Phase transition for fractional stochastic partial differential equations in a bounded domain.

by

Ngartelbaye Guerngar

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 3, 2019

Keywords: SPDE, nonlinear stochastic heat equation, white noise, colored noise, Lyapunov exponents, noise level

Approved by

Erkan Nane, Chair, Associate Professor, Department of Mathematics and Statistics
Jerzy Szulga, Professor, Department of Mathematics and Statistics
Ming Liao, Professor, Department of Mathematics and Statistics
Dmitry Glotov, Associate Professor, Department of Mathematics and Statistics
William Powell, Professor, Department of Music
George Flower, Professor, Dean of Graduate School

Abstract

In this dissertation, we study several stochastic partial differential equations (SPDEs) in the open and bounded domain D, subset of $\mathbb{R}^d$ for $d \geq 1$, driven by a multiplicative noise. We are interested in bounds and asymptotic properties of the random field solution.

We study the nonlinear stochastic fractional heat equation driven by three types of noise. Existence and uniqueness of the solution is proved using a Picard iteration scheme. Upper and lower bounds on all p^{th} moments, for $p \geq 2$, of the solution are obtained when the noise is spatially homogeneous (or spatially colored) with the space correlation function given by the Riesz kernel and when the noise is space-time homogeneous, with the time correlation function given by the fractional Brownian motion (fBm) while the space correlation function is given by the Riesz kernel in space. We also show that under exterior boundary conditions, in the long run, the p^{th}-moment of the solution grows exponentially fast for large values of the noise level. However, for small values of the noise level, we observe eventually an exponential decay of the p^{th}-moment of this solution.

Acknowledgments

First and foremost, I give thanks to the Lord God for giving me the wisdom and strength to complete this journey.

I would like to thank my advisor Dr. Erkane Nane for his patience with every attempt at understanding what I presented to him. Because of his motivational ability, I have made an expansive growth in my mathematical maturity. I would also like to thank Professor Jerzy Szulga, Professor Ming Liao and Dr. Dmitry Glotov for agreeing to be on my committee. Many thanks to Professor William Powell for taking some time off his busy schedule to be my University Reader. I don't forget Professor Ash Abebe for enhancing my interest in Statistics and always being there when I needed some advice. I would also like to thank my Mathematics and Statistics professors for providing me with inspiration and the necessary "tools" along the way.

I would like to thank my family: my late father Guerngar, my mother Blandine, my brothers Roland, Rodrigue and Anicet, my sisters Rosine and Benedicte, and my adorable nephew and nieces Stephane, Stella, Alexia, Aaricia, Tracy and Jenny, for the unwavering support throughout my educational odyssey albeit the distance. I also thank my girlfriend Lucy for her emotional support during those hard times.

I would also like to thank the late Dr. Gnedbaye Allahtan Victor, a mentor and great friend of mine–sadly, he will never read this milestone accomplishment.

Finally, to you the reader, I would also like to thank you, for taking the time to look at my thoughts and work.

Table of Contents

Acronyms

i.e that is

w.r.t with respect to

$\mathbb{P}$ probability measure

a.s almost surely

resp. respectively

c, C a positive constant whose value is irrelevant

CDF cumulative distribution function

$\mathbb{E}$ expected value, mathematical expectation

Cov covariance

Var variance

RF random field

GRF Gaussian randon field

fBm fractional Browinan motion

r.v random variable

Id identity map

iff if and only if

PDE partial differential equation

SPDE stochastic partial differential equation

PAM Parabolic Anderson Model

w.l.o.g without loss of generality

$\boldsymbol{a \wedge b}, \mathbf{min}(\boldsymbol{a}, \boldsymbol{b})$ minimum of a and b

$\boldsymbol{a \vee b}, \mathbf{max}(\boldsymbol{a}, \boldsymbol{b})$ maximum of a and b

$\boldsymbol{A^c}$ complement of the set A in $\mathbb{R}^d$

$\mathbf{1}_A$ characteristic function of A

$\boldsymbol{\partial_t}$ partial derivative w.r.t the variable t

$|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$: the Euclidean norm of the vector $x \in \mathbb{R}^d$

$\boldsymbol{B_R(0)} = \{x \in \mathbb{R}^d : |x| < R\}$: the open ball of radius R centered at the origin

$|\boldsymbol{A}|$ volume of the set A, Lebesgue measure of A

$\boldsymbol{C_0^\infty}$ set of infinitely-differentiable functions that have compact supports

$\boldsymbol{\Gamma(\cdot)}$ Euler's gamma function

$\boldsymbol{B(\cdot,\cdot)}$ Euler's beta function

$\mathscr{L}(\mathbb{R}^{\boldsymbol{d}})$ ring of all Borel-measurables subsets of $\mathbb{R}^d$ that have finite Lebesgue measure.

$\mathscr{S}(\mathbb{R}^{\boldsymbol{d}})$ set of all simple functions defined on $\mathbb{R}^d$

$\hat{\boldsymbol{f}}$ Fourier transform of the function f

$\boldsymbol{\delta_0}$ Dirac delta distribution

$\boldsymbol{\sigma(X)}$ $\sigma-$ field generated by the process X

$\boldsymbol{L^p}$ Lebesgue space

$\boldsymbol{H_n}$ n-th Hermite polynomial

This thesis is dedicated to the memories of my father Guerngar and Gnedbaye A. Victor.

Chapter 1

Introduction

Stochastic Partial Differential Equations(SPDEs) have been studied a lot recently due to many challenging open problems in the area but also due to their deep applications in disciplines that range from applied mathematics, statistical mechanics, and theoretical physics, to theoretical neuroscience, theory of complex chemical reactions [including polymer science], fluid dynamics, and mathematical finance, see for example [23] for an extensive list of literature devoted to the subject. On the other hand, SPDEs driven by a random noise which is white in time but colored in space have increasingly received a lot of attention recently, following the foundational work of [11]. One difference with SPDEs driven by space-time white noise is that they can be used to model more complex physical phenomena which are subject to random perturbations. Two phenomena of interest are usually observed when studying these SPDEs: ”intermittency” and ”phase transitions”. See for example [1], [2], [3], [4], [16] , [19] and [20] for the former and [14], [18], [19], [26] and [38] for the latter.

In this thesis, we consider the fractional stochastic heat equation driven by a space-time colored noise on $D := B_R(0)$, the open ball of radius R centered at the origin in $\mathbb{R}^d,\ d \geq 1$, with zero exterior Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial}{\partial t} u_t(x) = -(-\Delta)^{\alpha/2} u_t(x) + \xi\sigma(u_t(x))\dot{F}(t,x) \quad x \in D,\ \ t > 0, \\ u_t(x) = 0 \quad x \in D^c, \end{cases} \tag{1.0.1}$$

where $\alpha \in (0,2]$, $-(-\Delta)^{\alpha/2}$ is the L^2-generator of a symmetric $\alpha-$stable process in $\mathbb{R}^d$ "killed" upon exiting the domain D and can be written in the form

$$-(-\Delta)^{\frac{\alpha}{2}}u(x) = c(\alpha,d)\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}}\frac{\big(u(y)-u(x)\big)dy}{|y-x|^{d+\alpha}}.$$

The initial condition $u_0(\cdot)$ is a bounded and nonnegative function. The coefficient ξ denotes the level of the noise; $\sigma:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz function satisfying some growth conditions. When $\sigma = Id$, the identity map, the resulting equation is called the Parabolic Anderson Model (PAM) and has been studied extensively in [1, 2, 4]. The mean zero Gaussian process $\dot{F}$ is a space-time *colored noise*, i.e

$$\mathbb{E}\Big(\dot{F}(t,x)\dot{F}(s,y)\Big) = \gamma(t-s)\Lambda(x-y), \tag{1.0.2}$$

where $\gamma:\mathbb{R}\to\mathbb{R}_+$ and $\Lambda:\mathbb{R}^d\to\mathbb{R}_+$ are general nonnegative and nonnegative definite (generalized) functions satisfying some integrability conditions. The Fourier transform of the latter, $\hat{\Lambda} = \mu$ is a tempered measure. When $\gamma = \delta_0$ and $\Lambda = \delta_0$, where δ_0 is the Delta-Dirac distribution, the noise is said to be a space-time *white noise*.

When $\gamma = \delta_0$, following Dalang [11], it is well-known that, if the spectral measure satisfies condition (1.0.3) (known as Dalang's condition):

$$\int_{\mathbb{R}^d}\frac{\mu(\zeta)}{1+|\zeta|^\alpha} < \infty, \tag{1.0.3}$$

then there exists a unique random field solution of (1.0.1). We provide the proof of existence and uniqueness of the random field solution in Chapter 5.

Some examples of space correlation functions satisfying condition (1.0.3) include

- **Space-time white noise:** $\Lambda = \delta_0$ in which case $\mu(d\zeta) = d\zeta$ and (1.0.3) holds only when $\alpha > d$ which implies $d = 1$ and $1 < \alpha \leq 2$.

- **Riesz Kernel:** $\Lambda(x) = |x|^{-\beta},\ 0<\beta<d$. Here, $\mu(d\zeta) = c|\zeta|^{-(d-\beta)}d\zeta$ and (1.0.3) holds whenever $\beta < \alpha$.

- **Bessel kernel:** $\Lambda(x) = \int_0^\infty y^{\frac{\eta-d}{2}} e^{-y} e^{-\frac{|x|^2}{4y}} dy$. In this case, $\mu(d\zeta) = c(1+|\zeta|^2)^{-\frac{\eta}{2}} d\zeta$ and (1.0.3) implies $\eta > d - \alpha$.

- **Fractional Kernel:** $\Lambda(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$. $\mu(\zeta) = c\prod_{i=1}^d |x_i|^{1-2H_i} d\zeta$ and (1.0.3) holds whenever $\sum_{i=1}^d H_i > d - \frac{\alpha}{2}$.

We refer the interested reader to [17] for more examples of such functions.

Since stable Lévy processes will be mentioned many times throughout this thesis, we give a brief description of symmetric stable Lévy motions (processes) in the next few lines. For more general and detailed results about stable processes, we refer the reader to [35]. The next few definitions give an exposition to the Theory of Probability.
Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space and let $\mathcal{B}$ be the Borel $\sigma-$ field on $[0,\infty)$.

Definition 1.0.1. *A stochastic process, denoted $X_t(\omega)$ or just X_t, is a map: $\mathbb{R}_+ \times \Omega \to \mathbb{R}$, that is measurable w.r.t the product $\sigma-$ field $\mathcal{B} \times \mathcal{F}$.*

Definition 1.0.2. *Two (stochastic) processes X_t and Y_t defined on the same index set T are versions of one another if $X_t = Y_t$ a.s for each $t \in T$.*
In the special case that $T = \mathbb{R}^d$ or $\mathbb{R}_+$, we note that two continuous or right-continuous versions X and Y of the same process are indistinguishable, in the sense that $X \equiv Y$. In general, the latter notion is clearly stronger.

Definition 1.0.3. *Fix an arbitrary index set $T \subset \bar{\mathbb{R}}$. A filtration on T is a nondecreasing family of $\sigma-$ fields $\mathcal{F}_t \subset \mathcal{F},\ t \in T$.*
We say that a process X is adapted to the filtration $\{\mathcal{F}_t\}_{t\in T}$ if X_t is $\mathcal{F}_t-$ measurable for every $t \in T$.

Definition 1.0.4. *A (continuous-time) martingale is an integrable and adapted process M satisfying, for all $s \leq t$,*

$$M_s = \mathbb{E}\big[M_t|\mathcal{F}_s\big] \ \ a.s.$$

Definition 1.0.5. *A stochastic process* $\{X_t\}_{t\geq 0}$ *is a one-dimensional Brownian motion or Wiener process (started at* 0*) if*

(1) $X_0 = 0$ *a.s.*

(2) $X_t - X_s$ *is a mean zero Gaussian r.v with Var*$(X_t - X_s) = |t-s|$

(3) for all $s < t$, $X_t - X_s$ *is independent of* $\sigma(X_r; r \leq s)$*, where* $\sigma(X_r; r \leq s)$ *is the smallest* $\sigma-$ *field w.r.t which each* X_r *is measurable,* $r \leq s$.

Note also that for a Brownian motion, the covariance function is given by

$$Cov\Big(X_t, X_s\Big) = \frac{1}{2}\Big(t + s - |t-s|\Big) \quad s, t > 0. \tag{1.0.4}$$

Definition 1.0.6. *A r.v* X *is said to be symmetric* $\alpha-$ *stable if there are parameters* $0 < \alpha \leq 2$ *and* $\tau \geq 0$ *such that its characteristic function is given by*

$$\mathbb{E}e^{i\theta X} = e^{-\tau^{\alpha}|\theta|^{\alpha}},$$

and we will denote that by $X \sim S(\alpha, \tau)$*. Note that when* $\alpha = 2$, X *is a Gaussian r.v.*

Definition 1.0.7. *Let* $0 < \alpha \leq 2$*. A stochastic process* $\{X_t\}_{t\geq 0}$ *is called symmetric* α*-stable Lévy motion (process) if*

(1) $X_0 = 0$ *a.s.*

(2) X *has independent increments.*

(3) $X_t - X_s \sim S\big(\alpha, |t-s|^{1/\alpha}\big)$ *for all* $s, t \geq 0$

When $\alpha = 2$, *this process correspond to the famed Wiener process or Browian motion.*

Understanding the noise is essential in solving our main problem. Therefore, we give a few details about the noise below.

Let $\mathscr{L}(\mathbb{R}^d)$ denote the ring of all Borel-measurables subsets of $\mathbb{R}^d$ that have finite Lebesgue measure.

Definition 1.0.8 (Wiener, 1923)**.** *A white noise on $\mathbb{R}^d$ is a mean-zero set-indexed Gaussian random field (GRF) $\{W(A)\}_{A\in\mathscr{L}(\mathbb{R}^d)}$ with covariance*

$$\mathbb{E}\Big(W(A_i)W(A_j)\Big) = |A_i \cap A_j| \quad (A_l \in \mathscr{L}(\mathbb{R}^d)),$$

where $|\cdot|$ denotes the d-dimensional Lebesgue measure.

It is not hard to show that a *white noise* exists and is an $L^2(\Omega)$-valued countably-additive measure on $\mathscr{L}(\mathbb{R}^d)$. It is actually possible to construct interesting processes from a white noise, as shown in the next few examples. We also refer the reader to [23] for more examples.

Example 1.0.9. *Let W denote a white noise on $\mathbb{R}$. Then the map $t \mapsto B_t := W([0,t])$ is a one-dimensional Brownian motion for all $t > 0$. Clearly, B is a mean-zero Gaussian process since W is. Therefore, we only have to check if its covariances function satisfies* (1.0.4)*. W.l.o.g pick $0 < s \leq t$. Then*

$$\mathbb{E}\Big(B_tB_s\Big) = \Big|[0,t] \cap [0,s]\Big| = s.$$

and this proves the desired result.

Another interesting process that can be constructed from a *white noise* a Brownian sheet.

Definition 1.0.10 (Čencov, 1956)**.** *A Brownian sheet indexed by $\mathbb{R}^d$ is a mean-zero Gaussian random field $\{B(x)\}_{x\in\mathbb{R}^d}$ with covariance*

$$\mathbb{E}\Big[B(x)B(y)\Big] = \prod_{i=1}^{d} \min\Big(|x_i|, |y_i|\Big)\boldsymbol{1}_{[0,\infty)}(x_iy_i), \;\; x, y \in \mathbb{R}^d.$$

One way to show the existence of a Brownian sheet is to check that the identity in the definition above is indeed a covariance function on $\mathbb{R}^d$. But the next example gives a more informative method for proving the existence of such process.

Example 1.0.11. *Let W denote a white noise on $\mathbb{R}^d$ and define $B(x) := W\big(R(x)\big)$ for all $x \in \mathbb{R}^d$, where $R(x)$ denotes the smallest aligned hypercube in $\mathbb{R}^d$ that contains the origin and x as its two extremal vertices, and whose faces are parallel to the axes- for example,*

$B(x) = W\Big[[0,x_1]\times[0,x_2]\times\cdots\times[0,x_d]\Big]$ *if* $x_1, x_2, \cdots, x_d > 0$. *Then, B defines a Brownian sheet. Note also that a one-dimensional Brownian sheet is just a Brownian Motion.*

The following Proposition shows that a Brownian sheet is the CDF of a white noise.

Proposition 1.0.12. *[23] Let B denote the Brownian sheet that we just constructed from the white noise* W. *Then for all* $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\int \phi dW = (-1)^d \int_{\mathbb{R}^d} \frac{\partial^d \phi(x)}{\partial x_1 \cdots \partial x_d} B(x) dx \;\; a.s,$$

where the mixed derivative is interpreted in a "generalized sense".

Because a *white noise* is an $L^2(\Omega)$-valued measure, it makes sense to imagine integrating various functions against it (like in the Proposition above, for example). This turns out to be the case, and the resulting $L^2(\Omega)$-valued integral is called a *Wiener integral*. The construction of a *Wiener integral* is modeled on the the Lebesgue integration. We follow ideas from [23].

Fact: we can identify the $L^2(\Omega)$-valued measure W with an $L^2(\Omega)$-valued integral as follows:

Recall that a function $h : \mathbb{R}^d \to \mathbb{R}$ is *simple*–in the Lebesgue's sense– if

$$h(x) := \sum_{i=1}^n c_i \mathbf{1}_{A_i}(x), \;\; \text{where} \;\; A_i \in \mathscr{L}(\mathbb{R}^d) \;\; \text{are disjoint and} \;\; c_i \in \mathbb{R}.$$

Let $\mathscr{S}(\mathbb{R}^d)$ denote the collection of all simple functions on $\mathbb{R}^d$.

- Pick $h \in \mathscr{S}(\mathbb{R}^d)$. Then,

$$W(h) := \int h dW := \int h(x) W(dx) := \sum_{i=1}^n c_i W(A_i)$$

is linear in h (a.s), a GRF indexed by all elementary functions h. Moreover it is centered and

$$\|W(h)\|^2_{L^2(\Omega)} := \mathbb{E}\big(|W(h)|^2\big) = \|h\|^2_{L^2(\mathbb{R}^d)}. \tag{1.0.5}$$

Identity (1.0.5) is known as the *Wiener isometry.*

Next, define

$$W(h) := \int h dW := \int h(x)W(dx) \text{ for all } h \in L^2\big(\mathbb{R}^d\big) \text{ by density.}$$

The stochastic integral $\int hdW$ is called the *Wiener integral* of the non-random function $h \in L^2(\mathbb{R}^d)$ against the *white noise* W. The following facts hold:

- $$L^2(\mathbb{R}^d) \ni h \mapsto W(h) \in L^2(\Omega)$$

  is a GRF, a linear isometry ;

- $\mathbb{E}W(h) = 0$ for all $h \in L^2(\mathbb{R}^d)$;

- for all $g, h \in L^2(\mathbb{R}^d)$,

  $$\text{Cov}\bigg(\int gdW, \int hdW\bigg) = \int g(x)h(x)dx.$$

- If $h \in L^2(\mathbb{R}^d)$ and $A \in \mathscr{L}(\mathbb{R}^d)$, then we may write the definite stochastic integral of h on A as follows:

  $$\int_A hdW := \int_A h(x)W(dx) = \int h\mathbf{1}_A dW.$$

In this way, we can think of the *Wiener isometry* (1.0.5) as the assertion that the stochastic integration map $h \mapsto \int hdW$ is a linear isometric embedding $L^2(\mathbb{R}^d) \hookrightarrow L^2(\Omega)$.

Next, as we shall see later in this thesis, the solution of (1.0.1) is a random field (RF). Thus, we are interested in constructing an Itô-like integral to handle our computations. That is, we wish to construct an Itô-like stochastic integral $\int h\Psi dW$, where $h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ is non-random and Ψ is a "nice random field." Now we sketch a construction of the "Walsh stochastic integral" w.r.t the *white noise* W.This construction is taken from [23] and follows ideas from [11, 37]. For this reason, the resulting integral is sometimes referred to as the *"Walsh-Dalang" integral*. We have the following facts:

- If $h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, then the Gaussian process

$$0 < t \mapsto X_t(h) := \int_{(0,t)\times\mathbb{R}^d} hdW$$

is a continuous $L^2(\Omega)$-*martingale*.

- $(t, x) \mapsto \Psi_t(x)$ is an *elementary random field* if there exist $0 \le a < b$ and an $\mathcal{F}_a$-measurable r.v $X \in L^2(\Omega)$ and a non-random, bounded, and measurable function $\phi \in L^2(\mathbb{R}^d)$ such that

$$\Psi_t(x) = X\mathbf{1}_{(a,b]}(t)\phi(x), \quad t > 0,\ x \in \mathbb{R}^d.$$

- A random field Ψ is *simple* if there exist elementary random fields $\Psi^{(1)}$, ..., $\Psi^{(n)}$ with disjoint supports such that $\Psi = \sum_{i=1}^n \Psi^{(i)}$.

- If $h = h_t(x)$ and Ψ is elementary, then it is natural to define the stochastic integral

$$\int_{(a,b]\times\mathbb{R}^d} h\Psi dW := X \int_{(a,b]\times\mathbb{R}^d} h_t(x)\phi(x)W(dt, dx).$$

Note that h is non-random, and the stochastic integral $\int h\Psi dW$ is Wiener and it is well-defined iff $h_t(x)\phi(x) \in L^2([a, b] \times \mathbb{R}^d)$. We can then approximate h by simple functions–in the sense of Lebesgue–in order to see that in this case, the stochastic integral $\int h\Psi dW$ is $\mathcal{F}_b-$ measurable.

- If Ψ is a simple random field, then

$$\int h\Psi dW := \sum_{i=1}^n \int h\Psi^{(i)} dW.$$

The defining properties of Wiener integrals imply readily that the preceding integral is well defined. Moreover,

- for every simple RF Ψ,

$$\mathbb{E} \int h\Psi dW = 0$$

- and more significantly, the following property holds:

$$\mathbb{E}\left(\left|\int h\Psi dW\right|^2\right) = \int_0^\infty ds \int_{\mathbb{R}^d} dy[h_s(y)]^2\mathbb{E}\big(|\Psi_s|^2\big). \tag{1.0.6}$$

The identity (1.0.6) is a Hilbert-space isometry–known as the *Walsh isometry*– and has a character that is similar to the *Itô isometry* in the theory of ordinary stochastic integration.

Now that we know the meaning of stochastic integrals appearing in the study of SPDEs, we can now proceed to explore the problem of interest. But not before the following definition.

Definition 1.0.13. *Assume $\gamma = \delta_0$. Following Walsh [37], a random field $\{u_t(x)\}_{t>0,x\in D}$ is called a mild solution of* (1.0.1) *in the Walsh-Dalang sense if*

1. *$u_t(x)$ is jointly measurable in $t \geq 0$ and $x \in D$;*
2. *for all $(t, x) \in \mathbb{R}_+ \times D$, the stochastic integral*

$$\int_0^t \int_D p_D(t-s, x, y)\sigma\big(u_s(y)\big)F(dy, ds)$$

is well-defined in $L^2(\Omega)$; Moreover,

$$\sup_{t>0}\sup_{x\in D}\mathbb{E}|u_t(x)|^p < \infty, \text{ for all } p \geq 2; \tag{1.0.7}$$

3. *The following integral equation holds in $L^2(\Omega)$:*

$$u_t(x) = \big(\mathcal{G}_D u_0\big)_t(x) + \xi\int_0^t \int_D p_D(t-s, x, y)\sigma\big(u_s(y)\big)F(dy, ds), \tag{1.0.8}$$

where

$$\big(\mathcal{G}_D u_0\big)_t(x) := \int_D p_D(t, x, y)u_0(y)dy$$

and $p_D(t, x, y)$ denotes the Dirichlet heat kernel of the stable Lévy process. It is the transition density of the stable Lévy process killed in the exterior of D and the stochastic integral is understood in the Walsh-Dalang sense (extended Itô sense).

Because the Dirichlet heat kernel will play a major role in the proof of our main results, we give a few details about it. We define the "killed process":

$$X_t^D = \begin{cases} X_t & t < \tau_D \\ 0 & t \geq \tau_D, \end{cases}$$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the first exiting time. Next, define

$$r^D(t, x, y) := \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t].$$

Then

$$p_D(t, x, y) = p(t, x, y) - r^D(t, x, y),$$

where $p(t, \cdot)$ is the transition density of the "unkilled process" X_t. Note that $p(t, x, y)$ is also written $p(t, x - y)$ in some literature.

When $\alpha = 2$, X_t corresponds to a Brownian motion $(B_t)_{t\geq 0}$ with variance $2t$, and in this case $p(t, \cdot)$ is explicitely given by

$$p(t, x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}} \quad \text{for all } x, y \in \mathbb{R}^d. \tag{1.0.9}$$

When $\alpha \in (0, 2)$, then X_t coincides with an α-stable Lévy process given by $X_t = B_{S_t}$, where $(S_t)_{t\geq 0}$ is an $\alpha/2$-stable subordinator with Lévy measure

$$\nu(dx) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)} x^{-1-\alpha/2} \mathbf{1}_{\{x>0\}} dx.$$

No explicit expression is known for $p(t, \cdot)$ in this case, but the following approximation holds:

$$C_1 \min\left(t^{-d/\alpha}, \frac{t}{|x - y|^{\alpha+d}}\right) \leq p(t, x, y) \leq C_2 \min\left(t^{-d/\alpha}, \frac{t}{|x - y|^{\alpha+d}}\right) \tag{1.0.10}$$

for some positive constants C_1 and C_2. See for example [9] and the references therein. One important property of the heat kernel $p(.)$ is the Chapman-Kolmogorov identity (also known as the semigroup property), i.e

$$\int_{\mathbb{R}^d} p(t,x,z)p(s,y,z)dz = p(t+s,x,y) \text{ for all } x,y \in \mathbb{R}^d \text{ and } s,t>0. \tag{1.0.11}$$

It is an easy fact that $p_D(.)$ also satisfies the Chapman-Kolmogorov identity. Recall that the Dirichlet heat kernel $p_D(t,x,y)$ has the spectral decomposition

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\phi_n(y), \quad \text{for all } x,y \in D, \ t>0,$$

where $\{\phi_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(D)$ and $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq ...$ is a sequence of positive numbers satisfying, for all $n \geq 1$:

$$\begin{cases} -(-\Delta)^{\alpha/2}\phi_n(x) = -\mu_n\phi_n(x) & x \in D \\ \phi_n(x) = 0 & x \in D^c. \end{cases}$$

For all $n \geq 1$, it is well-known that

$$c_1 n^{\alpha/d} \leq \mu_n \leq c_2 n^{\alpha/d} \tag{1.0.12}$$

for some constants $c_1, c_2 > 0$. See for example [7, Theorem 2.3], for more details. Moreover by [8, Theorem 4.2], for all $x \in D$,

$$c^{-1}(R-|x|)^{\alpha/2} \leq \phi_1(x) \leq c(R-|x|)^{\alpha/2}, \quad \text{for some} \quad c>1. \tag{1.0.13}$$

For example, when $\alpha = 2$ and $D = (-1,1)$, we get for $n = 1, 2, ...$

$$\phi_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right) \text{ and } \mu_n = \left(\frac{n\pi}{2}\right)^2.$$

The following assumptions will be needed when proving the main results:

Assumption 1.0.14. $\gamma : \mathbb{R} \to \mathbb{R}_+$ *is locally integrable.*

Assumption 1.0.15. *There exist constants* C_1 *and* C_2 *and* $0 < \beta < \alpha \wedge d$ *such that for all* $x \in \mathbb{R}^d$,

$$C_1|x|^{-\beta} \leq \Lambda(x) \leq C_2|x|^{-\beta}.$$

Assumption 1.0.16. *There exist positive constants* l_σ *and* L_σ *such that for all* $x \in \mathbb{R}^d$,

$$l_\sigma|x| \leq \sigma(x) \leq L_\sigma|x|.$$

Assumption 1.0.17. *There is* $\epsilon \in \left(0, \frac{R}{2}\right)$ *such that*

$$\inf_{x \in D_\epsilon} u_0(x) > 0,$$

where $D_\epsilon := B_{R-\epsilon}(0)$

Assumption 1.0.18. *There exists a positive constant* K_R *such that*

$$\inf_{x \in D_\epsilon} \Lambda(x) \geq K_R$$

When the noise term is a space-time colored noise, the definition provided above for the random field solution doesn't apply. Therefore, a new approach is necessary in understanding the solution. The following definition provides a different interpretation of the solution. Unfortunately, this approach works only for the PAM.

Definition 1.0.19. *Assume* $\sigma = Id$. *An adapted random field* $u := \{u_t(x)\}_{t>0, x \in D}$ *such that* $\mathbb{E}[u_t(x)]^2 < \infty$ *for all* (t, x) *is a mild solution to* (1.0.1) *in the Skorohod sense if for any* $(t, x) \in \mathbb{R}_+ \times D$, *the process* $\{p_D(t-s, x, y)u_s(y)\boldsymbol{1}_{[0,t]}(s) : s \geq 0,\ y \in D\}$ *is Skorohod integrable with respect to the Gaussian differential* $F(\delta s, \delta y)$ *and the following integral equation holds:*

$$u_t(x) = \left(\mathcal{G}_D u_0\right)_t(x) + \xi \int_0^t \int_D p_D(t-s, x, y)u_s(y)F(\delta s, \delta y). \tag{1.0.14}$$

It is well-known that a unique mild solution (1.0.14) exists in the Skorohod sense provided that the time correlation γ is locally integrable and the space correlation Λ satisfies condition (1.0.3). One of the time correlation functions that has received a lot of attention lately is the correlation function of the so-called fractional Brownian motion (of index H.)

Definition 1.0.20 (Mandelbrot Van Ness, 1968)**.** *A Fractional Brownian Motion with index H, denoted fBm(H), is a centered Gaussian process* $\{X_t\}_{t\geq 0}$ *with* $X_0 = 0$ *and satisfying*

$$\mathbb{E}\big|X_t - X_s\big|^2 = |t-s|^{2H}, \;\; s,t \geq 0.$$

When a fBm(H) exists, its covariance is given by

$$\mathrm{Cov}(X_t, X_s) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).$$

Therefore, a fBm(H) exists iff $H \in (0,1)$. Note that a fBm(1/2) is a Brownian motion.

For the sake of simplicity and uniformity with other authors, we consider the time-correlation function as

$$\gamma(r) = C_H|r|^{2H-2}, \quad for \;\; H \in \big(1/2, 1\big) \text{ and } C_H = H(2H-1). \tag{1.0.15}$$

We refer the interested reader to [1] and the references therein for more information about this function. When handling the mild solution in the Skorohod sense, we shall make use of the Wiener-chaos expansion.

Recall that the covariance given by (1.0.2) is a mere formal notation. Let $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of test functions on $\mathbb{R}_+ \times \mathbb{R}^d$. Then on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a family of centered Gaussian r.v indexed by the test function $\Big\{F(\varphi), \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\Big\}$ with covariance

$$\mathbb{E}\big[\dot{F}(\varphi)\dot{F}(\psi)\big] = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(t,x)\psi(s,y)\gamma(t-s)\Lambda(x-y)dxdydtds. \tag{1.0.16}$$

We write equation (1.0.16) formally as (1.0.2). Let $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ w.r.t the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(t,x)\psi(s,y)\gamma(t-s)\Lambda(x-y)dxdydtds.$$

The mapping $\varphi \mapsto F(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}$. We denote this map by

$$F(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t,x)F(dt,dx), \ \ \varphi \in \mathcal{H}.$$

Note that if $\varphi, \psi \in \mathcal{H}$,

$$\mathbb{E}\big[\dot{F}(\varphi)\dot{F}(\psi)\big] = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

Furthermore, $\mathcal{H}$ contains the space of measurable functions φ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |\varphi(t,x)\varphi(s,y)|\gamma(t-s)\Lambda(x-y)dxdydtds < \infty.$$

For $n \geq 0$, denote by $\mathbf{H}_n$ the n^{th} Wiener-chaos of F. Recall that $\mathbf{H}_0$ is just $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(F(h)),\ h \in \mathcal{H},\ \|h\|_{\mathcal{H}} = 1\}$ where H_n is the n^{th} Hermite polynomial. For $n \geq 1$, we denote by $\mathcal{H}^{\otimes n}$(resp.$\mathcal{H}^n$) the n^{th} tensor product (resp. the n^{th} symmetric tensor product) of $\mathcal{H}$. Then, the mapping $I_n(h^{\otimes n}) = H_n(F(h))$ can be extended to a linear isometry between $\mathcal{H}^n$ (equipped with the modified norm $\sqrt{n!}\|.\|_{\mathcal{H}^{\otimes n}}$) and $\mathbf{H}_n$, see for example [29] and [31] and the references therein.

Consider now a random variable $X \in L^2(\Omega)$ measurable with respect to the $\sigma-$field $\mathcal{F}^F$ generated by F. This random variable can be expressed as

$$X = \mathbb{E}[X] + \sum_{n=1}^{\infty} I_n(f_n),$$

where the series converges in $L^2(\Omega)$ and the elements $f_n \in \mathcal{H}^n,\ n \geq 1$ are determined by X. This identity is known as the Wiener-chaos expansion. Please refer to [29] and [31] for a complete description on the matter.

With everything set, we can now state state and prove or main results. But before we jump into this important aspect of this work, we give a detailed organization of this dissertation. The rest of thesis is organized as follows: in Chapter 2, we state our main results when equation 1.0.1 is driven by a space-time white, Chapter 3 is devoted to case where the noise is white in time and colored in space while Chapter 4 deals with the space-time colored noise case. Finally, all our main results including the existence-uniqueness result are proved in Chapter 5. We close this dissertation with concluding remarks in Chapter 6. Also throughout this work, the letter c (upper or lower case) with or without a subscript is a positive constant whose value is not of primary interest for this dissertation.

Chapter 2

Fractional Stochastic heat equation driven by a space-time white noise.

We first study the fractional stochastic heat equation (1.0.1) driven by a space-time white noise, i.e, we look at the equation

$$\begin{cases} \frac{\partial}{\partial t}u_t(x) = -(-\Delta)^{\alpha/2}u_t(x) + \xi\sigma(u_t(x))\dot{W}(t,x) \quad x \in D, \;\; t > 0, \\ u_t(x) = 0 \quad x \in D^c, \end{cases} \tag{2.0.1}$$

where $0 < \alpha < 2$ and the noise term $\dot{W}$ has the following covariance structure:

$$\mathbb{E}\Big(\dot{F}(t,x)\dot{F}(s,y)\Big) = \delta_0(t-s)\delta_0(x-y), \tag{2.0.2}$$

and δ_0 is the Delta-Dirac distribution.

In [18], the authors considered the following stochastic heat equation,

$$\partial_t u_t(x) = \mathcal{L}u_t(x) + \xi\sigma(u_t(x))\dot{F}(t,x), \tag{2.0.3}$$

where $\mathcal{L} = \Delta$ is the Dirichlet Laplacian on $B_R(0)$, the ball of radius R centered at the origin. Under some appropriate conditions, it was shown that the long time behaviour of the solution is dependent on the *noise level*, that is on the values of ξ. More precisely, it was shown that for large values of ξ, the moments of the solution grow exponentially with time while for small values of ξ, the moments decay exponentially. In [30], the author found explicit bounds for the p^{th} moments of the solution to (2.0.1) with $0 < \alpha \leq 2$, also proving the dichotomy phenomenon described above. In this chapter, we extend the results obtained in [18] for the fractional Laplacian operator. We first obtain the results for the second moment of the random field solution and then extend it to higher order moments (greater than 2). We are also interested in the effect of the noise level ξ on the [expected] energy of the solution.

2.1 Main results

Theorem 2.1.1. *Suppose that σ satisfies Assumption 1.0.16. Let $u_t(x)$ be the unique mild solution of equation* (2.0.1)*, then there exists $\xi_0 > 0$ such that for all $\xi < \xi_0$ and $x \in D$,*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < 0.$$

Fix $\varepsilon > 0$, then there exists $\xi_1 > 0$ such that for all $\xi > \xi_1$ and $x \in D_\epsilon$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty.$$

Remark 2.1.2. *It is not hard to see that $\xi_0 \le \xi_1$. Otherwise there will be an obvious contradiction in Theorem 2.1.1. In Remark 5.3.3, we provide some estimates for ξ_0 and ξ_1 defined above.*

As in [18, 24], we define the [expected] energy of the solution at time t by the following quantity,

$$\mathcal{E}_t(\xi) = \left(\mathbb{E}\|u_t\|^2_{L^2(D)}\right)^{1/2}. \tag{2.1.1}$$

The study of $\mathcal{E}_t(\xi)$ as ξ gets large was initiated in [24, 25]. In [24], it was shown that $\mathcal{E}_t(\xi)$ grows like ce^{ξ^4} as ξ gets large. However, the next corollary shows that $\mathcal{E}_t(\xi)$ exhibits a behavior similar to that of the second moment of the solution of (2.0.1).

Corollary 2.1.3. *With ξ_0 and ξ_1 as in Theorem 2.1.1, we have*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < 0 \quad \textit{for all} \quad \xi < \xi_0$$

and

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < \infty \quad \textit{for all} \quad \xi > \xi_1.$$

The following Theorem provides an extension of the results in Theorem 2.1.1 to higher moments:

Theorem 2.1.4. *Suppose that σ satisfies Assumption 1.0.16. If u_t is the unique mild solution to* (2.0.1)*, then for all $p \geq 2$, there exists $\xi_0(p) > 0$ such that for all $\xi < \xi_0(p)$ and $x \in D$,*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < 0.$$

On the other hand, for all $\epsilon > 0$, there exists $\xi_1(p) > 0$ such that for all $\xi > \xi_1(p)$ and $x \in D_\epsilon$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < \infty.$$

In the next Chapter, we consider the main equation driven by a noise that is white in time and colored in space. Many similarities with the space-time white noise case are observed.

Chapter 3

Fractional Stochastic heat equation driven by a spatially-colored noise

In this section we consider equation (1.0.1) driven by a noise that is white in time but colored in space, i.e, we are looking at:

$$\begin{cases} \frac{\partial}{\partial t} u_t(x) = -(-\Delta)^{\alpha/2} u_t(x) + \xi \sigma(u_t(x)) \dot{F}(t,x) \quad x \in D, \ \ t > 0, \\ u_t(x) = 0 \quad x \in D^c, \end{cases} \tag{3.0.1}$$

where $0 < \alpha < 2$ and the noise term $\dot{F}$ has the following covariance structure:

$$\mathbb{E}\Big(\dot{F}(t,x)\dot{F}(s,y)\Big) = \delta_0(t-s)\Lambda(x-y) \tag{3.0.2}$$

We can state the main results in this section.

3.1 Main results

Theorem 3.1.1. *Suppose that Λ satisfies Assumption 1.0.18 and σ satisfies Assumption 1.0.16. Let $u_t(x)$ be the unique mild solution to* (3.0.1)*, then there exists $\xi_2 > 0$ such that for all $\xi < \xi_2$ and $x \in D$*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < 0.$$

On the other hand, for all $\epsilon > 0$, there exists $\xi_3 > 0$ such that for all $\xi > \xi_3$ and $x \in D_\epsilon$

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty.$$

As defined previously, see for example [18, 24], we define the [expected] energy of the solution (of equation 3.0.1) at time t by

$$\mathcal{E}_t(\xi) = \left(\mathbb{E}\|u_t\|^2_{L^2(D)}\right)^{1/2}. \tag{3.1.1}$$

Unsurprisingly, the dichotomy phenomenon is again observed with the energy of the solution, just like in the case where equation 1.0.1 is driven by a space-time white noise. We provide this result below:

Corollary 3.1.2. *With ξ_2 and ξ_3 as in Theorem 3.1.1, we have*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t}\log \mathcal{E}_t(\xi) < 0 \quad \textit{for all} \quad \xi < \xi_0$$

and

$$0 < \liminf_{t\to\infty} \frac{1}{t}\log \mathcal{E}_t(\xi) < \infty \quad \textit{for all} \quad \xi > \xi_1.$$

More importantly, Theorem 3.1.1 can also be extended to the case $p > 2$, offering again another similarity with the space-time white noise case.

Theorem 3.1.3. *Suppose that Λ satisfies Assumption 1.0.18 and σ satisfies Assumption 1.0.16. Let $u_t(x)$ be the unique mild solution to* (3.0.1)*, then for all $p \geq 2$ there exists $\xi_2(p) > 0$ such that for all $\xi < \xi_2(p)$ and $x \in D$*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^p < 0.$$

On the other hand, for all $\epsilon > 0$, there exists $\xi_3(p) > 0$ such that for all $\xi > \xi_3(p)$ and $x \in D_\epsilon$

$$0 < \liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^p < \infty.$$

We conclude this section with special case of equation 3.0.1 when the space correlation function satisfies Assumption 1.0.15. In this case, explicit bounds for the p^{th} moments of the solution are found. The next Theorem provide these details:

Theorem 3.1.4. *Suppose that Λ satisfies Assumption 1.0.15 and σ satisfies Assumption 1.0.16. Then for all $t > 0$ and $p \geq 2$, there exist positive constants $c_1, c_2(\alpha, \beta, d, l_\sigma), C_1$ and $C_2(\alpha, \beta, d, L_\sigma)$ such that for all $\xi > 0$ and $\delta > 0$,*

$$c_1^p e^{pt\left(c_2 \xi^{\frac{2\alpha}{\alpha-\beta}} - \mu_1\right)} \leq \inf_{x \in D_\epsilon} \mathbb{E}|u_t(x)|^p \leq \sup_{x \in D} \mathbb{E}|u_t(x)|^p \leq C_1^p e^{pt\left(C_2 \xi^{\frac{2\alpha}{\alpha-\beta}} z_p^{\frac{2\alpha}{\alpha-\beta}} - (1-\delta)\mu_1\right)},$$

where z_p is the constant in the Burkhölder-Davis-Gundy's inequality.

This theorem also provides an extension to [15] where similar bounds were obtained but only for the second moments of the solution to equation (1.0.1). This theorem also shows that the rate at which the moments of the solution to equation (1.0.1) exponentially grow or decay depends explicitly on the non-local operator $-(-\Delta)^{\alpha/2}$, the noise level ξ and the noise term via the quantity $\xi^{\frac{2\alpha}{\alpha-\beta}}$. This result provides an extension to [30] where the author used equation (1.0.1) with $\sigma = Id$, an essential assumption when using the Wiener-Chaos expansion in the proofs. However, the proof we provide for this theorem uses a different argument. Moreover, this Theorem implies Theorems 3.1.1 and 3.1.3.

Note that in Theorem 3.1.4, when $\xi < \Big(\mu_1/C(p, \delta)\Big)^{\frac{\alpha-\beta}{2\alpha}}$, then the solution u is not weakly-intermittent. However, quite the opposite situation occurs for the same random field u when $\xi > \Big(\mu_1/C_1(p)\Big)^{\frac{\alpha-\beta}{2\alpha}}$.

Chapter 4

Fractional Stochastic heat equation driven by a space-time colored noise

In this section, we consider equation (1.0.1) driven by a space-time colored noise, whose covariance function is given by (1.0.2). Several papers have examined SPDEs driven by a space-time colored noise: we cite for example [1, 2, 4, 20, 21]. The time correlation function considered in most cases is the fBm(H) and the space-correlation functions include the Riez Kernel, the fractional Brownian motion (please see P. 2-3 of this thesis). However, all these problems were considered on the spatial variable space $\mathbb{R}^d$. In our case, the time correlation function is the fBm(H) and the space correlation function is the Riez kernel. The Theorem below provides the details.

4.1 Main result

Theorem 4.1.1. *Assume $\sigma(x) = x$, γ satisfies assumption 1.0.14 and Λ satisfies Assumption 1.0.15. Then for all $t > 0$ and $p \geq 2$, there exist constants C_1 and $C_2(\alpha, \beta)$ such that for all $\xi > 0$ and $\delta > 0$,*

$$\sup_{x \in D} \mathbb{E}|u_t(x)|^p \leq C_1^p e^{C_2 p\left((p-1)^{\frac{\alpha}{\alpha-\beta}} t^{\frac{2H\alpha-\beta}{\alpha-\beta}} \xi^{\frac{2\alpha}{\alpha-\beta}} - (\mu_1 - \delta)t\right)}$$

With all the Assumptions in Theorem 4.1.1 valid, We also conjecture that

$$\liminf_{t\to\infty} \frac{1}{t^\rho} \log \mathbb{E}|u_t(x)|^2 > 0 \quad \text{for all } x \in D_\epsilon, \quad \text{where } \rho = \frac{2H\alpha - \beta}{\alpha - \beta} > 1.$$

Though this bound might not be very sharp, to the best of our knowledge, this dissertation is the first ever to examine the moments of the solution of SPDEs driven by such type of noise in bounded domains. Notice again the dependence of moments with the noise level. Note that Similar results were obtained in [1, 20] but on the whole Euclidean space $\mathbb{R}^d$.

Define the p^{th} upper Liapounov moment of the random field $u := \{u_t(x)\}_{t>0,x\in D}$ at $x_0 \in D$ as

$$\overline{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x_0)|^p \text{ for all } p \in (0, \infty).$$

Following [16], the random field u is said to be *weakly intermittent* if:

$$\text{for all } x \in D,\ \ \overline{\gamma}(2) > 0 \text{ and } \overline{\gamma}(p) < \infty \text{ for all } p \in (2, \infty).$$

It is said to be *fully intermittent* if:

$$p \mapsto \frac{\overline{\gamma}(p)}{p} \text{ is strictly increasing for all } p \geq 2 \text{ and } x \in D.$$

If $\overline{\gamma}(1) = 0$ and $u \geq 0$, then weak intermittency implies full intermittency. Intuitively, full intermittency shows that for $p > q$,

$$\limsup_{t\to\infty} \frac{\|u(t,x)\|_p}{\|u(t,x)\|_q} = \infty,$$

where $\|u\|_p$ denotes the norm in $L^p(\Omega)$. In other words, for $p > q$, asymptotically, the p^{th} moment of u is significantly larger than its q^{th} moment. This suggests that the random field u may take very large values with small (but significant) probabilities, and therefore it develops high peaks, when t is large. We refer to [6, 16] for a detailed explanation of this phenomenon. Theorem 4.1.1 combined with the conjecture right below shows in fact that u is weakly-intermittent for all $\xi > 0$ since $\frac{2H\alpha-\beta}{\alpha-\beta} > 1$.

We are now ready to prove our main results.

Chapter 5

Proofs of the main results

5.1 Existence and uniqueness for the solution

The proof for the existence of a solution of equation (1.0.1) driven by a space-time white noise or when the noise is white in time and colored in space follows a Picard's iteration scheme. We follow ideas from [15, 23]. The details are provided below:

Let $u_t^0(x) = \mathcal{G}_D u_0(x)$ and for $n \geq 1$,

$$u_t^{n+1}(x) = \Big(\mathcal{G}_D u_0\Big)_t(x) + \xi \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s^n(y)\big)F(ds,dy).$$

The stochastic integral is well defined even when the correlation function is restricted to D. This fact actually follows from Walsh [37]. Let $D_n(t,x) = u_t^{n+1}(x) - u_t^n(x)$. It follows that

$$D_n(t,x) = \xi \int_0^t \int_D p_D(t-s,x,y)\Big[\sigma\big(u_s^n(y)\big) - \sigma\big(u_s^{n-1}(y)\big)\Big]F(ds,dy).$$

Now using Burkhölder's inequality and Assumption 1.0.16, we get

$$\begin{aligned}\mathbb{E}\big|D_n(t,x)\big|^p \leq C_p \xi^p L_\sigma^p \Bigg[\int_0^t \int_{D^2} & p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z) \\ & \times \sup_{y\in D}\mathbb{E}\big[D_{n-1}(s,y)\big]^2 dydzds\Bigg]^{p/2}.\end{aligned}$$

Now let

$$H_n(t,x) = \sup_{x\in D}\sup_{0<t\le T} \mathbb{E}\big|D_n(t,x)\big|^p \text{ and } F(t) = \int_0^t \int_{D^2} p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z)dydzds.$$

Note that

$$\begin{aligned} F(t) \le & \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} p(t-s,x,y)p(t-s,x,z)\Lambda(y-z)dydzds \\ \le & c\int_{\mathbb{R}^d} \frac{\mu(\zeta)}{1+|\zeta|^\alpha}. \end{aligned}$$

Thus, $F(t) < \infty$ whenever Dalang's condition (1.0.3) holds. Note also that, combining the semigroup property of the Dirichlet kernel with Lemma A.0.3,

$G(t) = \int_{D^2} p_D(t-s,x,y)p_D(t-s,x,z)dydz < \infty$ for $0 \le t \le T$. We now use Hölder's inequality to obtain

$$H_n(t,x) \le C_p \xi^p L_\sigma^p F(t)^{p/2-1} \int_0^t H_{n-1}(s)ds. \tag{5.1.1}$$

Thus, by Gronwall's lemma, Lemma A.0.9, we have $\sum_{n=1}^\infty H_n(t) < \infty$. Therefore, $u_t^n(x)$ converges in $L^2(P)$ to some $u_t(x)$ for each t and x. This also proves that

$$\lim_{n\to\infty} \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big)W(ds,dy) = \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big)W(ds,dy).$$

where the convergence holds in $L^2(P)$. This proves that u is a solution to (1.0.1) when the noise term is either a space-time white noise or white in time and colored in space.

For the uniqueness, assume that u and v are both solution of (1.0.1) driven by a space-time white noise and noise that is white in time and colored in space, both satisfying the integrability condition (1.0.7). We show that one these solutions is a modification of the other.

Let $D(x,t) = u_t(x) - v_t(x)$. Then,

$$D(t,x) = \xi \int_0^t \int_D p_D(t-s,x,y)\Big[\sigma\big(u_s(y)\big) - \sigma\big(v_s(y)\big)\Big] F(ds,dy).$$

Using Assumption 1.0.16, we get

$$\mathbb{E}\Big|D(t,x)\Big|^2 \le \xi^2 L_\sigma^2 \int_0^t \int_{D^2} p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z) \sup_{y\in D}\mathbb{E}\Big|D(s,y)\Big|^2 dydzds.$$

Now, set $H(t) = \sup_{0<s\le t}\sup_{x\in D}\mathbb{E}\big|D(s,x)\big|^2$. It follows that

$$H(t) \le C\xi^2 L_\sigma^2 \int_0^t H(s)N(t-s)ds,$$

where

$$N(r) \le \begin{cases} c_1 r^{-1/\alpha}, & \text{when } \Lambda = \delta_0 \\ c_2 r^{-\beta/\alpha} & \text{when } \Lambda \text{ satisfies Assumption 1.0.15.} \end{cases}$$

Now choose and fix some $q \in (1,2)$ and let r be its conjugate exponent, i.e $q^{-1} + r^{-1} = 1$. Next, Apply Hölder's inequality to find that there exists some constant $A = A_T(\xi, L_\sigma)$, such that uniformly for all $0 \le t \le T$,

$$H(t) \le A\bigg(\int_0^t H^r(s)ds\bigg)^{1/r}.$$

Finally, apply Gronwall's Lemma with $a_1 = a_2 = \cdots = H^r$ to find that $H(t) \equiv 0$ and this concludes the proof.

With the existence-uniqueness result out of the way, we can now focus on proving our main results. The following estimates on the Dirichlet heat kernel will be used significantly when proving Theorems 2.1.1, 2.1.4 and 3.1.1.

5.2 Some estimates

The estimates in this section are mainly taken from [9]. They will be used a lot in the proof of Theorems 2.1.1, 2.1.4, 3.1.1 and 3.1.3.

-
$$p_D(t,x,y) \le c_1(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}). \tag{5.2.1}$$

  We will often use the above inequality in the form of $p_D(t,x,y) \le c_1 p(t,x-y)$.

- Fix $\epsilon > 0$ and let $x,\, y \in D_\epsilon$, then for all $t \le \epsilon^\alpha$,

$$p_D(t,x,y) \ge c_2(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}). \tag{5.2.2}$$

- There exist $t_0 > 0$ and $\mu_1 > 0$ such that,

$$c_1 e^{-\mu_1 t} \le p_D(t,x,y) \le c_2 e^{-\mu_1 t} \quad \text{for} \quad t \ge t_0. \tag{5.2.3}$$

  The upper bound is valid for any $x,\, y \in D$ while the lower bound is only valid for $x,\, y \in D_\epsilon$ with $\epsilon > 0$.

We first consider the case where equation 1.0.1 is driven by a space-time white noise.

5.3 The space-time white noise case

Recall that equation 1.0.1 driven by a space-time white noise has a unique solution only when $d = 1$ and $1 < \alpha \leq 2$. So we will assume that these two conditions hold throughout this section. Also, [18] have covered the case $\alpha = 2$, so we will focus only on the case $1 < \alpha < 2$.

Lemma 5.3.1. *There exists a constant $K_{\upsilon,\mu_1,\alpha}$ depending only on υ, μ_1 and α such that for all $\upsilon \in (0, \mu_1)$ and $x \in D$, we have*

$$\int_0^\infty e^{\upsilon t} p_D(t, x, x) dt \leq K_{\upsilon,\mu_1,\alpha}.$$

Proof. We begin by writing

$$\int_0^\infty e^{\upsilon t} p_D(t, x, x) dt = \int_0^{t_0} e^{\upsilon t} p_D(t, x, x) dt + \int_{t_0}^\infty e^{\upsilon t} p_D(t, x, x) dt,$$

where t_0 is as in (5.2.3). Now using (5.2.1), we have

$$\int_0^{t_0} e^{\upsilon t} p_D(t, x, x) dt \leq c_3 \int_0^{t_0} e^{\upsilon t} t^{-\frac{1}{\alpha}} dt.$$

It is now clear that the above integral has an upper bound depending on υ . Next, since $\upsilon < \mu_1$, we can use (5.2.3) to write

$$\begin{aligned}\int_{t_0}^\infty e^{\upsilon t} p_D(t, x, x) dt &\leq c_5 \int_{to}^\infty e^{-(\mu_1 - \upsilon)t} dt \\ &\leq \frac{c_6}{\mu_1 - \upsilon}.\end{aligned}$$

Combining the estimates, we obtain the result. □

Lemma 5.3.2. *Let $\upsilon \in (0, \mu_1)$ and $x \in D$. Then there exists a constant $c_{\mu_1,\alpha}$ depending on μ_1 and α such that for all $t > 0$*

$$\int_D e^{\upsilon t} p_D(t, x, y) dy \leq c_{\mu_1,\alpha}.$$

Proof. Again fix t_0 as in (5.2.3).

For $0 < t < t_0$, we have

$$\int_D e^{\upsilon t} p_D(t,x,y)dy \le e^{\upsilon t}\int_{\mathbb{R}^d} p(t,x,y)dy$$
$$\le e^{\upsilon t_0}.$$

And for $t \ge t_0$ we use (5.2.3) to get

$$\int_D e^{\upsilon t} p_D(t,x,y)dy \le c_2 e^{-(\mu_1-\upsilon)t_0}.$$

The result now easily follows from the two inequalities above. $\square$

We are now ready to prove our main results in this section.

Proof of Theorem 2.1.1. Using (1.0.8) and the Walsh isometry, we have

$$\mathbb{E}\big|u_t(x)\big|^2 = \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 + \xi^2\int_0^t\int_D {p_D}^2(t-s,x,y)\mathbb{E}\big|\sigma(u_s(y))\big|^2 dyds, \tag{5.3.1}$$

from which we obtain

$$\mathbb{E}|u_t(x)|^2 \ge \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2. \tag{5.3.2}$$

Using Assumption 1.0.17, we have for $\epsilon > 0$ small enough,

$$\begin{aligned}(\mathcal{G}_D u_0)_t(x) &= \int_D u_0(y)p_D(t,x,y)dy\\ &\ge \int_{D_\epsilon} u_0(y)p_D(t,x,y)dy\\ &\ge c_1 e^{-\mu_1 t},\end{aligned}$$

whenever $t \ge t_0$ with t_0 as in (5.2.3) and $x \in D_\epsilon$. This immediately gives

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}|u_t(x)|^2 > -\infty \quad for \quad x\in D_\epsilon.$$

We now look at the upper bound. We will assume that $\upsilon \in (0, 2\mu_1)$. From (5.3.1) and Assumption 1.0.16, we have

$$\begin{aligned}\mathbb{E}|u_t(x)|^2 &\le \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 + \xi^2 L_\sigma^2 \int_0^t \int_D {p_D}^2(t-s,x,y)\mathbb{E}\big|u_s(y)\big|^2 dyds \\ &:= I_1 + I_2.\end{aligned}$$

Using Lemma 5.3.2, we have

$$\begin{aligned}I_1 &\le c_2 e^{-\upsilon t}\left|\int_D e^{\frac{\upsilon t}{2}} p_D(t,x,y)dy\right|^2 \\ &\le c_3 e^{-\upsilon t}.\end{aligned}$$

We then look at the second term I_2. Using Chapman-Kolmogorov's identity and Lemma 5.3.1, we have

$$\begin{aligned}I_2 &= \xi^2 L_\sigma^2 e^{-\upsilon t}\int_0^t \int_D e^{\upsilon(t-s)} p_D^2(t-s,x,y) e^{\upsilon s}\mathbb{E}|u_s(y)|^2 dyds \\ &\le \xi^2 L_\sigma^2 e^{-\upsilon t}\sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2 \int_0^t \int_D e^{\upsilon(t-s)} p_D^2(t-s,x,y)dyds \\ &\le \xi^2 L_\sigma^2 e^{-\upsilon t}\sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2 \int_0^t e^{\upsilon s} p_D(2s,x,x)ds \\ &\le K_{\upsilon,\mu_1,\alpha}\xi^2 L_\sigma^2 e^{-\upsilon t}\sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2.\end{aligned}$$

Combining the inequalities above, we have

$$\sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2 \le c_3 + K_{\upsilon,\mu_1,\alpha}\xi^2 L_\sigma^2 \sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2. \tag{5.3.3}$$

We now choose ξ_0 such that for $\xi \le \xi_0$, we have $K_{\upsilon,\mu_1,\alpha}\xi^2 L_\sigma^2 < \frac{1}{2}$. This immediately gives

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}|u_t(x)|^2 < 0.$$

We have thus proved the first half of the theorem. For the second half, we look at the following Laplace transform,

$$I_\upsilon := \int_0^\infty e^{-\upsilon t} \inf_{x\in D_\epsilon} \mathbb{E}|u_t(x)|^2 \, dt.$$

Again, using (5.3.1) and Assumption 1.0.16, we have

$$\mathbb{E}|u_t(x)|^2 \geq \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 + \xi^2 l_\sigma^2 \int_0^t \int_D {p_D}^2(t-s,x,y)\mathbb{E}|u_s(y)|^2 dyds.$$

From the inequality above, we have $I_\upsilon \geq J_1 + J_2$, where J_1 and J_2 are respectively the Laplace transforms of the first and second term of the display above. We look at J_1 first. Note that for fixed $\epsilon > 0$,

$$\begin{aligned}\inf_{x\in D_\epsilon} (\mathcal{G}_D u_0)_t(x) &\geq \int_{D_\epsilon} u_0(y) p_D(t,x,y)dy \\ &\geq c_4 \inf_{x,y\in D_\epsilon} p_D(t,x,y).\end{aligned}$$

Using (5.2.3), for $t \geq t_0$, we have

$$\begin{aligned}J_1 &\geq \int_{t_0}^\infty e^{-\upsilon t} \inf_{x\in D_\epsilon} \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 dt \\ &\geq \frac{c_3 e^{-(\upsilon+2\mu_1)t_0}}{\upsilon + 2\mu_1}.\end{aligned}$$

For the second term, we obtain

$$\begin{aligned}J_2 &\geq \xi^2 l_\sigma^2 I_\upsilon \int_{t_0}^\infty e^{-\upsilon s} \inf_{x\in D_\epsilon} p_D^2(s,x,y)dy \\ &\geq c_4 \xi^2 l_\sigma^2 I_\upsilon \frac{e^{-(\upsilon+2\mu_1)t_0}}{\upsilon + 2\mu_1}.\end{aligned}$$

Combining the inequalities above yields

$$I_\upsilon \geq \frac{c_3 e^{-(\upsilon+2\mu_1)t_0}}{\upsilon + 2\mu_1} + c_4 \xi^2 l_\sigma^2 I_\upsilon \frac{e^{-(\upsilon+2\mu_1)t_0}}{\upsilon + 2\mu_1}.$$

We can now choose ξ_1 large enough so that for all $\xi \geq \xi_1$, we have

$$I_\upsilon \geq \frac{c_5 e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+2\mu_1} + 2I_\upsilon, \tag{5.3.4}$$

which gives us $I_\upsilon = \infty$. This proves

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 > 0.$$

The fact that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty$$

easily follows from the ideas in [16, 23]. We leave it to the reader to fill in the details. □

Remark 5.3.3. *From the proof above, we can get some estimates for ξ_0 and ξ_1. In fact, from inequality 5.3.3 , it is enough to set $\xi_0 = \left(2K_{\upsilon,\mu_1} L_\sigma^2\right)^{-1/2}$. In a similar way, choosing $\xi_1 = \left(\frac{2(\upsilon+2\mu_1)e^{t_0(\upsilon+2\mu_1)}}{c_5 l_\sigma^2}\right)^2$ will suffice in Theorem 2.1.1.*

Proof of Corollary 2.1.3. The proof follows essentially from Theorem 2.1.1 and the definition of the energy of the solution together with the following estimate

$$|D_\epsilon| \inf_{x\in D_\epsilon} \mathbb{E}|u_t(x)|^2 \leq \int_D \mathbb{E}|u_t(x)|^2 dx \leq |D| \sup_{x\in D} \mathbb{E}|u_t(x)|^2.$$

□

We now prove the result for higher moments. The proof is similar to that of Theorem 2.1.1.

Proof of Theorem 2.1.4. Since $\frac{p}{2} \geq 1$, using Jensens inequality we get

$$\left(\mathbb{E}|u_t(x)|^2\right)^{p/2} \leq \mathbb{E}|u_t(x)|^p. \tag{5.3.5}$$

This means that $\mathbb{E}|u_t(x)|^p$ will also grow exponentially if $\mathbb{E}|u_t(x)|^2$ does. So by Theorem 2.1.1, it follows that for all $\xi > \xi_1(p)$, and $x \in D_\epsilon$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p.$$

Again following the ideas of Foondun and Khoshnevisan [16], we can prove that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < \infty.$$

Next, in order to prove

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < 0,$$

we choose $\upsilon \in (0, 2\mu_1)$ and define the following norm:

$$\|u\|_{p,\upsilon,\alpha} = \sup_{t>0} \sup_{x\in D} e^{\frac{\upsilon p t}{2}} \mathbb{E}|u_t(x)|^p.$$

Its clear that if we show $\|u\|_{p,\upsilon,\alpha} < \infty$, the result will follow.

Since $p \geq 2$, we have

$$\begin{aligned}
|u_t(x)|^p =& \left| \big(\mathcal{G}_D u_0\big)_t(x) + \xi \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big) W(dy,ds) \right|^p \\
\leq& C_p \left[\left| \big(\mathcal{G}_D u_0\big)_t(x) \right|^p + \xi^p \left| \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big) W(dy,ds) \right|^p \right].
\end{aligned}$$

Now using the Burkhölder's inequality combined with Assumption 1.0.16, we get

$$\begin{aligned}
\mathbb{E}|u_t(x)|^p \leq& z_p \left[\left| \big(\mathcal{G}_D u_0\big)_t(x) \right|^p + \xi^p L_\sigma^p \left(\int_0^t \int_D p_D^2(t-s,x,y) \mathbb{E}|u_s(y)|^2 dyds \right)^{p/2} \right] \\
:=& K_1 + K_2.
\end{aligned}$$

Since u_0 is bounded, using Lemma 5.3.2,

$$
\begin{aligned}
K_1 &\le c_6 e^{-\frac{pvt}{2}} \left| \int_D e^{\frac{vt}{2}} p_D(t,x,y)dy \right|^p \\
&\le c_7 e^{-\frac{pvt}{2}}.
\end{aligned}
$$

Next, using again the Chapman-Kolmogorov's identity combined with Lemma 5.3.1, we have

$$
\begin{aligned}
K_2 =& \xi^p L_\sigma^p \Bigg(\int_0^t \int_D e^{v(t-s)} p_D^2(t-s,x,y) e^{-v(t-s)} \mathbb{E}|u_s(y)|^2 dyds \Bigg)^{p/2} \\
\le& \|u\|_{p,v,\alpha} \xi^p L_\sigma^p e^{-\frac{pvt}{2}} \left(\int_0^t \int_D e^{vs} p_D^2(t-s,x,y) dyds \right)^{p/2} \\
\le& \|u\|_{p,v,\alpha} \xi^p L_\sigma^p e^{-\frac{pvt}{2}} \left(\int_0^\infty e^{vs} p_D(2s,x,x) ds \right)^{p/2} \\
\le& \|u\|_{p,v,\alpha} \xi^p L_\sigma^p e^{-\frac{pvt}{2}} K_{v,\mu_1,\alpha}^{\frac{p}{2}}.
\end{aligned}
$$

Combining the estimates on K_1 and K_2, we get

$$
\mathbb{E}|u_t(x)|^p \le c_7 e^{-\frac{pvt}{2}} + \|u\|_{p,v,\alpha} \xi^p L_\sigma^p e^{-\frac{pvt}{2}} K_{v,\mu_1,\alpha}^{\frac{p}{2}},
$$

from which it follows that

$$
\|u\|_{p,v,\alpha} \le c_7 + c_8 \|u\|_{p,v,\alpha} \xi^p L_\sigma^p . K_{v,\mu_1,\alpha}^{\frac{p}{2}}
$$

So if we choose ξ small enough so that $c_8 \xi^p L_\sigma^p . K_{v,\mu_1,\alpha}^{\frac{p}{2}} < 1$, then we will have $\|u\|_{p,v,\alpha} < \infty$.

Finally, combining (5.3.2) and (5.3.5), we get

$$
\mathbb{E}|u_t(x)|^p \ge \Big(\mathbb{E}|u_t(x)|^2 \Big)^{p/2} \ge \Big((\mathcal{G}_D u_0)_t(x) \Big)^p.
$$

But it is well know that $(\mathcal{G}_D u_0)_t(x)$ decays exponentially fast for large time, it follows that $\mathbb{E}|u_t(x)|^p$ cannot decay faster, so by a similar argument to the case $p = 2$, we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p > -\infty,$$

for all $\xi < \xi_0(p)$ and $x \in D$. □

5.4 The spatially homogeneous noise case

While one can expect that the proof of Theorem 3.1.1 follows easily from that of Theorem 2.1.4, the noise term is now colored in space. Thus the proof is harder and requires a new idea.

Lemma 5.4.1. *Let* $\upsilon \in (0,\, 2\mu_1)$*. Then there exists a constant* c_{υ,μ_1} *depending on* υ *and* μ_1 *such that*

$$\int_0^\infty e^{\upsilon t} \int_{D^2} p_D(t, x_1, y_1) p_D(t, x_2, y_2) \Lambda(y_1 - y_2)\, dt dy_1 dy_2 \le c_{\upsilon,\mu_1}, \tag{5.4.1}$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in D$.

Proof. We again use (5.2.3). So we fix t_0 accordingly. We begin by splitting the integral as follows:

$$\begin{aligned}
\int_0^\infty e^{\upsilon t} \int_{D^2} & p_D(t, x_1, y_1) p_D(t, x_2, y_2) \Lambda(y_1 - y_2)\, dt dy_1 dy_2 \\
&= \int_0^{t_0} e^{\upsilon t} \int_{D^2} p_D(t, x_1, y_1) p_D(t, x_2, y_2) \Lambda(y_1 - y_2)\, dt dy_1 dy_2 \\
&\qquad + \int_{t_0}^\infty e^{\upsilon t} \int_{D^2} p_D(t, x_1, y_1) p_D(t, x_2, y_2) \Lambda(y_1 - y_2)\, dt dy_1 dy_2 \\
&:= I_1 \\
&\qquad + I_2.
\end{aligned}$$

I_1 can be bounded as follows: we use (5.2.1) to obtain

$$
\begin{aligned}
I_1 &\le e^{\upsilon t_0} \int_0^{t_0} e^{-\upsilon t} e^{\upsilon t} \int_{D^2} p_D(t, x_1, y_1) p_D(t, x_2, y_2) \Lambda(y_1 - y_2) dt dy_1 dy_2 \\
&\le e^{2\upsilon t_0} \int_0^{\infty} e^{-\upsilon t} \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t, x_1, y_1) p(t, x_2, y_2) \Lambda(y_1 - y_2) dt dy_1 dy_2 \\
&\le c_1 e^{2\upsilon t_0}.
\end{aligned}
$$

The last inequality holds since Dalang's condition (1.0.3) holds here.

For I_2, we use (5.2.3) to write

$$
\begin{aligned}
I_2 &\le \int_{t_0}^{\infty} e^{\upsilon t} \sup_{y_1, y_2 \in D} p_D(t, x_1, y_1) p_D(t, x_2, y_2)\, dt \int_{D^2} \Lambda(y_1 - y_2) dy_1 dy_2 \\
&\le c_2 \int_0^{\infty} e^{-(2\mu_1 - \upsilon)t} dt.
\end{aligned}
$$

Combining the above estimates yields the desired result. □

Lemma 5.4.2. *Fix $\epsilon > 0$. Then, there exist $t_0 > 0$ and a constant c_{υ, μ_1, t_0} such that for all $\upsilon > 0$,*

$$
\int_0^{\infty} e^{-\upsilon t} p_D(t, x_1, y_1) p_D(t, x_2, y_2) dt \ge c_{\upsilon, \mu_1, t_0},
$$

whenever $x_1,\ x_2,\ y_1,\ y_2 \in D_\epsilon$. The constant c_{υ, μ_1, t_0} depends on υ, μ_1 and t_0.

Proof. Using (5.2.3), we have

$$
\begin{aligned}
\int_0^{\infty} e^{-\upsilon t} p_D(t, x_1, y_1) p_D(t, x_2, y_2) dt &\ge \int_{t_0}^{\infty} e^{-\upsilon t} p_D(t, x_1, y_1) p_D(t, x_2, y_2) dt \\
&\ge c_3 \int_{t_0}^{\infty} e^{-\upsilon t} e^{-2\mu_1 t} dt \\
&= \frac{c_3 e^{-(\upsilon + 2\mu_1) t_0}}{\upsilon + 2\mu_1}.
\end{aligned}
$$

□

Proof of Theorem 3.1.1. Using the mild formulation of the solution and Assumption 1.0.16, we obtain

$$\begin{aligned}
\mathbb{E}\big|u_t(x)\big|^2 &= \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 + \xi^2 \int_0^t \int_{D^2} p_D(t-s,x,y_1)p_D(t-s,x,y_2)\Lambda(y_1,y_2) \\
&\qquad\qquad\qquad\qquad \times \mathbb{E}|\sigma(u_s(y_1))\sigma(u_s(y_2))|dy_1dy_2ds \\
&\le \Big|(\mathcal{G}_D u_0)_t(x)\Big|^2 + \xi^2 L_\sigma^2 \int_0^t \int_{D^2} p_D(t-s,x,y_1)p_D(t-s,x,y_2)\Lambda(y_1-y_2) \\
&\qquad\qquad\qquad\qquad \times \mathbb{E}|u_s(y_1)u_s(y_2)|dy_1dy_2ds \\
&= I_1 + I_2.
\end{aligned}$$

Pick $\upsilon \in (0,\, 2\mu_1)$ and t_0 as in (5.2.3). As is the proof of Theorem 2.1.1, we have

$$I_1 \le c_1 e^{-\upsilon t} \quad \text{whenever} \quad t > t_0.$$

We now bound I_2 using Lemma 5.4.1.

$$\begin{aligned}
I_2 &\le \xi^2 L_\sigma^2 e^{-\upsilon t} \sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2 \int_0^\infty e^{\upsilon t}\int_{D^2} p_D(t,x,y_1)p_D(t,x,y_2)\Lambda(y_1-y_2)dy_1dy_2dt \\
&\le c_2\xi^2 L_\sigma^2 e^{-\upsilon t} \sup_{t>0}\sup_{x\in D} e^{\upsilon t}\mathbb{E}|u_t(x)|^2.
\end{aligned}$$

Using the two bounds above, we can use the arguments of the first part of the proof of Theorem 2.1.1 to show that

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^2 < 0.$$

The first part of our first theorem also gives us

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^2 > -\infty \quad \text{for} \quad x \in D_\epsilon.$$

We now turn our attention to the final part of the proof. To this aim, fix $\upsilon, \epsilon > 0$ and consider the following Laplace transform:

$$J_\upsilon := \int_0^\infty e^{-\upsilon t} \inf_{x,y\in D_\epsilon} \mathbb{E}|u_t(x)u_t(y)|dt.$$

From the mild solution, we have

$$\mathbb{E}\big(u_t(x_1)u_t(x_2)\big) = (\mathcal{G}_D u_0)_t(x_1)(\mathcal{G}_D u_0)(x_2) + \xi^2 \int_0^t \int_{D^2} p_D(t-s,x_1,y_1) \\ \times p_D(t-s,x_2,y_2)\Lambda(y_1-y_2)\mathbb{E}\Big(\sigma\big(u_s(y_1)\big)\sigma\big(u_s(y_2)\big)\Big)dy_1dy_2ds. \quad (5.4.2)$$

Now using again Assumption 1.0.16, we have

$$\mathbb{E}\Big(|u_t(x_1)u_t(x_2)|\Big) \geq \Big|(\mathcal{G}_D u_0)_t(x_1)(\mathcal{G}_D u_0)(x_2)\Big| \\ + \xi^2 l_\sigma^2 \int_0^t \int_{D^2} p_D(t-s,x_1,y_1)p_D(t-s,x_2,y_2)\Lambda(y_1-y_2)\mathbb{E}\Big|u_s(y_1)u_s(y_2)\Big|dy_1dy_2ds \\ := J_1 + J_2.$$

We bound J_2 first by using Assumption 1.0.18.

$$J_2 \geq \xi^2 l_\sigma^2 K_R \int_0^t \int_{D_\epsilon^2} p_D(t-s,x_1,y_1)p_D(t-s,x_2,y_2)\mathbb{E}\big(|u_s(y_1)u_s(y_2)|\big)dy_1dy_2ds \\ \geq c_3\xi^2 l_\sigma^2 K_R \int_0^t \inf_{y_1,\,y_2\in D_\epsilon} p_D(t-s,x_1,y_1)p_D(t-s,x_2,y_2)\mathbb{E}(|u_s(y_1)u_s(y_2)|)ds.$$

Using these estimates, we have

$$J_\upsilon \geq \tilde{J}_1 + \tilde{J}_2,$$

where $\tilde{J}_1$ and $\tilde{J}_2$ are the Laplace transforms of J_1 and J_2 respectively. As in the proof of Theorem 2.1.1, we have

$$\begin{aligned}\tilde{J}_1 &\geq \int_0^\infty e^{-\upsilon t}\big|(\mathcal{G}_D u_0)_t(x_1)(\mathcal{G}_D u_0)(x_2)\big|dt\\ &\geq \frac{c_4 e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+2\mu_1} \quad \text{for} \quad x_1,\, x_2 \in D_\epsilon.\end{aligned}$$

$\tilde{J}_2$ can be estimated using Lemma 5.4.2 as follows.

$$\tilde{J}_2 \geq c_5 \xi^2 l_\sigma^2 K_R J_\upsilon \frac{e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+2\mu_1}.$$

We therefore have

$$J_\upsilon \geq \frac{c_4 e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+\mu_1} + c_5 \xi^2 l_\sigma^2 K_R J_\upsilon \frac{e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+2\mu_1}.$$

Therefore there exists $\xi_3 > 0$ such that we have $J_\upsilon = \infty$ for all $\xi \geq \xi_3$. Using the ideas above, we have

$$\int_0^\infty e^{-\upsilon t}\mathbb{E}|u_t(x)|^2 dt \geq c_5 K_R J_\upsilon \frac{e^{-(\upsilon+2\mu_1)t_0}}{\upsilon+2\mu_1} \quad \text{for} \quad x \in D_\epsilon.$$

Therefore for all $\xi \geq \xi_3$, we obtain

$$\int_0^\infty e^{-\upsilon t}\mathbb{E}|u_t(x)|^2 dt = \infty,$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^2 > 0 \quad \text{for} \quad x \in D_\epsilon.$$

Again the ideas of [16] give

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}|u_t(x)|^2 < \infty.$$

The first step of theorem is therefore proved. $\square$

The proof of Corollary 3.1.2 is omitted since it is similar to that of Corollary 2.1.3

Proof of Theorem 3.1.3. We follow the same ideas as in the one dimensional case.

By Jensens inequality, we have

$$\left(\mathbb{E}|u_t(x)|^2\right)^{p/2} \leq \mathbb{E}|u_t(x)|^p. \tag{5.4.3}$$

This means that $\mathbb{E}|u_t(x)|^p$ will also grow exponentially if $\mathbb{E}|u_t(x)|^2$ does. So by Theorem 3.1.1, it follows that for all $\xi > \xi_3(p)$, and $x \in D_\epsilon$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p.$$

The same Jensens inequality and Theorem 3.1.1 shows that for ξ small enough and $x \in D$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p > -\infty.$$

Again following the ideas of Foondun and Khoshnevisan [16], we can prove that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < \infty.$$

Finally, to prove

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^p < 0,$$

we choose $\upsilon \in (0, 2\mu_1)$ and define the following norm:

$$\|u\|_{p,\upsilon,\alpha} = \sup_{t>0} \sup_{x\in D} e^{\frac{\upsilon pt}{2}} \mathbb{E}|u_t(x)|^p.$$

Its clear that if we show $\|u\|_{p,\upsilon,\alpha} < \infty$, the result will follow.

Since $p \geq 2$, we have

$$\begin{aligned}|u_t(x)|^p =&\left|\big(\mathcal{G}_D u_0\big)_t(x) + \xi \int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big)F(dy,ds)\right|^p \\ \leq& C_p\left[\left|\big(\mathcal{G}_D u_0\big)_t(x)\right|^p + \xi^p\left|\int_0^t \int_D p_D(t-s,x,y)\sigma\big(u_s(y)\big)F(dy,ds)\right|^p\right].\end{aligned}$$

Now using the Burkhölders inequality combined with Assumption 1.0.16, we get

$$\begin{aligned}\mathbb{E}|u_t(x)|^p \leq& z_p\Bigg[\left|\big(\mathcal{G}_D u_0\big)_t(x)\right|^p \\ &+ \xi^p L_\sigma^p\left(\int_0^t \int_{D^2} p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z)\mathbb{E}|u_s(y)u_s(z)|dydzds\right)^{p/2}\Bigg] \\ := \quad & K_1 \\ &+ K_2.\end{aligned}$$

In our quest for upper bounds for K_1 and K_2 , we use the boundedness of u_0 and Lemma 5.3.2 to get,

$$\begin{aligned}K_1 \leq& c_1 e^{-\frac{p\upsilon t}{2}}\left|\int_D e^{\frac{\upsilon t}{2}} p_D(t,x,y)dy\right|^p \\ \leq& c_2 e^{-\frac{p\upsilon t}{2}} \text{ provided that } \upsilon \in (0, 2\mu_1).\end{aligned}$$

Next, using Lemma 5.4.1, we have

$$
\begin{aligned}
K_2 =&\xi^p L_\sigma^p \Bigg(\int_0^t \int_{D^2} e^{\upsilon(t-s)} p_D(t-s,x,y) p_D(t-s,x,z) \Lambda(y-z) e^{-\upsilon(t-s)} \mathbb{E}|u_s(y)u_s(z)| dydzds \Bigg)^{p/2} \\
=&\xi^p L_\sigma^p e^{-\frac{p\upsilon t}{2}} \Bigg(\int_0^t \int_{D^2} e^{\upsilon(t-s)} p_D(t-s,x,y) p_D(t-s,x,z) \Lambda(y-z) e^{\upsilon s} \sup_{w\in D} \mathbb{E}|u_s(w)|^2 dydzds \Bigg)^{p/2} \\
\leq&\|u\|_{p,\upsilon,\alpha} \xi^p L_\sigma^p e^{-\frac{p\upsilon t}{2}} \Bigg(\int_0^t \int_{D^2} e^{\upsilon s} p_D(s,x,y) p_D(s,x,z) \Lambda(y-z) dydzds \Bigg)^{p/2} \\
\leq&\|u\|_{p,\upsilon,\alpha} \xi^p L_\sigma^p e^{-\frac{p\upsilon t}{2}} c_{\upsilon,\mu_1}^{\frac{p}{2}}.
\end{aligned}
$$

Combining the estimates on K_1 and K_2, we get

$$
\mathbb{E}|u_t(x)|^p \leq c_2 e^{-\frac{p\upsilon t}{2}} + \|u\|_{p,\upsilon,\alpha} \xi^p L_\sigma^p e^{-\frac{p\upsilon t}{2}} c_{\upsilon,\mu_1}^{\frac{p}{2}},
$$

which yield

$$
\|u\|_{p,\upsilon,\alpha} \leq c_2 + \|u\|_{p,\upsilon,\alpha} \xi^p L_\sigma^p . c_{\upsilon,\mu_1}^{\frac{p}{2}}.
$$

So if we choose ξ small enough so that $\xi^p L_\sigma^p c_{\upsilon,\mu_1}^{\frac{p}{2}} < 1$, then we will have $\|u\|_{p,\upsilon,\alpha} < \infty$. and this concludes the proof.

□

We shall need the following estimates to prove Theorem 3.1.4 (and Theorem 4.1.1). The first two follow from applications of Theorems A.0.5 and A.0.4.

Proposition 5.4.3. *Fix $\epsilon \in (0, \frac{1}{2})$. Then for any $x, y \in D_\epsilon$ such that $|x-y| < t^{1/\alpha}$, we have*

$$
p_D(t,x,y) \geq ct^{-d/\alpha} e^{-\mu_1 t} \quad \textit{for all } t > 0
$$

and some positive constant c.

Proof. We first prove the Lemma for $\alpha = 2$. Assume $|x-y| < \sqrt{t}$. We apply Theorem A.0.5 to get

$$\begin{aligned}
p_D(t,x,y) \geq & C_1 \min\left(1, \frac{\phi_1(x)\phi_1(y)}{1 \wedge t}\right) e^{-\mu_1 t} \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{1 \wedge t^{d/2}} \\
\geq & C_2 e^{-\mu_1 t}\left\{ \min\left(1, \frac{\epsilon^2}{t}\right) \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{t^{d/2}} \mathbf{1}_{\{t<1\}} + \min\left(1, \epsilon^2\right) e^{-c_1 \frac{|x-y|^2}{t}} \mathbf{1}_{\{t \geq 1\}} \right\} \\
= & C_2 e^{-\mu_1 t}\left\{ \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{t^{d/2}} \mathbf{1}_{\{t<\epsilon^2\}} + \epsilon^2 \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{t^{1+d/2}} \mathbf{1}_{\{\epsilon^2 \leq t<1\}} + \min\left(1, \epsilon^2\right) e^{-c_1 \frac{|x-y|^2}{t}} \mathbf{1}_{\{t \geq 1\}} \right\} \\
\geq & C_3 e^{-\mu_1 t} t^{-d/2} \left\{ \mathbf{1}_{\{t<\epsilon^2\}} + \frac{\epsilon^2}{t} \mathbf{1}_{\{\epsilon^2 \leq t<1\}} + c(\epsilon) t^{d/2} \mathbf{1}_{\{t \geq 1\}} \right\} \\
\geq & C_4 e^{-\mu_1 t} t^{-d/2}.
\end{aligned}$$

Note the use of (1.0.13) in the second inequality above since $x, y \in D_\epsilon$. This proves the inequality for $\alpha = 2$.

Now suppose $0 < \alpha < 2$. Assuming $|x-y| < t^{1/\alpha}$, we apply Theorem A.0.4 to get

$$\begin{aligned}
p_D(t,x,y) \geq & C_1 e^{-\mu_1 t}\left[\min\left(1, \frac{\phi_1(x)}{\sqrt{t}}\right) \min\left(1, \frac{\phi_1(y)}{\sqrt{t}}\right) \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \mathbf{1}_{\{t<1\}} + \phi_1(x)\phi_1(y) \mathbf{1}_{\{t \geq 1\}} \right] \\
\geq & C_2 e^{-\mu_1 t}\left\{ \min\left(1, \frac{\epsilon^2}{t}\right) \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \mathbf{1}_{\{t<1\}} + \epsilon^\alpha \mathbf{1}_{\{t \geq 1\}} \right\} \\
\geq & C_3 e^{-\mu_1 t}\left\{ \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \mathbf{1}_{\{t<\epsilon^\alpha\}} + \frac{\epsilon^\alpha}{t} \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \mathbf{1}_{\{\epsilon^\alpha \leq t<1\}} \right. \\
& \left. + \epsilon^\alpha \mathbf{1}_{\{t \geq 1\}} \right\} \\
= & C_3 e^{-\mu_1 t} t^{-d/\alpha} \left\{ \min\left(1, \left(\frac{t^{1/\alpha}}{|x-y|}\right)^{\alpha+d}\right) \mathbf{1}_{\{t<\epsilon^\alpha\}} + \frac{\epsilon^\alpha}{t} \min\left(1, \left(\frac{t^{1/\alpha}}{|x-y|}\right)^{\alpha+d}\right) \mathbf{1}_{\{\epsilon^\alpha \leq t<1\}} \right. \\
& \left. + \epsilon^\alpha t^{d/\alpha} \mathbf{1}_{\{t \geq 1\}} \right\} \\
= & C_3 e^{-\mu_1 t} t^{-d/\alpha} \left\{ \mathbf{1}_{\{t<\epsilon^\alpha\}} + \frac{\epsilon^\alpha}{t} \mathbf{1}_{\{\epsilon^\alpha \leq t<1\}} + \epsilon^\alpha t^{d/\alpha} \mathbf{1}_{\{t \geq 1\}} \right\} \\
\geq & C_4 e^{-\mu_1 t} t^{-d/\alpha}.
\end{aligned}$$

Again note the use of (1.0.13) in the second inequality above. This concludes the proof. $\square$

Lemma 5.4.4. *For all $\delta > 0$, there exists $c_2(\delta) > 0$ such that for all $x,\ w \in D$ and $s, t > 0$,*

$$\int_{D\times D} p_D(t,x,y)p_D(s,w,z)\Lambda(y-z)dydz \le c_2 e^{-(1-\delta)\mu_1(t+s)}\Big(s+t\Big)^{-\beta/\alpha}$$

Proof. As usual, we first prove the result for $\alpha = 2$. By Theorem A.0.5, we have

$$\begin{aligned}
&\int_{D\times D} p_D(t,x,y)p_D(s,w,z)\Lambda(y-z)dydz \\
&\le C_1 e^{-\mu_1(t+s)}\int_{D\times D}\min\left(1, \frac{\phi_1(x)\phi_1(y)}{1\wedge t}\right)\min\left(1, \frac{\phi_1(w)\phi_1(z)}{1\wedge s}\right)\frac{e^{-c_1\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}}\frac{e^{-c_2\frac{|w-z|^2}{s}}}{1\wedge s^{d/2}}\Lambda(y-z)dydz \\
&\le C_2 e^{-\mu_1(t+s)}\Bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t,x,y)p(s,w,z)\Lambda(y-z)dydz\mathbf{1}_{\{t<1,s<1\}} + \int_{\mathbb{R}^d} p(t,x,y)\Lambda(y-z)dy\mathbf{1}_{\{t<1,s\ge1\}} \\
&\qquad\qquad + \int_{\mathbb{R}^d} p(s,w,z)\Lambda(y-z)dz\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\quad = C_2 e^{-\mu_1(t+s)}\Bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t+s,x-w,y)\Lambda(y)dy\mathbf{1}_{\{t<1,s<1\}} + \int_{\mathbb{R}^d} p(t,x,y)\Lambda(y-z)dy\mathbf{1}_{\{t<1,s\ge1\}} \\
&\qquad\qquad + \int_{\mathbb{R}^d} p(s,w,z)\Lambda(y-z)dz\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\quad \le C_3 e^{-\mu_1(t+s)}\Bigg\{c_1(t+s)^{-\beta/2}\mathbf{1}_{\{t<1,s<1\}} \ + \ c_2 t^{-\beta/2}\mathbf{1}_{\{t<1,s\ge1\}} \ + \ c_3 s^{-\beta/2}\mathbf{1}_{\{t\ge1,s<1\}} \ + \ c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\quad = C_3 e^{-\mu_1(t+s)}(t+s)^{-\beta/2}\Bigg\{c_1\mathbf{1}_{\{t<1,s<1\}} + c_2\Big(1+\frac{s}{t}\Big)^{\beta/2}\mathbf{1}_{\{t<1,s\ge1\}} + c_3\Big(1+\frac{t}{s}\Big)^{\beta/2}\mathbf{1}_{\{t\ge1,s<1\}} \\
&\qquad\qquad + (t+s)^{\beta/2}\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\quad \le C_5 e^{-(1-\delta)\mu_1(t+s)}(t+s)^{-\beta/2} \ \text{ for all } \delta > 0.
\end{aligned}$$

Note the use of (1.0.9) in the second inequality, the Chapman-Kolmogorov identity (1.0.11) in the first integral in the third inequality and Proposition A.0.8 in the fourth inequality.

The proof for the case $0 < \alpha < 2$ follows a very similar argument. By Theorem A.0.4, we have

$$\begin{aligned}
&\int_{D\times D} p_D(t,x,y)p_D(s,w,z)\Lambda(y-z)dydz \\
&\quad\le C_1 e^{-\mu_1(t+s)}\Bigg\{\int_{D\times D}\min\left(t^{-d/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\min\left(s^{-d/\alpha},\frac{s}{|w-z|^{\alpha+d}}\right)\Lambda(y-z)dydz\mathbf{1}_{\{t<1,s<1\}} \\
&\qquad\qquad + \int_{D\times D}\min\left(t^{-d/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\Lambda(y-z)dydz\mathbf{1}_{\{t<1,s\ge1\}} \\
&\qquad\qquad\qquad + \int_{D\times D}\min\left(s^{-d/\alpha},\frac{t}{|w-z|^{\alpha+d}}\right)\Lambda(y-z)dydz\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,s\ge1\}}\Bigg\} \\
&\le C_2 e^{-\mu_1(t+s)}\Bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t,x,y)p(s,w,z)\Lambda(y-z)dydz\mathbf{1}_{\{t<1,s<1\}} + \int_{\mathbb{R}^d} p(t,x,y)\Lambda(y-z)dy\mathbf{1}_{\{t<1,s\ge1\}} \\
&\qquad\qquad + \int_{\mathbb{R}^d} p(s,w,z)\Lambda(y-z)dz\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&= C_2 e^{-\mu_1(t+s)}\Bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t+s,x-w,y)\Lambda(y)dy\mathbf{1}_{\{t<1,s<1\}} + \int_{\mathbb{R}^d} p(t,x,y)\Lambda(y-z)dy\mathbf{1}_{\{t<1,s\ge1\}} \\
&\qquad\qquad + \int_{\mathbb{R}^d} p(s,w,z)\Lambda(y-z)dz\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\le C_4 e^{-\mu_1(t+s)}\Big\{c_1(t+s)^{-\beta/\alpha}\mathbf{1}_{\{t<1,s<1\}} + c_2 t^{-\beta/\alpha}\mathbf{1}_{\{t<1,s\ge1\}} + c_3 s^{-\beta/\alpha}\mathbf{1}_{\{t\ge1,s<1\}} + c\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Big\} \\
&\quad\le C_5 e^{-\mu_1(t+s)}(t+s)^{-\beta/\alpha}\Bigg\{c_1\mathbf{1}_{\{t<1,s<1\}} + c_2\Big(1+\frac{s}{t}\Big)^{\beta/\alpha}\mathbf{1}_{\{t<1,s\ge1\}} + c_3\Big(1+\frac{t}{s}\Big)^{\beta/\alpha}\mathbf{1}_{\{t\ge1,s<1\}} \\
&\qquad\qquad + (t+s)^{\beta/\alpha}\mathbf{1}_{\{t\ge1,\ s\ge1\}}\Bigg\} \\
&\le C_6 e^{-(1-\delta)\mu_1(t+s)}(t+s)^{-\beta/\alpha} \quad \text{for all } \delta > 0.
\end{aligned}$$

Again notice the use of (1.0.10) in the second inequality, the semigroup property (1.0.11) in the first integral in the third inequality and Proposition A.0.8 in the fourth inequality. This concludes the proof. □

Lemma 5.4.5. *Suppose $a \geq 0$ and $\zeta > -1$. Then*

$$
\begin{aligned}
I_n^{\zeta}(a,b) :=& \int_{\{a<r_1<r_2<\cdots<r_n<b\}} \Big[(r_2-r_1)(r_3-r_2)\cdots(b-r_n)\Big]^{\zeta} dr_1 dr_2 \cdots dr_n \\
=& \frac{\Gamma(1+\zeta)^{n+1}(b-a)^{n(1+\zeta)}}{\Gamma\big(n(1+\zeta)+1\big)},
\end{aligned}
$$

where $\Gamma(.)$ is the Euler's gamma function.

Proof. We shall consider two cases here:

When $a = 0$, this is just [4, Lemma 3.5].

Assume now that $a > 0$, then integrating iteratively yields:

starting with

$$
\int_a^{r_2} (r_2 - r_1)^{\zeta} dr_1 = \frac{(r_2 - a)^{1+\zeta}}{1+\zeta}.
$$

Next,

$$
\begin{aligned}
\int_a^{r_3} (r_2-a)^{1+\zeta}(r_3-r_2)^{\zeta} dr_2 =& \int_0^{r_3-a} r_2^{1+\zeta}(r_3-a-r_2)^{\zeta} dr_2 \\
=& (r_3-a)^{2(1+\zeta)} B\Big((1+\zeta)+1, \zeta+1\Big),
\end{aligned}
$$

where we have used successively the change of variables $r_2 \to r_2 - a$ and $r_2 \to \frac{r_2}{r_3-a}$ and $B(.,.)$ is the Euler's Beta function, i.e

$$
B(c,d) = \int_0^1 u^{c-1}(1-u)^{d-1} du, \;\; c > 0, d > 0.
$$

Continuing this way, we end up with

$$
\begin{aligned}
I_n^{\zeta}(a,b) =& \frac{1}{1+\zeta}\Bigg[B\Big((1+\zeta)+1,\zeta+1\Big)B\Big(2(1+\zeta)+1,\zeta+1\Big)\cdots B\Big((n-2)(1+\zeta),\zeta+1\Big) \\
&\qquad\qquad\qquad\qquad\qquad \times \int_a^b (r_n-a)^{(n-1)(1+\zeta)}(b-r_n)^{\zeta}dr_n\Bigg] \\
=& \frac{(b-a)^{n(1+\zeta)}}{1+\zeta}\Bigg[B\Big((1+\zeta)+1,\zeta+1\Big)B\Big(2(1+\zeta)+1,\zeta+1\Big)\cdots B\Big((n-2)(1+\zeta),\zeta+1\Big) \\
&\qquad\qquad\qquad\qquad\qquad \times B\Big((n-1)(1+\zeta),\zeta+1\Big)\Bigg].
\end{aligned}
$$

The fact that $\Gamma(z+1) = z\Gamma(z)$ for all $z > 0$ together with $B(c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}$ concludes the proof. $\square$

The following result is essential for the proof of the lower bound in Theorem 3.1.4.

Proposition 5.4.6. *Fix* $\epsilon > 0$. *Let* u *be the solution of* (1.0.1) *with* $\gamma = \delta_0$. *Then for all* $x \in D_\epsilon$, *we have*

$$
\mathbb{E}|u_t(x)|^2 \geq ce^{-2\mu_1 t}\sum_{n=1}^{\infty}\Big(C\xi l_\sigma\Big)^{2n}\left(\frac{t^n}{n!}\right)^{\left(\frac{\alpha-\beta}{\alpha}\right)},
$$

for some positive constants c *and* $C = C(\alpha,\beta,d)$

Proof. By squaring the mild solution (1.0.8), we get

$$
\mathbb{E}|u_t(x)|^2 = \big(\mathcal{G}u_0\big)_t^2(x) + \xi^2\int_0^t\int_{D^2} p_D(t-s,x,y)p_D(t-s,x,z)\mathbb{E}\big|\sigma\big(u_s(y)\big)\sigma\big(u_s(z)\big)\big|\Lambda(y-z)dydzds.
$$

Now using Assumption 1.0.16, we get

$$
\mathbb{E}|u_t(x)|^2 \geq \big(\mathcal{G}u_0\big)_t^2(x) + \xi^2 l_\sigma^2\int_0^t\int_{D_\epsilon^2} p_D(t-s,x,y)p_D(t-s,x,z)\mathbb{E}\big|u_s(y)u_s(z)\big|\Lambda(y-z)dydzds.
$$

But we also have from the mild solution and Assumption 1.0.16 that

$$
\begin{aligned}
\mathbb{E}\big|u_s(y)u_s(z)\big| \geq & \mathbb{E}\big[u_s(y)u_s(z)\big] \\
& \geq \big(\mathcal{G}u_0\big)_s(y)\big(\mathcal{G}u_0\big)_s(z) + \xi^2 l_\sigma^2 \int_0^s \int_{D_\epsilon^2} p_D(s-s_1,y,y_1)p_D(s-s_1,z,z_1)\mathbb{E}\big|u_{s_1}(y_1)u_{s_1}(z_1)\big| \\
& \qquad \times \Lambda(y_1-z_1)dy_1dz_1ds_1.
\end{aligned}
$$

Thus, combining this inequality with the previous one, we get

$$
\begin{aligned}
\mathbb{E}|u_t(x)|^2 \geq & \big(\mathcal{G}u_0\big)_t^2(x) + \xi^2 l_\sigma^2 \int_0^t \int_{D_\epsilon^2} p_D(t-s,x,y)p_D(t-s,x,z)\Big|\big(\mathcal{G}u_0\big)_s(y)\big(\mathcal{G}u_0\big)_s(z)\Big|\Lambda(y-z)dydzds \\
& + \Big(\xi^2 l_\sigma^2\Big)^2 \int_0^t \int_{D_\epsilon^2} p_D(t-s,x,y)p_D(t-s,x,z) \int_0^s \int_{D_\epsilon^2} p_D(s-s_1,y,y_1)p_D(s-s_1,z,z_1) \\
& \times \mathbb{E}\Big|u_{s_1}(y_1)u_{s_1}(z_1)\Big|\Lambda(y_1-z_1)dy_1dz_1ds_1dydzds.
\end{aligned}
$$

Continuing this iteration and possibly relabeling the variables, we end up with

$$
\begin{aligned}
\mathbb{E}|u_t(x)|^2 \geq & \big(\mathcal{G}u_0\big)_t^2(x) + \sum_{n=1}^\infty \Big(\xi^2 l_\sigma^2\Big)^n \int_0^t \int_{D_\epsilon^2} \int_0^{s_1} \int_{D_\epsilon^2} \int_0^{s_2} \int_{D_\epsilon^2} \cdots \int_0^{s_{n-1}} \int_{D_\epsilon^2} \big(\mathcal{G}u_0\big)_{s_n}(y_n)\big(\mathcal{G}u_0\big)_{s_n}(z_n) \\
& \times \prod_{i=1}^n p_D(s_{i-1}-s_i,y_i,y_{i-1})p_D(s_{i-1}-s_i,z_i,z_{i-1})\Lambda(x_i-y_i)dy_idz_ids_i \\
\geq & \big(\mathcal{G}u_0\big)_t^2(x) + \sum_{n=1}^\infty \Big(\xi^2 l_\sigma^2\Big)^n \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n-1}} \int_{D_\epsilon^{2n}} \big(\mathcal{G}u_0\big)_{s_n}(y_n)\big(\mathcal{G}u_0\big)_{s_n}(z_n) \\
& \times \prod_{i=1}^n p_D(s_{i-1}-s_i,y_i,y_{i-1})p_D(s_{i-1}-s_i,z_i,z_{i-1})\Lambda(x_i-y_i)dy_idz_ids_i
\end{aligned}
$$

where we have set $y_0 := x =: z_0$ and $s_0 := t$. Now for $x \in D_\epsilon$, and for $i = 1, 2, ..., n$, choose x_i and y_i such that

$$
y_i \in B\left(x, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right) \cap B\left(y_{i-1}, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right)
$$

and

$$z_i \in B\left(x, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right) \cap B\left(z_{i-1}, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right)$$

so that

$$|z_i - z_{i-1}| < (s_{i-1}-s_i)^{1/\alpha} \quad \text{and} \quad |y_i - y_{i-1}| < (s_{i-1}-s_i)^{1/\alpha}.$$

Furthermore,

$$|z_i - y_i| < (s_{i-1}-s_i)^{1/\alpha}.$$

These estimates will ensure that, for all $i = 1, 2, ..., n$,

$$p_D(s_{i-1}-s_i, y_i, y_{i-1}) \geq C_1(s_{i-1}-s_i)^{-d/\alpha} e^{-\mu_1(s_{i-1}-s_i)},$$

$$p_D(s_{i-1}-s_i, z_i, z_{i-1}) \geq C_2(s_{i-1}-s_i)^{-d/\alpha} e^{-\mu_1(s_{i-1}-s_i)}$$

and

$$\Lambda(y_i - z_i) \geq C_3(s_{i-1}-s_i)^{-\beta/\alpha}$$

for some positive constants C_1, C_2 and C_3, thanks to Proposition 5.4.3 and Assumption 1.0.15. Moreover, since the initial solution u_0 is bounded, using Lemma A.0.2, we get

$$\big(\mathcal{G}u_0\big)_{s_n}(y_n)\big(\mathcal{G}u_0\big)_{s_n}(z_n) \geq C_4 e^{-2\mu_1 s_n}.$$

Combining these estimates yields

$$\begin{aligned}\mathbb{E}|u_t(x)|^2 \geq & C_5 e^{-2\mu_1 t} \sum_{n=1}^{\infty} \left(\xi^2 l_\sigma^2\right)^n \int_{\Theta_n(t)} \int_{A_1\times B_1} \int_{A_2\times B_2} \\ & \cdots \int_{A_n\times B_n} \prod_{i=1}^{n} (s_{i-1}-s_i)^{-\beta/\alpha}(s_{i-1}-s_i)^{-2d/\alpha} dy_i dz_i ds_i\end{aligned}$$

Where $\Theta_n(t) := \Big\{(s_0, s_1, ..., s_{n-1}) \in \mathbb{R}^n_+ : s_0 > s_1 > ... > s_{n-1}\Big\}$,
$A_i := \left\{y_i \in B\left(x, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right) \cap B\left(y_{i-1}, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right)\right\}$
and $B_i := \left\{z_i \in B\left(x, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right) \cap B\left(z_{i-1}, \frac{(s_{i-1}-s_i)^{1/\alpha}}{3}\right)\right\}$.

It is not hard to see that Volume$(A_i)\wedge$Volume$(B_i) \geq C_6(s_{i-1}-s_i)^{d/\alpha}$ for all $i = 1, 2, ..., n$. Taking into account the latter gives

$$\begin{aligned}
\mathbb{E}|u_t(x)|^2 \geq& C_7 e^{-2\mu_1 t} \sum_{n=1}^{\infty} \left(\xi^2 l_\sigma^2\right)^n \int_{\Theta_n(t)} \prod_{i=1}^{n} (s_{i-1} - s_i)^{-\beta/\alpha} ds_i \\
=& C_8 e^{-2\mu_1 t} \sum_{n=1}^{\infty} \frac{\left(C_9 \xi^2 l_\sigma^2\right)^n t^{n(1-\beta/\alpha)}}{\Gamma\Big(n(1-\beta/\alpha)+1\Big)}, \;\; C_9 = C_9(\alpha, \beta) > 0,
\end{aligned}$$

where we have used Lemma 5.4.5 with $a = 0$ and $b = t$. Finally applying Stirling's approximation A.0.1 from Proposition A.0.6 yields the desired result. $\square$

Proof of Theorem 3.1.4. For the upper bound, we combine the Burkh$\ddot{o}$lder-Davis-Gundy's, Minkowski's and Jensen's inequalities after taking the p^{th} power of the mild solution to get

$$\begin{aligned}
&\mathbb{E}|u_t(x)|^p \\
&\quad\leq 2^{p-1}\Bigg\{\Big((\mathcal{G}_D u_0)_t(x)\Big)^p \\
&\qquad\qquad + \xi^p z_p^p \left(\int_0^t \int_{D^2} p_D(t-s,x,y) p_D(t-s,x,z) \Lambda(y-z) \mathbb{E}|\sigma(u_s(y))\sigma(u_s(z))| dydzds\right)^{p/2}\Bigg\} \\
&\quad\leq 2^{p-1}\Bigg\{\Big((\mathcal{G}_D u_0)_t(x)\Big)^p \\
&\qquad\qquad + \xi^p z_p^p \left(\int_0^t \Big(\sup_{y\in D} \mathbb{E}|\sigma(u_s(y))|^p\Big)^{2/p} \int_{D^2} p_D(t-s,x,y) p_D(t-s,x,z) \Lambda(y-z) dydzds\right)^{p/2}\Bigg\}
\end{aligned}$$

Where z_p is as in Theorem 3.1.4, See for example [16]. Note that we have also used the following fact straight from Hölder's inequality:

$$\mathbb{E}|\sigma(u_s(y))\sigma(u_s(z))| \leq \left[\left(\mathbb{E}\big|\sigma\big(u_s(y)\big)\big|^2\right)^{1/2}\left(\mathbb{E}\big|\sigma\big(u_s(z)\big)\big|^2\right)^{1/2}\right]$$
$$\leq \sup_{y\in D}\mathbb{E}\big|\sigma(u_s(y))\big|^2.$$

Because u_0 is bounded, using Assumption 1.0.16 and Lemma A.0.3, we get

$$\int_0^t\left(\sup_{y\in D}\mathbb{E}|\sigma(u_s(y))|^p\right)^{2/p}\int_{D^2}p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z)dydzds$$
$$\leq L_\sigma^2\int_0^t\left(\sup_{y\in D}\mathbb{E}|u_s(y)|^p\right)^{2/p}\int_{D^2}p_D(t-s,x,y)p_D(t-s,x,z)\Lambda(y-z)dydzds$$
$$\leq L_\sigma^2\int_0^t f(s)e^{-(2-\delta)\mu_1(t-s)}(t-s)^{-\beta/\alpha}ds,$$

where $f(t) := \left(\sup_{x\in D}\mathbb{E}|u_t(x)|^p\right)^{2/p}$. Thus, defining a new function $F(t) := e^{(2-\delta)\mu_1 t}f(t)$, we get for all $t > 0$,

$$F(t) \leq c_1 + c_2\xi^2 z_p^2\int_0^t F(s)(t-s)^{-\beta/\alpha}ds.$$

Finally applying Proposition A.0.1 with $\rho = 1-\beta/\alpha$ yields the desired upper bound.

For the lower bound, we combine Proposition 5.4.6 and Proposition A.0.7 with $\upsilon = \frac{\alpha-\beta}{\alpha} > 0$, together with Jensen's inequality to get the expected bound. □

5.5 The space-time colored noise case

We would like to point out that at the time this thesis was written, we were only able to prove the upper bound result in Theorem 4.1.1 but we conjecture that the lower bound is true. Thus, we only provide the proof for the upper bound in this section. The existence-uniqueness result is proved along the way.

Proof of Theorem 4.1.1. The solution to (1.0.1) (when $\sigma = Id$) when it exists, has the following Wiener-chaos expansion in $L^2(\Omega)$:

$$u_t(x) = \sum_{n=0}^{\infty} \xi^n I_n\Big(h_n(.,t,x)\Big), \tag{5.5.1}$$

where I_0 is the identity map on $\mathbb{R}$ and I_n denotes the multiple Wiener integral with respect to F in $\mathbb{R}_+^n \times D^n$ for any $n \geq 1$, and for any $(t_1, ..., t_n) \in \mathbb{R}_+^n$, $x_1, ..., x_n \in D$, and for each (t, x), $h_n(., t, x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$. To find an explicit expression for the kernels, we follow ideas from [21, Section 4.1] as follow:

Substituting equation (5.5.1) into the Skorohod integral in equation (1.0.14), we get

$$\begin{aligned}\int_0^t \int_D p_D(t-s,x,y)u_s(y)F(\delta s,\delta y) &= \sum_{n=0}^{\infty} \int_0^t \int_D I_n\big(p_D(t-s,x,y)h_n(.,s,y)\big)F(\delta s,\delta y)\\ &= \sum_{n=0}^{\infty} I_{n+1}\Big(p_D(t-s,\widetilde{x,y)}h_n(.,s,y)\Big),\end{aligned}$$

where $p_D(t-s,\widetilde{x,y)}h_n(.,s,y)$ is the symmetrization of the function $p_D(t-s,x,y)h_n(s_1,x_1,s_2,x_2,\cdots,s_n,x_n,s,y)$ in the variables $(s_1,x_1),(s_2,x_2),\cdots,(s_n,x_n),(s,y)$, i.e.,

$$\begin{aligned}p_D(t-s,\widetilde{x,y)}h_n(.,s,y) = \frac{1}{n+1}\Big[&p_D(t-s,x,y)h_n(s_1,x_1,s_2,x_2,\cdots,s_n,x_n,s,y)\\ &+\sum_{i=1}^{n} p_D(t-s_j,x,y_j)h_n(s_1,x_1,s_2,x_2,\cdots,s_{j-1},x_{j-1},s,y,s_{j+1},x_{j+1},\cdots,s_n,x_n,s_j,\end{aligned}$$

Hence, equation (1.0.14) is equivalent to equation (5.5.1) with $h_0(t,x) = (\mathcal{G}u_0)_t(x)$ and

$$h_{n+1}(.,t,x) = p_D(t-s,\widetilde{x,y)}h_n(.,s,y). \tag{5.5.2}$$

The adaptability property of the R.F u implies that $h_n(s_1, x_1, s_2, x_2, \cdots, s_n, x_n, s, y) = 0$ if $s_j > t$ for some j. This leads to the following formula for the kernels h_n, for all $n \geq 1$:

$$h_n(t_1, x_1, ..., t_n, x_n, t, x) = \frac{1}{n!} \prod_{i=1}^{n} p_D(t_{\tau(i+1)} - t_{\tau(i)}, x_{\tau(i+1)}, x_{\tau(i)})(\mathcal{G}u_0)_{t_{\tau(1)}}(x_{\tau(1)}), \quad (5.5.3)$$

where τ denotes the permutation of $\{1, 2, \cdots, n\}$ such that $0 < t_{\tau(1)} < t_{\tau(2)} < \cdots < t_{\tau(n)} < t$, with $t_{\tau(n+1)} := t$ and $x_{\tau(n+1)} := x$.

This shows that there exists a unique solution to equation (1.0.14) and the kernels of its Wiener-chaos expansion are given by (5.5.3). In order to show the existence of a solution, it suffices to check that the kernels defined in (5.5.3) determine an adapted random field satisfying the conditions of Definition 1.0.19. This is equivalent to showing that for all (t, x) we have

$$\sum_{n=0}^{\infty} \xi^{2n} n! \|h_n(., t, x)\|^2_{\mathcal{H}^{\otimes n}} < \infty. \quad (5.5.4)$$

In which case,

$$\mathbb{E}|u_t(x)|^2 = \sum_{n=0}^{\infty} \xi^{2n} n! \|h_n(., t, x)\|^2_{\mathcal{H}^{\otimes n}}.$$

To show (5.5.4), we start with

$$\begin{aligned} n!\|h_n(., t, x)\|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{n!} \int_{[0,t]^{2n}} \int_{D^{2n}} \prod_{i=1}^{n} p_D(t_{\tau(i+1)} - t_{\tau(i)}, x_{\tau(i+1)}, x_{\tau(i)})(\mathcal{G}u_0)_{t_{\tau(1)}}(x_{\tau(1)}) \\ \times \prod_{i=1}^{n} p_D(s_{\iota(i+1)} - s_{\iota(i)}, y_{\iota(i+1)}, y_{\iota(i)})(\mathcal{G}u_0)_{s_{\iota(1)}}(y_{\iota(1)}) \prod_{i=1}^{n} \gamma(t_i - s_i) \prod_{i=1}^{n} \Lambda(x_i - y_i) d\mathbf{x} d\mathbf{y} d\mathbf{t} d\mathbf{s} \end{aligned} \quad (5.5.5)$$

Notice that, for simplicity, we write $d\mathbf{t} = dt_1...dt_n$, $d\mathbf{s} = ds_1...ds_n$, $d\mathbf{x} = dx_1...dx_n$ and $d\mathbf{y} = dy_1...dy_n$ and the permutations τ and ι of $\{1, 2, \cdots, n\}$ are such that

$$0 < t_{\tau(1)} < t_{\tau(2)} < \cdots < t_{\tau(n)} < t \quad \text{and} \quad 0 < s_{\iota(1)} < s_{\iota(2)} < \cdots < s_{\iota(n)} < t$$

with $t_{\tau(n+1)} = s_{\iota(n+1)} = t$ and $x_{\tau(n+1)} = y_{\iota(n+1)} = x$.

Now apply Lemma 5.3.1 iteratively to get

$$n!\|h_n(.,t,x)\|^2_{\mathcal{H}^{\otimes n}} \le \frac{1}{n!} C_1 e^{-(2-\delta)\mu_1 t} \int_{[0,t]^{2n}} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n \Big(t_{\tau(i+1)} + s_{\iota(i+1)} - (t_{\tau(i)} + s_{\iota(i)})\Big)^{-\beta/\alpha} d\mathbf{t} d\mathbf{s}.$$

It follows that

$$n!\|h_n(.,t,x)\|^2_{\mathcal{H}^{\otimes n}} \le \frac{1}{n!} C_1 e^{-(2-\delta)\mu_1 t} \int_{[0,t]^{2n}} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n \Big(t_{\tau(i+1)} - t_{\tau(i)}\Big)^{-\beta/\alpha} d\mathbf{t} d\mathbf{s}.$$

We first take care of the integrals $\int_{[0,t]^n} \prod_{i=1}^n \gamma(t_i - s_i) d\mathbf{s}$. For $i = 1, 2, \cdots n$,

$$\begin{aligned} \int_0^t \gamma(t_i - s_i) ds_i &= \int_{t_i - t}^{t_i} \gamma(r) dr \\ &\le \int_{-t}^0 \gamma(r) dr + \int_0^t \gamma(r) dr. \end{aligned}$$

since γ satisfies Assumption 1.0.14. Therefore, setting $\kappa(t) := \int_0^t [\gamma(-r) + \gamma(r)] dr$, we have:

$$\begin{aligned} \int_{[0,t]^{2n}} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n \Big(t_{\tau(i+1)} - t_{\tau(i)}\Big)^{-\beta/\alpha} d\mathbf{t} d\mathbf{s} &\le \kappa(t)^n \int_{[0,t]^n} \prod_{i=1}^n \Big(t_{\tau(i+1)} - t_{\tau(i)}\Big)^{-\beta/\alpha} d\mathbf{t} \\ &\le \frac{\kappa(t)^n n! C_1^{n+1} t^{n(1-\beta/\alpha)}}{\Gamma\Big(n(1-\beta/\alpha)+1\Big)}, \quad C_1 = C_1(\alpha, \beta). \end{aligned}$$

Note the use of Proposition 5.4.5 with $a = 0$ and $b = t$ in the second inequality. Now using Stirling's approximation (A.0.1) from Proposition A.0.6, we have for $n = 0, 1, 2, \cdots$

$$\sum_{n=0}^\infty \xi^{2n} n! \|h_n(.,t,x)\|^2_{\mathcal{H}^{\otimes n}} \le C_2 e^{-(2-\delta)\mu_1 t} \sum_{n \ge 0} \frac{\Big(C_1 \xi^2 \kappa\Big)^n t^{n(1-\beta/\alpha)}}{(n!)^{1-\beta/\alpha}}.$$

This proves (5.5.4).

Moreover, using Minkowski's inequality and the equivalence of norms in a fixed Wiener chaos space, it follows that

$$\begin{aligned}\Big(\mathbb{E}|u_t(x)|^p\Big)^{1/p} &\le \sum_{n=0}^{\infty}(p-1)^{n/2}\xi^n\Big(n!\|h_n(.,t,x)\|^2_{\mathcal{H}^{\otimes n}}\Big)^{1/2}\\ &\le C_2 e^{-(1-\delta)\mu_1 t}\sum_{n=0}^{\infty}(p-1)^{n/2}\xi^n\frac{\kappa^{n/2}t^{n(\alpha-\beta)/2\alpha}}{(n!)^{(\alpha-\beta)/2\alpha}}.\end{aligned}$$

Finally, using Proposition A.0.6 with $\nu = (\alpha-\beta)/2\alpha$ and with

$$\kappa(t) = 2C_H\int_0^t r^{2H-2} = C_2 t^{2H-1}$$

yields the desired upper bound in Theorem 4.1.1 . $\square$

Chapter 6

Concluding remarks

In this thesis, we investigate the SPDE (1.0.1) driven by three types of noise: a space-time white noise, a spatially-colored noise and a space-time colored noise. In the first two cases, a phase transition phenomenon was observed for the p^{th} moments of the solution, with $p \geq 2$; while another physical phenomenon– intermittency– was observed for the third type of noise. We also noted that the moments of the solution of the equation driven by the first two types of noise exhibit a similar behavior.

In the near future, in addition to our current research, we plan to investigate the following problems: inverse problems for SPDEs, inverse problems for PDEs and numerical approaches to solve these problems.

References

[1] Balan R. M and Conus D. *A note of intermittency for the fractional heat equation.*, Stat. Prob. lett. 95(2014) 6-14.

[2] Balan R. M and Conus D. *Intermittency for the wave and heat equations with fractional noise in time.* The Annals of Probability, 2016, Vol. 44, No. 2, 1488-1534.

[3] Balan R. M, Jolis M and Quer-Sardanyons L. *Intermittency for the Hyperbolic Anderson Model with rough noise in space.* Stoch. Proc. Appl. Volume 127, Issue 7, July 2017, Pages 2316-2338.

[4] Balan R. M and Tudor C. A. *Stochastic Heat Equation with Multiplicative Fractional-Colored Noise.* J. Theor. Probab. (2010) 23: 834-870.

[5] Bass R.F. *Probabilistic techniques in Analysis.* Probability and its Applications. Springer-Verlag(1995).

[6] Bertini L and Cancrini N. *The stochastic heat equation: Feyman-Kac formula and intermittency.* J. Stat. Phys. 78 13771401. MR1316109

[7] Blumenthal R. M and Getoor R. K. *Asymptotic distribution of eigenvalues for a class of Markov operators.* Pac. J. Math. 9, 399-408.

[8] Chen Z-Q and Song R. *Intrinsic Ultracontractivity and Conditional Gauge for Symmetric Stable Processes.*, J. Funct. Anal. 150, 204-239(1997).

[9] Chen Z-Q, Kim P and Song R. *Heat kernel estimates for the Dirichlet fractional laplacian.*, J. Eur. Math. Soc. 12, 1307-1329(2008).

[10] Chung K. L and Zhao Z. *From Brownian Motion to Schrödinger's Equation.* A series of Comprehensive Studies in Mathematics 312. Springer (1995).

[11] Dalang R. C. *Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDEs.* Electron. J. Prob. (1996) 4, 1-29.

[12] Dalang R.C, Koshnevisan D, Mueler C, Nualart D and Xiao Y. *A minicourse on stochastic partial differential equations.* Lecture Notes in Mathematics, vol. 1962. Springer, Berlin (2009)

[13] Evans L.C. *Partial Differential Equations.* 2nd Ed, vol 19 Graduate Studies in Mathematics (2010).

[14] Foondun M, Guerngar N and Nane E. *Some properties of non-linear fractional stochastic heat equation on bounded domains.* Chaos, Solitons and Fractals, 102 (2017) 86-93.

[15] Foondun M, Liu W and Omaba M. *Moments bonds for a class of fractional stochastic heat equations.* The Annals of Probability 2017, Vol. 45, No. 4, 2131-2153

[16] Foondun M and Koshnevisan D. *Intermittence and non-linear parabolic stochastic partial differential equations.* Elec. J. Prob. Vol. 14 (2009) No. 21 548-568

[17] Foondun M and Koshnevisan D. *On the stochastic heat equation with spatially-colored random forcing.* Trans Ameri. Math. Soc. Vol. 365 (2013), No. 1 409-458

[18] Foondun M and Nualart E. *On the behavior of the stochastic heat equations on bounded domains.* ALEA Lat. Am. J. Prob Math. Stat. (2015) 12, 551-571.

[19] Guerngar N and Nane E. *Moment bounds of a class of stochastic heat equation driven by space-time colored noise in bounded domains : https://arxiv.org/pdf/1801.00878.pdf* (submitted for publication).

[20] Hu Y, Huang J, Nualart D and Tindel S. *Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency.*, Electron. J. Probab. 20 (2015), no. 55, 1-50.

[21] Hu Y and Nualart D. *Stochastic heat equation driven by fractional noise and local time.* Probab Theory Relat Fields (2009) 143: 285-328.

[22] Kallenberg O. *Foundations of modern probability.* 2nd Ed. Probability and its Applications, Springer, NY(2001).

[23] Koshnevisan D. *Analysis of stochastic partial differential equations.* CBMS Regional Conference series in Mathematics, (2010) 119. Published for CBMS, Washington, DC, by AMS.

[24] Khoshnevisan D and Kim K. *Non-linear noise excitation and intermittency under high disorder.* Proc. Am. Math. Soc. 143(9), 40734083 (2015)

[25] Khoshnevisan D and Kim K. *Nonlinear noise excitation of intermittent stochastic PDEs and the topology of LCA groups.* The Annals of Probability 2015, Vol. 43, No. 4, 19441991. DOI: 10.1214/14-AOP925

[26] Liu W, Tian K and Foondun M,. *On some properties of a class of fractional stochastic heat equations.*, J Theor Probab (2016) DOI 10.1007/s10959-016-0684-6

[27] Marcus M. B and Rosen J. *Markov processes, Gaussian processes and local times.* Cambridge studies in advanced mathematics 100 (2006).

[28] Meerschaert M. M and Sikorskii A. Stochastic models for fractional calculus. Studies in Mathematics 43 DeGruyter (2012)

[29] Nualart D. *Malliavin calculus.* (2006) Second edition, Springer

[30] Nualart E. *Moments bonds for some fractional stochastic heat equations on the ball.* Electron. Commun. Probab. 23 (2018), no. 41, 112.

[31] Di Nunno G, Øksendal B and Proske F. *Malliavin Calculus for Lévy Processes with Applications to Finance.* 2009 Springer-Verlag

[32] Podlubny I. *Fractional Differential Equations.* Mathematics in Science and Engineering, vol 198, Academic Press (1999).

[33] Revuz D and Yor M. Continuous martingales and Brownian Motions. 3rd Ed. A series of Comprehensive Studies in Mathematics, vol 293. Springer(1999).

[34] Riahi L. *Estimates for Dirichlet heat kernels, intrinsic ultracontractivity and expected exit time on Lipschitz domains.* Comm Math Ana 15 (2013), 115-130.

[35] Samorodnitsky G and Taqqu M. *Stable non-Gaussian random processes.* Stochastic Modeling. Chapman & Hall/CRC (1994).

[36] Sato K-I. *Lévy processes and infinitely divisible distributions.* Cambridge studies in advanced mathematics 68 (1999).

[37] Walsh J.B. *An introduction to Stochastic Partial Differential Equations*. Ecole d'été de probabilités de Saint-Flour, XIV-1984, Lecture notess in Math. Springer, Berlin 1180, 265-439.

[38] Xie B. *some effects of the noise intensity upon non-linear stochastic heat equations on [0,1]. Stoch Proc Appl 126(2016), 1184-1205.*

Appendix A

Some useful results

We compile in this section some results from other authors that we have used in our paper.

Proposition A.0.1. *[15, Proposition 2.5] Let $\rho > 0$ and suppose that $f(t)$ is a locally integrable function satisfying*

$$f(t) \leq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s) ds \quad \textit{for all} \;\; t > 0,$$

where c_1 is some positive constant. Then, we have

$$f(t) \leq c_2 e^{c_3 \left(\Gamma(\rho)\kappa\right)^{1/\rho} t} \quad \textit{for all} \quad t > 0,$$

for some positive constants c_2 and c_3.

Lemma A.0.2. *[30, Proposition 3.1] For any $\epsilon \in (0, \frac{1}{2})$, there exist positive constants $c_1(\epsilon)$ such that for all $x,\ w \in D_\epsilon$ and $t > 0$,*

$$a) \int_{D_\epsilon} p_D(t,x,y) dy \geq c_1 e^{-\mu_1 t}.$$

If we further impose $|x - w| \leq t^{1/\alpha}$, then there exists a positive constant $c_2(\epsilon)$ such that

$$b) \int_{D_\epsilon \times D_\epsilon} p_D(t,x,y) p_D(t,w,z) \Lambda(y-z) dy dz \geq c_2 e^{-2\mu_1 t} t^{-\beta/\alpha}.$$

Lemma A.0.3. *[30, Proposition 3.2] For all $\delta > 0$, there exists $c_2(\delta) > 0$ such that for all $x,\ w \in D$ and $t > 0$,*

$$a) \int_D p_D(t,x,y)dy \le ce^{-\mu_1 t}$$

$$b) \int_{D\times D} p_D(t,x,y)p_D(t,w,z)\Lambda(y-z)dydz \le c_2 e^{-(2-\delta)\mu_1 t} t^{-\beta/\alpha}.$$

Theorem A.0.4. *[9, theorem 1.1] Assume $\alpha \in (0,2)$. There exists a positive constant C such that for all $x, y \in D$ and $t > 0$,*

$$C^{-1}e^{-\mu_1 t}\left[\min\left(1,\frac{\phi_1(x)}{\sqrt{t}}\right)\min\left(1,\frac{\phi_1(y)}{\sqrt{t}}\right)\min\left(t^{-d/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\boldsymbol{1}_{\{t<1\}}+\phi_1(x)\phi_1(y)\boldsymbol{1}_{\{t\ge 1\}}\right]$$
$$\le p_D(t,x,y) \le$$
$$Ce^{-\mu_1 t}\left[\min\left(1,\frac{\phi_1(x)}{\sqrt{t}}\right)\min\left(1,\frac{\phi_1(y)}{\sqrt{t}}\right)\min\left(t^{-d/\alpha},\frac{t}{|x-y|^{\alpha+d}}\right)\boldsymbol{1}_{\{t<1\}}+\phi_1(x)\phi_1(y)\boldsymbol{1}_{\{t\ge 1\}}\right]$$

Theorem A.0.5. *[34, theorem 2.2] Asume $\alpha = 2$. Then there exist positive constants c_1, C_1, c_2 and C_2 such that for all $x, y \in D$ and $t > 0$,*

$$C_1 \min\left(1,\frac{\phi_1(x)\phi_1(y)}{1\wedge t}\right)e^{-\mu_1 t}\frac{e^{-c_1\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}} \le p_D(t,x,y) \le C_2 \min\left(1,\frac{\phi_1(x)\phi_1(y)}{1\wedge t}\right)e^{-\mu_1 t}\frac{e^{-c_2\frac{|x-y|^2}{t}}}{1\wedge t^{d/2}}$$

Proposition A.0.6. *[2, Lemma A.1] For any $\nu > 0$,*

$$\sum_{k=0}^{\infty}\frac{x^k}{(k!)^\nu} \le C_1 e^{c_1 x^{1/\nu}}, \ \ x > 0,$$

for some constants $c_1(\nu)$ and $C_1(\nu) > 0$.

Moreover,

$$\Gamma\big(n\tau+1\big) \sim C_n (n!)^\tau, \ \ \tau > 0, \tag{A.0.1}$$

where C_n is such that $\lambda^{-n} \le C_n \le \lambda^n$ for some $\lambda(\alpha,\beta) > 1$.

Proposition A.0.7. *[3, Lemma 5.2] For any $\nu > 0$,*

$$\sum_{k=1}^{\infty} \frac{x^k}{(k!)^{\upsilon}} \geq c_1 e^{c_2 x^{1/\upsilon}}, \quad x > 0,$$

for some constants $c_1(\upsilon) > 0$ and $c_2(\upsilon) > 0$.

Proposition A.0.8. *[1, Lemma 2.2] For any $t > 0$ and $w \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} e^{-t|\upsilon|^{\alpha}} |\omega - \upsilon|^{-d+\beta} d\upsilon \leq K_{d,\alpha,\beta} t^{-\beta/\alpha},$$

where

$$K_{d,\alpha,\beta} := \sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\upsilon|^{-d+\beta}}{1 + |\omega - \upsilon|^{\alpha}} d\upsilon.$$

Lemma A.0.9. *[12, Lemma 6.5] Suppose $a_1, a_2, \cdots : [0,T] \to \mathbb{R}_+$ are measurable and non-decreasing. Suppose also that there exist a constant A such that for all integers $n \geq 1$, and all $t \in [0,T]$,*

$$a_{n+1}(t) \leq A \int_0^t a_n(s) ds.$$

Then,

$$a_n(t) \leq a_1(T) \frac{(At)^{n-1}}{(n-1)!} \text{ for all } n \geq 1 \text{ and } t \in [0,T].$$

Appendix B

Curriculum Vitae

Serge Ngartelbaye Guerngar was born on June 15 , 1987 in Doba, Logone Oriental province, Chad. He grew up in the capital city– N'djamena– when the family moved there permanently in 1994.

From 1994 to 1998, he attented *Descartes* elementary school in N'djamena, Chad, where he obtained the Certificat d'Etudes Primaires (CEP) in 1998.

From 1998 to 2004, he attended *Sacred-Heart* middle/high school in N'djamena, Chad where he received the Brevet d'Etudes du Premier Cycle (BEPC) in 2002 and the Baccalauréat de l'enseignement du second degré (BAC) in 2004.

From 2005 to 2011, he attended the *University of N'djamena* in N'djamena, Chad, where he received in 2007 a Diplôme Universitaire d'Etudes Scientifiques (Associate degree) in Mathematics and Physics, in 2008 a Bachelor's in Mathematics and a Maitrise es Mathematiques in 2011.

After a "sabbatical", in which he taught Mathematics for 13th graders at *Les Etoiles Brillantes* high school, he was accepted in 2012, into the Postgraduate diploma program in Mathematics at the Abdus Salam International Center for Theoretical Physics in Trieste, Italy. After completing the program, he was accepted in 2013 as a Graduate Teaching Assistant (GTA) in the PhD program in Mathematics at Auburn University in Auburn, AL. As he was completing his PhD in Mathematics, his love for Statistics became hard to contain; this is why he decided to get a Master's in Probability and Statistics along the way.

For the past 6 years, as a GTA, he designed and taught several undergraduate Mathematics and

Statistics courses both lower and upper divisions. He was also a lab assistant for graduate level Statistics courses. For his hard work, he received several awards, grants and fellowships:

Awards, grants, fellowships

- December 2018, Auburn University, Department of Math. and Statistics excellence in Research award recipient
- December 2018, Auburn University, Department of Math. and Statistics Don and Sandy Logan Fellowship recipient
- December 2017, Auburn University, Department of Math. and Statistics Don and Sandy Logan Fellowship recipient
- November 2017, Auburn University, Graduate School Graduate Research and Travel Fellowship recipient
- October 2017, Auburn University, College of Sciences and Mathematics Travel Award recipient
- January 2017, American Mathematical Society (AMS), Graduate Students Travel Grant recipient
- December 2016, Auburn University, Department of Math. and Statistics Benneth Memorial Fellowship recipient

Under the supervision of Dr. Erkan Nane, his first research topic concerned the phase transition phenomena for moments of solutions of SPDEs. Then, he studied explicit bounds for SPDEs driven by several types of noise. These results constitute the main material for this thesis. The author presented these results in several conferences (see below for a description):

Talks/Presentations delivered

- *Some properties of the solution of SPDEs in bounded domains* at the AMS Special Session on Orthogonal Polynomials, Quantum Probability, and Stochastic Analysis, the **Joint Mathematics Meetings,** January 13, 2018, San Diego, CA

- *Support dose selection in oncology trials by predicting receptor occupancy* capstone project for team 5 at **Math-to-Industry boot camp II,** July 27, 2017, the Institute of Mathematics and its applications, University of Minnesota Twin cities, Minneapolis, MN

- *Large time behavior for the solution of the fractional stochastic heat equation in bounded domains* at the AMS special session on *Stochastic Processes and Modeling*, the **Joint Mathematics Meetings,** January 5, 2017, Atlanta, GA.

- *Large time behavior of the solution of SPDEs* at the Auburn University, **Department of mathematics and Statistics Stochastic seminar,** December 1, 2016, Auburn, AL

- *Inquiry Based Learning with Technology in Precalculus: Algebra* (Poster, co-authored with Dr. Regina Greiwe J. and X. Meng) at **Auburn University's Conversations in Celebration of Teaching (CCT 2016),** January 29, 2016, Auburn, AL.

Meetings/conferences attended

- AMS Spring Southeastern Sectional Meeting, March 15-17, 2019, Auburn University, Auburn, AL

- Joint Mathematics Meeting, January 10-13, 2018, San Diego, CA

- Math-to-Industry boot camp II, June 19-July 28, 2017, the Institute of Mathematics and its Applications (IMA), University of Minnesota Twin cities, Minneapolis, MN

- Joint Mathematics meeting (JMM) 2017, January 4-7, 2017, Atlanta, GA

- Stochastic Partial Differential Equations workshop at Simons Center for Geometry and Physics, Stony Brook University, May 16-20, 2016, Stony Brook, NY

- Conversations in Celebration of Teaching (CCT 2016), January 29, 2016, Auburn University, Auburn, AL.

- AMS southeastern sectional meeting, University of Alabama in Huntsville, March 27-29, 2015, Huntsville, AL

Articles in preparation

1. *Moment bounds of a class of stochastic heat equation driven by space-time colored noise in bounded domains : https://arxiv.org/pdf/1801.00878.pdf (Jointly with Erkan Nane).* **submitted for publication**.

2. *Inverse problem for a three-parameter space-time fractional diffusion equation* (Jointly with Erkan Nane and Suleyman Ulusoy) *arXiv:1810.01543.* **submitted for publication**.

3. *Inverse stochastic partial differential equations in bounded domains* (jointly with Erkan Nane), **in preparation**

4. *Comparison principles for the fractional stochastic heat equation in bounded domains* (jointly with LeChen and Erkan Nane) **in preparation**

5. *Simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation* $\partial_t^\beta u = -(-\Delta)^{\alpha/2}u - (-\Delta)^{\gamma/2}u$ (Jointly with Erkan Nane, Ramazan Tinaztepe and Suleyman Ulusoy) **in preparation.**