Intermittency properties of space-time fractional stochastic partial differential equations

by

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Abstract

This dissertation focuses on the analyses of the non-linear time-fractional stochastic reaction-diffusion equations of the type

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^{1-\beta}[b(u) + \sigma(u) \dot{F}(t,x)] \tag{0.0.1}$$

in $(d+1)$ dimensions, where $\nu > 0, \beta \in (0,1)$, $\alpha \in (0,2]$ and d is a positive integer. The operator ∂_t^β is the Caputo fractional derivative while $-(-\Delta)^{\alpha/2}$ is the generator of an isotropic α-stable Lévy process and $I^{1-\beta}$ is the Riesz fractional integral operator. The forcing noise denoted by $\dot{F}(t,x)$ is a Gaussian or white noise. These equations might be used as a model for materials with random thermal memory.

The first part of the dissertation studies *intermittency fronts* for the solution of the stochastic equation of Eq.(0.0.1) when $b \equiv 0$. Under some appropriate conditions on the parameters we prove that solutions to the initial value problem of Eq.(0.0.1) with nonempty measurable initial function with compact support and strictly positive on an open subset of $(0,\infty)^d$ have positive intermittency lower front. Furthermore, we also identified the parameters regions ensuring that the solutions to the initial value problem of Eq.(0.0.1) with the same condition on the initial function also have finite intermittency upper front. Our results recovers as particular cases some known results in the literature. For example, Mijena and Nane proved in [48] that : (i) absolute moments of the solutions of this equation grow exponentially; and (ii) the distances to the origin of the farthest high peaks of those moments grow exactly linearly with time. The last result was proved under the assumptions $\alpha = 2$ and $d = 1$. Here, we extend this result to the case $\alpha = 2$ and $d \in \{1,2,3\}$.

Next, we study the phenomena of finite-time blow up and non-existence of solutions of (0.0.1). In particular, when the term $\sigma(u)$ satisfies $\sigma(u) \geqslant |x|^{1+\gamma}$ for some positive number γ, we prove that solution to the initial value problem of Eq.(0.0.1) with strictly positive initial distribution have infinite second moment for t large enough. We derive non-existence (blow-up) of global random field solutions

under some additional conditions, most notably on b, σ and the initial condition. Our results complement those of P. Chow in [19], [20], and Foondun et al. in [29], [32] among others.

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Chapter 1

Introduction

Recently time-fractional diffusion equations were studied by researchers in many applied and theoretical fields of science and engineering. These equations are related with anomalous diffusions or diffusions in non-homogeneous media, with random fractal structures; see, for instance, [44]. A typical form of the time fractional diffusion equations is

$$\partial_t^\beta u = \nu \Delta u \tag{1.0.1}$$

with $\nu > 0$, $\beta \in (0, 1)$, where Δu is the Laplace transform of u and $\partial_t^\beta u$ denotes its Caputo fractional derivative defined by

$$\partial_t^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u_r(x)}{\partial r} \frac{dr}{(t-r)^\beta} \tag{1.0.2}$$

where $\Gamma(1-\beta)$ is the Gamma function evaluated at $1-\beta$. The expression $\partial_t^\beta u$ is referred to as the Caputo fractional derivative due to the pioneering work by Caputo [9]. We note that the Laplace transform of $\partial_t^\beta u_t(x)$ is

$$\int_0^\infty e^{-st} \partial_t^\beta u_t(x)\, dt = s^\beta \tilde{u}_s(x) - s^{\beta-1} u_0(x), \tag{1.0.3}$$

where $\tilde{u}_s(x) = \int_0^\infty e^{-st} u_t(x) dt$ and incorporates the initial value in the same way as the first derivative. For any $\gamma > 0$, define the fractional integral by

$$I_t^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau.$$

Then, for every $\gamma > 0$, and $g \in L^\infty(\mathbb{R}_+)$ or $g \in C(\mathbb{R}_+)$, we have the following relation

$$\partial_t^\gamma I_t^\gamma g(t) = g(t).$$

Hence the Caputo fractional derivative ∂_t^β is the left inverse of Riesz fractional integral I^β. It is well known that the stochastic solutions to fractional diffusion equations (1.0.1) can be realized through time-change by inverse stable subordinators and therefore we obtain time-changed processes. A couple of recent works in this field are [43, 44, 46, 53]. For some deep and rigorous mathematical approaches to time fractional diffusion (heat type) equations see [38, 51, 52, 57]. A natural extension of the time-fractional diffusion equation (1.0.1) is a stochastic partial differential equation of the form

$$\partial_t^\beta u_t(x) = \Delta u_t(x) + \dot{W}(t,x); \;\; u_t(x)|_{t=0} = u_0(x), \tag{1.0.4}$$

where $\dot{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$.

Mijena and Nane [47] have given an argument using the time fractional Duhamel's principle to obtain the following equation:

$$\begin{aligned} &\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I_t^{1-\beta}[\sigma(u)\dot{W}(t,x)], \;\; t>0,\, x \in \mathbb{R}^d; \\ &u_t(x)|_{t=0} = u_0(x), \end{aligned} \tag{1.0.5}$$

where the initial datum u_0 is $L^p(\Omega)$-bounded ($p \geqslant 2$), that is,

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}[|u_0(x)|^p] < \infty, \tag{1.0.6}$$

$-(-\Delta)^{\alpha/2}$ is the fractional Laplacian with $\alpha \in (0,2]$, and $\dot{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$, modeling the random effects. The fractional integral above in equation (1.0.5) when $\sigma(u) = 1$ for functions $\phi \in L^2(\mathbb{R}^d)$ is defined as

$$\int_{\mathbb{R}^d} \phi(x) I_t^{1-\beta}[\dot{W}(t,x)]dx = \frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}^d}\int_0^t (t-\tau)^{-\beta}\phi(x)W(d\tau,dx)$$

it is well defined only when $0 < \beta < 1/2$. It is a type of Rieman-Liouville process.

In reality, the environments of many living organisms are spatially and temporally heterogeneous. It would be nice to consider the equation (1.0.5) with the space-time white noise without the fractional integral. For related time fractional stochastic equations with different noise terms see [11, 12, 13, 35].

In this dissertation we study several dynamics features of solutions to

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^{1-\beta}[b(u) + \sigma(u)\overset{\cdot}{F}(t,x)] \tag{1.0.7}$$

in $(d+1)$ dimensions, where $\nu > 0, \beta \in (0,1)$, $\alpha \in (0,2]$ and d is a positive integer. The forcing noise denoted by $\overset{\cdot}{F}(t,x)$ is a Gaussian or white noise. We note that (1.0.5) is a particular case of (1.0.7) The noise $\overset{\cdot}{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$, which is assumed to be adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t, t \geqslant 0\}$ is right continuous.

Let $G_t(x)$ denote the heat kernel of the time fractional heat type equation

$$\partial_t^\beta G_t(x) = -\nu(-\Delta)^{\alpha/2} G_t(x). \tag{1.0.8}$$

We say that an $\mathcal{F}_t$-adapted random field $\{u(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d\}$ is said to be a mild solution of (1.0.5) with initial value u_0 if the following integral equation is fulfilled

$$u_t(x) = \int_{\mathbb{R}^d} u_0(y) G_t(x-y) dy + \int_0^t \int_{\mathbb{R}^d} \sigma(u_r(y)) G_{t-r}(x-y) W(drdy). \tag{1.0.9}$$

Let T be a fixed positive number, and let $B_{T,p}$ denote the family of all $\mathcal{F}_t$-adapted random fields $\{u_t(x),\ t \in [0,\ T],\ x \in \mathbb{R}^d\}$ satisfying

$$\sup_{x \in \mathbb{R}^d} \sup_{t \in [0,\ T]} \mathbb{E}\left[|u_t(x)|^p\right] < \infty, \tag{1.0.10}$$

with the convention that $B_{T,2} = B_T$. It is easy to check that for each fixed T and p, $B_{T,p}$ is a Banach space.

The existence and uniqueness of the solution to (1.0.5) has been studied by Mijena and Nane [47] under global Lipchitz conditions on σ, using the white noise approach of Walsh [56].

In this dissertation, among others, we prove the following results.

(i) Let $L_\sigma := \inf_{z\in\mathbb{R}^d} |\frac{\sigma(z)}{z}|$. If $b(u) \equiv 0$, $L_\sigma < \infty$, $d < \min\{2, \beta^{-1}\}\alpha$ and $\inf_{z\in\mathbb{R}^d} u_0(z) > 0$, then the solution $u_t(x)$ of (1.0.7) has a positive intermittency lower front and a finite inttermittency upper front (see Theorem 2.1.3 for more detail and (2.1.3) for the definition of intermittency front.)

The study of blow-up or non-existence of solutions has attracted a number of researches, because they are very useful to applied researchers. In this regard, Mueller and Sowers in [49, 50] prove that the space-time white noise driven stochastic heat equation with Dirichlet boundary condition will blow up in finite time with positive probability, if $\sigma(u) = u^\gamma$ with $\gamma > 3/2$. Bonder and Groisman in [8] also prove the finite time blow-up for almost every initial data when nonnegative convex drift function satisfying $\int^\infty 1/f < \infty$ is taken into consideration. We refer the reader to [6, 19, 20, 27, 33, 29, 40, 41] for more information on the blow-up phenomenon in the deterministic setting. In this dissertation, we prove among other results that

(ii) provided that the initial function is bounded below, the second moment will eventually be infinite for white noise driven equations.

We also prove that

(iii) Suppose that the correlation function f is given by

$$f(x,\, y) = \frac{1}{|x-y|^\omega} \quad \text{with} \quad \omega < d \wedge (\alpha\beta^{-1}).$$

Then for $\kappa > 0$, there exists a positive number $\tilde{t}$ such that for all $t \geqslant \tilde{t}$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}|u_t(x)|^2 = \infty.$$

Chapter 2

Intermittency fronts for space-time fractional stochastic partial differential equations in $(d+1)$ dimensions

2.1 Definitions and Mains Results

In this chapter, we study intermittency fronts for the solution of the stochastic equation (1.0.5). We adopt the definition given in [37, Chapter 7]: The random field $u_t(x)$ is called intermittent if $\inf_{z\in\mathbb{R}^d}|\sigma(z)|>0$, and $\eta_k(x)/k$ is strictly increasing for $k\geqslant 2$ for all $x\in\mathbb{R}^d$, where

$$\eta_k(x) := \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}(|u_t(x)|^k). \tag{2.1.1}$$

The following observation of Carmona and Molchanov [10, Theorem 3.1.2] gives a sufficient condition for intermittency: see [37, Proposition 7.2] for a proof of the next proposition.

Proposition 2.1.1. *If $\eta(k)<\infty$ for all sufficiently large k, then the function η is well-defined and convex on $(0,\infty)$. Moreover, If $\eta(k_0)>0$ for some $k_0>1$, then $k\to k^{-1}\eta(k)$ is strictly increasing on $[k_0,\infty)$*

Theorem 2.1.2 ([48])**.** *Let $d<\min\{2,\beta^{-1}\}\alpha$. If $\inf_{z\in\mathbb{R}^d}|u_0(z)|>0$, then*

$$\inf_{x\in\mathbb{R}^d}\eta_2(x)\geqslant\left[C^*(L_\sigma)^2\Gamma(1-\beta d/\alpha)\right]^{\frac{1}{(1-\beta d/\alpha)}}$$

where

$$L_\sigma := \inf_{z\in\mathbb{R}^d}|\sigma(z)/z|. \tag{2.1.2}$$

Therefore, the solution $u_t(x)$ of (1.0.5) is weakly intermittent when $\inf_{z\in\mathbb{R}^d}|u_0(z)|>0$ and $L_\sigma>0$.

There is a huge literature on the study of intermittency of SPDEs, see, for example, [28, 37].

According to the previous theorem the solution develops tall peaks over time which means that $t \to \sup_{x\in\mathbb{R}^d} \mathbb{E}|u_t(x)|^2$ grows exponentially rapidly with t. There appears another phenomena called intermittency fronts, that the distances of the farthest peaks of the moments of the solution to (1.0.5) grow linearly with time as θt: if θ is sufficiently small, then the quantity $\sup_{|x|>\theta t} \mathbb{E}|u_t(x)|^2$ grows exponentially quickly as $t \to \infty$; whereas the preceding quantity vanishes exponentially rapidly if θ is sufficiently large.

Here, we define for every $p \geqslant 2$ and every $\theta \geqslant 0$,

$$\mathscr{L}_p(\theta) := \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\theta t} \log \mathbb{E}\left(|u_t(x)|^p\right). \tag{2.1.3}$$

We can think of $\theta_{L_p} > 0$ as an *intermittency lower front* if $\mathscr{L}_p(\theta) < 0$ for all $\theta > \theta_{L_p}$, and of $\theta_{U_p} > 0$ as an *intermittency upper front* if $\mathscr{L}_p(\theta) > 0$ whenever $\theta < \theta_{U_p}$.

The following is our main theorem which establishes bounds for θ_{L_p} and θ_{U_p} that extend the results of [22] and [48] to the case of $p \geqslant 2$ and $\alpha = 2$ and $d \in \{1, 2, 3\}$ for time fractional SPDEs with crucial nontrivial changes to the methods in [22, 37].

Theorem 2.1.3. *Suppose that* $d < \min\{2, \beta^{-1}\}\alpha,\ \alpha = 2,\ p \geqslant 2$ *and measurable initial function* $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ *is bounded, has compact support, and is strictly positive in an open subset of* $(0,\infty)^d$*, and* σ *satisfies* $\sigma(0) = 0$*. Then the time fractional stochastic heat equation* (1.0.5) *has a positive intermittency lower front. In fact,*

$$\mathscr{L}_p(\theta) < 0 \ \text{ if } \theta > \frac{p^2}{4}\left(\frac{4\nu}{p}\right)^{1/\beta} (Lip_\sigma c_0)^{2\left(\frac{2-\beta}{2-\beta d}\right)}. \tag{2.1.4}$$

In addition, under the cone condition $L_\sigma > 0$*−where* L_σ *was defined in* (2.1.2)*-there exists* $\theta_0 > 0$ *such that*

$$\mathscr{L}_p(\theta) > 0 \ \text{ if } \theta \in (0, \theta_0). \tag{2.1.5}$$

That is, in this case, the stochastic heat equation has a finite intermittency upper front.

This theorem in the case of $d = 1, p = 2$ was proved by Mijena and Nane [48]. In the parabolic Anderson model which is the stochastic heat equation (1.0.5), when $\beta = 1$ and $\sigma(x) = cx$, it is now known that there exists a sharp intermittency front, namely $\theta_{L_2} = \theta_{U_2}$, see the work of Chen and Dalang [14]. It would be nice to consider equality of $\theta_{L_p} = \theta_{U_p}$ for (1.0.5) when $\beta \in (0,1)$. We will carry out this project in the near future.

Next we want to give an outline of this chapter. In section 2, we recall some preliminary results on the subject from the literature. Hence proofs of the results here can be found in the literature, in particular see references therein. Next, we established some useful results that we used in the proof of our main results.

2.2 Preliminaries

In this section we give some results about the heat kernel $G_t(x)$ of the time fractional heat type equation (1.0.8), and mention some basic facts about the integral (mild) solution of (1.0.5) in the sense of Walsh [56]. We know that $G_t(x)$ is the density function of $X(E_t)$, where X is an isotropic α-stable Lévy process in $\mathbb{R}^d$ and $E_t = \inf\{u : \; D(u) > t\}$, is the first passage time of a β-stable subordinator $D = \{D_r, \, r \geqslant 0\}$, or the inverse stable subordinator of index β: see, for example, Bertoin [7] for properties of these processes, Baeumer and Meerschaert [5] for more on time fractional diffusion equations, and Meerschaert and Scheffler [45] for properties of the inverse stable subordinator E_t.

Let $p_{X(s)}(x)$ and $f_{E_t}(s)$ be the densities of $X(s)$ and E_t, respectively. Then the Fourier transform of $p_{X(s)}(x)$ is given by

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x} p_{X(s)}(x) dx = e^{-s\nu|\xi|^\alpha}, \tag{2.2.1}$$

and

$$f_{E_t}(s) = t\beta^{-1} s^{-1-1/\beta} g_\beta(ts^{-1/\beta}), \tag{2.2.2}$$

where $g_\beta(\cdot)$ is the density function of D_1. The function $g_\beta(u)$ [cf. Meerschaert and Straka (2013)] is infinitely differentiable on the entire real line, with $g_\beta(s) = 0$ for $s \leqslant 0$.

By using (2.2.2) and change of variable we can show that

$$\mathbb{E}(D_1^{-\beta k}) = \mathbb{E}(E_1^k) = \int_0^\infty w^{-\beta k} g_\beta(w) dw \tag{2.2.3}$$

By conditioning, we have

$$G_t(x) = \int_0^\infty p_{X(s)}(x) f_{E_t}(s) ds. \tag{2.2.4}$$

Lemma 2.2.1 (Lemma 2.1 in [47])**.** *For $d < 2\alpha$,*

$$\int_{\mathbb{R}^d} G_t^2(x)dx = C^* t^{-\beta d/\alpha} \tag{2.2.5}$$

where $C^* = \frac{(\nu)^{-d/\alpha} 2\pi^{d/2}}{\alpha\Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty z^{d/\alpha - 1}(E_\beta(-z))^2 dz.$

Lemma 2.2.2 (Lemma 2.2 in [48])**.** *For $\lambda \in \mathbb{R}^d$ and $\alpha = 2$,*

$$\int_{\mathbb{R}^d} e^{\lambda \cdot x} G_s(x) dx = E_\beta(\nu|\lambda|^2 s^\beta).$$

We barrow the following definition from [28]: let Φ be a random field, and for every $\gamma > 0$ and $k \in [2, \infty)$ define

$$\mathcal{N}_{\gamma,k}(\Phi) := \sup_{t \geqslant 0} \sup_{x \in \mathbb{R}^d} \left(e^{-\gamma t} ||\Phi_t(x)||_k\right) := \sup_{t \geqslant 0} \sup_{x \in \mathbb{R}^d} \left(e^{-\gamma t} \left[\mathbb{E}|\Phi_t(x)|^k\right]^{1/k}\right). \tag{2.2.6}$$

If we identify a.s.-equal random fields, then every $\mathcal{N}_{\gamma,k}$ becomes a norm. Moreover, $\mathcal{N}_{\gamma,k}$ and $\mathcal{N}_{\gamma',k}$ are equivalent norms for all $\gamma, \gamma' > 0$ and $k \in [2, \infty)$. Finally, we note that if $\mathcal{N}_{\gamma,k}(\Phi) < \infty$ for some $\gamma > 0$ and $k \in [2, \infty)$, then $\mathcal{N}_{\gamma,2}(\Phi) < \infty$ as well, thanks to Jensen's inequality.

Definition 2.2.3. *We denote by $\mathcal{L}^{\gamma,2}$ the completion of the space of all simple random fields in the norm $\mathcal{N}_{\gamma,2}$.*

We next recall the **Walsh-Dalang Integral** briefly: We use the Brownian filtration $\{\mathcal{F}_t\}$ and the Walsh-Dalang integrals as follows

- $(t, x) \to \Phi_t(x)$ is an elementary random field when $\exists 0 \leqslant a < b$ and an $\mathcal{F}_a$-measurable $X \in L^2(\Omega)$ and $\phi \in L^2(\mathbb{R}^d)$ such that

$$\Phi_t(x) = X 1_{[a,b]}(t)\phi(x) \quad (t > 0, x \in \mathbb{R}^d).$$

- If $h = h_t(x)$ is non-random and Φ is elementary, then

$$\int h\Phi d\xi := X\int_{(a,b)\times\mathbb{R}^d} h_t(x)\phi(x)\xi(dtdx).$$

- The stochastic integral is Wiener's; well defined iff $h_t(x)\phi(x) \in L^2([a,b]\times\mathbb{R}^d)$.

- We have Walsh isometry,

$$\mathbb{E}\left(\left|\int h\Phi d\xi\right|^2\right) = \int_0^\infty \int_{\mathbb{R}^d} dy[h_s(y)]^2\mathbb{E}(|\Phi_s(y)|^2).$$

Given a random field $\Phi := \{\Phi_t(x)\}_{t\geqslant 0, x\in\mathbb{R}^d}$ and space-time noise $\dot{W}$, we define the [space-time] *stochastic convolution* $G \circledast \Phi$ to be the random field that is defined as

$$(G\circledast\Phi)_t(x) := \int_{(0,t)\times\mathbb{R}^d} G_{t-s}(y-x)\Phi_s(y)W(dsdy),$$

for $t > 0$ and $x \in \mathbb{R}^d$, and $(G\circledast\Phi)_0(x) := 0$.

2.2.1 Some Useful Lemmas

We start this subsection with a very important and non trivial result. The next Lemma provides an estimate which allows us to overcome some difficulties in the proof of the main result.

Lemma 2.2.4. *For $\beta \in (0,\ 1)$, $k \in \mathbb{N}\cup\{0\}$ and $d \in \{1,2,3\}$, define*

$$a_k^d(\beta) := \mathbb{E}(D_1^{-\beta(k-\frac{d}{4})}) = \int_0^\infty w^{-\beta(k-\frac{d}{4})} g_\beta(w)dw.$$

Then

$$0 < \ a_k^d(\beta) \ \leqslant 3\frac{\Gamma(1+k)}{\Gamma(1+\beta k)}, \quad \textit{for} \quad k \geqslant 1. \tag{2.2.7}$$

Proof. First observe that

$$\begin{aligned} a_k^d(\beta) &= \int_0^\infty \frac{g_\beta(w)}{w^{\beta(k-\frac{d}{4})}}dw \\ &= \int_0^1 \frac{g_\beta(w)}{w^{\beta(k-\frac{d}{4})}}dw + \int_1^\infty \frac{g_\beta(w)}{w^{\beta(k-\frac{d}{4})}}dw \end{aligned}$$

Since for every $k \geqslant 1$ we have

$$\frac{1}{w^{\beta(k-\frac{d}{4})}} \leqslant \begin{cases} \frac{1}{w^{\beta k}} & \text{if } 0 < w < 1 \\ \frac{1}{w^{\beta(k-1)}} & \text{if } w > 1. \end{cases}$$

Using the uniqueness of Laplace Transform and Remark 3.1 in [45], we can easily show that $\mathbb{E}(D_1^{-\beta k})) = \mathbb{E}(E_t^k) = \frac{\Gamma(1+k)}{\Gamma(1+\beta k)}$. Then we have that

$$\begin{aligned} a_k^d(\beta) &\leqslant \int_0^\infty \frac{g_\beta(w)}{w^{\beta k}}dw + \int_1^\infty \frac{g_\beta(w)}{w^{\beta(k-1)}}dw \\ &\leqslant \int_0^\infty \frac{g_\beta(w)}{w^{\beta k}}dw + \int_0^\infty \frac{g_\beta(w)}{w^{\beta(k-1)}}dw \\ &= \frac{\Gamma(1+k)}{\Gamma(1+\beta k)} + \frac{\Gamma(k)}{\Gamma(1+\beta(k-1))} \end{aligned} \tag{2.2.8}$$

But

$$\frac{1}{\Gamma(1+\beta(k-1))} = \frac{1+\beta(k-1)}{\Gamma(1+\beta(k-1)+1)}$$

Using D. kershaw inequality $1/\Gamma(x+1) < 1/(x+1/2)^{1-\lambda}\Gamma(x+\lambda)$ for $x = \beta(k-1)+1$ and $\lambda = \beta$, we obtain that

$$\frac{1}{\Gamma(1+(\beta(k-1)+1))} \leqslant \frac{1}{(\frac{3}{2}+\beta(k-1))^{1-\beta}\Gamma(1+\beta k)}.$$

Since $\frac{1}{(\frac{3}{2}+\beta(k-1))^{1-\beta}} \leqslant 1$ and $1+\beta(k-1) \leqslant 2k$, it follows that

$$\frac{1}{\Gamma(1+\beta(k-1))} \leqslant \frac{2k}{\Gamma(1+\beta k)}.$$

Multiplying both sides of the last expression by $\Gamma(k)$, we get

$$\begin{aligned}\frac{\Gamma(k)}{\Gamma(1+\beta(k-1))} &\leqslant 2\frac{k\Gamma(k)}{\Gamma(1+\beta k)}\\ &= 2\frac{\Gamma(1+k)}{\Gamma(1+\beta k)}.\end{aligned}$$

Adding $\frac{\Gamma(1+k)}{\Gamma(1+\beta k)}$ to both side of the last expression, and combine with inequality(2.2.8) give the proof of inequality (2.2.7).

□

Lemma 2.2.5. *For $\beta \in (0,\ 1)$, $k \in \mathbb{N}\cup\{0\}$ and $d \in \{1,2,3\}$ define*

$$a_k^d(\beta) = \int_0^\infty w^{-\beta(k-\frac{d}{4})} g_\beta(w)dw = \mathbb{E}(D_1^{-\beta(k-\frac{d}{4})}).$$

Then

$$\frac{a_k^d(\beta)\sqrt{\Gamma(1+\beta(2k-\frac{d}{2}))}}{k!} \leqslant \frac{3\sqrt{2\beta(k-\frac{d}{4})}\sqrt{\Gamma(2\beta(1-\frac{d}{4}))}}{\Gamma(1+\beta)} 2^{\beta(k-1)} \quad \text{for} \quad k \geqslant 1. \tag{2.2.9}$$

Proof. Using the Duplication formula $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})/\sqrt{\pi}$, we obtain that

$$\Gamma\left(1+\beta\left(2k-\frac{d}{2}\right)\right) = \beta(2k-\frac{d}{2})\left[\frac{2^{2\beta(k-\frac{d}{4})-1}\Gamma\left(\beta(k-\frac{d}{4})\right)\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}{\sqrt{\pi}}\right] \tag{2.2.10}$$

This combining with (2.2.7) yields that

$$\frac{a_k^d(\beta)\sqrt{\Gamma(1+\beta(2k-\frac{d}{2}))}}{k!} \leqslant 3\sqrt{\beta(2k-\frac{d}{2})}\left[\frac{2^{\beta(k-\frac{d}{4})-\frac{1}{2}}\sqrt{\Gamma(\beta(k-\frac{d}{4}))}\sqrt{\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}}{(\pi)^{\frac{1}{4}}\Gamma(1+\beta k)}\right] \tag{2.2.11}$$

Using the relationship between the Beta and Gamma functions $\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x,y)$ with $x = \beta k + \frac{1}{2} - \frac{\beta d}{4}$ and $y = \frac{1}{2} + \frac{\beta d}{4}$, we obtain that

$$\begin{aligned}\frac{\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{\beta d}{4})}{\Gamma(1+\beta k)} &= B\left(\beta(k-\frac{d}{4})+\frac{1}{2},\frac{1}{2}+\frac{\beta d}{4}\right)\\ &= \int_0^1 t^{\beta k+\frac{1}{2}-\frac{\beta d}{4}-1}(1-t)^{\frac{1}{2}+\frac{\beta d}{4}}dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 t^{\beta k} t^{\frac{1}{2}-\frac{\beta d}{4}-1}(1-t)^{\frac{1}{2}+\frac{\beta d}{4}}dt \\
&\leqslant \int_0^1 t^{\beta} t^{\frac{1}{2}-\frac{\beta d}{4}-1}(1-t)^{\frac{1}{2}+\frac{\beta d}{4}}dt \\
&= \int_0^1 t^{\beta+\frac{1}{2}-\frac{\beta d}{4}-1}(1-t)^{\frac{1}{2}+\frac{\beta d}{4}}dt \\
&= B\left(\beta+\frac{1}{2}-\frac{\beta d}{4}, \frac{1}{2}+\frac{\beta d}{4}\right) \\
&= \frac{\Gamma(\beta+\frac{1}{2}-\frac{\beta d}{4})\Gamma(\frac{1}{2}+\frac{\beta d}{4})}{\Gamma(1+\beta)}
\end{aligned}
$$

It follows that

$$
\frac{\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}{\Gamma(1+\beta k)} \leqslant \frac{\Gamma(\beta(1-\frac{d}{4})+\frac{1}{2})}{\Gamma(1+\beta)} \tag{2.2.12}
$$

On the other hand, by repeating again the previous arguments with $x = \beta(k - \frac{d}{4})$ and $y = 1 + \frac{\beta d}{4}$ we obtain that

$$
\frac{\Gamma(\beta(k-\frac{d}{4}))}{\Gamma(1+\beta k)} \leqslant \frac{\Gamma(\beta(1-\frac{d}{4}))}{\Gamma(1+\beta)}. \tag{2.2.13}
$$

Inequalities (2.2.12) and (2.2.13) combined give

$$
\frac{\Gamma(\beta(k-\frac{d}{4}))\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}{[\Gamma(1+\beta k)]^2} \leqslant \frac{\Gamma(\beta(1-\frac{d}{4})+\frac{1}{2})\Gamma(\beta(1-\frac{d}{4}))}{[\Gamma(1+\beta)]^2}. \tag{2.2.14}
$$

Duplication formula with $x = \beta(1 - \frac{d}{4})$ gives

$$
\Gamma(\beta(1-\frac{d}{4})+\frac{1}{2})\Gamma(\beta(1-\frac{d}{4})) = 2^{1-2\beta(1-\frac{d}{4})}\sqrt{\pi}\Gamma(2\beta(1-\frac{d}{4})). \tag{2.2.15}
$$

Combining (2.2.14),(2.2.15) and we obtain that

$$
\frac{\Gamma(\beta(k-\frac{d}{4}))\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}{[\Gamma(1+\beta k)]^2} \leqslant \frac{2^{1-2\beta(1-\frac{d}{4})}\sqrt{\pi}\Gamma(2\beta(1-\frac{d}{4}))}{[\Gamma(1+\beta)]^2}.
$$

Taking square root of both side of the last expression, we get

$$
\frac{2^{\beta(1-\frac{d}{4})-\frac{1}{2}}\sqrt{\Gamma(\beta(k-\frac{d}{4}))}\sqrt{\Gamma(\beta(k-\frac{d}{4})+\frac{1}{2})}}{(\pi)^{\frac{1}{4}}\Gamma(1+\beta k)} \leqslant \frac{\sqrt{\Gamma(2\beta(1-\frac{d}{4}))}}{\Gamma(1+\beta)}. \tag{2.2.16}
$$

Inequalities (2.2.14) and (2.2.16) complete the proof of (2.2.9). □

The next lemma will also be needed in the proof of our main theorem in the next section.

Lemma 2.2.6. *For every* $\beta \in (0,\ 1)$ *and* $n, k \in \mathbb{N} \cup \{0\}$, *and* $d \in \{1, 2, 3\}$ *satisfying the assumption of Proposition 2.3.2, define*

$$b_{k,n}(\beta) = \int_0^t e^{-\gamma s} s^{\beta(k+n-\frac{d}{2})} ds.$$

Then

$$b_{k,n}(\beta) \leqslant [b_{k,k}(\beta)]^{\frac{1}{2}} [b_{n,n}(\beta)]^{\frac{1}{2}} \tag{2.2.17}$$

and

$$b_{k,k}(\beta) \leqslant \left(\frac{1}{\gamma}\right)^{1+2\beta(k-\frac{d}{4})} \Gamma(1 + \beta(2k - \frac{d}{2})). \tag{2.2.18}$$

Proof. Proof of inequality (2.2.17):

$$\begin{aligned} b_{k,n}(\beta) &= \int_0^t e^{-\gamma s} s^{\beta(k+n-\frac{d}{2})} ds \\ &= \int_0^t \left(e^{-\frac{\gamma s}{2}} s^{\beta(k-\frac{d}{4})}\right) \left(e^{-\frac{\gamma s}{2}} s^{\beta(n-\frac{d}{4})}\right) ds \\ &\leqslant \left(\int_0^t e^{-\gamma s} s^{2\beta(k-\frac{d}{4})} ds\right)^{\frac{1}{2}} \left(\int_0^t e^{-\gamma s} s^{2\beta(n-\frac{d}{4})} ds\right)^{\frac{1}{2}} \\ &= [b_{k,k}(\beta)]^{\frac{1}{2}} [b_{n,n}(\beta)]^{\frac{1}{2}} . \end{aligned}$$

Proof of inequality (2.2.18):

$$\begin{aligned} b_{k,k}(\beta) &= \int_0^t e^{-\gamma s} s^{2\beta k - \frac{\beta d}{2}} ds \\ &= \int_0^{\gamma t} e^{-w} \left(\frac{w}{\gamma}\right)^{2\beta k - \frac{\beta d}{2}} \frac{dw}{\gamma} \\ &= \left(\frac{1}{\gamma}\right)^{2(\beta k - \frac{d}{4})+1} \int_0^{\gamma t} e^{-w} w^{\beta(2k-\frac{d}{2})} dw \\ &\leqslant \left(\frac{1}{\gamma}\right)^{2(\beta k - \frac{d}{4})+1} \Gamma(\beta(2k - \frac{d}{2}) + 1). \end{aligned}$$

$\square$

2.3 Intermittency fronts

Here we prove our main result on the intermittency fronts for the solution of equation (1.0.5). Our results generalize the work of Mejina and Nane see Theorem 4.1 in [48]. In [48] the authors proved the result for $d = 1,\ p = 2$ and $\alpha = 2$. With the aid of Lemma 2.2.4, Lemma 2.2.5 and Lemma 2.2.6 we are able to overcome the difficulties in their methods and extend the result to $d \in \{1, 2, 3\}$ and $\alpha = 2$. Furtheremore, using Burkholder-Davis-Gundy (BDG) inequaility and Minkowski inequality for integrals, we are able to extend these results to higher moments, that is for $p > 2$. Assume that $\sigma(\cdot)$ in (1.0.5) satisfies the following global Lipschitz condition, i.e. there exists a generic positive constant Lip_σ such that:

$$|\sigma(x) - \sigma(y)| \leqslant \text{Lip}_\sigma \|x - y\| \quad \text{for all} \ \ x,\, y \in \mathbb{R}^d. \tag{2.3.1}$$

Clearly, (2.3.1) implies the uniform linear growth condition of $\sigma(\cdot)$. Recall the definition of $\mathscr{L}(\theta)$ from (2.1.3).

2.3.1 Intermittency fronts for the second moment (case p=2)

We first state a proposition that implies that the solution of equation (1.0.5) is square integrable over time in the language of partial differential equations.

Proposition 2.3.1 (Proposition 4.2 in [48]). *Assume that $\alpha \in (0, 2]$, and $d < \min\{2, \beta^{-1}\}\alpha$, then $u_t \in L^2(\mathbb{R})$ a.s. for all $t \geqslant 0$; in fact, for any fixed $\epsilon \in (0, 1)$ and $t \geqslant 0$,*

$$\mathbb{E}\left(||u_t||^2_{L^2(\mathbb{R}^d)}\right) \leqslant \epsilon^{-1}||u_0||^2_{L^2(\mathbb{R}^d)} \exp\left(\left[\frac{C^*\Gamma(1-\beta d/\alpha)Lip^2_\sigma}{1-\epsilon}\right]^{\frac{1}{1-\beta d/\alpha}} t\right) \tag{2.3.2}$$

The proof of Theorem 2.1.3 requires the following "weighted stochastic Young's inequality" which is an extension of Proposition 8.3 in [37].

Proposition 2.3.2. *Let $\alpha = 2$ and $d < \min\{2, \beta^{-1}\}\alpha$. Define for all $\gamma > 0, c \in \mathbb{R}^d$, and $\Phi \in \mathcal{L}^{\beta,2}$,*

$$\mathcal{N}_{\gamma,c}(\Phi) := \sup_{t\geqslant 0}\sup_{x\in\mathbb{R}^d}\left[e^{-\gamma t + c.x}\mathbb{E}\left(|\Phi_t(x)|^2\right)\right]^{1/2}.$$

Then,

$$\mathcal{N}_{\gamma,c}(G \circledast \Phi) \leqslant C_d(c,\gamma,\beta)\mathcal{N}_{\gamma,c}(\Phi) \ \ \textit{for all} \ \left(\frac{\gamma}{2}\right)^{\beta} > \frac{\nu\|c\|^2}{2},$$

where $C_d(c,\gamma,\beta)$ *is a finite constant that depends on* $d, \|c\|, \gamma,$ *and* β*.*

Using Lemma 2.2.4, Lemma 2.2.5, Lemma 2.2.6 and the last two propositions, we are now ready to give the proof of Theorem 2.1.3.

Proof of Proposition 2.3.2. Using $p_u(y) = \frac{e^{-\frac{\|y\|^2}{4u\nu}}}{(4\pi u\nu)^{\frac{d}{2}}}$, direct computations yield

$$\int_{\mathbb{R}^d} e^{-c.y}\left[p_u(y)\right]^2 dy = \frac{e^{\frac{u\nu\|c\|^2}{2}}}{(8\pi u\nu)^{\frac{d}{2}}}.$$

Observe that

$$\begin{aligned}
[G_s(y)]^2 &= \int_0^\infty p_u(y) f_{E_s}(u)du \int_0^\infty p_v(y) f_{E_s}(v)dv \qquad (2.3.3)\\
&= \int_0^\infty \int_0^\infty p_u(y)p_v(y) f_{E_s}(u) f_{E_s}(v)dudv.
\end{aligned}$$

We use Holder's inequality to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-c.y}\left[G_s(y)\right]^2 dy\\
&= \int_{\mathbb{R}^d}\int_0^\infty\int_0^\infty \left(e^{-\frac{c.y}{2}}p_u(y)f_{E_s}(u)\right)\left(e^{-\frac{c.y}{2}}p_v(y)f_{E_s}(v)\right)dudvdy\\
&= \int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\left[\left(e^{-\frac{c.y}{2}}p_u(y)\right)\left(e^{-\frac{c.y}{2}}p_v(y)\right)dy\right] f_{E_s}(u)f_{E_s}(v)dudv\\
&\leqslant \int_0^\infty\int_0^\infty\left(\int_{\mathbb{R}^d} e^{-c.y}\left[p_u(y)\right]^2 dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d} e^{-c.y}\left[p_v(y)\right]^2 dy\right)^{\frac{1}{2}} f_{E_s}(u)f_{E_s}(v)dudv\\
&= \left[\int_0^\infty\left(\int_{\mathbb{R}^d} e^{-c.y}\left[p_u(y)\right]^2 dy\right)^{\frac{1}{2}} f_{E_s}(u)du\right]^2\\
&= \left[\int_0^\infty\left(\frac{e^{\frac{u\nu\|c\|^2}{2}}}{(8\pi u\nu)^{\frac{d}{2}}}\right)^{\frac{1}{2}} f_{E_s}(u)du\right]^2\\
&= \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\int_0^\infty \frac{e^{\frac{u\nu\|c\|^2}{4}}}{u^{\frac{d}{4}}} f_{E_s}(u)du\right]^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{\nu\|c\|^2}{4}\right)^k}{k!}\int_0^{\infty}u^{k-\frac{d}{4}}f_{E_s}(u)du\right]^2\\
&= \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{\nu\|c\|^2}{4}\right)^k}{k!}s^{\beta(k-\frac{d}{4})}\int_0^{\infty}w^{-\beta(k-\frac{d}{4})}g_\beta(w)dw\right]^2\\
&= \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{\nu\|c\|^2}{4}\right)^k}{k!}s^{\beta(k-\frac{d}{4})}a_k^d(\beta)\right]^2\\
&\leqslant \frac{s^{-\frac{\beta d}{2}}}{(8\pi\nu)^{\frac{d}{2}}}\sum_{k,n=0}^{\infty}\frac{a_k^d(\beta)a_n^d(\beta)}{n!k!}\left(\frac{\nu\|c\|^2}{4}\right)^{k+n}s^{\beta(k+n)}
\end{aligned}
\tag{2.3.4}
$$

where

$$
a_k^d(\beta)=\int_0^{\infty}w^{-\beta(k-\frac{d}{4})}g_\beta(w)dw
$$

From the inequality (2.3.4), we obtain that

$$
\begin{aligned}
&\int_0^t e^{-\gamma s}ds\int_{\mathbb{R}^d}e^{-c.y}\left[G_s(y)\right]^2dy\\
&\leqslant \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\sum_{k,n=0}^{\infty}\frac{a_k^d(\beta)a_n^d(\beta)}{n!k!}\left(\frac{\nu\|c\|^2}{4}\right)^{k+n}\int_0^t s^{\beta(k+n-\frac{d}{2})}e^{-\gamma s}ds\\
&= \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\sum_{k,n=0}^{\infty}\frac{a_k^d(\beta)a_n^d(\beta)}{n!k!}\left(\frac{\nu\|c\|^2}{4}\right)^{k+n}b_{n,k}(\beta)
\end{aligned}
\tag{2.3.5}
$$

Where

$$
b_{n,k}(\beta)=\int_0^t s^{\beta(k+n-\frac{d}{2})}e^{-\gamma s}ds.
$$

Combining inequality (2.3.5) and inequality (2.2.17) of Lemma 2.2.6 we obtain that

$$
\begin{aligned}
&\int_0^t e^{-\gamma s}\left[\int_{\mathbb{R}^d}e^{-c.y}\left[G_s(y)\right]^2dy\right]ds\\
&\leqslant \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\sum_{k,n=0}^{\infty}\frac{a_k^d(\beta)a_n^d(\beta)}{n!k!}\left(\frac{\nu\|c\|^2}{4}\right)^{k+n}\left[b_{k,k}(\beta)\right]^{\frac{1}{2}}\left[b_{n,n}(\beta)\right]^{\frac{1}{2}}
\end{aligned}
$$

$$= \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\sum_{k=0}^{\infty}\frac{a_k^d(\beta)}{k!}\left(\frac{\nu\|c\|^2}{4}\right)^k [b_{k,k}(\beta)]^{\frac{1}{2}}\right]^2$$

Using inequality (2.2.18) of Lemma 2.2.6, the last inequality can be improved to

$$\begin{aligned}
&\int_0^t e^{-\gamma s}\left[\int_{\mathbb{R}^d} e^{-c.y}\,[G_s(y)]^2\,dy\right] ds \\
\leqslant\ & \frac{1}{(8\pi\nu)^{\frac{d}{2}}}\left[\sum_{k=0}^{\infty}\frac{a_k^d(\beta)\sqrt{\Gamma(1+\beta(2k-\frac{d}{2}))}}{k!}\left(\frac{\nu\|c\|^2}{4}\right)^k\left(\frac{1}{\gamma}\right)^{\frac{1}{2}+\beta(k-\frac{d}{4})}\right]^2
\end{aligned} \tag{2.3.6}$$

Next, using inequality (2.2.9) of Lemma 2.2.5 and the fact that $\sqrt{2k-\frac{d}{2}} \leqslant 2^k$ for every $k \geqslant 1$, inequality (2.3.6) becomes

$$\begin{aligned}
&\int_0^t e^{-\gamma s}\left[\int_{\mathbb{R}^d} e^{-c.y}\,[G_s(y)]^2\,dy\right] ds \\
\leqslant\ & \frac{\gamma^{\frac{\beta d}{2}-1}}{(8\pi\nu)^{\frac{d}{2}}}\left[a_0^d(\beta)\sqrt{\Gamma(1-\frac{\beta d}{2})}+\frac{3\sqrt{\beta\Gamma(2\beta(1-\frac{d}{4}))}}{2^\beta\Gamma(1+\beta)}\sum_{k=1}^{\infty}\left(\frac{2^\beta\nu\|c\|^2}{4\gamma^\beta}\right)^k\sqrt{2k-\frac{d}{2}}\right]^2 \\
\leqslant\ & \frac{\gamma^{\frac{\beta d}{2}-1}}{(8\pi\nu)^{\frac{d}{2}}}\left[a_0^d(\beta)\sqrt{\Gamma(1-\frac{\beta d}{2})}+\frac{3\sqrt{\beta\Gamma(2\beta(1-\frac{d}{4}))}}{2^\beta\Gamma(1+\beta)}\sum_{k=1}^{\infty}\left(\frac{2^{\beta+1}\nu\|c\|^2}{4\gamma^\beta}\right)^k\right]^2 \\
\leqslant\ & \frac{M^2\gamma^{\frac{\beta d}{2}-1}}{(8\pi\nu)^{\frac{d}{2}}}\left[1+\sum_{k=1}^{\infty}\left(\frac{2^{\beta-1}\nu\|c\|^2}{\gamma^\beta}\right)^k\right]^2 \\
=\ & \left[\frac{M\gamma^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}}\sum_{k=0}^{\infty}\left(\frac{2^{\beta-1}\nu\|c\|^2}{\gamma^\beta}\right)^k\right]^2
\end{aligned} \tag{2.3.7}$$

where

$$M = \max\left\{a_0^d(\beta)\sqrt{\Gamma(1-\frac{\beta d}{2})},\frac{3\sqrt{\beta\Gamma(2\beta(1-\frac{d}{4}))}}{2^\beta\Gamma(1+\beta)}\right\}$$

The last series converges if and only if $\left(\frac{\gamma}{2}\right)^{\beta} > \frac{\nu\|c\|^2}{2}$. Therefore from inequality (2.3.7), we obtain that

$$\begin{aligned} & e^{-\gamma t+c.x}\mathbb{E}(|(G \circledast \Phi)_t(x)|^2) \\ = \; & e^{-\gamma t+c.x}\int_0^t \left[\int_{\mathbb{R}^d} [G_{t-s}(y-x)]^2\, \mathbb{E}(|\Phi_s(y)|^2) dy\right] ds \\ \leqslant \; & [\mathcal{N}_{\gamma,c}(\Phi)]^2 \int_0^t e^{-\gamma s}\int_{\mathbb{R}^d} e^{-c.y}\,[G_s(y)]^2\, dyds \\ \leqslant \; & [\mathcal{N}_{\gamma,c}(\Phi)]^2 \left[\frac{M\gamma^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}}\sum_{k=0}^{\infty}\left(\frac{2^{\beta-1}\nu\|c\|^2}{\gamma^{\beta}}\right)^k\right]^2 \end{aligned} \tag{2.3.8}$$

The right-hand side is independent of (x,t). Therefore, by optimizing over (x,t) and then square roots of both side , we get

$$\mathcal{N}_{\gamma,c}(G \circledast \Phi) \leqslant C_d(c,\gamma,\beta)\mathcal{N}_{\gamma,c}(\Phi)$$

with

$$C_d(c,\gamma,\beta) = \frac{M\gamma^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}}\sum_{k=0}^{\infty}\left(\frac{2^{\beta-1}\nu\|c\|^2}{\gamma^{\beta}}\right)^k = \frac{M\gamma^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}\left(1-\frac{2^{\beta-1}\nu\|c\|^2}{\gamma^{\beta}}\right)}. \tag{2.3.9}$$

Which complete the proof of Proposition 2.3.2. □

The next corollary is a generalization of Corollary 4.4 in [48].

Corollary 2.3.3. *If* $\|c\|^{1/\beta-d/2} > Lip_{\sigma}\sqrt{\frac{M^2}{(2\nu)^{\frac{1}{\beta}-\frac{d}{2}}(1-2^{\beta-2})^2(8\pi\nu)^{\frac{d}{2}}}}$, *then the solution to the tfspde* (1.0.5) *for* $\alpha = 2$ *satisfies*

$$\mathbb{E}(|u_t(x)|^2) \leqslant A(\|c\|,\beta)\exp\left(-\|c\|\|x\| + (2\nu c^2)^{1/\beta}t\right), \tag{2.3.10}$$

simultaneously for all $x \in \mathbb{R}^d$ *and* $t \geqslant 0$, *where* $A(\|c\|,\beta)$ *is a finite constant that depends only on* $\|c\|$ *and* β.

Proof. The proof generalizes some of the ideas used in the proof of Corollary 4.4 in [48]. Recall that for all $\gamma > 0$

$$\begin{aligned} \mathcal{N}_{\gamma,c}(u^{(n+1)}) \; & \leqslant \; [\mathcal{N}_{\gamma,c}(G_t * u_0)] + [\mathcal{N}_{\gamma,c}(G \circledast \sigma(u^{(n)})] \\ & \leqslant \; [\mathcal{N}_{\gamma,c}(G_t * u_0)] + C_d(\|c\|,\gamma,\beta)[\mathcal{N}_{\gamma,c}(\sigma(u^{(n)}))], \end{aligned}$$

using Proposition 2.3.2. Because $\sigma(z) \leqslant \text{Lip}_\sigma |z|$ for all $z \in \mathbb{R}^d$,

$$\mathcal{N}_{\gamma,c}(\sigma(u^{(n)}) \leqslant \text{Lip}_\sigma \mathcal{N}_{\gamma,c}(u^{(n)}).$$

Also,

$$\begin{aligned} e^{-\gamma t + c.x}(G_t * |u_0|)(x) &= e^{-\gamma t}\int_{\mathbb{R}^d} G_t(y-x)e^{-c.(y-x)}e^{cy}|u_0(y)|dy \\ &\leqslant e^{-\gamma t}\mathcal{N}_{0,c}(u_0)\int_{\mathbb{R}^d} e^{c.z}G_t(z)dz \\ &= e^{-\gamma t}E_\beta(\nu\|c\|^2 t^\beta)\mathcal{N}_{0,c}(u_0). \end{aligned} \tag{2.3.11}$$

We take $\gamma^\beta := 2\nu\|c\|^2$ to see that for all integers $k \geqslant 0$

$$e^{-\gamma t}E_\beta(\nu\|c\|^2 t^\beta) = \sum_{k=0}^{\infty} \frac{\nu^k \|c\|^{2k} t^{\beta k} e^{-\gamma t}}{\Gamma(1+\beta k)} = \sum_{k=0}^{\infty} \frac{\nu^k\|c\|^{2k}}{\gamma^{\beta k}} \frac{u^{\beta k}e^{-u}}{\Gamma(1+\beta k)} \leqslant 2,$$

since $\frac{u^{\beta k}e^{-u}}{\Gamma(1+\beta k)} < 1$. From equation 2.3.8

$$C_d(\|c\|,\gamma,\beta) = \frac{M\gamma^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}\left(1 - \frac{2^{\beta-1}\nu\|c\|^2}{\gamma^\beta}\right)} = \frac{M(2\nu)^{\frac{d}{4}-\frac{1}{2\beta}}\|c\|^{\frac{d}{2}-\frac{1}{\beta}}}{(1-2^{\beta-2})(8\pi\nu)^{\frac{d}{4}}}.$$

and by our assumption

$$C_d(\|c\|,\gamma,\beta)\text{Lip}_\sigma < 1.$$

For every $\theta \in [0, 2\pi]$ and $\varphi \in [0,\pi]$, we define $c(\theta,\varphi)$ [1] by the expression

$$c(\theta,\varphi) := \|c\|(\cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), \sin(\varphi)).$$

Since for every $\theta \in [0,2\pi]$ and $\varphi \in [0,\pi]$, we have $\|c(\theta,\varphi)\| = \|c\|$, then for all integers $n \geqslant 0$ we have

$$\mathcal{N}_{(2\nu\|c\|^2)^{1/\beta},c(\theta,\varphi)}(u^{(n+1)}) \leqslant 2\mathcal{N}_{0,c(\theta,\varphi)}(u_0) + C_d(\|c\|,\gamma,\beta)\text{Lip}_\sigma \mathcal{N}_{(2\nu\|c\|^2)^{1/\beta},c(\theta,\varphi)}(u^{(n)}). \tag{2.3.12}$$

[1] For $d = 2$ we define $c(\theta) := \|c\|(\cos(\theta), \sin(\theta))$ for every $\theta \in [0, 2\pi]$ while for the case $d = 1$ we consider $c(\theta) := |c|\cos(\theta)$ where $\theta \in \{0, \pi\}$

Since u_0 has compact support, there is some constant $R > 0$ such that $u_0(x) = 0$ whenever $\|x\| \geqslant R$. Hence we obtain that

$$\begin{aligned}
\sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \mathcal{N}_{0,c(\theta,\varphi)}(u_0) : &= \sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \sup_{x\in\mathbb{R}^d} e^{c(\theta,\varphi).x} u_0(x) \\
&= \sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \sup_{x\in supp(u_0)} e^{c(\theta,\varphi).x} u_0(x) \\
&\leqslant \sup_{x\in supp(u_0)} e^{\|c(\theta,\varphi)\|\|x\|} u_0(x) \\
&= \sup_{x\in supp(u_0)} e^{\|c\|\|x\|} u_0(x) \\
&\leqslant e^{R\|c\|}\|u_0\|_\infty.
\end{aligned}$$

Since $e^{R\|c\|}\|u_0\|_\infty < \infty$, it follows from inequality (2.3.12) that

$$\sup_{n\geqslant 0} \sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \mathcal{N}_{(2\nu\|c\|^2)^{1/\beta},c(\theta,\varphi)}(u^{(n+1)}) < \infty.$$

Since $u_t^{(n+1)}(x)$ converges to $u_t(x)$ in $L^2(\Omega)$ as $n \to \infty$, Fatou's lemma implies that

$$\sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \mathcal{N}_{(2\nu\|c\|^2)^{1/\beta},c(\theta,\varphi)}(u) < \infty.$$

Since every $x \in \mathbb{R}^d$ can be written as $x = \|x\|(\cos(\phi_x)\cos(\theta_x), \cos(\phi_x)\sin(\theta_x), \sin(\phi_x))$ and the preceding supremum is independent of θ and ϕ, then in particular for $\theta = \theta_x$ and $\phi = \phi_x$ we obtain that $c(\theta_x, \phi_x).x = \|c\|\|x\|$. The corollary follows readily from this fact. □

We are ready to prove Theorem 2.1.3. We do this in two steps adapting the method in [, Chapter 8] with crucial nontrivial changes: First we derive (2.1.4); and then we establish (2.1.5).

Proof of (2.1.4)**.** Since u_0 has compact support, it follows that $|u_0(x)| = O(e^{\|c\|\|x\|})$ for all $\|c\| > 0$. Therefore, we may apply Corollary 2.3.3 to an arbitrary $\|c\|^{1/\beta-d/2} > \text{Lip}_\sigma\sqrt{\frac{M^2}{(2\nu)^{\frac{1}{\beta}-\frac{d}{2}}(1-2^{\beta-2})^2(8\pi\nu)^{\frac{d}{2}}}} :=$

$\text{Lip}_\sigma c_0$ in order to see that

$$\begin{aligned}\mathscr{L}(\theta) &= \limsup_{t\to\infty}\frac{1}{t}\sup_{\|x\|>\theta t}\log\mathbb{E}\left(|u_t(x)|^2\right)\\ &\leqslant -\sup_{\|c\|>(\text{Lip}_\sigma c_0)^{2\beta/(2-\beta d)}}\left[\theta\|c\|-(2\nu\|c\|^2)^{1/\beta}\right]\\ &\leqslant -\left[\theta(\text{Lip}_\sigma c_0)^{2\beta/(2-\beta d)})-(2\nu)^{1/\beta}(\text{Lip}_\sigma c_0)^{4/(2-\beta d)})\right], \end{aligned} \tag{2.3.13}$$

obtained by setting $\|c\|:=(\text{Lip}_\sigma c_0)^{2\beta/(2-\beta d)}$ in the maximization problem of the first line of preceding display. The right-most quantity is strictly negative when

$$\theta > (2\nu)^{1/\beta}(\text{Lip}_\sigma c_0)^{2(\frac{2-\beta}{2-\beta d})};$$

this proves(2.1.4). □

Proof of (2.1.5)**.** We have that

$$\begin{aligned}&\mathbb{E}(|u_t(x)|^2)\\ \geqslant\ & |(G_t*u_0)(x)|^2+L_\sigma^2\int_0^t ds\int_{\mathbb{R}^d}dy[G_{t-s}(y-x)]^2\mathbb{E}(|u_s(y)|^2).\end{aligned} \tag{2.3.14}$$

For all $t>0$ and $x\in\mathbb{R}^d$. Define $\mathbb{K}^+_{\theta t}:=\mathbb{R}^{d-1}\times[\theta t,\infty)$, $\mathbb{K}^-_{\theta t}:=\mathbb{R}^{d-1}\times(-\infty,-\theta t]$ and $\mathbb{K}_{\theta t}=\mathbb{K}^+_{\theta t}\cup\mathbb{K}^-_{\theta t}$ for every $t>0$ and $\theta>0$. Then if $x,y\in\mathbb{R}^d, 0\leqslant s\leqslant t$, and $\theta\geqslant 0$, we have

$$1_{\mathbb{K}^+_{\theta t}}(x)\geqslant 1_{\mathbb{K}^+_{\theta(t-s)}}(x-y)\cdot 1_{\mathbb{K}^+_{\theta s}}(y).$$

This is a consequence of the triangle inequality. Therefore,

$$\begin{aligned}&\int_{\mathbb{K}^+_{\theta t}}\int_0^t ds\int_{\mathbb{R}^d}dy[G_{t-s}(y-x)]^2\mathbb{E}(|u_s(y)|^2)\\ =\ &\int_0^t ds\int_{\mathbb{R}^d\times\mathbb{R}^d}dy[G_{t-s}(y-x)]^2\mathbb{E}(|u_s(y)|^2)1_{\mathbb{K}^+_{\theta t}}(x)dydx\\ \geqslant\ &\int_0^t ds\left(\int_{\mathbb{K}^+_{\theta(t-s)}}[G_{t-s}(y)]^2dy\right)\left(\int_{\mathbb{K}^+_{\theta s}}\mathbb{E}(|u_s(y)|^2)dy\right)\end{aligned} \tag{2.3.15}$$

This and (2.3.14) together show that the function

$$M_+(t) := \int_{\mathbb{K}^+_{\theta t}} \mathbb{E}(|u_t(y)|^2)dy \tag{2.3.16}$$

satisfies the following renewal inequality:

$$M_+(t) \geqslant \int_{\mathbb{K}^+_{\theta t}} |(G_t * u_0)(x)|^2 dx + L_\sigma^2 (T * M_+)(t), \tag{2.3.17}$$

with

$$T(t) := \int_{\mathbb{K}^+_{\theta t}} [G_t(z)]^2 dz.$$

Because of symmetry we can write $T(t) = \int_{\mathbb{K}^-_{\theta t}} [G_t(z)]^2 dz$. Therefore, a similar argument shows that the function

$$M_-(t) := \int_{\mathbb{K}^-_{\theta t}} \mathbb{E}(|u_s(y)|^2)dy,$$

satisfies the following renewal inequality:

$$M_-(t) \geqslant \int_{\mathbb{K}^-_{\theta t}} |(G_t * u_0)(x)|^2 dx + L_\sigma^2 (T * M_-)(t). \tag{2.3.18}$$

Define

$$M(t) := \int_{\mathbb{K}_{\theta t}} \mathbb{E}(|u_t(y)|^2)dy = M_+(t) + M_-(t),$$

Define $\mathcal{L}\phi$ to be the Laplace transform of any measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. That is,

$$(\mathcal{L}\phi)(\lambda) = \int_0^\infty e^{-\lambda t}\phi(t)dt \quad (\lambda \geqslant 0).$$

Then, we have the following inequality of Laplace transforms: For every $\lambda \geqslant 0$,

$$\begin{aligned}(\mathcal{L}M)(\lambda) &= (\mathcal{L}M_+)(\lambda) + (\mathcal{L}M_-)(\lambda) \\ &\geqslant \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{K}_{\theta t}} dx |(G_t * u_0)(x)|^2 + L_\sigma^2 (\mathcal{L}T)(\lambda)(\mathcal{L}M)(\lambda).\end{aligned} \tag{2.3.19}$$

Since

$$\begin{aligned}\lim_{\theta\to 0}\int_{\mathbb{K}_{\theta t}} |(G_t)(x)|^2\,dx &= \int_{\mathbb{R}^d} |(G_t)(x)|^2\,dx \\ &= C^* t^{-\frac{\beta d}{\alpha}}\end{aligned}$$

where the second equality follows from Lemma 2.2.1. On the other hand we have that

$$(\mathcal{L}T)(0) = \int_0^\infty dt \int_{\mathbb{K}_{\theta t}} [G_t(z)]^2 dz.$$

Since

$$\int_0^\infty C^* t^{-\frac{\beta d}{\alpha}} dt = \infty,$$

then we obtain that

$$\lim_{\theta\to 0}(\mathcal{L}T)(0) = \infty.$$

Therefore, there exists $\theta_0 > 0$ such that $(\mathcal{L}T)(0) > L_\sigma^{-2}$ whenever $\theta \in (0, \theta_0)$. This and dominated convergence theorem together imply that there, in turn, will exist $\lambda_0 > 0$ such that $(\mathcal{L}T)(\lambda) > L_\sigma^{-2}$ whenever $\theta \in (0, \theta_0)$ and $\lambda \in (0, \lambda_0)$. Since $u_0 > 0$ on a set of positive measure, it follows readily that

$$\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{K}_{\theta t}^+} dx |(G_t * u_0)(x)|^2 > 0,$$

for all $\theta, \lambda \geqslant 0$, including $\theta \in (0, \theta_0)$ and $\lambda \in (0, \lambda_0)$. Therefore, (2.3.19) implies that

$$(\mathcal{L}M)(\lambda) = \infty \quad \text{for } \theta \in (0, \theta_0) \text{ and } \lambda \in (0, \lambda_0). \tag{2.3.20}$$

Combining this with the fact that

$$\int_{|y|>\theta t} \mathbb{E}(|u_t(y)|^2) dy \geqslant \int_{\mathbb{K}_{\theta t}} \mathbb{E}(|u_t(y)|^2) dy = M(t),$$

one can deduce from this and the definition of M that

$$\limsup_{t\to\infty} e^{-\lambda t}\int_{|y|>\theta t}\mathbb{E}(|u_t(y)|^2)dy=\infty,$$

whenever $\theta\in(0,\theta_0)$ and $\lambda\in(0,\lambda_0)$. This and the already-proven first part (2.1.4) together show that

$$\limsup_{t\to\infty} e^{-\lambda t}\int_{\theta t<|y|<\gamma t}\mathbb{E}(|u_t(y)|^2)dy=\infty,$$

whenever $\theta\in(0,\theta_0)$, $\lambda\in(0,\lambda_0)$ and $\gamma>(2\nu)^{1/\beta}(\mathrm{Lip}_\sigma c_0)^{2(\frac{2-\beta}{2-\beta d})}$. Since the last integral is not greater than $(\gamma-\theta)t\sup_{|x|>\theta t}\mathbb{E}(|u_t(x)|^2)$, it follows that

$$\mathscr{L}_2(\theta)=\limsup_{t\to\infty}\frac{1}{t}\sup_{|x|>\theta t}\log\mathbb{E}(|u_t(x)|^2)\geqslant\lambda_0, \tag{2.3.21}$$

for $\theta\in(0,\theta_0)$. This proves (2.1.5) and hence the theorem.

□

2.3.2 Intermittency fronts for higher moments (case p>2)

In this section, we shall generalize all the results of the previous for $p\geqslant 2$. Since the previous results cover the case $p=2$, we only consider the case $p>2$ in this section.

Proposition 2.3.4. *Let $\alpha=2$ and $d<\min\{2,\beta^{-1}\}\alpha$. Define for all $\gamma>0, c\in\mathbb{R}^d$, and $\Phi\in\mathcal{L}^{\beta,2}$,*

$$\mathcal{N}_{\gamma,c,p}(\Phi):=\sup_{t\geqslant 0}\sup_{x\in\mathbb{R}^d}\left[e^{-\gamma t+c.x}\mathbb{E}\left(|\Phi_t(x)|^p\right)\right]^{1/p}.$$

Then,

$$\mathcal{N}_{\gamma,c,p}(G\circledast\Phi)\leqslant C_d(c,\gamma,\beta,p)\mathcal{N}_{\gamma,c,p}(\Phi)\;\;\textit{for all}\;\left(\frac{\gamma}{p}\right)^\beta>\frac{2\nu\|c\|^2}{p^2},$$

where $C_d(c,\gamma,\beta,p)$ is a finite constant that depends on $d,\|c\|,\gamma,\beta$ and p.

Proof. Using Burkholder-Davis-Gundy inequality, there is a constant $K_p>0$ depending on p such that

$$\mathbb{E}(|(G\circledast\Phi)_t(x)|^p)\leqslant K_p\mathbb{E}\left[<G\circledast\Phi(x)>_t^{\frac{p}{2}}\right] \tag{2.3.22}$$

where

$$< G \circledast \Phi(x) >_t = \int_{[0,t]\times\mathbb{R}^d} \left[G_{t-s}(y-x)\Phi_s(y)\right]^2 dsdy$$

denotes the quadratic variation of $G \circledast \Phi$. Now , using Minkowski inequality for integrals, we have

$$\mathbb{E}(< G \circledast \Phi(x) >_t^{\frac{p}{2}}) \leqslant \left[\int_{[0,t]\times\mathbb{R}^d} \left[G_{t-s}(y-x)\right]^2 \|\Phi_s\|_p^2 dsdy\right]^{\frac{p}{2}} \tag{2.3.23}$$

Therefore

$$\begin{aligned} e^{-\gamma t + c.x}\mathbb{E}(|(G \circledast \Phi)_t(x)|^p) &\leqslant K_p \left(\int_{[0,t]\times\mathbb{R}^d} e^{\frac{-2\gamma t}{p}+\frac{2c.x}{p}} \left[G_{t-s}(y-x)\right]^2 \|\Phi_s\|_p^2 dsdy\right)^{\frac{p}{2}} \\ &\leqslant K_p \left[\mathcal{N}^2_{\gamma,c,p}(\Phi) \int_0^t e^{\frac{-2\gamma s}{p}} \int_{\mathbb{R}^d} e^{\frac{-2c.y}{p}} \left[G_s(y)\right]^2 dyds\right]^{\frac{p}{2}} \\ &= K_p \left[\mathcal{N}_{\gamma,c,p}(\Phi)\right]^p \left[\int_0^t e^{\frac{-2\gamma s}{p}} \int_{\mathbb{R}^d} e^{\frac{-2c.y}{p}} \left[G_s(y)\right]^2 dyds\right]^{\frac{p}{2}} \end{aligned} \tag{2.3.24}$$

Taking $\tilde{\gamma} = \frac{2\gamma}{p}$, $\tilde{c} = \frac{2c}{p}$, by inequality (2.3.7) we have that

$$e^{-\gamma t + c.x}\mathbb{E}(|(G \circledast \Phi)_t(x)|^p) \leqslant K_p \left[\mathcal{N}_{\gamma,c,p}(\Phi)\right]^p \left[\frac{M\tilde{\gamma}^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}} \sum_{k=0}^{\infty} \left(\frac{2^{\beta-1}\nu\|\tilde{c}\|^2}{\tilde{\gamma}^\beta}\right)^k\right]^p. \tag{2.3.25}$$

Since the R.H.S. of the preceding line is independent of (x,t), optimizing over (x,t) and taking the pth root of both sides, we obtain:

$$\left[e^{-\gamma t + c.x}\mathbb{E}(|(G \circledast \Phi)_t(x)|^p)\right]^{\frac{1}{p}} \leqslant K_p^{\frac{1}{p}} \left[\mathcal{N}_{\gamma,c,p}(\Phi)\right] \left[\frac{M\tilde{\gamma}^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}} \sum_{k=0}^{\infty} \left(\frac{2^{\beta-1}\nu\|\tilde{c}\|^2}{\tilde{\gamma}^\beta}\right)^k\right]. \tag{2.3.26}$$

The last series converges for $\left(\frac{\tilde{\gamma}}{2}\right)^\beta > \frac{\nu\|\tilde{c}\|^2}{2}$, which is equivalent to $\left(\frac{\gamma}{p}\right)^\beta > \frac{2\nu\|c\|^2}{p}$. By letting $C_p(c,\gamma,\beta,p)$ = $K_p^{\frac{1}{p}} \left[\frac{M\tilde{\gamma}^{\frac{\beta d}{4}-\frac{1}{2}}}{(8\pi\nu)^{\frac{d}{4}}} \sum_{k=0}^{\infty} \left(\frac{2^{\beta-1}\nu\|\tilde{c}\|^2}{\tilde{\gamma}^\beta}\right)^k\right]$, we have that $\mathcal{N}_{\gamma,c,p}(G\circledast\Phi) \leqslant C_p(c,\gamma,\beta,p)\mathcal{N}_{\gamma,c,p}(\Phi)$, which completes the proof.

□

The next Corollary generalizes some of the ideas used in the proof of Corollary 4.4 in [48].

Corollary 2.3.5. *If* $\|c\|^{1/\beta-d/2} > (\frac{2}{p})^{\frac{d}{2}-\frac{1}{\beta}} Lip_\sigma\sqrt{\frac{M^2}{(2\nu)^{\frac{1}{\beta}-\frac{d}{2}}(1-2^{\beta-2})^2(8\pi\nu)^{\frac{d}{2}}}}$*, for all* $2 < p < \infty$ *then the solution to the tfspde* (1.0.5) *for* $\alpha = 2$ *satisfies*

$$\mathbb{E}(|u_t(x)|^p) \leqslant A_p(\|c\|, \beta) \exp\left(-\frac{2}{p}\|c\|\|x\| + \left(\frac{4}{p}\nu c^2\right)^{1/\beta} t\right), \tag{2.3.27}$$

simultaneously for all $x \in \mathbb{R}^d$ *and* $t \geqslant 0$*, where* $A_p(\|c\|, \beta)$ *is a finite constant that depends only on* $\|c\|$*,*p *and* β*.*

Proof. Using the same argument as in the proof of Corollary 2.3.3 and equation (2.3.12), we have that

$$\sup_{\theta\in[0,2\pi]} \sup_{\varphi\in[0,\pi]} \mathcal{N}_{(2\nu\|\tilde{c}\|^2)^{1/\beta},\tilde{c}(\theta,\varphi)}(u) < \infty,$$

with $\tilde{c} = \frac{2c}{p}$. The corollary follows readily from this fact. □

We are ready to prove Theorem 2.1.3 for $p > 2$. We do this in two steps adapting the method in [37, Chapter 8] with crucial nontrivial changes: First we derive (2.1.4); and then we establish (2.1.5).

Proof of (2.1.4)**.** Since u_0 has compact support, it follows that $|u_0(x)| = O(e^{\|c\|\|x\|})$ for all $\|c\| > 0$. Therefore, we may apply Corollary 2.3.5 to an arbitrary $\|c\|^{1/\beta-d/2} > \left(\frac{2}{p}\right)^{\frac{d}{2}-\frac{1}{\beta}} \text{Lip}_\sigma\sqrt{\frac{M^2}{(2\nu)^{\frac{1}{\beta}-\frac{d}{2}}(1-2^{\beta-2})^2(8\pi\nu)^{\frac{d}{2}}}} :=$ $(\frac{2}{p})^{\frac{d}{2}-\frac{1}{\beta}}\text{Lip}_\sigma c_0$ in order to see that

$$\begin{aligned}
\mathscr{L}_p(\theta) &= \limsup_{t\to\infty} \frac{1}{t} \sup_{\|x\|>\theta t} \log \mathbb{E}\left(|u_t(x)|^p\right) \\
&\leqslant -\sup_{\|c\|>\frac{p}{2}(\text{Lip}_\sigma c_0)^{2\beta/(2-\beta d)}} \left[\frac{2}{p}\theta\|c\| - \left(\frac{4\nu}{p}\nu\|c\|^2\right)^{1/\beta}\right] \\
&\leqslant -\left[\theta(\text{Lip}_\sigma c_0)^{2\beta/(2-\beta d)}) - \frac{p^2}{4}\left(\frac{4\nu}{p}\right)^{\frac{1}{\beta}} (\text{Lip}_\sigma c_0)^{4/(2-\beta d)})\right],
\end{aligned} \tag{2.3.28}$$

obtained by setting $\|c\| := \frac{p}{2}\left(\text{Lip}_\sigma c_0\right)^{2\beta/(2-\beta d)}$ in the maximization problem of the first line of preceding display. The right-most quantity is strictly negative when

$$\theta > \frac{p^2}{4}\left(\frac{4\nu}{p}\nu\|c\|^2\right)^{1/\beta} (\text{Lip}_\sigma c_0)^{2\left(\frac{2-\beta}{2-\beta d}\right)};$$

this proves(2.1.4). □

Proof of inequality (2.1.5). The fact that $(\mathbb{E}(|u_t(x)|^p))^{\frac{2}{p}} \geqslant \mathbb{E}(|u_t(x)|^2)$, inequality (2.3.21) and the non-decreasing property of $\mathscr{L}_p(\theta)$ for $p \geqslant 2$, implies the proof of (2.1.5). □

Chapter 3

Blow-up results for space-time fractional stochastic partial differential equations

3.1 Motivation and main results

The study of blow-up or non-existence of solutions has attracted a number of researches, because they are very useful to applied researchers. In this regard, Mueller and Sowers in [49, 50] prove that the space-time white noise driven stochastic heat equation with Dirichlet boundary condition will blow up in finite time with positive probability, if $\sigma(u) = u^\gamma$ with $\gamma > 3/2$. Bonder and Groisman in [8] also prove the finite time blow-up for almost every initial data when nonnegative convex drift function satisfying $\int^\infty 1/f < \infty$ is taken into consideration. We refer the reader to [6, 19, 20, 27, 33, 29, 40, 41] for more information on the blow-up phenomenon in the deterministic setting.

We can study the natural extension of the time-fractional diffusion equation $\partial_t^\beta u = \nu\Delta u$ with $\beta \in (0,1)$ where Δu denotes the Laplacian of u and $\partial_t^\beta u$ denotes the Caputo fractional derivative of u defined by (1.0.2)

$$\partial_t^\beta u_t(x) = \Delta u_t(x) + \dot{W}(t,x); \;\; u_t(x)|_{t=0} = u_0(x), \tag{3.1.1}$$

where $\dot{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$.

The "correct" form of (3.1.1) can be obtained using **time fractional Duhamel's principle** [55] as follows. Consider the time-fractional PDE with force term $f(t,x)$

$$\partial_t^\beta u_t(x) = \Delta u_t(x) + f(t,x); \;\; u_t(x)|_{t=0} = u_0(x), \tag{3.1.2}$$

whose solution is given by Duhamel's principle. The role of the external force $f(t,x)$ to the output can be seen as

$$\partial_t^{\beta} V(r,t,x) = \Delta V(r,t,x); \;\; V(r,r,x) = \partial_t^{1-\beta} f(t,x)|_{t=r}, \tag{3.1.3}$$

with solution

$$V(t,r,x) = \int_{\mathbb{R}^d} G_{t-r}(x-y)\partial_r^{1-\beta} f(r,y)dy,$$

where $G_t(x)$ is the fundamental solution of $\partial_t^{\beta} u = \Delta u$. Thus (3.1.2) has solution

$$u(t,x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y)\partial_r^{1-\beta} f(r,y)dydr.$$

Now we will write the mild (integral) solution of (3.1.1) using time fractional Duhamel's principle as the form (informally):

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y)\partial_r^{1-\beta}[\dot{W}\,(r,y)]dydr. \end{aligned} \tag{3.1.4}$$

For $\gamma > 0$, we define the Riesz fractional integral by

$$I^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau)d\tau.$$

The Caputo fractional derivative ∂_t^{β} is the left inverse of Riesz fractional integral I^{β}. That means, for every $\beta \in (0,1)$, and $h \in L^{\infty}(\mathbb{R}_+)$ or $h \in C(\mathbb{R}_+)$

$$\partial_t^{\beta} I^{\beta} h(t) = h(t).$$

The fractional derivative of the noise term in (3.1.4) can now be removed as follows. Consider the time fractional PDE with a force given by $f(t,x) = I^{1-\beta} h(t,x)$, then by the time fractional Duhamel's principle the mild solution to (3.1.2) will be given by

$$u_t(x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y)\partial_r^{1-\beta} I^{1-\beta} h(r,y)dydr.$$
$$= \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y)h(r,y)dydr.$$

Hence, the preceding discussion suggest that the "correct" time fractional stochastic PDE is the following model problem:

$$\partial_t^\beta u_t(x) = \Delta u_t(x) + I^{1-\beta}[\dot{W}\ (t,x)];\ \ u_t(x)|_{t=0} = u_0(x). \tag{3.1.5}$$

The fractional integral above in equation (3.1.5) for functions $\phi \in L^2(\mathbb{R}^d)$ is defined as

$$\int_{\mathbb{R}^d} \phi(x) I^{1-\beta}[\dot{W}\ (t,x)]dx = \frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}^d} \int_0^t (t-\tau)^{-\beta} \phi(x) W(d\tau, dx),$$

by using the Walsh isometry.

By the Duhamel's principle, mentioned above, mild (integral) solution of (3.1.5) will be (informally):

$$u_t(x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y)W(dydr). \tag{3.1.6}$$

Next we want to give a **Physical motivation** to study time fractional stochastic PDEs which is adapted from [16]. The time fractional stochastic PDEs studied in this chapter may arise naturally by considering the heat equation in a material with thermal memory. Let $u_t(x), k(t,x)$ and $\vec{H}\ (t,x)$ denote the body temperature, internal energy and flux density, respectively. Then the relations

$$\begin{aligned} \partial_t k(t,x) &= -div\ \vec{H}\ (t,x), \\ k(t,x) = \beta u_t(x),\ \ \vec{H}\ (t,x) &= -\lambda \nabla u_t(x), \end{aligned} \tag{3.1.7}$$

yield the classical heat equation $\beta \partial_t u = \lambda \Delta u$.

The speed of heat flow is infinite according to the law of classical heat equation, but since the heat flow can be disrupted by the response of the material the propagation speed can be finite. In a material with thermal memory we might have

$$k(t,x) = \bar{\beta} u_t(x) + \int_0^t n(t-s) u_s(x) ds,$$

holds with some appropriate constant $\bar{\beta}$ and kernel n. In most cases we would have $n(t) = \Gamma(1-\beta_1)^{-1} t^{-\beta_1}$ for $\beta_1 \in (0,1)$. The convolution gives the fact that the nearer past affects the present more. If in addition the internal energy also depends on past random effects, then

$$\begin{aligned} k(t,x) = \bar{\beta} u_t(x) + \int_0^t n(t-s) u_s(x) ds \\ + \int_0^t l(t-s) h(s, u_s(x)) W(ds, x), \end{aligned} \tag{3.1.8}$$

where W is "white noise" modeling the random effects. Take $l(t) = \Gamma(2-\beta_2)^{-1} t^{1-\beta_2}$ for $\beta_2 \in (0,1)$, then after differentiation (3.1.8) gives

$$\partial_t^{\beta_1} u = div \vec{F} + \frac{1}{\Gamma(2-\beta_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\beta_2} h(s, u_s(x)) W(ds, x). \tag{3.1.9}$$

A version of equation (3.1.9) was studied recently by L. Chen and his co-authors: see, for example, [12]

Mijena and Nane [47] proposed to study a class of space-time fractional stochastic heat type equation as a physical model for heat in a material with random thermal memory. In the current chapter, we consider the following related space-time fractional stochastic reaction-diffusion type equations in $(d+1)$ dimension

$$\partial_t^{\beta} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^{1-\beta}[b(u) + \sigma(u) \dot{F}(t,x)] \quad t > 0 \quad \text{and} \quad x \in \mathbb{R}^d, \tag{3.1.10}$$

where $\nu > 0, \beta \in (0,1), \alpha \in (0,2]$. The operator ∂_t^{β} is the Caputo fractional derivative while $-(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian, the generator of a α-stable Lévy process and $I^{1-\beta}$ is the Riesz fractional integral operator. The forcing noise denoted by $\dot{F}(t,x)$ is a Gaussian noise and will be taken to be white in time and possibly colored in space. The initial condition will always be assumed to be a non-negative bounded deterministic function. The functions σ and b are locally Lipschitz functions.

The fractional integral of the noise term in equation (3.1.10) is not used to get a simple integral solution. A physical important reason to take the fractional integral of the noise in equation (3.1.10): Apply the fractional derivative of order $1-\beta$ to both sides of the equation (3.1.10) to see the forcing function, in the traditional units x/t: see, for example, Meerschaert et al [42]. In this chapter, we work on a deterministic time fractional equation with an external force, but the same physical principle should apply for the stochastic equations too.

Recently a numerical approximation of solutions to space-time fractional stochastic equations was established in [34]. Versions of equation (3.1.10) with or without the fractional integral of the noise term was the subject of some papers recently: see, for example, [1, 2, 11, 16, 12, 23].

Using the time fractional Duhamel's principle, as mentioned above, a mild solution to (3.1.10) in the sense of Walsh [56] is any u which is adapted to the filtration generated by the Gaussian noise and satisfies the following evolution equation

$$u_t(x) = (\mathcal{G}u)_t(x)+\int_{\mathbb{R}^d}\int_0^t G_{t-s}(x-y)b(u_s(y))\mathrm{d}s\,\mathrm{d}y+\int_{\mathbb{R}^d}\int_0^t G_{t-s}(x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y), \quad (3.1.11)$$

where

$$(\mathcal{G}u)_t(x) := \int_{\mathbb{R}^d} G_t(x-y)u_0(y)\,\mathrm{d}y, \quad (3.1.12)$$

and $G_t(\cdot)$ denotes the density of the time changed process X_{E_t}. More explanation about this process is given in Section 2.

The existence and uniqueness of the solution to (3.1.10) with the space-time white noise when $d < (2 \wedge \beta^{-1})\alpha$ has been studied by Mijena and Nane [47] under global Lipschitz condition on σ, using the white noise approach of Walsh [56]. Foondun and Nane [30] and Foondun et al. [31] established existence of solutions of space-time fractional equations with space colored noise. Asogwa and Nane [4], Mijena and Nane [48], show that if σ is globally Lipschitz, then for every non-negative measurable bounded initial function with non-empty compact support, solution to (3.1.10) is defined for all time and the distances to the origin of the farthest high peaks of absolute moments of solutions grow exactly linearly with time. See [4, 48] for more details. In this chapter, we will be concerned with the moments of the solution.

If we further have

$$\sup_{x\in[0,\,L]^d}\sup_{t\in[0,\,T]}\mathbb{E}|u_t(x)|^k<\infty \quad \text{for all} \quad L>0 \quad \text{and} \quad k\in[2,\,\infty], \tag{3.1.13}$$

then we say that u is a *random field* solution on $[0,T]$. Usually, existence is proved under the assumption that σ is globally Lipschitz. But this can be proved under the local Lipschitz condition as well. We can see this by defining

$$\tau_N := \inf\{t>0, \sup_{x\in\mathbb{R}^d}|u_t(x)|>N\},$$

then we have $|\sigma(u_s(x))-\sigma(u_s(y))|\leqslant K_N|u_s(x)-u_s(y)|$ for any $s\leqslant \min(T,\tau_N)$, where K_N is a constant dependent on N. Following the techniques in [37], [47] and [56], we can prove existence and uniqueness of a local solution in $(0,\min(T,\tau_N))$ provided that $0<\alpha<2$ and $d<(2\wedge\beta^{-1})\alpha$; two conditions which will be in force whenever we are dealing with the above equation.

When (3.1.10) has a solution $u_t(x)$ which is defined on $\mathbb{R}^d\times(0,T)$ for every $T>0$, we say that the solution is global. The main aim of this chapter is to show that under some additional conditions on the initial condition and the functions σ and b, (3.1.10) cannot have global random field solutions. The failure of global solutions usually manifests itself via the 'blow up' of certain quantities involving the solution.

In this chapter, we will work with white and space colored noise driven equations. First, we will look at the following equation driven by the space-time white noise

$$\partial_t^\beta u_t(x)=-\nu(-\Delta)^{\alpha/2}u_t(x)+I^{1-\beta}[\sigma(u)\stackrel{\cdot}{W}(t,x)] \quad t>0 \quad \text{and} \quad x\in\mathbb{R}^d, \tag{3.1.14}$$

where $\stackrel{\cdot}{W}(t,x)$ is a space-time white noise. We will also look at equations driven by noise colored in space of the following type

$$\partial_t^\beta u_t(x)=-\nu(-\Delta)^{\alpha/2}u_t(x)+I^{1-\beta}[\sigma(u)\stackrel{\cdot}{F}(t,x)] \quad t>0 \quad \text{and} \quad x\in\mathbb{R}^d, \tag{3.1.15}$$

where $\dot{F}(t,x)$ is a space colored noise. The corresponding mild solution in the sense of Walsh [56] is given by

$$u_t(x) = (\mathcal{G}u)_t(x) + \int_{\mathbb{R}^d}\int_0^t G_{t-s}(x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y). \tag{3.1.16}$$

We will again be interested in the random field solution. But for this equation, we will need to impose some conditions on the noise term. We have

$$\mathbb{E}[\dot{F}(s,x)\dot{F}(t,y)] = \delta_0(t-s)f(x,\,y),$$

where $f(x,\,y) \leqslant g(x-y)$ and g is a locally integrable function on $\mathbb{R}^d$ with a possible singularity at 0 satisfying

$$\int_{\mathbb{R}^d}\frac{\hat{g}(\xi)}{1+|\xi|^\alpha}\mathrm{d}\xi < \infty,$$

where $\hat{g}$ denotes the Fourier transform of g.

It is worth mentioning that not a lot of work has been done in this type of problems for space-time fractional stochastic partial differential equations.

Assumption 3.1.1. *The function σ is a locally Lipschitz function satisfying the following growth condition. There exist a $\gamma > 0$ such that*

$$\sigma(x) \geqslant |x|^{1+\gamma} \quad \textit{for all} \quad x \in \mathbb{R}^d. \tag{3.1.17}$$

Now we are ready to state our findings in detail. For the first couple of our results, we will assume that the initial condition is bounded below by a positive constant given below

$$\inf_{x\in\mathbb{R}^d} u_0(x) := \kappa. \tag{3.1.18}$$

Theorem 3.1.1. *Let $d < (2\wedge\beta^{-1})\alpha$. Suppose that $\kappa > 0$ and u_t be the solution to* (3.1.14)*. Then there exists a $t_0 > 0$ such that for all $x \in \mathbb{R}^d$,*

$$\mathbb{E}|u_t(x)|^2 = \infty \quad \textit{whenever} \quad t \geqslant t_0.$$

This theorem states that provided that the initial function is bounded below, the second moment will eventually be infinite for white noise driven equations.

Remark 3.1.2. *We can also get a blow up for the following equation that was considered by Chen et al [**?**] for any $\gamma > 0$ and $d < 2\alpha + \frac{\alpha}{\beta}\min(2\gamma - 1, 0)$.*

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^\gamma[\sigma(u)\stackrel{\cdot}{W}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d, \tag{3.1.19}$$

In this case the corresponding mild solution in the sense of Walsh [56] is given by

$$u_t(x) = (\mathcal{G}u)_t(x) + \int_{\mathbb{R}^d}\int_0^t H_{t-s}(x-y)\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y). \tag{3.1.20}$$

*Where $H(t,x)$ is given by the time fractional derivative of $G(t,x)$. Using Lemma 5.5 in [**?**] we can get finite time blow up as in the proof of Theorem 3.1.1.*

Here the nonlinear renewal inequality becomes

$$P(s) \geqslant C + C_1 \int_0^t P(s)^{1+\eta}(t-s)^{2(\beta+\gamma-1)-\mathrm{d}\beta/\alpha} ds$$

where $P(s) = \inf_{x\in\mathbb{R}^d} \mathbb{E}|u_s(x)|^2$ and $2(\beta+\gamma-1) - \mathrm{d}\beta/\alpha > 0$. Note that when $\gamma = \beta - 1$ this condition becomes the same condition in Theorem 3.1.1, $d < (2 \wedge \beta^{-1})\alpha$.

We have a slightly more complicated picture for equations with colored noise. We will assume the following non-degeneracy condition on the spatial correlation of the noise.

Assumption 3.1.2. *For fixed $R > 0$, there exists a positive number K_f such that*

$$\inf_{x,\,y\in B(0,\,R)} f(x,\,y) \geqslant K_f.$$

Since we mostly set $R = 1$ when using this condition, the dependence of K_f on R is not necessarily specified. The above assumption is also very mild. There are a lot of examples including the Riesz Kernel, exponential kernel, Ornstein-Uhlenbeck-type kernels, Poisson kernels and Cauchy kernels; see, for example, Example 1.4 in [29] for more details.

Theorem 3.1.3. *Let u_t be the solution to* (3.1.15) *and suppose that Assumption 3.1.2 holds. Fix $t_0 > 0$, then there exists a positive number κ_0 such that for all $\kappa \geqslant \kappa_0$, and $x \in \mathbb{R}^d$ we have*

$$\mathbb{E}|u_t(x)|^2 = \infty, \quad \textit{whenever} \quad t \geqslant t_0.$$

To establish non-existence of the second moment, in contrast to Theorem 3.1.1, we require that the initial condition is large enough. This is a result of the spatially correlated nature of the noise, which induces some extra dissipation effect. In fact, even in the case of the corresponding linear equation($\sigma(u) \propto u$), it is known that for some correlation functions, their moments might not grow exponentially fast. See for instance [17] and [36]. However, if we consider the case when the correlation function is given by the Riesz Kernel, we have the following stronger result concerning the solution to (3.1.15).

Theorem 3.1.4. *Suppose that the correlation function f is given by*

$$f(x, y) = \frac{1}{|x-y|^{\omega}} \quad \textit{with} \quad \omega < d \wedge (\alpha\beta^{-1}).$$

Then for $\kappa > 0$, there exists a positive number $\tilde{t}$ such that for all $t \geqslant \tilde{t}$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}|u_t(x)|^2 = \infty.$$

Remark 3.1.5. *We can also get a blow up for the following equation that was considered by Chen et al [12] for any $\gamma > 0$ (where we also need to add the condition from Chen et al [12]about d and other parameters!)*

$$\partial_t^{\beta} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^{\gamma}[\sigma(u) \dot{F}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d, \tag{3.1.21}$$

In this case the corresponding mild solution in the sense of Walsh [56] is given by

$$u_t(x) = (\mathcal{G}u)_t(x) + \int_{\mathbb{R}^d} \int_0^t H_{t-s}(x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y). \tag{3.1.22}$$

*Where $H_t(x)$ is given by the time fractional derivative of $G_t(x)$. Using Plancharel theorem and equations (4.8) in [**?**] we can get finite time blow up as in the proof of Theorem 3.1.4. We need to work on the following lower bound*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{t-s}(x_1-y_1)G_{t-s}(x_2-y_2)f(y_1,\,y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \geqslant c(t-s)^{2\beta+2\gamma-2-\beta w/\alpha}.$$

Hence the nonlinear renewal inequality in this case is

$$H(t)\geqslant C + C\int_0^t H(s)^{1+\eta}(t-s)^{2\beta+2\gamma-2-\beta w/\alpha}$$

where $H(s)=\inf_{x\in\mathbb{R}^d,y\in\mathbb{R}^d}\mathbb{E}|u_s(x)u_s(y)|$.

It is also important to mention that all the results established so far in this work are obtained under the assumption that the initial function is bounded below away from zero. In fact, as we shall see from the next result, this condition can be weakened.

Assumption 3.1.3. *Suppose that initial condition is non-negative and satisfies the following,*

$$\int_{B(0,\,1)} u_0(x)\,\mathrm{d}x := K_{u_0} > 0.$$

We have taken $B(0,\,1)$ as a matter of convenience.

Theorem 3.1.6. *Let $d<(2\wedge\beta^{-1})\alpha$, and u_t be the solution to* (3.1.14). *Then, under Assumption 3.1.3, there exists a $t_0\geqslant 0$ such that for all $t\geqslant t_0$ and $x\in\mathbb{R}^d$,*

$$\mathbb{E}|u_t(x)|^2=\infty \quad \textit{whenever} \quad K_{u_0}\geqslant K,$$

where K is some positive constant.

We have a similar result for the equation driven by space colored noise.

Theorem 3.1.7. *Let u_t be the solution to* (3.1.15). *Then, under Assumptions 3.1.2 and 3.1.3, there exists a $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ and $x \in \mathbb{R}^d$,*

$$\mathbb{E}|u_t(x)|^2 = \infty \quad \textit{whenever} \quad K_{u_0} \geqslant K,$$

where K is a positive constant.

It should be noted that the constant K appearing in the above two results need not be the same. The concept of our method involves obtaining non-linear renewal-type inequalities whose solutions blow up in finite time. We adapt the methods in [29] with crucial changes to suit to the space-time fractional equations. This method is soft and can be adapted to study a wider class of equations. For the colored-noise case, a crucial quantity we study is $\mathbb{E}|u_t(x)u_t(y)|$ instead of $\mathbb{E}|u_t(x)|^2$; and a good control of the deterministic term $(\mathcal{G}u)_t(x)$ is crucial in getting the non-existence of the solutions. Our methods depend on crucial heat kernel estimates for short times and use the fact that we can restart the solution at a later time. We will explain these methods in the proof of our results.

Our next theorem, extends those of [19] , [20] and [29]. Fix $R > 0$. We will study the equations above in the ball $B(0,\, R)$ with Dirichlet boundary conditions. We will need the following assumption.

Assumption 3.1.4. *We assume that the initial condition u_0 is a non-negative function whose support, denoted by S_{u_0} satisfies $B(0,\, R/2) \subset S_{u_0}$ such that $\inf_{x\in B(0,\, R/2)} u_0(x) > \tilde{\kappa}$ for some positive constant $\tilde{\kappa}$.*

Theorem 3.1.8. *Fix $R > 0$ and consider*

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + I^{1-\beta}[\sigma(u)\stackrel{\cdot}{F}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in B(0,\, R). \tag{3.1.23}$$

Here $-(-\Delta)^{\alpha/2}$ denotes the generator of a α-stable Lévy process killed upon exiting the ball $B(0,\, R)$. The noise $\dot{F}$, when not space-time white noise is taken to be spatially colored with correlation function satisfying all the conditions stated above. Fix $\epsilon > 0$, then there exist $t_0 > 0$ and $K > 0$, such that for $K_{u_0} > K$,

$$\mathbb{E}|u_t(x)|^2 = \infty \quad \textit{for all} \quad t \geqslant t_0 \quad \textit{and} \quad x \in B(0,\, R-\epsilon).$$

Fujita in [33] showed that the only global solution to

$$\partial_t u_t(x) = \Delta u_t(x) + u(x)^{1+\lambda} \quad \text{for} \quad t > 0 \quad x \in \mathbb{R}^d,$$

with initial condition u_0 and $\lambda > 0$ is the trivial one for $\lambda < 2/d$. In the case $\lambda > 2/d$, the global solution exist for small enough u_0. A good way to look at this result is that for large λ, the quantity $u^{1+\lambda}$ becomes much smaller when the initial condition is small and the dissipative effect of the Laplacian prevents the solution to grow too big for blow-up to happen. But, when λ is close to zero, regardless of the size of the initial condition, the dissipative effect of the Laplacian cannot prevent blow up of the solution. For the reaction-diffusion type space-time fractional stochastic equations, we work with the first moment $\mathbb{E}(|u_t(x)|)$. There is still an interplay between the dissipative effect of the operator and the forcing term and we are able to shed light only on part of the true picture. We show that if the initial condition is large enough then there is no global solution. It might very well be just like for the deterministic case, if the non-linearity is high enough, then for small initial condition, there exist global solutions. See the survey papers [26, 39] for blow-up results for the deterministic equations.

Next we want to state our non-existence results for reaction-diffusion type equations.

Assumption 3.1.5. *The function b is locally Lipschitz satisfying the following growth condition. There exist a $\eta > 0$ such that*

$$b(x) \geqslant |x|^{1+\eta} \quad \textit{for all} \quad x \in \mathbb{R}^d. \tag{3.1.24}$$

Theorem 3.1.9. *Suppose that σ is globally Lipschitz and b satisfies the conditions in Assumption 3.1.5. Consider*

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I^{1-\beta}[b(u_t(x)) + \sigma(u) \dot{F}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d. \tag{3.1.25}$$

Here $-(-\Delta)^{\alpha/2}$ denotes the generator of α-stable Lévy process. The noise $\dot{F}$, when not space-time white noise is taken to be spatially colored. Then (3.1.25) *has no random field solution in the following cases:*

1: $\inf_{x\in\mathbb{R}^d} u_0(x) > \kappa > 0$ *and* $\eta > 0$.

2: $||u_0||_{L^1(\mathbb{R}^d)} > 0$, *and* $\beta d\eta/\alpha < 1$.

When $\beta = 1$, a version of this theorem with $\alpha = 2$ was considered by Chow [20] and a version with $\alpha \in (1, 2)$ was considered by Foondun and Parshad [32].

The mild solution of equation (3.1.25) is given in the sense of Walsh [56] as follows:

$$u_t(x) = (\mathcal{G}u)_t(x) + \int_{\mathbb{R}^d}\int_0^t b(u_s(y))G_{t-s}(x-y)\mathrm{d}s\,\mathrm{d}y + \int_{\mathbb{R}^d}\int_0^t G_{t-s}(x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y). \quad (3.1.26)$$

Remark 3.1.10. *We can also get a blow up for the following equation that was considered by Chen et al [12] for any $\gamma > 0$ and $d < 2\alpha + \frac{\alpha}{\beta}\min(2\gamma - 1, 0)$.*

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + I^\gamma[b(u_t(x)) + \sigma(u)\stackrel{\cdot}{F}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d, \qquad (3.1.27)$$

In this case the corresponding mild solution in the sense of Walsh [56] is given by

$$u_t(x) = (\mathcal{G}u)_t(x) + \int_{\mathbb{R}^d}\int_0^t b(u_s(y))H_{t-s}(x-y)\mathrm{d}s\,\mathrm{d}y + \int_{\mathbb{R}^d}\int_0^t H_{t-s}(x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y). \quad (3.1.28)$$

Where $H_t(x)$ is given by the time fractional derivative of $G_t(x)$. Using Equation (4.7) in [12] for $\xi = 0$ to get $\int_{\mathbb{R}^d} H_t(x)dx = t^{\beta+\gamma-1}$ we can get finite time blow up as in the proof of Theorem 3.1.9.

Here the nonlinear renewal inequality becomes

$$F(s) \geqslant C + C_1\int_0^t F(s)^{1+\eta}(t-s)^{\beta+\gamma-1}ds$$

where $F(s) = \inf_{x\in\mathbb{R}^d}\mathbb{E}|u_s(x)|^2$.

Theorem 3.1.11. *Suppose that σ is globally Lipschitz and b satisfies the conditions in Assumption 3.1.5. Fix $R > 0$ and consider*

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + I^{1-\beta}[b(u_t(x)) + \sigma(u)\stackrel{\cdot}{F}(t,x)] \quad t > 0 \quad \textit{and} \quad x \in B(0,\,R). \quad (3.1.29)$$

Here $-(-\Delta)^{\alpha/2}$ denotes the generator of α-stable Lévy process killed upon exiting the ball $B(0,\,R)$. The noise $\dot{F}$, when not space-time white noise is taken to be spatially colored. Let ϕ_1 be the first eigenfunction *of the fractional Laplacian with Dirichlet exterior boundary condition in the ball* $B = B(0, R)$. *Then* (3.1.29) *has no random field solution in the following cases:*

1: $\int_B u_0(x)\phi_1(x) := K_{u_0,\phi_1} > 0$, *and* $\beta(1+\eta) \leqslant 1$

2: $\beta(1+\eta) > 1$ *and for large* $K_{u_0,\phi_1} > 0$.

The mild solution of equation (3.1.29) is given in the sense of Walsh [56] as follows:

$$\begin{aligned} u_t(x) = (\mathcal{G}_B u)_t(x) &+ \int_{B(0,R)} \int_0^t b(u_s(y)) G_B(t-s, x, y) \mathrm{d}s\, \mathrm{d}y \\ &+ \int_{B(0,R)} \int_0^t G_B(t-s, x, y) \sigma(u_s(y)) F(\mathrm{d}s\, \mathrm{d}y), \end{aligned} \tag{3.1.30}$$

where $G_B(t, x, y)$ is the density of X_{E_t} killed on the exterior of B.

In this chapter, we will denote the ball of radius R by $B = B(0,\, R)$. For $x \in \mathbb{R}^d$, $|x|$ will be the magnitude of x. The letter c and c^* with or without subscripts will denote a constant whose value is not relevant.

The outline of this chapter is the following. Preliminary notions and results needed for the proofs of the main results are presented in Section 3.2. Section 3.3 contains the proofs of Theorem 3.1.1 and 3.1.3. Theorem 3.1.4 is proved in Section 3.4. In Section 3.5 we give the proofs of Theorems 3.1.6, 3.1.7, and 3.1.8. Finally, in section 3.6 we present the proof of the remaining results. We list a couple of results we need in the appendix, section 3.7.

3.2 Preliminaries

Now we are ready to give results that are used in the proof of our main results. Let $\alpha \in (0, 2)$. Let X_t denote a symmetric α-stable Lévy process with density function denoted by $p(t,\, x)$. This is characterized through the Fourier transform which is given by

$$\widehat{p(t,\, \xi)} = e^{-t\nu|\xi|^\alpha}. \tag{3.2.1}$$

Let $D = \{D_r,\, r \geqslant 0\}$ denote a β-stable subordinator with $\beta \in (0,1)$ and E_t be its first passage time. It is known that the density of the time changed process X_{E_t} is given by $G_t(x)$. By conditioning, we have

$$G_t(x) = G(t,x) = \int_0^\infty p(s,\, x) f_{E_t}(s) \mathrm{d}s, \tag{3.2.2}$$

where

$$f_{E_t}(x) = t\beta^{-1} x^{-1-1/\beta} g_\beta(t x^{-1/\beta}),$$

where $g_\beta(\cdot)$ is the density function of D_1 and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leqslant 0$. Moreover,

$$g_\beta(u) \sim K(\beta/u)^{(1-\beta/2)/(1-\beta)} \exp\{-|1-\beta|(u/\beta)^{\beta/(\beta-1)}\} \quad \text{as } u \to 0+, \tag{3.2.3}$$

and

$$g_\beta(u) \sim \frac{\beta}{\Gamma(1-\beta)} u^{-\beta-1} \quad \text{as } u \to \infty. \tag{3.2.4}$$

We will also need the following estimates given in Lemma 2.1 in [30],

$$c_1\left(t^{-\beta d/\alpha} \wedge \frac{t^\beta}{|x|^{d+\alpha}}\right) \leqslant G_t(x), \tag{3.2.5}$$

and

$$G_t(x) \leqslant c_2\left(t^{-\beta d/\alpha} \wedge \frac{t^\beta}{|x|^{d+\alpha}}\right), \tag{3.2.6}$$

when $\alpha > d$.

Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $p_D(t,x,y)$ denote the heat kernel of the equation (3.1.14) when $\beta = 1$ and $\sigma = 0$. This is the space fractional diffusion equation with Dirichlet exterior boundary conditions. A well known fact is that

$$p_D(t,x,y) \leqslant p(s,x,y) \text{ for all } x,y \in D, t > 0. \tag{3.2.7}$$

Let $G_D(t,x,y)$ denote the heat kernel of the equation (3.1.14) when $\sigma = 0$. This is the space-time fractional diffusion equation with Dirichlet exterior boundary conditions. Using the representation from Meerschaert et al. [18] and [43]

$$G_D(t,x,y) = \int_0^\infty p_D(s,x,y) f_{E_t}(s) ds.$$

and using equation (3.2.7) we get

$$G_D(t,x,y) \leqslant G(t,x,y) = G_t(x-y) \text{ for all } x,y \in D, t > 0. \tag{3.2.8}$$

The next proposition is crucial in proving the lower bounds in Theorem 3.1.8.

Proposition 3.2.1 (Proposition 2.1 in [31]). *Fix $\epsilon > 0$, then there exists $t_0 > 0$ such that for all $x, y \in B(0, R-\epsilon)$ and for all $t < t_0$ and $|x-y| < t^{\beta/\alpha}$ we have*

$$G_B(t,x,y) \geqslant Ct^{-\beta d/\alpha},$$

for some constant $C > 0$.

For notational convenience, we set

$$(\mathcal{G}u)_t(x) := \int_{\mathbb{R}^d} G_t(x-y) u_0(y)\, \mathrm{d}y.$$

and

$$(\tilde{\mathcal{G}}u)_t(x) := \int_{\mathbb{R}^d} p(t,\, x-y) u_0(y)\, \mathrm{d}y.$$

Proposition 3.2.2. *Let $x \in B(0,\, 1)$ and Assumption 3.1.3 holds. Then there exists a positive number t_0 such that for $t \in (0,\, t_0]$, we have*

$$(\mathcal{G}u)_{t+t_0}(x) \geqslant cK_{u_0},$$

where

$$K_{u_0} := \int_{B(0,\,1)} u_0(x)\, \mathrm{d}x > 0. \tag{3.2.9}$$

Proof. By definition and Proposition 2.1 in [29], we have

$$
\begin{aligned}
(\mathcal{G}u)_{t+t_0}(x) &= \int_{\mathbb{R}^d} G_{t+t_0}(x-y)u_0(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_0^\infty p(s,\, x-y) f_{E_{t+t_0}}(s)\,\mathrm{d}s\, u_0(y)\mathrm{d}y \\
&= \int_0^\infty (\tilde{\mathcal{G}}u)_s(x) f_{E_{t+t_0}}(s)\,\mathrm{d}s \\
&\geqslant c_1 K_{u_0}\int_{t_0}^{2t_0} f_{E_{t+t_0}}(s)\,\mathrm{d}s \\
&= c_1 K_{u_0}\int_{t_0}^{2t_0} (t+t_0)\beta^{-1}s^{-1-1/\beta} g_\beta((t+t_0)s^{-1/\beta})\,\mathrm{d}s \\
&= c_1 K_{u_0}\int_{\frac{t+t_0}{(2t_0)^{1/\beta}}}^{\frac{t+t_0}{t_0^{1/\beta}}} g_\beta(u)\,\mathrm{d}u \\
&\geqslant c_1 K_{u_0}\int_{\frac{2t_0}{(2t_0)^{1/\beta}}}^{\frac{t_0}{t_0^{1/\beta}}} g_\beta(u)\,\mathrm{d}u \\
&= c_2 K_{u_0},
\end{aligned}
$$

where c_2 depends on t_0. The last equality before the last inequality follows by substitution. □

Define

$$
(\mathcal{G}_B u)_t(x) := \int_{B(0,\,R)} G_B(t,x,y)u_0(y)\,\mathrm{d}y.
$$

The following proposition will be used in the proof of Theorem 3.1.8.

Proposition 3.2.3. *Let $t \leqslant \left(\frac{R}{2}\right)^\alpha$ and $R > 0$. Under Assumption 3.1.4, we have*

$$
(\mathcal{G}_B u)_t(x) \geqslant c, \quad \textit{for all} \quad x \in B(0,\, R/2),
$$

where c is some positive constant.

Proof. By a simple conditioning and Proposition 2.2 in [29], we have

$$
\begin{aligned}
(\mathcal{G}_B u)_t(x) &= \int_{B(0,\,R)} G_B(t,x,y)u_0(y)\,\mathrm{d}y \\
&= \int_0^\infty (\tilde{\mathcal{G}}_B u)_s(x) f_{E_t}(s)\,\mathrm{d}s \\
&\geqslant \int_0^{(R/2)^\alpha} (\tilde{\mathcal{G}}_B u)_s(x) f_{E_t}(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant c_1 \int_0^{(R/2)^\alpha} f_{E_t}(s)\,\mathrm{d}s \\
&= c_1 \int_0^{(R/2)^\alpha} t\beta^{-1} s^{-1-1/\beta} g_\beta(ts^{-1/\beta})\,\mathrm{d}s \\
&= c_1 \int_{t/(R/2)^{\alpha/\beta}}^{\infty} g_\beta(u)\,\mathrm{d}u \\
&\geqslant c_1 \int_{(R/2)/(R/2)^{\alpha/\beta}}^{\infty} g_\beta(u)\,\mathrm{d}u \\
&= c.
\end{aligned}
$$

□

Proposition 3.2.4. *Suppose that* $t \leqslant \left(\frac{R}{2}\right)^\alpha$ *and* $R > 0$*. Then for all* $x_1,\, x_2 \in B(0,\, R)$*, we have*

$$
\int_{B(0,\,R)\times B(0,\,R)} G_{t-s}(x_1 - y_1) G_{t-s}(x_2 - y_2) f(y_1,\, y_2)\,\mathrm{d}y_1 \mathrm{d}y_2 \geqslant cK_f,
$$

where $s \leqslant t$ *and* c *is some positive constant.*

Proof. By assumption 3.1.2, we observe

$$
\begin{aligned}
&\int_{B(0,\,R)\times B(0,\,R)} G_{t-s}(x_1 - y_1) G_{t-s}(x_2 - y_2) f(y_1,\, y_2)\,\mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad \geqslant K_f \int_{B(0,\,R)\times B(0,\,R)} G_{t-s}(x_1 - y_1) G_{t-s}(x_2 - y_2)\,\mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned}
$$

Let

$$
\mathcal{A}_i := \{y_i \in B(0,\, R); |x_i - y_i| \leqslant (t-s)^{\beta/\alpha}\}, \quad \text{for} \quad i = 1, 2.
$$

Since $t \leqslant \left(\frac{R}{2}\right)^\alpha$ we observe that $|\mathcal{A}_i| = c|t-s|^{\beta d/\alpha}$ for some constant c. Using the estimates given by (3.2.5) for $G_t(x)$, we have

$$
\begin{aligned}
&K_f \int_{B(0,\,R)\times B(0,\,R)} G_{t-s}(x_1 - y_1) G_{t-s}(x_2 - y_2)\,\mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad \geqslant K_f \int_{\mathcal{A}_1\times\mathcal{A}_2} G_{t-s}(x_1 - y_1) G_{t-s}(x_2 - y_2)\,\mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad \geqslant K_f \int_{\mathcal{A}_1\times\mathcal{A}_2} C(t-s)^{-2\beta d/\alpha}\,\mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad = c_2 K_f.
\end{aligned}
$$

This completes the proof. ☐

We need to introduce a version of the above result for the killed space-time fractional kernel using Proposition 3.2.1.

Proposition 3.2.5. *Let $R > 0$ and fix $\epsilon > 0$. Then for all $x_1,\, x_2 \in B(0,\, R-\epsilon)$ and $t \leqslant \left(\frac{R}{2}\right)^{\alpha}$, we have*

$$\int_{B(0,\,R)\times B(0,\,R)} G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)f(y_1,\, y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \geqslant cK_f,$$

where $s \leqslant t$ and c is some positive constant.

Proof. Assumption 3.1.2 gives

$$\begin{aligned}\int_{B(0,\,R)\times B(0,\,R)} & G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)f(y_1,\, y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \\ & \geqslant K_f \int_{B(0,\,R-\epsilon)\times B(0,\,R-\epsilon)} G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)\,\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

Since $t \leqslant \left(\frac{R}{2}\right)^{\alpha}$ if we set

$$\mathcal{A}_i := \{y_i \in B(0,\, R-\epsilon); |x_i - y_i| \leqslant (t-s)^{\beta/\alpha}\} \quad \text{for} \quad i = 1, 2,$$

then $|\mathcal{A}_i| = c_1|t-s|^{\beta d/\alpha}$ for some c_1. We therefore have using Proposition 3.2.1

$$\begin{aligned}\int_{B(0,\,R-\epsilon)\times B(0,\,R-\epsilon)} & G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \\ & \geqslant \int_{\mathcal{A}_1\times\mathcal{A}_2} G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \\ & \geqslant \int_{\mathcal{A}_1\times\mathcal{A}_2} C(t-s)^{-2\beta d/\alpha}\,\mathrm{d}y_1\mathrm{d}y_2 \\ & = c_2,\end{aligned}$$

This proves the required inequality. ☐

Remark 3.2.6. *Under the same assumption of Proposition 3.2.5, we clearly have*

$$\int_{B(0,\,R-\epsilon)\times B(0,\,R-\epsilon)} G_B(t-s, x_1-y_1)G_B(t-s, x_2-y_2)f(y_1,\, y_2)\,\mathrm{d}y_1\mathrm{d}y_2 \geqslant cK_f,$$

where $s \leqslant t$ and c is some positive constant.

In the next two propositions we will give the renewal inequalities needed to prove non-existence results.

Proposition 3.2.7. *Fix $T > 0$ and suppose that h is a non-negative function satisfying the following non-linear integral inequality,*

$$h(t) \geqslant C + D\int_0^t \frac{h(s)^{1+\gamma}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s, \quad \textit{for} \quad 0 < t \leqslant T,$$

where C, D and γ are positive numbers. Then for any $t_0 \in (0, T]$, there exists an C_0 such that for $C > C_0$

$$h(t) = \infty \quad \textit{whenever} \quad t \geqslant t_0.$$

Proof. Since $t \leqslant T$ the inequality reduces to

$$h(t) \geqslant C + \frac{D}{T^{d\beta/\alpha}}\int_0^t h(s)^{1+\gamma}\,\mathrm{d}s.$$

Thanks to comparison principle, it suffices to consider

$$h(t) = C + \frac{D}{T^{d\beta/\alpha}}\int_0^t h(s)^{1+\gamma}\,\mathrm{d}s, \quad \text{for} \quad t \leqslant T,$$

which is equivalent to the following non-linear ordinary differential equation,

$$\frac{h'(t)}{h(t)^{1+\gamma}} = \frac{D}{T^{d\beta/\alpha}},$$

with initial condition $h(0) = C$, whose solution is given by

$$\frac{1}{h(t)^{\gamma}} = \frac{1}{C^{\gamma}} - \frac{\gamma D t}{T^{d\beta/\alpha}}, \quad \text{for} \quad t \leqslant T.$$

Thus the blowup occurs at $t = \frac{T^{d\beta/\alpha}}{C^\gamma D\gamma}$. Choose $C > \left(\frac{T^{d\beta/\alpha}}{D\gamma t_0}\right)^{1/\gamma}$ for any fixed $t_0 < T$. The conclusion follows since $h(t)$ is increasing on $(0, \infty)$ and blow-up occurs before time t_0. $\square$

Next we will give a slightly modified renewal inequalities needed for the proof of our main results.

Proposition 3.2.8. *Let* $0 < (1+\gamma)d\beta/\alpha < 1$. *Suppose* h *is a non-negative function satisfying the following non-linear integral inequality,*

$$h(t) \geqslant C + D\int_0^t \frac{h(s)^{1+\gamma}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s, \quad \textit{for} \quad t > 0,$$

where C, D *and* γ *are positive numbers. Then for any* $C > 0$ *there exists* $t_0 > 0$ *such that* $h(t) = \infty$ *for all* $t \geqslant t_0$.

Proof. Since $0 < t - s \leqslant t$, we get

$$h(t) \geqslant C + D\int_0^t \frac{h(s)^{1+\gamma}}{t^{d\beta/\alpha}}\,\mathrm{d}s, \quad \text{for} \quad t > 0.$$

Now let $q(t) := t^{d\beta/\alpha}h(t)$ and since we can always assume $t_0 > 1$, the above inequality becomes

$$q(t) \geqslant C + D\int_0^t \frac{q(s)^{1+\gamma}}{s^{(1+\gamma)d\beta/\alpha}}\,\mathrm{d}s, \quad \text{for} \quad t \geqslant 1.$$

We only need to consider the following ordinary differential equation,

$$\frac{q'(t)}{[q(t)]^{1+\gamma}} = \frac{D}{t^{(1+\gamma)d\beta/\alpha}}, \quad \text{for} \quad t \geqslant 1,$$

with initial condition $q(1) = C$. The solution of this equation is given by

$$\frac{1}{q(t)^\gamma} = \frac{1}{C^\gamma} + \frac{\gamma D}{1 - \frac{(1+\gamma)d\beta}{\alpha}} - \frac{\gamma D t^{1-\frac{(1+\gamma)d\beta}{\alpha}}}{1 - \frac{(1+\gamma)d\beta}{\alpha}}, \quad \text{for } t \geqslant 1.$$

Since $(1+\gamma)d\beta/\alpha < 1$ the blowup occurs when t is equal to t_0 given by

$$t_0 := \left(\frac{1 - \frac{(1+\gamma)d\beta}{\alpha}}{\gamma D}\left(\frac{1}{C^\gamma} + \frac{\gamma D}{1 - \frac{(1+\gamma)d\beta}{\alpha}}\right)\right)^{1/(1-\frac{(1+\gamma)d\beta}{\alpha})}.$$

Thus, $h(t) = \infty$ for $t \geqslant t_0$ since $h(t)$ is increasing on $(0, \infty)$.

$\square$

Remark 3.2.9. *The above Proposition 3.2.8 is also true when h satisfies*

$$h(t) \geqslant Ct^{-d\beta/\alpha} + D\int_0^t \frac{h(s)^{1+\gamma}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s, \text{ for } t > 0.$$

In this case t_0 is given by

$$t_0 := \left(\frac{1 - \frac{(1+\gamma)d\beta}{\alpha}}{\gamma D C^{\gamma}}\right)^{1/(1-\frac{(1+\gamma)d\beta}{\alpha})}.$$

Remark 3.2.10. *The above proposition holds when $(1+\gamma)/\alpha \geqslant 1$ as well. This is because we can always write $\gamma = \gamma_0 + (\gamma - \gamma_0)$ so that $\gamma_0 < \gamma$ and $(1+\gamma_0)/\alpha < 1$. Now we use the fact that $h(t) > A$ for all $t > 0$ to reduce the integral inequality to*

$$h(t) \geqslant C + D\int_0^t \frac{h(s)^{1+\gamma}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s \geqslant C + DC^{\gamma-\gamma_0}\int_0^t \frac{h(s)^{1+\gamma_0}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s, \quad \text{for} \quad t > 0.$$

The result now follows by Proposition 3.2.8.

Next we state a general version of the proposition 3.2.8.

Proposition 3.2.11. *Let $0 < \theta$. Suppose h is a non-negative function satisfying the following non-linear integral inequality,*

$$h(t) \geqslant C + D\int_0^t \frac{h(s)^{1+\gamma}}{(t-s)^{\theta}}\,\mathrm{d}s, \quad \text{for} \quad t > 0$$

where C, D and γ are positive numbers. Then for any $C > 0$ there exists $t_0 > 0$ such that $h(t) = \infty$ for all $t \geqslant t_0$.

Proof. The proof is similar to the proof of Proposition 3.2.8 and Remark 3.2.10. So it is omitted here.

$\square$

Proof of Theorem 3.1.1. To start the proof of the theorem with the use of the mild formulation of the solution given by (3.7.2), then take second moment and use the Walsh isometry to get

$$\begin{aligned}\mathbb{E}|u_t(x)|^2 &= |(\mathcal{G}u)_t(x)|^2 + \int_0^t \int_{\mathbb{R}^d} G^2_{t-s}(x-y)\mathbb{E}|\sigma(u_s(y))|^2 \,\mathrm{d}y\,\mathrm{d}s \\ &:= I_1 + I_2.\end{aligned}$$

Using the fact that the initial condition is bounded below gives

$$I_1 \geqslant \kappa^2.$$

This follows since $\int_{\mathbb{R}^d} G_t(x-y)dy = 1$. By utilizing the growth condition on σ, Jensen's inequality, and Lemma 2.2.1 we bound I_2 as follows

$$\begin{aligned}I_2 &\geqslant \int_0^t \left(\inf_{x\in\mathbb{R}^d} \mathbb{E}|u_s(x)|^2\right)^{\mathbf{1}+\gamma} \int_{\mathbb{R}^d} G^2_{t-s}(x-y)\,\mathrm{d}y\,\mathrm{d}s \\ &\geqslant C^* \int_0^t \left(\inf_{x\in\mathbb{R}^d} \mathbb{E}|u_s(x)|^2\right)^{\mathbf{1}+\gamma} \frac{1}{(t-s)^{\beta d/\alpha}}\mathrm{d}s.\end{aligned}$$

If we set

$$P(s) := \inf_{x\in\mathbb{R}^d} \mathbb{E}|u_s(x)|^2,$$

the inequalities becomes

$$P(t) \geqslant C + C^* \int_0^t \frac{P(s)^{1+\gamma}}{(t-s)^{d\beta/\alpha}}\,\mathrm{d}s.$$

Proposition 3.2.7 now completes the proof of the theorem. □

For the proof of Theorem 3.1.3 we need the following proposition.

Proposition 3.3.1. *Suppose that there exists a* $\kappa_0 > 0$ *and* $t_0 > 0$ *such that the lower bound of* u_0 *in* (3.1.18) *satisfies* $\kappa > \kappa_0$*. Then for all* $t_0 < t \leqslant (1/2)^\alpha$*, we have*

$$\mathbb{E}|u_t(x)u_t(y)| = \infty \quad \textit{for all}, \quad x,\, y \in B(0,\, 1).$$

Proof. By the mild formulation (3.7.2) we observe

$$\begin{aligned}
&\mathbb{E}|u_t(x)u_t(y)| \\
&\quad \geqslant \mathcal{G}u_t(x)\mathcal{G}u_t(y) + \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} G_{t-s}(x-z)G_{t-s}(y-w)f(z,w)\left(\mathbb{E}|u_s(z)u_s(w)|\right)^{1+\gamma} \mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\quad := I_1 + I_2.
\end{aligned}$$

First consider the term I_1. Using the fact that the initial condition is bounded below by κ gives

$$I_1 \geqslant \kappa^2.$$

By Proposition 3.2.4 for $t < \left(\frac{1}{2}\right)^\alpha$ the second part becomes

$$\begin{aligned}
I_2 &\geqslant \int_0^t \left(\inf_{z,\, w\in B(0,\,1)} \mathbb{E}|u_s(z)u_s(w)|\right)^{1+\gamma} \int_{B(0,\,1)\times B(0,\,1)} G_{t-s}(x-z)G_{t-s}(y-w)f(z,w)\,\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\geqslant c_1K_f \int_0^t \left(\inf_{z,\, w\in B(0,\,1)} \mathbb{E}|u_s(z)u_s(w)|\right)^{1+\gamma} \mathrm{d}s.
\end{aligned}$$

Letting

$$H(s) := \inf_{x,\, y\in B(0,\,1)} \mathbb{E}|u_s(x)u_s(y)|,$$

and combining the above estimates, we have

$$H(t) \geqslant c^2\kappa^2 + c_1K_f \int_0^t H(s)^{1+\gamma}\,\mathrm{d}s, \quad \text{for} \quad t \leqslant \left(\frac{1}{2}\right)^\alpha.$$

By taking κ big enough, we can make sure that t_0 is as small as we wish by Proposition 3.2.7. This finishes the proof of the proposition. □

Proof of Theorem 3.1.3. We can now easily prove the theorem. From the mild formulation and Proposition 3.3.1, we have

$$
\begin{aligned}
&\mathbb{E}|u_t(x)|^2 \\
&\quad \geqslant c^2\kappa^2 + \int_0^t \int_{B(0,1)\times B(0,1)} G_{t-s}(x-y)G_{t-s}(x-w)f(y,w)\left(\mathbb{E}|u_s(y)u_s(w)|\right)^{1+\gamma} \mathrm{d}y\mathrm{d}w\mathrm{d}s \\
&\quad \geqslant c^2\kappa^2 + \int_{t_0}^t \int_{B(0,1)\times B(0,1)} G_{t-s}(x-y)G_{t-s}(x-w)f(y,w)\left(\mathbb{E}|u_s(y)u_s(w)|\right)^{1+\gamma} \mathrm{d}y\mathrm{d}w\mathrm{d}s \\
&\quad = \infty,
\end{aligned}
$$

when κ is large. □

3.4 Proof of Theorem 3.1.4

Before presenting the proof of our theorem, we need to give some important results given in the propositions bellow. In the remainder of this section, u_t will be the solution to (3.1.15) and the correlation function is always given by the Riesz kernel, that is

$$f(x,y) = \frac{1}{|x-y|^{\omega}}, \quad \omega < d \wedge (\alpha\beta^{-1}).$$

In this case the colored noise converges to the white noise when $\omega \to 1$.

Proposition 3.4.1. *For $x,\, y \in B(0,\, t^{\beta/\alpha})$, there exists a constant c such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} G_t(x-z)G_t(y-w)f(z,w)\,\mathrm{d}z\mathrm{d}w \geqslant \frac{c}{t^{\omega\beta/\alpha}}.$$

Proof. By the bounds given by (3.2.5) we observe

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} G_t(x-z)G_t(y-w)f(z,w)\,\mathrm{d}z\mathrm{d}w \\
&\qquad\qquad \geqslant \int_{B(0,\,t^{\beta/\alpha})\times B(0,\,t^{\beta/\alpha})} G_t(x-z)G_t(y-w)f(z,w)\,\mathrm{d}z\mathrm{d}w \\
&\qquad\qquad \geqslant \frac{c_1}{t^{2d\beta/\alpha}} \int_{B(0,\,t^{\beta/\alpha})\times B(0,\,t^{\beta/\alpha})} f(z,w)\,\mathrm{d}z\mathrm{d}w
\end{aligned}
$$

$$\geqslant \frac{c_1}{t^{2d\beta/\alpha}} \frac{c^*}{t^{\omega\beta/\alpha}} \int_{B(0,\, t^{\beta/\alpha}) \times B(0,\, t^{\beta/\alpha})} \mathrm{d}z\mathrm{d}w$$
$$= \frac{c}{t^{\omega\beta/\alpha}}.$$

The last inequality follows since $|z - w| \leqslant 2t^{\beta/\alpha}$ for $z,\, w \in B(0,\, t^{\beta/\alpha})$. □

The following proposition now easily follows by the last result.

Proposition 3.4.2. *For fixed $t > 0$, we have*

$$\mathbb{E}|u_t(x)u_t(y)| \geqslant ct^{(\alpha-\omega\beta)/\alpha}, \quad \textit{for all} \quad x,\, y \in B(0,\, t^{\beta/\alpha}),$$

where c is some constant.

Proof. Since initial condition is non-negative and $\mathbb{E}|u_t(x)u_t(y)| \geqslant \kappa$, then by the above proposition, we get

$$\mathbb{E}|u_t(x)u_t(y)| \geqslant \kappa^{1+\lambda} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} G_s(x-z)G_s(y-w)f(z,w)\, \mathrm{d}z\mathrm{d}w\mathrm{d}s$$
$$\geqslant ct^{(\alpha-\omega\beta)/\alpha}.$$

□

To give the proof of Theorem 3.1.4 we will need the following proposition.

Proposition 3.4.3. *Fix $t > 0$ and let $t_0 \leqslant t/3$. Then for $x,\, y \in B(0,\, t^{\beta/\alpha})$, we have*

$$\int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)\mathbb{E}|u_s(z)u_s(w)|^{1+\gamma} f(z,w)\, \mathrm{d}z\, \mathrm{d}w \geqslant ct^{2(\alpha-\omega\beta)/\alpha},$$

for some constant c.

Proof. First observe that, if $s \geqslant (t+t_0)/2$, then $s \geqslant t-s+t_0$ and also, if $s \leqslant 3(t+t_0)/4$, then $s \leqslant 3(t-s+t_0)$. Using this, the fact that $\mathbb{E}|u_t(x)u_t(y)| \geqslant \kappa$ and Proposition 3.4.2, we write

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)\mathbb{E}|u_s(z)u_s(w)|^{1+\gamma} f(z,w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \\
&\quad\geqslant \kappa^\gamma \int_0^t s^{(\alpha-2\omega\beta)/\alpha} \int_{B(0,\, s^{\beta/\alpha})\times B(0,\, s^{\beta/\alpha})} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \\
&\quad\geqslant \kappa^\gamma \int_{(t+t_0)/2}^{3(t+t_0)/4} s^{(\alpha-2\omega)/\alpha} \\
&\quad\times \int_{B(0,\,(t+t_0-s)^{\beta/\alpha})\times B(0,\,(t+t_0-s)^{\beta/\alpha})} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s.
\end{aligned}$$

Note that

$$\begin{aligned}
|x-z| &\leqslant t^{\beta/\alpha} + (t-s+t_0)^{\beta/\alpha} \\
&\leqslant (t-s+t_0+s)^{\beta/\alpha} + (t-s+t_0)^{\beta/\alpha} \\
&\leqslant c_1(t-s+t_0)^{\beta/\alpha},
\end{aligned}$$

for some constant c_1. The last inequality is true since $f(t) = t^{\beta/\alpha}$ is increasing for $t > 0$ and $s \leqslant 3(t-s+t_0)$ in our last integral above. By the bound on $G_t(x)$ in (3.2.5), we get the following not sharp bound which is sufficient for our needs:

$$\begin{aligned}
&\int_{(t+t_0)/2}^{3(t+t_0)/4} s^{(\alpha-2\omega\beta)/\alpha} \int_{B(0,\,(t+t_0-s)^{\beta/\alpha})\times B(0,\,(t+t_0-s)^{\beta/\alpha})} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \\
&\qquad\geqslant \int_{(t+t_0)/2}^{3(t+t_0)/4} s^{(\alpha-2\omega\beta)/\alpha} c_1(t-s+t_0)^{-d\beta/\alpha} \\
&\qquad\times \int_{B(0,\,(t+t_0-s)^{\beta/\alpha})\times B(0,\,(t+t_0-s)^{\beta/\alpha})} \mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \\
&\qquad\geqslant c_2 t^{2(\alpha-\omega\beta)/\alpha}.
\end{aligned}$$

□

Proof of Theorem 3.1.4. With the above propositions at hand, we are ready to give the proof of our theorem. By the mild formulation, the fact that initial condition is bounded below and change of variables give

$$
\begin{aligned}
&\mathbb{E}|u_{T+t}(x)u_{T+t}(y)| \\
&\qquad \geqslant \kappa^2 + \int_0^{T+t}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{T+t-s}(x-z)G_{T+t-s}(y-w)\mathbb{E}|u_s(z)u_s(w)|^{1+\lambda} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\qquad = \kappa^2 + \int_0^{T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{T+t-s}(x-z)G_{T+t-s}(y-w)\mathbb{E}|u_s(z)u_s(w)|^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\qquad + \int_0^{t}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{t-s}(x-z)G_{t-s}(y-w)\mathbb{E}|u_{T+s}(z)u_{T+s}(w)|^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\qquad \geqslant \kappa^2 + \int_0^{T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{T+t-s}(x-z)G_{T+t-s}(y-w)\mathbb{E}|u_s(z)u_s(w)|^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\qquad + \int_0^{t}\int_{B(0,1)\times B(0,1)} G_{t-s}(x-z)G_{t-s}(y-w)\mathbb{E}|u_{T+s}(z)u_{T+s}(w)|^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s.
\end{aligned}
\tag{3.4.1}
$$

Take $T \gg 1$ and $t \leqslant T/3$, so that we can use the previous Proposition to bound the second term. To bound the third term, we use similar arguments as in the proof of Theorem 3.1.3. If we let

$$Q(s) := \inf_{x,\, y\in B(0,\,1)} \mathbb{E}|u_{T+s}(x)u_{T+s}(y)|,$$

we observe

$$Q(t) \geqslant \kappa^2 + cT^{2(\alpha-\omega\beta)/\alpha} + c_1\int_0^t Q(s)^{1+\gamma}\,\mathrm{d}s.$$

It suffices to consider, the following differential equation

$$\frac{Q'(t)}{Q(t)^{1+\gamma}} = c_1,$$

with initial condition $Q(0) = \kappa^2 + cT^{2(\alpha-\omega\beta)/\alpha} := A$. Solving this equation, we get

$$\frac{1}{Q(t)^{1+\gamma}} = \frac{1}{A^\gamma} - c_1\gamma t.$$

The blow up occurs at $t = \frac{1}{c_1\gamma A^\gamma}$. That means, as long as κ is strictly positive, we will have blow up of Q for any fixed small time; we just need to take T large enough. To finish the proof we use the mild formulation and the above result to write

$$
\begin{aligned}
&\mathbb{E}|u_{T+t}(x)|^2 \\
&\geqslant c^2\kappa^2 + \int_0^{T+t}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{T+t-s}(x-z)G_{T+t-s}(y-w)(\mathbb{E}|u_s(z)u_s(w)|)^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\geqslant c^2\kappa^2 + \int_T^{T+t}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_{T+t-s}(x-z)G_{T+t-s}(y-w)(\mathbb{E}|u_s(z)u_s(w)|)^{1+\gamma} f(z,w)\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&= \infty,
\end{aligned}
\tag{3.4.2}
$$

when κ is large. □

3.5 Proof of Theorems 3.1.6, 3.1.7 and 3.1.8

The following proposition is crucial in the proof of Theorem 3.1.6.

Proposition 3.5.1. *Under Assumption 3.1.3, there exist $t^*,\ K > 0$ such that for all $t \geqslant t^*$, we have*

$$\inf_{x\in B(0,1)} \mathbb{E}|u_t(x)|^2 = \infty, \quad \textit{for} \quad K_{u_0} > K.$$

Proof. By Walsh isometry, we have

$$\mathbb{E}|u_t(x)|^2 = |(\mathcal{G}u)_t(x)|^2 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}^2(x-y)\mathbb{E}|\sigma(u_s(y))|^2 \,\mathrm{d}y\,\mathrm{d}s.$$

We can always assume t^* to be large. Otherwise, there is nothing to prove. So instead of looking at time t, we will look at $t+t_0$ and fix $t_0 > 0$ later. We have

$$
\begin{aligned}
\mathbb{E}|u_{t+t_0}(x)|^2 &= |(\mathcal{G}u)_{t+t_0}(x)|^2 + \int_0^{t+t_0}\int_{\mathbb{R}^d} G_{t+t_0-s}^2(x-y)\mathbb{E}|\sigma(u_s(y))|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&\geqslant |(\mathcal{G}u)_{t+t_0}(x)|^2 + \int_{t_0}^{t+t_0}\int_{\mathbb{R}^d} G_{t+t_0-s}^2(x-y)\mathbb{E}|\sigma(u_s(y))|^2 \,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

By substituting $S = s - t_0$ in the second part, we obtain

$$\mathbb{E}|u_{t+t_0}(x)|^2 \geqslant |(\mathcal{G}u)_{t+t_0}(x)|^2 + \int_0^t \int_{\mathbb{R}^d} G_{t-S}^2(x-y)\mathbb{E}|\sigma(u_{S+t_0}(y))|^2 \,\mathrm{d}y\,\mathrm{d}S$$

$$:= I_1 + I_2.$$

We will assume that $t < 1$ for most of the rest of the proof. We find a lower bound on I_1 first. Let $x \in B(0,\, 1)$, then we fix t_0 as in Proposition 3.2.2. This gives us

$$I_1 \geqslant cK_{u_0}^2,$$

where the constant c depends on t_0. We now look at the second term:

$$\begin{aligned} I_2 &\geqslant \int_0^t \left(\inf_{y \in B(0,\,1)} \mathbb{E}|u_{S+t_0}(y)|^2 \right)^{1+\gamma} \int_{B(0,\,1)} G_{t-S}^2(x-y)\,\mathrm{d}y\,\mathrm{d}S \\ &\geqslant c_1 \int_0^t \left(\inf_{y \in B(0,\,1)} \mathbb{E}|u_{S+t_0}(x)|^2 \right)^{1+\gamma} \frac{1}{(t-S)^{d\beta/\alpha}}\,\mathrm{d}S \end{aligned}$$

For the last inequality, we used the fact that $t < 1$, the fact that $\{y \in B(0,1) :\ |x-y| < t^{\beta/\alpha}\} \subset \{y \in B(0,1) :\ |x-y| < 1\}$, and the inequality (3.2.5).

Letting $R(S) := \inf_{x \in B(0,\,1)} \mathbb{E}|u_{S+t_0}(x)|^2$, we obtain

$$R(t) \geqslant cK_{u_0}^2 + c_1 \int_0^t \frac{R(S)^{1+\gamma}}{(t-S)^{d\beta/\alpha}}\,\mathrm{d}S \quad \text{for} \quad t \leqslant 1.$$

Now by Proposition 3.2.7 we have the desired result. □

Proof of Theorem 3.1.6. Let $t > t^*$ where t^* is as given in the above proposition. The proof of the theorem now follows from Walsh isometry, Jensen's inequality and Proposition 3.5.1

$$\begin{aligned} \mathbb{E}|u_t(x)|^2 &\geqslant |(\mathcal{G}u)_t(x)|^2 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}^2(x-y) \left(\mathbb{E}|u_s(y))|^2\right)^{1+\gamma} \mathrm{d}y\,\mathrm{d}s \\ &\geqslant |(\mathcal{G}u)_t(x)|^2 + \int_{t^*}^t \int_{B(0,\,1)} G_{t-s}^2(x-y) \left(\mathbb{E}|u_s(y))|^2\right)^{1+\gamma} \mathrm{d}y\,\mathrm{d}s \\ &= \infty. \end{aligned}$$

This follows since the first term of the above display is strictly positive for any $x \in \mathbb{R}^d$. □

Proposition 3.5.2. *Suppose that Assumptions 3.1.2 and 3.1.3 hold. Let u_t be the solution to* (3.1.15). *Then, there exists a $t^* > 0$ such that for all $t \geqslant t^*$, we have*

$$\inf_{x,\, y \in B(0,\,1)} \mathbb{E}|u_t(x)u_t(y)| = \infty, \quad \textit{whenever} \quad K_{u_0} > K,$$

where K is some positive constant.

Proof. We can always assume that t^* to be large like in proof of Proposition 3.5.1. So instead of looking at time t, we will look at $t + t_0$ and fix $t_0 > 0$ later. From the mild formulation and appropriate change of variables as in Proposition 3.5.1, we obtain

$$\begin{aligned}
&\mathbb{E}|u_{t+t_0}(x)u_{t+t_0}(y)| \\
&\quad \geqslant (\mathcal{G}u)_{t+t_0}(x)(\mathcal{G}u)_{t+t_0}(y) \\
&\quad + \int_0^{t+t_0} \int_{\mathbb{R}^d \times \mathbb{R}^d} G_{t+t_0-s}(x-z)G_{t+t_0-s}(y-w)f(z-w)\mathbb{E}|\sigma(u_s(z))\sigma(u_s(w))| \,\mathrm{d}z\mathrm{d}w\mathrm{d}s \\
&\quad \geqslant (\mathcal{G}u)_{t+t_0}(x)(\mathcal{G}u)_{t+t_0}(y) \\
&\quad + \int_0^{t} \int_{B(0,1)\times B(0,1)} G_{t-s}(x-z)G_{t-s}(y-w)f(z-w)\mathbb{E}|u_{s+t_0}(z)u_{s+t_0}(w)|^{1+\gamma} \,\mathrm{d}z\mathrm{d}w\mathrm{d}s.
\end{aligned}$$

The proof essentially follows the same idea as in Proposition 3.5.1. The key idea is to take t_0 as in Proposition 3.2.2 and set

$$G(s) := \inf_{x,\, y \in B(0,\,1)} \mathbb{E}|u_{s+t_0}(x)u_{s+t_0}(y)|.$$

By following the ideas in Proposition 3.3.1, we get

$$G(t) \geqslant cK_{u_0}^2 + c_1 K_f \int_0^t G(s)^{1+\gamma}\mathrm{d}s,$$

valid for a suitable range of t. Now we have the desired result using Proposition 3.2.7. □

Proof of Theorem 3.1.7. With the above Proposition, the proof of theorem is now very similar to that of Theorem 3.1.6. Again by Walsh isometry, we have

$$\mathbb{E}|u_t(x)|^2$$

$$\geqslant |\mathcal{G}u)_t(x)|^2$$
$$+ \int_{t^*}^{t} \int_{B(0,1)\times B(0,1)} G_{t-s}(x-z)G_{t-s}(y-w)f(z-w)(\mathbb{E}|u_s(z)u_s(w)|)^{1+\gamma}\,\mathrm{d}z\mathrm{d}w\mathrm{d}s.$$

Proposition 3.5.2 completes the proof since the first term of the above inequality is strictly positive for any $x \in \mathbb{R}^d$. □

To prove Theorem 3.1.8 we will follow a similar pattern of the proof of the previous results. We emphasize that in the case of (3.1.23), the mild solution in the sense of Walsh [56] is given by

$$u_t(x) = (\mathcal{G}_B u)_t(x) + \int_{\mathbb{R}^d}\int_0^t G_B(t-s, x-y)\sigma(u_s(y))F(\mathrm{d}s\,\mathrm{d}y). \tag{3.5.1}$$

Proof of Theorem 3.1.8. Before giving the proof of our theorem we need the following result.

$$\begin{aligned}
&\mathbb{E}|u_t(x)u_t(y)| \\
&\quad \geqslant (\mathcal{G}_B u)_t(x)(\mathcal{G}_B u)_t(y) \\
&\quad + \int_0^t \int_{B(0,\,R)\times B(0,\,R)} G_B(t-s, x-z)G_B(t-s, y-w)f(z-w)\left(\mathbb{E}|u_s(z))u_s(w)|\right)^{1+\gamma}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \\
&\quad := I_1 + I_2.
\end{aligned}$$

We look at I_1 first. By Proposition 3.2.3, if $x,\, y \in B(0,\, R/2)$ and t is small enough, we have $I_1 \geqslant c_1\kappa^2$. We now turn our attention to the second term.

$$\begin{aligned}
I_2 \geqslant &\int_0^t \left(\inf_{x,\,y\in B(0,\,R/2)} \mathbb{E}|u_s(x)u_s(y)|\right)^{1+\gamma} \\
&\times \int_{B(0,\,R/2)\times B(0,\,R/2)} G_B(t-s, x-z)G_B(t-s, y-w)f(z-w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s.
\end{aligned}$$

Fix $\epsilon = R/4$ and Proposition 3.2.5 with $t \leqslant \left(\frac{R}{4}\right)^{\alpha}$ to obtain

$$\begin{aligned}
&\int_{B(0,\,R/2)\times B(0,\,R/2)} G_B(t-s, x-z)G_B(t-s, y-w)f(z-w)\,\mathrm{d}z\,\mathrm{d}w \\
&\qquad \geqslant c_1 K_f.
\end{aligned}$$

We then have

$$I_2 \geqslant c_2 K_f \int_0^t \left(\inf_{x,\, y \in B(0,\, R/2)} \mathbb{E}|u_s(x)u_s(y)| \right)^{1+\gamma} \mathrm{d}s.$$

If we let

$$H(s) := \inf_{x,\, y \in B(0,\, R/2)} \mathbb{E}|u_s(x)u_s(y)|,$$

then we get

$$H(t) \geqslant c_1\kappa^2 + c_2 K_f \int_0^t H(s)^{1+\gamma}\, \mathrm{d}s.$$

By comparison principle, it is enough to consider

$$\frac{H'(t)}{H(t)^{1+\gamma}} = c_2 K_f,$$

with initial condition $c_1\kappa^2$. Hence the blowup occurs at $t = \frac{1}{(c_1\kappa^2)^\gamma \gamma c_2 K_f}$. Fix any $t_0 < \left(\frac{R}{2}\right)^\alpha$ and take $\kappa_0 > \frac{1}{c_1^{0.5}(\gamma c_2 K_f t_0)^{1/2\gamma}}$ such that for $\kappa > \kappa_0$, $H(s) = \infty$ for all $s \geqslant t_0$. Using the above result we can easily prove our result. Observe that

$$\begin{aligned}
&\mathbb{E}|u_t(x)|^2 \\
&\quad \geqslant |(\mathcal{G}_B u)_t(x)|^2 \\
&\quad + \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} G_B(t-s, x-z) G_B(t-s, x-w) f(z-w) \left(\mathbb{E}|(u_s(z))(u_s(y))|\right)^{1+\gamma} \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}s \\
&\quad = \infty.
\end{aligned}$$

This is true since all the relevant terms involved in the above inequality are positive. □

Proof of Theorem 3.1.11. Let $B := B(0, R)$. Suppose by contradiction that there is a random field solution of equation (3.1.29). This means that

$$\sup_{t\geqslant 0}\sup_{x\in B}\mathbb{E}|u_t(x)|^2 < \infty.$$

Then since $\mathbb{E}|u_t(x)| \leqslant (\mathbb{E}|u_t(x)|^2)^{1/2}$ we have

$$\sup_{t\geqslant 0}\sup_{x\in B}\mathbb{E}|u_t(x)| < \infty.$$

The eigenfunctions $\{\phi_n : n \in \mathbb{N}\}$ of fractional Laplacian $-(-\Delta)^{\alpha/2}$ in B form an orthonormal basis for $L^2(B)$. It is well-known that the first eigenfunction $\phi_1(x) > 0$ for all $x \in B$. Now we also have

$$\sup_{t\geqslant 0} Q(t) := \sup_{t\geqslant 0}\int_B \mathbb{E}[|u_t(x)|]\phi_1(x)\mathrm{d}x \leqslant \sup_{t\geqslant 0}\sup_{x\in B}\mathbb{E}|u_t(x)|\int_B \phi_1(x)\mathrm{d}x < \infty. \tag{3.6.1}$$

Next, we start with taking expectation of both sides of equation (3.1.30) to get

$$\mathbb{E}[u_t(x)] = (\mathcal{G}_B u_0)_t(x) + \int_{B(0,R)}\int_0^t \mathbb{E}[b(u_s(y))]G_B(t-s,x,y)\mathrm{d}s\,\mathrm{d}y. \tag{3.6.2}$$

Multiply both sides of (3.6.2) by the first eigenfunction $\phi_1(x)$ of $-(-\Delta)^{\alpha/2}$ on the ball B and integrating over B we get

$$\begin{aligned}
Q(t) &:= \int_B \mathbb{E}[|u_t(x)|]\phi_1(x)\mathrm{d}x \\
&\geqslant \int_B (\mathcal{G}_B u_0)_t(x)\phi_1(x)\mathrm{d}x + \int_B \left[\int_B\int_0^t \mathbb{E}[b(u_s(y))]G_B(t-s,x-y)\mathrm{d}s\,\mathrm{d}y\right]\phi_1(x)\mathrm{d}x \\
&\geqslant \int_B (\mathcal{G}_B u_0)_t(x)\phi_1(x)\mathrm{d}x + \int_B \left[\int_B\int_0^t \mathbb{E}[|u_s(y)|^{1+\eta}]G_B(t-s,x-y)\mathrm{d}s\,\mathrm{d}y\right]\phi_1(x)\mathrm{d}x \\
&\geqslant \int_B (\mathcal{G}_B u_0)_t(x)\phi_1(x)\mathrm{d}x + \int_B \left[\int_B\int_0^t \Big(\mathbb{E}[|u_s(y)|\Big)^{1+\eta}G_B(t-s,x-y)\mathrm{d}s\,\mathrm{d}y\right]\phi_1(x)\mathrm{d}x, \\
&= I_1 + I_2.
\end{aligned} \tag{3.6.3}$$

where the last inequality follows from Jensen's inequality.

We only give the proof in one of the cases below for the convenience of the reader. For other cases, see Asogwa et al. [3]

The eigenfunctions $\{\phi_n : n \in \mathbb{N}\}$ of fractional Laplacian $-(-\Delta)^{\alpha/2}$ in B form an orthonormal basis for $L^2(B)$. We have an eigenfunction expansion of the kernel

$$G_B(t,x,y) = \sum_{n=1} E_\beta(-\mu_n t^\beta)\phi_n(x)\phi_n(y). \tag{3.6.4}$$

See, for example, Chen et al. [18] and Meerschaert et al. [43]. From this equation we can easily get

$$\int_B G_B(t,x,y)\phi_1(x)\mathrm{d}x = E_\beta(-\mu_1 t^\beta)\phi_1(y) \tag{3.6.5}$$

It is a well-know fact that $\phi_1(x) > 0$ for $x \in B$. Now consider first I_1, since $u_0(x) \geqslant \kappa$ by assumption we obtain

$$\begin{aligned} I_1 &\geqslant \kappa \int_B \int_B G_B(t,x,y)\phi_1(x)\mathrm{d}y\mathrm{d}x = \kappa \int_B E_\beta(-\mu_1 t^\beta)\phi_1(y)\mathrm{d}y \\ &= \kappa E_\beta(-\mu_1 t^\beta)\int_B \phi_1(y)dy = C_{\kappa,\phi_1} E_\beta(-\mu_1 t^\beta) \end{aligned} \tag{3.6.6}$$

Next we estimate I_2. By Fubini theorem and equation (3.6.5)

$$I_2 = \int_0^t \int_B E_\beta(-\mu_1(t-s)^\beta)\phi_1(y)\mathbb{E}[|u_s(y)|^{1+\eta}]\mathrm{d}y\,\mathrm{d}s \tag{3.6.7}$$

Applying the Jensen's inequality twice using the fact that $0 < A := \int_B \phi_1 \mathrm{d}x < \infty$, and by using the fact that the Mittag-leffler function is a decreasing function, we get

$$
\begin{aligned}
I_2 &\geqslant \int_0^t E_\beta(-\mu_1(t-s)^\beta) A\left[\int_B \mathbb{E}[|u_s(y)|]\frac{\phi_1(y)}{A}\mathrm{d}y\right]^{1+\eta}\mathrm{d}s \\
&= A^{-\eta}\int_0^t E_\beta(-\mu_1(t-s)^\beta)\left[\int_B \mathbb{E}[|u_s(y)|]\phi_1(y)\mathrm{d}y\right]^{1+\eta}\mathrm{d}s \\
&\geqslant A^{-\eta}E_\beta(-\mu_1 t^\beta)\int_0^t \left[\int_B \mathbb{E}[|u_s(y)|]\phi_1(y)\mathrm{d}y\right]^{1+\eta}\mathrm{d}s
\end{aligned}
\tag{3.6.8}
$$

We have the uniform estimate of Mittag-Leffler function in [54, Theorem 4]

$$
\frac{1}{1+\Gamma(1-\beta)t} \leqslant E_\beta(-t) \leqslant \frac{1}{1+\Gamma(1+\beta)^{-1}t} \text{ for any } t>0. \tag{3.6.9}
$$

Using equation (3.6.9) we get

$$
\begin{aligned}
Q(t) &= \int_B \mathbb{E}[|u_t(x)|]\phi_1(x)dx \\
&\geqslant C_1\frac{1}{1+\mu_1\Gamma(1-\beta)t^\beta} + C_2\frac{1}{1+\mu_1\Gamma(1-\beta)t^\beta}\int_0^t Q(s)^{1+\eta}\,\mathrm{d}s
\end{aligned}
\tag{3.6.10}
$$

Hence for $t \geqslant 1$ we get

$$
\begin{aligned}
Q(t) &= \int_B \mathbb{E}[|u_t(x)|]\phi_1(x)dx \\
&\geqslant C_3 t^{-\beta} + C_4 t^{-\beta}\int_1^t Q(s)^{1+\eta}\,\mathrm{d}s
\end{aligned}
\tag{3.6.11}
$$

Set $P(t) = t^\beta Q(t)$ and multiply both sides of equation (3.6.11) by t^β to get

$$
\begin{aligned}
P(t) &= t^\beta \int_B \mathbb{E}[|u_t(x)|]\phi_1(x)dx \\
&\geqslant C_3 + C_4\int_1^t \frac{(s^\beta Q(s))^{1+\eta}}{s^{\beta(1+\eta)}}\,\mathrm{d}s \\
&= C_3 + C_4\int_1^t \frac{P(s)^{1+\eta}}{s^{\beta(1+\eta)}}\,\mathrm{d}s
\end{aligned}
\tag{3.6.12}
$$

Now we have three cases: When $\beta(1+\eta) < 1$, $\beta(1+\eta) > 1$ and $\beta(1+\eta) = 1$. We only give the proof in the first case $\beta(1+\eta) < 1$. In this case it is enough to consider the following equation

$$\frac{P'(t)}{P^{1+\eta}(t)} = \frac{C_4}{t^{\beta(1+\eta)}}, \;\; t > 1 \;\text{ and }\; P(1) = C_3.$$

This equation has a solution that satisfies

$$P^{-\eta}(t) = C_3^{-\eta} - \frac{\eta C_4}{1-\beta(1+\eta)}\left(t^{1-\beta(1+\eta)} - 1\right) \tag{3.6.13}$$

this blows up at $t = t_0 > 1$ that makes the right hand side of the last equation zero;

$$t_0^{1-\beta(1+\eta)} = 1 + \frac{C_3^{-\eta}}{\eta C_4}(1-\beta(1+\eta))$$

Since the solution $P(t)$ is a non-decreasing function, $P(t) = \infty$ for all $t \geqslant t_0$.

The other cases are handled similarly.

Hence by Theorem 3.7.2 $V(t,x) = \mathbb{E}[|u_t(x)|]$ blows up in finite time. This is a contradiction to inequality (3.6.1).

□

Proof of Theorem 3.1.9. Now suppose by contradiction that there is a random field solution of equation (3.1.25). This means that

$$\sup_{t\geqslant 0}\sup_{x\in\mathbb{R}^d} \mathbb{E}|u_t(x)|^2 < \infty.$$

Then since $\mathbb{E}|u_t(x)| \leqslant (\mathbb{E}|u_t(x)|^2)^{1/2}$ we have

$$\sup_{t\geqslant 0}\sup_{x\in B(0,R)} \mathbb{E}|u_t(x)| < \infty. \tag{3.6.14}$$

Now we start with taking expectation of the both sides of equation (3.1.26) to get

$$\mathbb{E}[u_t(x)] = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy + \int_{\mathbb{R}^d}\int_0^t \mathbb{E}[b(u_s(y))]G(t-s,x,y)\mathrm{d}s\,\mathrm{d}y. \tag{3.6.15}$$

Hence by the Jensen's inequality we get

$$\begin{aligned}\mathbb{E}[|u_t(x)|] &\geqslant (G_t * u_0)(x) + \int_{\mathbb{R}^d}\int_0^t [\mathbb{E}|u_s(y)|]^{1+\eta} G(t-s,x,y)\mathrm{d}s\,\mathrm{d}y \\ &= I + II.\end{aligned} \tag{3.6.16}$$

We give the proof when $\inf_{x\in\mathbb{R}^d} u_0(x) \geqslant \kappa$: in this case, $I \geqslant \kappa$. Since $G(t-s,x,y)$ is a probability density function on $\mathbb{R}^d$ we get

$$II \geqslant \int_0^t F(s)^{1+\eta}\mathrm{d}s \tag{3.6.17}$$

where $F(s) = \inf_{y\in\mathbb{R}^d}\mathbb{E}|u_s(y)|$. Hence we obtain

$$F(t) \geqslant \kappa + \int_0^t F(s)^{1+\eta}\mathrm{d}s$$

Now if $\kappa > 0$, then blow up happens at some $t_0 = \kappa^{-\eta}/\eta$ for any $\eta > 0$. Hence $V(t,x) = \mathbb{E}[|u_t(x)|]$ blows up in finite time. Hence we have a contradiction to equation (3.6.14).

The other case in the theorem is more complicated and it follows from Theorem 3.7.1 by making the following observation: From equation (3.6.15), the function $V(t,x) = \mathbb{E}[|u_t(x)|]$ is a super solution of the following deterministic equation (this follows by using the Fractional Duhamels' principle in the reverse order!)

$$\begin{aligned}&\partial_t^\beta V(t,x) = -\nu(-\Delta)^{\alpha/2}V(t,x) + I^{1-\beta}[(V(t,x))^{1+\eta}] \quad t>0 \quad \text{and} \quad x\in\mathbb{R}^d; \\ &V(0,x) = u_0(x), \quad x\in\mathbb{R}^d.\end{aligned} \tag{3.6.18}$$

By Theorem 3.7.1 $V(t,x) = \mathbb{E}[|u_t(x)|]$ blows up in finite time. Hence we have a contradiction to equation (3.6.14).

□

3.7 Appendix

In this section, we consider the following space-time fractional reaction-diffusion type equations in $(d+1)$ dimension:

$$\begin{aligned} \partial_t^\beta V(t,x) &= -\nu(-\Delta)^{\alpha/2}V(t,x) + I^{1-\beta}[b(V(t,x))] \quad t>0 \quad \text{and} \quad x\in\mathbb{R}^d, \\ V(0,x) &= V_0(x) \;\; x\in\mathbb{R}^d. \end{aligned} \tag{3.7.1}$$

The operator $-(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian, the generator of a α-stable Lévy process. The initial condition will always be assumed to be a non-negative bounded deterministic function. The function b is a locally Lipschitz function.

For every given $T>0$, a mild solution to (3.7.1) on $(0,T)$ is any V that satisfies the following evolution equation–this is also called the mild/integral solution of equation (3.7.1)– which follows by the fractional Duhamel's principle [55]

$$V(t,x) = \int_{\mathbb{R}^d} G_t(x-y)V_0(y)\,\mathrm{d}y + \int_{\mathbb{R}^d}\int_0^t G_{t-s}(x-y)b(V(s,y))\mathrm{d}s\,\mathrm{d}y, \tag{3.7.2}$$

for $0<t<T$ where $G_t(\cdot)$ denotes the density of the time changed process X_{E_t}.

Theorem 3.7.1 (Theorem 1.1 in Asogwa et al. [3])**.** *Suppose that $0<\eta\leqslant\alpha/\beta d$ and $V_0\not\equiv 0$, then there is no global solution to* (3.7.1) *in the sense that there exists a $t_0>0$ such that $V(t,\,x)=\infty$ for all $t>t_0$ and $x\in\mathbb{R}^d$.*

Next result gives conditions for non-existence of global mild solutions in bounded domains.

Theorem 3.7.2 (Theorem 1.4 in Asogwa et al. [3])**.** *Suppose b satisfies the conditions in Assumption 3.1.5. Fix $R>0$ and consider*

$$\begin{aligned} \partial_t^\beta V(t,x) &= -\nu(-\Delta)^{\alpha/2}V(t,x) + I^{1-\beta}[b(V(t,x))] \quad t>0 \quad \textit{and} \quad x\in B(0,\,R), \\ V(t,x) &= 0 \;\; x\in B(0,\,R)^C \\ V(0,x) &= V_0(x) \;\; x\in B(0,\,R). \end{aligned} \tag{3.7.3}$$

Here $-(-\Delta)^{\alpha/2}$ *denotes the generator of* α*-stable Lévy process killed upon exiting the ball* $B(0, R)$*. Suppose that* $0 < \eta < 1/\beta - 1$*, then there is no global solution to* (3.7.3) *whenever* $K_{V_0,\phi_1} := \int_B V_0(x)\phi_1(x)dx > 0$*. For any* $\eta > 0$*, there is no global solution whenever* $K_{V_0,\phi_1} > 0$ *is large enough.*

The mild solution of equation (3.7.3) is given by using the fractional Duhamel's principle [55] as follows

$$V(t,x) = \int_{B(0,R)} G_B(t,x,y)V_0(y)\mathrm{d}y + \int_{B(0,R)} \int_0^t b(V(s,y))G_B(t-s,x,y)\mathrm{d}s\,\mathrm{d}y, \tag{3.7.4}$$

where $G_B(t,x,y)$ is the density of $X(E_t)$ killed on the exterior of B.

Chapter 4

On Going and Future Work

4.1 Large space, fixed time properties:space-time white noise

Recently, Conus, Joseph and Khoshnevisan [21] established the large x-behavior of $u_t(x)$ at a fixed time $t > 0$. Consider the parabolic Anderson model

$$\partial_t u(t,x) = \frac{\kappa}{2}\Delta u(t,x) + u\,\dot{W}\,(t,x);\;\; u(0,x) = 1, \tag{4.1.1}$$

Outside a P-null set $u_t(x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^d$.

This follows from Mueller's Comparison principle. Hence we can replace $\log|u_t(x)|$ by $\log u_t(x)$.

The random field $h_t(x) := \log u_t(x)$ is the "Cole-Hopf solution to the KPZ equation" of statistical mechanics, Kardar, Parisi and Zhang (1986):

$$\partial_t h = \frac{\kappa}{2}\partial_x^2 u(t,x) - [\partial_x h]^2 + \dot{W}\,(t,x).$$

Since $u_t(x)$ is nonnegative we get that $||u_t(x)||_1 = \mathbb{E}(u_t(x)) = 1$ for all $t \geqslant 0$ and $x \in \mathbb{R}$. Hence by Fatou's lemma we get

$$\mathbb{E}(\liminf_{|x|\to\infty} |u_t(x)|) \leqslant 1.$$

Therefore $\liminf_{|x|\to\infty} |u_t(x)|$ is finite a.s.

As a corollary we obtain

$$\liminf_{|x|\to\infty} \frac{\log|u_t(x)|}{(\log|x|)^{2/3}} = 0 \;\text{ a.s.}$$

Large space, fixed time properties: spatially colored noise

$$\partial_t u(t,x) = \frac{\kappa}{2}\Delta u(t,x) + u\ \dot{F}\ (t,x);\ \ u(0,x) = u_0(x), \tag{4.1.2}$$

where $\dot{F}(t,x)$ is a spatially colored noise with $x \in \mathbb{R}^d$.

Theorem 4.1.1 (Conus, Joseph, Khoshnevisan, and Shiu; 2012)**.** *If $f(x) = ||x||^{-\gamma}$ for some $\gamma \in (0, d\wedge 2)$ then for every $t > 0$ there exists positive and finite constant C_1, C_2–depending only on (t, d, γ)– such that*

$$\frac{C_1}{\kappa^{\gamma/(4-\gamma)}} < \limsup_{||x||\to\infty} \frac{\log|u_t(x)|}{(\log||x||)^{2/(4-\gamma)}} < \frac{C_2}{\kappa^{\gamma/(4-\gamma)}} \ \text{ a.s.}$$

(i) **Large-Space Asymptotic behavior :** We intend to extend the work of Conus, Joseph and Khoshnevisan [21] to fractional SPDE. In this direction we have the following preliminary results

Theorem 4.1.2. *There exists a constant $L > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\bigg(|u_t(x)|^k\bigg) \leqslant L^k \exp(Lk^{1+\alpha/(\alpha-\beta d)}t).$$

For every fixed t, we get

$$E\left[\exp\left(\gamma\left(\log_+|u_t(x)|\right)^{\frac{2\alpha-\beta d}{\alpha}}\right)\right] < \infty \textit{ for some } \gamma > 0. \tag{4.1.3}$$

Theorem 4.1.3.

$$-\infty < \liminf_{z\to\infty} \frac{\log P(|u_t(x)| > z)}{(\log z)^{(2\alpha-\beta d)/\alpha}} \leqslant \limsup_{z\to\infty} \frac{\log P(|u_t(x)| > z)}{(\log z)^{(2\alpha-\beta d)/\alpha}} < 0,$$

uniformly for all $x \in \mathbb{R}$.

Proposition 4.1.4. *Let $d < \min\{2, \beta^{-1}\}\alpha$. If $\epsilon_0 := \inf_{z\in\mathbb{R}^d}\sigma(z) > 0$, then for all $t > 0$,*

$$\inf_{x\in\mathbb{R}^d} E(|u_t(x)|^{2k}) \geqslant (o(1) + \sqrt{2})(\mu_t k)^k \quad \textit{as} \ \ k \to \infty, \tag{4.1.4}$$

where the ”$o(1)$” *term depends only on* k*, and*

$$\mu_t := \frac{2}{e} \cdot \frac{\epsilon_0^2 C^* t^{1-(\beta d)/\alpha}}{1-(\beta d)/\alpha}. \tag{4.1.5}$$

Proposition 4.1.5. *Let* $d < \min\{2, \beta^{-1}\}\alpha$*. If there exists* $\epsilon_0 > 0$ *such that* $\sigma(x) \geqslant \epsilon_0$ *for all* $x \in \mathbb{R}^d$*, then there exists a universal constant* $C \in (0, \infty)$ *that depends only on Lip, and* $\sup_{z \in \mathbb{R}^d} |u_0(z)|$ *such that for all* $t > 0$*,*

$$\liminf_{\lambda \to \infty} \frac{1}{\lambda^{2+\frac{2\alpha}{\alpha-\beta d}}} \inf_{x \in \mathbb{R}^d} \log P(|u_t(x) \geqslant \lambda) \geqslant \frac{-C}{\epsilon_0^{2+\frac{2\alpha}{\alpha-\beta d}} t^{\frac{\alpha-\beta d}{\alpha}}} \tag{4.1.6}$$

4.2 Growth Indices

For $\beta = 1$, $\frac{1}{2} < \alpha < 1$ and $d = 1$, Le Chen and Robert C. Dalang [15] proved that the distances to the origin of the farthest high peaks of the absolute moments of solutions of (3.1.2) grow exponentially with time. So, we intend to extend this result to the case $\beta \in (0, 1)$, $\alpha \in (0, 2)$, and $d < \min\{2, \beta^{-1}\}\alpha$.

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