

# **A Method for Obtaining New Conservation Quantities and a Solution Method for the N-Body Collision**

by

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## Abstract

Exact, numerical, and perturbative methods are commonly used to solve dynamical systems of equations. Many systems' solutions cannot be written in terms of elementary functions thus numerical and perturbative solutions take over and provide only approximate solutions (even though perturbative solutions are, in fact, series representations of the exact solutions, but truncating higher order terms only provide approximate solutions). Therefore, if possible, solving for unique, exact solutions should be of utmost importance when determining the dynamics of a system. In general, to solve a dynamical system, there must be a sufficient number of invariant equations about symmetries, or conservation quantities, to reduce the degrees of freedom of the system.

One of the most renown dynamical systems is the n-body problem; this thesis will aim to provide a sufficient number of conservation quantities for the special case of the n-body problem that involves only simultaneous elastic collisions of free particles by analytical and experimental methods as well as present general formulations of newly theorized conservation quantities associated with any dynamical system. This thesis also presents the proof of existence of analytical solutions in the three-body collision stated above using Bezout's theorem.

## Acknowledgments

I have come across a time on this planet going somewhere, who knows where, where reflection of what this thesis means but only a small contribution to comprehensive human knowledge; we can confidently assume this to be of infinitesimal impact relative to what is relevant and influential on much grander, universal scales. Despite its insignificance (believe me, I have tried to maximize its potential impact on the scientific world), like any naturally inquisitive human does, I attempt to find answers for my questions, and in this thesis was an attempt at one question I had. Clarification of the perspective is key to being humble given this spring that life has bestowed upon me in this remarkable time in human history. Inspiration for writing this paper didn't come from what the accomplishment could potentially be, but it rather was for the opportunity to contribute to human scientific knowledge using my personal strengths, in which case, from the reader's perspective, hopefully will be presented with much elegance.

With that being said, I'm able to pursue these predictably intriguing answers in a formal document such as this because there are people in my life who have sacrificed their valuable time and efforts to make this possible, and I'd like to express my appreciation for them. First, I'd like to give a formal thanks to my adviser, Dr. Hartfield, for believing in my efforts and allowing me to have this opportunity. I'd also like to thank my parents for supporting me throughout my academic life and their willingness to give up part of their

life for mine; it is a blessing to have been raised under their guidance. Last, but certainly not least, I'd like to thank my brother and my close friends for supporting me throughout this journey as well. It is truly a blessing to have experienced this, and inexplicable gratitude towards those who are deserving was hopefully conveyed.

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## Nomenclature

$m$	mass
$m_0$	rest mass
$n$	number of bodies or degrees of freedom
$t$	time
$p$	momentum or conjugate momentum
$q$ or $x$	position
$\dot{q}$ or $v$	velocity
$\ddot{q}$ or $a$	acceleration
$\vec{F}$	force vector
$\mathcal{H}$	Hamiltonian
$\Omega$	constants of motion
$\gamma$	Lorentz factor
$\beta$	percent speed of light
$c$	speed of light
$\mathcal{M}$	Momentergy
$C_n$	constants of motion
$W$	work
$KE$	kinetic energy
$U$	potential energy



$\mathcal{L}$	Lagrangian
$\mathcal{S}$	action
$E$	total energy for a specified system
$\alpha$	symmetry associated with momentenergy
$\mathcal{A}$	Absement

## Chapter 1

### INTRODUCTION

The  $n$ -body problem has been a long-standing unsolved problem in mathematics and physics due to the nonintegrability in which the number of bodies in the system is greater than two. Much of the  $n$ -body problem (particularly, the three-body problem) has been studied by Poincaré, who in 1887, along with Bruns, showed that there is no closed form solution to the gravitational three-body problem in current mathematical theory. The nonintegrability criterion for a dynamical system can be explained several ways, one of which is the Liouville definition - for every function of the phase space, there exists a maximal set of Poisson commuting invariants. Several proofs showing the lack of a sufficient number of commuting invariants have followed the work of Bruns and Poincaré, further proving the nonintegrability characteristic of such a system [1,7]. The two-body problem using the general formulation of Newtonian equations of motion has been completely solved using the momentum, energy, center of mass, and angular momentum constants [5]; however, generally, with the existence of an external field (for instance, the gravitational field in the celestial case), the solution of the  $n$ -body problem, where  $n > 2$ , cannot be written in terms of elementary functions.

In all cases of the proofs, however, a potential field is externally applied to the system. With a field applied to the system, a new term arises in the Hamiltonian of the system: the potential term. The potential term is the source of the nonintegrability mainly because of its meromorphic nature. It is of utmost relevance to maintain a stance that, in

this thesis, there will not be a closed form solution to the three-body problem presented, but rather the tools to possibly solve it in the future, or at least uncover certain dynamics that have not been investigated to this point. However, many systems have not been explored to which experimental and analytical (to some degree) results are achievable. One such system is a simultaneous elastic collision of three free particles which will be the system used as an example in this thesis.

Kinetic (or elastic) collision theory, one in which there is no change in kinetic energy [2], is important in understanding ionization or dissociation via different types of collisions in the atomic, molecular, nuclear, and electric fields, as well as various aspects of kinetic gas theory [4], aside from being an important general physics and math problem. As with any dynamical system, one of the steps to solving the problem is reducing the degrees of freedom of the system by using constants of motion [3,6,8]. Since the two-body elastic collision already has a well-known solution, it is the  $n$ -body elastic collision that is of most interest. If we look at current collision theory, the only constants of motion for elastic collisions useful for solving the problem are the linear momentum and energy constants; generalized angular momentum is available but is only useful in providing conditions under which collisions occur [4]. Current theory on the  $n$ -body collision does not provide enough constants of motion to describe a system with positions and velocities relative to a nonaccelerating origin, and it uses an approximate constant for calculating when collisions occur; specifically, it assumes the squared angular momentum term,

$$\Lambda^2 = \frac{1}{2} \sum_{ij} (\Lambda_{ij})^2 \tag{1.1}$$

is small [4]. To solve the many-body collision, we must provide more constants that are not approximations and can be applied in the inertial frame, namely first integrals, in order

to sufficiently reduce the degrees of freedom of the system [6]. Although the following theory can be applied to any  $n$ -body elastic collision, the three-body collision will be the primary focus of this thesis.

Analytically, this paper will introduce the concept behind where additional constants of motion come from, if they exist, and what physical revelations come from them when applied mathematically. Upon examining the results of the particular system this thesis will explore (the three body collision) and comparing them to the known current collision theory, the conservation quantities used in determining the solutions will be accessed for their mathematical validity.

The layout of the paper is as follows. First there will be a discussion of the use of Hamiltonian mechanics in the theory, as the phase space variables will be used throughout, and we will use the Hamiltonian to prove the constants are actually constant. Then derivations of the conservation quantities and reasoning behind the derivations in Newtonian mechanics will be presented along with a relativistic correction and an application to free particles. Following this chapter is an explanation of how the foregoing ties to Noether's theorem with new symmetries along with an attempt to derive general potentials using results of Noether's theorem. Then the three-body simultaneous elastic collision analysis is explored. With the goal of providing formulations of new integrals for the  $n$ -body problem, much discussion of physical systems, albeit gravitational or not, must follow. The physical relation of the theorized quantities to each system applicable is important to understanding the system's progression through time, space, rotation, and other possible symmetries.

## Chapter 2

### HAMILTONIAN MECHANICS FORMALISM USAGE

Let us begin by defining a classical closed system (for now to be holonomic) in a non-accelerating frame of reference with  $n$  degrees of freedom and  $n$  generalized (canonical) position coordinates  $q_1, q_2, \dots, q_n$  and  $n$  conjugate momenta  $p_1, p_2, \dots, p_n$ . The phase space of a system is defined as the space of all possible states of the system, and an arbitrary set of coordinates  $(q_j, p_j)$  corresponds to exactly one point in the phase space [16]. The system's Hamiltonian,  $\mathcal{H}$ , depends on the  $2n$  number of phase space coordinates  $(q_j, p_j)$  and possibly a time coordinate  $t$  and can be written as

$$\mathcal{H}(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n, t) \equiv \mathcal{H}(q_j, p_j, t) \quad \text{where } j = 1, 2, \dots, n \quad 2.1$$

Generally, the Hamiltonian equates to the total energy of a closed system because in closed systems there is a time-invariant Hamiltonian that can be written, namely,  $\frac{\partial \mathcal{H}}{\partial t} = 0$ , which, if held true, means that the Hamiltonian equals the total energy and is conserved throughout the dynamical process [15]. Let us assume non-relativistic motion in a closed system to which the total energy or Hamiltonian, or the sum of all the kinetic energies and potential energies (if an external field is applied), has the general form

$$\mathcal{H} = \sum_{j=1}^n \left( \frac{p_j^2}{2m_j} + V(q_j) \right) \quad 2.2$$

The Poisson bracket operation,  $\{A, B\}$ , takes the functions of phase space and time  $A(q_j, p_j, t)$  and  $B(q_j, p_j, t)$  and outputs a new function. The Poisson bracket  $\{A, B\}$  is defined as

$$\{A, B\} = \sum_{j=1}^n \left( \frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} \right) \quad 2.3$$

It is known that a function  $\Omega(q_j, p_j, t)$  is a constant of motion if and only if  $\{\Omega, H\} = 0$  for all points in the phase space [15] (already knowing that from the formulation of  $\Omega$  to be discussed later in this theory, it does not explicitly depend on time  $t$ , thus  $\frac{\partial \Omega}{\partial t} = 0$ ). In other words, the expression on the right-hand side of the equation of the time-evolution of  $\Omega$  on a symplectic manifold,

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial t} + \{\Omega, \mathcal{H}\} \quad 2.4$$

must vanish (equal zero). There are several other ways of determining new constants of motion, including by experiment and observation or possibly by finding conjugate variables in the Lagrangian. For the sake of consistency, the Hamiltonian proof for constancy of the constants of motion will be used in this theory. It should be noted that canonical coordinates will be used throughout this thesis, as it will be used in the principle of least action formulation and the proof of commuting with the Hamiltonian. At this point, a sufficient enough discussion of Hamiltonian mechanics has been presented.

## Chapter 3

### NEWTONIAN MECHANICS DERIVATION

The Newtonian mechanics derivation of the integrals of motion is the most simplistic derivation of this thesis yet shows that constants of motion can be derived from a force-based formulation. An energy based derivation can be carried out, but for this chapter, and throughout the thesis, we will deal with just Newtonian mechanics (for the derivation itself), which is exactly the inherently force-based methodology we are looking for. Newton's definition of force will be manipulated to carry out the derivation, and we will assume a nonrelativistic velocity for a particle of interest, so relativistic effects are negligible. However, it is necessary to include such effects at velocities close to  $c$ , so a relativistic correction will be written in the succeeding chapter, but the explored case for this thesis will assume non-relativistic particles. Assuming knowledge of Newton's laws of motion, the second law of motion states that for a constant mass, non-accelerating frame of reference, closed system (where  $\dot{\vec{q}}$  is a velocity coordinate)

$$\vec{F} = m\vec{a} = m\frac{d\dot{\vec{q}}}{dt} = \frac{d\vec{p}}{dt} \quad 3.1$$

Work is defined as the change in the kinetic energy of a rigid body under the assumption that the body is not subject to an external field (the body is free). From Newton's second law, under this assumption, the formulation for work

$$W = \Delta KE \quad 3.2$$

Work can also be formulated a different way. Work of a force  $\vec{F}$  is the line integral of its scalar tangential component along the path of its application point. Inserting the equation for force into the work equation yields

$$W = \Delta KE = \int \vec{F} \cdot d\vec{q} = \int \frac{d\vec{p}}{dt} \cdot d\vec{q} = \int \frac{d\vec{q}}{dt} \cdot \frac{d\vec{p}}{d\dot{q}} \cdot d\dot{q} \quad 3.3$$

Here it is easy to see that the final term of the above formulation is an alternative form for a difference in the kinetic energies. We want to express work as an integral over velocity for the reason of staying consistent with the definition of term *first integrals*. Here, we will define first integrals as constants of motion that are independent of time, and therefore only explicitly depend on phase space coordinates and their derivatives. A similar expression of Eqn. 3.3 can be written for the change in momentum (impulse) of a rigid body,

$$\Delta \vec{p} = \int \frac{d\vec{p}}{d\dot{q}} \cdot d\dot{q} \quad 3.4$$

The proposed theory begins with a discussion of how the above equations are similar. More specifically, the change in the kinetic energy (work) and the change in the momentum (impulse) expressions are very similar; notice that the kinetic energy integrand only differs from the momentum integrand by multiplying by a time derivative of position.

The postulate at hand introduces the simple concept that these integrals are directly related to each other, and successive kinetic (not potential) terms can be found by multiplying the integrand by  $\dot{q}^{n-1}$  terms, for  $n$  number of bodies. Because we are assuming there is no external field applied to the system (or in the case of a particle accelerated from rest, that the particle is very slowly accelerated, thus negligible energy loss due to radiation), the potential terms are nonexistent in the Hamiltonian, so the sum of the kinetic energies and the sum of the momenta over all bodies are conserved quantities, hence is the



reason we are able to manipulate the Newtonian equations in this manner. Essentially, we have neglected all perturbative terms, such as dissipative forces, etc. Also, because the  $(n - 1)^{\text{th}}$  integral (and any of the integrals from 0 to  $n - 1$  for that matter) will not be explicitly dependent on time, if the quantity commutes with the Hamiltonian in the Poisson bracket, it is a conserved quantity. In general, to find the kinetic terms for an  $n$  degree of freedom system (where  $n$  particles simultaneously collide) and expressing it as a system of equations involving initial and final velocities (denoted by primes), Eqn. 3.3 can be summed over all bodies, since we are initially postulating that it is a constant of motion (also an integral of motion in this case), so the integrals can be expressed as

$$\text{For } i = 0, 1, 2, \dots, n - 1 \quad \sum_{j=1}^n \int_0^{\dot{q}_j} m_{o,j} \dot{q}_j^i d\dot{q}_j = \sum_{j=1}^n \int_0^{\dot{q}_j'} m_{o,j} \dot{q}_j'^i d\dot{q}_j' \quad 3.5$$

where  $m_o$  is the rest mass of a particle (in general, the mass of a body, in the nonrelativistic case here), and  $i$  indicates the power of the velocity term.

Eqn. 3.5 can also be expressed in terms of initial and final momenta,

$$\text{For } i = 0, 1, 2, \dots, n - 1 \quad \sum_{j=1}^n \int_0^{p_j} \left( \frac{\vec{p}_j}{m_{o,j}} \right)^i d\vec{p}_j = \sum_{j=1}^n \int_0^{p_j'} \left( \frac{\vec{p}_j'}{m_{o,j}} \right)^i d\vec{p}_j' \quad 3.6$$

Prior to this point,  $\Omega$  has been an arbitrary function, but we can now define  $\Omega$  from the equation above, expressed as

$$\Omega_{i+1} = \sum_{j=1}^n \int_0^{\dot{q}_j} m_{o,j} \dot{q}_j^i d\dot{q}_j \quad 3.7$$

The limits of the integrals are zero and a certain velocity  $\dot{q}$  because the kinetic energy of one time, where the velocity of the particle equals  $\dot{q}$ , is found from integrating the above formula with these limits, so only the kinetic energy of the body when it had the velocity of interested will be computed. This is quite intuitive. We can also prove the limits deduced from a statement earlier: the particle is slowly accelerated from rest; therefore, we can take the velocities at time  $t_o$  and  $t_f$  to be zero and  $\dot{q}$ . It should be noted that the

nonrelativistic integrals formulated above can remain indefinite and will produce the same set of equations; this is not the case for relativistic particles, as we will later see. It should also be noted that although we are denoting the  $\dot{\vec{q}}^{n-1}$  terms as vectors for generality (this will be expanded on in Chapter 7), we must take the norm of the vectors to obtain the correct scalar values of  $\Omega$ 's (we will see later that this will equate to a one-dimensional scenario which is what will be analyzed, and we are only allowed to do this for the 1D case). The expansion of the  $\Omega$  terms for arbitrary  $i$ 's yields the equations

$$\Omega_{i+1} = \Omega_{1,2,\dots,n} = \begin{cases} \vec{\Omega}_1 = m_o \dot{\vec{q}} \\ \Omega_2 = \frac{1}{2} m_o \dot{q}^2 \\ \vec{\Omega}_3 = \frac{1}{3} m_o \dot{\vec{q}}^3 \\ \Omega_4 = \frac{1}{4} m_o \dot{q}^4 \\ \dots \end{cases} \quad 3.8$$

Is there any physical meaning to  $\Omega$  terms where  $i > 1$ ? That definition is a bit complex for this section of the theory, but we can start from a dimensional viewpoint, and give a simple answer. For example, the first integral after energy,  $\Omega_3$ , which we can now identify as *momentergy* (Latin for *movēre* (movement) and *energia* (energy) or Greek *ergon* (work)) for brevity, or  $\mathcal{M}$ , has units of  $kg \frac{m^3}{s^3}$ . To perhaps better understand the physical meaning, the simple discussion of momentum, which has the units of  $kg \frac{m}{s}$ , should be brought to attention. We can imply that the integral equation above starts from a mass unit,  $\Omega_0$  (m), which is a scalar quantity. Using the definition above for momentum as the integral of mass with respect to velocity, the integral is giving the scalar quantity of mass a direction of movement, thus a way of describing the mass's motion in space. Similarly, energy can have motion in space as well, since mass is inherently a form of equating the amount of energy in a system, by a factor of  $c^2$ , from Einstein's relation  $E = m_o c^2$ . Its motion can also be described by a similar velocity multiplication (although not true in the

field case) by the scalar quantity of energy, thus momentergy can generally be thought of as a characterization of energy's motion through space, though this view is quite crudely simplistic, and possibly partially incorrect.

These equations seem very simple, but as  $n$  increases, the solutions become more difficult to attain. Let us use the example of the three-body system. Obviously, assuming the three bodies are rigid and moving about a nonaccelerating one-dimensional frame of reference, where  $v$  is the one-dimensional velocity and primed coordinates represent the velocity in a second (or final) state, the integrals in Eqn. 3.5 can be expanded out to produce the following system of equations:

$$\begin{aligned}
 m_1 v_1 + m_2 v_2 + m_3 v_3 &= m_1 v_1' + m_2 v_2' + m_3 v_3' \\
 \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 &= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + \frac{1}{2} m_3 v_3'^2 \\
 \frac{1}{3} m_1 v_1^3 + \frac{1}{3} m_2 v_2^3 + \frac{1}{3} m_3 v_3^3 &= \frac{1}{3} m_1 v_1'^3 + \frac{1}{3} m_2 v_2'^3 + \frac{1}{3} m_3 v_3'^3
 \end{aligned}
 \tag{3.9}$$

These equations should be familiar, as they are just conservation of momentum, kinetic energy, and kinetic momentergy. These equations will be later discussed in Chapter 8, but we must continue with developing the theory beforehand. As one can see, these do not describe relative positions as the field terms are nonexistent due to a zero denominator, but positions can be solved since there is no acceleration of the particles and can be written as a time variant solution using the relation  $x = vt$ . So, the postulate in this thesis at this point can only be applied to free, nonrelativistic, rigid bodies that are simultaneously interacting with each other (elastically colliding), shown in Fig. 1 below

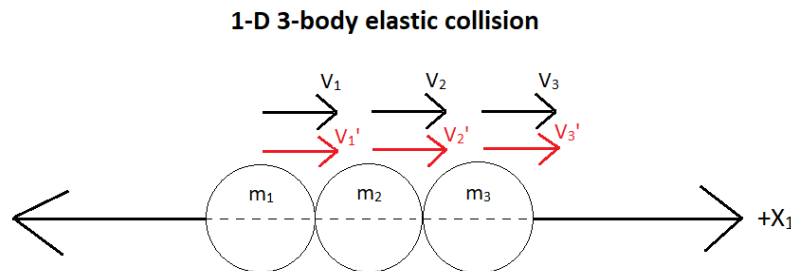


FIG. 1. One dimensional 3-body simultaneous collision

A closed form of the solutions to this set of equations might exist, however compact formulas for each final velocity term are very difficult to attain due to the amount of algebra needed to solve this system of equations simultaneously. However, this system of equations can be solved computationally quite easily. The result will yield six possible sets of answers, with each corresponding to an initial configuration of the bodies prior to interaction (more on this discussion in Chapter 8 - 10). For any  $n$ -body system, a similar pattern can be shown. The two-body elastic collision is a well-defined process based on momentum and kinetic energy, thus momentery is not needed, and the formulas describing the interaction are quite simple. However, for systems where  $n > 2$ , higher order integrals can be used to solve the system analytically, and the solutions become more complex.

It is here that we want to prove that  $\Omega$  terms whose  $i = 2$  will commute with the Hamiltonian of a system, and thus result in an entirely new set of symmetries in the evolution of a system. Because we have already assumed the particles are free, the formulation of the constants do not explicitly depend on time, and we are in a one-dimensional coordinate system, the Hamiltonian of the system is only a function of conjugate momenta and can be defined as

$$\mathcal{H}(p_i) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} \quad 3.10$$

The system's momentery can also be expressed similarly by the expression

$$\mathcal{M}(p_i) = \frac{p_1^3}{3m_1^2} + \frac{p_2^3}{3m_2^2} + \frac{p_3^3}{3m_3^2} \quad 3.11$$

Expanding the terms of the Poisson bracket  $\{\mathcal{M}, \mathcal{H}\}$  yields

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{M}}{\partial p_1} = \frac{p_1^2}{m_1^2}, \quad \frac{\partial \mathcal{M}}{\partial p_2} = \frac{p_2^2}{m_2^2}, \quad \frac{\partial \mathcal{M}}{\partial p_3} = \frac{p_3^2}{m_3^2} \\ \frac{\partial \mathcal{M}}{\partial q_1} = 0, \quad \frac{\partial \mathcal{M}}{\partial q_2} = 0, \quad \frac{\partial \mathcal{M}}{\partial q_3} = 0 \\ \frac{\partial \mathcal{H}}{\partial p_1} = \frac{p_1}{m_1}, \quad \frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m_2}, \quad \frac{\partial \mathcal{H}}{\partial p_3} = \frac{p_3}{m_3} \\ \frac{\partial \mathcal{H}}{\partial q_1} = 0, \quad \frac{\partial \mathcal{H}}{\partial q_2} = 0, \quad \frac{\partial \mathcal{H}}{\partial q_3} = 0 \end{array} \right. \quad 3.12$$

and the Poisson bracket becomes

$$\{\mathcal{M}, \mathcal{H}\} = \left( 0 \times \frac{p_1}{m_1} - \frac{p_1^2}{m_1^2} \times 0 \right) + \left( 0 \times \frac{p_2}{m_2} - \frac{p_2^2}{m_2^2} \times 0 \right) + \left( 0 \times \frac{p_3}{m_3} - \frac{p_3^2}{m_3^2} \times 0 \right) \quad 3.13$$

This is of course means  $\{\mathcal{M}, \mathcal{H}\} = 0$ , which very simply proves its commutation with the Hamiltonian in the Poisson bracket, thus proving that it remains constant in the evolution of a system.

There is a deeper physics problem here than just proving if momentergy is a constant of motion by its commuting with the Hamiltonian in the Poisson bracket. Considering the simultaneous collision of particles, comparing the cases of 2 and 3-body collision, there is an interesting phenomenon occurring in the physical world that is inconsistent with the mathematics presented above. In examining the 2-body elastic collision, if we use the formulas already derived in practice for solving such an interaction, we notice that momentergy is *not* constant. But in the case of the 3-body simultaneous elastic collision, we are making the astounding claim that momentergy *is* constant. Why is this so? Well, there are two parts to this answer that is even more astounding than the fact that momentergy *isn't* constant in a 2-body elastic collision, and it is here where the conglomeration of these claims ripe for explanation is presented.

Part one of the answer involves not including the potential term. This will be further investigated in Chapter V, but the basic underlying presumption here is that when there is

a potential term included in momentergy, it will always be a constant because there is simultaneous interaction between the potentials between all three bodies, and the balance of the kinetic terms with the potential terms will make it constant. But what happens in the case of free bodies?

The second part of the answer has much more profound implications to the physical world, and we will assume no potential for example. In nature, it is statistically an extremely rare event for three particles to simultaneously collide when compared to the amount of linear collisions that occur in nature, as one can find from Smith's paper. Nonetheless, we must assume that tertiary collisions occur naturally in the physical world for this part of the theory to have any implication at all, albeit a rare occurrence. When two particles or bodies (elastic in this case so semantics don't matter) interact, we can only assume that their "inertial terms" are what is interacting. If we think about it from a Newtonian perspective, two particles colliding instantaneously exchange a force equal on both sides, thus the same change in momentum on a linear time scale, which is an inertial, or kinetic-like term since it involves only mass and velocity. However, since there are only two objects, we can only consider the first two integrals from equation (2) because that is the extent of the transfer of any inertial force. There is no tertiary inertial force exchange to consider so introducing the momentergy constant generates an over-constrained system, where there are too many equations relative to the unknowns, or in the case of using momentergy as one of two constants of motion to solve the problem, it will be too high of order to consider, since only second order inertial exchanges are to be considered. We can be sure there is a more rigorous mathematical proof behind this, but an undeniably novice explanation of it in the physical sense is sufficient for the point at which this thesis is.

## Chapter 4

### STANDARD RELATIVISTIC CORRECTION

A reformulation accounting for relativistic effects under standard Lorentz transformation formulations can be derived directly from the Newtonian formulation above. It should be noted, though, that the following formulation assumes a very slowly accelerated particle in an electromagnetic field, thus negligible energy loss due to radiation, so we can model these particles as free. We could also assume the absence of a field, in which case, for the purposes of the derivation, is arguably the same assumption. Once this assumption is taken to be true, we can proceed with this part of the thesis. Upon examining the final form of Eqn. 3.3, we must first take the derivative of momentum with respect to velocity. It is known from relativistic kinematics that the momentum of a particle is

$$\vec{\Omega}_1 = \gamma(\dot{\vec{q}})m_o\dot{\vec{q}} \quad 4.1$$

where  $\gamma(\dot{\vec{q}})$  is the particle's Lorentz factor, or correction factor of any coordinate system, defined as [3]

$$\gamma(\dot{\vec{q}}) = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{and} \quad \beta = \frac{\dot{\vec{q}}}{c} \quad 4.2$$

The derivative of relativistic momentum with respect to velocity can be carried out to produce the expression

$$\frac{d\vec{\Omega}_1}{d\dot{\vec{q}}} = \frac{d(\gamma(\dot{\vec{q}})m_o\dot{\vec{q}})}{d\dot{\vec{q}}} = m_o\gamma(\dot{\vec{q}})^3 \quad 4.3$$

It is here where the difference in the relativistic and nonrelativistic formulations arise. The final term of the above formulation is referred to as the longitudinal mass of a particle, or the mass parallel to the direction of a force applied (in the Lorentz ether theory),

which is composed of its rest mass and its cubed Lorentz factor. This is intuitive; thinking back to the nonrelativistic definition of the work of a force (the line integral of its scalar tangential component along the path of its application point), the only component of the mass that is acted upon by the force *is* the mass parallel to that of the force. Inserting this expression into the nonrelativistic form Eqn. 3.3, we can obtain the system of integrals

$$\text{For } i = 0, 1, 2, \dots, n-1 \quad \sum_{j=1}^n \int_0^{\dot{q}_j} m_{o,j} \gamma(\dot{\vec{q}}_j)^3 \dot{\vec{q}}_j^i d\dot{\vec{q}}_j = \sum_{j=1}^n \int_0^{\dot{q}_j} m_{o,j} \gamma(\dot{\vec{q}}_j')^3 \dot{\vec{q}}_j'^i d\dot{\vec{q}}_j' \quad 4.4$$

where

$$\Omega_{i+1} = \sum_{j=1}^n \int_0^{\dot{q}_j} m_{o,j} \gamma(\dot{\vec{q}}_j)^3 \dot{\vec{q}}_j^i d\dot{\vec{q}}_j \quad 4.5$$

These equations expanded out for arbitrary  $n$ 's yield relativistic momentum ( $\vec{\Omega}_1$ ), relativistic kinetic energy ( $\Omega_2$ ), relativistic kinetic momentery ( $\Omega_3$ ),  $\Omega_4$ , etc. As stated earlier in the text, definite integrals do, in fact, change the final equations resultant from the integration; this is due to including the rest mass as part of the derivation. In the indefinite case, the equations simply yield *total* terms (of course, excluding field terms); for example, removing the limits of the integral of  $\Omega_2$  yields the relativistic equation for total energy  $E = \gamma(\dot{\vec{q}}) m_o c^2$ , not kinetic energy. However, this is only case for even numbered  $n$ 's. As stated earlier in this theory, it is the limits of zero and a velocity of interest that yields kinetic terms. For even numbered  $n$ 's for all  $\Omega$ 's, however, the total term must be used when comparing initial and final configurations. The expansion of the left-hand side of Eqn. 4.4 for arbitrary  $n$ 's yields the equations

$$\Omega_{i+1} = \Omega_{1,2,\dots,n} = \begin{cases} \vec{\Omega}_1 = \gamma(\dot{\vec{q}}) m_o \dot{\vec{q}} \\ \Omega_2 = \gamma(\dot{\vec{q}}) m_o c^2 \\ \vec{\Omega}_3 = \gamma(\dot{\vec{q}}) \dot{\vec{q}} m_o c^2 + i \ln \left( i\beta + \frac{1}{\gamma(\dot{\vec{q}})} \right) m_o c^3 \\ \Omega_4 = \left( \gamma(\dot{\vec{q}}) + \frac{1}{\gamma(\dot{\vec{q}})} \right) m_o c^4 \\ \dots \end{cases} \quad 4.6$$



The expansion of (3) without limits yields the total formulations

$$\Omega_{i+1} = \Omega_{1,2,\dots,n} = \begin{cases} \vec{\Omega}_1 = \gamma(\vec{q}) m_o \vec{q} \\ \Omega_2 = \left( \gamma(\vec{q}) - 1 \right) m_o c^2 \\ \vec{\Omega}_3 = \gamma(\vec{q}) \dot{\vec{q}} m_o c^2 + i l n \left( i \beta + \frac{1}{\gamma(\vec{q})} \right) m_o c^3 \\ \Omega_4 = \left( \gamma(\vec{q}) + \frac{1}{\gamma(\vec{q})} - 2 \right) m_o c^4 \\ \dots \end{cases} \quad 4.7$$

As stated above,  $\vec{\Omega}_1$  and  $\Omega_2$  are already known equations, thus  $\Omega$ 's whose  $i > 1$ , or  $n > 2$ , are of particular interest. It shall be postulated, as before, that  $\Omega$  terms are conserved only up to the expansion of Eqn. 4.4 (or Eqn. 3.5 in the nonrelativistic case) where  $n = i+1$ , in simultaneous collisions of  $n$  rigid particles, where the particles are to be slowly accelerated from rest, or ideally free, but we now have a relativistic correction for particle collisions. To postulate this, however, one must do two things: prove that all  $\Omega$ 's commute with the Hamiltonian of the system regarded, therefore proving its time translation invariance property, and show the dynamical meaning and importance of  $\Omega$  terms where  $n > 2$ , for in these cases, the quantity is a postulate itself; this was discussed in detail in Chapter 3.

## Chapter 5

### GENERAL POTENTIAL INCLUSION AND ASSOCIATED SYMMETRY

In the sections above, we have not included the potential term for obvious reasons. For the purposes of this thesis, the kinetic terms, since completely integrable, have been used as example. However, it is important to include the potential as most dynamical systems exhibit some deviation of its motion from a potential field. It is important to note that, in actuality, on a quantum level, there is no such thing as a free particle with definite energy even modeled without a potential; nonetheless, we will neglect this aspect of dynamics of particles in this chapter, so we will only consider classical mechanics and ideal cases. The goal for this section is to obtain expressions that include potential field terms.

The representation of constants that include potential effects begins with the elementary equation of motion for particles, assuming no dissipative forces

$$m\ddot{q} = -\frac{\partial U}{\partial q} \quad 5.1$$

or in terms of the time derivative of momentum,

$$\dot{p} = -\frac{\partial U}{\partial q} \quad 5.2$$

where  $U$  is the potential function.

The same successive velocity multiplication process from Chapter 3 begins with multiplying each side by  $n$  velocity terms and then carrying out a normal time integration.

The constants that arise from this, after rearranging the equation, can be expressed as

$$C_n = \int \left( m\ddot{q} + \frac{\partial U(q)}{\partial q} \right) \dot{q}^{n-1} dt \quad 5.3$$

Moving the partial derivatives outside of the potential integral term yields

$$\frac{\partial}{\partial q} \int U(q) \dot{q}^{n-1} dt = \int \frac{\partial U(q)}{\partial q} \dot{q}^{n-1} dt \quad 5.4$$

which holds for any  $n$ , assuming we have a potential that is a function of positions  $q(t)$ , so

each  $C_n$  constant can be expressed as

$$C_n = \begin{cases} C_1 = m\dot{q} + \frac{\partial}{\partial q} \int U(q) dt \\ C_2 = \frac{1}{2}m\dot{q}^2 + \frac{\partial}{\partial q} \int U(q) \dot{q} dt \\ C_3 = \frac{1}{3}m\dot{q}^3 + \frac{\partial}{\partial q} \int U(q) \dot{q}^2 dt \\ \dots \\ C_n = \frac{1}{n}m\dot{q}^n + \frac{\partial}{\partial q} \int U(q) \dot{q}^{n-1} dt \end{cases} \quad 5.5$$

$C_2$  is the total energy of the system (which will also be the Hamiltonian if the potential is independent of time). Carrying out the second term's integration will just yield the system's potential function. With  $C_1$ , this is the full form of a particle's total momentum (not to be confused with linear momentum, which is the first term). This might not be what one expects however, as a particle's linear momentum can sometimes be generally thought of as its total momentum, but there are very clear distinctions. A particle's linear momentum is not conserved unless the second term of this expression becomes zero from the nature of the formulation of the potential. For example, if we use the gravity field model potential,

$$U(q_{ij}) = -\frac{Gm_i m_j}{q_{ij}} \quad 5.6$$

it is easy to see that the terms will cancel after taking the spatial partial derivative (which subsequently equals the force between the bodies) and then summing over all  $i$ 's and  $j$ 's.

Hence, upon integrating the zero, we obtain another constant which can be absorbed by  $C_1$ , thus making the linear momentum of the gravitational case a constant.

As one can see from close investigation of  $C_3$ , which is the full general form of momentergy, the second term cannot be integrated and expressed in terms of elementary functions which limits its usefulness. However,  $C_4$  can be integrated and expressed in elementary functions. After integration by parts,  $C_4$  becomes

$$C_4 = \frac{1}{4}m\dot{q}^4 + U(q)\dot{q}^2 - 2 \int U(q)\dot{q}\ddot{q}dt \quad 5.7$$

and after substituting in Eq. 5.1, we can obtain

$$C_4 = \frac{1}{4}m\dot{q}^4 + U(q)\dot{q}^2 + \frac{2}{m} \int U(q)\frac{dU(q)}{dt}dt \quad 5.8$$

which yields

$$C_4 = \frac{1}{4}m\dot{q}^4 + U(q)\dot{q}^2 + \frac{U(q)^2}{m} \quad 5.9$$

Although  $C_3$  cannot be integrated, an interesting and useful revelation arises from the manipulation of the odd numbered constants by substituting in the total energy or Hamiltonian. It shall be postulated that odd numbered  $C_n$ 's equal zero, and only holds true when we add a potential function that is time independent in the equations of motion.

Using the fact that total energy,

$$C_2 = E = \mathcal{H} = \frac{1}{2}m\dot{q}^2 + U(q) \quad 5.10$$

is constant through time assuming the potential independent of time, and substituting in the conjugate momenta that would be in the Hamiltonian for the respective mass and velocity, we can now solve for  $U(q)$  and use Hamilton's equations of motion to carry out the proof that odd numbered  $C_n$ 's equal zero:

$$\begin{aligned}
C_3 = \mathcal{M} &= \frac{1}{3}m\dot{q}^3 + \int \frac{\partial U(q)}{\partial q} \dot{q}^2 dt \\
&= \frac{1}{3}m\dot{q}^3 + \int \frac{\partial (\mathcal{H} - \frac{1}{2}m\dot{q}^2)}{\partial q} \dot{q}^2 dt \\
&= \frac{1}{3}m\dot{q}^3 + \int -\dot{p}\dot{q}^2 dt \\
&= \frac{1}{3}m\dot{q}^3 + \int -m\ddot{q}\dot{q}^2 dt \\
\implies \mathcal{M} &= \frac{1}{3}m\dot{q}^3 - \frac{1}{3}m\dot{q}^3 = 0
\end{aligned} \tag{5.11}$$

This of course means that if we integrate momentergy again with respect to time, we get another constant, or

$$\int \mathcal{M} dt = C \tag{5.12}$$

A similar derivation can be carried out for any odd numbered  $C_n$ .

$C_3$  has some interesting characteristics that should be noted. The kinetic momentergy term (first term) of the equation is exactly what is attained from the integral expression Eqn. 3.5 earlier in the theory. The potential momentergy term (the second term containing the integral which we will denote as  $\Phi$ ) can possibly restrict the problem from becoming integrable due to many potential models having intrinsic meromorphic nature, although there are some potential models that can be integrated (keep in mind it if the indefinite integral we would want to attain). Therefore, a part of this theory includes two new concepts: (I) the kinetic term of the theorized class of first integrals ( $\Omega$ 's) of the equations of motion can always be obtained in terms of elementary functions, and (II) the potential terms will serve as possible restrictions on the problem due to their (usually) meromorphic nature, so it is the factor for integrability criterion within the equation.

A more specific discussion of these constants follows. Momentergy can reveal a new symmetry in a dynamical system using Noether's theorem. Noether's theorem states

that for every differentiable symmetry of a dynamical system, there is some constant of motion associated with the symmetry. It must be distinguished here that the converse is not necessarily true [3]. The converse is only true, for the purposes of this paper, because we are dealing with a general conserved quantity on the phase space which is conserved for all possible initial conditions [17]. An important fact of the symmetries and associated constants of motion is that when multiplied together, the units equal an action, which is a path or trajectory of a body to which the result is a real number.

The most useful form of this theorem for this paper's sake is to first take the mathematical definition of an action,

$$\begin{aligned} \mathcal{S} &= \int_{t_1}^{t_2} \mathcal{L} dt \\ &= \int_{t_1}^{t_2} (p\dot{q} - \mathcal{H}) dt \end{aligned} \tag{5.13}$$

or the abbreviated action, according to Maupertuis' principle,

$$\mathcal{S}_0 = \int_{q_1}^{q_2} p \cdot dq \tag{5.14}$$

One can easily deduce upon investigating the variables involved in each integral that the integrand and the variable the integrand is to be integrated with respect to are conjugate variables. It is also important to note that once each integral is carried out, the resulting value, the action, has units of  $J \cdot s$ . With this information, we can determine what the symmetry associated with momentergy would be.

Since we know that momentergy has units  $kg \frac{m^3}{s^3}$ , and we know that an action must have units  $J \cdot s$ , we can conclude that the conjugate variable or momentergy, or the symmetry associated with it, must have units of  $\frac{s^2}{m}$ , which has the units reciprocal to that of acceleration, but does not mean the symmetry itself is the reciprocal of acceleration. We

can formulate what the action integral, roughly speaking, would look like, setting the new symmetry arbitrarily to the variable  $\alpha$ :

$$\mathcal{S}_2 = \int_{\alpha_1}^{\alpha_2} (\mathcal{M} - 2\Phi) d\alpha \quad 5.15$$

which by definition must be minimized for a particle's trajectory and would result in an invariant Lagrangian under a small perturbation  $\alpha = \alpha' = \alpha + \delta\alpha$  if the Lagrangian were a function of  $\alpha$ . However, it is a bit more complicated than that.

To show this complexity, let us use the fact that conjugate variables do not commute in the Poisson bracket and produce the value of 1 or -1. This will lead to a partial differential equation in which one can, in theory, solve for the missing variable:

$$\begin{aligned} \mathcal{M} &= \frac{1}{3}m\dot{q}^3 + \int \frac{\partial U(q)}{\partial q} \dot{q}^2 dt \\ &= \frac{p^3}{3m^2} + \int \frac{\partial U(q)}{\partial q} \left(\frac{p}{m}\right)^2 dt \\ &\quad \alpha = ? \\ 1 &= \frac{\partial \mathcal{M}}{\partial q} \frac{\partial \alpha}{\partial p} - \frac{\partial \mathcal{M}}{\partial p} \frac{\partial \alpha}{\partial q} \end{aligned} \quad 5.16$$

One can clearly see from the partial differential equation, there is no closed form solution to  $\alpha$ . Therefore, it is natural to believe that the forms for momentergy and  $\alpha$  exist outside of the realms of current mathematical theory and notation, but as we can prove the commutation of momentergy in the Poisson bracket with the Hamiltonian, there must be this symmetry, regardless of whether or not its form can be expressed in terms of elementary functions.

## Chapter 6

### EXPLORING AN INTEGRABLE MOMENTERGY POTENTIAL TERM

In this chapter, we shall briefly discuss an integrable potential momentergy. In the last chapter, it was discussed that when formulating momentergy with a potential term, it must equal zero and also must be a constant through time.

Although already a widely known solution is available, the problem chosen is a simple mass falling to the surface of the Earth because of the integrability of the potential term; since the solution is widely known, it is not a new method to solving the system that is of interest, but rather the behavior of the constants and proof they are indeed constant. The system is shown in Fig. 2 below.

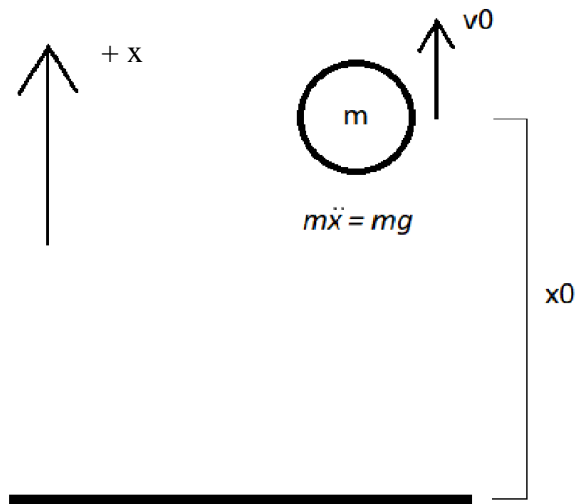


FIG. 2. Schematic of a rigid mass falling to a flat surface on the surface of the earth



Taking the equation of motion  $m\ddot{x} = mg$ , or  $m\ddot{x} = -mg$  if we allow the gravitational acceleration constant to be a positive value, we can find the first few integrals using Eqns. 5.5.

$$C_n = \begin{cases} C_1 = m\dot{x} + mgt = 0 \\ C_2 = \frac{1}{2}m\dot{x}^2 + mgx \\ C_3 = \frac{1}{3}m\dot{x}^3 + mgx\dot{x} + mg^2\mathcal{A} \\ C_4 = \frac{1}{4}m\dot{x}^4 + mgx\dot{x}^2 + mg^2x^2 \\ \dots \end{cases} \quad 6.1$$

where  $\mathcal{A}$  is the mass's absement, which is the integral of position with respect to time [13],

$$\mathcal{A} = \int x dt \quad 6.2$$

Although the above form is a bit ambiguous, it can easily be found for this system by the known solutions. If we plot the system's momentergy with respect to time assuming arbitrarily we have a mass of 50 kg, an initial position of 1000 m and an initial velocity of 0.1 m/s, we get the following plot:

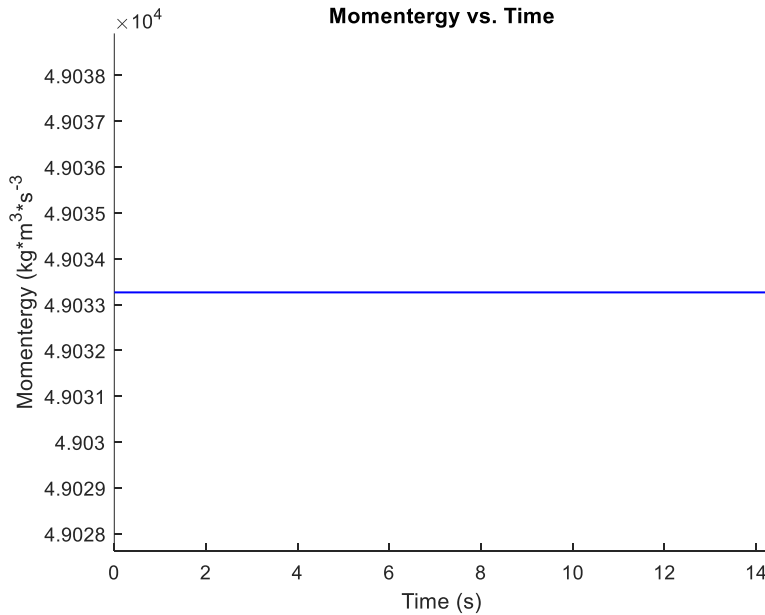


FIG. 3. Momentergy vs. Time

The plot clearly shows that momentergy is constant through time. The value of momentergy is not equal to zero, however, because the absement, position, and velocity formulations are time dependent.

## Chapter 7

### EXPLORING THE N-BODY GRAVITATIONAL PROBLEM

As stated earlier, to solve any dynamical system analytically, one must provide enough constants of motion to sufficiently reduce the degrees of freedom of the system. With this goal in mind, let us again use Eqns. 5.5 to derive constants of motion.

Under assumptions that space-time is nearly flat, and that pressure is small compared to density, Einstein's field equations reduce to the Newtonian equations of motions for gravity, i.e.  $\frac{d^2\vec{R}}{dt^2} = -\nabla\phi$ . With this reduction, let's apply the theorem to Newton's 3D gravitational equations of motion, and use Newton's dot notation to represent total derivatives of time. We will define new coordinates in a non-accelerating frame of reference for consistency with traditionally used celestial coordinates, where  $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n$  are the Cartesian position coordinates. Assuming the bodies are moving around the center of mass  $\mathbf{C}$  (we know it is an inertial point) the equations of motion can be expressed as [19]

$$\sum_{i=1}^n m_i \ddot{\mathbf{R}}_i = G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij} \quad 7.1$$

, where  $i \neq j$ . To obtain the integrals of motion, the set of equations.

$$\text{For } k = 0, 1, 2, \dots, n-1 \quad \Omega_{k+1} = \int \left( \sum_{i=1}^n m_i \ddot{\mathbf{R}}_i - G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij} \right) \cdot \dot{\mathbf{R}}_i^k dt \quad 7.2$$

are used, where the  $k$ th term denotes the number of successive velocity dot products. It is important to note here that the vector multiplications must be of the same kind to create a form that is conducive for integration. Summing over all the masses and accounting for the symmetry condition  $\mathbf{r}_{ij} = -\mathbf{r}_{ji}$ , where  $\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i$ , then integrating with respect to time (though the derivatives in the equations are total derivatives of time, so the integration can

be thought of as a total time integration), where  $k = 0$ , the equation becomes the momentum integral of motion,  $\vec{\Omega}_1$ , expressed as

$$\vec{\Omega}_1 = \sum_{i=1}^n m_i \dot{\vec{R}}_i \quad 7.3$$

The energy integral of motion,  $\Omega_2$ , where  $k = 1$ , can be expressed as

$$\Omega_2 = \frac{1}{2} \sum_{i=1}^n m_i (\dot{\vec{R}}_i \cdot \dot{\vec{R}}_i) - \frac{1}{2} G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{\|\vec{r}_{ij}\|} \quad 7.4$$

One can find angular momentum and the center of mass integrals from equation 7.1 as well, and there are two other equations that can reduce the degrees of freedom of the  $n$ -body problem (one deals with the semi-major axis and the other a time equation), but since the first system unable to be solved with increasing  $n$  (which is obviously the 3-body problem) has 18 degrees of freedom in the 3D case, there are still 6 degrees of freedom to be accounted for, so the need for additional constants of motion arises.

We want to consider the momentergy integral of motion,  $\Omega_3$ , i.e. when  $k = 2$ .

Accounting for the relation  $\vec{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i$ , the integral can be formulated as

$$\Omega_3 = \frac{1}{3} m_i \dot{\vec{R}}_i \cdot \dot{\vec{R}}_i \cdot \dot{\vec{R}}_i + \frac{1}{2} G \int \sum_{i=1}^n \sum_{j=1}^n m_i m_j \frac{\vec{r}_{ij}}{\|\vec{r}_{ij}\|^3} \cdot \dot{\vec{r}}_{ij} \cdot (\dot{\vec{R}}_i + \dot{\vec{R}}_j) \quad 7.5$$

It is obvious here where the impasse has occurred. It is not unexpected that the potential momentergy term cannot be integrated in terms of elementary functions as this was a revelation from Eqns. 5.5. The bigger issue here is the successive dot products. Current mathematical notation and theory does not allow for a “vector triple dot product”, as this would be a dot product between a vector and a scalar. At this point, it should be made aware of a comment made in Chapter 3. Since we are dealing with 3D vectors, we cannot simply just take the norm of the vector then cube it as this only works in 1D (since the vector itself only has one component). To make any sense of what such an operation might be, we can use tensor notation.

The definition of a dot product between two vectors using tensor notation is

$$\vec{u} \cdot \vec{v} = g_{\mu\nu} u^\mu v^\nu \quad 7.6$$

[18] where  $g_{\mu\nu}$  is the covariant metric tensor, which for all intents and purposes would be treated as a 3x3 array when using vectors with three components. So, if we wanted to naively expand the idea into a dot product of three vectors, one could write

$$\vec{u} \cdot \vec{v} \cdot \vec{w} = C_{\mu\nu\rho} u^\mu v^\nu w^\rho \quad 7.7$$

where  $C_{\mu\nu\rho}$  would be treated as a 3x3x3 array, but it is unclear what the components would be. Therefore, the usefulness of momentenergy for solving the  $n$ -body problem is currently nonexistent even though, in theory, it is constant.

## Chapter 8

### ANALYSIS OF THE SYSTEM OF EQUATIONS

The derivations of the constants and associated proofs now lead to the analysis of the one-dimensional three-body simultaneous elastic collision. Assuming we have a system depicted by Fig. 1, we will set the system of equations up like Eqns. 3.9, and simultaneously solve this system for the primed velocity coordinates. A postulate is now stated as such: a closed form of the solutions to this set of equations exists, however compact formulas for each of the final velocity terms are very difficult to attain due to the amount of algebra needed to solve this system of equations simultaneously. More on this subject matter will be in the succeeding chapters. However, this system of equations can be solved computationally quite easily. The results yield six possible sets of answers, with each corresponding to an initial configuration of the bodies prior to interaction. For any  $n$ -body system, a similar pattern can be shown, but attention is focused on the three-body case for the remainder of the thesis, as stated earlier.

In order to fully understand the system of equations in Eqns. 3.9, one must be familiar with the theory behind polynomial equations, i.e., knowing how many solutions it has, overdetermined, underdetermined, etc., and the theory on this subject is quite developed. We know that the equations can be categorized as a system of polynomial equations as they must be simultaneously solved, and they can be written in the form  $f_1 = 0, \dots, f_i = 0$  (Eqns. 9.2), where  $f_i$  are just polynomials in multiple variables with coefficients over some field  $K$  (in our case the final velocities  $v_i'$  would be the independent

variables and the masses  $m_i$  would be the coefficients) [10]. To classify this system of equations, we must know its dimensionality, i.e., whether it is inconsistent, zero-dimensional, or positive dimensional. To do this, normally one would compute the Gröbner basis [9] to find another set of polynomials that have the same solutions, but this can be circumvented by applying simple logic to the problem.

We know that in collision theory, the 2-body collision will have solutions corresponding to each possible initial configuration of the bodies. Thus, in the 3-body case, this will hold true, assuming we only know momentum and energy (we know these must be constant). We know it *must* have solutions if this is the case, assuming we don't know if they are unique or not. Let us determine its uniqueness again using logic.

The system is classified as zero-dimensional as the number of independent variables equals the number of equations [10]. Now the question arises: how many solutions does this system have? Again, we know the 3-body case *must* have at least as many solutions as possible configurations of the initial state, so the zero-solutions (inconsistent) case is eliminated. The question to ask now is if the number of solutions is infinite or finite (whether the system of zero or positive dimensional).

According to Bezout's theorem [11], any well-behaved system (number of equations equals number of variables) can only have as many solutions as the orders of the system successively multiplied, or  $d_1, \dots, d_n$  degrees of the systems can have at most  $d_1 \dots d_n$  solutions, meaning that when applied to the system Eqns. 3.9, we find it can have at most six solutions. This will automatically classify it as a zero-dimensional well-behaved system of polynomial equations.

## Chapter 9

### SYMBOLIC COMPUTATION IN MATLAB

Given that we now know that the system of equations in Eqns. 3.9 will yield six solutions, each solution can be found using any numerical or symbolic method of choice, and the six solutions can be matched to each possibility of the system's initial configuration and will arise from the iterative process quite easily.

The method we chose is to use MATLAB's symbolic computation tool for solutions to the problem. Using this tool, it is easy to set the independent variables and coefficients and simultaneously solve all three equations. This will generate at most six solutions – one being the initial state of the particles which, by definition, must be a solution; physically this means no collision occurred. The other five solutions are unique and assume some simultaneous collision has occurred. Again, this can be scaled to any n-body collision. The code for finding the two and three-body collision solutions in MATLAB is provided in the Appendix.



## Chapter 10

### POSSIBILITY OF THE EXISTENCE OF AN ANALYTIC SOLUTION

Although the sections above explain how the one-dimensional simultaneous elastic collision of rigid bodies (specifically the three-body case) can be solved using symbolic computation, it is important to explore the possibility of an analytical solution, one which can be written in terms of elementary functions and requires no numerical computation.

The method used for determining if an analytical solution exists had to be an unconventional one, as both MATLAB's symbolic computation software and Wolfram Mathematica could not handle the amount of algebra needed to produce analytical solutions to the system of equations. As for manually solving the system of equations in the nonrelativistic three-body case only, it is here I will postulate that it is, in fact, possible to obtain solutions to them by regular chains, which is a method of finding analytical solutions of zero-dimensional systems of polynomial equations. It must be noted that the form required to obtain the solutions of equations cannot easily be represented as polynomials  $f_i = 0$ , producing the triangular system of equations [14]

$$\begin{cases} f_1(x_1) = 0 \\ f_2(x_1, x_2) = 0 \\ \dots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases} \quad 9.1$$

from the system of equations

$$\text{For } \sigma = 1, 2, \dots, n \quad C_\sigma - \sum_{\rho=1}^n \frac{1}{\sigma} m_\rho v_\rho^\sigma = 0 \quad 9.2$$

where  $C_\sigma$  are the constants obtained from the summing the left hand side of the individual

equations in Eqns. 3.5. So, in order to prove the postulate that there is an analytical solution to the nonrelativistic one-dimensional three-body simultaneous elastic collision, we will be exploring the solutions obtained from the symbolic computation method. In order to accurately access the solutions, one must consider some extreme specific cases of the system: all masses are the same, all initial velocities are the same, one initial velocity is zero, two of the initial velocities are zero, and a general combination. We can always exclude the initial velocity configuration solution because in the process of solving the system of equations using regular chains, if done correctly, the initial case (bolded) will be eliminated, as this simply means no collision occurred at all, thus is not useful. We will be using logic from Bezout's theorem to show that as long as the number of unique solutions does not exceed four, then the system of equations has an analytical solution, since anything above four will fall into the realm of the Abel Ruffini theorem, stating that there is no solution to polynomials whose degree is greater than four [12]. Below are the tables of the extreme cases as well as a general case.

	Mass 1	Mass 2	Mass 3	Initial Velocity 1	Initial Velocity 2	Initial Velocity 3	Final Velocity 1	Final Velocity 2	Final Velocity 3
Solution 1	5	5	5	-1	4	10	10	4	-1
Solution 2	5	5	5	-1	4	10	4	10	-1
Solution 3	5	5	5	-1	4	10	10	-1	4
Solution 4	5	5	5	-1	4	10	-1	10	4
Solution 5	5	5	5	-1	4	10	4	-1	10
<b>Solution 6</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>-1</b>	<b>4</b>	<b>10</b>	<b>-1</b>	<b>4</b>	<b>10</b>

TABLE 1. Same Masses

As one can see in Table 1, for the case of all bodies having the same mass, regardless of magnitude and of the magnitude and direction of the velocities, there appears to be six unique solutions; however, taking a closer look, it is clear that there are actually only two unique solutions, solution two and three, for when all three particles collide simultaneously (which of course means there are only two initial configurations of the masses that allow for all three masses to collide), as solutions one, four, and five show that only two particles have collided, thus when all bodies have the same mass, for this specific case, the system of equations can be reduced to a second order polynomial.

It is intuitive to think about the case in which all bodies have the same initial velocity: there is no other solution besides the initial velocity solution as the bodies never collided in the first place and never will, no matter the configuration of the bodies initially.

	Mass 1	Mass 2	Mass 3	Initial Velocity 1	Initial Velocity 2	Initial Velocity 3	Final Velocity 1	Final Velocity 2	Final Velocity 3
Solution 1	7	5	3	-2	0	6	-0.36292	-2.30171	6.01633
Solution 2	7	5	3	-2	0	6	-1.94876	4.80014	-2.11979
Solution 3	7	5	3	-2	0	6	-1.94876	4.80014	-2.11979
Solution 4	7	5	3	-2	0	6	3.93022	-2.84928	-3.08837
Solution 5	7	5	3	-2	0	6	3.93022	-2.84928	-3.08837
<b>Solution 6</b>	<b>7</b>	<b>5</b>	<b>3</b>	<b>-2</b>	<b>0</b>	<b>6</b>	<b>-2</b>	<b>0</b>	<b>6</b>

TABLE 2. One velocity is zero

Table 2 explores the case in which all three masses are different, regardless of magnitude, but one mass is at rest prior to any collisions, regardless of the magnitude or direction of the other velocities. Solutions two and three are identical, as well as solutions four and five. Thus, there are only three unique solutions in this specific case, so the system

of equations can be reduced to a third order polynomial. If all the masses are the same, we get the same pattern of results as Table 1 shows.

	Mass 1	Mass 2	Mass 3	Initial Velocity 1	Initial Velocity 2	Initial Velocity 3	Final Velocity 1	Final Velocity 2	Final Velocity 3
Solution 1	7	5	3	-2	0	0	-1.14434	0.28869	-2.47768
Solution 2	7	5	3	-2	0	0	0.07768	-2.15535	-1.25566
Solution 3	7	5	3	-2	0	0	0.0861	-1.5139	-2.3444
Solution 4	7	5	3	-2	0	0	-0.61943	-2.21943	0.47773
<b>Solution 5</b>	<b>7</b>	<b>5</b>	<b>3</b>	<b>-2</b>	<b>0</b>	<b>0</b>	<b>-2</b>	<b>0</b>	<b>0</b>
Solution 6	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A

TABLE 3. Two velocities are zero

Table 3 shows the case where two of the velocities in the initial state are zero, regardless of the magnitude or direction of the other velocity, and all masses are different, regardless of magnitude. Note that MATLAB's symbolic computation software only gave five solutions, with four of the five being unique. Thus, in this specific case, the system of equations can be reduced to a fourth order polynomial, which is still within the realms of polynomials that can be solved in terms of elementary functions. If all the masses are the same, it can be shown that we only get two unique solutions which leads to a second order polynomial.

	Mass 1	Mass 2	Mass 3	Initial Velocity 1	Initial Velocity 2	Initial Velocity 3	Final Velocity 1	Final Velocity 2	Final Velocity 3
Solution 1	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	2.3111	-4.08761	-3.42729
Solution 2	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	2.3112	-3.6577	-4.30234
Solution 3	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	1.69993	-4.83465	5.14796
Solution 4	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	0.99516	5.00306	-6.71302

Solution 5	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	-0.58284	6.86431	7.72321
Solution 6	56.9757	10.0312	4.9346	2.3112	-3.6577	-4.30234	-0.58284	6.86431	7.72321

TABLE 4. A General Case

Table 4 shows a general case that had the most unique solutions out of all tests of general cases. It is clear that there are four unique solutions corresponding to a polynomial of order four that can be found from the system of equations.

It is important to note that these tables only show samples of the specific cases, but there were several other tests done for each case which showed results that only went up to a maximum of four unique solutions; thus, from Bezout's theorem, since there are only a maximum of four unique solutions, the corresponding polynomial can only be of order four since we are dealing with a zero-dimensional system of polynomials. The solutions of the general quartic polynomial is well known and can be written in terms of elementary functions. So, the postulate that the nonrelativistic one-dimensional three-body simultaneous elastic collision has an analytical solution is backed by strong evidence.

## Chapter 11

### CONCLUSION

In this thesis we have identified the need for additional constants of motion for general cases as well as for the one-dimensional simultaneous  $n$ -body elastic collision, as this was unavailable in current collision theory. We derived them and found they are adequate for a sufficient reduction of the order of the system in efforts to solve it. It was clear that the higher order constants  $\Omega_3, \Omega_4$ , etc. commuted with the Hamiltonian and thus must be constant in the evolution of the system, whether a collision occurred or not, and that with each constant there is an associated symmetry. The purpose of this study was to explore new constants of motion for any dynamical system and was refined to solve the  $n$ -body collision as this system was most applicable given the method for finding a new constant because the constants needed to solve the problem can be written in terms of elementary functions. With this information, the possibility of using this method for other dynamical systems might hold some merit for good potential in future studies involving many-body collisions and perhaps even the  $n$ -body problem with any field.

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## APPENDIX

The following is the MATLAB code for generating all possible solutions, using MATLAB's symbolic computation software, of the one-dimensional two and three-body nonrelativistic simultaneous elastic collisions. Four or more body collisions can be coded in a similar fashion.

```
clc
clear
close all

% ----- INPUTS -----

prompt = {'Enter Mass 1 (in g)', 'Enter Mass 1 Velocity (in m/s)', 'Enter Mass 2 (in g)', 'Enter Mass 2
Velocity (in m/s)', 'Enter Mass 3 (in g)', 'Enter Mass 3 Velocity (in m/s)'};

promptTitle = ('INITIAL CONDITIONS');

num_lines = 1;

defaultAns = {'5', '-1', '2', '4', '1', '-10'}; % Default Answers

inputAns = inputdlg(prompt, promptTitle, num_lines, defaultAns);

m1 = str2double(inputAns{1});
v1 = str2double(inputAns{2});
m2 = str2double(inputAns{3});
v2 = str2double(inputAns{4});
m3 = str2double(inputAns{5});
v3 = str2double(inputAns{6});

name = inputAns{6}
```

```

% ----- Solve the Systems of Equations -----

tic

syms v1prime v2prime v3prime

S = vpasolve([((m1*v1 + m2*v2 + m3*v3) == ((m1*v1prime) + (m2*v2prime) + (m3*v3prime))),
(((1/2)*m1*(v1^2)) + ((1/2)*m2*(v2^2)) + ((1/2)*m3*(v3^2))) == (((1/2)*m1*(v1prime^2)) +
((1/2)*m2*(v2prime^2)) + ((1/2)*m3*(v3prime^2))), (((1/3)*m1*(v1^3)) + ((1/3)*m2*(v2^3)) +
((1/3)*m3*(v3^3))) == (((1/3)*m1*(v1prime^3)) + ((1/3)*m2*(v2prime^3)) + ((1/3)*m3*(v3prime^3)))]),
[v1prime, v2prime, v3prime]);

V1prime = S.v1prime;
V2prime = S.v2prime;
V3prime = S.v3prime;

syms v11prime v22primeT = vpasolve([m1*v1 + m2*v2 == m1*v11prime + m2*v22prime,
(((1/2)*m1*(v1^2)) + ((1/2)*m2*(v2^2))) == (((1/2)*m1*(v11prime^2)) + ((1/2)*m2*(v22prime^2)))]),
[v11prime, v22prime]);

V11prime = T.v11prime;
V22prime = T.v22prime;

toc

% ----- Print Solutions -----

fprintf('First Pair Solution (Two Balls).....[%9.15f %9.15f] (m/s)\n', V11prime(1),V22prime(1))
fprintf('Second Pair Solution (Two Balls).....[%9.15f %9.15f] (m/s)\n', V11prime(2),V22prime(2))
fprintf('First Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n',
V1prime(1),V2prime(1),V3prime(1))
fprintf('Second Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n',
V1prime(2),V2prime(2),V3prime(2))

```

```
fprintf("Third Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n",
V1prime(3),V2prime(3),V3prime(3))
fprintf("Fourth Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n",
V1prime(4),V2prime(4),V3prime(4))
fprintf("Fifth Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n",
V1prime(5),V2prime(5),V3prime(5))
fprintf("Sixth Pair Solution (Three Balls).....[%9.15f %9.15f %9.15f] (m/s)\n",
V1prime(6),V2prime(6),V3prime(6))
```