Edge-Regular Graphs with $\lambda = 2$

by

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Abstract

A graph G is *edge-regular* with parameters n, d, and λ if $|V(G)| = n$, the degree of every vertex of G is d, and for any pair of adjacent vertices u and v, $|N_G(u) \cap N_G(v)| = \lambda$. We say such graphs are in $ER(n, d, \lambda)$.

In this dissertation we examine properties of edge-regular graphs, especially those with $d = 6$ and $\lambda = 2$. In particular, multiple infinite families of graphs in $ER(n, 6, 2)$ are exhibited, and it is shown that $ER(n, 6, 2)$ contains a connected graph for each $n \geq 12$.

Several ways of obtaining edge-regular graphs from old ones are discussed. These come in the form of a graph transformation called the triangle graph, in addition to multiple graph products.

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Chapter 1

Introduction

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G, $E(G)$ is the set of edges of G, and two elements of $V(G)$, say, u and v, are adjacent in G iff uv is in $E(G)$; that is, uv is an edge in G. We mainly concern ourselves with finite, simple graphs. *Finite* graphs are those with a finite number of vertices, and *simple* graphs are graphs with no repeated edges or edges from a vertex to itself, called a *loop*. The set of vertices adjacent to a vertex v in G is called the *open neighbor set of* v, and is denoted by $N_G(v) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighbor set of* v is similarly defined: $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in $V(G)$ is $d_G(v) = |N_G(v)|$. A graph G is said to be d*−regular* if every vertex of G has degree d. A *subgraph* of G is a graph $H = (V(H), E(H))$ where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *graph induced by a set of vertices* $S \subseteq V(G)$ is the graph $G[S] = (S, E(G[S]))$, where $E(G[S]) = \{uv : u, v \in S \text{ and } uv \in E(G)\}$. The *complete graph on* n *vertices* is the graph K_n such that $|V(K_n)| = n$, and for which every pair of vertices is adjacent. The *clique number of* G, denoted $\omega(G)$, is the size of the largest complete subgraph in G. In the notations d_G and N_G, if the underlying graph is obvious, the subscript G may be omitted.

If G, H are graphs, then $G + H$ is the disjoint union of G and H. mG is understood to be the disjoint union of m copies G. The $join$ of two graphs of G and H, denoted $G \vee H$, is the graph formed by taking disjoint copies of G and H and adding edges so that every vertex in $V(G)$ is adjacent to every vertex in $V(H)$.

A graph G is *edge-regular* with parameters n, d, and λ if $|V(G)| = n$, the degree of every vertex of G is d, and for any pair of adjacent vertices u and v, $|N_G(u) \cap N_G(v)| = \lambda$. We say such graphs are in $ER(n,d,\lambda)$. Edge-regular graphs are called *strongly-regular* if there exists μ such that for any pair of distinct non-adjacent vertices u, v in $V(G)$, $|N_G(u) \cap N_G(v)| = \mu$. Graphs with the aforementioned parameters are said to be in $SR(n,d,\lambda,\mu)$.

A graph G is a *regular clique assembly* with parameters n, d, and k if the following hold:
1. G is d-regular on n vertices with $k = \omega(G) \geq 2$;
2. every maximal clique in G is maximum; and
3. each edge in $E(G)$ belongs to exactly one maximal clique of G.
If G is such a graph, we say $G \in RCA(n,d,k)$.

Theorem 1.1 (Bragan et al, 2017). *For all integers* $n > d > \lambda \geq 0$, $ER(n,d,\lambda) \supseteq RCA(n,d,\lambda+2)$ *with equality if either* $\lambda \in \{0,1\}$ *or* $ER(n,d,\lambda) = \varnothing$.

It is already known that $ER(n,d,0) = RCA(n,d,2)$ consists of triangle-free d-regular graphs on n vertices. $ER(n,d,1) = RCA(n,d,3)$ has been described in [1]. Also, edge-regular graphs for which $d = \lambda + k$ for $k \in \{1,2,3\}$ have been fully described in [3].

Our goal is to describe, as best as possible, $RCA(n,d,4)$ and $ER(n,d,2)$. To do this succinctly, we later define the concepts of *spectra* for both regular clique assemblies and edge-regular graphs.

In Chapter 2 we investigate regular clique assemblies, in particular those with $d = 6$ and $\lambda = 2$. By construction, we show that $RCAS_c(6,4) = \{n : RCA(n,6,4) \text{ contains a connected graph}\} = \{16\} \cup \{20,22,24,...\}$.

In Chapter 3 we explore a different type of edge-regular graph for which $\lambda = 2$ - graphs for which the open neighbor set of any vertex induces not a disjoint union of complete graphs, as in the RCA case, but a disjoint union of cycles. The uniqueness of such graphs in $ER(n,3,2)$, $ER(n,4,2)$, $ER(n,5,2)$, and $ER(12,6,2)$ are proved. As before, special attention is given to the $d = 6$ case, and so two new infinite families of edge-regular graphs for which the open neighbor set of any vertex induces a C_6 are shown. Considering these constructions, we see that the *spectrum of* $ER(n,6,2) = S_2^c(6) = \{n : ER(n,6,2) \text{ contains a connected graph}\} = \{12,13,14,...\}$. Even stronger, this set is also equal to $\{n : ER(n,6,2)$ contains a connected graph, and for which $G[N_G(v)] \simeq C_6$ for all $v \in V(G)\}$.

In Chapter 4 we briefly discuss a new graph transformation, which we call the triangle graph. Taking the triangle graph of RCAs yields new edge-regular graphs that are otherwise not so easy to find. In particular, the triangle graph of a complete graph has a nice local structure similar to what interests us in chapters 2 and 3.

In Chapter 5, graph products of edge-regular graphs are investigated. In particular, we interest ourselves in when the Cartesian product, the tensor product, the strong product, and the lexicographic product of two edge-regular graphs is also edge-regular. In addition to the triangle graph, these graph products effortlessly produce examples of edge-regular graphs with higher values of d and λ, which is usually a difficult task.

Chapter 2

Regular Clique Assemblies

Recall that a graph G is a *regular clique assembly* with parameters n, d, and k if the following hold:

1. G is d-regular on n vertices with $k = \omega(G) \geq 2$;

2. every maximal clique in G is maximum; and

3. each edge in $E(G)$ belongs to exactly one maximal clique of G.

If G is such a graph, we say $G \in RCA(n,d,k)$.

Let $\overline{RCA}(d,k) = \bigcup\limits_{n>0} RCA(n,d,k)$. It is shown in [1] that if $G \in RCA(n,d,k)$ and $v \in V(G)$, then the number of K_ks in G is $\frac{nd}{k(k-1)}$, and $G[N_G(v)] \simeq \frac{d}{k-1}K_{k-1}$. The converse also holds: if $k-1|d$ and $G[N_G(v)] \simeq \frac{d}{k-1}K_{k-1}$ for every v in $V(G)$, then $G \in RCA(n,d,k)$. Also, recall that if $ER(n,d,\lambda) \neq \varnothing$, then $ER(n,d,\lambda) \supseteq RCA(n,d,\lambda+2)$ with equality if $\lambda = 0$ or 1. If $k = 4$ then $3|d$, so the smallest value d for which an $RCA(n,d,4)$ could be nonempty is 3. K_4, being the only connected graph in $\bigcup\limits_{n} ER(n,3,2)$, is the only connected graph in $\overline{RCA}(3,4)$. Therefore, we shall begin our search for graphs in $\overline{RCA}(d,4)$ with $d = 6$. By the previous result of Bragan, if $RCA(n,6,4) \neq \varnothing$, then n must be even and at least 12. Also, since $k-1|d$, if $RCA(n,d,4) \neq \varnothing$ for some n, then $3|d$. We begin our study of regular clique assemblies using the scaffold method.

2.1 The Scaffold Method for $RCA(n,6,4)$

We begin with what we call the primary scaffold of $RCA(n,6,4)$.

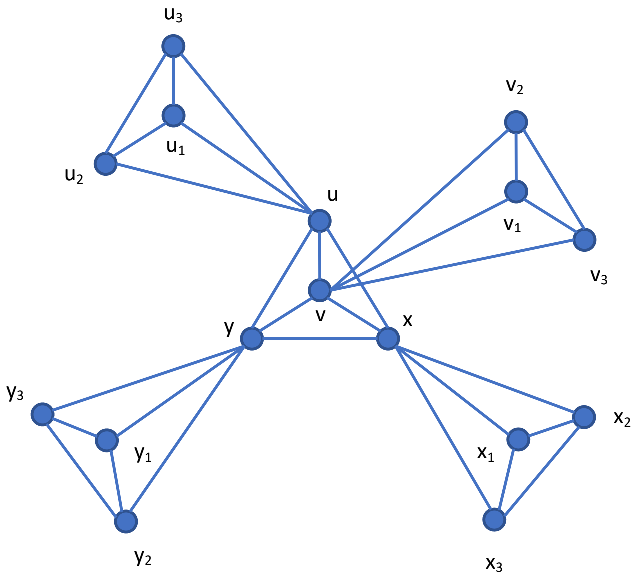

Figure 2.1: Primary Scaffold of $\overline{RCA}(6,4)$.

We say a graph G is a *scaffold* of $\overline{RCA}(d,k)$ if G is a graph on n vertices and $v \in V(G)$ implies that $G[N_G(v)] \simeq rK_{k-1}$ for some $r = r(v) = \{1, 2, ..., \frac{d}{k-1}\}$. The set of $\overline{RCA}(d,k)$ scaffolds will be denoted $SC(d,k)$. The set of $\overline{RCA}(d,k)$ scaffolds on n vertices will be denoted $SC(n,d,k)$. A vertex in such a scaffold is *finished* if $r = \frac{d}{k-1}$. If $1 \leq r < \frac{d}{k-1}$ then the vertex is *unfinished*. Finally, we define the *spectrum of* $RCA(d,k)$ to be $RCAS_c(n,d,k) = \{n : RCA(n,d,k)$ contains a connected graph$\}$. The *spectrum of* $ER(n,d,\lambda)$ is similarly defined: $S^c_\lambda(d) = \{n : ER(n,d,\lambda)$ contains a connected graph$\}$.

In the $d = 6, k = 4$ case, the possible degrees of the vertices are 3 and 6, so the unfinished vertices are of degree 3, and the finished vertices are of degree 6.

In general, a *scaffold* of $RCA(n,6,4)$ is a (not necessarily induced) connected subgraph of a graph in $\overline{RCA}(6,4)$ with the properties that any two vertices adjacent in the scaffold have exactly two common neighbors in the scaffold, any two nonadjacent vertices have at most one common neighbor in the scaffold. There are two types of vertices that a scaffold of $\overline{RCA}(d,4)$ may have - finished vertices and unfinished vertices. Finished vertices are those of degree 6 (labeled u, v, x, and y in the primary scaffold), and unfinished vertices are those of degree 3. Finished vertices are called that because, as a vertex of a graph in $\overline{RCA}(6,4)$, that particular vertex has the required degree. Unfinished vertices need a degree of 3 larger than their current degree to be a part of a graph in $\overline{RCA}(6, 4)$, and an unfinished vertex can become finished by the addition of 3 more edges in the scaffold which are adjacent to it.

The primary scaffold of $\overline{RCA}(6, 4)$, which we call the primary $(6, 4)$ scaffold, may be *finished*, that is, be made into a graph in $\overline{RCA}(6, 4)$, by adding edges, and possibly vertices, to the scaffold. In particular, the primary scaffold may be finished by adding edges to create K_4s out of the following sets of four vertices - $\{u_i, v_i, x_i, y_i\}$, $1 \leq i \leq 3$, resulting in a graph in $RCA(16, 6, 4)$. This graph is also the Cartesian product of K_4 with itself.

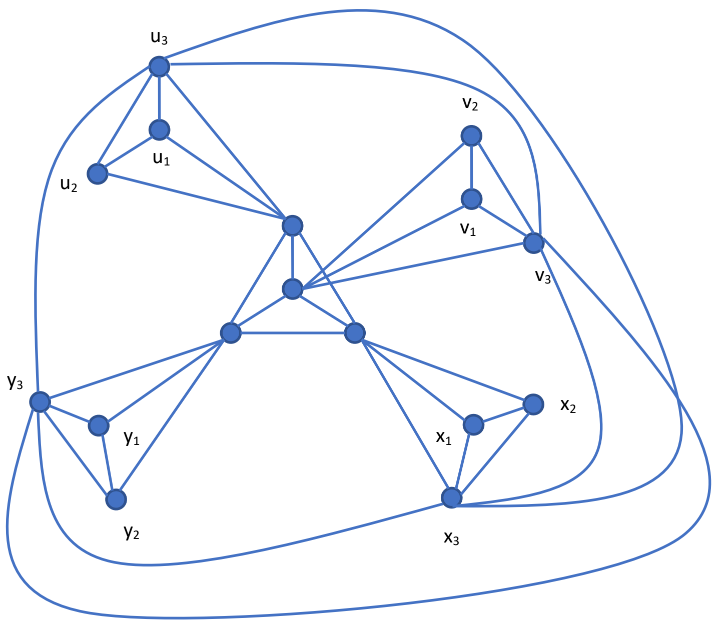

Figure 2.2: A graph in $RCA(16, 6, 4)$. Not shown are the edges of K_4s $u_1v_1x_1y_1$ and $u_2v_2x_2y_2$.

Instead of immediately finishing the primary scaffold, it is also possible to *extend* the primary scaffold, that is, add vertices and edges to create a new scaffold in $SC(6, 4)$. There are four main "methods" of extending not just the primary scaffold, but any scaffold subgraph in $\overline{RCA}(d, 4)$.

Method M1 consists of adding to the scaffold a new K_3, and adding edges so that each vertex in the new K_3 is adjacent to a previously unfinished vertex.

Method M2 consists of adding a new K_2 to the scaffold, and adding edges so that the new K_2 and two previously unfinished vertices that were at least distance 3 apart in the original scaffold form a K_4. It's necessary in M2, in addition to in M3 and M4, that the previously unfinished vertices be distance at least 3 apart from each other. If u, v are unfinished vertices in a scaffold that are distance 1 apart, then u, v share 2 common neighbors in the scaffold. Applying M2 to these vertices would result in $|N(u) \cap N(v)| = 4$, contradicting the fact that the eventual finished graph will be in $RCA(n, 6, 4) \subset ER(n, 6, 2)$. Additionally, suppose u, v are unfinished vertices that are distance 2 apart in the scaffold. Then u, v share 1 common vertex. Similar to before, applying M2 to u and v will force $|N(u) \cap N(v)| = 3$, making it impossible to obtain a graph in $RCA(n, 6, 4)$. A similar argument can be used to justify the unfinished vertices needing to be sufficiently far apart when applying M0 and M3 to a scaffold.

Method M3 consists of adding to the scaffold a new vertex, and adding edges so that the new vertex and three previously unfinished vertices, each pair distance at least 3 apart, form a K_4.

If we stick with this convention, we can also define M4 to be the process of adding edges to a scaffold so that four unfinished vertices, each pair at least distance 3 apart, form a K_4. Finally, M0 is the addition of a new K_4 to the scaffold without the addition of any other edges. It should be observed that M4 was applied three times to finish the primary scaffold, which resulted in a graph in $RCA(16, 6, 4)$.

2.1.1 Multiple Applications of M1

Let P be the primary scaffold of $RCA(n, 6, 4)$. P has 12 unfinished (degree 3) vertices and 4 finished (degree 6) vertices. Each application of M1 to the primary scaffold increases the number of vertices by 3, so m_1M1, the application of M1 to the scaffold m_1 times, results in the scaffold having $16 + 3m_1$ vertices. Since the number of vertices of a graph in $\overline{RCA}(d, 4)$ must be even, so must m_1.

It is possible to apply M1 four times to the primary scaffold to obtain a finishable scaffold subgraph of $RCA(28, 6, 4)$. The simplest way to do this is by applying M1 to the vertices u_1, v_1, x_1, and y_1. The new vertices produced by applying M1 to u_1 are $u_{1,1}, u_{1,2}$, and

$u_{1,3}$. Similarly for v_1, x_1, and y_1. After this, M4 can be applied 5 times to the vertex sets $\{u_i, v_i, x_i, y_i\}, i = 2, 3$, and $\{u_{1,i}, v_{1,i}, x_{1,i}, y_{1,i}\}, i = 1, 2, 3$, in order to finish the scaffold and obtain a graph in $RCA(28, 6, 4)$.

This same pattern can be repeated to obtain finishable scaffolds of $RCA(16 + 12m_1, 6, 4)$, for m_1 a positive multiple of 4. Suppose M1 is applied m_1 times to u_1, v_1, x_1, y_1, $u_{1,1}, v_{1,1}, x_{1,1}, y_{1,1}$, ..., $u_{1,1,...,1}, v_{1,1,...,1}, x_{1,1,...,1}, y_{1,1,...,1}$, where $\{1, 1, ..., 1\}$ is a string of $\frac{m_1}{4}$ 1's. After applying these M1's to the primary scaffold, there are $12 + 2m_1$ unfinished vertices, and so $3 + \frac{1}{2}m_1$ applications of M4 are needed to finish the scaffold. As in the previous example, M4 will be applied to the vertex sets $\{u_1, v_1, x_1, y_1\}$, $\{u_2, v_2, x_2, y_2\}$, $\{u_{1,2}, v_{1,2}, x_{1,2}, y_{1,2}\}$, $\{u_{1,3}, v_{1,3}, x_{1,3}, y_{1,3}\}$, $\{u_{1,1,2}, v_{1,1,2}, x_{1,1,2}, y_{1,1,2}\}$, $\{u_{1,1,3}, v_{1,1,3}, x_{1,1,3}, y_{1,1,3}\}$, $\{u_{1,...,1,1}, v_{1,...,1,1}, x_{1,...,1,1}, y_{1,...,1,1}\}$, $\{u_{1,...,1,2}, v_{1,...,1,2}, x_{1,...,1,2}, y_{1,...,1,2}\}$, and $\{u_{1,...,1,3}, v_{1,...,1,3}, x_{1,...,1,3}, y_{1,...,1,3}\}$, where $1, ..., 1, 1$ is a string of $\frac{m_1}{4} + 1$ 1's, and $1, ..., 1, 2$ and $1, ..., 1, 3$ are strings of $\frac{m_1}{4}$ 1's followed by a 2 or 3, respectively.

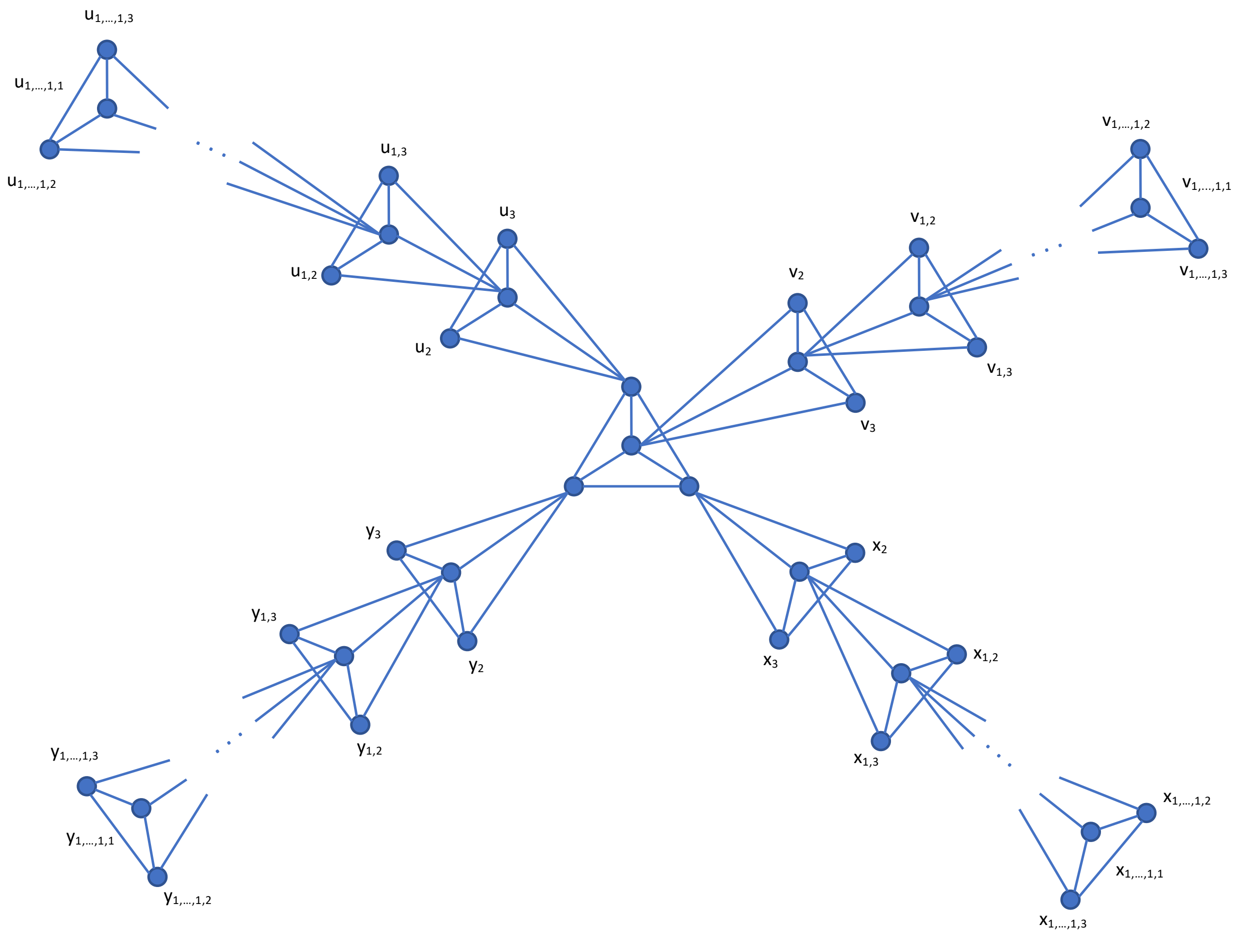

Figure 2.3: Scaffold in $SC(n, 6, 4)$ for $n = 16 + 12m$.

Using the above construction shows that $\{16 + 12m : m \geq 1\}$ are values of n for which $RCA(n, 6, 4) \neq \varnothing$. However, these are not the only permissible n values.

2.1.2 Using Other Methods

Aside from using strictly M1's to build scaffolds, it is possible to use M2's, M3's, or M4's, possibly in combination with M1's, to obtain graphs in $RCA(n, 6, 4)$ for values of n other than $n = 16 + 12k$.

Consider again the primary scaffold subgraph of $\bigcup_n RCA(n, 6, 4)$. We may apply M2 four times in order to get a scaffold of $RCA(24, 6, 4)$. M2 can be applied to vertex sets $\{u_3, v_2\}$, $\{v_3, x_2\}$, $\{x_3, y_2\}$, and $\{y_3, x_2\}$. We label the vertices of the new K_2's as $\{u'_3, v'_2\}$, $\{v'_3, x'_2\}$, $\{x'_3, y'_2\}$, and $\{y'_3, x'_2\}$, respectively.

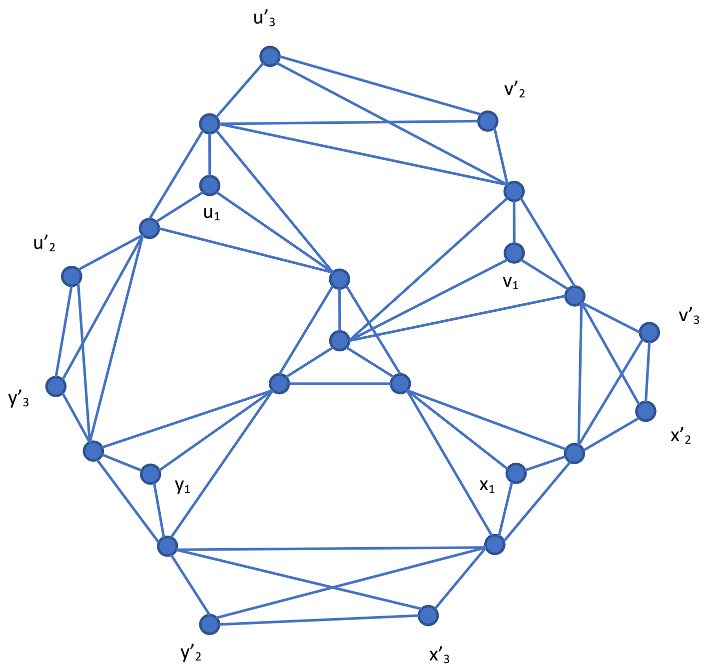

Figure 2.4: Scaffold of $RCA(24,6,4)$.

Now, M0 can be applied three times to finish the scaffold. We add all possible edges to vertex sets $\{u_1, v_1, x_1, y_1\}$, $\{u_2', v_2', x_2', y_2'\}$, and $\{u_3', v_3', x_3', y_3'\}$. The resulting graph is in $RCA(24,6,4)$.

Further, the application of 4 M1's to the vertices u_1, v_1, x_1, and y_1 of the unfinished scaffold shown above yields a finishable scaffold of $RCA(36,6,4)$. Continuing in this manner allows us to generate graphs in $RCA(24+12m,6,4)$ for all $m \geq 1$.

M3 can be applied four times to the primary scaffold to generate a scaffold of $RCA(20,6,4)$. Let M3 be applied to vertex sets $\{u_1, v_1, x_1\}$, $\{v_2, x_2, y_2\}$, $\{x_3, y_3, u_3\}$, and $\{u_2, v_3, y_1\}$. Name the new vertices arising from these applications of M3 y_4, u_4, v_4, and x_4, respectively. Now, applying M0 to these four new vertices yields a graph in $RCA(20,6,4)$.

Before finishing this scaffold, as you might guess, we can apply M1 to the four new vertices u_4, v_4, x_4, and y_4, and then finish the scaffold as before, or apply M1 a multiple of 4 more times before finishing the resulting scaffold. This yields graphs in $RCA(20+12m,6,4)$.

There is only one way to apply M1 twice to the primary scaffold so that the new scaffold may be finished in order to obtain a graph in $RCA(22, 6, 4)$. In particular, M1 must be applied to two neighbors in the primary scaffold. Suppose M1 is applied to u_2 and u_3 in the primary scaffold. Let's call the new vertices $u_{2,1}, u_{2,2}, u_{2,3}$ and $u_{3,1}, u_{3,2}, u_{3,3}$, respectively. This new scaffold may be finished by adding edges so that the following vertex sets induce $K_4's$: $\{u_{2,1}, u_{3,1}, v_1, x_1\}$, $\{u_{2,2}, u_{3,2}, x_2, y_2\}$, $\{u_{2,3}, u_{3,3}, v_2, y_1\}$, and $\{u_1, v_3, x_3, y_3\}$. This yields a graph in $RCA(22, 6, 4)$.

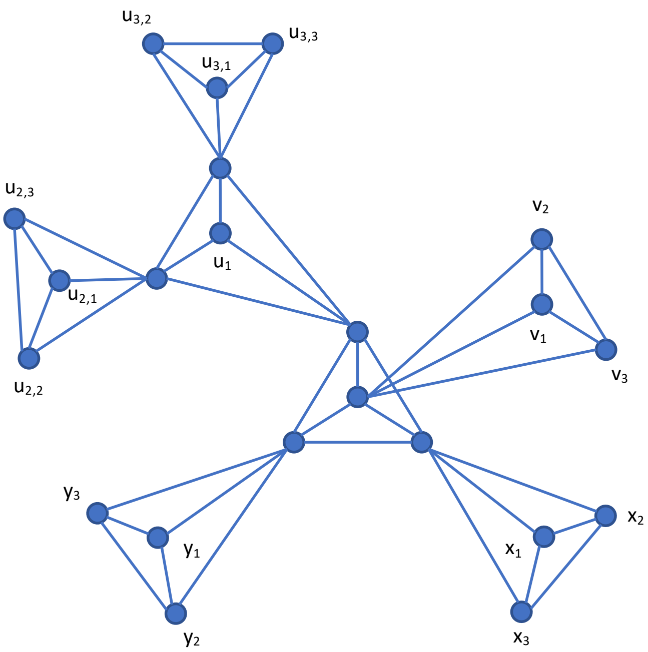

Figure 2.5: Scaffold of $RCA(22, 6, 4)$.

Naturally, instead of finishing the aforementioned scaffold of $RCA(22, 6, 4)$, we can apply 4M1 to u_1, v_1, x_1, and y_1 and finish the resulting scaffold to obtain a graph in $RCA(34, 6, 4)$. Multiple applications of 4M1 to the scaffold gives us graphs in $RCA(22 + 12k, 6, 4)$.

Applying $M1$ to two neighbors, as in the $RCA(22, 6, 4)$ case, is occasionally necessary to fill out the spectrum of $RCA(n, 6, 4)$. This particular type of application of 2M1 will be denoted 2M1*.

One such case is for $RCA(26+12k,6,4)$. Take the primary scaffold with 2M1* applied to u_2 and u_3. Then apply M2 to $\{v_3,x_2\}$ and to $\{x_3,y_2\}$ to create new vertices $\{v'_3,x'_2\}$ and $\{x'_3,y'_2\}$, respectively.

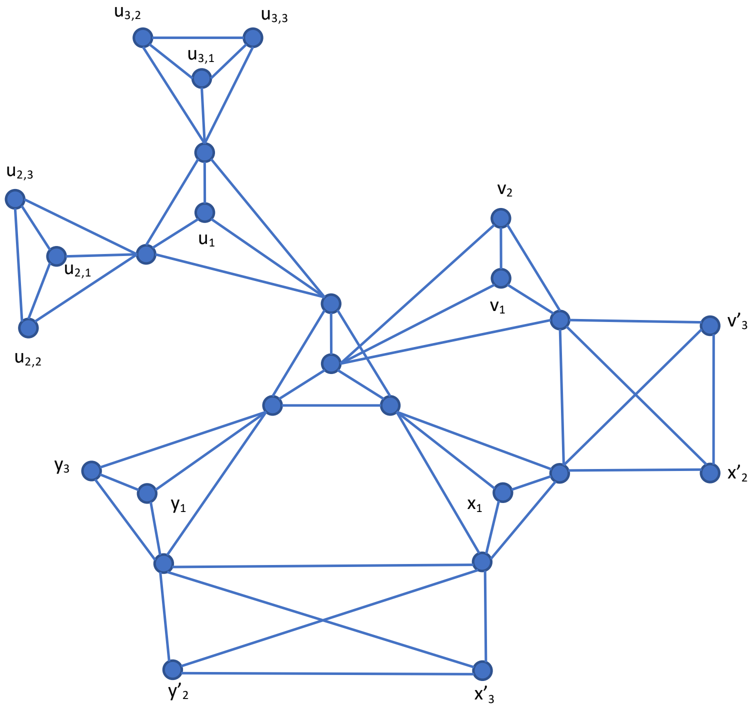

Figure 2.6: Scaffold of $RCA(26,6,4)$.

If we add edges to this new scaffold to form K_4's out of vertex sets $\{u_1,v_1,x_1,y_1\}$, $\{u_{2,2},u_{3,2},x'_2,y'_2\}$, $\{u_{2,3},u_{3,3},v'_3,x'_3\}$, and $\{u_{2,1},u_{3,1},v_2,y_3\}$, we finish the scaffold and obtain a graph in $RCA(26,6,4)$. Again, if we apply M1 to u_1,v_1,x_1, and y_1 in the previous scaffold, it is easy to add edges to get a graph in $RCA(38,6,4)$. Multiple applications of 4M1 in this manner gives us graphs in $RCA(26+12k,6,4)$.

Consider again the scaffold of $RCA(22,6,4)$. We apply M2 four times to $\{u_{2,2},y_3\}$, $\{u_{3,3},v_2\}$, $\{v_3,x_2\}$, and $\{v_3,x_2\}$, calling the new vertices $\{u'_{2,2},y'_3\}$, $\{u'_{3,3},v'_2\}$, $\{v'_3,x'_2\}$, and $\{v'_3,x'_2\}$, respectively. The resulting scaffold is shown below.

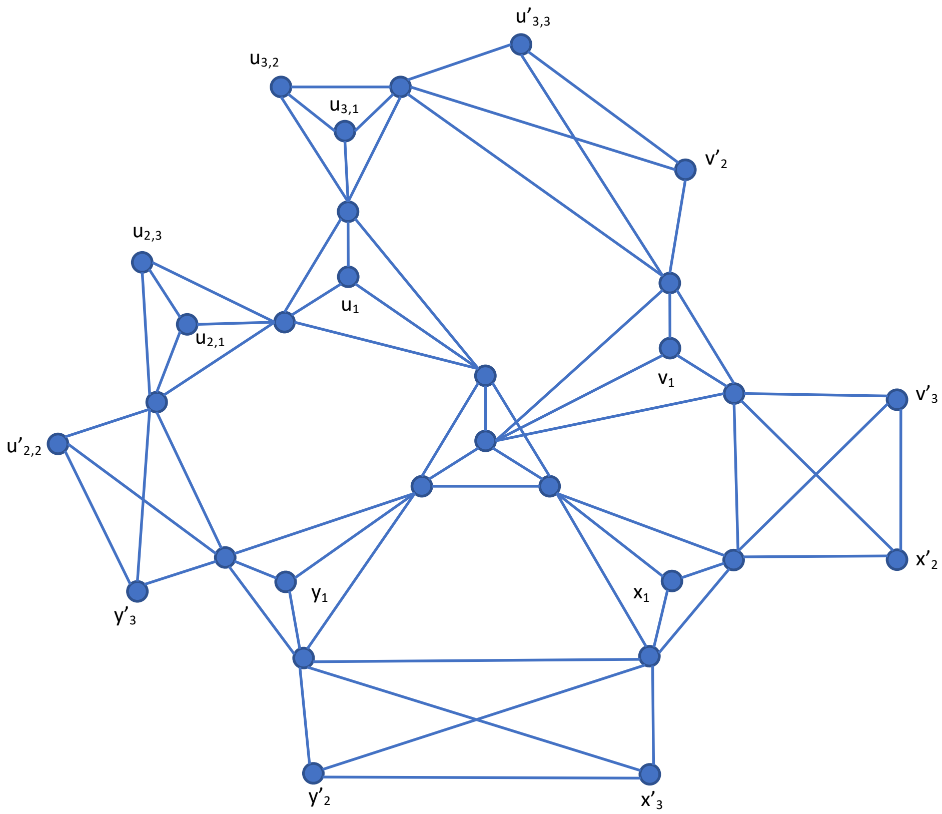

Figure 2.7: Scaffold of $RCA(30, 6, 4)$.

Now, we can finish this scaffold by adding all possible edges to the vertex sets $\{u'_{2,2}, v'_2, x'_2, y'_2\}$, $\{u'_{2,3}, u'_{3,2}, x_1, y_1\}$, $\{u_{2,1}, u_{3,1}, v_1, x'_3\}$, and $\{u_1, u'_{3,3}, v'_3, y'_3\}$. The resulting graph is in $RCA(30, 6, 4)$. Yet again, if we apply M1 to each of the vertices $u'_{2,2}, v'_2, x'_2$, and y'_2, the resulting scaffold of $RCA(42, 6, 4)$ can be finished. Doing this multiple times yields graphs in $RCA(30+12k, 6, 4)$.

Using these constructions tells us that $RCAS_c(6, 4) \supseteq \{16\} \cup \{20, 22, 24, \ldots\}$. Recall that if $ER(n, d, \lambda) \neq \varnothing$ then $n \geq 3(d - \lambda)$. This means that it's possible that 12, 14, or 18 are also members of $RCAS_c(6, 4)$. We now prove that this is not the case.

Suppose $G \in RCA(n, 6, 4)$ for some n, and pick an arbitrary K_4 from G, say, $uvxy$. The closed neighbor set of that K_4, $N_G(u, v, x, y)$, induces the primary $(6, 4)$ scaffold, which has 16 vertices. Thus, 12 and 14 cannot be in $RCAS_c(6, 4)$.

Now, consider the primary scaffold of $RCA(6, 4)$. If $RCA(18, 6, 4) \neq \varnothing$, then there must be two other vertices, w_1 and w_2 which are not adjacent to u, v, x, or y. w_1 must be adjacent to six other vertices. Suppose w_1 is part of a K_4 with, say, u_1, v_1, and x_1. w_1 must also be part of another K_4 with three of the remaining vertices. If w_2 and y_1 are two of those vertices, then w_1 has no other choice for a third neighbor without violating the $\lambda = 2$ condition. Therefore, 18 is also not in $RCAS_c(6,4)$, resulting in the following theorem.

Theorem 2.1. $RCAS_c(6,4) = \{16\} \cup \{20, 22, 24, ...\}$.

Chapter 3

Edge-regular graphs with other local structure properties

While regular clique assemblies are edge-regular graphs for which the open neighbor set of each vertex induces a disjoint union of complete graphs of the same order, there is another natural restriction on the subgraph induced by the open neighbor set of any vertex that will be satisfied only by edge-regular graphs with $\lambda = 2$. That restriction is that the open neighbor sets induce cycles. If $G \in ER(n, d, 2)$ and the open neighbor set of any vertex induces a cycle, the length of the cycle must be equal to d. We start by exhibiting some examples of such graphs. Since the shortest cycle in a simple graph is of length 3, we start with that case.

3.1 $G[N(v)] \simeq C_d$ for $3 \leq d \leq 5$

It is easy to see that $G = K_4$ is the only connected graph in $ER(n, 3, 2)$ for which $G[N_G(v)] \simeq C_3$ for each v in $V(G)$.

Similarly, $G = K_6 - M$, where M is a perfect matching of K_6, is the only connected graph in $ER(n, 4, 2)$ for which $G[N_G(v)] \simeq C_4$ for each v in $V(G)$. To see this, we construct such a graph. First, notice that a necessary subgraph of G must be a wheel with, say, vertex v in the center, and a 4-cycle $wxyz$ on the outside. Vertex x currently has degree 3, so we must find another neighbor for it. If w's fourth neighbor is z, then x and z would share v, w, and y as neighbors, violating the $\lambda = 2$ condition. So the fourth neighbor of x must be a new vertex, say, a. a must also be adjacent to w and y to satisfy $G[N_G(x)] \simeq C_4$, so we add the edges aw and ay. Now a must be adjacent to precisely one more neighbor which is adjacent to both w and y. The only possibility for this last vertex is w. So by adding the edge az, we obtain the aforementioned graph in $ER(6, 4, 2)$.

In the same vein, and with a little more work, we can construct a graph G in $ER(n, 5, 2)$ for which $G[N_G(v)] \simeq C_5$ for each v in $V(G)$. Again, notice that since the open neighbor set of a vertex v induces a cycle, say, $uwxyz$, we construct G starting with this wheel. Vertex x currently has degree 3, so it must be adjacent to 2 more vertices. These vertices cannot be u or z as it would make the number of vertices adjacent to both v and x more than 2. So x must be adjacent to 2 more vertices, say, a and b. To satisfy $G[N_G(x)] \simeq C_5$, we also add the edges aw, ab, and by. Vertex y has degree 4 and must be adjacent to a new vertex, say, c. Then we add the edges bc and cz. w must be adjacent to a new vertex d, and d must also be adjacent to a and u. u must be adjacent to a new vertex e, which is also adjacent to d and z. Now, since z has degree 5, we must complete the cycle induced by its neighbors by adding edge ce to the graph. a has degree 4 and needs a new vertex, say, f, to be adjacent to. Then, to complete the cycle around a, edges bf and df must be added. Finally, to complete the cycles around b and d, we must add edges cf and ef, respectively. This completes G, which is in $ER(12, 5, 2)$. We call this graph I, the *icosahedron graph*.

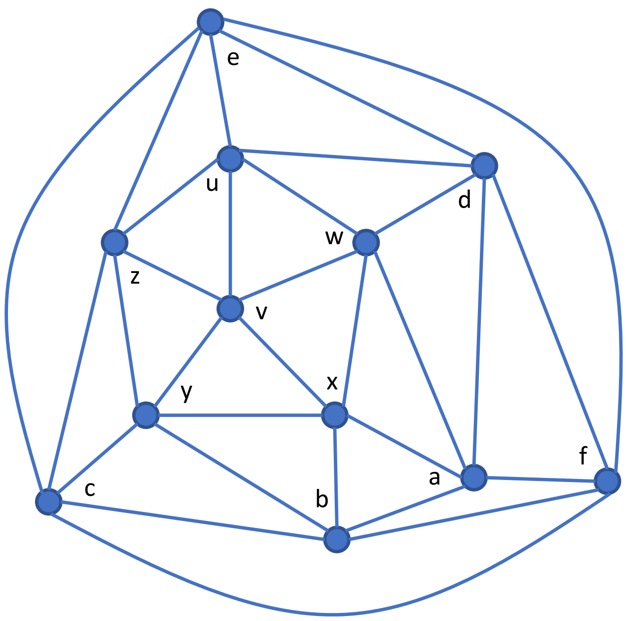

Figure 3.1: The icosahedron graph.

These constructions also show that K_4, $K_6 - M$, and I are the only connected graphs in $ER(n, d, 2)$, for $d = 3, 4, 5$, respectively, regardless (or, as we say in Alabama, irregardless) of what we want the open neighbor set of any vertex to look like. The situation is much more interesting for the $d = 6$ case.

3.2 $G[N(v)] \simeq C_6$

$G = K_{4,4,4} - \{$the edges of a 2-factor consisting of disjoint triangles$\} \in ER(12, 6, 2)$ and satisfies $G[N_G(v)] \simeq C_6$ for each v in $V(G)$. This has been shown to be the only graph in $ER(12, 6, 2)$ [2], and until recently was the only known edge-regular graph whose open neighbor sets induce a C_6. However, as we will show, this is far from the only graph in $ER(n, 6, 2)$ with that property. The following two constructions exhibit infinite families of edge-regular graphs satisfying the aforementioned property.

3.2.1 First construction

We defined the class of graphs, for now called "New" graphs, as follows:

Let m, n be positive integers, both greater than or equal to 3, and at least one greater than or equal to 4. We define $New(m, n)$ to be the graph uniquely determined by m, n as follows. The vertex set of $New(m, n)$ consists of all ordered pairs (i, j) from Z_mxZ_n. Two vertices (i, j) and (i', j') are adjacent in $New(m, n)$ if and only if one of the following three conditions holds:

1) $i' = i + 1$ and $j' = j$

2) $i' = i$ and $j' = j + 1$

3) $i' = i + 1$ and $j' = j + 1$

Theorem 3.1. *$G = New(m, n) \in ER(mn, 6, 2)$ and $G[N_G(v)] \simeq C_6$ for all v in $V(G)$.*

Proof. Clearly $|V(New(m, n))| = mn$.

Each vertex (i, j) is adjacent to the following six vertices: $(i - 1, j)$ and $(i + 1, j)$ by adjacency condition 1; $(i, j - 1)$ and $(i, j + 1)$ by adjacency condition 2; and $(i - 1, j - 1)$ and $(i + 1, j + 1)$ by adjacency condition 3.

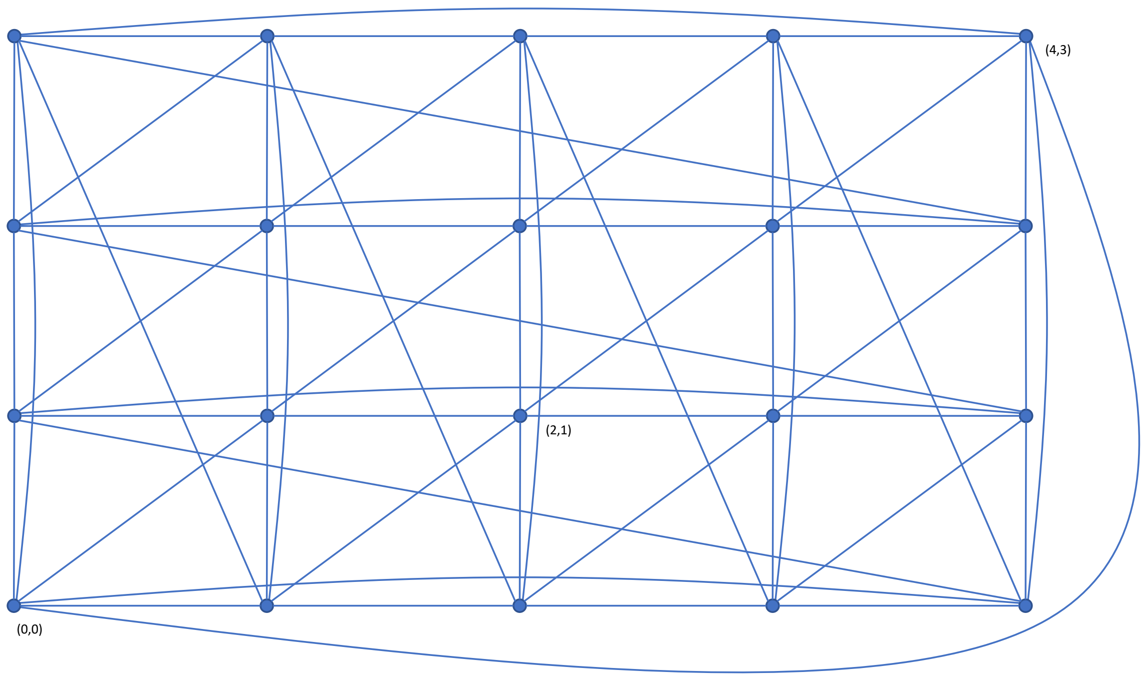

Figure 3.2: The graph $New(5, 4)$ with three labeled vertices.

To show $\lambda = 2$, suppose (i, j) and (i', j') are adjacent.

If $(i', j') = (i-1, j)$ then both (i, j) and (i', j') are adjacent to $(i-1, j-1)$ and $(i, j+1)$.

If $(i', j') = (i+1, j)$ then both (i, j) and (i', j') are adjacent to $(i+1, j+1)$ and $(i, j-1)$.

If $(i', j') = (i, j-1)$ then both (i, j) and (i', j') are adjacent to $(i-1, j-1)$ and $(i+1, j)$.

If $(i', j') = (i, j+1)$ then both (i, j) and (i', j') are adjacent to $(i-1, j)$ and $(i+1, j+1)$.

If $(i', j') = (i-1, j-1)$ then both (i, j) and (i', j') are adjacent to $(i-1, j)$ and $(i, j-1)$.

If $(i', j') = (i+1, j+1)$ then both (i, j) and (i', j') are adjacent to $(i, j+1)$ and $(i+1, j)$.

The above paragraph also shows that the neighbor set of any vertex in $New(m, n)$ induces a C_6. □

3.2.2 Second construction

For the next construction, first consider the Paley graph on 13 vertices, $P(13)$. $P(13)$ has vertex set $V(P(13)) = Z_{13}$, and a vertex v in $V(P(13))$ is adjacent to $v \pm 1$, $v \pm 3$, and $v \pm 4$. $P(13)$ is in $SR(13, 6, 2, 3)$. The neighbors of v induce the 6-cycle $(v+1, v-3, v-4, v-1, v+3, v+4)$.

$P(13)$ is only one example of the class of Paley graphs $P(q)$. There are many generalizations of Paley graphs, but one generalization in particular, called the *Paley-like graphs*, is useful to us.

Let a, b, and n be positive integers with $a < b$ and $n \geq 3(a+b)+1$. We define the *Paley-like graph* $P(a, b, n)$ to be the graph on the vertex set Z_n, with vertex v adjacent to all vertices which differ from v by a, b, and $a + b$. Using this notation, the familiar Paley graph on 13 vertices is given by $P(1, 3, 13)$.

Theorem 3.2. *If* $b \geq 3$*, then* $G = P(1, b, n)$ *is a connected graph in* $ER(n, 6, 2)$ *and* $G[N(v)] \simeq C_6$ *for all* v *in* $V(G)$.

Proof. $P(1, b, n)$ has n vertices.

Let $v \in V(P(1, b, n)) = Z_n$. Since v is adjacent to $v \pm 1$, $v \pm b$, and $v \pm (1+b)$, $d \leq 6$. We need to check that no two of those six vertices are the same. It suffices to check that the "farthest" vertices from v are different. These farthest vertices are $v + (1 + b)$ and $v - (1 + b)$. Since $n \geq 3(1+b)+1$, the difference in the labels of these two vertices is at least $(1+b)+(1+b) = 2(1+b)$, or $3(1 + b) + 1 - 2(1 + b) = b$. Since their difference is not 0, these vertices are different, and so $d = 6$.

We list the vertices to which v is adjacent as follows: $N(v) = \{v + 1, v - b, v - (1 + b), v - 1, v + b, v + (1 + b)\}$. Listing v's neighbors this way suggests that the open neighbor set of v induces a C_6 as long as we can show that each vertex in $N(v)$ is adjacent to exactly two other vertices in $N(v)$.

$v+1$ is adjacent to $v-b$ and $v+(1+b)$ as permissible differences are $1+b$ and b, respectively. $v + 1$ is not adjacent to $v + b$ because $b - 1 \notin \{1, b, 1 + b\}$. Also, $v + 1$ is not adjacent to $v - 1$ or $v - (1 + b)$ as $2, 2 + b \notin \{1, b, 1 + b\}$ either.

$v + b$ is adjacent to $v - 1$ and $v + (1 + b)$ as their differences are $1 + b$ and 1, respectively. $v + b$ is not adjacent to $v - b$ or $v - (1 + b)$ as their distances are $2b$ and $2b + 1$, respectively.

$v + (1 + b)$ is adjacent to $v + 1$ and $v + b$ as their differences are b and 1, respectively. $v - 1$ and $v - b$ are not neighbors of v as their differences are $2 + b$ and $1 + 2b$, respectively. Also,

$v-(1+b)$ is not a neighbor of $v+(1+b)$ as their difference is either $2(1+b)$ or, as we've shown before, at least $2+b$.

By symmetry, analogous results hold for $v-1, v-b$, and $v-(1+b)$. This shows that $G[N(v)] \simeq C_6$ and, as a consequence, $\lambda = 2$.

Finally, the subgraph of $P(1,b,n)$ with only the edges resulting from the difference of 1 is a Hamiltonian cycle on all n vertices, so $P(1,b,n)$ is connected. □

The construction for $P(1,3,n)$ fills in the rest of the spectrum of $ER(n,6,2)$, resulting in the following theorem:

Theorem 3.3. $S_2^c(6) = \{n \mid n \geq 12\}$. *Further, for each integer $n \geq 12$, there exists a connected graph in $ER(n,6,2)$ for which the open neighbor set of each vertex induces a C_6.*

$a = 1$ is not the only value that yields such a graph. The previous result generalizes nicely.

Theorem 3.4. *Let a, b be positive integers with $a < b$, $b \geq 3$, and $b \neq 2a$. If at least one of $gcd(a,n)$, $gcd(b,n)$, or $gcd(a+b,n)$ is equal to 1, then $P(a,b,n)$ is a connected graph in $ER(n,6,2)$ with the property that $G[N(v)] \simeq C_6$ for all v in $V(P(a,b,n))$.*

Proof. This proof follows the previous proof, with a few addenda.

That $|N(v)| = 6$ for any vertex v is clear. The condition that $b \neq 2a$ is necessary to make sure that $G[N(v)] \simeq C_6$ and does not merely contain C_6 as an induced subgraph. $b \neq 2a$ assures us that $v-a$ and $v+a$ are not adjacent. If $v+a$ and $v+b$ were adjacent, or if $v-a$ and $v-b$ were adjacent, that would imply that $b-a \in \{a, b, a+b\}$; that is, $b-a = a$. $b \neq 2a$ takes care of that possibility, too.

Finally, if $gcd(a,n) = 1$, then the cycle $\{a, 2a, 3a, ..., (n-1)a, 0\}$ includes all n vertices. Similarly for $gcd(b,n) = 1$ or $gcd(a+b,n) = 1$. Thus, $P(a,b,n)$ is connected, provided at least one of those is true. □

Chapter 4

Triangle graphs

Let G be a regular clique assembly. Define the triangle transformation $T : G \to T(G)$ as follows: T sends K_3s in G to vertices in $T(G)$, and vertices are adjacent in $T(G)$ iff the corresponding K_3s share an edge in G.

Theorem 4.1. *If* $G \in RCA(n,d,k) \subseteq ER(n,d,\lambda)$*, where* $k = \lambda + 2 \geq 4$*, then* $T(G) \in ER(n',d',\lambda')$*, where* $n' = \frac{nd\lambda}{6}$*,* $d' = 3(\lambda - 1)$*, and* $\lambda' = \lambda = k - 2$*.*

Proof. We find n' by counting the number of triangles in G. G has $\frac{nd}{2}$ edges and each edge in G is part of λ triangles. So there are $\frac{nd\lambda}{2}$ edge-triangle pairs in G. But this counts each edge 3 times, so there are $\frac{nd\lambda}{6}$ triangles in G.

If uvw is a triangle in G, then d' is the number of triangles in G sharing an edge with uvw. That is, d' is the number of vertices in G adjacent to uv, uw, or vw. u and v share $\lambda - 1$ neighbors that are not w, u and w share $\lambda - 1$ neighbors that are not v, and v and w share $\lambda - 1$ neighbors that are not u. Therefore, $d' = 3(\lambda - 1)$.

Suppose uvw and uvx are triangles in G with $w \neq x$. Since $k \geq 4$, the triangles uwx and vwx share an edge with uvw and an edge with uvx. Additionally, there are $\lambda - 2$ triangles with edge uv. So $\lambda' = 2 + (\lambda - 2) = \lambda$. □

Corollary 4.1.1. $T(K_n) \in ER(\binom{n}{3}, 3(n-3), n-2)$ *for* $n \geq 4$*.*

Corollary 4.1.2. *The only connected regular clique assembly* G *for which* $G \simeq T(G)$ *is* $G = K_4$*.*

Proof. Set $n = \frac{nd\lambda}{6}$ and $d = 3(\lambda - 1)$. □

Proposition 4.1.1. *If $H = T(K_n)$ and $n \geq 4$, then $H[N_H(v)] \simeq K_{n-3} \square K_3$ for all $v \in V(H)$.*

Proof. Let $H = T(K_n)$, and let $v \in V(H)$ be the vertex defined by the triangle xyz in G. There are three types of vertices in $N_H(v)$: vertices whose first, second, or third coordinate is one of the $n-3$ vertices in G that are not x, y, or z, respectively. Let the vertices of the first, second, and third type belong to T_1, T_2, and T_3, respectively. A vertex in any T_i is adjacent to every other vertex in that T_i, so each T_i induces a K_{n-3}. Additionally, each vertex in T_i is adjacent to exactly one vertex in each T_j, $j \neq i$, inducing $n-3$ copies of K_3. □

This construction can be used to generate edge-regular graphs, with arbitrarily large λ, having that structure.

Chapter 5

Products of Edge-Regular Graphs

In this chapter we consider the Cartesian product, the tensor product, the strong product, and the lexicographic product on edge-regular graphs.

5.1 The Cartesian Product

Let G_1, G_2 be graphs. The Cartesian product of G_1 and G_2, $G_1 \square G_2$, is defined by: $V(G_1 \square G_2) = V(G_1)\text{x}V(G_2)$, and $(u,v) \sim (u',v')$ in $G_1\text{x}G_2$ iff either $u = u'$ and $v \sim v'$, or $u \sim u'$ and $v = v'$. Let $H = G_1 \square G_2$.

Since $V(H) = V(G_1)\text{x}V(G_2)$, $|V(H)| = |V(G_1) \square G_2| = |V(G_1)||V(G_2)| = n_1 n_2$.

If $(u,v) \in V(G_1)\text{x}V(G_2) = V(H)$, then (u,v) is adjacent, in H, to, and only to, the $d_{G_1}(u)$ pairs (u',v), $u' \in N_{G_1}(u)$, and the $d_{G_2}(v)$ pairs (u,v'), $v' \in N_{G_2}(v)$. Thus, if G_i is d_i regular, $i = 1,2$, then H is $d_1 + d_2$ regular.

Suppose $(u,v) \simeq (u',v')$ in H. Then either $u = u'$ and $v \simeq v'$, or $u \simeq u'$ and $v = v'$. In the former case, (u,v) and (u',v') have as many common neighbors in H as do v and v' in G_2. In the other case, (u,v) and (u',v') have as many common neighbors in H as do u and u' in G_1. So, for the Cartesian product of G_1 and G_2 to be edge-regular, we need $\lambda_1 = \lambda_2$.

Theorem 5.1. *If $G_1 \in ER(n_1,d_1,\lambda)$ and $G_2 \in ER(n_2,d_2,\lambda)$, then $G_1 \square G_2 \in ER(n_1n_2, d_1 + d_2, \lambda)$.*

5.2 The Tensor Product

Let G_1, G_2 be graphs. The tensor product of G_1 and G_2, $G_1 \mathrm{x} G_2$, is defined by: $V(G_1 \mathrm{x} G_2) = V(G_1) \mathrm{x} V(G_2)$, and $(u, v) \sim (u', v')$ in $G_1 \mathrm{x} G_2$ iff $u \sim u'$ in G_1 and $v \sim v'$ in G_2.

Let $G_1 \in ER(n_1, d_1, \lambda)$ and $G_2 \in ER(n_2, d_2, \lambda)$, and let $H = G_1 \mathrm{x} G_2$.

Since $V(H) = V(G_1) \mathrm{x} V(G_2)$, $|V(H)| = |V(G_1) \mathrm{x} G_2| = |V(G_1)||V(G_2)| = n_1 n_2$.

Suppose $(u, v) \in V(H)$. Then $d_H(u, v) = |N_H(u, v)| = |\{(u', v') : (u', v') \sim (u, v) \text{ in } H\}| = |\{(u', v') : u' \sim u \text{ in } G_1 \text{ and } v' \sim v \text{ in } G_2\}| = d_1 d_2$.

Let $(u, v), (u', v') \in V(H)$ and $(u, v) \sim (u', v')$. This means $u \sim u'$ in G_1 and $v \sim v'$ in G_2. $N_{G_1}(u) \cap N_{G_1}(u') = \{u_1, u_2, ..., u_{\lambda_1}\}$ and $N_{G_2}(v) \cap N_{G_2}(v') = \{v_1, v_2, ..., u_{\lambda_2}\}$. So $N_H(u, v) \cap N_H(u', v') = \{(u_i, v_j) : 1 \leq i \leq \lambda_1, 1 \leq j \leq \lambda_2\}$, and so $|N(u, v) \cap N(u', v')| = \lambda_1 \lambda_2$. This results in the following theorem:

Theorem 5.2. *If $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$, then $G_1 x G_2 \in ER(n_1 n_2, d_1 d_2, \lambda_1 \lambda_2)$.*

Proposition 5.2.1. *The tensor product of two graphs is regular iff the factor graphs are regular.*

Proof. The if direction was already proved.

Suppose G_1 is regular of degree d and G_2 is not regular. That is, G_2 has two vertices, say, v and v', with different degrees t and t', respectively. The vertices (u, v) and (u, v') in $G_1 \mathrm{x} G_2$ have degrees dt and dt', respectively. The same result holds if neither G_1 nor G_2 is regular. □

Proposition 5.2.2. *If $G_1 x G_2$ is edge-regular, then G_1 and G_2 are also edge-regular.*

Proof. Suppose $G_1 \in ER(n_1, d_1, \lambda_1)$ and G_2 is a d_2-regular graph on n_2 vertices, but is not edge-regular. G_2 has two pairs of adjacent vertices, say, $v \sim v'$ and $w \sim w'$, such that $|N_{G_2}(v) \cap N_{G_2}(v')| = t$ and $|N_{G_2}(w) \cap N_{G_2}(w')| = t'$, with $t \neq t'$. Let $H = G_1 \mathrm{x} G_2$. If $u \sim u'$ in G_1, then $|N_H(u, v) \cap N_H(u', v')| = \lambda_1 t$ and $|N_H(u, w) \cap N_H(u', w')| = \lambda_1 t'$. Thus, H is not edge-regular. □

5.3 The Strong Product

Define the strong product of two graphs G_1, G_2, $G_1 \boxtimes G_2$, to be the graph with vertex set $V(G_1)\text{x}V(G_2)$, and $(u,v) \sim (u',v')$ in $G_1 \boxtimes G_2$ iff $u = u'$ and $v \sim v'$, or $v = v'$ and $u \sim u'$, or $u \sim u'$ and $v \sim v'$.

Let $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$, and let $H = G_1 \boxtimes G_2$. Since $V(H) = V(G_1)\text{x}V(G_2)$, $|V(H)| = |V(G_1)\text{x}V(G_2)| = |V(G_1)||V(G_2)| = n_1 n_2$.

Suppose $(u,v) \in V(H)$. Then $d_H(u,v) = |\{(u',v') : u' = u \text{ and } v' \sim v\}| + |\{(u',v') : u' \sim u \text{ and } v = v'\}| + |\{(u',v') : u' \sim u \text{ and } v' \sim v\}| = d_2 + d_1 + d_1 d_2$.

To find λ_H, let $(u,v), (u',v') \in V(H)$ and $(u,v) \sim (u',v')$. There are three cases for the adjacency of (u,v) and (u',v'), and some further subcases for the adjacency of (u,v) and (u',v') to a third vertex (u'',v'').

1. $u = u'$ and $v \sim v'$

   (a) $u'' = u = u'$ and $v'' \sim v$ and $v'' \sim v'$

   There is 1 choice for u'' and λ_2 choices for v'' so there are λ_2 possible (u'',v'').

   (b) $u'' \sim u$ and $u'' \sim u'$ and $v'' = v$ or $v'' = v'$

   There are d_1 choices for u'' and 2 choices for v'' so there are $2d_1$ possible (u'',v'').

   (c) $u'' \sim u$ and $u'' \sim u'$ and $v'' \sim v$ and $v'' \sim v'$

   There are d_1 choices for u'' and λ_2 choices for v'' so this case contributes $d_1\lambda_2$ possible (u'',v'').

   (u,v) and (u',v') have $\lambda_2 + 2d_1 + d_1\lambda_2$ common neighbors in H.

2. $u \sim u'$ and $v = v'$

   (a) $u'' \sim u$ and $u'' \sim u'$ and $v'' = v = v'$

   There are λ_1 choices for u'' and 1 choice for v'' so there are λ_1 choices for (u'',v'')

   (b) $u'' = u$ or $u'' = u'$ and $v'' \sim v$ or $v'' \sim v'$

   There are 2 choices for u'' and d_2 choices for v'' so there are $2d_2$ choices for (u'',v'')

(c) $u'' \sim u$ or $u'' \sim u'$ and $v'' \sim v$ and $v'' \sim v'$ There are λ_1 choices for u'' and d_2 choices for v'' so there are $\lambda_1 d_2$ choices for (u'', v'')

(u, v) and (u', v') have $\lambda_1 + 2d_2 + d_2\lambda_1$ common neighbors in H.

3. $u \sim u'$ and $v \sim v'$

   (a) $u'' \sim u$ and $u'' \sim u'$ and $v'' = v$ or $v'' = v'$

   There are λ_1 choices for u'' and 2 choices for v'' so there are $2\lambda_1$ choices for (u'', v'').

   (b) $u'' = u$ or $u'' = u$ and $v'' \sim v$ and $v'' \sim v'$

   There are 2 choices for u'' and λ_2 choices for v'' so there are $2\lambda_2$ choices for (u'', v'').

   (c) $u'' = u$ and $v'' = v'$

   There is 1 choice for u'' and 1 choice for v'' so there is 1 choice for (u'', v'').

   (d) $u'' = u'$ and $v'' = v$

   There is 1 choice for u'' and 1 choice for v'' so there is 1 choice for (u'', v'').

   (e) $u'' \sim u$ and $u'' \sim u'$ and $v'' \sim v$ and $v'' \sim v'$

   There are λ_1 choices for u'' and λ_2 choices for v'' so there are $\lambda_1\lambda_2$ choices for (u'', v'').

   In this case, in total, (u, v) and (u', v') have $2\lambda_1 + 2\lambda_2 + 2 + \lambda_1\lambda_2$ common neighbors in H.

If we can get the permissible values of λ_H to agree, then we can conclude that $H \in ER(n_1 n_2, d_1 + d_2 + d_1 d_2, \lambda_H)$.

$$\lambda_H = d_1\lambda_2 + 2d_1 + \lambda_2 = d_2\lambda_1 + 2d_2 + \lambda_1 = \lambda_1\lambda_2 + 2\lambda_1 + 2\lambda_2 + 2$$

$$(d_1 + 1)(\lambda_2 + 2) - 2 = (d_2 + 1)(\lambda_1 + 2) - 2 = (\lambda_1 + 2)(\lambda_2 + 2) - 2$$

$$(d_1 + 1)(\lambda_2 + 2) = (d_2 + 1)(\lambda_1 + 2) = (\lambda_1 + 2)(\lambda_2 + 2)$$

These equalities imply that $d_1 = \lambda_1 + 1$ and $d_2 = \lambda_2 + 1$, which in turn imply that $G_1 \simeq K_{\lambda_1+2}$ and $G_2 \simeq K_{\lambda_2+2}$. Thus, if $G_1 \boxtimes G_2$ is edge-regular, then $G_1 \simeq K_{n_1}$ and $G_2 \simeq K_{n_2}$. On the other hand, if $G_1 \simeq K_{n_1}$ and $G_2 \simeq K_{n_2}$, then $G_1 \boxtimes G_2 \in ER(n_1n_2, n_1n_2 - 1, n_1n_2 - 2)$. That is, $K_{n_1} \boxtimes K_{n_2} \simeq K_{n_1n_2}$.

Theorem 5.3. *A strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 is edge-regular iff $G_1 = K_{n_1}$ and $G_2 = K_{n_2}$ for some n_1, n_2.*

5.4 The Lexicographic Product

Define the lexicographic product of two graphs G_1 and G_2, $G_1[G_2]$, to be the graph with vertex set $V(G_1)$x$V(G_2)$, and $(u, v) \sim (u', v')$ in $G_1[G_2]$ iff $u \sim u'$, or $u = u'$ and $v \sim v'$.

Let $G_1 \in ER(n_1, d_1, \lambda)$ and $G_2 \in ER(n_2, d_2, \lambda)$, and let $H = G_1[G_2]$.

Since $V(H) = V(G_1)$x$V(G_2)$, $|V(H)| = |V(G_1)$x$V(G_2)| = |V(G_1)||V(G_2)| = n_1n_2$.

Suppose $(u, v) \in V(H)$. Then $d_H(u, v) = |\{(u', v') : u' \sim u\}| + |\{(u', v') : u' = u$ and $v' \sim v\}| = d_1n_2 + d_2$.

To find λ_H, let $(u, v), (u', v') \in V(H)$ and $(u, v) \sim (u', v')$. There are three cases for the adjacency of (u, v) and (u', v'), and some further subcases for the adjacency of (u, v) and (u', v') to a third vertex (u'', v'').

1. $u \sim u'$ and $v = v'$

   (a) $u'' \sim u$ and $u'' \sim u'$

   There are λ_1 choices for u'' and n_2 choices for v'' so there are $\lambda_1 n_2$ choices for (u'', v'')

   (b) $u'' = u$ and $v'' \sim v = v'$

   There is 1 choice for u'' and d_2 choices for v'' so there are d_2 choices for (u'', v'')

   (c) $u'' = u'$ and $v'' \sim v = v'$ There is 1 choice for u'' and d_2 choices for v'' so there are d_2 choices for (u'', v'')

(u, v) and (u', v') have $2d_2 + \lambda_1 n_2$ common neighbors.

2. $u \sim u'$ and $v \sim v'$

   (a) $u'' = u$

   There is 1 choice for u'' and d_2 choices for v'' so there are d_2 possible (u'', v'').

   (b) $u'' = u'$

   There is 1 choice for u'' and d_2 choices for v'' so there are d_2 possible (u'', v'').

   (c) $u'' \sim u$ and $u'' \sim u'$

   There are λ_1 choices for u'' and n_2 choices for v'' so this case contributes $\lambda_1 n_2$ possible (u'', v'').

   (u, v) and (u', v') have $2d_2 + \lambda_1 n_2$ common neighbors.

3. $u \sim u'$ and $v \neq v'$ and $v \nsim v'$

   (a) $u'' = u$

   There is 1 choice for u'' and d_2 choices for v'' so there are d_2 possible (u'', v'').

   (b) $u'' = u'$

   There is 1 choice for u'' and d_2 choices for v'' so there are d_2 possible (u'', v'').

   (c) $u'' \sim u$ and $u'' \sim u'$

   There are λ_1 choices for u'' and n_2 choices for v'' so this case contributes $\lambda_1 n_2$ possible (u'', v'').

   (u, v) and (u', v') have $2d_2 + \lambda_1 n_2$ common neighbors.

4. $u = u'$ and $v \sim v'$

   (a) $u'' \sim u = u'$

   There are d_1 choices for u'' and n_2 choices for v'' so there are $d_1 n_2$ possible (u'', v'').

(b) $u'' = u = u'$ and $v'' \sim v$ and $v'' \sim v'$

There is 1 choice for u'' and λ_2 choices for v'' so there are λ_2 possible (u'', v'').

(u, v) and (u', v') have $d_1 n_2 + \lambda_2$ common neighbors.

Theorem 5.4. *If* $G_1 \in ER(n_1, d_1, \lambda_1)$ *and* $G_2 \in ER(n_2, d_2, \lambda_2)$ *then* $G_1[G_2] \in ER(n_1 n_2, d_1 n_2 + d_2, \lambda)$ *if* $\lambda = 2d_2 + \lambda_1 n_2 = d_1 n_2 + \lambda_2$.

Corollary 5.4.1. *Suppose* $G_1 \in ER(n_1, d_1, \lambda_1)$*,* $G_2 \in ER(n_2, d_2, \lambda_2)$*, and* $G_1[G_2]$ *is edge-regular.*

1. *If* $G_1 = K_{n_1}$ *then* $d_1 = \lambda_1 + 1$*, and so* $d_2 = \frac{n_2 + \lambda_2}{2}$.
2. *If* $G_2 \simeq K_{n_2}$ *then* $G_1 \simeq K_{n_1}$*, and* $K_{n_1}[K_{n_2}] \simeq K_{n_1 n_2} \in ER(n_1 n_2, n_1 n_2 - 1, n_1 n_2 - 2)$.

5.5 Subgraphs induced by open neighbor sets

In this section we explore what the subgraph induced by the open neighbor set of any vertex in a product graph looks like.

Proposition 5.4.1. *Suppose* $G_1 \in ER(n_1, d_1, \lambda)$ *and* $G_2 \in ER(n_2, d_2, \lambda)$*. Also suppose* $G_1[N_{G_1}(u)] \simeq H_1$ *for all* $u \in V(G_1)$ *and* $G_2[N_{G_2}(v)] \simeq H_2$ *for all* $v \in V(G_2)$*. If* $H = G_1 \square G_2$ *then* $H[N_H(u, v)] \simeq H_1 + H_2$ *for all* (u, v) *in* $V(H)$*.*

Proof. Let $H = G_1 \square G_2$ be the Cartesian product of G_1 and G_2, and $(u, v) \in V(H)$. The open neighbor set of (u, v) consists of all vertices of the form $N((u, v)) = \{(u', v') : u' = u \text{ in } G_1 \text{ and } v' \sim v \text{ in } G_2 \text{ or } u' \sim u \text{ in } G_1 \text{ and } v' = v \text{ in } G_2\} = \{(u', v') : u' = u \text{ in } G_1 \text{ and } v' \sim v \text{ in } G_2\} \cup \{(u', v') : u' \sim u \text{ in } G_1 \text{ and } v' = v \text{ in } G_2\} = (N_{G_1}(u) \text{x} \{v\}) \cup (\{u\} \text{x} N_{G_2}(v))$. So the induced neighbor set of (u, v) is $H[N_H((u, v))] \simeq G_1[N_{G_1}(u)] + G_2[N_{G_2}(v)]$. □

This need not be edge-regular. However, if $\lambda = 2$, we can at least say the following:

Theorem 5.5. *If* $G \in ER(n, d, 2)$ *and* $G[N_G(v)] \simeq H$ *for all* v *in* $V(G)$*, then* H *is a disjoint union of cycles, not necessarily of equal length.*

Proof. Suppose $G \in ER(n_G, d_G, \lambda_G)$, and $H \simeq G[N_G(v)]$ for some v in $V(G)$. $|V(H)| = d_G$ since $|N_G(v)| = d_G$. If $v' \in V(G)$ and $v' \sim v$ in G, then v and v' have λ_G neighbors in common in G. So the degree of v in H is equal to λ_G. If $v \sim v'$ in G, then those two vertices have common neighbor set $\{v_i : 1 \le i \le \lambda_G\}$. So $d_H = |N_H(v')| = \lambda_G$. It follows that when $\lambda_G = 2$, H is 2-regular, and so H must be a disjoint union of cycles. □

While the above theorem shows that the open neighbor set of a vertex of a graph in $ER(n, d, 2)$ induces a disjoint union of edge-regular graphs (in particular, cycles), we cannot conclude that this disjoint union is edge-regular. If $G = K_4 \square I$, for example, then $G \in ER(48, 8, 2)$. But the open neighbor set of a vertex in G induces $C_3 + C_5$, which is not edge-regular.

If $\lambda > 2$, then H need not be a disjoint union of other edge-regular graphs. For example, consider $T(K_5) \in ER(10, 6, 3)$ and vertex $a \in V(T(K_5))$. $H = T(K_5)[N_{T(K_5)}(a)]$ is a 3-regular graph on 6 vertices, but H contains a pair of adjacent vertices b, c with one common neighbor i, while c, d is another pair of adjacent vertices in H having two common neighbors e, f.

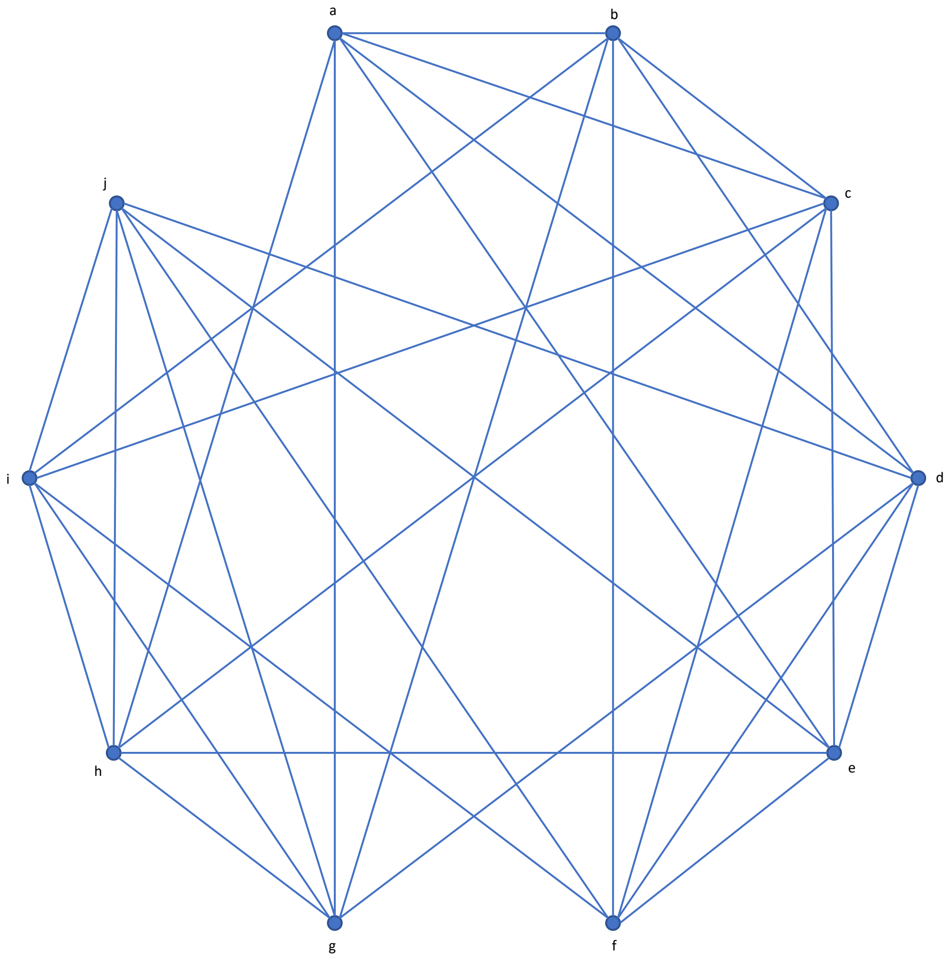

Figure 5.1: The graph $T(K_5)$.

Chapter 6

Future Work

6.1 Chapter 2

Can every graph in $RCA(n, 6, 4)$ be obtained by the scaffold building process?

How useful is the scaffold method for building graphs in $RCA(n, 3t, 4)$ for $t > 2$?

6.2 Chapter 3

Do there exist graphs G in $ER(n, d, 2)$, with $d > 6$, such that $G[N_G(v)] \simeq C_d$ for all v in $V(G)$?

6.3 Chapter 4

For which graphs G in $ER(n, d, \lambda)\backslash RCA(n, d, \lambda + 2)$ is $T(G)$ also edge-regular?

6.4 Chapter 5

It is known that the complete graph on n vertices, K_n, is in $ER(n, n-1, n-2)$, and that the Turán graph $T(mp, p)$, the complete regular p-partite graph on mp vertices, is in $ER(mp, m(p-1), m(p-2))$. $n-1 = \frac{n+(n-2)}{2}$ and $m(p-1) = \frac{mp+m(p-2)}{2}$. For which other graphs in $ER(n, d, \lambda)$ is $d = \frac{n+\lambda}{2}$? Answering this would give us more understanding of the lexicographic product as it relates to edge-regular graphs.

By taking the Cartesian product of $K_4 \in ER(4,3,2)$ with a graph in $ER(n,6,2)$, we see that $S_2^c(9) \supseteq \{4t : t \geq 12\}$. What else can we say about $S_2^c(9)$ and, in general, $S_2^c(3t)$ for $t > 2$?

Similarly, by taking the Cartesian products of $K_6 \in ER(6,5,4)$, $K_{2,2,2,2} \in ER(8,6,4)$, $K_{4,4,4} \in ER(12,8,4)$, and $T(K_6) \in ER(20,9,4)$, we get that $S_4^c(d) \neq \varnothing$ for all $d \geq 5$, except possibly $d = 7$. Two obvious questions arise from this observation: Is $S_4^c(7) = \varnothing$, and what are $S_4^c(d)$ equal to for all permissible values of d?

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