_{2,\sup}[\max_{t_n\le s\le t_n+T}(\|G^2(\cdot,s)\|_{C^0(\bar\Omega)}+\|\tilde G^2(\cdot,s)\|_{C^0(\bar\Omega)})]\}\|\phi_n^2(\cdot,s)\|_{L^p(\Omega)}ds\\
&+Cb_{1,\sup}\max_{t_n\le s\le t_n+T}\|G^1(\cdot,s)\|_{C^0(\bar\Omega)}\int_{t_n}^t\|\phi_n^2(\cdot,s)\|_{L^p(\Omega)}ds\\
&+Cb_{1,\sup}\max_{t_n\le s\le t_n+T}\|\tilde G^2(\cdot,s)\|_{C^0(\bar\Omega)}\int_{t_n}^t\|\phi_n^1(\cdot,s)\|_{L^p(\Omega)}ds.
\end{aligned}
$$

Therefore there exists a positive constant C_0 independent of n such that

$$
\begin{aligned}
&\|\phi_n^1(\cdot,t+t_n)\|_{L^p(\Omega)}+\|\phi_n^1(\cdot,t+t_n)\|_{L^p(\Omega)}\\
&\le\|u_n-\tilde u_n\|_{L^p(\Omega)}+C_0\int_0^t(t-s)^{-\epsilon-\frac12}(\|\phi_n^1(\cdot,s+t_n)\|_{L^p(\Omega)}+\|\phi_n^1(\cdot,s+t_n)\|_{L^p(\Omega)})ds
\end{aligned}
$$

for all $t\in[0,T]$. This together with the generalized Gronwall's inequality (see [17, page 6]) implies that

$$
\lim_{n\to\infty}(\|\phi_n^1(\cdot,t)\|_{L^p(\Omega)}+\|\phi_n^1(\cdot,t)\|_{L^p(\Omega)})=0
$$

uniformly in $t\in[t_n,t_n+T]$ for all $1\le p<\infty$. This implies that

$$
\lim_{n\to\infty}\|w(\cdot,t;t_n,u_n,v_n))-w(\cdot,t;t_n;\tilde u_n,v_n)\|_{C^1(\bar\Omega)}=0
$$

uniformly in $t\in[t_n,t_n+T]$. Note that $v(x,t;t_n,\tilde u_n,v_n)\le\bar A_2+\epsilon_0$ for $t\in[t_n,t_n+T]$ and by Lemma 3.4(1), $u(x,t;t_n,\tilde u_n,v_n)\le\epsilon_0$ for $t\in[t_n,t_n+T]$. Hence

$$
w(\cdot,t;t_n;\tilde u_n,v_n)\le\frac{k}{\lambda}\epsilon_0+\frac{l}{\lambda}(\bar A_2+\epsilon_0)
$$

for all $t\in[t_n,t_n+T]$ and $x\in\Omega$. It then follows that for any $\epsilon>0$,

$$
w(\cdot,t;t_n;u_n,v_n)\le(\frac{k}{\lambda}+\epsilon)\epsilon_0+\frac{l}{\lambda}(\bar A_2+\epsilon_0)
$$

for all $t \in [t_n, t_n + T]$, $x \in \Omega$, and $n \gg 1$. Then by the arguments of Lemma 3.5, $\inf u(\cdot, t_n + T; t_n, u_n) \geq A_{1,n}$, which is a contradiction. Therefore, $m_0 \neq 0$.

By $m_0 \neq 0$ and comparison principle for parabolic equations, without loss of generality, we may assume that

$$\liminf_{n \to \infty} \|e^{-At} u_n\|_{C^0(\bar{\Omega})} > 0 \quad \forall\, t \in [0, T].$$

This implies that there is $0 < T_0 < T$ and $\delta_\infty > 0$ such that

$$\sup_{x \in \bar{\Omega}} u(x, t_n + T_0; t_n, u_n, v_n) \geq \delta_\infty$$

for all $n \gg 1$. By a priori estimates for parabolic equations, without loss of generality, we may assume that

$$u(\cdot, t_n + T_0; t_n, u_n, v_n) \to u_0^*, \quad v(\cdot, t_n + T_0; t_n, u_n, v_n) \to v_0^*$$

and

$$u(\cdot, t_n + T; t_n, u_n, v_n) \to u^*, \quad v(\cdot, t + n + T; t_n, u_n, v_n) \to v^*$$

as $n \to \infty$. Without loss of generality, we may also assume that

$$a_i(t + t_n, x) \to a_i^*(t, x), \quad b_i(t + t_n, \cdot) \to b_i^*(t, x)$$

as $n \to \infty$ locally uniformly in $(t, x) \in \mathbb{R} \times \bar{\Omega}$. Then we have

$$u^*(x) = u^*(x, T; T_0, u_0^*, v_0^*), \quad v^*(x) = v^*(x, T; t_0, u_0^*, v_0^*)$$

and

$$\inf_{x \in \bar{\Omega}} u^*(x) = 0, \quad \inf_{x \in \bar{\Omega}} v^*(x) \geq 0,$$

where $(u^*(x,t;T_0,u_0^*,v_0^*), v^*(x,t;T_0,u_0^*,v_0^*), w(x,t;T_0,u_0^*,v_0^*))$ is the solution of (3.1) with $a_i(t,x)$ and $b_i(t,x)$ being replaced by $a_i^*(t,x)$ and $b_i^*(t,x)$, and

$$(u^*(x,T_0;T_0,u_0^*,v_0^*), v^*(x,T_0;T_0,u_0^*,v_0^*)) = (u_0^*(x), v_0^*(x))$$

. By comparison principle, we must have $u_0^* \equiv 0$. But $\sup u_0^* \geq \delta_\infty$. This is a contradiction.

(1)(ii) It can be proved by the similar arguments as those in (1)(i).

(2) Follows by similar arguments as those in (1). □

Remark 3.11. *Consider* (3.11) *and assume* (3.12). *By the arguments of Lemma 3.7, the following holds. Let ϵ_0 and $\delta_0 = \delta_0(\epsilon_0)$ be such that Remark 3.8 and Remark 3.9 hold with $\epsilon = \epsilon_0$ and $\delta = \delta_0$. There is $\underline{A}_1^2 > 0$ such that for any $t_0 \in \mathbb{R}$ and $0 < u_0 < \bar{A}_1 + \epsilon_0$, for any $\underline{A}_1 \leq \underline{A}_1^2$, if $\inf_{x\in\bar{\Omega}} u_0(x) \geq \underline{A}_1$, then $\inf_{x\in\bar{\Omega}} u(x, T+t_0; t_0, u_0) \geq \underline{A}_1$.*

Let

$$\underline{\mathrm{A}}_1 = \min\{\underline{\mathrm{A}}_1^1, \underline{\mathrm{A}}_1^2\}, \quad \underline{\mathrm{A}}_2 = \min\{\underline{\mathrm{A}}_2^1, \underline{\mathrm{A}}_2^2\}$$

and

$$\underline{\mathrm{B}}_1 = \min\{\underline{\mathrm{B}}_1^1, \underline{\mathrm{B}}_1^2\}, \quad \underline{\mathrm{B}}_2 = \min\{\underline{\mathrm{B}}_2^1, \underline{\mathrm{B}}_2^2\}.$$

Note that the constants $\underline{\mathrm{A}}_1$, $\underline{\mathrm{A}}_2$, $\underline{\mathrm{B}}_1$ and $\underline{\mathrm{B}}_2$ depend on T and ϵ_0.

Lemma 3.8. *(1) Assume (H6). Let ϵ_0 and $\delta_0 = \delta_0(\epsilon_0)$ be such that Lemma 3.4(1) and Lemma 3.5(1) hold with $\epsilon = \epsilon_0$ and $\delta = \delta_0$. For any $0 < u_0 < \bar{A}_1 + \epsilon_0$ and $0 < v_0 < \bar{A}_2 + \epsilon_0$, the following holds.*

(i) If $\inf_{x\in\bar{\Omega}} u_0(x) \geq \underline{A}_1$, then

$$\underline{A}_1 \leq u(x, t+t_0; t_0, u_0, v_0) \leq \bar{A}_1 + \epsilon_0 \quad \forall\, t \geq T,\ \ x \in \bar{\Omega}. \tag{3.40}$$

(ii) If $\inf_{x\in\bar{\Omega}} v_0(x) \geq \underline{A}_2$, then

$$\underline{A}_2 \leq v(x, t+t_0; t_0, u_0, v_0) \leq \bar{A}_2 + \epsilon_0 \quad \forall\, t \geq T,\ \ x \in \bar{\Omega}.$$

(2) Assume (H7). Let ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Lemma 3.4(2) and Lemma 3.5(2) hold with* $\epsilon = \epsilon_0$ *and* $\delta = \delta_0$*. For any* $0 < u_0 < \bar{B}_1 + \epsilon_0$ *and* $0 < v_0 < \bar{B}_2 + \epsilon_0$*, the following hold.*

(i) If $\inf_{x\in\bar{\Omega}} u_0(x) \geq \underline{B}_1$*, then*

$$\underline{B}_1 \leq u(x, t + t_0; t_0, u_0, v_0) \leq \bar{B}_1 + \epsilon_0 \quad \forall\, t \geq T,\ \ x \in \bar{\Omega}.$$

(ii) If $\inf_{x\in\bar{\Omega}} v_0(x) \geq \underline{B}_2$*, then*

$$\underline{B}_2 \leq v(x, t + t_0; t_0, u_0, v_0) \leq \bar{B}_2 + \epsilon_0 \quad \forall\, t \geq T,\ \ x \in \bar{\Omega}.$$

Proof. (1)(i) First of all, by Lemma 3.7(1), we have

$$\underline{\mathrm{A}}_1 \leq u(x, T + t_0; t_0, u_0, v_0) \leq \bar{A}_1 + \epsilon_0 \quad \forall\, x \in \bar{\Omega}.$$

Note that we have

$$\text{either } \sup_{x\in\bar{\Omega}} u(x, T + t_0; t_0, u_0, v_0) > \delta_0 \text{ or } \sup_{x\in\bar{\Omega}} u(x, T + t_0; t_0, u_0, v_0) \leq \delta_0.$$

In the former case, if $\sup_{x\in\bar{\Omega}} u(x, t + T + t_0; t_0, u_0, v_0) > \delta_0$ for all $0 \leq t \leq T$, by Lemma 3.6, (3.40) holds for all $T \leq t \leq 2T$. If there is $t^* \in (T, 2T)$ such that $\sup_{x\in\bar{\Omega}} u(x, t + t_0; t_0, u_0, v_0) > \delta_0$ for $T \leq t \leq t^*$ and $\sup_{x\in\bar{\Omega}} u(x, t^* + t_0; t_0, u_0, v_0) = \delta_0$, then by Lemma 3.6, (3.40) holds for all $T \leq t \leq t^*$, which together with Lemma 3.5 implies that (3.40) also holds for all $t^* \leq t \leq 2T$. In the later case, by Lemma 3.5, (3.40) also holds for all $T \leq t \leq 2T$. Therefore, in any case, (3.40) also holds for all $T \leq t \leq 2T$. Repeating the above process, we have that (3.40) also holds for all $t \geq T$.

(1)(ii) It can be proved by the similar arguments as those in (1)(i).

(2) It follows from the similar arguments as those in (1). □

Remark 3.12. *Consider* (3.11) *and assume* (3.12)*. By the arguments of Lemma 3.8, the following holds. Let* ϵ_0 *and* $\delta_0 = \delta_0(\epsilon_0)$ *be such that Remark 3.8 and Remark 3.9 hold with* $\epsilon = \epsilon_0$

and $\delta = \delta_0$. For any $0 < u_0 < \bar{A}_1 + \epsilon_0$, if $\inf_{x\in\bar\Omega} u_0(x) \ge \underline{A}_1$, then

$$\underline{A}_1 \le u(x, t+t_0; t_0, u_0) \le \bar{A}_1 + \epsilon_0 \quad \forall\, t \ge T,\;\; x \in \bar\Omega.$$

We now prove Theorem 3.2.

Proof of Theorem 3.2. (1) Let ϵ_0 and $\delta_0 = \delta_0(\epsilon_0)$ be such that Lemma 3.4(1) and Lemma 3.5(1) hold with $\epsilon = \epsilon_0$ and $\delta = \delta_0$. Let $\underline{A}_1$, $\bar{A}_1$, $\underline{A}_2$, and $\bar{A}_2$ be as in Lemma 3.8(1). By the assumption that $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, and comparison principle for parabolic equations, without loss of generality, we may assume that $\inf_{x\in\bar\Omega} u_0(x) > 0$ and $\inf_{x\in\bar\Omega} v_0(x) > 0$.

First, by Theorem 3.1, there is $T_1 = T_1(u_0, v_0, \epsilon_0)$ such that

$$u(x, t+t_0; t_0, u_0, v_0) \le \bar{A}_1 + \epsilon_0,\;\; v(x, t+t_0; t_0, u_0, v_0) \le \bar{A}_2 + \epsilon_0 \quad \forall\, t \ge T_1, \quad x \in \bar\Omega.$$

Observe that if $\sup_{x\in\Omega} u(x, t+t_0; t_0, u_0, v_0) < \delta_0$, then

$$\begin{aligned}
u_t &= d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big)\\
&= d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})v - \frac{\chi_1\lambda}{d_3}w\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w +\\
&\quad u\Big(a_0(t,x) - (a_1(t,x) - \frac{\chi_1 k}{d_3})u - (a_2(t,x) - \frac{\chi_1 l}{d_3})(\bar{A}_2+\epsilon_0) - \frac{\chi_1\lambda}{d_3}(\frac{k}{\lambda}\delta_0 + \frac{l}{\lambda}(\bar{A}_2+\epsilon_0))\Big)\\
&\ge d_1\Delta u - \chi_1\nabla u\cdot\nabla w + u\Big(a_0(t,x) - a_2(t,x)(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_1(t,x) - \frac{\chi_1 k}{d_3})u\Big).
\end{aligned}$$

Let $\tilde{u}(t;\tilde{u}_0)$ be the solution of

$$\tilde{u}_t = \tilde{u}\Big(a_{0,\inf} - a_{2,\sup}(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})\tilde{u}\Big)$$

with $\tilde{u}(0;\tilde{u}_0) = \tilde{u}_0 \in (0, \delta_0)$. We have $\tilde{u}(t)$ is monotonically increasing in $t \ge 0$ and

$$\lim_{t\to\infty} \tilde{u}(t;\tilde{u}_0) = \frac{a_{0,\inf} - a_{2,\sup}(\bar{A}_2+\epsilon_0) - \frac{\chi_1 k}{d_3}\epsilon_0}{(a_{1,\sup} - \frac{\chi_1 k}{d_3})} > \delta_0. \tag{3.41}$$

Observe also that

$$\inf_{t_0\in\mathbb{R}}\inf_{x\in\bar\Omega} u(x,T+t_0;t_0,u_0,v_0)>0. \tag{3.42}$$

Indeed, we have either $\sup u_0<\delta_0$ or $\sup u_0\ge\delta_0$. If $\sup u_0<\delta_0$, we have by Lemma 3.5 (i) that $\inf_{x\in\bar\Omega}u(x,T+t_0;t_0,u_0,v_0)\ge\inf u_0>0$ for all $t_0\in\mathbb{R}$ and then (3.42) follows. If $\sup u_0\ge\delta_0$, but (3.42) does not hold, then there are $t_{0n}\in\mathbb{R}$ and $x_n\in\bar\Omega$ such that

$$\lim_{n\to\infty}u(x_n,T+t_{0n};t_{0n},u_0,v_0)=0.$$

Let $a_i^n(t,x)=a_i(t+t_{0n},x)$ and $b_i^n(t,x)=b_i(t+t_{0n},x)$ for $i=0,1,2$. Then

$$\begin{aligned}&(u(x,t+t_{0n};t_{0n},u_0,v_0),v(x,t+t_{0n};t_{0n},u_0,v_0),w(x,t+t_{0n};t_{0n},u_0,v_0))\\&=(u^n(x,t;u_0,v_0),v^n(x,t;u_0,v_0),w^n(x,t;u_0,v_0))\end{aligned}$$

for $t\ge0$, where $(u^n(x,t;u_0,v_0),v^n(x,t;u_0,v_0),w^n(x,t;u_0,v_0))$ is the solution of (3.1) with a_i and b_i ($i=0,1,2$) being replaced by a_i^n and b_i^n ($i=0,1,2$) and

$$(u^n(x,0;u_0,v_0),v^n(x,0;u_0,v_0))=(u_0(x),v_0(x)).$$

Without loss of generality, we may assume that

$$\lim_{n\to\infty}a_i^n(t,x)=a_i^\infty(t,x),\quad \lim_{n\to\infty}b_i^n(t,x)=b_i^\infty(t,x)$$

uniformly in $x\in\bar\Omega$ and t in bounded sets of $\mathbb{R}$, and

$$\lim_{n\to\infty}x_n=x_\infty.$$

Then

$$\begin{aligned}&\lim_{n\to\infty}(u^n(x,t;u_0,v_0),v^n(x,t;u_0,v_0),w^n(x,t;u_0,v_0))\\&=(u^\infty(x,t;u_0,v_0),v^\infty(x,t;u_0,v_0),w^\infty(x,t;u_0,v_0))\end{aligned}$$

uniformly in $x \in \bar{\Omega}$ and t in bounded set of $[0, \infty)$, where $(u^{\infty}(x, t; u_0, v_0), v^{\infty}(x, t; u_0, v_0), w^{\infty}(x, t; u_0, v_0))$ is the solution of (3.1) with a_i and b_i $(i = 0, 1, 2)$ being replaced by a_i^{∞} and b_i^{∞} $(i = 0, 1, 2)$ and $(u^{\infty}(x, 0; u_0, v_0), v^{\infty}(x, 0; u_0, v_0)) = (u_0(x), v_0(x))$. It then follows that

$$\inf u_0(x) > 0 \quad \text{and} \quad u^{\infty}(x_{\infty}, T; u_0, v_0) = 0,$$

which is a contradiction. Hence if $\sup u_0 \geq \delta_0$, (3.42) also holds.

Note that we have either $\sup_{x \in \Omega} u(x, T + t_0; t_0, u_0, v_0) \geq \delta_0$ or $\sup_{x \in \Omega} u(x, T + t_0; t_0, u_0, v_0) < \delta_0$. If $\sup_{x \in \Omega} u(x, T + t_0; t_0, u_0, v_0) < \delta_0$, by (3.41), (3.42), and comparison principle for parabolic equations, there are $T_2(u_0, v_0, \epsilon_0) \geq T$ and $T \leq \tilde{T}_2(u_0, v_0, \epsilon_0) \leq T_2(u_0, v_0, \epsilon_0)$ such that

$$\sup_{x \in \Omega} u(x, \tilde{T}_2(u_0, v_0, \epsilon_0) + t_0; t_0, u_0, v_0) = \delta_0.$$

Hence, in either case, there is $\tilde{T}_2(u_0, v_0, \epsilon_0) \in [T, T_2(u_0, v_0, \epsilon_0)]$ such that

$$\sup_{x \in \Omega} u(x, \tilde{T}_2(u_0, v_0, \epsilon_0) + t_0; t_0, u_0, v_0) \geq \delta_0.$$

This together with Lemma 3.6 implies that

$$\inf_{x \in \Omega} u(x, \tilde{T}_2(u_0, v_0, \epsilon_0) + t_0; t_0, u_0, v_0) \geq \underline{A}_1.$$

Then by Lemma 3.8(1),

$$\underline{A}_1 \leq u(x, t + t_0; t_0, u_0, v_0) \leq \bar{A}_1 + \epsilon_0 \quad \forall\, t \geq \max\{T_1(u_0, v_0, \epsilon_0), T + T_2(u_0, v_0, \epsilon_0)\}. \tag{3.43}$$

Similarly, we can prove that there are $\tilde{T}_1(u_0, v_0, \epsilon_0) > 0$ and $\tilde{T}_2(u_0, v_0, \epsilon_0) \geq T$ such that

$$\underline{A}_2 \leq v(x, t + t_0; t_0, u_0, v_0) \leq \bar{A}_2 + \epsilon_0 \quad \forall\, t \geq \max\{\tilde{T}_1(u_0, v_0, \epsilon_0), T + \tilde{T}_2(u_0, v_0, \epsilon_0)\}.$$

This together with Theorem 3.1 and (3.43) implies that for any $\epsilon > 0$, there is t_{ϵ, u_0, v_0} such that (3.10) holds.

(2) It follows from the similar arguments as those in (1). $\square$

Corollary 3.2. *Consider* (3.11) *and assume* (3.12)*. There is* $\underline{A}_1$ *such that for any* $\epsilon > 0$*,* $t_0 \in \mathbb{R}$*,* $u_0 \in C^0(\bar{\Omega})$ *with* $u_0 \geq 0$*, and* $u_0 \not\equiv 0$*, there exists* t_{ϵ,u_0} *such that*

$$\underline{A}_1 \leq u(x,t;t_0,u_0) \leq \bar{A}_1 + \epsilon$$

for all $x \in \bar{\Omega}$ *and* $t \geq t_0 + t_{\epsilon,u_0}$*, where* $(u(x,t;t_0,u_0), w(x,t;t_0,u_0))$ *is the global solution of* (3.11) *with* $u(x,t_0;t_0,u_0) = u_0(x)$ *(see Corollary 3.2).*

Proof. It follows from Remarks 3.8-3.12 and the arguments of Theorem 3.2. $\square$

3.5 Coexistence

In this section, we study the existence of coexistence states in (3.1) and prove Theorem 3.3.

We first prove a lemma.

Lemma 3.9. *Consider*

$$\begin{cases} u_t = u\big(a_0(t) - a_1(t)u - a_2(t)v\big) \\ v_t = v\big(b_0(t) - b_1(t)u - b_2(t)v\big). \end{cases} \tag{3.44}$$

Assume (3.5) *is satisfied. Then there is a positive entire solution* $(u^{**}(t), v^{**}(t))$ *of* (3.44)*. Moreover, for any* $u_0, v_0 > 0$ *and* $t_0 \in \mathbb{R}$*,*

$$(u(t;t_0,u_0,v_0), v(t;t_0,u_0,v_0)) - (u^{**}(t), v^{**}(t)) \to 0$$

as $t \to \infty$*, where* $(u(t;t_0,u_0,v_0)$*,* $v(t;t_0,u_0,v_0))$ *is the solution of* (3.44) *with* $(u(t_0;t_0,u_0,v_0)$*,* $v(t_0;t_0,u_0,v_0)) = (u_0,v_0)$*. In addition, if* $a_i(t)$ *and* $b_i(t)$ *are almost periodic, then so is* $(u^{**}(t)$*,* $v^{**}(t))$*.*

Proof. First, let

$$s_1 = \frac{b_{2,\inf}a_{0,\inf} - a_{2,\sup}b_{0,\sup}}{b_{2,\inf}a_{1,\sup} - a_{2,\sup}b_{1,\inf}}, \quad r_1 = \frac{b_{2,\sup}a_{0,\sup} - a_{2,\inf}b_{0,\inf}}{b_{2,\sup}a_{1,\inf} - a_{2,\inf}b_{1,\sup}},$$

and

$$r_2 = \frac{a_{1,\inf} b_{0,\inf} - b_{1,\sup} a_{0,\sup}}{a_{1,\inf} b_{2,\sup} - b_{1,\sup} a_{2,\inf}}, \quad s_2 = \frac{a_{1,\sup} b_{0,\sup} - b_{1,\inf} a_{0,\inf}}{a_{1,\sup} b_{2,\inf} - b_{1,\inf} a_{2,\sup}}.$$

Then

$$0 < s_1 \le r_1 \quad \text{and} \quad 0 < r_2 \le s_2.$$

Next, for given $t_0 \in \mathbb{R}$ and $u_0, v_0 \in \mathbb{R}$, if $0 < u_0 \le r_1$ and $v_0 \ge r_2$, by [1, Lemma 3.1], we have

$$0 < u(t; t_0, u_0, v_0) \le r_1 \quad \text{and} \quad v(t; t_0, u_0, v_0) \ge r_2 \quad \forall\, t \ge t_0. \tag{3.45}$$

And if $u_0 \ge s_1$ and $0 < v_0 \le s_2$, by [1, Lemma 3.2] again,

$$u(t; t_0, u_0, v_0) \ge s_1 \quad \text{and} \quad 0 < v(t; t_0, u_0, v_0) \le s_2 \quad \forall\, t \ge t_0. \tag{3.46}$$

We now start with the proof of existence of positive entire solutions of (3.44) by the so called pullback method. Fix $u_0, v_0 \in \mathbb{R}$ such that $s_1 \le u_0 \le r_1$ and $s_2 \le v_0 \le r_2$. For $n \in \mathbb{N}$, let $t_n = -n$, $u_n = u(0; t_n, u_0, v_0)$ and $v_n = v(0; t_n, u_0, v_0)$. Then by (3.45) and (3.46), we have

$$s_1 \le u_n \le r_1 \text{ and } s_2 \le v_n \le r_2 \quad \forall\, n \in \mathbb{N}.$$

Therefore there exists $u_0^0, v_0^0 \in \mathbb{R}$ such that up to subsequence $u_n \to u_0^0$ as $n \to \infty$ and $v_n \to v_0^0$ as $n \to \infty$. And so

$$s_1 \le u_0^0 \le r_1 \text{ and } s_2 \le v_0^0 \le r_2.$$

Furthermore $(u(t; t_n, u_0, v_0), v(t; t_n, u_0, v_0)) \to (u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0))$ as $n \to \infty$. Again by (3.45) and (3.46), we have

$$s_1 \le u(t; 0, u_0^0, v_0^0) \le r_1 \text{ and } s_2 \le u(t; 0, u_0^0, v_0^0) \le r_2 \quad \forall\, t \ge 0.$$

We claim that $(u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0))$ has backward extension. Indeed for each $m \in \mathbb{N}$, we define for $n > m$, $u_n^m = u(-m; t_n, u_0, v_0)$ and $v_n^m = v(-m; t_n, u_0, v_0)$. Then by similar arguments as before there exist $u_0^m, v_0^m \in \mathbb{R}$ such that up to subsequence $u_n^m \to u_0^m$ as $n \to \infty$ and $v_n^m \to v_0^m$ as $n \to \infty$, $s_1 \le u_0^m \le r_1$ and $s_2 \le v_0^m \le r_2$, and $(u(t; t_n, u_0, v_0), v(t; t_n, u_0, v_0)) \to (u(t; -m, u_0^m, v_0^m), v(t; -m, u_0^m, v_0^m))$ as $n \to \infty$. It follows that

$$(u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0)) = (u(t; -m, u_0^m, v_0^m), v(t; -m, u_0^m, v_0^m))$$

for all $t \ge 0$. Thus $(u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0))$ has backward extension up $t \ge -m$, for each $m \in \mathbb{N}$. This show that $(u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0))$ is defined for all $t \in \mathbb{R}$ and moreover we have $s_1 \le u(t; 0, u_0^0, v_0^0) \le r_1$ and $s_2 \le v(t; 0, u_0^0, v_0^0) \le r_2$, for all $t \in \mathbb{R}$. Hence $u(t; 0, u_0^0, v_0^0), v(t; 0, u_0^0, v_0^0))$ is a positive entire solution of (3.44).

Finally, we prove the stability of positive entire solutions and the almost periodicity of positive entire solutions when the coefficients are almost periodic. Let $(u^{**}(t), v^{**}(t))$ be a positive entire solution of (3.44) and let $u_0, v_0 > 0$ and $t_0 \in \mathbb{R}$. It follows from [1, Theorem 1] that

$$(u(t; t_0, u_0, v_0), v(t; t_0, u_0, v_0)) - (u^{**}(t), v^{**}(t)) \to 0 \quad \text{as } \to \infty.$$

By [22, Theorem C], when $a_i(t)$ and $b_i(t)$ ($i = 0, 1, 2$) are almost periodic in t, then positive entire solutions of (3.44) are unique and almost periodic. The lemma thus follows. □

We now prove Theorem 3.3. Let $T > 0$ be fixed and $\underline{A}_i$, $\bar{A}_i$, $\underline{B}_i$, and $\bar{B}_i$ ($i = 1, 2$) be as in the previous section.

Proof of Theorem 3.3. (1) We first prove the existence of positive entire solutions. Let $u_0, v_0 \in C^0(\bar{\Omega})$ be such that $0 < \underline{A}_1 \le u_0(x) \le \bar{A}_1$ and $0 < \underline{A}_2 \le v_0(x) \le \bar{A}_2$. By Theorem 3.1(1) and Lemma 3.8(1),

$$0 < \underline{A}_1 \le u(x, t + t_0; t_0, u_0, v_0) \le \bar{A}_1 \quad \text{and} \quad 0 < \underline{A}_2 \le v(x, t + t_0; t_0, u_0, v_0) \le \bar{A}_2$$

for all $x \in \bar{\Omega}$, $t \ge T$, and $t_0 \in \mathbb{R}$. For $n \in \mathbb{N}$ with $n > T$, set $t_n = -n$, $u_n = u(\cdot, 0; t_n, u_0, v_0)$ and $v_n = v(\cdot, 0; t_n, u_0, v_0)$. Then by parabolic regularity there exist $t_{n_k} \in \mathbb{N}$, u_0^{**}, $v_0^{**} \in C^0(\bar{\Omega})$

such that

$$u_{n_k} \to u_0^{**} \quad \text{and} \quad v_{n_k} \to v_0^{**} \quad \text{in } C^0(\bar{\Omega}).$$

We have $u(\cdot,t;t_{n_k},u_0,v_0) = u(\cdot,t;0,u(\cdot,0;t_{n_k},u_0,v_0),v(\cdot,0;t_{n_k},u_0,v_0))$, and $v(\cdot,t;t_{n_k},u_0,v_0) = v(\cdot,t;0,u(\cdot,0;t_{n_k},u_0,v_0),v(\cdot,0;t_{n_k},u_0,v_0))$. Thus for $t \geq 0$ we have

$$(u(\cdot,t;t_{n_k},u_0,v_0),v(\cdot,t;t_{n_k},u_0,v_0)) \to (u(\cdot,t;0,u_0^{**},v_0^{**}),v(\cdot,t;0,u_0^{**},v_0^{**})) \text{ in } C^0(\bar{\Omega})\times C^0(\bar{\Omega}).$$

Moreover

$$0 < \underline{\mathrm{A}}_1 \leq u(x,t;0,u_0^{**},v_0^{**}) \leq \bar{A}_1 \quad \text{and} \quad 0 < \underline{\mathrm{A}}_2 \leq v(x,t;0,u_0^{**},v_0^{**}) \leq \bar{A}_2 \quad \forall\, x \in \Omega,\ t \geq 0.$$

We now prove that $(u(\cdot,t;0,u_0^{**},v_0^{**}),v(\cdot,t;0,u_0^{**},v_0^{**}))$ has backward extension. In order to prove that, fix $m \in \mathbb{N}$ and define $u_n^m = u(\cdot,-m;t_n,u_0,v_0)$ and $v_n^m = v(\cdot,-m;t_n,u_0,v_0)$ for all $n > m+T$. Then by parabolic regularity, without loss of generality, we may assume that there exist $u_m^{**},\ v_m^{**} \in C^0(\bar{\Omega})$ such that

$$u_{n_k}^m \to u_m^{**} \quad \text{and} \quad v_{n_k}^m \to v_m^{**} \quad \text{in } C^0(\bar{\Omega}).$$

Furthermore we have $u(\cdot,t;t_{n_k},u_0,v_0) = u(\cdot,t;-m,u(\cdot,-m;t_{n_k},u_0,v_0),v(\cdot,-m;t_{n_k},u_0,v_0))$, and $v(\cdot,t;t_{n_k},u_0,v_0) = u(\cdot,t;-m,u(\cdot,-m;t_{n_k},u_0,v_0),v(\cdot,-m;t_{n_k},u_0,v_0))$. Therefore we have

$$(u(\cdot,t;t_{n_k},u_0,v_0),v(\cdot,t;t_{n_k},u_0,v_0)) \to (u(\cdot,t;-m,u_m^{**},v_m^{**}),v(\cdot,t;-m,u_m^{**},v_m^{**}))$$

in $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ for all $t \geq -m$, which implies that $(u(\cdot,t;0,u_0^{**},v_0^{**}),v(\cdot,t;0,u_0^{**},v_0^{**}))$ has backward extension in the sense that

$$(u(\cdot,t;0,u_0^{**},v_0^{**}),v(\cdot,t;0,u_0^{**},v_0^{**})) = (u(\cdot,t;-m,u_m^{**},v_m^{**}),v(\cdot,t;-m,u_m^{**},v_m^{**}))$$

for all $t > -m$ and $m \in \mathbb{N}$. Moreover

$$0 < \underline{A}_1 \leq u(\cdot, t; -m, u_m^{**}, v_m^{**}) \leq \bar{A}_1 \quad \text{and} \quad 0 < \underline{A}_2 \leq v(\cdot, t; -m, u_m^{**}, v_m^{**}) \leq \bar{A}_2,$$

$\forall x \in \Omega,\ t \geq -m$. Set $u^{**}(x,t) = u(x,t;0,u_0^{**},v_0^{**})$, $v^{**}(x,t) = v(x,t;0,u_0^{**},v_0^{**})$, and $w^{**} = (-\Delta + I)^{-1}(ku^{**} + lv^{**})$. Then $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$ is a positive bounded entire solution of (3.1).

(i) Assume that $a_i(t+T,x) = a_i(t,x)$ and $b_i(t+T,x) = b_i(t,x)$ for $i = 0,1,2$. Set

$$E(T) = \{(u_0,v_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}) \,|\, 0 < \underline{A}_1 \leq u_0(x) \leq \bar{A}_1 \,\text{and}\, 0 < \underline{A}_2 \leq v_0(x) \leq \bar{A}_2\}.$$

Note that E is nonempty, closed, convex and bounded subset of $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$. Define the map $\mathcal{T}(T) : E(T) \to C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ by

$$\mathcal{T}(T)(u_0,v_0) = (u(\cdot,T;0,u_0,v_0), v(\cdot,T;0,u_0,v_0)).$$

Note that $\mathcal{T}(T)$ is well defined, $\mathcal{T}(T)E(T) \subset E(T)$, and continuous by continuity with respect to initial conditions. Moreover by regularity and Arzella-Ascoli's Theorem, $\mathcal{T}(T)$ is completely continuous and therefore by Schauder fixed point there exists $(u_T,v_T) \in E(T)$ such that $(u(\cdot,T;0,u_T,v_T), v(\cdot,T;0,u_T,v_T)) = (u_T,v_T)$. Then $((u(\cdot,t;0,u_T,v_T), v(\cdot,t;0,u_T,v_T),$ $w(\cdot,t;0,u_T,v_T)))$ is a positive periodic solution of (3.1) with periodic T.

(ii) Assume that $a_i(t,x) \equiv a_i(x)$ and $b_i(t,x) \equiv a_i(x)$ $(i = 0,1,2)$. In this case, each $\tau > 0$ is a period for a_i and b_i . By (i), there exist $(u^\tau, v^\tau) \in E(\tau)$ such that $(u(\cdot,t;0,u^\tau,v^\tau), v(\cdot,t;0,u^\tau,v^\tau),$ $w(\cdot,t;0,u^\tau,v^\tau))$ is a positive periodic solution of (3.1) with period τ.

Observe that $C^0(\bar{\Omega}) \subset L^p(\Omega)$ for any $1 \leq p < \infty$. Choose $p > 1$ and $\alpha \in (1/2,1)$ are such that $X^\alpha \hookrightarrow C^1(\bar{\Omega})$, where $X^\alpha = D(A^\alpha)$ with the graph norm $\|u\|_\alpha = \|A^\alpha u\|_{L^p(\Omega)}$ and $A = I - \Delta$ with domain $D(A) = \{u \in W^{2,p}(\Omega) \,|\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$.

Note that there is $\tilde{M} > 0$ such that for each $\tau > 0$ and $(u_0,v_0) \in E(\tau)$, $\|u(\cdot,t;0,u_0,v_0)\|_\alpha + \|v(\cdot,t;0,u_0,v_0)\|_\alpha \leq \tilde{M}$ for each $1 \leq t \leq 2$. Let $\tau_n = \frac{1}{n}$, then there exists $u_n, v_n \in E(\frac{1}{n})$ such that $(u(\cdot,t;0,u_n,v_n), v(\cdot,t;0,u_n,v_n), w(\cdot,t;0,u_n,v_n))$ is periodic with period τ_n and

$$\|u_n\|_\alpha + \|v_n\|_\alpha = \|u(\cdot,N\tau_n;0,u_n,v_n)\|_\alpha + \|v(\cdot,N\tau_n;0,u_n,v_n)\|_\alpha \le \tilde{M},$$

where N is such that $1 \le N\tau_n \le 2$.

We claim that there is $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|u_n(\cdot)\|_{C^0(\bar{\Omega})} \ge \delta_1 \quad \forall\, n \ge 1. \tag{3.47}$$

and

$$\|v_n(\cdot)\|_{C^0(\bar{\Omega})} \ge \delta_2 \quad \forall\, n \ge 1. \tag{3.48}$$

Since the proof of (3.47) and (3.48) are similar, we only prove (3.47). Suppose by contradiction that (3.47) does not hold. Then there exists n_k such that $\|u_{n_k}\|_{C^0(\bar{\Omega})} < \frac{1}{n_k}$ for every $k \ge 1$. Let k_0 such that $\frac{1}{n_k} < \delta_0$ for all $k \ge k_0$. By Lemma 3.4 and the proof of Lemma 3.5, we get that $u(\cdot,t;0,u_{n_k},v_{n_k}) \ge u(t;\inf u_{n_k})$ for all $t > 0$ and $k \ge k_0$, where $u(t;\inf u_{n_k})$ is the solution of

$$u_t = u\Big(a_{0,\inf} - a_{2,\sup}\bar{A}_2 - \frac{\chi_1 k}{d_3}\epsilon_0 - (a_{1,\sup} - \frac{\chi_1 k}{d_3})u\Big)$$

with $u(0;\inf u_{n_k}) = \inf u_{n_k}$. Let $\delta_* = \frac{a_{0,\inf} - a_{2,\sup}\bar{A}_2 - \frac{\chi_1 k}{d_3}\epsilon_0}{2(a_{1,\sup} - \frac{\chi_1 k}{d_3})}$ and choose k large enough such that $\frac{1}{n_k} < \delta_*$. There is $t_0 > 0$ such that $u(t;\inf u_{n_k}) > \delta_*$ for all $t \ge t_0$. Then we have

$$u_{n_k}(x) = u(\cdot, m\tau_{n_k}; 0, u_{n_k}, u_{n_k}) \ge u(m\tau_{n_k};\inf u_{n_k}) > \delta^*$$

for all $m \in \mathbb{N}$ satisfying that $m\tau_{n_k} > t_0$. This is a contradiction. Therefore, (3.47) holds.

By (3.5) and Arzela-Ascoli theorem, there exist $\{n_k\}$, $(u^{**}, v^{**}) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ such that (u_{n_k}, u_{n_k}) converges to (u^{**}, v^{**}) in $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$. By (3.47) and (3.48), we have that $\|u^{**}(\cdot)\|_{C^0(\bar{\Omega})} \ge \delta_1$ and $\|v^{**}(\cdot)\|_{C^0(\bar{\Omega})} \ge \delta_2$. We claim that $(u(\cdot,t;0,u^{**},v^{**}), v(\cdot,t;0,u^{**},v^{**}),$ $w(\cdot,t;0,u^{**},v^{**}))$ is a steady state solution of (3.1), that is,

$$u(\cdot,t;0,u^{**},v^{**}) = u^{**}(\cdot) \quad \text{and} \quad v(\cdot,t;0,u^{**},v^{**}) = v^{**}(\cdot) \quad \text{for all}\, t \ge 0. \tag{3.49}$$

In fact, let $\epsilon > 0$ be fix and let $t > 0$. Note that

$$[n_k t]\tau_{n_k} = \frac{[n_k t]}{n_k} \le t \le \frac{[n_k t] + 1}{n_k} = ([n_k t] + 1)\tau_{n_k}.$$

Then, we can choose k large enough such that

$$|u(x,t;0,u^{**},v^{**}) - u(x,t;0,u_{n_k},v_{n_k})| < \epsilon, \quad |u_{n_k}(x) - u^{**}(x)| < \epsilon, \quad |v_{n_k}(x) - v^{**}(x)| < \epsilon,$$

$$|v(x,t;0,u^{**},v^{**}) - v(x,t;0,u_{n_k},v_{n_k})| < \epsilon, \quad |v(x,\frac{[n_k t]}{n_k};0,u_{n_k},v_{n_k}) - v(x,t;0,u_{n_k},v_{n_k})| < \epsilon,$$

$$|u(x,\frac{[n_k t]}{n_k};0,u_{n_k},v_{n_k}) - u(x,t;0,u_{n_k},v_{n_k})| < \epsilon.$$

for all $x \in \bar{\Omega}$. We then have

$$\begin{aligned}|u(x,t;0,u^{**},v^{**}) - u^{**}| &\le |u(x,t;0,u^{**},v^{**}) - u(x,t;0,u_{n_k},v_{n_k})| + |u_{n_k}(x) - u^{**}(x)| \\ &\quad + |u(x,t;0,u_{n_k},v_{n_k}) - u(x,[n_k t]\tau_{n_k};0,u_{n_k},v_{n_k})| < 3\epsilon \quad \forall\ x \in \bar{\Omega},\end{aligned}$$

and

$$\begin{aligned}|v(x,t;0,u^{**},v^{**}) - v^{**}| &\le |v(x,t;0,u^{**},v^{**}) - v(x,t;0,u_{n_k},v_{n_k})| + |v_{n_k}(x) - v^{**}(x)| \\ &\quad + |v(x,t;0,u_{n_k},v_{n_k}) - v(x,[n_k t]\tau_{n_k};0,u_{n_k},v_{n_k})| < 3\epsilon \quad \forall\ x \in \bar{\Omega}.\end{aligned}$$

Letting $\epsilon \to 0$, (3.49) follows.

(iii) Note that solutions of the following system,

$$\begin{cases} u_t = u(a_0(t) - a_1(t)u - a_2(t)v) \\ v_t = v(b_0(t) - b_1(t)u - b_2(t)v) \\ 0 = ku(t) + lv(t) - \lambda w(t) \end{cases}$$

are spatially homogeneous solutions $(u(t), v(t), w(t))$ of (3.1). By (H4) and Remark 3.3, (3.5) is satisfied. (iii) then follows by Lemma 3.9.

(2) It follows from the similar arguments as those in (1). □

3.6 Extinction of One of the Species

In this section, our aim is to find conditions on the parameters which guarantee the extinction of one of the species. First we prove a lemma.

Assume (H3) or (H4). For given $u_0, v_0 \in C(\bar{\Omega})$ with $u_0 \geq 0$ and $v_0 \geq 0$, let

$$L_1(t_0,u_0,v_0) = \limsup_{t\to\infty}(\max_{x\in\bar{\Omega}} u(x,t;t_0,u_0,v_0)),\ \ l_1(t_0,u_0,v_0) = \liminf_{t\to\infty}(\min_{x\in\bar{\Omega}} u(x,t;t_0,u_0,v_0)),$$

and

$$L_2(t_0,u_0,v_0) = \limsup_{t\to\infty}(\max_{x\in\bar{\Omega}} v(x,t;t_0,u_0,v_0)),\ \ l_2(t_0,u_0,v_0) = \liminf_{t\to\infty}(\min_{x\in\bar{\Omega}} v(x,t;t_0,u_0,v_0)).$$

If no confusion occurs, we may write $L_i(t_0,u_0,v_0)$ and $l_i(t_0,u_0,v_0)$ as L_i and l_i $(i = 1,2)$ respectively. By Theorem 3.1 we have

$$0 \leq l_1 \leq L_1 < \infty, \quad 0 \leq l_2 \leq L_2 < \infty.$$

Furthermore, using the definition of $\limsup$ and of $\liminf$, and elliptic regularity, we get that given $\epsilon > 0$, there exists $T_\epsilon > 0$ such that

$$l_1 - \epsilon \leq u(x,t) \leq L_1 + \epsilon, \quad l_2 - \epsilon \leq v(x,t) \leq L_2 + \epsilon, \quad \forall\ t > T_\epsilon. \tag{3.50}$$

Lemma 3.10. *(1) Assume $a_{1,\inf} > \frac{k\chi_1}{d_3}$ and $a_{2,\inf} \geq \frac{l\chi_1}{d_3}$. Then*

$$L_1 \leq \frac{\{a_{0,\sup} - a_{2,\inf} l_2\}_+}{a_{1,\inf} - \frac{\chi_1 k}{d_3}}. \tag{3.51}$$

(2) Assume $b_{2,\inf} > \frac{l\chi_2}{d_3}$. *Then*

$$L_2 \le \frac{\left\{b_{0,\sup} - \frac{\chi_2 l}{d_3} l_2 + \left(b_{1,\inf} - k\frac{\chi_2}{d_3}\right)_- L_1\right\}_+}{b_{2,\inf} - \frac{\chi_2 l}{d_3}}, \tag{3.52}$$

and

$$l_2 \ge \frac{\left\{b_{0,\inf} - \left(\left(b_{1,\sup} - k\frac{\chi_2}{d_3}\right)_+ + k\frac{\chi_2}{d_3}\right) L_1 - \frac{\chi_2 l}{d_3} L_2\right\}_+}{b_{2,\sup} - \frac{\chi_2 l}{d_3}}. \tag{3.53}$$

Proof. (1) From the first equation of (3.1), (3.50), and the fact that $a_{2,\inf} \ge \frac{\chi_1 l}{d_3}$, we have

$$\begin{aligned}
&u_t - d_1\Delta u + \chi_1 \nabla u \cdot \nabla w \\
&= u\left\{a_0(t,x) - (a_1(t,x) - \frac{\chi_1}{d_3}k)u - (a_2(t,x) - l\frac{\chi_1}{d_3})v - \frac{\chi_1}{d_3}\lambda w\right\} \\
&\le u\left\{a_{0,\sup} - (a_{1,\inf} - \frac{\chi_1}{d_3}k)u - a_{2,\inf} l_2 + \left(a_{2,\sup} + k\frac{\chi_1}{d_3}\right)\epsilon\right\}
\end{aligned}$$

for $t \ge T_\epsilon$, and thus since $a_{1,\inf} > \frac{\chi_1 k}{d_3}$, (3.51) follows from parabolic comparison principle.

(2) From the second equation of (3.1) and (3.50), we have that

$$\begin{aligned}
&v_t - d_2\Delta v + \chi_2 \nabla v \cdot \nabla w \\
&= v\left\{b_0(t,x) - (b_2(t,x) - \frac{\chi_2}{d_3}k)v - (b_1(t,x) - k\frac{\chi_2}{d_3})u - \frac{\chi_2}{d_3}\lambda w\right\} \\
&\le v\left\{b_{0,\sup} - (b_{2,\inf} - \frac{\chi_2}{d_3}k)v + \left(b_{1,\inf} - k\frac{\chi_2}{d_3}\right)_- L_1 - l\frac{\chi_2}{d_3} l_2\right\} \\
&+ v\left((k+l)\frac{\chi_2}{d_3} + \left(b_{1,inf} - k\frac{\chi_2}{d_3}\right)_-\right)\epsilon
\end{aligned}$$

for $t \ge T_\epsilon$, and (3.52) follows from parabolic comparison principle.

Similarly, we have

$$
\begin{aligned}
&v_t - d_2\Delta v + \chi_2 \nabla v\cdot\nabla w\\
&= v\left\{b_0(t,x) - (b_2(t,x) - \frac{\chi_2}{d_3}k)v - (b_1(t,x) - k\frac{\chi_2}{d_3})u - \frac{\chi_2}{d_3}\lambda w\right\}\\
&\geq v\left\{b_{0,\inf} - (b_{2,\sup} - \frac{\chi_2}{d_3}k)v - \Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ L_1 - k\frac{\chi_2}{d_3}L_1\right\}\\
&- l\frac{\chi_2}{d_3}L_2 - v\Big(l\frac{\chi_2}{d_3} + v\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+\Big)\epsilon
\end{aligned}
$$

for $t \geq T_\epsilon$, and (3.53) thus follows from parabolic comparison principle. □

Now we prove Theorem 3.4.

Proof of Theorem 3.4. We first prove that $L_1 = 0$.

Suppose by contradiction that $L_1 > 0$. Then by (3.51) and (3.13), we have

$$
l_2 < \frac{a_{0,\sup}}{a_{2,\inf}}. \tag{3.54}
$$

By (3.14), we have

$$
\begin{aligned}
a_{2,\inf}\Big(b_{0,\inf}(b_{2,\inf} - l\frac{\chi_2}{d_3}) - b_{0,\sup}\frac{\chi_2}{d_3}l\Big) &\geq a_{0,\sup}\Big((b_{2,\inf} - l\frac{\chi_2}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - (l\frac{\chi_2}{d_3})^2\Big)\\
&= a_{0,\sup}\Big((b_{2,\inf} - l\frac{\chi_2}{d_3})b_{2,\sup} - l\frac{\chi_2}{d_3}b_{2,\inf}\Big)\\
&\geq a_{0,\sup}(b_{2,\inf} - 2l\frac{\chi_2}{d_3})b_{2,\sup}.
\end{aligned}
$$

This together with the fact that $a_{2,\inf}\big(b_{0,\inf}(b_{2,\inf} - l\frac{\chi_2}{d_3}) - b_{0,\sup}\frac{\chi_2}{d_3}l\big) \leq a_{2,\inf}b_{0,\sup}(b_{2,\inf} - 2l\frac{\chi_2}{d_3})$, we get

$$
a_{2,\inf}b_{0,\sup}(b_{2,\inf} - 2l\frac{\chi_2}{d_3}) \geq a_{0,\sup}(b_{2,\inf} - 2l\frac{\chi_2}{d_3})b_{2,\sup},
$$

which combines with $b_{2,\inf} - 2l\frac{\chi_2}{d_3} > 0$ implies

$$
a_{2,\inf}b_{0,\sup} \geq a_{0,\sup}b_{2,\sup} \geq a_{0,\sup}2l\frac{\chi_2}{d_3}.
$$

Therefore

$$b_{0,\sup} - \frac{\chi_2 l}{d_3} l_2 > b_{0,\sup} - \frac{\chi_2 l}{d_3}\frac{a_{0,\sup}}{a_{2,\inf}} \geq 0. \tag{3.55}$$

From (3.53), we get

$$\frac{l\chi_2}{d_3} L_2 \geq b_{0,\inf} - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) L_1 - (b_{2,\sup} - \frac{\chi_2}{d_3} l) l_2.$$

Thus, from (3.51) and $L_1 > 0$, we get

$$\frac{l\chi_2}{d_3} L_2 \geq b_{0,\inf} - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) \frac{\{a_{0,\sup} - a_{2,\inf} l_2\}}{a_{1,\inf} - \frac{\chi_1 k}{d_3}} - (b_{2,\sup} - \frac{\chi_2}{d_3} l) l_2.$$

Therefore

$$\begin{aligned}\frac{l\chi_2}{d_3}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) L_2 \geq{}& b_{0,\inf}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{0,\sup} \\ &- \Big((a_{1,\inf} - \frac{\chi_1 k}{d_3})(b_{2,\sup} - \frac{\chi_2}{d_3} l) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{2,\inf}\Big) l_2.\end{aligned}$$

It follows from the last inequality, (3.52), and (3.55) that

$$\begin{aligned}&\frac{l\chi_2}{d_3}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) \frac{\Big\{b_{0,\sup} - \frac{\chi_2 l}{d_3} l_2 + \Big(b_{1,\inf} - k\frac{\chi_2}{d_3}\Big)_- L_1\Big\}}{b_{2,\inf} - \frac{\chi_2 l}{d_3}} \\ &\geq b_{0,\inf}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{0,\sup} \\ &- \Big((a_{1,\inf} - \frac{\chi_1 k}{d_3})(b_{2,\sup} - \frac{\chi_2}{d_3} l) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{2,\inf}\Big) l_2.\end{aligned}$$

Therefore from (3.51), we get

$$\begin{aligned}&\frac{l\chi_2}{d_3}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) \frac{\Big\{b_{0,\sup} - \frac{\chi_2 l}{d_3} l_2 + \Big(b_{1,\inf} - k\frac{\chi_2}{d_3}\Big)_- \frac{\big\{a_{0,\sup} - a_{2,\inf} l_2\big\}}{a_{1,\inf} - \frac{\chi_1 k}{d_3}}\Big\}}{b_{2,\inf} - \frac{\chi_2 l}{d_3}} \\ &\geq b_{0,\inf}(a_{1,\inf} - \frac{\chi_1 k}{d_3}) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{0,\sup} \\ &- \Big((a_{1,\inf} - \frac{\chi_1 k}{d_3})(b_{2,\sup} - \frac{\chi_2}{d_3} l) - \Big(\Big(b_{1,\sup} - k\frac{\chi_2}{d_3}\Big)_+ + k\frac{\chi_2}{d_3}\Big) a_{2,\inf}\Big) l_2.\end{aligned}$$

Thus

$$\Big\{\underbrace{(a_{1,\inf}-\frac{\chi_1 k}{d_3})\Big[(b_{2,\inf}-\frac{\chi_2 l}{d_3})(b_{2,\sup}-\frac{\chi_2 l}{d_3})-(l\frac{\chi_2}{d_3})^2\Big]}_{B_1}\Big\}l_2$$
$$-\Big\{\underbrace{\Big[\Big(\Big(b_{1,\sup}-k\frac{\chi_2}{d_3}\Big)_+ +k\frac{\chi_2}{d_3}\Big)(b_{2,\inf}-\frac{\chi_2 l}{d_3})+\frac{l\chi_2}{d_3}\Big(b_{1,\inf}-k\frac{\chi_2}{d_3}\Big)_-\Big]a_{2,\inf}}_{B_2}\Big\}l_2$$
$$\geq \underbrace{\left(b_{0,\inf}(b_{2,\inf}-\frac{\chi_2 l}{d_3})-l\frac{\chi_2}{d_3}b_{0,sup}\right)\left(a_{1,\inf}-\frac{\chi_1 k}{d_3}\right)}_{A_1}$$
$$-\underbrace{\Big[\Big(\Big(b_{1,\sup}-k\frac{\chi_2}{d_3}\Big)_+ +k\frac{\chi_2}{d_3}\Big)(b_{2,\inf}-\frac{\chi_2 l}{d_3})+\frac{l\chi_2}{d_3}\Big(b_{1,\inf}-k\frac{\chi_2}{d_3}\Big)_-\Big]a_{0,\sup}}_{A_2},$$

which is equivalent to

$$Bl_2 \geq A, \tag{3.56}$$

with $B = B_1 - B_2$ and $A = A_1 - A_2$. Note that (3.15) yields that $A > 0$. This combined with (3.56) implies that $B > 0$. Therefore, inequality (3.56) becomes

$$l_2 \geq \frac{A}{B}.$$

Then thanks to equation (3.54), we get

$$B > \frac{a_{2,\inf}}{a_{0,\sup}}A.$$

That means

$$a_{0,\sup}(a_{1,\inf}-\frac{\chi_1 k}{d_3})\Big[(b_{2,\inf}-\frac{\chi_2 l}{d_3})(b_{2,\sup}-\frac{\chi_2 l}{d_3})-(l\frac{\chi_2}{d_3})^2\Big]$$
$$-a_{0,\sup}\Big[\Big(\Big(b_{1,\sup}-k\frac{\chi_2}{d_3}\Big)_+ +k\frac{\chi_2}{d_3}\Big)(b_{2,\inf}-\frac{\chi_2 l}{d_3})+\frac{l\chi_2}{d_3}\Big(b_{1,\inf}-k\frac{\chi_2}{d_3}\Big)_-\Big]a_{2,\inf}$$
$$> a_{2,\inf}\left(b_{0,\inf}(b_{2,\inf}-\frac{\chi_2 l}{d_3})-l\frac{\chi_2}{d_3}b_{0,sup}\right)\left(a_{1,\inf}-\frac{\chi_1 k}{d_3}\right)$$
$$-\Big[\Big(\Big(b_{1,\sup}-k\frac{\chi_2}{d_3}\Big)_+ +k\frac{\chi_2}{d_3}\Big)(b_{2,\inf}-\frac{\chi_2 l}{d_3})+\frac{l\chi_2}{d_3}\Big(b_{1,\inf}-k\frac{\chi_2}{d_3}\Big)_-\Big]a_{0,\sup}a_{2,\inf}.$$

Thus

$$a_{0,\sup}\Big[(b_{2,\inf}-\frac{\chi_2 l}{d_3})(b_{2,\sup}-\frac{\chi_2 l}{d_3})-(l\frac{\chi_2}{d_3})^2\Big]>a_{2,\inf}\left(b_{0,\inf}(b_{2,\inf}-\frac{\chi_2 l}{d_3})-l\frac{\chi_2}{d_3}b_{0,\sup}\right),$$

which contradicts to (3.14). Hence $L_1=0$.

Next, we prove (3.17) and (3.18). Since $L_1=0$, we get from (3.52) and (3.53) respectively that

$$L_2\le\frac{b_{0,\sup}-\frac{\chi_2 l}{d_3}l_2}{b_{2,\inf}-\frac{\chi_2 l}{d_3}}.$$

and

$$l_2\ge\frac{b_{0,\inf}-\frac{\chi_2 l}{d_3}L_2}{b_{2,\sup}-\frac{\chi_2 l}{d_3}}.$$

(3.17) then follows. Furthermore (3.18) follows from (3.16), (3.17) and elliptic comparison principle.

Finally, assume that (3.19) has a unique positive entire solution $(v^*(x,t;\tilde b_0,\tilde b_2),w^*(x,t;\tilde b_0,\tilde b_2))$ for any $(\tilde b_0,\tilde b_2)\in H(b_0,b_2)$. We claim that (3.20) holds. Indeed, if (3.20) does not hold. Then there are $\tilde\epsilon_0>0$ and $t_n\to\infty$ such that

$$\|v(\cdot,t_n+t_0;t_0,u_0,v_0)-v^*(\cdot,t_n+t_0;b_0,b_2)\|_\infty\ge\tilde\epsilon_0\quad\forall\ n=1,2,\cdots.$$

Without loss of generality, we may assume that

$$\lim_{n\to\infty}(b_0(t+t_n+t_0,x),b_2(t+t_n+t_0,x))=(\tilde b_0(t,x),\tilde b_2(t,x))$$

and

$$\begin{aligned}&\lim_{n\to\infty}(u(x,t+t_n+t_0;t_0,u_0,v_0),v(x,t+t_n+t_0;t_0,u_0,v_0),w(x,t+t_n+t_0;t_0,u_0,v_0))\\&=(0,\tilde v(x,t),w(x,t))\end{aligned}$$

locally uniformly in $(t,x)\in\mathbb{R}\times\bar\Omega$. Then $(\tilde v(x,t),\tilde w(x,t))$ is a positive entire solution of (3.19) and

$$\|\tilde v(\cdot;0)-v^*(\cdot,0;\tilde b_0,\tilde b_2)\|_\infty\geq\tilde\epsilon_0,$$

which is a contradiction. Hence (3.20) holds. □

3.7 Optimal Attracting Rectangle and Proof of Theorem 3.5

In this section, we construct optimal attracting rectangles for (3.1) and prove Theorem 3.5. We first prove two important lemmas.

Lemma 3.11. *Consider* (3.1). *For given* $u_0,v_0\in C^0(\bar\Omega)$ *with* $u_0,v_0\geq 0$, *let* $\overline{u}_0=\max_{x\in\bar\Omega}u_0(x)$, $\underline{u}_0=\min_{x\in\bar\Omega}u_0(x)$, $\overline{v}_0=\max_{x\in\bar\Omega}v_0(x)$, $\underline{v}_0=\min_{x\in\bar\Omega}v_0(x)$.

(1) Assume (H8). Let $\underline{r}_1^0=\underline{r}_2^0=0$, $\bar r_1^0=\bar A_1$, $\bar r_2^0=\bar A_2$, *and*

$$\begin{cases}\bar r_1^n=\frac{a_{0,\sup}-a_{2,\inf}\underline{r}_2^{n-1}-k\frac{\chi_1}{d_3}\underline{r}_1^{n-1}}{a_{1,\inf}-k\frac{\chi_1}{d_3}}\\ \bar r_2^n=\frac{b_{0,\sup}-b_{1,\inf}\underline{r}_1^{n-1}-k\frac{\chi_1}{d_3}\underline{r}_2^{n-1}}{b_{2,\inf}-l\frac{\chi_2}{d_3}}\\ \underline{r}_1^n=\frac{a_{0,\inf}-a_{2,\sup}\bar r_2^{n}-k\frac{\chi_1}{d_3}\bar r_1^{n}}{a_{1,\sup}-k\frac{\chi_1}{d_3}}\\ \underline{r}_2^n=\frac{b_{0,\inf}-b_{1,\sup}\bar r_1^{n}-l\frac{\chi_2}{d_3}\bar r_2^{n}}{b_{2,\sup}-l\frac{\chi_2}{d_3}}\end{cases}$$

for $n=1,2,\cdots$. *Then*

$$\begin{cases}0<\underline{r}_1^{n-1}\leq\underline{r}_1^n\leq\bar r_1^n\leq\bar r_1^{n-1}\leq\bar A_1\\ 0<\underline{r}_2^{n-1}\leq\underline{r}_2^n\leq\bar r_2^n\leq\bar r_2^{n-1}\leq\bar A_2\end{cases}\tag{3.57}$$

for $n=2,\cdots$, *and for any given* $u_0,v_0\in C^0(\bar\Omega)$ *with* $\inf u_0>0$, $\inf v_0>0$, $\epsilon>0$, *and* $n\in\mathbb{N}$ *with* $n\geq 1$, *there exists* $t^n_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}\geq t^{n-1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ *(*$t^0_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}=0$*) such that*

$$\begin{cases}\underline{r}_1^n-\epsilon\leq u(x,t;t_0,u_0,v_0)\leq\bar r_1^n+\epsilon\\ \underline{r}_2^n-\epsilon\leq v(x,t;t_0,u_0,v_0)\leq\bar r_2^n+\epsilon,\end{cases}\tag{3.58}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ *and* $t \geq t_0 + t^n_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$.

(2) Assume (H9). Let $\underline{s}_1^0 = \underline{s}_2^0 = 0$, $\bar{s}_1^0 = \bar{B}_1$, $\bar{s}_2^0 = \bar{B}_2$, *and*

$$
\begin{cases}
\bar{s}_1^n = \frac{\left(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2^{n-1} - k\frac{\chi_1}{d_3}\underline{s}_1^{n-1}\right)(b_{2,\inf} - l\frac{\chi_2}{d_3})}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{l\chi_1}{d_3}\left(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1^{n-1} - l\frac{\chi_2}{d_3}\underline{s}_2^{n-1}\right)}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\
\bar{s}_2^n = \frac{\left(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1^{n-1} - l\frac{\chi_2}{d_3}\underline{s}_2^{n-1}\right)(a_{1,\inf} - k\frac{\chi_1}{d_3})}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{k\chi_2}{d_3}\left(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2^{n-1} - k\frac{\chi_1}{d_3}\underline{s}_1^{n-1}\right)}{(a_{1,\inf} - k\frac{\chi_1}{d_3})(b_{2,\inf} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\
\underline{s}_1^n = \frac{\left(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{s}_2^n - k\frac{\chi_1}{d_3}\bar{s}_1^n\right)(b_{2,\sup} - l\frac{\chi_2}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{l\chi_1}{d_3}\left(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{s}_1^n - l\frac{\chi_2}{d_3}\bar{s}_2^n\right)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} \\
\underline{s}_2^n = \frac{\left(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})\bar{s}_1^n - l\frac{\chi_2}{d_3}\bar{s}_2^n\right)(a_{1,\sup} - k\frac{\chi_1}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{k\chi_2}{d_3}\left(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})\bar{s}_2^n - k\frac{\chi_1}{d_3}\bar{s}_1^n\right)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}}
\end{cases}
$$

for $n = 1, 2, \cdots$. *Then*

$$
\begin{cases}
0 < \underline{s}_1^{n-1} \leq \underline{s}_1^n \leq \bar{s}_1^n \leq \bar{s}_1^{n-1} \leq \bar{B}_1 \\
0 < \underline{s}_2^{n-1} \leq \underline{s}_2^n \leq \bar{s}_2^n \leq \bar{s}_2^{n-1} \leq B_2
\end{cases}
\tag{3.59}
$$

for $n = 2, \cdots$, *and for any given* $u_0, v_0 \in C^0(\bar{\Omega})$ *with* $\inf u_0 > 0$, $\inf v_0 > 0$, $\epsilon > 0$, *and* $n \in \mathbb{N}$ *with* $n \geq 1$, *there exists* $t^n_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^{n-1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ *(*$t^0_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} = 0$*) such that*

$$
\begin{cases}
\underline{s}_1^n - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{s}_1^n + \epsilon \\
\underline{s}_2^n - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{s}_2^n + \epsilon,
\end{cases}
\tag{3.60}
$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ *and* $t \geq t_0 + t^n_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$.

Proof. (1) First of all, note that $\bar{r}_1^1 = \bar{r}_1^0$ and $\bar{r}_2^1 = \bar{r}_2^0$, and by (H8), $0 < \underline{r}_1^1 \leq \bar{r}_1^1$ and $0 < \underline{r}_2^1 \leq \bar{r}_2^1$. (3.57) then follows from the definition of $\bar{r}_i^n$ and $\underline{r}_n^n$ $(i = 1, 2)$ directly.

We then prove (3.58). We do so by induction.

First we claim that there exists $t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq 0$ such that

$$
\begin{cases}
\underline{r}_1^1 - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{r}_1^1 + \epsilon \\
\underline{r}_2^1 - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{r}_2^1 + \epsilon
\end{cases}
\tag{3.61}
$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \geq t_0 + t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$.

In fact, from the first and third equations of (3.1), we get

$$u_t \leq d_1 \Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\sup} - (a_{1,\inf} - k\frac{\chi_1}{d_3})u\big). \tag{3.62}$$

Let $u(t; t_0, \overline{u}_0)$ be the solution of

$$u' = u\big(a_{0,\sup} - (a_{1,\inf} - k\frac{\chi_1}{d_3})u\big)$$

with $u(t_0; t_0, \overline{u}_0) = \overline{u}_0$. Then by solving, we get

$$u(t; t_0, \overline{u}_0) = \frac{c_0 a}{c_0 b - e^{-a(t-t_0)}} \quad \forall\ t \geq t_0,$$

where $a = a_{0,\sup}$, $b = a_{1,\inf} - k\frac{\chi_1}{d_3}$, and $c_0 = \frac{\overline{u}_0}{b\overline{u}_0 - a}$. (Actually $u(t; t_0, \overline{u}_0) > 0$ for all $t > t_0 - \frac{\ln(c_0 b)}{a}$ and blows up in backward time at $t^* = t_0 - \frac{\ln(c_0 b)}{a} < t_0$.) It then follows from parabolic comparison principle that

$$u(x, t; t_0, u_0, v_0) \leq \frac{c_0 a}{c_0 b - e^{-a(t-t_0)}} \quad \forall\ t \geq t_0, \quad \forall\ t_0 \in \mathbb{R}.$$

Thus

$$u(x, t + t_0; t_0, u_0, v_0) \leq u(t + t_0; t_0, \overline{u}_0) = \frac{c_0 a}{c_0 b - e^{-at}} \quad \forall\ t \geq 0, \quad \forall\ t_0 \in \mathbb{R}.$$

Therefore there is $t^1_{\epsilon,\overline{u}_0} > 0$ such that

$$u(x, t; t_0, u_0, v_0) \leq \bar{r}^1_1 + \epsilon\ \forall t \geq t_0 + t^1_{\epsilon,\overline{u}_0}, \quad \forall\ t_0 \in \mathbb{R}. \tag{3.63}$$

Similarly using the second and third equation of (3.1), there exists $t^1_{\epsilon,\overline{v}_0} > 0$ such that

$$v(x, t; t_0, u_0, v_0) \leq \bar{r}^1_2 + \epsilon\ \forall\ t \geq t_0 + t^1_{\epsilon,\overline{v}_0}, \quad \forall\ t_0 \in \mathbb{R}. \tag{3.64}$$

Choose $0 < \tilde{\epsilon} \le \epsilon$ such that

$$\frac{a_{0,\inf} - a_{2,\sup}\bar{r}_2^1 - k\frac{\chi_1}{d_3}\bar{r}_1^1 - \tilde{\epsilon}\big(a_{2,\sup} + k\frac{\chi_1}{d_3}\big)}{a_{1,\sup} - \frac{k\chi_1}{d_3}} - \tilde{\epsilon} \ge \underline{r}_1^1 - \epsilon.$$

Let $t^1_{\tilde{\epsilon},\overline{u}_0,\overline{v}_0} = \max\{t_{\tilde{\epsilon},\overline{u}_0}, t_{\tilde{\epsilon},\overline{v}_0}\}$. Then for $t \ge t^1_{\tilde{\epsilon},\overline{u}_0,\overline{v}_0}$, from (3.63), (3.64), the first and third equations of (3.1), we get

$$u_t \ge d_1\Delta u - \chi_1\nabla w\cdot\nabla u + u\big(a_{0,\inf} - a_{2,\sup}\bar{r}_2^1 - k\frac{\chi_1}{d_3}\bar{r}_1^1 - (a_{1,\sup} - k\frac{\chi_1}{d_3})u - \tilde{\epsilon}\big(a_{2,\sup} + k\frac{\chi_1}{d_3}\big)\big).$$

Thus similar arguments as those lead to (3.63) implies that there is $t^1_{\epsilon,\underline{u}_0,\overline{u}_0,\overline{v}_0} \ge t^1_{\epsilon,\overline{u}_0,\overline{v}_0}$ such that

$$\underline{r}_1^1 - \epsilon \le u(x,t;t_0,u_0,v_0)\ \forall\ t \ge t_0 + t^1_{\epsilon,\underline{u}_0,\overline{u}_0,\overline{v}_0}, \quad \forall\ t_0 \in \mathbb{R}. \tag{3.65}$$

Similarly, from (3.63), (3.64), the second and third equation of (3.1) and similar arguments as those lead to (3.63), there is $t^1_{\epsilon,\underline{v}_0,\overline{u}_0,\overline{v}_0} \ge t^1_{\epsilon,\overline{v}_0,\overline{v}_0}$ such that

$$\underline{r}_2^1 - \epsilon \le v(x,t;t_0,u_0,v_0)\ \forall\ t \ge t_0 + t^1_{\epsilon,\underline{u}_0,\overline{u}_0,\overline{v}_0}, \quad \forall\ t_0 \in \mathbb{R}. \tag{3.66}$$

Choose $t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} = \max\{t_{\epsilon,\underline{u}_0,\overline{u}_0,\overline{v}_0}, t_{\epsilon,\underline{v}_0,\overline{u}_0,\overline{v}_0}\}(\ge 0)$. Then (3.61) follows from (3.63), (3.64), (3.65) and (3.66).

Next, assume that for any $\epsilon > 0$, there is $t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \ge t^{k-1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ ($k \ge 2$) such that

$$\begin{cases} \underline{r}_1^k - \epsilon \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^k + \epsilon \\ \underline{r}_2^k - \epsilon \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^k + \epsilon \end{cases} \tag{3.67}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \ge t_0 + t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. We claim that there is there is $t^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \ge t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ ($k \ge 2$) such that

$$\begin{cases} \underline{r}_1^{k+1} - \epsilon \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^{k+1} + \epsilon \\ \underline{r}_2^{k+1} - \epsilon \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^{k+1} + \epsilon \end{cases}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \geq t_0 + t^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$.

In fact, choose $0 < \tilde{\epsilon} \leq \epsilon$ such that

$$\frac{a_{0,\sup} - a_{2,\inf}\underline{\mathrm{r}}_2^k - k\frac{\chi_1}{d_3}\underline{\mathrm{r}}_1^k + \tilde{\epsilon}\big(a_{2,\inf} + k\frac{\chi_1}{d_3}\big)}{a_{1,\inf} - \frac{k\chi_1}{d_3}} + \tilde{\epsilon} \leq \bar{r}_1^{k+1} + \epsilon$$

and

$$\frac{a_{0,\inf} - a_{2,\sup}\bar{r}_2^{k+1} - k\frac{\chi_1}{d_3}\bar{r}_1^{k+1} - \tilde{\epsilon}\big(a_{2,\sup} + k\frac{\chi_1}{d_3}\big)}{a_{1,\sup} - \frac{k\chi_1}{d_3}} - \tilde{\epsilon} \geq \underline{\mathrm{r}}_1^{k+1} - \epsilon.$$

We have that for $t \geq t_0 + t^k_{\tilde{\epsilon},\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$,

$$u_t \leq d_1\Delta u - \chi_1\nabla w\cdot\nabla u + u\big(a_{0,\sup} - a_{2,\inf}\underline{\mathrm{r}}_2^k - k\frac{\chi_1}{d_3}\underline{\mathrm{r}}_1^k - (a_{1,\inf} - k\frac{\chi_1}{d_3})u - \tilde{\epsilon}\big(a_{2,\inf} + k\frac{\chi_1}{d_3}\big)\big)$$

Then there is $\tilde{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ such that for $t \geq \tilde{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$

$$u(x,t;t_0,u_0,v_0) \leq \bar{r}^{k+1} + \epsilon \quad \forall\ x \in \bar{\Omega},\ \forall\ t_0 \in \mathbb{R}\ \forall\ t \geq t_0 + \tilde{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}.$$

This implies that for $t \geq t_0 + \tilde{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$,

$$u_t \geq d_1\Delta u - \chi_1\nabla w\cdot\nabla u + u\big(a_{0,\inf} - a_{2,\sup}\bar{r}_2^{k+1} - k\frac{\chi_1}{d_3}\bar{r}_1^{k+1} - (a_{1,\sup} - k\frac{\chi_1}{d_3})u - \tilde{\epsilon}\big(a_{2,\sup} + k\frac{\chi_1}{d_3}\big)\big).$$

It then follows that there is $\bar{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ such that for $t \geq t_0 + \bar{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$,

$$\underline{\mathrm{r}}_1^{k+1} - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{r}_1^{k+1} + \epsilon.$$

Similarly, we can prove that there is $\hat{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ such that for $t \geq t_0 + \hat{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$,

$$\underline{\mathrm{r}}_2^{k+1} - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{r}_2^{k+1} + \epsilon.$$

The claim then follows with $t^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} = \max\{\bar{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}, \hat{t}^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}\}$.

Now, by induction, (3.58) holds for all $n \geq 1$. This completes the proof of (1).

(2) It can be proved by the similar arguments as those in (1). We outline some idea in the following.

First of all, note that $\bar{s}_1^1 = \bar{s}_1^0 = \bar{B}_1$ and $\bar{s}_2^1 = \bar{s}_2^0 = \bar{B}_1$, and by (H9), $0 < \underline{s}_1^1 \leq \bar{s}_1^1$ and $0 < \underline{s}_2^1 \leq \bar{s}_2^1$. (3.59) then follows from the definition of $\bar{s}_i^n$ and $\underline{s}_i^n$ $(i = 1, 2)$ directly.

We prove (3.60) by induction.

To this end, we first claim that there exists $t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq 0$ such that

$$\begin{cases} \underline{s}_1^1 - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{s}_1^1 + \epsilon \\ \underline{s}_2^1 - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{s}_2^1 + \epsilon \end{cases} \tag{3.68}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \geq t_0 + t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$.

In fact, note that

$$\begin{cases} u_t \leq d_1\Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\sup} - (a_{2,\inf} + l\frac{\chi_1}{d_3})\underline{s}_2^0 - k\frac{\chi_1}{d_3}\underline{s}_1^0 - (a_{1,\inf} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\big) \\ v_t \leq d_2\Delta v - \chi_2 \nabla w \cdot \nabla v + v\big(b_{0,\sup} - (b_{1,\inf} + k\frac{\chi_2}{d_3})\underline{s}_1^0 - l\frac{\chi_2}{d_3}\underline{s}_2^0 - (b_{2,\inf} - l\frac{\chi_2}{d_3})v + k\frac{\chi_2}{d_3}u\big). \end{cases}$$

Then for any $\epsilon > 0$, there is $\bar{t}^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq 0$ such that

$$\begin{cases} u(x,t;t_0,u_0,v_0) \leq \bar{s}_1^1 + \epsilon \\ v(x,t;t_,u_0,v_0) \leq \bar{s}_2^1 + \epsilon \end{cases}$$

for all $x \in \Omega$, $t_0 \in \mathbb{R}$, and $t \geq t_0 + \bar{t}^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. This implies that for any $\tilde{\epsilon} > 0$, $t \geq t_0 + \bar{t}^1_{\tilde{\epsilon},\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$,

$$\begin{cases} u_t \geq d_1\Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})(\bar{s}_2^1 + \tilde{\epsilon}) - k\frac{\chi_1}{d_3}(\bar{s}_1^1 + \tilde{\epsilon}) - (a_{1,\sup} - k\frac{\chi_1}{d_3})u + l\frac{\chi_1}{d_3}v\big) \\ v_t \geq d_2\Delta v - \chi_2 \nabla w \cdot \nabla v + v\big(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})(\bar{s}_1^1 + \tilde{\epsilon}) - l\frac{\chi_2}{d_3}(\bar{s}_2^1 + \tilde{\epsilon}) - (b_{2,\sup} - l\frac{\chi_2}{d_3})v + k\frac{\chi_2}{d_3}u\big). \end{cases}$$

Choose $0 < \tilde{\epsilon} < \epsilon$ such that

$$\begin{cases} \frac{\left(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})(\bar{s}_2^1 + \tilde{\epsilon}) - k\frac{\chi_1}{d_3}(\bar{s}_1^1 + \tilde{\epsilon})\right)(b_{2,\sup} - l\frac{\chi_2}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{l\chi_1}{d_3}\left(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})(\bar{s}_1^1 + \tilde{\epsilon}) - l\frac{\chi_2}{d_3}(\bar{s}_2^1 + \tilde{\epsilon})\right)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} > \underline{s}_1^1 - \epsilon \\ \frac{\left(b_{0,\inf} - (b_{1,\sup} + k\frac{\chi_2}{d_3})(\bar{s}_1^1 + \tilde{\epsilon}) - l\frac{\chi_2}{d_3}(\bar{s}_2^1 + \tilde{\epsilon})\right)(a_{1,\sup} - k\frac{\chi_1}{d_3})}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} + \frac{\frac{k\chi_2}{d_3}\left(a_{0,\inf} - (a_{2,\sup} + l\frac{\chi_1}{d_3})(\bar{s}_2^1 + \tilde{\epsilon}) - k\frac{\chi_1}{d_3}(\bar{s}_1^1 + \tilde{\epsilon})\right)}{(a_{1,\sup} - k\frac{\chi_1}{d_3})(b_{2,\sup} - l\frac{\chi_2}{d_3}) - lk\frac{\chi_1\chi_2}{d_3^2}} > \underline{s}_2^1 - \epsilon. \end{cases}$$

Then there is $t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq \overline{t}^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ such that

$$\begin{cases} u(x,t;t_0,u_0,v_0) \geq \underline{s}_1^1 - \epsilon \\ v(x,t;t,u_0,v_0) \geq \underline{s}_2^1 - \epsilon \end{cases}$$

for $x \in \Omega$, $t_0 \in \mathbb{R}$, and $t \geq t_0 + t^1_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. The claim (3.68) then follows.

Next, assume that for any $\epsilon > 0$, there is $t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^{k-1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ ($k \geq 2$) such that

$$\begin{cases} \underline{s}_1^k - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{s}_1^k + \epsilon \\ \underline{s}_2^k - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{s}_2^k + \epsilon \end{cases}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \geq t_0 + t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. By the similar arguments as in (1), there is there is $t^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0} \geq t^k_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$ ($k \geq 2$) such that

$$\begin{cases} \underline{s}_1^{k+1} - \epsilon \leq u(x,t;t_0,u_0,v_0) \leq \bar{s}_1^{k+1} + \epsilon \\ \underline{s}_2^{k+1} - \epsilon \leq v(x,t;t_0,u_0,v_0) \leq \bar{s}_2^{k+1} + \epsilon \end{cases}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \geq t_0 + t^{k+1}_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. (3.60) then follows by induction and (2) is thus proved. □

Lemma 3.12. *Consider* (3.1).

(1) Assume (H8). For any given $n \in \mathbb{N}$, $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$ and $t_0 \in \mathbb{R}$, if

$$\underline{r}_1^n \leq u_0 \leq \bar{r}_1^n \text{ and } \underline{r}_2^n \leq v_0 \leq \bar{r}_2^n, \tag{3.69}$$

then

$$\underline{r}_1^n \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^n \text{ and } \underline{r}_2^n \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^n \;\forall\; t \ge t_0. \tag{3.70}$$

(2) Assume (H9). For any given $n \in \mathbb{N}$*,* $u_0, v_0 \in C^0(\bar{\Omega})$ *with* $u_0, v_0 \ge 0$ *and* $u_0, v_0 \not\equiv 0$ *and* $t_0 \in \mathbb{R}$*, if*

$$\underline{s}_1^n \le u_0 \le \bar{s}_1^n \text{ and } \underline{s}_2^n \le v_0 \le \bar{s}_2^n,$$

then

$$\underline{s}_1^n \le u(x,t;t_0,u_0,v_0) \le \bar{s}_1^n \text{ and } \underline{s}_2^n \le v(x,t;t_0,u_0,v_0) \le \bar{s}_2^n \;\forall\; t \ge t_0.$$

Proof. (1) For given $n \in \mathbb{N}$, suppose (3.69) holds. We prove (3.70) holds in two steps.

Step 1. We prove in this step that the following holds for $k = 1$,

$$\underline{r}_1^k \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^k \text{ and } \underline{r}_2^k \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^k \;\; \forall\; t \ge t_0. \tag{3.71}$$

Recall that (3.62) reads as

$$u_t \le d_1 \Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\sup} - (a_{1,\inf} - k\frac{\chi_1}{d_3})u\big).$$

Thus, by parabolic comparison principe and $\bar{u}_0 \le \bar{r}_1^n \le \bar{r}_1^1$, we get that

$$0 \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^1 \quad \forall\; t \ge t_0. \tag{3.72}$$

Similarly, by parabolic comparison principe and $\bar{v}_0 \le \bar{r}_2^n \le \bar{r}_2^1$, we can get that

$$0 \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^1 \quad \forall\; t \ge t_0. \tag{3.73}$$

Therefore, for $t \geq t_0$,

$$u_t \geq d_1 \Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\inf} - a_{2,sup}\bar{r}_2^1 - k\frac{\chi_1}{d_3}\bar{r}_1^1 - (a_{1,\sup} - k\frac{\chi_1}{d_3})u)\big).$$

By parabolic comparison principe and $\underline{r}_1^1 \leq \underline{r}_1^n \leq \underline{u}_0$, we have that

$$\underline{r}_1^1 \leq u(x,t;t_0,u_0,v_0) \quad \forall\ t \geq t_0. \tag{3.74}$$

Similarly, by parabolic comparison principe and $\underline{r}_2^1 \leq \underline{r}_2^n \leq \underline{v}_0$, we have that

$$\underline{r}_2^1 \leq v(x,t;t_0,u_0,v_0) \quad \forall\ t \geq t_0.$$

This together with (3.72), (3.73), and (3.74) implies that (3.71) holds for $k = 1$.

Step 2. Suppose that (3.71) holds for $k = 1, 2, \cdots, l$ ($l \leq n-1$), we prove that (3.71) holds for $k = l+1$.

Indeed since (3.71) holds for $1 \leq k \leq l$, for $t \geq t_0$, we get from the first and third equation of (3.1) that

$$u_t \leq d_1 \Delta u - \chi_1 \nabla w \cdot \nabla u + u\big(a_{0,\sup} - a_{2,\inf}\underline{r}_2^l - k\frac{\chi_1}{d_3}\underline{r}_1^l - (a_{1,\inf} - k\frac{\chi_1}{d_3})u\big).$$

Thus, by parabolic comparison principe and $\bar{u}_0 \leq \bar{r}_1^n \leq \bar{r}_1^{l+1}$, we get that

$$u(x,t;t_0,u_0,v_0) \leq \bar{r}_1^{l+1} \quad \forall t \geq t_0. \tag{3.75}$$

Similarly, from the second and third equation of (3.1) and parabolic comparison principe, we get since $\bar{v}_0 \leq \bar{r}_2^n \leq \bar{r}_2^{l+1}$ that

$$v(x,t;t_0,u_0,v_0) \leq \bar{r}_2^{l+1} \quad \forall t \geq t_0. \tag{3.76}$$

Next again from the first and third equation of (3.1) that

$$u_t \geq d_1\Delta u - \chi_1\nabla w\cdot\nabla u + u\big(a_{0,\inf} - a_{2,sup}\bar{r}_2^l - k\frac{\chi_1}{d_3}\bar{r}_1^l - (a_{1,\sup} - k\frac{\chi_1}{d_3})u\big).$$

Therefore by parabolic comparison principe we get since $\underline{r}_1^{l+1} \leq \underline{r}_1^n \leq \underline{u}_0$ that

$$\underline{r}_1^{l+1} \leq u(x,t;t_0,u_0,v_0) \quad \forall t\geq t_0. \tag{3.77}$$

Similarly, from the second and third equation of (3.1) and parabolic comparison principe, we get since $\underline{r}_2^{l+1} \leq \underline{r}_2^n \leq \underline{u}_0$ that

$$\underline{r}_2^{l+1} \leq v(x,t;t_0,u_0,v_0) \quad \forall t\geq t_0.$$

This together with (3.75), (3.76), and (3.77) implies that (3.71) holds for $k=l+1$. (3.70) then follows by induction.

(2) It can be proved by the similar arguments as those in (1). □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. (1) First of all, from (3.57), the sequences $\underline{r}_1^n$ and $\underline{r}_2^n$ are nondecreasing bounded sequences of nonnegative real numbers and the sequences $\bar{r}_1^n$ and $\bar{r}_2^n$ non-increasing bounded sequences of nonnegative real numbers. Thus there exist real numbers $0<\underline{r}_1\leq\bar{r}_1\leq\bar{A}_1$ and $0<\underline{r}_2\leq\bar{r}_2\leq\bar{A}_2$ such that

$$\begin{cases}\lim_{n\to\infty}\underline{r}_1^n = \underline{r}_1, \quad \lim_{n\to\infty}\bar{r}_1^n = \bar{r}_1,\\ \lim_{n\to\infty}\underline{r}_2^n = \underline{r}_2, \quad \lim_{n\to\infty}\bar{r}_2^n = \bar{r}_2.\end{cases}$$

Combining this with the definition of $\bar{r}_i^n$ and $\underline{r}_i^n$ $(i=1,2)$, we get

$$\bar{r}_1 = \frac{a_{0,\sup} - a_{2,\inf}\underline{r}_2 - k\frac{\chi_1}{d_3}\underline{r}_1}{a_{1,\inf} - k\frac{\chi_1}{d_3}},$$

$$\bar{r}_2 = \frac{b_{0,\sup} - b_{1,\inf}\underline{r}_1 - k\frac{\chi_1}{d_3}\underline{r}_2}{b_{2,\inf} - l\frac{\chi_2}{d_3}},$$

$$\underline{r}_1 = \frac{a_{0,\inf} - a_{2,\sup}\bar{r}_2 - k\frac{\chi_1}{d_3}\bar{r}_1}{a_{1,\sup} - k\frac{\chi_1}{d_3}},$$

and

$$\underline{r}_2 = \frac{b_{0,\inf} - b_{1,\sup}\bar{r}_1 - l\frac{\chi_2}{d_3}\bar{r}_1}{b_{2,\sup} - l\frac{\chi_2}{d_3}}.$$

Hence $(\bar{r}_1, \bar{r}_2, \underline{r}_1, \underline{r}_2)$ is the unique solution of (3.22).

Next, we prove (3.25). By (3.58) and (3.7), for any $\epsilon > 0$, we can choose N such

$$\begin{cases} \underline{r}_1 - 2\epsilon \le \underline{r}_1^N - \epsilon \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^N + \epsilon \le \bar{r}_1 + 2\epsilon \\ \underline{r}_2 - 2\epsilon \le \underline{r}_2^N - \epsilon \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^N + \epsilon \le \bar{r}_2 + 2\epsilon, \end{cases}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$ and $t \ge t_0 + t^N_{\epsilon,\overline{u}_0,\overline{v}_0,\underline{u}_0,\underline{v}_0}$. Thus (3.10) holds.

Now suppose that (3.24) holds. We prove (3.25). Assume that

$$\underline{r}_1 \le u_0 \le \bar{r}_1 \text{ and } \underline{r}_2 \le v_0 \le \bar{r}_2.$$

Since the sequences $\underline{r}_1^n$ and $\underline{r}_2^n$ are nondecreasing bounded sequences of nonnegative real numbers and the sequences $\bar{r}_1^n$ and $\bar{r}_2^n$ non-increasing bounded sequences of nonnegative real numbers, from (3.7), we get for $n \in \mathbb{N}$ that

$$\underline{r}_1^n \le \underline{r}_1 \le u_0 \le \bar{r}_1 \le \bar{r}_1^n \text{ and } \underline{r}_2^n \le \underline{r}_2 \le v_0 \le \bar{r}_2 \le \bar{r}_2^n.$$

By Lemma 3.12,

$$\underline{r}_1^n \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1^n \text{ and } \underline{r}_2^n \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2^n \ \forall n \in \mathbb{N}\ , t \ge t_0.$$

Then as $n \to \infty$, we get

$$\underline{r}_1 \le u(x,t;t_0,u_0,v_0) \le \bar{r}_1 \text{ and } \underline{r}_2 \le v(x,t;t_0,u_0,v_0) \le \bar{r}_2 \ \forall t \ge t_0.$$

Thus (3.25) holds.

(2) It follows from the similar arguments as those in (1). □

3.8 Uniqueness and Stability of Coexistence States and Proof of Theorem 3.6

In this section, we establish the nonlinear stability and uniqueness of entire solutions of system (3.1) and prove Theorem 3.6 and Corollary 3.1.

We first prove Theorem 3.6(3).

Proof of Theorem 3.6(3). Recall that (3.8) implies (H4) (see Remark 3.7(2)). For given $t_0 \in \mathbb{R}$ and $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0(x), v_0(x) \geq 0$, $u_0(\cdot), v_0(\cdot) \not\equiv 0$, let $(u(\cdot, t; t_0, u_0, v_0), v(\cdot, t; t_0, u_0, v_0), w(\cdot, t; t_0, u_0, v_0))$ be the solution of (3.1) given by Theorem 3.1(2). Note that $(u(\cdot, t; t_0, u_0, v_0), v(\cdot, t; t_0, u_0, v_0), w(\cdot, t; t_0, u_0, v_0))$ exists for all $t > t_0$ and without loss of generality, we may assume that $u_0(x), v_0(x) > 0$ for all $x \in \bar{\Omega}$.

Let $(u^{**}(t), v^{**}(t), w^{**}(t))$ be a spatially homogeneous coexistence state of (3.1) (see Remark 3.7(1)). We first prove that (3.29) and (3.30) hold.

To this end, let $(\overline{u}(t), \underline{u}(t), \overline{v}(t), \underline{v}(t))$ be as in Lemma 3.3. Then by Lemma 3.3, we have

$$\underline{u}(t) \leq u(x, t; t_0, u_0, v_0) \leq \bar{u}(t), \quad \underline{v}(t) \leq v(x, t; t_0, u_0, v_0) \leq \bar{v}(t) \quad \forall\, x \in \bar{\Omega},\ t \geq t_0. \tag{3.78}$$

We claim that for any $\epsilon > 0$, there is $t_{\epsilon, u_0, v_0, t_0} > 0$ such that

$$\underline{u}(t) - \epsilon \leq u^{**}(t) \leq \bar{u}(t) + \epsilon, \quad \underline{v}(t) - \epsilon \leq v^{**}(t) \leq \bar{v}(t) + \epsilon \quad \forall\, t \geq t_0 + t_{\epsilon, u_0, v_0, t_0}. \tag{3.79}$$

Indeed let $(u^1(t), v^1(t))$ be the solution of (3.44) with $(u^1(t_0), v^1(t_0)) = (\underline{u}_0, \overline{v}_0)$. Note that $(\overline{u}(t), \underline{v}(t))$ satisfies

$$\begin{cases} \overline{u}_t \geq \overline{u}(t)(a_0(t) - a_1(t)\overline{u}(t) - a_2(t)\underline{v}(t)) \\ \underline{v}_t \leq \underline{v}(t)(b_0(t) - b_1(t)\overline{u}(t) - b_2(t)\underline{v}(t)). \end{cases}$$

Then by comparison principle for two species competition systems,

$$u^1(t) \le \overline{u}(t) \quad \text{and} \quad v^1(t) \ge \underline{v}(t) \quad \text{for all} \quad t \ge t_0. \tag{3.80}$$

Similarly, let $(u^2(t), v^2(t))$ be the solution of (3.44) with $(u^2(t_0), v^2(t_0)) = (\overline{u}_0, \underline{v}_0)$. Note that

$$\begin{cases} \underline{u}_t \le \underline{u}(t)(a_0(t) - a_1(t)\underline{u}(t) - a_2(t)\overline{v}(t)) \\ \overline{v}_t \ge \overline{v}(t)(b_0(t) - b_1(t)\underline{u}(t) - b_2(t)\overline{v}(t)) \end{cases}$$

By comparison principle for two species competition systems again,

$$u^2(t) \ge \underline{u}(t) \quad \text{and} \quad v^2(t) \le \overline{v}(t) \quad \text{for all} \quad t \ge t_0. \tag{3.81}$$

By Lemma 3.9,

$$\lim_{t\to\infty} \big(|u^i(t) - u^{**}(t)| + |v^i(t) - v^{**}(t)|\big) = 0 \quad \text{for} \quad i = 1, 2.$$

This implies that for any $\epsilon > 0$, there is $t_{\epsilon,u_0,v_0,t_0} > 0$ such that

$$u^2(t) - \epsilon \le u^{**}(t) \le u^1(t) + \epsilon, \quad v^1(t) - \epsilon \le v^{**}(t) \le v^2(t) + \epsilon \quad \forall\ t \ge t_0 + t_{\epsilon,u_0,v_0,t_0}.$$

This together with (3.80) and (3.81) implies (3.79).

By (3.78) and (3.79), to show (3.29) and (3.30), it suffices to show $0 \le \ln\frac{\overline{u}(t)}{\underline{u}(t)} + \ln\frac{\overline{v}(t)}{\underline{v}(t)} \longrightarrow 0$ as $t \to \infty$. Assume that $t > t_0$. By (3.33), we have

$$\begin{cases} \frac{\overline{u}'}{\overline{u}} = \frac{\chi_1}{d_3}\big(k\overline{u} + l\overline{v} - k\underline{u} - l\underline{v}\big) + \big[a_0(t) - a_1(t)\overline{u} - a_2(t)\underline{v}\big] \\ \frac{\underline{u}'}{\underline{u}} = \frac{\chi_1}{d_3}\big(k\underline{u} + l\underline{v} - k\overline{u} - l\overline{v}\big) + \big[a_0(t) - a_1(t)\underline{u} - a_2(t)\overline{v}\big] \\ \frac{\overline{v}'}{\overline{v}} = \frac{\chi_2}{d_3}\big(k\overline{u} + l\overline{v} - k\underline{u} - l\underline{v}\big) + \big[b_0(t) - b_2(t)\overline{v} - b_1(t)\underline{u}\big] \\ \frac{\underline{v}'}{\underline{v}} = \frac{\chi_2}{d_3}\big(k\underline{u} + l\underline{v} - k\overline{u} - l\overline{v}\big) + \big[b_0(t) - b_2(t)\underline{v} - b_1(t)\overline{u}\big]. \end{cases}$$

This together with (3.5) implies that

$$\frac{d}{dt}\Big(\ln\frac{\overline{u}}{\underline{u}} + \ln\frac{\overline{v}}{\underline{v}}\Big) = \frac{\overline{u}'}{\overline{u}} - \frac{\underline{u}'}{\underline{u}} + \frac{\overline{v}'}{\overline{v}} - \frac{\underline{v}'}{\underline{v}} \le -\min\{\alpha_1, \beta_1\}\left((\overline{u} - \underline{u}) + (\overline{v} - \underline{v})\right) \le 0, \quad (3.82)$$

where

$$0 < \alpha_1 = \inf_{t\in\mathbb{R}}\{a_1(t) - b_1(t) - 2k\frac{\chi_1 + \chi_2}{d_3}\},$$

and

$$0 < \beta_1 = \inf_{t\in\mathbb{R}}\{b_2(t) - a_2(t) - 2l\frac{\chi_1 + \chi_2}{d_3}\}.$$

Thus by integrating (3.82) over (t_0, t), we get

$$0 \le \ln\frac{\overline{u}(t)}{\underline{u}(t)} + \ln\frac{\overline{v}(t)}{\underline{v}(t)} \le \ln\frac{\overline{u}_0}{\underline{u}_0} + \ln\frac{\overline{v}_0}{\underline{v}_0}, \quad \text{and then} \quad \frac{\overline{u}(t)\overline{v}(t)}{\underline{u}(t)\underline{v}(t)} \le \frac{\overline{u}_0\overline{v}_0}{\underline{u}_0\underline{v}_0}.$$

We have by mean value theorem that

$$-\left((\overline{u} - \underline{u}) + (\overline{v} - \underline{v})\right) \le -\underline{u}\Big(\ln\frac{\overline{u}}{\underline{u}}\Big) - \underline{v}\Big(\ln\frac{\overline{v}}{\underline{v}}\Big)$$

Therefore

$$\frac{d}{dt}\Big(\ln\frac{\overline{u}}{\underline{u}} + \ln\frac{\overline{v}}{\underline{v}}\Big) \le -\Big(\min\{\alpha_1, \beta_1\}\Big)\Big(\min\{\alpha_2, \beta_2\}\Big)\Big(\ln\frac{\overline{u}}{\underline{u}} + \ln\frac{\overline{v}}{\underline{v}}\Big), \quad (3.83)$$

where

$$0 < \alpha_2 := \alpha_{2,t_0,u_0,v_0} = \inf_{t\geq t_0} \overline{u}(t)\frac{\underline{u}_0\underline{v}_0}{\overline{u}_0\overline{v}_0},$$

and

$$0 < \beta_2 := \beta_{2,t_0,u_0,v_0} = \inf_{t\geq t_0} \overline{v}(t)\frac{\underline{u}_0\underline{v}_0}{\overline{u}_0\overline{v}_0}.$$

By letting $\epsilon_{0,t_0,u_0,v_0} = \big(\min\{\alpha_1,\beta_1\}\big)\big(\min\{\alpha_2,\beta_2\}\big)$, we have $\epsilon_{0,t_0,u_0,v_0} > 0$ and

$$0 \leq \ln\frac{\overline{u}}{\underline{u}} + \ln\frac{\overline{v}}{\underline{v}} \leq \left(\ln\frac{\overline{u}_0}{\underline{u}_0} + \ln\frac{\overline{v}_0}{\underline{v}_0}\right) e^{-\epsilon_{0,t_0,u_0,v_0}(t-t_0)} \to 0 \quad \text{as} \quad t\to\infty.$$

Hence (3.29) and (3.30) hold.

Next, we show that (3.1) has a unique spatially homogeneous coexistence state. Suppose that $(u_i^*(t), v_i^*(t), w_i^*(t))$ $(i = 1,2)$ are spatially homogeneous coexistence states of (3.1). Let $u_{01} = \max\{\sup_{t\in\mathbb{R}} u_1^*(t), \sup_{t\in\mathbb{R}} u_2^*(t)\}$, $v_{01} = \min\{\inf_{t\in\mathbb{R}} v_1^*(t), \inf_{t\in\mathbb{R}} v_2^*(t)\}$, $u_{02} = \min\{\inf_{t\in\mathbb{R}} u_1^*(t), \inf_{t\in\mathbb{R}} u_2^*(t)\}$, and $v_{02} = \max\{\sup_{t\in\mathbb{R}} v_1^*(t), \sup_{t\in\mathbb{R}} v_2^*(t)\}$. For any $t_0 \in \mathbb{R}$, let $(u_i(t), v_i(t)) = (u(t;t_0,u_{0i},v_{0i}), v(t;t_0,u_{0i},v_{0i}))$ be the solution of (3.44) with

$$(u(t_0;t_0,u_{0i},v_{0i}), v(t_0;t_0,u_{0i},v_{0i})) = (u_{0i}, v_{0i})$$

$(i = 1,2)$. By comparison principle for two species competition systems,

$$u_2(t) \leq u_i^*(t) \leq u_1(t) \quad \text{and} \quad v_1(t) \leq v_i^*(t) \leq v_2(t) \tag{3.84}$$

for $i = 1,2$ and $t \geq t_0$. By the definition of coexistence states, there are $0 < \delta < K$ such that

$$\delta \leq u_i^*(t) \leq K, \quad \delta \leq v_i^*(t) \leq K \tag{3.85}$$

for $i = 1,2$ and all $t \in \mathbb{R}$. By the similar arguments of (3.82), we have

$$\frac{d}{dt}\ln\frac{u_1(t)}{u_2(t)} + \frac{d}{dt}\ln\frac{v_2(t)}{v_1(t)} \leq -\min\{\tilde{\alpha}_1,\tilde{\beta}_1\}\big(u_1(t) - u_2(t) + v_2(t) - v_1(t)\big)$$

for $t \geq t_0$, where

$$\tilde{\alpha}_1 = \inf_{t\in\mathbb{R}}(a_1(t) - b_1(t)), \quad \tilde{\beta}_1 = \inf_{t\in\mathbb{R}}(b_2(t) - a_2(t)).$$

Let

$$\tilde{\alpha}_2 = \inf_{t\in\mathbb{R}} u_1^*(t)\frac{u_{02}v_{01}}{u_{01}v_{02}}, \quad \tilde{\beta}_2 = \inf_{t\in\mathbb{R}} v_2^*(t)\frac{u_{02}v_{01}}{u_{01}v_{02}}$$

and $\tilde{\epsilon}_0 = \big(\min\{\tilde{\alpha}_1, \tilde{\beta}_1\}\big)\big(\min\{\tilde{\alpha}_2, \tilde{\beta}_2\}\big)$. Then by the similar arguments of (3.83), we have

$$0 \leq \ln\frac{u_1(t+t_0)}{u_2(t+t_0)} + \ln\frac{v_2(t+t_0)}{v_1(t+t_0)} \leq \left(\ln\frac{u_{01}}{u_{02}} + \ln\frac{v_{02}}{v_{01}}\right)e^{-\tilde{\epsilon}_0 t}$$

for $t \geq t_0$. This together with (3.85) implies that

$$0 \leq \ln\frac{u_1(t+t_0)}{u_2(t+t_0)} + \ln\frac{v_2(t+t_0)}{v_1(t+t_0)} \leq 2\ln\Big(\frac{K}{\delta}\Big)e^{-\tilde{\epsilon}_0 t}.$$

Therefore

$$\lim_{t\to\infty}\ln\frac{u(t+t_0;t_0;u_{01},v_{01})}{u(t+t_0;t_0,u_{02},v_{02})} + \ln\frac{v(t+t_0;t_0;u_{02},v_{02})}{v(t+t_0;t_0,u_{01},v_{01})} = 0$$

uniformly in $t_0 \in \mathbb{R}$. It then follows from (3.84) that $u_1^*(t) \equiv u_2^*(t)$ and $v_1^*(t) \equiv v_2^*(t)$. Indeed let $t \in \mathbb{R}$ be given. It follows from (3.84) that

$$\begin{aligned}
|u_1^*(t) - u_2^*(t)| &= |\tilde{u}^*(t)\ln\Big(\frac{u_1^*(t)}{u_2^*(t)}\Big)| \quad \text{(for some } \tilde{u}^*(t) \text{ between } u_1^*(t) \text{ and } u_2^*(t))\\
&\leq \max\{|u_1^*(t)|, |u_2^*(t)|\}|\ln\Big(\frac{u_1^*(t)}{u_2^*(t)}\Big)|\\
&\leq K\ln\Big(\frac{u_1(t)}{u_2(t)}\Big)\\
&\leq 2K\ln\Big(\frac{K}{\delta}\Big)e^{-\tilde{\epsilon}_0(t-t_0)}, \ \forall t_0 \leq t.
\end{aligned}$$

And similarly

$$|v_1^*(t) - v_2^*(t)| \leq 2K\ln\Big(\frac{K}{\delta}\Big)e^{-\tilde{\epsilon}_0(t-t_0)}, \ \forall t_0 \leq t.$$

Therefore as $t_0 \to -\infty$, we get $|u_1^*(t) - u_2^*(t)| = |v_1^*(t) - v_2^*(t)|$. Hence (3.1) has a unique spatially homogeneous coexistence state. □

Next, we prove Theorem 3.6(1) and (2).

Proof of Theorem 3.6(1) and (2). For given $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0, v_0 \geq 0$, let $\overline{u}_0 = \max_{x\in\bar{\Omega}} u_0(x)$, $\underline{u}_0 = \min_{x\in\bar{\Omega}} u_0(x)$, $\overline{v}_0 = \max_{x\in\bar{\Omega}} v_0(x)$, $\underline{v}_0 = \min_{x\in\bar{\Omega}} v_0(x)$.

(1) By Theorem 1.2(1) and Remark 1.3(2), (3.1) has coexistence states. Let $(u^{**}(x,t), v^{**}(x,t), w^{**}(x,t))$ be a coexistence state of (3.1). Let $q_1(t)$, $Q_1(t)$, $q_2(t)$ and $Q_2(t)$ be as in Theorem 3.6 (1), By (3.26),

$$\mu = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t \max\{q_1(\tau) - Q_1(\tau), q_2(\tau) - Q_2(\tau)\}d\tau < 0.$$

Fix $0 < \epsilon < -\mu$. Then, for given $u_0, v_0 \in C^0(\bar{\Omega})$ with $\inf u_0 > 0$, $\inf v_0 > 0$, there exists $T_{\epsilon,u_0,v_0} := T_{\epsilon,\bar{u}_0,\bar{v}_0,\underline{u}_0,\underline{v}_0} > 0$ such that for any $t_0 \in \mathbb{R}$,

$$\underline{r}_1 - \epsilon \leq u(\cdot, t_0+t; t_0; u_0, v_0) \leq \bar{r}_1 + \epsilon,\ \ \underline{r}_1 - \epsilon \leq u^{**}(x,t) \leq \bar{r}_1 + \epsilon\ \ \forall x \in \bar{\Omega},\ \ t \geq t_0 + T_{\epsilon,u_0,v_0},$$

$$\underline{r}_2 - \epsilon \leq v(\cdot, t_0+t; t_0; u_0, v_0) \leq \bar{r}_2 + \epsilon,\ \ \underline{r}_2 - \epsilon \leq v^{**}(x,t) \leq \bar{r}_2 + \epsilon\ \ \forall x \in \bar{\Omega},\ \ t \geq t_0 + T_{\epsilon,u_0,v_0},$$

and

$$\int_{t_0}^{t_0+t} \max\{q_1(s) - Q_1(s), q_2(s) - Q_2(s)\}ds \leq (\mu_1 + \epsilon)t,\ \forall\ t \geq t_0 + T_{\epsilon,u_0,v_0}.$$

To simplify the notation, set $u(t) = u(\cdot,t;t_0;u_0,v_0)$, $v(t) = v(\cdot,t;t_0;u_0,v_0)$, $u^{**}(t) = u^{**}(\cdot,t)$, and $v^{**}(t) = v^{**}(\cdot,t)$. Let $\psi = u - u^{**}$ and $\phi = v - v^{**}$. Then ψ satisfies

$$\begin{aligned}\psi_t &= d_1\Delta\psi - \chi_1\nabla\cdot(\psi\nabla w) - \chi_1\nabla\cdot(u^{**}\nabla(w - w^{**}))\\ &\quad + \psi\Big(a_0(t,x) - a_1(t,x)(u+u^{**}) - a_2(t,x)v\Big) - a_2(t,x)u^{**}\phi, \end{aligned} \tag{3.86}$$

and ϕ satisfies

$$\begin{aligned}\phi_t &= d_2\Delta\phi - \chi_2\nabla\cdot(\phi\nabla w) - \chi_2\nabla\cdot(v^{**}\nabla(w - w^{**}))\\ &\quad + \psi\Big(b_0(t,x) - b_1(t,x)u - b_2(t,x)(v+v^{**})\Big) - b_1(t,x)v^{**}\psi. \end{aligned} \tag{3.87}$$

We first prove that $\int_\Omega \big(\psi^2+\phi^2\big)dx \to 0$ as $t\to\infty$ uniformly in $t_0\in\mathbb{R}$. To this end, by multiplying (3.86) by ψ_+ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi_+^2 + d_1\int_\Omega |\nabla\psi_+|^2 = \chi_1\int_\Omega \psi_+\nabla\psi_+\cdot\nabla w + \chi_1\int_\Omega u^{**}\nabla\psi_+\cdot\nabla(w-w^{**})$$
$$+\int_\Omega \psi_+^2\big(a_0(t,x)-a_1(t,x)(u+u^{**})-a_2(t,x)v\big)-\int_\Omega a_2(t,x)u^{**}\psi_+\phi$$

for a.e $t>t_0$ (see [30, (4.6)]) for the reasons to have the above equality). Then by integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi_+^2 + d_1\int_\Omega |\nabla\psi_+|^2 \le -\frac{\chi_1}{2}\int_\Omega \psi_+^2\Delta w + \chi_1\int_\Omega u^{**}\nabla\psi_+\cdot\nabla(w-w^{**})$$
$$+\int_\Omega \psi_+^2\big(a_{0,\sup}(t)-a_{1,\inf}(t)(u+u^{**})-a_{2,\inf}(t)v\big)$$
$$-\int_\Omega a_2(t,x)u^{**}\psi_+\phi_+ + \int_\Omega a_2(t,x)u^{**}\psi_+\phi_- \tag{3.88}$$

for a.e $t>t_0$.

We have from the third equation of (3.1) that

$$-\frac{\chi_1}{2}\int_\Omega \psi_+^2\Delta w = \frac{\chi_1}{2d_3}\int_\Omega \psi_+^2(ku+lv-\lambda w), \tag{3.89}$$

and by Young's inequality

$$\chi_1\int_\Omega u^{**}\nabla\psi_+\cdot\nabla(w-w^{**}) \le d_1\int_\Omega |\nabla\psi_+|^2 + \frac{\chi_1^2(u^{**}_{\sup})^2}{4d_1}\int_\Omega |\nabla(w-w^{**})|^2. \tag{3.90}$$

We claim that

$$\int_\Omega |\nabla(w-w^{**})|^2 \le \frac{k^2}{2\lambda d_3}\int_\Omega \psi^2 + \frac{l^2}{2\lambda d_3}\int_\Omega \phi^2. \tag{3.91}$$

Indeed since (u,v,w) and (u^{**},v^{**},w^{**}) are both solutions of (3.1), from the third equation of (3.1) we get

$$0=d_3\Delta(w-w^{**})+k(u-u^{**})+l(v-v^{**})-\lambda(w-w^{**}) = d_3\Delta(w-w^{**})+k\psi+l\phi-\lambda(w-w^{**}).$$

By multiplying this last equation by $w-w^{**}$ and integrating over Ω, we get by Green's Theorem

$$0 = -d_3 \int_\Omega |\nabla(w - w^{**})|^2 + k \int_\Omega \psi(w - w^{**}) + l \int_\Omega \psi(w - w^{**}) - \lambda \int_\Omega (w - w^{**})^2.$$

By Young's inequality we get

$$d_3 \int_\Omega |\nabla(w - w^{**})|^2 + \lambda \int_\Omega (w - w^{**})^2 \le \frac{k^2}{2\lambda} \int_\Omega \psi^2 + \frac{l^2}{2\lambda} \int_\Omega \phi^2 + \lambda \int_\Omega (w - w^{**})^2,$$

and the claim thus follows.

By (3.88)-(3.91), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \psi_+^2 \le & \frac{\chi_1}{2d_3}\int_\Omega \psi_+^2(ku + lv - \lambda w) + \frac{(k\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \psi^2 + \frac{(l\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \phi^2 \\ & + \int_\Omega \psi_+^2\big(a_{0,\sup}(t) - a_{1,\inf}(t)(u + u^{**}) - a_{2,\inf}(t)v\big) - a_{2,\inf}(t)\int_\Omega u^{**}\psi_+\phi_+ \\ & + a_{2,\sup}(t)u_{\sup}^{**}\int_\Omega \psi_+\phi_-\end{aligned}$$

for a.e $t > t_0$. Thus by Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \psi_+^2 \le & \frac{\chi_1}{2d_3}\int_\Omega \psi_+^2(ku + lv - \lambda w) + \frac{(k\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \psi^2 + \frac{(l\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \phi^2 \\ & + \int_\Omega \psi_+^2\big(a_{0,\sup}(t) - a_{1,\inf}(t)(u + u^{**}) - a_{2,\inf}(t)v\big) + \frac{a_{2,\sup}(t)u_{\sup}^{**}}{2}\int_\Omega \psi_+^2 \\ & + \frac{a_{2,\sup}(t)u_{\sup}^{**}}{2}\int_\Omega \phi_-^2 - a_{2,\inf}(t)\int_\Omega u^*\psi_+\phi_+ \end{aligned} \tag{3.92}$$

for a.e $t > t_0$.

Similarly, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \psi_-^2 \le & \frac{\chi_1}{2d_3}\int_\Omega \psi_-^2(ku + lv - \lambda w) + \frac{(k\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \psi^2 + \frac{(l\chi_1 u_{\sup}^{**})^2}{8\lambda d_1 d_3}\int_\Omega \phi^2 \\ & + \int_\Omega \psi_-^2\big(a_{0,\sup}(t) - a_{1,\inf}(t)(u + u^{*}) - a_{2,\inf}(t)v\big) + \frac{a_{2,\sup}(t)u_{\sup}^{**}}{2}\int_\Omega \psi_-^2 \\ & + \frac{a_{2,\sup}(t)u_{\sup}^{**}}{2}\int_\Omega \phi_+^2 - a_{2,\inf}(t)\int_\Omega u^{**}\psi_-\phi_- \end{aligned} \tag{3.93}$$

for a.e $t > t_0$. By adding (3.92) and (3.93), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \psi^2 \\
&\le \int_\Omega \psi^2\Big(\frac{\chi_1}{2d_3}(ku+lv-\lambda w)+\frac{(k\chi_1 u^{**}_{\sup})^2}{4\lambda d_1 d_3}+a_{0,\sup}(t)-a_{1,\inf}(t)(u+u^{**})-a_{2,\inf}(t)v\Big)\\
&+\frac{a_{2,\sup}(t)u^{**}_{\sup}}{2}\int_\Omega \psi^2+\Big(\frac{(l\chi_1 u^{**}_{\sup})^2}{4\lambda d_1 d_3}+\frac{a_{2,\sup}(t)u^{*}_{\sup}}{2}\Big)\int_\Omega \phi^2 - a_{2,\inf}(t)\int_\Omega u^*\big(\psi_+\phi_+ + \psi_-\phi_-\big)
\end{aligned}
$$

for a.e. $t > t_0$.

Similarly we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \phi^2 \\
&\le \int_\Omega \phi^2\Big(\frac{\chi_2}{2d_3}(ku+lv-\lambda w)+\frac{(l\chi_2 v^{**}_{\sup})^2}{4\lambda d_2 d_3}+b_{0,\sup}(t)-b_{2,\inf}(t)(v+v^{*})-b_{1,\inf}(t)u\Big)\\
&+\frac{b_{1,\sup}(t)v^{**}_{\sup}}{2}\int_\Omega \phi^2+\Big(\frac{(k\chi_2 v^{**}_{\sup})^2}{4\lambda d_2 d_3}+\frac{b_{1,\sup}(t)v^{**}_{\sup}}{2}\Big)\int_\Omega \psi^2 - b_{1,\inf}(t)\int_\Omega v^{**}\big(\psi_+\phi_+ + \psi_-\phi_-\big)
\end{aligned}
$$

for a.e. $t > t_0$. By adding the last two inequalities, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\psi^2+\phi^2\big) \\
&\le \int_\Omega \psi^2\Big(\frac{\chi_1}{2d_3}(ku+lv-\lambda w)+\frac{k^2}{4\lambda d_3}\big(\frac{(\chi_1 u^{**}_{\sup})^2}{d_1}+\frac{(\chi_2 v^{**}_{\sup})^2}{d_2}\big)+a_{0,\sup}(t)-a_{1,\inf}(t)(u+u^{**})\Big)\\
&+\Big(-a_{2,\inf}(t)v+\frac{a_{2,\sup}(t)u^{**}_{\sup}+b_{1,\sup}(t)v^{**}_{\sup}}{2}\Big)\int_\Omega \psi^2\\
&+\int_\Omega \phi^2\Big(\frac{\chi_2}{2d_3}(ku+lv-\lambda w)+\frac{l^2}{4\lambda d_3}\big(\frac{(\chi_1 u^{**}_{\sup})^2}{d_1}+\frac{(\chi_2 v^{**}_{\sup})^2}{d_2}\big)+b_{0,\sup}(t)-b_{2,\inf}(t)(v+v^{**})\Big)\\
&+\Big(-b_{1,\inf}(t)u+\frac{a_{2,\sup}(t)u^{**}_{\sup}+b_{1,\sup}(t)v^{**}_{\sup}}{2}\Big)\int_\Omega \phi^2
\end{aligned}
$$

for a.e. $t > t_0$. Thus for $t \ge t_0 + T_{\epsilon,u_0,v_0}$, we have

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\psi^2+\phi^2\big) \le \Big(Q_1(t)-q_1(t)+K_1(t,\epsilon)\Big)\int_\Omega \psi^2+\Big(Q_2(t)-q_2(t)+K_2(t,\epsilon)\Big)\int_\Omega \phi^2,
$$

where

$$K_1(t,\epsilon) = \frac{\chi_1(k+l)}{d_3}\epsilon + \frac{k^2\epsilon}{4\lambda d_3}\Big(\frac{\chi_1^2}{d_1}(2\bar{r}_1+\epsilon) + \frac{\chi_2^2}{d_2}(2\bar{r}_2+\epsilon)\Big)$$
$$+ \epsilon\Big(2a_{1,\inf}(t) + a_{2,inf}(t) + \frac{a_{2,\sup}(t)+b_{1,\sup}(t)}{2}\Big),$$

and

$$K_2(t,\epsilon) = \frac{\chi_2(k+l)}{d_3}\epsilon + \frac{l^2\epsilon}{4\lambda d_3}\Big(\frac{\chi_1^2}{d_1}(2\bar{r}_1+\epsilon) + \frac{\chi_2^2}{d_2}(2\bar{r}_2+\epsilon)\Big)$$
$$+ \epsilon\Big(2b_{2,\inf}(t) + b_{1,inf}(t) + \frac{b_{1,\sup}(t)+a_{2,\sup}(t)}{2}\Big).$$

Therefore for $t \geq t_0 + T_{\epsilon,u_0,v_0}$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\psi^2+\phi^2\big) \leq (h(t)+K(t,\epsilon))\Big(\int_\Omega \big(\psi^2+\phi^2\big)\Big),$$

where

$$h(t) = \max\{Q_1(t)-q_1(t), Q_2(t)-q_2(t)\},$$

and

$$K(t,\epsilon) = |K_1(t,\epsilon)| + |K_2(t,\epsilon)|.$$

Note that $0 \leq \sup_{t\in\mathbb{R}} K(t,\epsilon) \to 0$ as $\epsilon \to 0$. Choose $\epsilon_0 \ll 1$ ($\epsilon_0 < -\mu$) such that

$$0 \leq \sup_{t\in\mathbb{R}} K(t,\epsilon) < \frac{-\mu-\epsilon_0}{2}.$$

By $\int_{t_0}^t h(s)ds \leq (\mu+\epsilon_0)(t-t_0)$ for $t \geq t_0 + T_{\epsilon,u_0,v_0}$, we have

$$\begin{aligned}
&\int_\Omega \big(\psi^2+\phi^2\big) \\
&\leq \Big(\int_\Omega \psi^2(t_0+T_{\epsilon,u_0,v_0}) + \phi^2(t_0+T_{\epsilon,u_0,v_0})\Big)e^{2(\mu+\epsilon_0)(t-t_0-T_{\epsilon,u_0,v_0})}e^{2(\frac{-\mu-\epsilon_0}{2})(t-t_0-T_{\epsilon,u_0,v_0})} \\
&\leq \Big(\int_\Omega \psi^2(t_0+T_{\epsilon,u_0,v_0}) + \phi^2(t_0+T_{\epsilon,u_0,v_0})\Big)e^{(\mu+\epsilon_0)(t-t_0-T_{\epsilon,u_0,v_0})} \quad \forall\ t > t_0+T_{\epsilon,u_0,v_0}.
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \|u(\cdot,t+t_0;t_0,u_0,v_0) - u^{**}(\cdot,t+t_0)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|\psi(t+t_0)\|^2_{L^2(\Omega)} = 0, \tag{3.94}$$

and

$$\lim_{t\to\infty} \|v(\cdot,t+t_0;t_0,u_0,v_0) - v^{**}(\cdot,t+t_0)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|\phi(t+t_0)\|^2_{L^2(\Omega)} = 0. \tag{3.95}$$

uniformly in $t_0 \in \mathbb{R}$.

It follows from (3.94) and (3.95) and similar arguments as in the proof [30, Theorem 1.4 (2)] that for any $u_0, v_0 \in C^0(\bar{\Omega})$ with $\inf u_0 > 0$, $\inf v_0 > 0$, we have

$$\lim_{t\to\infty} \|u(\cdot,t+t_0;t_0,u_0,v_0) - u^{**}(\cdot,t+t_0)\|_{L^\infty(\Omega)} = 0,$$

and

$$\lim_{t\to\infty} \|v(\cdot,t+t_0;t_0,u_0,v_0) - v^{**}(\cdot,t+t_0)\|_{L^\infty(\Omega)} = 0.$$

uniformly in $t_0 \in \mathbb{R}$. It then follows that (3.27) and (3.28) hold for any $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, $v_0 \geq 0$, and $u_0 \neq 0$, $v_0 \neq 0$.

Next, we prove that (3.1) has a unique positive entire solution. We are going to prove that in the following two steps.

Step 1. (3.1) has a unique positive entire solution (u^*, v^*, w^*) which satisfy

$$\underline{r}_1 \leq u^*(x,t) \leq \bar{r}_1 \text{ and } \underline{r}_2 \leq v^*(x,t) \leq \bar{r}_2 \ \forall x \in \bar{\Omega} \text{ and } t \in \mathbb{R}. \tag{3.96}$$

Suppose that $(u_1^*(x,t), v_1^*(x,t), w_1^*(x,t))$ and $(u_2^*(x,t), v_2^*(x,t), w_2^*(x,t))$ are two positive entire solutions of (3.1) that satisfy (3.96). We claim that

$$(u_1^*(x,t), v_1^*(x,t), w_1^*(x,t)) \equiv (u_2^*(x,t), v_2^*(x,t), w_2^*(x,t))$$

for any $t \in \mathbb{R}$. Indeed, Then by assumption (3.26), for given $\epsilon > 0$, there is $t_\epsilon > 0$ such that

$$\int_{t_0}^{t_0+t} \max\{q_1(s) - Q_1(s), q_2(s) - Q_2(s)\}ds \leq (\mu_1 + \epsilon)t,\ \forall t_0 \in \mathbb{R},\ t \geq t_0 + t_\epsilon.$$

Then by the arguments in the proof of (3.94) and (3.95), there is $\epsilon_0 >$ such that for any $t, t_0 \in \mathbb{R}$ with $t \geq t_0 + t_{\epsilon_0}$, we have

$$\begin{aligned} &\|u_1^*(\cdot,t) - u_2^*(\cdot,t)\|_{L^2(\Omega)} + \|v_1^*(\cdot,t) - v_2^*(\cdot,t)\|_{L^2(\Omega)} \\ &\leq (\int_\Omega (u_1^* - u_2^*)^2(t_0 + t_{\epsilon_0}) + (v_1^* - v_2^*)^2(t_0 + t_{\epsilon_0}))e^{(\mu+\epsilon_0)(t-t_0-t_{\epsilon_0})}. \end{aligned} \tag{3.97}$$

Moreover, by (3.96), we have

$$m = \min\{\underline{r}_1, \underline{r}_2\} \leq u_i^*(x,t) \leq M = \max\{\bar{r}_1, \bar{r}_2\} \quad \text{and} \quad m \leq v_i^*(x,t) \leq M, i = 1, 2.$$

By combining this with (3.97), we get

$$\begin{aligned} &\|u_1^*(\cdot,t) - u_2^*(\cdot,t)\|_{L^2(\Omega)} + \|v_1^*(\cdot,t) - v_2^*(\cdot,t)\|_{L^2(\Omega)} \\ &\leq 8M^2|\Omega|e^{(\mu+\epsilon_0)(t-t_0-t_{\epsilon_0})} \quad \forall t_0 \in \mathbb{R} \text{ and } t \geq t_0 + t_{\epsilon_0}. \end{aligned}$$

Now let $t \in \mathbb{R}$ be given. Choose $t_0 \in \mathbb{R}$ such $t_0 < t - t_{\epsilon_0}$. It then follows that

$$\begin{aligned} &\|u_1^*(\cdot,t) - u_2^*(\cdot,t)\|_{L^2(\Omega)} + \|v_1^*(\cdot,t) - v_2^*(\cdot,t)\|_{L^2(\Omega)} \\ &\leq 8M^2|\Omega|e^{(\mu+\epsilon_0)(t-t_0-t_{\epsilon_0})} \to 0 \quad \text{as} \quad t_0 \to -\infty. \end{aligned}$$

Thus we get by continuity of solution that $u_1^*(x,t) = u_2^*(x,t)$ and $v_1^*(x,t) = v_2^*(x,t)$ for all $x \in \bar{\Omega}$ and $t \in \mathbb{R}$.

Step 2. We claim that every positive entire solution of (3.1) satisfies (3.96). Indeed, let (u^*, v^*, w^*) be a positive entire solution of (3.1). Then for any given $\epsilon > 0$ there exists

$t_{t_\epsilon,u^*,v^*} := t_{\epsilon,\sup u^*,\sup v^*,\inf u^*,\inf v^*}$ such that

$$\begin{cases} \underline{r}_1 - \epsilon \le u^*(x,t;t_0,u^*(\cdot,t_0),v^*(\cdot,t_0)) \le \bar{r}_1 + \epsilon \\ \underline{r}_2 - \epsilon \le v^*(x,t;t_0,u^*(\cdot,t_0),v^*(\cdot,t_0)) \le \bar{r}_2 + \epsilon \end{cases} \tag{3.98}$$

for all $x \in \bar{\Omega}$, $t_0 \in \mathbb{R}$, and $t \ge t_0 + t_{\epsilon,u^*,v^*}$. Let $t \in \mathbb{R}$ be fix. We have $u^*(x,t) = u^*(x,t;t-t_{\epsilon,u^*,v^*},u^*(\cdot,t-t_{\epsilon,u^*,v^*}),v^*(\cdot,t-t_{\epsilon,u^*,v^*}))$ and $v^*(x,t) = v^*(x,t;t-t_{\epsilon,u^*,v^*},u^*(\cdot,t-t_{\epsilon,u^*,v^*}),v^*(\cdot,t-t_{\epsilon,u^*,v^*}))$. Then by (3.98) with $t_0 = t - t_{\epsilon,u^*,v^*}$, we get

$$\underline{r}_1 - \epsilon \le u^*(x,t) \le \bar{r}_1 + \epsilon \text{ and } \underline{r}_2 - \epsilon \le v^*(x,t) \le \bar{r}_2 + \epsilon.$$

And since ϵ is arbitrary, we get as $\epsilon \to 0$ that

$$\underline{r}_1 \le u^*(x,t) \le \bar{r}_1 \text{ and } \underline{r}_2 \le v^*(x,t) \le \bar{r}_2.$$

and thus the claim holds.

(2) It follows by the similar arguments as those in (2). □

Finally, we prove Corollary 3.1.

Proof of Corollary 3.1. First, note that in this case $\chi_1 = \chi_2 = 0$, condition (3.31) becomes condition (3.26) for the global stability and uniqueness of positive entire solution of (3.2). Recall that (3.31) reads as

$$\begin{cases} a_{0,\sup} + \frac{a_{2,\sup}}{2}\bar{r}_1 + \frac{b_{1,\sup}}{2}\bar{r}_2 < 2a_{1,\inf}\underline{r}_1 + a_{2,\inf}\underline{r}_2 \\ b_{0,\sup} + \frac{b_{1,\sup}}{2}\bar{r}_2 + \frac{a_{2,\sup}}{2}\bar{r}_1 < 2b_{2,\inf}\underline{r}_2 + b_{1,\inf}\underline{r}_1. \end{cases} \tag{3.99}$$

Note that $\underline{\mathrm{r}}_1 = \frac{a_{0,\inf}b_{2,\inf}-a_{2,\sup}b_{0,\sup}}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}}$, and $\underline{\mathrm{r}}_2 = \frac{a_{1,\inf}b_{0,\inf}-a_{0,\sup}b_{1,\sup}}{a_{1,\inf}b_{2,\sup}-a_{2,\inf}b_{1,\sup}}$ (see Remark 3.6(3)). Hence (3.99) is equivalent to

$$\begin{cases} \frac{a_{2,\sup}}{2}\bar{r}_1 + \frac{b_{1,\sup}}{2}\bar{r}_2 < -a_{0,\sup} + 2a_{1,\inf}\left(\frac{a_{0,\inf}b_{2,\inf}-a_{2,\sup}b_{0,\sup}}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}}\right) + a_{2,\inf}\underline{\mathrm{r}}_2 \\ \\ \frac{b_{1,\sup}}{2}\bar{r}_2 + \frac{a_{2,\sup}}{2}\bar{r}_1 < -b_{0,\sup} + 2b_{2,\inf}\left(\frac{a_{1,\inf}b_{0,\inf}-a_{0,\sup}b_{1,\sup}}{a_{1,\inf}b_{2,\sup}-a_{2,\inf}b_{1,\sup}}\right) + b_{1,\inf}\underline{\mathrm{r}}_1, \end{cases}$$

which is equivalent to

$$\begin{cases} \frac{a_{2,\sup}}{2}\bar{r}_1 + \frac{b_{1,\sup}}{2}\bar{r}_2 < \frac{-a_{0,\sup}a_{1,\sup}b_{2,\inf}+a_{0,\sup}a_{2,\sup}b_{1,\inf}+2a_{1,\inf}a_{0,\inf}b_{2,\inf}-2a_{1,\inf}a_{2,\sup}b_{0,\sup}}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}} + a_{2,\inf}\underline{\mathrm{r}}_2 \\ \\ \frac{b_{1,\sup}}{2}\bar{r}_2 + \frac{a_{2,\sup}}{2}\bar{r}_1 < \frac{-b_{0,\sup}a_{1,\inf}b_{2,\sup}+b_{0,\sup}a_{2,\inf}b_{1,\sup}+2b_{2,\inf}a_{1,\inf}b_{0,\inf}-2b_{2,\inf}a_{0,\sup}b_{1,\sup}}{a_{1,\inf}b_{2,\sup}-a_{2,\inf}b_{1,\sup}} + b_{1,\inf}\underline{\mathrm{r}}_1, \end{cases}$$

and so

$$\begin{cases} \frac{a_{2,\sup}}{2}\bar{r}_1 + \frac{b_{1,\sup}}{2}\bar{r}_2 < \frac{b_{2,\inf}\left(2a_{1,\inf}a_{0,\inf}-a_{0,\sup}a_{1,\sup}\right)-a_{2,\sup}\left(2a_{1,\inf}b_{0,\sup}-a_{0,\sup}b_{1,\inf}\right)}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}} + a_{2,\inf}\underline{\mathrm{r}}_2 \\ \\ \frac{b_{1,\sup}}{2}\bar{r}_2 + \frac{a_{2,\sup}}{2}\bar{r}_1 < \frac{a_{1,\inf}\left(2b_{2,\inf}b_{0,\inf}-b_{0,\sup}b_{2,\sup}\right)-b_{1,\sup}\left(2b_{2,\inf}a_{0,\sup}-b_{0,\sup}a_{2,\inf}\right)}{a_{1,\inf}b_{2,\sup}-a_{2,\inf}b_{1,\sup}} + b_{1,\inf}\underline{\mathrm{r}}_1. \end{cases}$$

Therefore (3.99) is equivalent to

$$\begin{cases} a_{2,\sup}\left(\frac{\bar{r}_1}{2} + \frac{2a_{1,\inf}b_{0,\sup}-a_{0,\sup}b_{1,\inf}}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}}\right) + \frac{b_{1,\sup}}{2}\bar{r}_2 < b_{2,\inf}\frac{2a_{1,\inf}a_{0,\inf}-a_{0,\sup}a_{1,\sup}}{a_{1,\sup}b_{2,\inf}-a_{2,\sup}b_{1,\inf}} + a_{2,\inf}\underline{\mathrm{r}}_2 \\ \\ b_{1,\sup}\left(\frac{\bar{r}_2}{2} + \frac{2b_{2,\inf}a_{0,\sup}-b_{0,\sup}a_{2,\inf}}{b_{2,\sup}a_{1,\inf}-b_{1,\sup}a_{2,\inf}}\right) + \frac{a_{2,\sup}}{2}\bar{r}_1 < a_{1,\inf}\frac{2b_{2,\inf}b_{0,\inf}-b_{0,\sup}b_{2,\sup}}{b_{2,\sup}a_{1,\inf}-b_{1,\sup}a_{2,\inf}} + b_{1,\inf}\underline{\mathrm{r}}_1. \end{cases} \tag{3.100}$$

Next, suppose that $2a_{1,\inf}a_{0,\inf} - a_{0,\sup}a_{1,\sup} > 0$ and $2b_{2,\inf}b_{0,\inf} - b_{0,\sup}b_{2,\sup} > 0$. If a_2 and b_1 are such small so that (3.5) and (3.100) hold, then conditions (3.5) and (3.26) hold and Corollary 3.1 follows from Theorem 3.6. □

Remark 3.13. *We discussed the conditions under which* (3.22)*, that is,*

$$\begin{cases} (a_{1,\inf} - k\frac{\chi_1}{d_3})\bar{r}_1 = a_{0,\sup} - a_{2,\inf}\underline{r}_2 - k\frac{\chi_1}{d_3}\underline{r}_1 \\ (b_{2,\inf} - l\frac{\chi_2}{d_3})\bar{r}_2 = b_{0,\sup} - b_{1,\inf}\underline{r}_1 - k\frac{\chi_1}{d_3}\underline{r}_2 \\ (a_{1,\sup} - k\frac{\chi_1}{d_3})\underline{r}_1 = a_{0,\inf} - a_{2,\sup}\bar{r}_2 - k\frac{\chi_1}{d_3}\bar{r}_1 \\ (b_{2,\sup} - l\frac{\chi_2}{d_3})\underline{r}_2 = b_{0,\inf} - b_{1,\sup}\bar{r}_1 - l\frac{\chi_2}{d_3}\bar{r}_2, \end{cases}$$

has a unique solution in the appendix of our paper [29, Appendix].

Chapter 4

Concluding Remarks and Future Works

The results obtained in this dissertation lead to many interesting and challenging open problems. In this last chapter, we will enumerate and discuss some of these interesting research problems.

Problem 1 : Can the results obtained in Chapter 1 be extended the to the known mathematically challenging case of full chemotaxis i.e $\tau > 0$?

System (1.1) with $\tau = d_2 = d_3 = 1$ reduces to

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)\int_\Omega u\Big), \quad x \in \Omega \\ v_t = \Delta v - v + u, \quad x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \end{cases} \tag{4.1}$$

Note that the existing methods on global existence of nonnegative solutions of (4.1) in the case that a_0 and a_1 are constants and $a_2 = 0$ such as [64, Theorem 0.1] can be adopted to the study of global solutions of (4.1) in the general case without much modification. Thus in the study of dynamics of system (4.1), a very challenging problem is to prove under the condition of global existence, the persistence and existence of positive entire solutions. In a recent ongoing work, I and Dr. Wenxian Shen were able to prove under the condition of global existence, the persistence and existence of positive entire solutions for system (4.1). Furthermore, one can consider working on the following open problems on dynamics of system (4.1).

- Is chemotaxis/heterogeneous environment good/bad for persistence/existence of entire solution? (see [62])

- When the coefficients $a_i(t,x)$ are periodic with period T, does system (4.1) have a positive periodic solution with period T?

- Uniqueness and stability of positive entire solutions.

The main challenge in this case is that unlike in system (2.1), u small does not guarantee any more that v and Δv are small. Actually because in this case, v also depends on the initial v_0, u small does guarantee that v and Δv are small only for t very large and that we basically proved and used to obtain our recent result on the persistence and existence of positive entire solutions for system (4.1) under the condition of global existence. When, Ω is convex, global existence of classical solutions of system (4.1) holds under the explicit parameter region $a_{1,\inf} > \frac{\chi n}{4}$ and $\inf_{t\in\mathbb{R}}\big(a_{1,\inf}(t) - (a_{2,inf}(t))_-\big) > 0$. No such explicit global existence parameter region exist up to today for system (4.1) even when the coefficients are constant.

Problem 2 : Can the results obtained in Chapter 2 and 3 be extended the to the known mathematically challenging case of full chemotaxis i.e $\tau > 0$?

More precisely, one can consider the system

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u\Big(a_0(t,x) - a_1(t,x)u - a_2(t,x)v\Big), \quad x\in\Omega \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + v\Big(b_0(t,x) - b_1(t,x)u - b_2(t,x)v\Big), \quad x\in\Omega \\ w_t = d_3\Delta w + ku + lv - \lambda w, \quad x\in\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases} \tag{4.2}$$

and address the following interesting open problems:

- Find natural parameter region for which global existence of classical solutions hold for system (4.2). (This should follow from existing global existence results for constant coefficients)

- Under the condition of global existence plus some further natural conditions, prove persistence and existence of positive entire solutions for system (4.2)

- Is chemotaxis/heterogeneous environment good/bad for persistence/existence of entire solution?

- When the coefficients $a_i(t, x)$ are periodic with period T, does system (4.2) have a positive periodic solution with period T?

- Existence of optimal attracting rectangle

- Uniqueness and stability of positive entire solutions.

Problem 3: Study the dynamics of two species chemotaxis with homogeneous/heterogeneous coefficients on unbounded domain.

For example, I, Dr. Rachidi Salako, and Dr. Shen are currently studying the existence of traveling wave solutions for the following two species chemotaxis models with constant coefficients on unbounded domain.

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + u_1(a_0 - a_1 u - a_2 v), \quad x\in\mathbb{R}^N,\ t>0, \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + v(b_0 - b_1 u - b_2 v), \quad x\in\mathbb{R}^N,\ t>0, \\ 0 = (\Delta - \lambda I)w + l_1 u + l_2 v, \quad x\in\mathbb{R}^N,\ t>0, \end{cases} \tag{4.3}$$

Problem 4 : Can movement of populations attracted/repelled by things like job opportunities, individual freedom, political stability,..., etc be modeled by certain variants of (discrete) Keller-Segel model? If yes can mathematics, big data, statistics and machine learning be combined to understand and predict such complex dynamics of population? I plan to work on these type of problems that combine mathematics and data science.

Chapter 5

References

[1] S. Ahmad, Convergence and ultimate bounds of solutions of the nonautonomous Volterra-Lotka competition equations, *J. Math. Anal. Appl.*, **127** (1987), no. 2, 377-387.

[2] H. Amann, Existence and regularity for semilinear parabolic evolution equations, *Annali della Scuola Normale Superiore di Pisa*, **11** (1984) 593-676.

[3] C. Alvarez and A. C. Lazek, An application of topological degree to the periodic competing species problem, *J. Austral. Morh. SIC. Ser. B* **28** (1986), 202-219.

[4] E. E. E.Arenas, A. Stevens, and J. J. L. Velzquez, Simultaneous finite time blow-up in a two-species model for chemotaxis, *Analysis*, **29**(2009), 317-338 .

[5] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler., Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, **25(9)** (2015), 1663-1763.

[6] T. Black, J. Lankeit, M. Mizukami, On the weakly competitive case in a two-species chemotaxis model, preprint (2016).

[7] R.S. Cantrell and C. Cosner, The effects of spatial heterogeneity in population dynamics, *J. Math. Biol.* **29** (1991), no. 4, 315-338.

[8] M.A.J. Chaplain and J.I. Tello, On the stability of homogeneous steady states of a chemotaxis system with logistic growth term, *Appl. Math. Lett.*, **57** (2016), 1-6.

[9] C. Conca, E. Espejo and K. Vilches, Remarks on the blow-up and global existence for a two species chemotactic Keller-Segel system in $\mathbb{R}^2$, *European J. Appl. Math.*, **22**(2011), 553-580.

[10] J.I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis. Advances in Mathematical Sciences and Applications, *Gakkotosho,Tokyo*, **5** (1995), 659-680.

[11] A.M. Fink, Almost Periodic Differential Equations, Lectures Notes in Mathematics, Vol. 377, Springer-Verlag, Berlin-New York, 1974.

[12] R. Fisher, The wave of advance of advantageous genes, *Ann. of Eugenics*, **7**(1937), 355-369.

[13] A. Friedman, Partial Differential Equations of Parabolic Types, Englewood Cliffs, NJ:Prentice-Hall, 1964.

[14] A. Friedman and J.I. Tello, Stability of solutions of chemotaxis equations in reinforced random walks, *J. Math. Anal. Appl.*, **272** (2002), 138-163.

[15] S. Fu and R. Ma, Existence of a global coexistence state for periodic competition diffusion systems, *Nonlinear Anal* **28** (1977), 1265-1271.

[16] H. Gajewski and K. Zacharias, Global behavior of a reaction-diffusion system modeling chemotaxis, *Math. Nachr.*, **195** (1998), 177-194.

[17] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, Berlin, Heidelberg, New York, 1977.

[18] M.A. Herrero, E. Medina, and J.J.L. Velzquez, Singularity patterns in a chemotaxis model, *Math. Ann.*, **306** (1996), 583-623.

[19] M.A. Herrero and J.J.L. Velzquez, A blow-up mechanism for a chemotaxis model, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV. Ser. 24 (1997), 633-683.

[20] M.A. Herrero and J.J.L. Velzquez, Finite-time aggregation into a single point in a reaction-diffusion system, *Nonlinearity*, **10** (1997), 1739-1754.

[21] P. Hess and H. Weinberger, Convergence to spatial-temporal clines in the Fisher equation with time-periodic fitness, *J. Math. Biol.* 28 (1990), 83-98.

[22] G. Hetzer and W. Shen, Convergence in almost periodic competition diffusion systems, *J. Math. Anal. Appl.* **262** (2001), 307-338.

[23] G. Hetzer and W. Shen, Uniform persistence, Coexistence, and Extinction in Almost Periodic/ Nonautonomous Competition Diffusion Systems, *SIAM J. MATH. ANAL.* **34(1)** (2002), 204-227.

[24] T. Hillen, K. J. Painter, A users guide to PDE models for chemotaxis , *Math. Biol.* **58** (2009) 183-217.

[25] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, *I. Jber. DMW*, **105** (2003), 103-165.

[26] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, *Eur. J. Appl. Math.*, **12** (2001), 159-177.

[27] D. Horstmann and M. Winkler, Boundedness vs.blow-up in a chemotaxis system, *J. Differential Equations*, **215** (2005), 52-107.

[28] M. Isenbach, Chemotaxis. Imperial College Press, London (2004).

[29] T.B Issa and R. Salako, Asymptotic dynamics in a two-species chemotaxis model with non-local terms, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), no. 10, 3839-3874.

[30] T.B Issa and W. Shen, Dynamics in chemotaxis models of parabolic-elliptic type on bounded domain with time and space dependent logistic sources, *SIAM J. Appl. Dyn. Syst.*, **16** (2017), no. 2, 926-973.

[31] T. B. Issa and W. Shen, Persistence, coexistence and extinction in two species chemotaxis models on bounded heterogeneous environments, *J. Dyn. Diff. Equat.*, (2018). https://doi.org/10.1007/s10884-018-9686-7

[32] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modeling chemotaxis, *Trans. Amer. Math. Soc.*, **329** (1992), 819-824.

[33] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399-415.

[34] E.F. Keller and L.A. Segel, A model for chemotaxis, *J.Theoret. Biol.*, **30** (1971), 225-234.

[35] A. Khelghati and K. Baghaei, Global existence and boundedness of classical solutions in a quasilinear parabolic-elliptic chemotaxis system with logistic source, *C. R. Math. Acad. Sci. Paris*, **353** (2015), no. 10, 913-917.

[36] A. Kolmogorov, I. Petrowsky, and N.Piscunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Bjul. Moskovskogo Gos. Univ.*, **1** (1937), 1-26.

[37] A. Kubo and J.I. Tello, Mathematical analysis of a model of chemotaxis with competition terms, *Differential Integral Equations*, **29** (2016), no. 5-6, 441-454.

[38] K. Kuto, K. Osaki, T. Sakurai, and T. Tsujikawa, Spatial pattern formation in a chemotaxis-diffusion-growth model. *Physica D* **241** (2012), 1629-1639.

[39] J. Lankeit, Chemotaxis can prevent thresholds on population density. *Discr. Cont. Dyn. Syst. B*, **20** (2015), 1499-1527.

[40] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source. *J. Differential Eq.*, **258** (2015), 1158-1191.

[41] D. A. Lauffenburger, Quantitative studies of bacterial chemotaxis and microbial population dynamics *Microbial. Ecol.*, **22**(1991), 175-85.

[42] T. Nagai, Blow-up of radially symmetric solutions of a chemotaxis system, *Adv. Math. Sci. Appl.*, **5** (1995), 581-601.

[43] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.*, **6** (2001), 37-55.

[44] M. Negreanu and J.I. Tello, On a competitive system under chemotaxis effects with non-local terms, *Nonlinearity*, **26** (2013), 1083-1103.

[45] M. Negreanu and J.I. Tello, Asysmptotic stability of a two species chemotaxis system with non-diffusive chemoattractant, *J. Differential Eq.*, **258** (2015), 1592-1617.

[46] M.N. Nkashama, Dynamics of logistic equations with non-autonomous bounded efficients, *Electronic Journal of Differential Equations*, **2000** (2000), No. 02, 1-8.

[47] K. Osaki and A. Yagi, Finite dimensional attractors for one-dimensional Keller-Segel equations, *Funkcialaj Ekvacioj*, **44** (2001), 441469.

[48] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[49] Painter, K.J., Hillen, T. Spatio-temporal chaos in a chemotaxis model. *Physica D*, **240**(2011), 363-375

[50] Painter, K.J., Mathematical models for chemotaxis1 and their applications in self-organisation phenomena. http://doi.org/10.1016/j.jtbi.2018.06.019. *Journal of Theoretical Biology* .

[51] R. B. Salako and W. Shen, Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on R^N, *J. Differential Equations*, **262(11)** (2017), 5635-5690.

[52] T. Senba, Blowup Behavior of radial solutions to Jager-Luckhaus system in high dimensional domains, *Funkcialaj Ekvacioj*, **48** (2005), 247-271.

[53] T. Senba and T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, *Methods Appl. Anal.*, **8** (2001), 349-367.

[54] W. Shen and Y. Yi, Convergence in almost periodic Fisher and Kolmogorov models, *J. Math. Biol.* **37** (1998), no. 1, 84-102.

[55] C. Stinner, J.I. Tello, and W. Winkler, Competive exclusion in a two-species chemotaxis, *J.Math.Biol.*. **68** (2014), 1607-1626.

[56] Y. Tao, M. Winkler Persistence of mass in a chemotaxis system with logistic source, *J. DifferentialEquations*, **259** (2015), no. 11, 6142-6161.

[57] J.I. Tello, Mathematical analysis and stability of a chemotaxis problem with a logistic growth term, *Math. Methods Appl. Sci.*, **27** (2004), 1865-1880.

[58] J.I. Tello and M.A.J. Chaplain, On the steady states and stability of solutions of a chemotaxis problem with a logistic growth term, Preprint.

[59] J.I. Tello and W. Winkler M, A chemotaxis system with logistic source, *Common Partial Diff. Eq.*, **32** (2007), 849-877.

[60] J.I. Tello and M. Winkler, Stabilization in two-species chemotaxis with a logistic source, *Nonlinearity*, **25** (2012), 1413-1425.

[61] G. Viglialoro, Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source, *J. Math. Anal. Appl.*, **439** (2016), no. 1, 197-212.

[62] Qi Wang and Jingyue Yang, Time-periodic and stable patterns of a two-competing-species Keller-Segel chemotaxis model: effect of cellular growth, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 3547-3574.

[63] Y. Wang, A quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type with logistic source, *J.Math. Anal.Appl.*, **441** (2016), 259-292.

[64] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, **35** (2010), no. 8, 1516-1537.

[65] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, **100** (2013), 748-767.

[66] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, *Journal of Mathematical Analysis and Applications*, **384** (2011), 261-272.

[67] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? *J. Nonlinear Sci.*, **24** (2014), 809-855.

[68] X.-Q. Zhao, Global attractivity in monotone and subhomogeneous almost periodic systems, *J. Differential Equations*, **187** (2003), 494-509.

[69] Q. Zhang, Y. Li. An attraction-repulsion chemotaxis system with logistic source, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift fr Angewandte Mathematik und Mechanik*, **96**(2016), 5, 570-584.