Interior Backus Problem with Expanded Data

by

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Abstract

We consider a special formulation of the Backus geomagnetic problem in two and higher dimension. Given a domain $\Omega \subset \mathbb{R}^n$, we seek to determine whether there exists a unique harmonic function u satisfying $|\nabla u|^2 = P$ and $\frac{\partial}{\partial \nu}(|\nabla u|^2) = q$ on the boundary $\partial\Omega$ for given functions P and q which we refer to as the Backus problem with expanded data. In two dimensional case, for a function u satisfying the Laplace equation and the above boundary conditions, we derive a system of ordinary differential equations for the tangential and normal components of ∇u with the coefficients in terms of P and q on the boundary. We study the explicit solutions of the ODE system and establish conditions for existence and uniqueness of solutions for the problem involving the PDE on bounded domains. To achieve this goal we introduce the notion of generalized Hilbert transform and use representation formulas for solutions of the boundary value problems. In addition, we perform numerical experiments to corroborate our well-posedness results. For the higher dimensional problem, our approach is markedly different. For harmonic functions u in $\mathbb{R}^{n+1}$, we derive a quasilinear elliptic equation with coefficients involving the expanded data for the Backus problem satisfied by the restriction of u on n-dimensional hyperplanes. The Leray-Schauder fixed point theory relates the solvability of the Dirichlet boundary value problem to apriori estimates for solutions of a related family of problems. This theory is not applicable to the derived equation directly due to the restriction on the gradient of admissible function. To work around this restriction, we introduce a regularization of the operator. The Apriori Estimates Program is fulfilled by establishing the comparison and maximum principles, which allows the estimation of $\sup\limits_{\Omega} |u|$ in terms of $\sup\limits_{\partial\Omega} |u|$ and additive constants; boundary gradient estimates, that is an estimation of $\sup\limits_{\partial\Omega} |Du|$ in terms of $\sup\limits_{\Omega} |u|$; interior gradient bounds, by which we estimate $\sup\limits_{\Omega} |Du|$ in terms of $\sup\limits_{\partial\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$; and Hölder estimates for the gradient, that is an estimation of $[Du]_{\alpha;\Omega}$ in terms of $\sup\limits_{\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$. We eventually obtain the existence of solutions of the regularized equation with Dirichlet boundary conditions.

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Chapter 1

Introduction

The geomagnetic and gravitational fields emanating from the Earth's interior hold essential information about seismic activities and the internal structure. Therefore detection and analysis of the geomagnetic and gravitational fields play an important role in geophysical exploration. The geophysical survey is the process of collecting systematic information and geophysical data for spatial studies. The data may be collected from above and below the Earth's surface or from aerial, orbital, or marine platforms and such surveys may use a great variety of sensing instruments. In geophysical surveys, the gravitational field in steady state can be measured with accelerometers and its strength has usually been easier to measure with such devices than its direction relative to the surface of the Earth because atmospheric refraction interferes with accurate measurements of the shape of the surface. The introduction of scalar devices for measuring the magnetic field, e.g. the nuclear precession magnetometer, has produced a similar situation in geomagnetism which provides an accurate measurement of the magnitude of the local magnetic field but gives no indication of its direction.

The geophysical problem was first considered by Backus [2]. He sought to use the survey data to determine the external field when such data consists of field magnitudes rather than field components. Backus introduced the following mathematical formulation of the non-linear boundary value problem: a function u known to be harmonic in a region Ω with smooth boundary $\partial\Omega$ needs to be determined from $|\nabla u|$, the magnitude of the gradient of u, known on $\partial\Omega$, rather than, for example, from $\frac{\partial u}{\partial \nu}$, the normal derivative of u as in the classical Neumann problem.

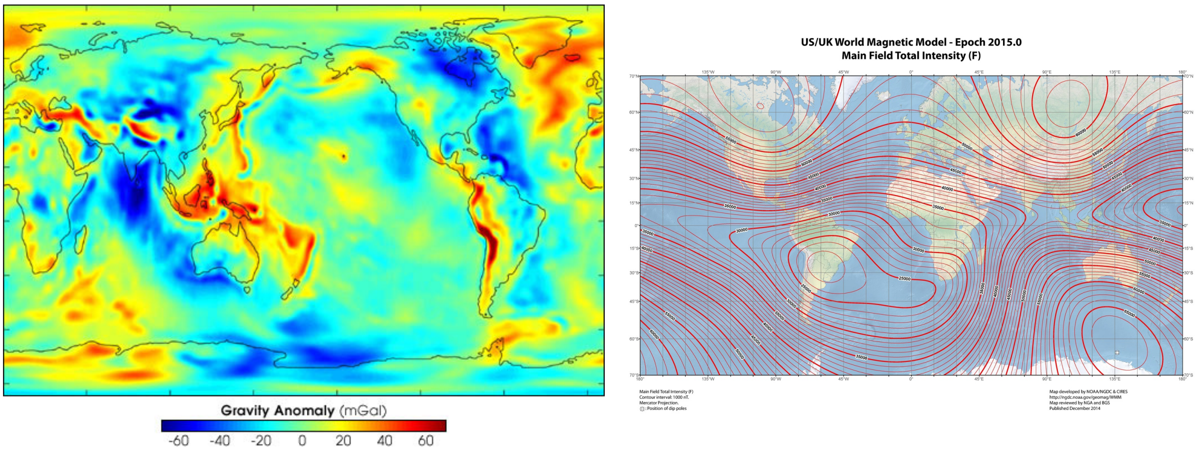


Figure 1.1: Gravitational and Magnetic Fields of the Earth

Backus considered three types of domain Ω: an open, bounded, connected set whose closure is simply connected, the exterior of the closure of such a set, and the half-plane. These three cases are referred to as the *interior*, *exterior* and *half-plane*. In the two-dimensional interior problem, Backus [2] showed that for any points $z_0, z_1, \cdots, z_n$ in Ω, there exists precisely one harmonic function u satisfying the boundary condition such that ∇u has a chosen direction at z_0, and u vanishes only at $z_1, \cdots, z_n$ in Ω. In dimension three or higher, no results on existence or uniqueness for interior problem were obtained by Backus. Meanwhile, in the exterior case, the solution can be shown to be unique under special conditions. One example is when Ω is the exterior of an open, bounded, connected set, function u is harmonic in Ω and vanishes at infinity, and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$, then u is uniquely determined in Ω by the values of $|\nabla u|$ on $\partial\Omega$. Another example is when Ω is the exterior of a sphere, u is a finite sum of exterior spherical harmonics, and $|\nabla u|$ is known on $\partial\Omega$, then u is uniquely determined in Ω except for the sign. Finally, if Ω is any type of domain, u is harmonic in Ω, and $|\nabla u|$ is known in Ω, then u is uniquely determined in Ω except for the sign. In the three-dimensional exterior case, the uniqueness is not guaranteed. Backus gave an example [3] of two functions u and v that are harmonic outside a solid sphere Ω, vanish at infinity, satisfy $|\nabla u| = |\nabla v|$ everywhere on $\partial\Omega$, and neither $u + v$ nor $u - v$ vanishes identically outside Ω.

Lieberman [23] discussed the boundary regularity of solutions of the fully nonlinear boundary value problem $F(x, u, Du, D^2u) = 0$ in Ω, $G(x, u, Du) = 0$ on $\partial\Omega$ for two-dimensional domains Ω. The Backus problem is a special case of this problem. The function F is assumed uniformly elliptic and G is assumed to depend in a nonvacuous manner on Du. Lieberman proved the continuity estimates for first and second derivatives of u under hypotheses for smoothness of F, G and Ω.

Jorge and Magnanini [16] [24] studied the Backus problem for the exterior gravitational field of the Earth further. They obtained the following uniqueness result: if two smooth functions u and v satisfy $\Delta u = \Delta v = 0$ in Ω; $\nabla u \cdot \nabla u = \nabla v \cdot \nabla v$ on $\partial\Omega$; u, v are regular at infinity; and $u(\bar{x}) = v(\bar{x})$ for $\bar{x} \in M = \{\bar{x} \in \partial\Omega \mid \frac{\partial}{\partial\nu}(u+v)(\bar{x}) = 0\}$, then $u = v$ in Ω. Once these conditions are satisfied, an explicit series solution for the geophysical problem was computed and the convergence of this series solution was proved.

Díaz, Díaz, and Otero [6] [7] considered a nonlinear oblique derivative interior boundary value problem suggested by the study of the Backus problem for the external gravitational potential of the Earth. They focused on the simplest case of a sphere: the unit ball in $\mathbb{R}^3$. For the boundary value in a special form of $\sqrt{(g^2 - |\nabla_s u|^2)_+}$, where $\nabla_s u$ denotes the tangential or surface gradient of u, they showed the existence and uniqueness of viscosity solutions. For a function that is harmonic outside a unit sphere Ω, vanishes at infinity and $|\nabla u|$ takes prescribed value g on Ω, the solution is not unique in general. Díaz, Díaz, and Otero [8] proved that the solution is unique with the additional property that the radial component of the gradient of u on Ω is nonpositive. If a solution u with this property exists, they showed that u is the maximal solution of the Backus problem. Otero [25] proposed an existence program for the Backus problem based on the establishment of a priori estimates in a Hölder space. Under certain hypothesis and simplifications, he obtained maximum modules and gradient bounds for the solutions.

Holota [14] discussed the linear gravimetric boundary value problem in the sense of the weak solution. He constructed a Sobolev weight space for an unbounded domain representing the exterior of the Earth and deduced the quantitative estimates for the trace theorem and equivalent norms. In the generalized formulation of the problem a special decomposition of the Laplace operator was used to express the oblique derivative in the boundary condition. The main result concerns the ellipticity of a bilinear form associated with the problem under consideration. He used the Lax-Milgram theorem to determine the existence, uniqueness and stability of the weak solution of the problem.

Among related results, Kaiser [17] considered the geomagnetic problem in which the direction of the gradient rather than its magnitude is assumed to be known on the boundary. Kaiser was interested in the nonlinear boundary value problem in the exterior $\hat{V}^d$ of a sphere S^{d-1} in two and three dimensions $(d = 2, 3)$. Given a direction field D: $S^{d-1} \to \mathbb{R}^d$, he sought to determine all harmonic vector fields B: $\hat{V}^d \to \mathbb{R}^d$ with asymptotic behavior $|B| = \mathcal{O}(|x|^{-\delta})$, $\delta \in \mathbb{N} \setminus \{d-1, d-2\}$ for $|x| \to \infty$, which are parallel to D on S^{d-1}. For $d = \delta = 3$, this problem is related to the problem of reconstructing the geomagnetic field outside the Earth from its directional data measured on the Earth's surface. For a fixed direction field D, the set of harmonic vector fields B forms a linear space $L(D)$. This space was described in the two-dimensional case and its dimension was estimated in the three-dimensional axisymmetric case. Introducing the rotation number ρ of a Hölder continuous direction field D with respect to S^1, in the case $d = 2$, Kaiser [17] showed that $\dim L(D) = \max\{2(\rho - \delta) + 1, 0\}$. Similarly, in the axisymmetric case for $d = 3$, he obtained the estimate $\dim L(D) \leq \max\{\rho - \delta + 2, 0\}$. Thus, in an axisymmetric setting with $\delta = 3$, uniqueness is guaranteed only for direction fields with $\rho = 2$. Kaiser and Neudert [18] characterized the solution space $V_D^{\perp}$ of the boundary value problem as orthogonal complement of a certain set of functions determined by the vector field D in an appropriate Hilbert space. They determined $V_D^{\perp}$ and its dimension $\dim V_D^{\perp}$ for vector field D in the case $d = 2$ and in the axisymmetric case $d = 3$.

Isakov [15] considered and stated the inverse problem of potential theory. Glotov [12] considered a problem related to an inverse problem for the Poisson equation with point sources on a disk or half plane in $\mathbb{R}^2$. He presented a method for converting the absolute value of the gradient and its normal derivative to the Cauchy data, i.e., values of both the solution and its normal derivative on the boundary of the half-plane. His approach is based on the study of a linear system of ordinary differential equations corresponding to this problem. In the case of the unit disk, the existence of periodic solutions is equivalent to a condition that corresponds to having a finite number of monopoles and dipoles in the domain. He introduced an additional constraint to guarantee the uniqueness of the solution (determined up to a constant). It was also shown that this method can not be easily extended to three dimension.

One advantage of the approach introduced in [12] is that it allows us to estimate the number of monopoles and dipoles in bounded domains. To fix the notation, let Ω be a bounded simply connected domain in $\mathbb{R}^2$ and consider the Poisson equation $\Delta u = f$ in Ω with the source term f of the form $f = \sum_{j=1}^{M} a_j \delta_{x^j} + \sum_{j=1}^{N} b_j D\delta_{y^j}$ for some $M,\ N \in \mathbb{N}$; $a_j \in \mathbb{R}$, $x^j \in \Omega$ for $j = 1, \cdots, M$ and $b^j \in \mathbb{R}^2$, $y^j \in \Omega$ for $j = 1, \cdots, N$, and where δ_x is the Dirac delta function at x. Such a representation of f corresponds to monopoles of magnitude a_j located at x^j and dipoles oriented and scaled by b_j and located at y^j. As shown in [12], we have the following estimate

$$-\frac{1}{2\pi}\int_{\partial\Omega} \frac{P_\nu}{2P}\, d\tau = -\frac{1}{2\pi}\int_0^{2\pi} \frac{q}{2P}\, dt \leq M + 2N \tag{1.1}$$

where $P = |Du|^2$ and $q = \frac{\partial}{\partial\nu}(|Du|^2)$ on $\partial\Omega$.

The problem that is considered in this dissertation is a variation of the interior Backus problem on bounded domains. In addition to the magnitude of $|\nabla u|$ on $\partial\Omega$, we assume that we are given the value of $\frac{\partial}{\partial\nu}(|\nabla u|^2)$ on the boundary as in [12].

Problem Statement. *Let Ω be a bounded simply connected domain in $\mathbb{R}^n$ with sufficiently smooth boundary. Given the values of P and q on $\partial\Omega$, we seek to determine whether there exists a unique harmonic function u defined in Ω (or a neighborhood of $\partial\Omega$) that satisfies*

$$\begin{cases} |Du|^2 = P, \\ \dfrac{\partial}{\partial\nu}(|Du|^2) = q, \end{cases} \quad \textit{on } \partial\Omega. \tag{1.2}$$

We refer to this problem as the Backus problem with expanded data.

Remark. *In the Backus problem with expanded data, we seek a harmonic function u, which implies from estimate* (1.1) *that the numbers of monopoles and dipoles of the inverse source problem are both zero. In order to guarantee that the given data* (P, q) *generates a harmonic* *function in $\mathbb{R}^2$, we require that the following estimate holds:*

$$-\frac{1}{2\pi}\int_0^{2\pi}\frac{q}{2P}\,dt \leq 0.$$

This dissertation is organized as follows.

In Section 2.1, we start with a harmonic function u in $\mathbb{R}^2$ satisfying the boundary conditions $|Du|^2 = P$ and $\frac{\partial}{\partial\nu}(|Du|^2) = q$, and derive a system of linear ordinary differential equations

$$\dot{z}(t) = A\cdot z(t) \tag{1.3}$$

for the tangential and normal derivatives $z(t) = (z_1(t), z_2(t))^T = (u_\tau(\gamma(t)), u_\nu(\gamma(t)))^T$, where γ represents the parametrization of the boundary of the general domain and the entries of the matrix A are functions of P and its derivatives, namely,

$$A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix},$$

where $a_1 = \dfrac{1}{2P}(P_x x' + P_y y')$ and $a_2 = \dfrac{1}{2P}(P_x y' - P_y x') + x'y'' - y'x''$. We have an explicit representation of the solutions of the ODE system in complex form as follows,

$$z(t) = z_1(t) + iz_2(t) = z_0 e^{\int_0^t (a_1+ia_2)d\tau}.$$

By matching the real and imaginary parts, we obtain

$$z_1(t) = z_0\sqrt{P(t)}\,\cos\int_0^t a_2(\tau)\,d\tau \quad\text{and}\quad z_2(t) = z_0\sqrt{P(t)}\,\sin\int_0^t a_2(\tau)\,d\tau.$$

In Sections 2.2 and 2.3, we rewrite the well-known representation formulas for the solutions of the Dirichlet and Neumann boundary value problems on the half-plane and the unit disk in terms of the derivatives of solutions with the purpose to highlight the relation between the representations of solutions for the two problems and motivate the introduction of the generalized Hilbert transform in Section 2.4.

In Section 2.4, with the help of the generalized Hilbert transform, we address the uniqueness of harmonic functions u arising from the solutions of the ODE system on the boundary of the general bounded domain. Specifically, we assume that (z_1, z_2) is a solution of the ODE system (1.3), and $\mathcal{H}z_2 = z_1$, where $\mathcal{H}$ represents the generalized Hilbert transform. Under this hypothesis, we show that there exists a unique (up to a phase) harmonic function u satisfying $|Du|^2 = P$ and $\frac{\partial}{\partial\nu}(|Du|^2) = q$ on the boundary.

In Chapter 3, we introduce a numerical method for solving the non-linear boundary value problem and present results of numerical experiments. An organizational workflow of experiments mainly consists of these three steps:

- Generate the data from harmonic functions.
- Solve the ODE system.
- Use the solution of the ODE to solve the PDE.

To solve the ODE system, we use a Matlab boundary value problem solver implemented in the function `bvp4c`. In order to represent the solutions z_1 and z_2 of the ODE system as functions of the parameter t and supply them as data for the PDE problem, we scale z_1 and z_2 to satisfy the condition $z_1^2 + z_2^2 = P$ and then use the cubic spline interpolation with periodic conditions (see Appendix B) to obtain continuous functions. In the last step, the PDE Toolbox in Matlab is used to solve the PDE problem using the Finite Element Method (FEM). The toolbox is designed to construct numerical solutions of problems on bounded domain in the two-dimensional plane.

Three harmonic functions are considered in the examples:

- $u_0 = y^2 - x^2$,
- $u_1 = 0.1(x^2 + y^2)^5 \cos(10 \tan^{-1}(\frac{y}{x}))$,
- $u_2 = u_0 + u_1$.

We measure the error in approximating the boundary data P with the solution of the ODE system u_τ and u_ν, namely we compute $||u_\tau^2 + u_\nu^2 - P||_{L^2(\partial\Omega)}$ and estimate its rate of convergence.

In order to estimate the error of the approximation, we tabulate the values of the error $u - u_e$ in the L^2-norm and H^1-norm for different values of the space discretization parameter. Here u_e denotes the exact values of the harmonic solution on the boundary mesh points and $u = \cos\alpha \cdot u_d + \sin\alpha \cdot v_d$ is an approximate solution with u_d and v_d being the solutions of Laplace's equation with Dirichlet boundary conditions $\int_0^t z_1(\theta)d\theta$ and $\int_0^t z_2(\theta)d\theta$, respectively and α being the optimal phase. We next estimate the rate of convergence in the L^2-norm and H^1-norm of $u - u_e$.

In Section 4.1, we start with a harmonic function u in $\mathbb{R}^{n+1}$, and obtain a quasilinear equation involving Du, the tangential gradient of u, in n-dimensional hyperplanes. Specifically, we show that Du satisfies

$$Qu := \operatorname{div} \frac{Du}{\sqrt{P - |Du|^2}} + \frac{1}{2}\frac{\sigma q}{P - |Du|^2} = 0, \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

This is a second order equation. We confirm its ellipticity by computing the eigenvalues of this operator.

By Leray-Schauder Theorem, the solvability of the Dirichlet problem, $Qu = 0$ in Ω, $u = \phi$ on $\partial\Omega$, in the space $C^{2,\alpha}(\bar{\Omega})$ is thus equivalent to the solvability of the equation $u = Tu$ in the Banach space $\mathcal{B} = C^{1,\beta}(\bar{\Omega})$, where $T : v \to u$ is defined by letting $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\bar{\Omega})$ of the linear Dirichlet problem. Theorem 4.3 in Section 4.1 reduces the solvability of the Dirichlet problem to the apriori estimates in the space $C^{1,\beta}(\bar{\Omega})$ of the solutions of a related family of problems. In practice, it is desirable to break the derivation of the following **Apriori Estimates Program** into four stages:

1. Estimation of $\sup_{\Omega} |u|$.
2. Estimation of $\sup_{\partial\Omega} |Du|$ in terms of $\sup_{\Omega} |u|$.
3. Estimation of $\sup_{\Omega} |Du|$ in terms of $\sup_{\partial\Omega} |Du|$ and $\sup_{\Omega} |u|$.
4. Estimation of $[Du]_{\beta;\Omega}$ in terms of $\sup_{\Omega} |Du|$ and $\sup_{\Omega} |u|$.

We point out some challenges in applying Theorem 4.3 to the quasilinear equation (1.4) and motivate the forthcoming regularization. The purpose of the regularization is to allow $|Du|$

to take arbitrarily large values. Given $\varepsilon > 0$, we consider the following equation in divergence form:

$$Q_\varepsilon u = \operatorname{div} \frac{Du}{\sqrt{\xi_\varepsilon(x, |Du|)}} + \frac{1}{2}\frac{\sigma q}{\zeta_\varepsilon(x, |Du|)} = 0, \tag{1.5}$$

where

$$\xi_\varepsilon(x,p) = \begin{cases} P - |p|^2 + \varepsilon|p| & |p|^2 \le P, \\ \dfrac{\varepsilon^2|p|^2}{|p|^2 - P + \varepsilon|p|} & |p|^2 > P, \end{cases}$$

and

$$\zeta_\varepsilon(x,p) = \begin{cases} P - |p|^2 + \varepsilon|p| & |p|^2 \le P, \\ \varepsilon|p| & |p|^2 > P. \end{cases}$$

Notice that $\xi_\varepsilon(x,p)$ is continuously differentiable as a function of x and p and $\xi_\varepsilon(x,p)$ is continuous as a function of x and p. Also we note that taking the limit as $\varepsilon \to 0$ in the coefficients ξ_ε and ζ_ε, we recover the operator Q in (1.4) for $|p|^2 \le P$. We prove in Lemma 4.4 that the operator Q_ε is uniformly elliptic in Ω.

We propose to study the well-posedness of the regularized quasilinear equation by following the Apriori Estimates Program for regularized quasilinear equation (1.5) with the intention to pass to the limit as $\varepsilon \to 0$ to obtain the well-posedness of equation (1.4).

In Sections 4.6 and 4.7, we develop the structure conditions and then obtain two versions of the maximum principle: one for the divergence form and another for the non-divergence form. These results allow the estimation of $\sup_{\Omega} |u|$ in terms of $\sup_{\partial\Omega} |u|$ and additive constants, which is Stage 1 in the Apriori Estimates Program.

In Section 4.8, we show that if $u \in C^2(\Omega)$ satisfies $Q_\varepsilon u = 0$ in Ω, then $w = D_k u$ is a solution of the linear elliptic equation

$$Lw = D_i(\bar{a}^{ij} D_j w) = -D_i f_k^i.$$

Using that L is strictly elliptic in Ω and has bounded coefficients, we verify the Hölder estimate

$$[Du]_{\alpha;\Omega} \le C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega, \Phi),$$

which is an estimation of $[Du]_{\alpha;\Omega}$ in terms of $\sup\limits_{\Omega}|Du|$ and $\sup\limits_{\Omega}|u|$. The constant C does not depend on ε and this is Stage 4 in the Apriori Estimates Program.

In Section 4.9, we impose an additional constraint on the domain Ω to satisfy an exterior sphere condition. With this assumption, for the constant

$$\mu = \sup_{x\in\Omega}\left\{\sqrt{P},\ \frac{2}{\sqrt{P}} + \frac{|DP(x)|}{\sqrt{P(x)^3}} + \frac{q(x)}{2\sqrt{P(x)^3}}\right\},$$

we derive the estimate

$$|p|\cdot\Lambda + |b| \leq \mu\cdot\mathscr{E}$$

that holds for all $(x,z,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$ with $|p|\geq\mu$. We therefore assert the boundary gradient estimate for general domains

$$\sup_{\partial\Omega}|Du| \leq C(n,\mu,\Phi,\delta),$$

which is an estimation of $\sup\limits_{\partial\Omega}|Du|$ in terms of $\sup\limits_{\Omega}|u|$ with the constant C not depending on ε. This is Stage 2 in the Apriori Estimates Program.

In Section 4.10, we calculate an explicit positive constant $\nu = \min\left\{\frac{\min(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\min(\lambda'_1,\lambda'_n)}{\lambda_1'^{\frac{1}{2}}(2\lambda'_n)^{\frac{3}{4}}}\right\}$ on $\mathbb{R}$ for which the estimate

$$\bar{a}^{ij}(x,p)\xi_i\xi_j = D_{p_j}\mathbf{A}^i(x,p)\xi_i\xi_j \geq \nu|\xi|^2$$

holds. In addition, we verify that

$$D_x\mathbf{A}_\varepsilon,\ \ B_\varepsilon = o(|p|) \quad \text{as } |p|\to\infty.$$

We confirm that the choice of function $\mathbf{A}_\varepsilon$ insures that the operator is uniformly elliptic in Ω in the sense that

$$|D_p\mathbf{A}_\varepsilon(x,p)| \leq \mu,$$

where

$$\mu = \max\left\{\frac{\max(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\max(\lambda'_1,\lambda'_n)}{\lambda_1'^{\frac{1}{2}}(2\lambda_n-2P)^{\frac{3}{4}}}\right\}$$

is a positive constant on $\mathbb{R}$. By verifying the more general condition

$$g(x,p) = |D_x \mathbf{A}_\varepsilon| + |B_\varepsilon| \leq \mu(1+|p|)^2,$$

we finally arrive at the global estimate

$$\sup_\Omega |Du| \leq C(n, \mu, \nu, \sup_\Omega |\mathbf{A}(x,p)|, \partial\Omega, |\phi|_{2:\Omega})$$

which is an estimation of $\sup\limits_\Omega |Du|$ in terms of $\sup\limits_{\partial\Omega} |Du|$ and $\sup\limits_\Omega |u|$. This is Stage 3 in the Apriori Estimates Program. We point out that unlike in the previous stages, we have not been able to establish that bound on $|Du|$ in Ω is independent of ε.

Based on the theorems we develop in the previous sections, in Section 4.11 we obtain the apriori estimate in terms of the boundary value of Dirichlet problem for the regularized operator.

Chapter 2

Backus Problem in Planar Domain

In this chapter we focus on the Backus problem with expanded data in planar domains. Suppose Ω is a bounded simply connected domain in $\mathbb{R}^2$ and u is a function in $C^2(\bar{\Omega})$ that satisfies $\Delta u = 0$ in Ω and let

$$\begin{cases} P = |Du|^2, \\ q = \dfrac{\partial}{\partial \nu}(|Du|^2), \end{cases} \tag{2.1}$$

At each point $x_0 \in \partial\Omega$, we select τ to be the unit vector tangent to $\partial\Omega$ at x_0 with the counter-clockwise orientation, and ν to be the outward pointing unit normal to $\partial\Omega$ at x_0. We parametrize the boundary $\partial\Omega$ by the arc length $\gamma : [0, T] \to \partial\Omega$, here T is the period of γ, so that $\gamma(t) = (x(t), y(t))$ in coordinate form and we set

$$z(t) = (z_1(t), z_2(t))^T = (u_\tau(\gamma(t)), u_\nu(\gamma(t)))^T.$$

2.1 Transformation to ODE

We start by transforming the values of $|Du|$ and $\dfrac{\partial}{\partial \nu}(|Du|^2)$ given on the boundary into the boundary data for linear problems. In the derived system of ordinary differential equations that affords this transformation, z_1 and z_2 will play the role of unknowns.

First, we expand z in Cartesian coordinates as follows

$$z_1(t) = \nabla u \cdot \tau = Du(x(t), y(t)) \cdot (x'(t), y'(t)) = u_x \cdot x' + u_y \cdot y',$$

$$z_2(t) = \nabla u \cdot \nu = Du(x(t), y(t)) \cdot (y'(t), -x'(t)) = u_x \cdot y' - u_y \cdot x'.$$

Taking the derivatives of $P = |Du|^2 = u_x^2 + u_y^2$ with respect to x and y, we have

$$P_x = 2(u_x u_{xx} + u_y u_{xy}),$$

$$P_y = 2(u_x u_{xy} + u_y u_{yy}).$$

Using the fact that u is harmonic, we get

$$u_x P_x - u_y P_y = 2(u_x^2 u_{xx} - u_y^2 u_{yy}) = 2P u_{xx},$$

$$u_y P_x + u_x P_y = 2(u_y^2 u_{xy} + u_x^2 u_{xy}) = 2P u_{xy},$$

or, equivalently,

$$u_{xx} = \frac{1}{2P}(u_x P_x - u_y P_y), \quad u_{xy} = \frac{1}{2P}(u_y P_x + u_x P_y), \quad u_{yy} = \frac{1}{2P}(u_y P_y - u_x P_x). \tag{2.2}$$

Taking the derivatives of $z_1(t)$ and $z_2(t)$ with respect to t and using (2.2), we have

$$\begin{aligned}
\dot{z}_1(t) &= u_{xx}(x')^2 + u_{xy}x'y' + u_x x'' + u_{xy}x'y' + u_{yy}(y')^2 + u_y y'' \\
&= \frac{1}{2P}(u_x P_x - u_y P_y)((x')^2 - (y')^2) + \frac{1}{P}(u_y P_x + u_x P_y)x'y' + u_x x'' + u_y y'' \\
&= \frac{1}{2P}(P_x x' + P_y y')(u_x x' + u_y y') - \frac{1}{2P}(P_x y' - P_y x')(u_x y' - u_y x') + u_x x'' + u_y y'',
\end{aligned}$$

and

$$\begin{aligned}
\dot{z}_2(t) &= u_{xx}x'y' + u_{xy}(y')^2 + u_x y'' - u_{xy}(x')^2 - u_{yy}x'y' - u_y x'' \\
&= -\frac{1}{2P}(u_y P_x + u_x P_y)((x')^2 - (y')^2) + \frac{1}{P}(u_x P_x - u_y P_y)x'y' + u_x y'' - u_y x'' \\
&= \frac{1}{2P}(P_x y' - P_y x')(u_x x' + u_y y') + \frac{1}{2P}(P_x x' + P_y y')(u_x y' - u_y x') + u_x y'' - u_y x''.
\end{aligned}$$

We rewrite the equations for $z(t) = (z_1(t), z_2(t))^T$ in the vector form as follows:

$$\dot{z}(t) = \tilde{A} \cdot z(t) + \tilde{b} \tag{2.3}$$

where

$$\tilde{A} = \frac{1}{2P}\begin{pmatrix} P_x x' + P_y y' & -(P_x y' - P_y x') \\ P_x y' - P_y x' & P_x x' + P_y y' \end{pmatrix} \text{ and } \tilde{b} = \begin{pmatrix} u_x x'' + u_y y'' \\ u_x y'' - u_y x'' \end{pmatrix} \tag{2.4}$$

The next step is to rewrite (2.4) in terms of local coordinators (τ, ν) at each point (x, y) on the boundary $\partial\Omega$. Taking the derivatives of P with respect to τ and ν respectively, we obtain

$$P_\tau = \nabla P \cdot \tau = \langle P_x, P_y\rangle \cdot \langle x', y'\rangle = P_x \cdot x' + P_y \cdot y',$$

$$P_\nu = \nabla P \cdot \nu = \langle P_x, P_y\rangle \cdot \langle y', -x'\rangle = P_x \cdot y' - P_y \cdot x'.$$

Recognizing P_τ and P_ν as the entries of the above matrix $\tilde{A}$, we can write it in terms of P, P_τ and P_ν as follows:

$$\tilde{A} = \frac{1}{2P}\begin{pmatrix} P_\tau & -P_\nu \\ P_\nu & P_\tau \end{pmatrix} = \frac{1}{2P}\begin{pmatrix} P_\tau & -q \\ q & P_\tau \end{pmatrix} \tag{2.5}$$

If $\partial\Omega$ is parametrized by the arc length, we represent $\tilde{b}$ in the form $\tilde{b} = \tilde{B} \cdot z$ with

$$\tilde{B} = \frac{1}{(x')^2 + (y')^2}\begin{pmatrix} x'x'' + y'y'' & y'x'' - x'y'' \\ x'y'' - y'x'' & x'x'' + y'y'' \end{pmatrix} = \begin{pmatrix} 0 & y'x'' - x'y'' \\ x'y'' - y'x'' & 0 \end{pmatrix}$$

then equation (2.3) can be rewritten in the homogeneous form

$$\dot{z}(t) = A \cdot z(t) \tag{2.6}$$

where

$$A = \tilde{A} + \tilde{B} = \begin{pmatrix} \frac{1}{2P}(P_x x' + P_y y') & -\frac{1}{2P}(P_x y' - P_y x') + y'x'' - x'y'' \\ \frac{1}{2P}(P_x y' - P_y x') + x'y'' - y'x'' & \frac{1}{2P}(P_x x' + P_y y') \end{pmatrix} \tag{2.7}$$

Next, we consider two special cases of Ω and compute the corresponding matrices A of the homogeneous systems.

Example 1. *Let Ω be the unit disk $B_1(0)$. We parametrize the boundary $\partial B_1(0)$ by $x(t) = \cos t$ and $y(t) = \sin t$, and simplify the matrix (2.7) to get*

$$A = \begin{pmatrix} \frac{1}{2P}(P_x x' + P_y y') & -\frac{1}{2P}(P_x y' - P_y x') - 1 \\ \frac{1}{2P}(P_x y' - P_y x') + 1 & \frac{1}{2P}(P_x x' + P_y y') \end{pmatrix} = \begin{pmatrix} \frac{P_\tau}{2P} & -\frac{q}{2P} - 1 \\ \frac{q}{2P} + 1 & \frac{P_\tau}{2P} \end{pmatrix}. \quad (2.8)$$

The domain in this example will be employed in the numerical studies in Chapter 3.

Example 2. *Let Ω be the half-plane $\mathbb{R}^2_+$. The boundary, i.e., x-axis can be parametrized by $x = t$ and $y = 0$. The matrix A is simplified to*

$$A = \frac{1}{2P} \begin{pmatrix} P_x & P_y \\ -P_y & P_x \end{pmatrix}. \quad (2.9)$$

Suppose u is a solution of the PDE problem (2.1). In general, $z(t) = (z_1(t), z_2(t)) = (u_\tau(\gamma(t)), u_\nu(\gamma(t)))$ are T-periodic if and only if

$$\int_0^T a_1(t)dt = 0 \text{ and } \int_0^T a_2(t)dt = 2n\pi \quad (2.10)$$

for some integer n, which is shown in [12], where

$$a_1 = \frac{1}{2P}(P_x x' + P_y y') \text{ and } a_2 = \frac{1}{2P}(P_x y' - P_y x') + x'y'' - y'x''$$

are the entries in the first column of matrix A defined in (2.7).

Theorem. *Given $P \in C^{1,\alpha}(\partial\Omega)$, $q \in C^\alpha(\partial\Omega)$, suppose (2.10) holds. Then $z = (z_1, z_2)$ defined by*

$$z_1(t) = z_0\sqrt{P(t)}\, \cos \int_0^t a_2(\tau)\, d\tau \quad \textit{and} \quad z_2(t) = z_0\sqrt{P(t)}\, \sin \int_0^t a_2(\tau)\, d\tau,$$

is a solution of (2.6).

Proof. We will show that the expressions of z_1 and z_2 are a solution of (2.6) by direct substitution into the equations. First we note that

$$\sqrt{P} = e^{\frac{1}{2}\ln P} = e^{\int_0^t \frac{P_\tau}{2P} d\tau} = e^{\int_0^t a_1(\tau) d\tau}.$$

Therefore,

$$\begin{aligned}
\dot{z}_1(t) &= z_0 e^{\int_0^t a_1(\tau)d\tau} a_1(t) \cdot \cos \int_0^t a_2(\tau)\, d\tau - z_0 \sqrt{P(t)} \sin \int_0^t a_2(\tau)\, d\tau \cdot a_2(t) \\
&= a_1(t) \cdot z_0 \sqrt{P(t)}\, \cos \int_0^t a_2(\tau)\, d\tau - a_2(t) \cdot z_0 \sqrt{P(t)}\, \sin \int_0^t a_2(\tau)\, d\tau \\
&= a_1(t) z_1(t) - a_2(t) z_2(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\dot{z}_2(t) &= z_0 e^{\int_0^t a_1(\tau)d\tau} a_1(t) \cdot \sin \int_0^t a_2(\tau)\, d\tau + z_0 \sqrt{P(t)} \cos \int_0^t a_2(\tau)\, d\tau \cdot a_2(t) \\
&= a_1(t) \cdot z_0 \sqrt{P(t)}\, \sin \int_0^t a_2(\tau)\, d\tau + a_2(t) \cdot z_0 \sqrt{P(t)}\, \cos \int_0^t a_2(\tau)\, d\tau \\
&= a_1(t) z_2(t) + a_2(t) z_1(t).
\end{aligned}$$

Thus z is a solution of the ODE system (2.6). □

Remark. *Suppose u satisfies $\Delta u = 0$ in Ω and $(z_1(t), z_2(t)) = (u_\tau(\gamma(t)), u_\nu(\gamma(t)))$. Then it has the following averaging properties:*

$$\int_0^T z_1(t) dt = 0, \quad \textit{and} \quad \int_0^T z_2(t) dt = 0. \tag{2.11}$$

where T is the period of γ.

The first equation is due to the fundamental theorem of line integrals and the second one is obtained from the divergence theorem.

In the next two sections, we rewrite the well-known representation formulas for the solutions of the Dirichlet and Neumann boundary value problems in the half-plane and the unit disk in terms of the derivatives of solutions. The purpose of these sections is to highlight the relation between the representations of solutions for the two problems and motivate the introduction of the generalized Hilbert transform in Section 2.4. A systematic study of the Dirichlet-to-Neumann map and the Neumann-to-Dirichlet map, in the framework of linear relations in Hilbert spaces, is presented in [4].

2.2 Linear boundary-value problems on half-plane

Suppose that u is a solution of Laplace's equation $\Delta u = 0$ with the Dirichlet boundary condition g on the half-plane $\mathbb{R}^2_+$, that is,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = g & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

This solution u is given by the Poisson formula for half-plane as follows for $x = (x_1, x_2) \in \mathbb{R}^2_+$:

$$u(x) = \frac{x_2}{\pi} \int_{\partial\mathbb{R}^2_+} \frac{g(y)}{|x-y|^2}\, dy. \tag{2.12}$$

For the Dirichlet boundary condition we set $g' = z_1$ and, assuming that z_1 vanishes at $-\infty$, we have $g(y_1) = \int_{-\infty}^{y_1} z_1(s)\, ds$. Then Poisson's formula (2.12) becomes

$$\begin{aligned} u(x) &= \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g(y_1)}{|x-y|^2}\, dy_1 \\ &= \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{1}{|x-y|^2} \int_{-\infty}^{y_1} z_1(s)\, ds\, dy_1 \\ &= \frac{x_2}{\pi} \int_{\mathbb{R}} z_1(s) \int_s^{\infty} \frac{1}{|x-y|^2}\, dy_1\, ds \end{aligned}$$

To compute the inside integral, we change the variable and let $x_1 - y_1 = x_2 \tan\theta$, then $|x-y|^2 = x_2^2 \tan\theta + x_2^2 = x_2^2 \sec^2\theta$. We have $dy_1 = -x_2 \sec^2\theta\, d\theta$ and

$$\int_s^{\infty} \frac{1}{|x-y|^2}\, dy_1 = \frac{1}{x_2}\Big[\frac{\pi}{2} + \tan^{-1}\Big(\frac{x_1 - s}{x_2}\Big)\Big].$$

Thus

$$u(x) = \int_{\mathbb{R}} \left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x_1 - y_1}{x_2}\right)\right] z_1(y_1)\, dy_1.$$

Let $t = \tan^{-1}\left(\frac{x_1 - y_1}{x_2}\right)$, which is equivalent to $\tan t = \frac{x_1 - y_1}{x_2}$. Taking the derivative with respect to x_2, we have $\sec^2 t \, \frac{dt}{dx_2} = -\frac{x_1 - y_1}{x_2^2}$ which implies that

$$\frac{dt}{dx_2} = -\frac{x_1 - y_1}{x_2^2 \sec^2 t} = -\frac{x_1 - y_1}{x_2^2} \cdot \frac{x_2^2}{|x - y|^2} = -\frac{x_1 - y_1}{|x - y|^2}.$$

Computing the derivative of u on $\partial \mathbb{R}^2_+$ with respect to ν, we have

$$u_\nu(x) = \lim_{x_2 \to 0} u_{x_2}(x) = -\frac{1}{\pi} \lim_{x_2 \to 0} \int_{\mathbb{R}} \frac{x_1 - y_1}{|x - y|^2}\, z_1(y_1)\, dy_1. \tag{2.13}$$

Next we consider the Neumann problem in the half-plane. Suppose that v is a solution of Laplace's equation $\Delta v = 0$ with the Neumann boundary condition, that is,

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial \nu} = z_2 & \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

This solution v is derived in Appendix A and given by Dini's formula for half-plane as follows:

$$v(x) = -2 \int_{\partial \mathbb{R}^2_+} \Phi(x, y) z_2(y)\, dy \tag{2.14}$$

where

$$\Phi(x, y) = \Phi(x - y) = -\frac{1}{2\pi} \log(|x - y|)$$

is the fundamental solution of Laplace equation so that formula (2.14) becomes

$$v(x) = \frac{1}{\pi} \int_{\mathbb{R}} \log(|x - y|) z_2(y_1)\, dy_1.$$

Taking the derivative of v on $\partial\mathbb{R}^2_+$ with respect to τ, we have

$$v_\tau(x) = \lim_{x_2\to 0} v_{x_1}(x) = \frac{1}{\pi}\lim_{x_2\to 0}\int_{\mathbb{R}} \frac{x_1 - y_1}{|x-y|^2}\, z_2(y_1)\, dy_1. \tag{2.15}$$

In summary, the representations of solutions of Dirichlet and Neumann problems yield

$$u_\nu(x_1, 0) = -\frac{1}{\pi}\lim_{x_2\to 0}\int_{\mathbb{R}} \frac{x_1 - y_1}{|x-y|^2}\, z_1(y_1)\, dy_1,$$

and

$$v_\tau(x_1, 0) = \frac{1}{\pi}\lim_{x_2\to 0}\int_{\mathbb{R}} \frac{x_1 - y_1}{|x-y|^2}\, z_2(y_1)\, dy_1.$$

If the solutions of the Dirichlet and Neumann problems are identical, i.e. $u = v$, then we must have

$$z_2 = -\frac{1}{\pi}\,\text{P.V.}\int_{\mathbb{R}} \frac{z_1(y)}{x-y}\, dy,$$

and

$$z_1 = \frac{1}{\pi}\,\text{P.V.}\int_{\mathbb{R}} \frac{z_2(y)}{x-y}\, dy.$$

Recalling the definition of Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}\,\text{P.V.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy,$$

we can write these relations as

$$\mathcal{H}z_1 = -z_2 \quad \text{and} \quad \mathcal{H}z_2 = z_1.$$

This connection gives us a motivation to use Hilbert transform to connect z_1 and z_2 as a condition for the existence and uniqueness of the solution.

2.3 Linear boundary-value problems on unit disk

In this section, we follow the same calculations as in the previous section to obtain the results on unit disk. Suppose that v is a solution of Laplace's equation $\Delta v = 0$ with the Neumann boundary condition on the unit ball $B_1(0)$, that is

$$\begin{cases} \Delta v = 0 & \text{in } B_1(0), \\ \dfrac{\partial v}{\partial \nu} = z_2 & \text{on } \partial B_1(0). \end{cases}$$

The solution is given by Dini's formula for the unit ball

$$v(x) = -\frac{1}{\pi} \int_{\partial B_1(0)} \log(|x-y|) g(y)\, dS(y). \tag{2.16}$$

The derivation of this formula also appears in Appendix A where $x = (x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$ is an arbitrary point in the unit disk $(0 \le \rho \le 1)$ and $y = (y_1, y_2) = (\cos t, \sin t)$ is a point on the unit circle. Moreover, $\mathbf{w} = \dfrac{\langle x_2, -x_1 \rangle}{|x|}$ is a unit vector orthogonal to the vector connecting the origin and the point x.

Suppose $\bar{x}(\bar{x}_1, \bar{x}_2)$ is a point on the vector $\mathbf{w}$, then $\overrightarrow{x\bar{x}} = c\mathbf{w}$ where c is a scalar, which implies

$$\langle \bar{x}_1 - x_1, \bar{x}_2 - x_2 \rangle = \frac{1}{|x|} \langle t x_2, -t x_1 \rangle,$$

and then

$$\bar{x}_1 = x_1 + t\,\frac{x_2}{|x|} \quad \text{and} \quad \bar{x}_2 = x_2 - t\,\frac{x_1}{|x|}.$$

Taking the derivative of v with respect to τ, we have

$$\begin{aligned}
Dv(x) \cdot w &= \lim_{t\to 0} \frac{v(x) - v(\bar{x})}{t|\mathbf{w}|} \\
&= \lim_{t\to 0} \frac{1}{\pi} \int_{\partial B_1(0)} \frac{\log(|\bar{x}-y|) - \log(|x-y|)}{t|\mathbf{w}|} z_2(y)\, dS(y) \\
&= \lim_{t\to 0} \frac{1}{2\pi} \int_{\partial B_1(0)} \left[\frac{\log \dfrac{\left(x_1 + t\,\dfrac{x_2}{|x|} - y_1\right)^2 + \left(x_2 - t\,\dfrac{x_1}{|x|} - y_2\right)^2}{(x_1-y_1)^2 + (x_2-y_2)^2}}{t|\mathbf{w}|} \right] z_2(y)\, dS(y) \\
&= \frac{1}{2\pi} \int_{\partial B_1(0)} \lim_{t\to 0} \left[\frac{\log \dfrac{\left(x_1 + t\,\dfrac{x_2}{|x|} - y_1\right)^2 + \left(x_2 - t\,\dfrac{x_1}{|x|} - y_2\right)^2}{(x_1-y_1)^2 + (x_2-y_2)^2}}{t|\mathbf{w}|} \right] z_2(y)\, dS(y)
\end{aligned}$$

$$= \frac{1}{2\pi} \int_{\partial B_1(0)} \lim_{t\to 0} \left\{ \frac{\log\left[1 + \dfrac{t^2\left(\dfrac{x_1^2+x_2^2}{|x|^2}\right) + 2t\left(\dfrac{x_1y_2 - x_2y_1}{|x|}\right)}{(x_1-y_1)^2+(x_2-y_2)^2}\right]}{t} \right\} z_2(y)\, dS(y)$$

$$= \frac{1}{2\pi} \int_{\partial B_1(0)} \lim_{t\to 0} \left[\frac{\dfrac{2t\left(\dfrac{x_1^2+x_2^2}{|x|^2}\right) + 2\left(\dfrac{x_1y_2 - x_2y_1}{|x|}\right)}{(x_1-y_1)^2+(x_2-y_2)^2}}{1 + \dfrac{t^2\left(\dfrac{x_1^2+x_2^2}{|x|^2}\right) + 2t\left(\dfrac{x_1y_2 - x_2y_1}{|x|}\right)}{(x_1-y_1)^2+(x_2-y_2)^2}} \right] z_2(y)\, dS(y)$$

$$= -\frac{1}{\pi} \int_{\partial B_1(0)} \frac{x_2y_1 - x_1y_2}{|x|\cdot|x-y|^2}\, z_2(y)\, dS(y)$$

next, we parametrize the boundary $\partial B_1(0)$ by angle θ to obtain

$$Dv(x)\cdot w = -\frac{1}{\pi}\int_0^{2\pi} \frac{1}{|\rho|} \frac{\rho \sin(\theta - t)}{\rho^2 + 1 - 2\rho\cos(\theta - t)}\, z_2(t)\, dt, \tag{2.17}$$

in the Principal Value sense. Letting $\rho \to 1$, we have

$$v_\tau(\theta) = \lim_{|x|\to 1} Dv(x)\cdot w = -\frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta - t}{2}\right) z_2(t)\, dt. \tag{2.18}$$

The integral in formula (2.18) is in the principal value sense and it represents the Hilbert transform $\mathcal{H}$ on the unit circle of z_2 [19]. If the solutions of the Dirichlet and Neumann problems are identical, then the relation in (2.18) can be written as $\mathcal{H}z_2 = z_1$. Similarly, starting with a solution of Laplace's equation with the Dirichlet boundary condition, we verify the relation $\mathcal{H}z_1 = -z_2$.

2.4 Generalized Hilbert Transform and the Main Result

We start by extending the notion of *Hilbert transform* to the general domain. Suppose that Ω is an arbitrary $C^{2,\alpha}$ bounded domain in $\mathbb{R}^2$. According to the standard elliptic theory [11], given $f \in C^{1,\alpha}(\bar{\Omega})$, there exists a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ of the following Dirichlet boundary value problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \int_0^t f(s)ds & \text{on } \partial\Omega. \end{cases}$$

We define the generalized Hilbert transform $\mathcal{H}$: $C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ by setting

$$\mathcal{H}f = \frac{\partial u}{\partial \nu}. \tag{2.19}$$

The main result of this section is the following existence and uniqueness theorem for the Backus problem with expanded data on bounded planar domain.

Theorem. *Suppose $P \in C^{1,\alpha}(\partial\Omega)$, $q \in C^{\alpha}(\partial\Omega)$. Let $z = (z_1, z_2)$ be a solution of the ODE system (2.6) where A is given by (2.7). Suppose that $z_1^2(t_0) + z_2^2(t_0) = P(t_0)$ for some t_0. In addition, assume that $\mathcal{H}z_1 = -z_2$, where $\mathcal{H}$ is the generalized Hilbert transform (2.19) on $\partial\Omega$. Then there exists a unique (up to a phase) function u that is harmonic in Ω and such that*

$$|Du|^2 = P \quad \textit{and} \quad \frac{\partial}{\partial \nu}|Du|^2 = q \quad \textit{on} \ \ \partial\Omega.$$

Proof. Let u be the unique solution of Laplace's equation with Dirichlet boundary condition generated by z_1,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = -\int_0^t z_1(s)ds & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

By construction $u_\tau = z_1$ on $\partial\Omega$ and, based on the assumption $\mathcal{H}z_1 = -z_2$, we have $u_\nu = z_2$ on $\partial\Omega$. Let $\omega(t) = z_1^2(t) + z_2^2(t)$ for $t \in [0, T]$. Then, from (2.7), we get

$$\dot{\omega}(t) = 2z_1(t)\dot{z_1}(t) + 2z_2(t)\dot{z_2}(t) = 2a_1(t)z_1(t)^2 + 2a_1(t)z_2(t)^2 = 2a_1(t)\omega(t) = \omega(t) \cdot \frac{P_\tau(t)}{P(t)}.$$

This is a separable ODE for ω. Combining with the initial condition $P(t_0) = z_1^2(t_0) + z_2^2(t_0)$, we have

$$|Du|^2 = u_\tau^2(t) + u_\nu^2(t) = z_1^2(t) + z_2^2(t) = P(t) \quad \text{on } \partial B_r(0) \quad \text{for } t \in [0, 2\pi].$$

Taking the derivatives of u_τ with respect to τ and ν, we have

$$u_{\tau\tau} = (x', y') \cdot \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2,$$

$$u_{\tau\nu} = (x', y') \cdot \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \begin{pmatrix} y' \\ -x' \end{pmatrix} = u_{xx}x'y' + u_{xy}y'^2 - u_{xy}x'^2 - u_{yy}x'y'.$$

Then

$$\dot{z}_1 = \dot{u}_\tau = \frac{d}{dt}(u_x x' + u_y y') = u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2 + u_x x'' + u_y y'' = u_{\tau\tau} + u_x x'' + u_y y''.$$

Similarly,

$$\dot{z}_2 = \dot{u}_\nu = \frac{d}{dt}(u_x y' - u_y x') = u_{xx}x'y' + u_{xy}y'^2 - u_{xy}x'^2 - u_{yy}x'y' = u_{\tau\nu} - u_y x'' + u_x y''.$$

Solving for $u_{\tau\tau}$ and $u_{\tau\nu}$, we get

$$u_{\tau\tau} = \frac{P_\tau}{2P} \cdot z_1 - \frac{q}{2P} \cdot z_2,$$

$$u_{\tau\nu} = \frac{q}{2P} \cdot z_1 + \frac{P_\tau}{2P} \cdot z_2,$$

and therefore

$$\begin{aligned}
\frac{\partial}{\partial \nu}|Du|^2 &= \frac{\partial}{\partial \nu}(u_\tau^2(t) + u_\nu^2(t)) \\
&= 2u_\tau u_{\tau\nu} + 2u_\nu u_{\nu\nu} \\
&= 2u_\tau u_{\tau\nu} - 2u_\nu u_{\tau\tau} \\
&= 2z_1\Big(\frac{q}{2P}\cdot z_1 + \frac{P_\tau}{2P}\cdot z_2\Big) - 2z_2\Big(\frac{P_\tau}{2P}\cdot z_1 - \frac{q}{2P}\cdot z_2\Big) \\
&= q.
\end{aligned}$$

Hence the solution u of (2.20) is the desired solution of the Backus problem with expanded data. The uniqueness follows from well-posedness of the Dirichlet problem.

□

Chapter 3

Numerical Studies

In this chapter, we report on the results of some numerical experiments with the goal to corroborate our well-posedness result in 2D. An organizational workflow of experiments mainly consists of the following three steps:

- Generate the data from harmonic functions.
- Solve the ODE system (2.6).
- Use the solution of the ODE to solve the PDE.

We use Matlab, specifically the Boundary Value Problem solver for ordinary differential equations and the Partial Differential Equation Toolbox to complete the task.

The domain Ω being considered in this chapter is the unit disk $B_1(0)$. To solve the system of ordinary differential equations, the function `bvp4c` with periodic boundary conditions may be considered as an alternative to `dsolve`. The former one has the function call `bvp4c(odefun, bcfun, solinit)` where `odefun` is a handle for the function that evaluates the right-hand side of the differential equations that is obtained from equation (2.6) and matrix A is defined in (2.7); `bcfun` is a handle for the function that computes the residual in the boundary conditions and, in our experiment, we enforce the periodicity of solutions, that is, we require that $z_1(0) - z_1(2\pi) = 0$ and $z_2(0) - z_2(2\pi) = 0$; `solinit` is a structure containing the initial guess for a solution and we choose the constant values $z_1 = 1$ and $z_2 = 0$.

In order to represent the solutions z_1 and z_2 of the ODE system as functions of the angle t and supply them as data for the PDE problem, we first scale z_1 and z_2 to satisfy the condition

$z_1^2 + z_2^2 = P$ and then use the cubic spline interpolation with periodic conditions. Suppose $s_i(t)$ is a cubic polynomial interpolation in each of the subintervals $[t_i, t_{i+1}]$. The conditions to be satisfied by the cubic spline $s_i(t)$ are as follows:

$$s_i(t_i) = f(t_i), \quad i = 0, \dots n-1,$$

$$s_i(t_{i+1}) = f(t_{i+1}), \quad i = 0, \dots n-1,$$

$$s_i'(t_{i+1}) = s_{i+1}'(t_{i+1}), \quad i = 0, \dots n-2,$$

$$s_i''(t_{i+1}) = s_{i+1}''(t_{i+1}), \quad i = 0, \dots n-2,$$

with two more conditions $s_0'(t_0) = s_{n-1}'(t_n)$ and $s_0''(t_0) = s_{n-1}''(t_n)$ arising from the periodicity. The details of the construction are provided in Appendix B.

In the next step, the Partial Differential Equation Toolbox uses the Finite Element Method (FEM) for problems defined on bounded domain in the two-dimensional plane with the equation in divergence form $-\nabla \cdot (c\nabla u) + au = f$. In the Finite Element Method, a complicated geometry of an arbitrary smooth domain is approximated by a collection of subdomains by generating a mesh. For instance, we can approximate the computational domain Ω with the union of triangles. The main idea in the Finite Element Method is to convert the original differential (strong) form of PDE

$$-\nabla \cdot (c\nabla u) + au = f$$

into an integral (weak) form

$$\int_\Omega ((c\nabla u) \cdot \nabla v + auv - fv)dx - \int_{\partial\Omega} (-qu + g)vds = 0,$$

where $v \in C_0^1(\Omega)$ is an arbitrary test function, and replace the infinite-dimensional linear problem with a finite-dimensional version by taking the finite-dimensional subspace to be a space of piecewise polynomial functions, that is, expand u in a basis of elements $\phi_1, \phi_2, \dots, \phi_N$ as follows

$$u(x) = \sum_{j=1}^{N} U_j \phi_j(x) \tag{3.1}$$

and obtain the equations

$$\sum_{j=1}^{N} \Big(\int_{\Omega} ((c\nabla\phi_j) \cdot \nabla\phi_i + a\phi_j\phi_i) dx + \int_{\partial\Omega} q\phi_j\phi_i ds \Big) U_j = \int_{\Omega} f\phi_i dx + \int_{\partial\Omega} g\phi_i dx. \tag{3.2}$$

that hold for $i = 1, 2, \ldots, N$. Using the notations

$$K_{i,j} = \int_{\Omega} ((c\nabla\phi_j) \cdot \nabla\phi_i dx,$$

$$M_{i,j} = \int_{\Omega} a\phi_j\phi_i dx, \quad Q_{i,j} = \int_{\partial\Omega} q\phi_j\phi_i ds,$$

$$F_i = \int_{\Omega} f\phi_i dx, \quad G_i = \int_{\partial\Omega} g\phi_i dx,$$

we rewrite the system in the matrix form

$$(K + M + Q)U = F + G. \tag{3.3}$$

We can apply this algorithm to solve the Dirichlet and Neumann boundary value problems. However, the finite element matrix for the Neumann problem is close to singular. We therefore add the condition $\int_{\Omega} u_n \; dx = \int_{\Omega} u_d \; dx$ to the finite element matrix to resolve the non-uniqueness, where u_n and u_d are solutions of Laplace's equation with Neumann and Dirichlet boundary conditions z_2 and $\int_0^t z_1(\theta) d\theta$.

Now we describe the results of numerical experiments. We consider three harmonic functions

- $u_0 = y^2 - x^2$,
- $u_1 = 0.1(x^2 + y^2)^5 \cos(10 \tan^{-1}(\frac{y}{x}))$,
- $u_2 = u_0 + u_1$.

The function u_0 is a harmonic polynomial of order 2 with unit magnitude; u_1 is the real part of a complex analytic function z^10 with the magnitude one order less than that of u_0 and relatively high frequency oscillations; u_2 models the combination of the background field of u_0 and the perturbation of u_1.

The boundary values $P = |Du|^2$ and $q = \frac{\partial}{\partial\nu}|Du|^2$ of such three functions are computed explicitly as follows:

u	$u_0 = y^2 - x^2$	$u_1 = 0.1(x^2+y^2)^5 \cos(10\tan^{-1}(\frac{y}{x}))$
P	4	1
q	0	18

u	$u_2 = u_0 + u_1$
P	$5 - 4\{(x^2-y^2)\cos(10\tan^{-1}(\frac{y}{x})) + 2xy\sin(10\tan^{-1}(\frac{y}{x}))\}$
q	$26 - 40\{(x^2-y^2)\cos(10\tan^{-1}(\frac{y}{x})) + 2xy\sin(10\tan^{-1}(\frac{y}{x}))\}$

In numerical experiments, P and q on the boundary are computed numerically.

In order to estimate the error of the approximation, we compare $u - u_e$ in the $L^2(\Omega)$-norm and $H^1(\Omega)$-norm where u_e denotes the exact values of the harmonic solution on the boundary mesh points and $u = \cos\alpha \cdot u_d + \sin\alpha \cdot v_d$ is an estimation, where u_d and v_d are solutions of Laplace's equation with Dirichlet boundary conditions $\int_0^t z_1(\theta)d\theta$ and $\int_0^t z_2(\theta)d\theta$, respectively and α is the optimal phase which is described later.

The $L^2(\Omega)$-norm of a function u in the form (3.1) can be computed as follows,

$$||u||_{L^2}^2 = \int_\Omega |u(x)|^2 dx = \int_\Omega [\sum_i U_i\phi_i(x)]^2 dx = \sum_{i,j}\langle\phi_i(x), \phi_j(x)\rangle U_iU_j = U^T MU,$$

where $M = \langle\phi_i(x), \phi_j(x)\rangle = \int_\Omega \phi_i(x)\phi_j(x)\,dx$ is the mass matrix.

Similarly, the $H^1(\Omega)$-norm of such function u can be represented as follows,

$$\begin{aligned}
||u||_{H^1}^2 &= ||\nabla u||_{L^2}^2 + ||u||_{L^2}^2 = \int_\Omega (|\nabla u|^2 + |u(x)|^2)dx \\
&= \int_\Omega [\sum_i U_i \nabla\phi_i(x)]^2 + [\sum_i U_i \phi_i(x)]^2 dx \\
&= \sum_{i,j} \langle \nabla\phi_i(x), \nabla\phi_j(x)\rangle U_i U_j + \sum_{i,j} \langle \phi_i(x), \phi_j(x)\rangle U_i U_j \\
&= U^T(K+M)U,
\end{aligned}$$

where $K = \langle \nabla\phi_i(x), \nabla\phi_j(x)\rangle = \int_\Omega \nabla\phi_i(x)\nabla\phi_j(x)dx$ is the stiffness matrix.

In particular, the error estimates take the form

$$||u-u_e||_{L^2} \approx [(u-u_e)^T M(u-u_e)]^{\frac{1}{2}},$$

and

$$||u-u_e||_{H^1} \approx [(u-u_e)^T (K+M)(u-u_e)]^{\frac{1}{2}}.$$

We note that the solution of (2.1) is not unique due to an additive constant and an arbitrary constant phase. We apply vertical and phase shifts to u before evaluating the error.

In order to determine the additive constant, we ensure that $\int_{B_1(0)} u_d dx = \int_{B_1(0)} u_e dx$ and $\int_{B_1(0)} v_d dx = \int_{B_1(0)} u_e dx$ by shifting u_d and v_d vertically. Suppose that $\int_{B_1(0)} u_e dx = \int_{B_1(0)} (u_d + c)dx$, then the additive constant c is calculated as follows

$$c = \frac{\int_{B_1(0)}(u_e - u_d)dx}{\int_{B_1(0)} 1\, dx} = \frac{\langle 1,\dots,1\rangle^T M(u_e-u_d)}{\langle 1,\dots,1\rangle^T M\langle 1,\dots,1\rangle}.$$

Similarly, if $\int_{B_1(0)} u_e dx = \int_{B_1(0)} (v_d + d)dx$, we have

$$d = \frac{\int_{B_1(0)}(u_e - v_d)dx}{\int_{B_1(0)} 1\, dx} = \frac{\langle 1,\dots,1\rangle^T M(u_e-v_d)}{\langle 1,\dots,1\rangle^T M\langle 1,\dots,1\rangle}.$$

We update u_d and v_d with $u_d + c$ and $v_d + d$, respectively.

To match the phases for u and u_e, we optimize the phase of u by minimizing the $L^2(\Omega)$-norm of the difference $(\cos\alpha \cdot u_d + \sin\alpha \cdot v_d) - u_e$ with respect to α in the interval $[-\pi, \pi]$. For calculated local minimizer α, we update $u = \cos\alpha \cdot u_d + \sin\alpha \cdot v_d$. In Table 3.1, we list the $L^2(\Omega)$-norms and $H^1(\Omega)$-norms of the three test functions u_e in the approximation by u.

Table 3.1: Approximation errors in $L^2(\Omega)$ and $H^1(\Omega)$

	u_0		u_1		u_2	
H_{max}	$L^2(\Omega)$	$H^1(\Omega)$	$L^2(\Omega)$	$H^1(\Omega)$	$L^2(\Omega)$	$H^1(\Omega)$
0.1	$4.35 \cdot 10^{-04}$	$1.05 \cdot 10^{-02}$	$1.05 \cdot 10^{-02}$	$4.92 \cdot 10^{-02}$	$6.60 \cdot 10^{-01}$	$2.42 \cdot 10^{-00}$
0.05	$1.18 \cdot 10^{-04}$	$6.02 \cdot 10^{-03}$	$1.16 \cdot 10^{-03}$	$9.37 \cdot 10^{-03}$	$1.66 \cdot 10^{-02}$	$6.75 \cdot 10^{-02}$
0.025	$2.77 \cdot 10^{-05}$	$2.86 \cdot 10^{-03}$	$4.55 \cdot 10^{-05}$	$4.18 \cdot 10^{-03}$	$3.38 \cdot 10^{-03}$	$1.61 \cdot 10^{-02}$
0.0125	$7.70 \cdot 10^{-06}$	$1.47 \cdot 10^{-03}$	$1.29 \cdot 10^{-05}$	$2.26 \cdot 10^{-03}$	$1.48 \cdot 10^{-05}$	$2.69 \cdot 10^{-03}$

With the decreasing mesh size, $L^2(\Omega)$-norms and $H^1(\Omega)$-norms become smaller and L^2-norms deceases much faster than H^1-norms. Notice that the error of function u_2 is significant when the mesh size is relatively large.

Next, we estimate the rate of convergence γ for L^2-norm and H^1-norm of $u - u_e$. Assume that the norm satisfies the relation

$$||u - u_e||_{L^2 \text{ or } H^1} \approx C \cdot (H_{max})^\gamma. \tag{3.4}$$

for some constants C and γ. To estimate γ, we take the natural logarithm on both sides in (3.4) to get

$$\log ||u - u_e||_{L^2 \text{ or } H^1} \approx \log C + \gamma \cdot \log H_{max}$$

Setting $y = \log ||u - u_e||$, $x = \log H_{max}$, $\beta_0 = \log C$ and $\beta_1 = \gamma$, we recover γ using linear regression for

$$y = \beta_0 + \beta_1 \cdot x + \epsilon,$$

where ϵ is the error term.

Starting with a set of 4 observed values of x and y given by (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4). Using the simple linear regression relation, these values form a system of linear equations and represent these equations in matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}$, $B = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$. The relation becomes $Y = XB$.

In Matlab, we compute B using the `mldivide` operator as $B = X\backslash Y$. The estimate of parameter γ is the second element in vector B. The estimated rates of convergence γ for the L^2-norm and H^1-normof $u - u_e$ are summarized in Table 3.2.

Table 3.2: Rates of convergence in $L^2(\partial\Omega)$ and $H^1(\partial\Omega)$

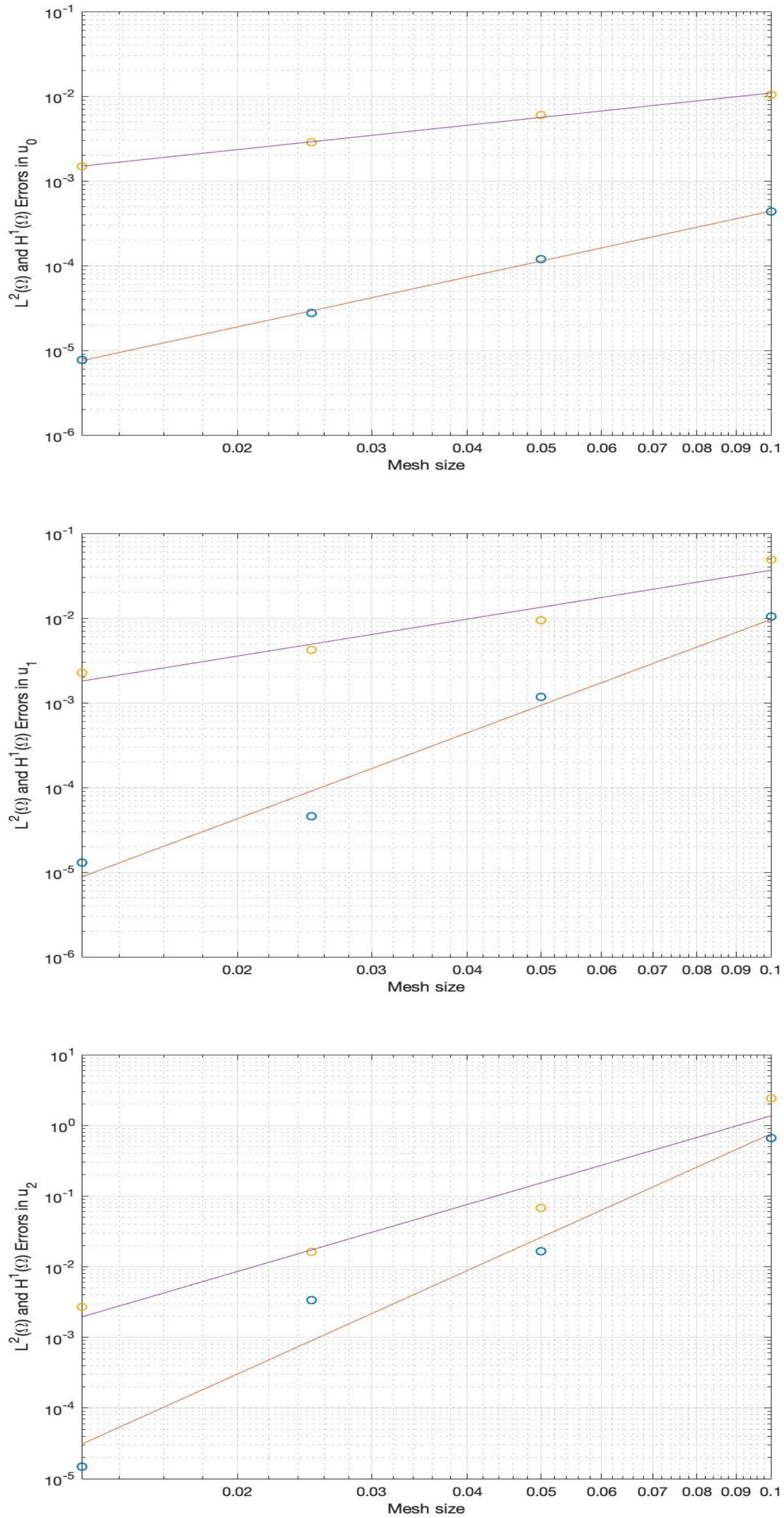
γ	u_0	u_1	u_2
$L^2(\partial\Omega)$	1.9549	3.3662	4.8629
$H^1(\partial\Omega)$	0.9558	1.4489	3.1510

The graphs in Figure 3.1 are log-log plots where mesh size is on the horizontal axis, the error is on vertical axis, and the slopes correspond to the rate of convergence. The slopes are greater for L^2-norm than H^1-norm.

Table 3.3: Approximation errors in $L^2(\partial\Omega)$

H_{max}	u_0	u_1	u_2
0.1	$4.0021 \cdot 10^{-04}$	$2.0656 \cdot 10^{-01}$	$8.2328 \cdot 10^{-00}$
0.05	$6.5553 \cdot 10^{-06}$	$1.9216 \cdot 10^{-02}$	$1.0405 \cdot 10^{-01}$
0.025	$4.8555 \cdot 10^{-07}$	$5.4880 \cdot 10^{-04}$	$4.3205 \cdot 10^{-02}$
0.0125	$3.6481 \cdot 10^{-07}$	$7.9137 \cdot 10^{-05}$	$7.9447 \cdot 10^{-06}$

Figure 3.1: Linear regression on errors and mesh size



We also measure the error in approximating the boundary data P with the solution of the ODE system u_τ and u_ν, namely, we compute $||u_\tau^2 + u_\nu^2 - P||_{L^2(\partial\Omega)}$ and estimate the rate of convergence for this norm.

Table 3.4: Rate of convergence in $L^2(\partial\Omega)$

γ	u_0	u_1	u_2
$\|\|u_\tau^2 + u_\nu^2 - p\|\|_{L^2(\partial\Omega)}$	3.4053	3.9180	6.1217

We note that, as expected, the solution of the ODE system is close to the boundary data P, especially with relatively small mesh size.

Chapter 4

Quasilinear Elliptic PDE for Backus Problem in Higher Dimension

4.1 Dimension Reduction

In this section, we derive an equation involving Du, the tangential gradient of u, in n-dimensional hyperplanes. Suppose

$$u \in C^2(\bar{\mathbb{R}}^{n+1}_+) \text{ satisfies } \Delta u = 0 \text{ in } \mathbb{R}^{n+1}_+.$$

Denote by Δ_k the Laplace operator in $\mathbb{R}^k$, namely, let

$$\Delta_k u = \sum_{j=1}^{k} D_{jj}u, \quad k = n, n+1.$$

We also denote

$$\Delta u = \Delta_n u,$$

so that the Laplace equation in $\mathbb{R}^{n+1}$ becomes

$$\Delta_{n+1}u = \Delta u + D_{n+1,n+1}u = 0.$$

Write $|Du|^2 = \sum_{j=1}^{n}(D_j u)^2$ and let

$$P := |Du|^2 + (D_{n+1}u)^2. \tag{4.1}$$

Taking partial derivatives with respect to x_i in equation (4.1), we obtain

$$2D_j u D_{ij} u + 2D_{n+1} u D_{n+1,i} u = D_i P, \quad i = 1, \ldots, n,$$
$$2D_i u D_{i,n+1} u + 2D_{n+1} u D_{n+1,n+1} u = D_{n+1} P =: q,$$

where, here and throughout, the summation from 1 to n is assumed for repeated indices.

Multiplying the last equation by $D_{n+1}u$ and replacing in it the terms involving the partial derivative with respect to x_{n+1}, we arrive at

$$D_i u(D_i P - 2D_j u D_{ij} u) + 2(P - |Du|^2)(-\Delta u) = qD_{n+1}u.$$

Rearranging the terms further, we obtain

$$(P - |Du|^2)\Delta u + D_i u D_j u D_{ij} u - \frac{1}{2} D_i P D_i u + \frac{q}{2}\sigma\sqrt{P - |Du|^2} = 0 \qquad (4.2)$$

where $\sigma = \operatorname{sign} D_{n+1}u$. Note that this equation involves explicitly only the partial derivatives with respect to $x_1, \ldots, x_n$.

We can transform (4.2) to the divergence form of this equation in $\mathbb{R}^n$:

$$Qu := \operatorname{div} \frac{Du}{\sqrt{P - |Du|^2}} + \frac{1}{2}\frac{\sigma q}{P - |Du|^2} = 0. \qquad (4.3)$$

Note that the left hand side of this equation is defined only when $|Du|^2 < P$.

4.2 Legendre Transform

We investigate the equation (4.2) in the case when $n = 2$ with a technique called Legendre transform. This technique is used to convert quasilinear systems of PDE into linear systems, by reversing the roles of the dependent and independent variables.

Assume that in region $\Omega \subset \mathbb{R}^2$, we can invert the relations

$$p_1 = u_{x_1}(x_1, x_2), \quad p_2 = u_{x_2}(x_1, x_2),$$

to solve for

$$x_1 = x_1(p_1, p_2), \quad x_2 = x_2(p_1, p_2).$$

Define

$$v(p) = \mathbf{x} \cdot p - u(\mathbf{x}(p)),$$

where $\mathbf{x} = (x_1, x_2)$ and $p = (p_1, p_2)$. Upon substituting into the equation (4.2), we arrive at

$$\begin{aligned}\left(P - p_1^2 - p_2^2\right) J v_{p_2 p_2} - 2p_1 p_2 J v_{p_1 p_2} + \left(P - p_1^2 - p_2^2\right) J v_{p_1 p_1} \\ - \frac{1}{2} P_{v_{p_1}} p_1 - \frac{1}{2} P_{v_{p_2}} p_2 + \frac{q}{2} \sigma \sqrt{P - p_1^2 - p_2^2} = 0. \quad (4.4)\end{aligned}$$

The detailed derivation of equation (4.4) appears in Appendix B.

Equation (4.4) is non-linear because of the presence of $J = u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2$. In addition, the boundary conditions are not easily transformed by this technique. For these reasons, the Legendre transform approach is not pursued further in this dissertation but we state equation (4.4) for the record.

4.3 Definitions

In this section, we recall the ellipticity of second order, quasilinear equations of the form $Qu = 0$ where

$$Qu = a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du), \quad (4.5)$$

and $x = (x_1, \ldots, x_n)$ is contained in a domain Ω of $\mathbb{R}^n$, $n \geq 2$. The coefficients of Q, namely the functions $a^{ij}(x, z, p)$ and $b(x, z, p)$ are assumed to be defined in $\mathscr{U} \subseteq \Omega \times \mathbb{R} \times \mathbb{R}^n$. We recall the following definitions:

Definitions [11, p.259]. *Let $\mathscr{U}$ be a subset of $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Denote by Λ the maximum eigenvalue of $[a^{ij}(x, z, p)]$ and by λ the minimum eigenvalue of $[a^{ij}(x, z, p)]$.*

- *The operator Q is elliptic in $\mathscr{U}$ if the coefficient matrix $[a^{ij}(x, z, p)]$ is positive definite for all $(x, z, p) \in \mathscr{U}$.*
- *The operator Q is uniformly elliptic in $\mathscr{U}$ if Λ / λ is bounded in $\mathscr{U}$.*

- *If Q is elliptic (uniformly elliptic) in the whole set $\Omega \times \mathbb{R} \times \mathbb{R}^n$, then we say that Q is elliptic (uniformly elliptic) in Ω.*

4.4 Leray-Schauder Estimates

In this section, we recall how the solvability of the classical Dirichlet problem for quasilinear equations is reduced to the establishment of certain apriori estimates for solutions. This reduction is achieved through the application of topological fixed point theorems in appropriate function spaces.

The Brouwer fixed point theorem can be extended to infinite dimensional spaces with primarily applications in Banach spaces as follows:

Theorem 4.1 (Schauder Fixed Point Theorem) [11, p.279]. *Let $\mathcal{G}$ be a closed convex set in a Banach space $\mathcal{B}$ and let T be a continuous mapping of $\mathcal{G}$ into itself such that the image $T\mathcal{G}$ is precompact. Then T has a fixed point.*

For later purposes we note the following extension of Theorem 4.1.

Theorem 4.2 (Schaefer's Fixed Point Theorem) [11, p.280]. *Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself, and suppose there exists a constant M such that*

$$||x||_{\mathcal{B}} < M$$

for all $x \in \mathcal{B}$ and $\sigma \in [0,1]$ satisfying $x = \sigma Tx$. Then T has a fixed point.

In order to apply Theorem 4.2 to the Dirichlet problem for quasilinear equations, we fix a number $\beta \in (0,1)$ and take the Banach space $\mathcal{B}$ to be the Hölder space $C^{1,\beta}(\bar{\Omega})$. Let Q be the operator given by (4.5) and Q is elliptic in $\bar{\Omega}$. We also assume, for some $\alpha \in (0,1)$, that the coefficients a^{ij}, $b \in C^{\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, that the boundary $\partial\Omega$ is of class $C^{2,\alpha}$, and $\phi \in C^{2,\alpha}(\bar{\Omega})$. For all $v \in C^{1,\beta}(\bar{\Omega})$, the operator $T : v \to u$ is defined by letting $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\bar{\Omega})$ of the linear Dirichlet problem,

$$a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) = 0 \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega. \tag{4.6}$$

The solvability of the Dirichlet problem, $Qu = 0$ in Ω, $u = \phi$ on $\partial\Omega$, in the space $C^{2,\alpha}(\bar{\Omega})$ is thus equivalent to the solvability of the equation $u = Tu$ in the Banach space $\mathcal{B} = C^{1,\beta}(\bar{\Omega})$. The equation $u = \sigma T u$ in $\mathcal{B}$ is equivalent to the Dirichlet problem

$$Q_\sigma u = a^{ij}(x, u, Du) D_{ij} u + \sigma b(x, u, Du) = 0 \text{ in } \Omega, \quad u = \sigma\phi \text{ on } \partial\Omega. \tag{4.7}$$

By applying Theorem 4.2, we arrive at the following criterion for existence.

Theorem 4.3 [11, p.281]. *Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that Q is elliptic in $\bar{\Omega}$ with coefficients a^{ij}, $b \in C^\alpha(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $0 < \alpha < 1$. Let $\partial\Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}$. Then, if for some $\beta > 0$ there exists a constant M, independent of u and σ, such that every $C^{2,\alpha}(\bar{\Omega})$ solution of the Dirichlet problems, $Q_\sigma u = 0$ in Ω, $u = \sigma\phi$ on $\partial\Omega$, $0 \le \sigma \le 1$, satisfies*

$$||u||_{C^{1,\beta}(\bar{\Omega})} < M,$$

it follows that the Dirichlet problems, $Qu = 0$ in Ω, $u = \phi$ on $\partial\Omega$, is solvable in $C^{2,\alpha}(\bar{\Omega})$.

Theorem 4.3 reduces the solvability of the Dirichlet problem $Qu = 0$ in Ω, $u = \phi$ on $\partial\Omega$ to the apriori estimates in the space $C^{1,\beta}(\bar{\Omega})$ of the solutions of a related family of problems. In practice it is desirable to break the derivation of the apriori estimates into four stages:

Apriori Estimates Program

1. Estimation of $\sup\limits_{\Omega} |u|$;
2. Estimation of $\sup\limits_{\partial\Omega} |Du|$ in terms of $\sup\limits_{\Omega} |u|$;
3. Estimation of $\sup\limits_{\Omega} |Du|$ in terms of $\sup\limits_{\partial\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$;
4. Estimation of $[Du]_{\beta;\Omega}$ in terms of $\sup\limits_{\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$.

We will consider these stages starting from Section 4.6 and establish the apriori estimates for a family of operators that approximate (4.3) and constitute its regularization.

4.5 Regularized Quasilinear Operator

The standard approach summarized in Theorem 4.2 is not applicable to the quasilinear operator defined in (4.3). For this operator, the admissible set is constrained by the domain of definition of the coefficients. Namely, the function u, for which the operator is defined, must have values $(x, u(x), Du(x))$ in the set

$$\mathscr{U} = \{(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : \, P(x) \geq |p|^2\}.$$

The set $\mathscr{U}$ is not a linear space. Moreover, the operator $T : v \to u$ where u is defined as the solution of (4.6) is expected to map $\mathscr{U}$ into itself. We see that this is not the case, at least for arbitrary chosen data in the following example.

Example. *Let $n = 1$, $P = 1$, $\sigma = 1$ and $q = 16\cos x$. Then (4.6) becomes*

$$u_{xx}(1 - v_x^2) + v_x^2 u_{xx} + 8\cos x\sqrt{1 - v_x^2} = u_{xx} + 8\cos x\sqrt{1 - v_x^2} = 0. \tag{4.8}$$

Notice that for $v = \sin x \in \mathscr{U}$, the solution of (4.8) is given by $u = \sin 2x$. However, we note that $u \notin \mathscr{U}$.

In view of this example and the degeneracy of equation (4.3), we recognize the need to regularize the problem and construct approximate solutions. The idea is that in an appropriate regime, the approximate solutions will converge to the solution of equation (4.3).

Thus we seek to preform the stages outlined in the Apriori Estimates Program for the regularized problem and amend the Program with an additional step of showing the convergence of the approximate solutions. We propose to study the following regularized form of (4.3). The purpose of this regularization is to expand the set of admissible functions u in equation (4.3) and include functions with arbitrarily large values of $|Du|$. Let $\varepsilon > 0$ and consider the following equation in divergence form:

$$Q_\varepsilon u = \operatorname{div} \frac{Du}{\sqrt{\xi_\varepsilon(x, |Du|)}} + \frac{1}{2}\frac{\sigma q}{\zeta_\varepsilon(x, |Du|)} = 0, \tag{4.9}$$

where

$$\xi_\varepsilon(x,p) = \begin{cases} P - |p|^2 + \varepsilon|p| & |p|^2 \le P, \\ \dfrac{\varepsilon^2|p|^2}{|p|^2 - P + \varepsilon|p|} & |p|^2 > P, \end{cases} \tag{4.10}$$

and

$$\zeta_\varepsilon(x,p) = \begin{cases} P - |p|^2 + \varepsilon|p| & |p|^2 \le P, \\ \varepsilon|p| & |p|^2 > P. \end{cases} \tag{4.11}$$

Notice that $\xi_\varepsilon(x,p)$ is continuously differentiable as a function of x and p and $\zeta_\varepsilon(x,p)$ is continuous as a function of x and p. Indeed,

$$\frac{\partial \xi_\varepsilon(x,p)}{\partial x} = \begin{cases} D_x P & |p|^2 \le P, \\ \dfrac{\varepsilon^2|p|^2 D_x P}{(|p|^2 - P + \varepsilon|p|)^2} & |p|^2 > P. \end{cases}$$

$$\frac{\partial \xi_\varepsilon(x,p)}{\partial p} = \begin{cases} -2|p| + \varepsilon & |p|^2 \le P, \\ \dfrac{\varepsilon^3|p|^2 - 2\varepsilon^3|p|P}{(|p|^2 - P + \varepsilon|p|)^2} & |p|^2 > P. \end{cases}$$

Therefore

$$\lim_{|p|^2 \to P^-} \frac{\partial \xi_\varepsilon}{\partial x} = D_x P = \lim_{|p|^2 \to P^+} \frac{\partial \xi_\varepsilon}{\partial x},$$

$$\lim_{|p|^2 \to P^-} \frac{\partial \xi_\varepsilon}{\partial p} = \varepsilon - 2\sqrt{P} = \lim_{|p|^2 \to P^+} \frac{\partial \xi_\varepsilon}{\partial p},$$

and

$$\lim_{|p|^2 \to P^-} \zeta_\varepsilon = \varepsilon\sqrt{P} = \lim_{|p|^2 \to P^+} \zeta_\varepsilon.$$

We note that taking the limit as $\varepsilon \to 0$ in the coefficients ξ_ε and ζ_ε, we recover the unregularized operator Q in (4.3) for $|p|^2 \le P$.

The corresponding non-divergence form of (4.3) is as follows:

If $|Du|^2 \leq P$,

$$\Big(P - |Du|^2 + \varepsilon|Du|\Big)\Delta u + \Big(1 - \frac{\varepsilon}{2|Du|}\Big)D_i u D_j u D_{ij} u \\ - \frac{1}{2}D_i P D_i u + \frac{q}{2}\sigma\sqrt{P - |Du|^2 + \varepsilon|Du|} = 0 \quad (4.12)$$

If $|Du|^2 > P$,

$$\Big(|Du|^2 - P + \varepsilon|Du|\Big) \cdot \varepsilon|Du| \cdot \Delta u + \Big(\frac{\varepsilon P}{|Du|} - \frac{\varepsilon^2}{2}\Big)D_i u D_j u D_{ij} u \\ - \frac{1}{2}D_i P \cdot \varepsilon|Du| \cdot D_i u + \frac{q}{2}\sigma \cdot \varepsilon|Du| \cdot \sqrt{|Du|^2 - P + \varepsilon|Du|} = 0 \quad (4.13)$$

Lemma 4.4. *The operator Q_ε defined as* (4.12) *and* (4.13) *is uniformly elliptic in Ω with*

$$\lambda = \begin{cases} \min\{P - |p|^2 + \varepsilon|p|, P + \frac{1}{2}\varepsilon|p|\} & |p|^2 \leq P, \\ \min\{(|p|^2 - P + \varepsilon|p|) \cdot \varepsilon|p|, \varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2\} & |p|^2 > P, \end{cases}$$

and

$$\Lambda = \begin{cases} \max\{P - |p|^2 + \varepsilon|p|, P + \frac{1}{2}\varepsilon|p|\} & |p|^2 \leq P, \\ \max\{(|p|^2 - P + \varepsilon|p|) \cdot \varepsilon|p|, \varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2\} & |p|^2 > P, \end{cases}$$

Proof. We first identify the coefficients of Q_ε as follows:

We compute for $|p|^2 \leq P$,

$$a^{ij}_\varepsilon(x,p) = \delta_{ij} \cdot \Big(P - |p|^2 + \varepsilon|p|\Big) + \Big(1 - \frac{\varepsilon}{2|p|}\Big)p_i \cdot p_j,$$

and

$$b_\varepsilon(x,p) = -\frac{1}{2}D_i P \cdot p_i + \frac{q}{2}\sigma\sqrt{P - |p|^2 + \varepsilon|p|}.$$

We compute for $|p|^2 > P$,

$$a_\varepsilon^{ij}(x,p) = \delta_{ij} \cdot \Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| + \Big(\frac{\varepsilon P}{|p|} - \frac{\varepsilon^2}{2}\Big) p_i \cdot p_j,$$

and

$$b_\varepsilon(x,p) = -\frac{1}{2} D_i P \cdot \varepsilon|p| \cdot p_i + \frac{q}{2}\sigma \cdot \varepsilon|p| \cdot \sqrt{|p|^2 - P + \varepsilon|p|}.$$

We claim that the eigenvalues of matrix $A = [a_\varepsilon^{ij}(x,p)]$, when $|p|^2 \leq P$, are given by

$$\lambda_1 = \cdots = \lambda_{n-1} = P - |p|^2 + \varepsilon|p| \text{ and } \lambda_n = P + \frac{1}{2}\varepsilon|p|$$

and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} -\dfrac{p_2}{p_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} -\dfrac{p_3}{p_1} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; v_{n-1} = \begin{bmatrix} -\dfrac{p_n}{p_1} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \; v_n = \begin{bmatrix} \dfrac{p_1}{p_n} \\ \dfrac{p_2}{p_n} \\ \vdots \\ \dfrac{p_{n-1}}{p_n} \\ 1 \end{bmatrix}.$$

Moreover,

$$\frac{\Lambda}{\lambda} = \frac{P + \frac{1}{2}\varepsilon|p|}{P - |p|^2 + \varepsilon|p|} \quad \text{or} \quad \frac{\Lambda}{\lambda} = \frac{P - |p|^2 + \varepsilon|p|}{P + \frac{1}{2}\varepsilon|p|},$$

and $\dfrac{\Lambda}{\lambda}$ is bounded in either case.

On the other hand, when $|p|^2 > P$, we claim the eigenvalues of matrix A are

$$\lambda_1 = \cdots = \lambda_{n-1} = \Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| \text{ and } \lambda_n = \varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2$$

and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} -\dfrac{p_2}{p_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} -\dfrac{p_3}{p_1} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ v_{n-1} = \begin{bmatrix} -\dfrac{p_n}{p_1} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \ v_n = \begin{bmatrix} \dfrac{p_1}{p_n} \\ \dfrac{p_2}{p_n} \\ \vdots \\ \dfrac{p_{n-1}}{p_n} \\ 1 \end{bmatrix}$$

In addition,

$$\frac{\Lambda}{\lambda} = \frac{\Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p|}{\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2} \quad \text{or} \quad \frac{\Lambda}{\lambda} = \frac{\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2}{\Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p|},$$

and $\dfrac{\Lambda}{\lambda}$ is bounded above by 1 in both cases.

To verify that these are indeed eigenvalues and eigenfunctions of matrix A with $p = (p_1, \ldots, p_n)$, we write

$$E = \begin{bmatrix} v_1 & v_2 & \ldots & v_{n-1} & v_n \end{bmatrix} = \begin{bmatrix} -\dfrac{p_2}{p_1} & -\dfrac{p_3}{p_1} & \ldots & -\dfrac{p_n}{p_1} & \dfrac{p_1}{p_n} \\ 1 & 0 & \ldots & 0 & \dfrac{p_2}{p_n} \\ 0 & 1 & \ldots & 0 & \dfrac{p_3}{p_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 1 \end{bmatrix}.$$

The diagonal of matrix D_i, $i = 1, 2$ consist of the eigenvalues of A. For $|p|^2 \leq P$,

$$D_1 = \begin{bmatrix} P - |p|^2 + \varepsilon|p| & 0 & \dots & 0 & 0 \\ 0 & P - |p|^2 + \varepsilon|p| & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P - |p|^2 + \varepsilon|p| & 0 \\ 0 & 0 & \dots & 0 & P + \dfrac{1}{2}\varepsilon|p| \end{bmatrix}.$$

For $|p|^2 > P$,

$$D_2 = \begin{bmatrix} \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & 0 & \dots & 0 & 0 \\ 0 & \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & 0 \\ 0 & 0 & \dots & 0 & \varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2 \end{bmatrix}.$$

Next, we compute

$$E{\cdot}D_1 = \begin{bmatrix} -\left(P - |p|^2 + \varepsilon|p|\right)\cdot\dfrac{p_2}{p_1} & -\left(P - |p|^2 + \varepsilon|p|\right)\cdot\dfrac{p_3}{p_1} & \dots & -\left(P - |p|^2 + \varepsilon|p|\right)\cdot\dfrac{p_n}{p_1} & \left(P + \dfrac{1}{2}\varepsilon|p|\right)\cdot\dfrac{p_1}{p_n} \\ P - |p|^2 + \varepsilon|p| & 0 & \dots & 0 & \left(P + \dfrac{1}{2}\varepsilon|p|\right)\cdot\dfrac{p_2}{p_n} \\ 0 & P - |p|^2 + \varepsilon|p| & \dots & 0 & \left(P + \dfrac{1}{2}\varepsilon|p|\right)\cdot\dfrac{p_3}{p_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P - |p|^2 + \varepsilon|p| & P + \dfrac{1}{2}\varepsilon|p| \end{bmatrix}$$

and

$$E{\cdot}D_2 = \begin{bmatrix} -\left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p|\cdot\dfrac{p_2}{p_1} & -\left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p|\cdot\dfrac{p_3}{p_1} & \dots & -\left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p|\cdot\dfrac{p_n}{p_1} & \left(\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2\right)\cdot\dfrac{p_1}{p_n} \\ \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & 0 & \dots & 0 & \left(\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2\right)\cdot\dfrac{p_2}{p_n} \\ 0 & \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & \dots & 0 & \left(\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2\right)\cdot\dfrac{p_3}{p_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \left(|p|^2 - P + \varepsilon|p|\right)\cdot\varepsilon|p| & \varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2 \end{bmatrix}$$

Moreover, we have

$$A_1 \cdot E = \begin{bmatrix} -\Big(P - |p|^2 + \varepsilon|p|\Big) \cdot \frac{p_2}{p_1} & -\Big(P - |p|^2 + \varepsilon|p|\Big) \cdot \frac{p_3}{p_1} & \dots & -\Big(P - |p|^2 + \varepsilon|p|\Big) \cdot \frac{p_n}{p_1} & \Big(P + \frac{1}{2}\varepsilon|p|\Big) \cdot \frac{p_1}{p_n} \\ P - |p|^2 + \varepsilon|p| & 0 & \dots & 0 & \Big(P + \frac{1}{2}\varepsilon|p|\Big) \cdot \frac{p_2}{p_n} \\ 0 & P - |p|^2 + \varepsilon|p| & \dots & 0 & \Big(P + \frac{1}{2}\varepsilon|p|\Big) \cdot \frac{p_3}{p_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P - |p|^2 + \varepsilon|p| & P + \frac{1}{2}\varepsilon|p| \end{bmatrix}$$

and

$$A_2 \cdot E = \begin{bmatrix} -\Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| \cdot \frac{p_2}{p_1} & -\Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| \cdot \frac{p_3}{p_1} & \dots & -\Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| \cdot \frac{p_n}{p_1} & \Big(\varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2\Big) \cdot \frac{p_1}{p_n} \\ \Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| & 0 & \dots & 0 & \Big(\varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2\Big) \cdot \frac{p_2}{p_n} \\ 0 & \Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| & \dots & 0 & \Big(\varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2\Big) \cdot \frac{p_3}{p_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Big(|p|^2 - P + \varepsilon|p|\Big) \cdot \varepsilon|p| & \varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2 \end{bmatrix}$$

so that

$$A_i \cdot E = E \cdot D_i, \quad i = 1, 2.$$

This identity confirms that the columns of E are the eigenvectors of A and with corresponding eigenvalues of the diagonal of D. Since all those eigenvalues are positive and $\frac{\Lambda}{\lambda}$ is bounded, the operator Q_ε is uniformly elliptic in Ω.

$\square$

Corollary 4.5. *The operator* Q *defined in* (4.3) *is elliptic in*

$$\mathscr{U} = \{(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : \, P(x) \geq |p|^2\},$$

but not uniformly ellptic.

Proof. By taking the limit as $\varepsilon \to 0$ in the coefficients ξ_ε and ζ_ε of the operator Q_ε, we conclude that the eigenvalues of A are

$$\lambda := \lambda_1 = \cdots = \lambda_{n-1} = P - |p|^2, \quad \text{and} \quad \Lambda := \lambda_n = P.$$

Since $\lambda > 0$, Q is elliptic in $\mathscr{U}$, but $\dfrac{\Lambda}{\lambda} = \dfrac{P}{P - |p|^2}$ is not guaranteed to be bounded. $\square$

4.6 Comparison and Maximum Principles for Divergence Form

In this sections, we pursue the estimates of solutions of $Q_\varepsilon u = 0$ using the divergence structure of this operator per Stage 1 in the Apriori Estimates Program.

Recall that the operator Q is of divergence form if there exists a differentiable vector function $\mathbf{A}(x, z, p) = (A^1(x, z, p), \cdots, A^n(x, z, p))$ and a scalar function $B(x, z, p)$ such that

$$Qu = \text{div } \mathbf{A}(x, u, Du) + B(x, u, Du), \quad \text{for } u \in C^2(\Omega). \tag{4.14}$$

Furthermore, a function u, weakly differentiable in Ω, satisfies $Qu \geq 0$ $(= 0, \leq 0)$ in Ω if the functions $A^i(x, u, Du)$, $B(x, u, Du)$ are locally integrable in Ω and

$$Q(u, \varphi) = \int_\Omega [\mathbf{A}(x, u, Du) \cdot D\varphi - B(x, u, Du) \cdot \varphi] \, dx \leq 0 \; (= 0, \; \geq 0) \tag{4.15}$$

for all non-negative $\varphi \in C_0^1(\Omega)$.

The divergence form of the quasilinear operator Q_ε in (4.9) is written out explicitly as follows:

$$Q_\varepsilon u = \text{div } \mathbf{A}_\varepsilon(x, Du) + B_\varepsilon(x, Du) = 0$$

where

$$\mathbf{A}_\varepsilon(x, p) = \begin{cases} \dfrac{p}{\sqrt{P - |p|^2 + \varepsilon|p|}} & |p|^2 \leq P, \\[2ex] \dfrac{p \cdot \sqrt{|p|^2 - P + \varepsilon|p|}}{\varepsilon|p|} & |p|^2 > P, \end{cases} \tag{4.16}$$

and

$$B_\varepsilon(x, p) = \begin{cases} \dfrac{1}{2} \dfrac{\sigma q}{P - |p|^2 + \varepsilon|p|} & |p|^2 \leq P, \\[2ex] \dfrac{1}{2} \dfrac{\sigma q}{\varepsilon|p|} & |p|^2 > P. \end{cases} \tag{4.17}$$

We notice that

- As shown in Section 4.5, $\mathbf{A}$ is continuously differentiable with respect to p,
- $\mathbf{A}_\varepsilon$ is independent of z,
- B_ε, being independent of z, is non-increasing in z for fixed $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$.

These observations allows us to conclude the following theorem that requires that the first condition and any one of the second or third conditions are satisfied.

Theorem 4.6 (Comparison Principle) [11, p.268]. *Suppose u, $v \in C^1(\overline{\Omega})$ satisfy $Q_\varepsilon u \geq 0$, $Q_\varepsilon v \leq 0$ in Ω and $u \leq v$ on $\partial\Omega$. Then it follows that $u \leq v$ in Ω.*

The comparison principle, Theorem 4.6, was proved in [28]. Certain structure conditions are essential for establishing the maximum principle for equation in divergence form. The first condition takes the form

$$p \cdot \mathbf{A}(x, p) = \begin{cases} \dfrac{|p|^2}{\sqrt{P - |p|^2 + \varepsilon|p|}}, & |p|^2 \leq P \\ & \qquad\qquad \geq |p|^\alpha - a^\alpha, \\ \dfrac{|p|\sqrt{|p|^2 - P + \varepsilon|p|}}{\varepsilon}, & |p|^2 > P \end{cases} \tag{4.18}$$

$x \in \Omega$, $p \in \mathbb{R}^n$ for some $\alpha \geq 1$, $a > 0$. For $|p|^2 \leq P$, the infimum of $\dfrac{|p|^2}{\sqrt{P - |p|^2 + \varepsilon|p|}}$ is attained at $|p|^2 = 0$, which yields $\dfrac{|p|^2}{\sqrt{P - |p|^2 + \varepsilon|p|}} \geq \dfrac{1}{\sqrt{P}}$. Therefore the inequality for the first case in (4.18) holds if $a = \sup_{x \in \Omega} \sqrt{P(x)}$.

On the other hand, for $|p|^2 > P$,

$$\frac{|p|\sqrt{|p|^2 - P + \varepsilon|p|}}{\varepsilon} \geq \frac{|p|\sqrt{\varepsilon|p|}}{\varepsilon} \geq \frac{|p|^{\frac{3}{2}}}{\sqrt{\varepsilon}} \geq |p|^{\frac{3}{2}} - a^{\frac{3}{2}}.$$

Hence the inequality for the second case in (4.18) holds if $a \geq 0$, $\alpha = \frac{3}{2}$ and $\varepsilon < 1$.

Combing the two cases, (4.18) holds if $a = \sup_{x \in \Omega} \sqrt{P(x)}$ and $\alpha = \frac{3}{2}$. For the sake of Lemma 4.7 and Theorem 4.8 below, we note that $\alpha > 1$.

The second structure condition is as follows:

$$B(x,p)\ \operatorname{sign} z = \begin{cases} \dfrac{1}{2}\dfrac{\sigma q}{P-|p|^2+\varepsilon|p|}, & |p|^2 \le P \\ & \qquad\qquad \le b^{\alpha-1} = b^{\frac{1}{2}}, \\ \dfrac{1}{2}\dfrac{\sigma q}{\varepsilon|p|}, & |p|^2 > P \end{cases} \tag{4.19}$$

$x \in \Omega$, $p \in \mathbb{R}^n$. For $|p|^2 \le P$, the supremum of $B(x,p)\ \operatorname{sign} z = \dfrac{1}{2}\dfrac{\sigma q}{P-|p|^2+\varepsilon|p|}$ is attained at either $|p|^2 = 0$ or $|p|^2 = P$, that is,

$$\frac{1}{2}\frac{q}{P} \le b^{\frac{1}{2}} \ \text{ and } \ \frac{1}{2}\frac{q}{\varepsilon\sqrt{P}} \le b^{\frac{1}{2}},$$

where $b = \sup\limits_{x\in\Omega}\left\{\dfrac{1}{4}\dfrac{q^2(x)}{P^2(x)}, \dfrac{1}{4}\dfrac{q(x)^2}{\varepsilon^2 P(x)}\right\}$.

For $|p|^2 > P$, the supremum of $B(x,p)\ \operatorname{sign} z = \dfrac{1}{2}\dfrac{\sigma q}{\varepsilon|p|}$ is attained at $|p|^2 = P$, that is,

$$\frac{1}{2}\frac{q}{\varepsilon\sqrt{P}} \le b^{\frac{1}{2}},$$

where $b = \sup\limits_{x\in\Omega}\left\{\dfrac{1}{4}\dfrac{q(x)^2}{\varepsilon^2 P(x)}\right\}$.

Hence the inequality (4.19) holds if $b = \sup\limits_{x\in\Omega}\left\{\dfrac{1}{4}\dfrac{q^2(x)}{P^2(x)}, \dfrac{1}{4}\dfrac{q(x)^2}{\varepsilon^2 P(x)}\right\}$.

Once the structure conditions (4.18) and (4.19) are verified to hold, the development below is along the lines of the derivation of global estimates for weak solutions of linear elliptic equations.

Lemma 4.7 [11, p.271]. *Let $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy $Q_\varepsilon u = 0$ in Ω. Then we have*

$$\sup_{\Omega} u \le C\{\|u^+\|_\alpha + a + b\} + \sup_{\partial\Omega} u^+,$$

where $C = C(n, \alpha, |\Omega|)$.

Using Lemma 4.7 and the fact that Q_ε satisfies the structure conditions (4.18) and (4.19) with $\alpha > 1$, we derive the following apriori estimates for solutions of $Q_\varepsilon u$.

Theorem 4.8 (Maximum Principle) [11, p.272]. *Let $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy $Q_\varepsilon u = 0$ in Ω. Then we have*

$$\sup_\Omega |u| \le C(a+b) + \sup_{\partial\Omega} |u|,$$

where $C = C(n, \alpha, |\Omega|)$.

The technique of proof was demonstrated in [27]. This is an estimation of $\sup\limits_\Omega |u|$ in terms of boundary data and an additive constant. It has a disadvantage that the constant term depends on ε. We still choose to include it here for the sake of comparison with the non-divergence case.

4.7 Comparison and Maximum Principles for Non-divergence Form

In this section, we obtain the estimate of $\sup\limits_\Omega |u|$ using the non-divergence structure of the operator Q_ε that is independent of ε. Operator Q_ε appears in the non-divergence form as follows:

$$Q_\varepsilon u = \begin{cases} \Big(P - |Du|^2 + \varepsilon|Du|\Big)\Delta u + \Big(1 - \dfrac{\varepsilon}{2|Du|}\Big) D_i u D_j u D_{ij} u - \dfrac{1}{2} D_i P D_i u \\ \qquad\qquad + \dfrac{q}{2}\sigma\sqrt{P - |Du|^2 + \varepsilon|Du|}, & |p|^2 \le P \\ \Big(|Du|^2 - P + \varepsilon|Du|\Big)\cdot\varepsilon|Du|\cdot\Delta u + \Big(\dfrac{\varepsilon P}{|Du|} - \dfrac{\varepsilon^2}{2}\Big) D_i u D_j u D_{ij} u \\ \qquad\qquad - \dfrac{1}{2} D_i P \cdot \varepsilon|Du| \cdot D_i u + \dfrac{q}{2}\sigma\cdot\varepsilon|Du|\cdot\sqrt{P - |Du|^2 + \varepsilon|Du|}, & |p|^2 > P \end{cases}$$

From this representation, we note that

- the operator Q_ε is locally elliptic with respect to either u or v;
- the coefficients a^{ij} are independent of z;
- the coefficient b, being independent of z, is non-increasing in z for each $(x,p) \in \Omega \times \mathbb{R}^n$;
- the coefficients $a^{ij}(x,p)$ are continuously differentiable with respect to the p variables in $\Omega \times \mathbb{R}^n$.

The comparison principle for linear operator has the following extension to quasilinear operator Q_ε. This extension requires that all of the above conditions hold.

Theorem 4.9 (Comparison Principle) [11, p.263]. *Let u, $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Q_\varepsilon u > Q_\varepsilon v$ in Ω, $u \le v$ on $\partial\Omega$. It then follows that $u < v$ in Ω.*

Using Theorem 4.9, we can derive the quasilinear extension of the apriori bound, which also illustrates the significance of the function $\mathscr{E}$. In the general form (4.5), $\mathscr{E}$ is defined as follows:

$$\mathscr{E}(x, z, p) = a^{ij}(x, z, p)p_i \cdot p_j. \tag{4.20}$$

Lemma 4.10. *Let $\mathscr{E}$ be the scalar function defined in (4.20) for the operator Q_ε, i.e.,*

$$\mathscr{E}(x,p) = \begin{cases} \Big(P - |p|^2 + \varepsilon|p|\Big)|p|^2 + \Big(1 - \dfrac{\varepsilon}{2|p|}\Big)|p|^4 = P|p|^2 + \dfrac{1}{2}\varepsilon|p|^3, & |p|^2 \le P, \\ \Big(|p|^2 - P + \varepsilon|p|\Big)\varepsilon|p|^3 + \Big(\dfrac{\varepsilon P}{|p|} - \dfrac{\varepsilon^2}{2}\Big)|p|^4 = \varepsilon|p|^5 + \dfrac{1}{2}\varepsilon^2|p|^4, & |p|^2 > P. \end{cases}$$

Then there exist non-negative constants μ_1 and μ_2 such that

$$\frac{b(x,p)\ \operatorname{sign} z}{\mathscr{E}(x,p)} \le \frac{\mu_1|p| + \mu_2}{|p|^2}, \quad (x,p) \in \Omega \times \mathbb{R}^n.$$

Proof. We compute

$$\frac{b(x,p)\cdot|p|^2}{\mathscr{E}(x,p)} = \begin{cases} \dfrac{-\dfrac{1}{2}D_iP\cdot p_i + \dfrac{q}{2}\sigma\sqrt{P - |p|^2 + \varepsilon|p|}}{P + \dfrac{1}{2}\varepsilon|p|}, & |p|^2 \le P, \\ \dfrac{-\frac{1}{2}D_iP\cdot\varepsilon|p|\cdot p_i + \dfrac{q}{2}\sigma\cdot\varepsilon|p|\cdot\sqrt{|p|^2 - P + \varepsilon|p|}}{\varepsilon|p|^3 + \dfrac{1}{2}\varepsilon^2|p|^2}, & |p|^2 > P. \end{cases}$$

When $|p|^2 \le P$,

$$\frac{-\dfrac{1}{2}D_iP\cdot p_i + \dfrac{q}{2}\sigma\sqrt{P - |p|^2 + \varepsilon|p|}}{P + \dfrac{1}{2}\varepsilon|p|} \le \frac{|DP|\cdot p_i}{2P} + \frac{q}{2}\frac{\sqrt{P + \varepsilon|p|}}{P + \dfrac{1}{2}\varepsilon|p|} \le \frac{|DP|}{2P}|p| + \frac{q}{2\sqrt{P}}.$$

When $|p|^2 > P$,

$$\frac{-\frac{1}{2}D_i P \cdot p_i + \dfrac{q}{2}\sigma\sqrt{|p|^2 - P + \varepsilon|p|}}{|p|^2 + \dfrac{1}{2}\varepsilon|p|} \leq \frac{|DP| \cdot |p|}{2|p|^2} + \frac{q}{2}\frac{\sqrt{|p|^2 + \varepsilon|p|}}{|p|^2 + \dfrac{1}{2}\varepsilon|p|} \leq \frac{|DP|}{2\sqrt{P}} + \frac{q}{2\sqrt{P}}.$$

If we choose $\mu_1 = \sup\limits_{x\in\Omega} \dfrac{|DP(x)|}{2P(x)}$ and $\mu_2 = \sup\limits_{x\in\Omega} \left\{ \dfrac{|DP(x)|}{2\sqrt{P(x)}} + \dfrac{q(x)}{2\sqrt{P(x)}} \right\}$,

then we obtain $\mu_1|p| + \mu_2 \geq \dfrac{b(x,p)\cdot|p|^2}{\mathscr{E}(x,p)}$, or equivalently, $\dfrac{b(x,p)\ \operatorname{sign} z}{\mathscr{E}(x,p)} \leq \dfrac{\mu_1|p| + \mu_2}{|p|^2}$.

$\square$

For the uniformly elliptic operators Q_ε, the inequality in Lemma 4.10 is equivalent to a condition of the form

$$\frac{b(x,p)\ \operatorname{sign} z}{\lambda_n} \leq \mu_1|p| + \mu_2 \quad (x,p) \in \Omega \times \mathbb{R}^n.$$

Based on Lemma 4.10, we arrive at the following Maximum Principle.

Theorem 4.11 (Maximum Principle) [11, p.264]. *Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Q_\varepsilon u = 0$ in Ω. We have*

$$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|u| + C\mu_2 \quad \textit{where } C = C(\mu_1,\ \operatorname{diam}\Omega).$$

This is another estimation of $\sup\limits_{\Omega}|u|$ in terms of the boundary data and an additive constant that does not depend on ε.

The derivation of estimates in Theorem 4.8 and Theorem 4.11 is done using the standard theory for elliptic equations. We would expect that the divergence structure affords stronger estimates but, in our case, the non-divergence structure yields a more favorable outcome.

4.8 Hölder Estimates for Divergence Form

In this section, we return to the divergence form of the elliptic equation (4.9):

$$Q_\varepsilon u = \text{div } \mathbf{A}_\varepsilon(x, Du) + B_\varepsilon(x, Du) = 0$$

where $\mathbf{A}_\varepsilon(x, p)$ and $B_\varepsilon(x, p)$ are as in (4.16) and (4.17). We copy the representations here for the reader's convenience.

$$\mathbf{A}_\varepsilon(x, p) = \begin{cases} \dfrac{p}{\sqrt{P - |p|^2 + \varepsilon|p|}} & |p|^2 \leq P, \\ \dfrac{p \cdot \sqrt{|p|^2 - P + \varepsilon|p|}}{\varepsilon|p|} & |p|^2 > P, \end{cases}$$

and

$$B_\varepsilon(x, p) = \begin{cases} \dfrac{1}{2} \dfrac{\sigma q}{P - |p|^2 + \varepsilon|p|} & |p|^2 \leq P, \\ \dfrac{1}{2} \dfrac{\sigma q}{\varepsilon|p|} & |p|^2 > P. \end{cases}$$

We note that the regularization that we choose ensures that the coefficients are sufficiently smooth across the transition $|p|^2 = P$ for all $\varepsilon > 0$, namely, $\mathbf{A}_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}^n)$ and $B_\varepsilon \in C^0(\overline{\Omega} \times \mathbb{R}^n)$.

In this section, we derive interior and global Hölder estimates for the derivatives of solutions of quasilinear elliptic equations (4.9) in a bounded domain Ω. These estimates correspond to Stage 4 in the Appriori Estimates Program. We follow the standard elliptic theory and outline the main steps in proving these estimates for the sake of completeness. Also the linear elliptic equation (4.21) below will reappear in the derivation of the interior gradient bounds in Section 4.10.

If $u \in C^2(\Omega)$ satisfies $Q_\varepsilon u = 0$ in Ω, we have

$$\int_\Omega \{\mathbf{A}_\varepsilon(x, Du) \cdot D\zeta - B_\varepsilon(x, Du) \cdot \zeta\} dx = 0 \qquad \forall \zeta \in C_0^1(\Omega)$$

Fixing k, replacing ζ by $D_k\zeta$, and integrating by parts, we then obtain

$$\int_\Omega \{(D_{p_j}\mathbf{A}^i_\varepsilon(x,Du)\cdot D_{jk}u + \delta_k\mathbf{A}^i_\varepsilon(x,Du)\cdot D_i\zeta + B_\varepsilon(x,Du)\cdot D_k\zeta\}dx = 0$$

where δ_k is differential operator defined by $\delta_k\mathbf{A}^i_\varepsilon(x,p) = D_{x_k}\mathbf{A}^i_\varepsilon(x,p)$.

Hence, writing

$$\bar{a}^{ij}(x) = D_{p_j}\mathbf{A}^i_\varepsilon(x,Du(x)), \qquad f^i_k(x) = \delta_k\mathbf{A}^i_\varepsilon(x,Du(x)) + \delta^i_k B_\varepsilon(x,Du(x)),$$

we have that the derivative $w = D_k u$ satisfies

$$\int_\Omega (\bar{a}^{ij}D_j w + f^i_k(x))\cdot D_i\zeta dx = 0 \qquad \forall\zeta\in C^1_0(\Omega) \tag{4.21}$$

that is, w is a solution of the linear elliptic equation

$$Lw = D_i(\bar{a}^{ij}D_j w) = -D_i f^i_k. \tag{4.22}$$

The operator L is required to be strictly elliptic in Ω, that is, there exists a constant λ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \forall x\in\Omega,\ \xi\in\mathbb{R}^n. \tag{4.23}$$

We also assume that L has bounded coefficients, that is, for some constants Λ, we have

$$\sum_{i,j}|a^{ij}(x)|^2 \leq \Lambda^2. \tag{4.24}$$

The following two theorems are used as prerequisites for obtaining Theorem 4.14 below.

Theorem 4.12 [11, p.200]. *Let operator L satisfy conditions* (4.23), (4.24), *and suppose that $f^i\in L^q(\Omega),\ i=1,\dots,n$ for some $q>n$. Then if u is a $W^{1,2}(\Omega)$ solution of the linear equation $Lu = D_i f$ in Ω, it follows that u is locally Hölder continuous in Ω, and for any ball*

$B_0 = B_{R_0}(y) \subset \Omega$ and $R \le R_0$ we have

$$\operatorname*{osc}_{B_R(y)} u \le CR^\alpha (R_0^{-\alpha} \sup_{B_0} |u| + k)$$

where $C = C(n, \Lambda/\lambda, q, R_0)$ and $\alpha = \alpha(n, \Lambda/\lambda, q)$ are positive constants, and $k = \lambda^{-1}\|\mathbf{f}\|_q$.

By taking $R_0 = d$ in Theorem 4.12 and estimating $\sup_{B_0} |u|$ in terms of its L^p norm, we have the following interior Hölder estimate for weak solutions of equation (4.22).

Theorem 4.13 [11, p.202]. *Let operator L satisfy conditions (4.23), (4.24), and suppose that $f^i \in L^q(\Omega)$, $i = 1, \dots, n$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ solution of equation $Lu = D_i f$ in Ω, we have for any $\Omega' \subset\subset \Omega$*

$$\|u\|_{C^\alpha(\bar{\Omega}')} := \sup_\Omega |u| + \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le C(\|u\|_{L^2(\Omega)} + k)$$

where $C = C(n, \Lambda/\lambda, q, d)$, $d = \operatorname{dist}(\Omega', \partial\Omega)$ and $\alpha = \alpha(n, \Lambda/\lambda, q) > 0$, and $k = \lambda^{-1}\|\mathbf{f}\|_q$.

In the linear elliptic equation (4.22), by replacing Ω if necessary by a strictly contained subdomain, we can assume that L is strictly elliptic in Ω and that the coefficients $\bar{a}^{ij}$, f_k^i are bounded, that is, the hypotheses of Theorem 4.12 and Theorem 4.13 are satisfied. The derivation of Theorem 4.14 and Theorem 4.15 are based on [21] and [22]. Accordingly, choosing λ_K, Λ_K, μ_K so that

$$\lambda_K \le \lambda(x,p), \quad \Lambda_K \ge D_{p_j}|\mathbf{A}_\varepsilon^i(x,p)|, \quad \mu_K \ge |\delta_k \mathbf{A}_\varepsilon^i(x,p)| + |\delta_k^i B_\varepsilon(x,p)|,$$

for all $x \in \Omega$ and $|z| + |p| \le K$, we obtain the following interior estimate.

Theorem 4.14 [11, p.320]. *Let $u \in C^2(\overline{\Omega})$ satisfy $Q_\varepsilon u = 0$ in Ω where Q_ε is elliptic in Ω and is of divergence form with $\mathbf{A}_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}^n)$, $B_\varepsilon \in C^0(\overline{\Omega} \times \mathbb{R}^n)$. Then for any for any $\Omega' \subset\subset \Omega$ we have*

$$[Du]_{\alpha;\Omega'} := \sup_{x,y\in\Omega',\ x\neq y} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha} \le Cd^{-\alpha},$$

where

$$C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \operatorname{diam}\Omega),$$

$$K = |u|_{1;\Omega} = \sup_{\Omega}(|u| + |Du|),$$

$$d = \operatorname{dist}(\Omega', \partial\Omega),\ \alpha = \alpha(n, \frac{\Lambda_K}{\lambda_K})$$

In order to extend Theorem 4.14 to a global Hölder estimate in Ω, we recall that Q_ε is elliptic in $\bar{\Omega}$ with $\mathbf{A}_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}^n)$, $B_\varepsilon \in C^0(\overline{\Omega} \times \mathbb{R}^n)$, that $\partial\Omega \in C^2$ and that $u = \phi$ on $\partial\Omega$ where $\phi \in C^2(\overline{\Omega})$.

By replacing u with $u - \phi$, we can assume without loss of generality that $u = 0$ on $\partial\Omega$. Since $\partial\Omega \in C^2$, there exists for each $x_0 \in \partial\Omega$ a ball $B = B(x_0)$ and a one-to-one mapping ψ from B onto an open set $D \subset \mathbb{R}^n$ such that

$$\psi(B \cap \Omega) \subset \mathbb{R}^n_+, \quad \psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+,$$

$$\psi \in C^2(B), \quad \psi^{-1} \in C^2(D).$$

Writing $y = \psi(x)$, $v(y) = u \circ \psi^{-1}(y)$, $B^+ = B \cap \Omega$, $D^+ = \psi(B^+)$, we have $D_{y_k} v = 0$ on $\partial D^+ \cap \partial\mathbb{R}^n_+$, $k = 1, \ldots, n-1$.

Then equation $Q_\varepsilon u = 0$ in B^+ is equivalent to the equation

$$\bar{Q}_\varepsilon v = D_{y_i} \bar{\mathbf{A}}^i_\varepsilon(x, Du) + \bar{B}_\varepsilon(x, Du) = 0 \tag{4.25}$$

in D^+ where $x = \psi^{-1}(y)$ and the functions $\bar{\mathbf{A}}_\varepsilon$ and $\bar{B}_\varepsilon$ are given by

$$\bar{\mathbf{A}}^i_\varepsilon = \frac{\partial y_i}{\partial x_r} \mathbf{A}^r_\varepsilon, \qquad \bar{B}_\varepsilon = -\frac{\partial}{\partial y_i}\left(\frac{\partial y_i}{\partial x_r}\right) \mathbf{A}^r_\varepsilon + B_\varepsilon.$$

The derivatives $w = D_{y_k} v$ $k = 1, \ldots n-1$ are generalized solutions in D^+ of the linear elliptic equation

$$Lw = D_i(\bar{a}^{ij} D_j w) = -D_i f^i_k.$$

The operator L is strictly elliptic in D^+ with bounded coefficients $\bar{a}^{ij}$ and f_k^i. We thus have for any $D' \subset\subset D$,

$$[w]_{\alpha;D'\cap D^+} = [D_{y_k}v]_{\alpha;D'\cap D^+} \leq C \quad k = 1,\dots,n-1, \tag{4.26}$$

where

$$C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega, d),\ K = |u|_{1;\Omega},\ d = \operatorname{dist}(D'\cap D^+, \partial D),\ \alpha = \alpha(n, \frac{\Lambda_K}{\lambda_K}).$$

The remaining derivative $D_{y_n}v$ can be estimated as follows. Let $y_0 \in D' \cap D^+$, $R \leq \frac{d}{3}$, $B_{2R} = B_{2R}(y_0)$, $\eta \in C_0^1(B_{2R})$ and let c be a constant such that

$$c = \begin{cases} w(y_0), & \text{if } B_{2R} \subset D^+ \\ 0, & \text{if } B_{2R} \cap \partial\mathbb{R}^n \neq \emptyset \end{cases}$$

Setting $\zeta = \eta^2(w - c)$ for $w = D_{y_k}v$, $k = 1,\dots,n-1$, then $\zeta \in W_0^{1,2}(D^+)$. By substitution into (4.21) with $\Omega = D+$, we then have

$$\int_{D+} (\bar{a}^{ij}D_jw + f_k^i)\cdot D_i\eta^2(w-c)\,dy = \int_{D+} (\bar{a}^{ij}D_jw + f_k^i)\cdot(\eta^2 D_iw + 2\eta(w-c)D_i\eta)\,dy = 0,$$

which implies

$$\begin{aligned}\int_{D^+} \eta^2\bar{a}^{ij}D_iwD_jw\,dy &= \int_{D^+} \{2\eta(w-c)\bar{a}^{ij}D_i\eta D_jw + \eta^2 f_k^iD_iw + 2\eta(w-c)f_k^iD_i\eta\}\,dy \\ &\leq \int_{D^+} \{|2\eta(w-c)\bar{a}^{ij}D_i\eta D_jw| + |\eta^2 f_k^iD_iw| + |2\eta(w-c)f_k^iD_i\eta|\}\,dy,\end{aligned}$$

so that, by the Schwarz inequality and the ellipticity of L, we obtain

$$\int_{D^+} \eta^2|Dw|^2\,dy \leq C\int_{D^+} (\eta^2 + |D\eta|^2(w-c)^2)\,dy$$

where $C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega)$.

Suppose η is chosen in a way such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_r(y_0)$ and $|D\eta| \leq \frac{2}{R}$. We thus obtain by (4.21),

$$\int_{B_R} |Dw|^2 \, dy \leq CR^{n-2}(R^2 + \sup_{B_{2R}}(w-c)^2) \leq CR^{n-2+2\alpha},$$

where $C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega, d)$. Therefore we have

$$\int_{B_R} |D_{ij}v|^2 \, dy \leq CR^{n-2+2\alpha} \tag{4.27}$$

provided $j \neq n$. We solve equation (4.25) for $D_{nn}v$ so that we can write

$$D_{nn}v = b^{ij}D_{ij}v + b, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n-1,$$

for certain functions b^{ij}, b bounded in terms of $D\psi$, K, $\dfrac{\Lambda_K}{\lambda_K}$ and $\dfrac{\mu_K}{\lambda_K}$. Hence by (4.27) we have

$$\int_{B_R} |D_{nj}v|^2 \, dy \leq CR^{n-2+2\alpha} \quad i = 1, \ldots, n,$$

we can conclude that the estimate (4.26) is also valid for $k = n$. Returning to the domain Ω by means of the mapping ψ^{-1}, we thus have

$$[Du]_{\alpha;B' \cap \Omega} \leq C, \tag{4.28}$$

for any concentric ball $B' \subset\subset B$, where $C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega, B')$. By choosing finitely many points $x_0 \in \partial\Omega$ and balls B' covering $\partial\Omega$, we obtain the following global Hölder estimate from Theorem 4.14 and inequality (4.28).

Theorem 4.15 [11, p.323]. *Let $u \in C^2(\overline{\Omega})$ satisfy $Q_\varepsilon u \geq 0$ in Ω where Q_ε is elliptic in $\overline{\Omega}$ and is of divergence form with $\mathbf{A}_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}^n)$, $B_\varepsilon \in C^0(\overline{\Omega} \times \mathbb{R}^n)$. Then if $\partial\Omega \in C^2$ and $u = \phi$ on $\partial\Omega$ where $\phi \in C^2(\overline{\Omega})$, we have the estimate*

$$[Du]_{\alpha;\Omega} := \sup_{x,y \in \Omega,\, x \neq y} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha} \leq C,$$

where

$$C = C(n, K, \frac{\Lambda_K}{\lambda_K}, \frac{\mu_K}{\lambda_K}, \Omega, \Phi),\ K = |u|_{1;\Omega},\ \Phi = |\phi|_{2;\Omega},\ \alpha = \alpha(n, \frac{\Lambda_K}{\lambda_K}, \Omega).$$

Theorem 4.15 is used to estimate $[Du]_{\alpha;\Omega}$ in terms of $\sup_\Omega |Du|$ and $\sup_\Omega |u|$ and the constant C in the estimate does not depend on ε. Since ellipticity and regularity of functions $\mathbf{A}_\varepsilon$ and B_ε are the only requirement for the Hölder estimates of the gradient, the standard elliptic theory applies.

4.9 Boundary Gradient Estimates

The boundary gradient estimates developed in this section are the implementation of Stage 2 of the Apriori Estimates Program. These estimates are tied through the classical maximum principle to natural choices of barrier functions discussed below.

Recall that the operator Q_ε defined in (4.9) is an elliptic operator of the form

$$Q_\varepsilon u = a^{ij}_\varepsilon(x, Du)D_{ij}u + b_\varepsilon(x, Du)$$

where $b_\varepsilon(x, p)$, being independent of z, is non-increasing in z. Suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Q_\varepsilon u = 0$ in Ω. Suppose that, in some neighborhood $\mathscr{N} = \mathscr{N}_{x_0}$ of a point $x_0 \in \partial\Omega$, there exist two functions $w^\pm = w^\pm_{x_0} \in C^2(\mathscr{N} \cap \Omega) \cap C^1(\mathscr{N} \cap \bar{\Omega})$ such that

(i) $\pm Q_\varepsilon w^\pm < 0$ in $\mathscr{N} \cap \Omega$

(ii) $w^\pm(x_0) = u(x_0)$

(iii) $w^-(x) \le u(x) \le w^+(x), \quad x \in \partial(\mathscr{N} \cap \Omega).$

It then follows from Theorem 4.9 that

$$w^-(x) \le u(x) \le w^+(x) \quad \text{for all } x \in \mathscr{N} \cap \Omega,$$

and hence by (ii)

$$\frac{w^-(x) - w^-(x_0)}{|x - x_0|} \leq \frac{u(x) - u(x_0)}{|x - x_0|} \leq \frac{w^+(x) - w^+(x_0)}{|x - x_0|}.$$

Consequently, the normal derivatives of $w^\pm$ and u satisfy

$$\frac{\partial w^-}{\partial v}(x_0) \leq \frac{\partial u}{\partial v}(x_0) \leq \frac{\partial w^+}{\partial v}(x_0). \tag{4.29}$$

We call the functions $w^\pm$ respectively upper and lower barriers at x_0 for the operator Q_ε and function u. Their existence at all points $x_0 \in \partial\Omega$, implies the desired boundary gradient estimate for u satisfying $Q_\varepsilon u = 0$.

Suppose that Ω satisfies an exterior sphere condition at a point $x_0 \in \partial\Omega$ so that there exists a ball $B = B_R(y)$ with $x_0 \in \bar{B} \cap \bar{\Omega} = \bar{B} \cap \partial\Omega$. Let us define the distance function $d(x) = \operatorname{dist}(x, \partial B)$ and set $w = \psi(d)$ where $\psi \in C^2\left[0, \infty\right)$ and $\psi' > 0$. We have for any $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$,

$$\begin{aligned} \bar{Q}_\varepsilon w &= a_\varepsilon^{ij}(x, u(x), Dw) D_{ij} w + b_\varepsilon(x, u(x), Dw) \\ &= \psi' a_\varepsilon^{ij} D_{ij} d + \frac{\psi''}{(\psi')^2} \mathscr{E} + b_\varepsilon \\ &\leq \frac{n-1}{R} \psi' \Lambda + \frac{\psi''}{(\psi')^2} \mathscr{E} + b_\varepsilon \end{aligned} \tag{4.30}$$

the last inequality follows from $tr(AB) \leq \Lambda(A) tr(B)$ where $A = [a_\varepsilon^{ij}]$ is a real symmetric matrix and $B = D_{ij} d$ is a positive semidefinite matrix.

Recall that, for the special choice (4.8) of operator Q_ε, we have

$$\mathscr{E}(x, p) = a^{ij}(x, p) p_i \cdot p_j = \begin{cases} P|p|^2 + \dfrac{1}{2}\varepsilon|p|^3 & |p|^2 \leq P \\ \\ \varepsilon|p|^5 + \dfrac{1}{2}\varepsilon^2|p|^4 & |p|^2 > P. \end{cases}$$

Moreover, we obtained the estimate in Section 4.7

$$b(x,p) \le \frac{\mu_1|p| + \mu_2}{|p|^2} \cdot \mathscr{E}(x,p) \qquad (x,p) \in \Omega \times \mathbb{R}^n,$$

with the choice of μ_1 and μ_2 as follows:

$$\mu_1 = \sup_{x\in\Omega} \frac{|DP(x)|}{2P(x)} \quad \text{and} \quad \mu_2 = \sup_{x\in\Omega} \Big\{ \frac{|DP(x)|}{2\sqrt{P(x)}} + \frac{q(x)}{2\sqrt{P(x)}} \Big\}.$$

Recall that in the case $|p|^2 > P$, $\Lambda = \lambda_1 = (|p|^2 - P + \varepsilon|p|) \cdot \varepsilon|p|$ or $\Lambda = \lambda_n = \varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2$. If $\Lambda = \lambda_1$,

$$\begin{aligned} |p| \cdot \Lambda + |b| &= |p| \cdot \Big(|p|^2 - P + \varepsilon|p| \Big) \varepsilon|p| + \frac{\mu_1|p| + \mu_2}{|p|^2} \cdot \mathscr{E}(x,p) \\ &\le \varepsilon|p|^4 + \varepsilon^2|p|^3 + \frac{\mu_1|p| + \mu_2}{|p|^2} \cdot \mathscr{E}(x,p). \end{aligned}$$

If $\Lambda = \lambda_n$,

$$\begin{aligned} |p| \cdot \Lambda + |b| &= |p| \cdot \Big(\varepsilon|p|^3 + \frac{1}{2}\varepsilon^2|p|^2 \Big) + \frac{\mu_1|p| + \mu_2}{|p|^2} \cdot \mathscr{E}(x,p) \\ &\le \varepsilon|p|^4 + \frac{1}{2}\varepsilon^2|p|^3 + \frac{\mu_1|p| + \mu_2}{|p|^2} \cdot \mathscr{E}(x,p). \end{aligned}$$

In both cases, we derive the following estimate

$$|p| \cdot \Lambda + |b| \le \frac{2}{|p|}\, \mathscr{E}(x,p) + \Big(\frac{\mu_1}{|p|} + \frac{\mu_2}{|p|^2} \Big)\, \mathscr{E}(x,p) \le \Big(\frac{2+\mu_1}{\sqrt{P}} + \frac{\mu_2}{P} \Big)\, \mathscr{E}(x,p).$$

Therefore for the constant $\mu = \sup_{x\in\Omega} \Big\{ \sqrt{P},\ \frac{2}{\sqrt{P}} + \frac{|DP(x)|}{\sqrt{P(x)^3}} + \frac{q(x)}{2\sqrt{P(x)^3}} \Big\}$, we have

$$|p| \cdot \Lambda + |b| \le \mu \cdot \mathscr{E} \tag{4.31}$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ with $|p| \ge \mu$.

Since $|p| = |Dw| = \psi'|Dd| = \psi'$, we have

$$\psi' \cdot \Lambda + |b| \leq \mu \cdot \mathscr{E} \tag{4.32}$$

provided $\psi' \geq \mu = \mu(M)$ where $M = \sup_{\Omega}|u|$. Using (4.30) and (4.32), we obtain

$$\bar{Q}_\varepsilon w < \Big(\frac{\psi''}{(\psi')^2} + \nu\Big)\,\mathscr{E}$$

where $\nu = \Big(1 + \dfrac{n-1}{R}\Big)\mu$. Consider the function ψ given by

$$\psi(d) = \frac{1}{\nu}\log(1 + kd)$$

and the neighborhood $\mathscr{N} = \mathscr{N}_{x_0} = \{x \in \bar{\Omega} \,|\, d(x) < a\}$, for some $a > 0$. Clearly $\psi'' = -\nu(\psi')^2$ in $\mathscr{N}$. Moreover,

$$\psi(a) = \frac{1}{\nu}\log(1 + ka) = M \qquad \text{if } \; ka = e^{\nu M} - 1$$

and

$$\begin{aligned} \psi'(d) = \frac{k}{\nu(1+kd)} &\geq \frac{k}{\nu(1+ka)} \quad \text{ in } \mathscr{N} \cap \Omega \\ &= \frac{k}{\nu e^{\nu M}} \\ &\geq \mu, \quad \text{if } k \geq \mu\nu e^{\nu M}. \end{aligned}$$

Consequently, if k and a are chosen to satisfy the relations

$$k \geq \mu\nu e^{\nu M}, \qquad ka = e^{\nu M} - 1,$$

the function $w^+ = \psi(d)$ is an upper barrier at x_0 for the operator $\bar{Q}_\varepsilon$ and the function u provided $u = 0$ on $\mathscr{N} \cap \partial\Omega$. Similarly the function $w^- = -\psi(d)$ is a corresponding lower barrier. Hence if also $Q_\varepsilon u = 0$ in Ω, we obtain from (4.29) the estimate

$$|Du(x_0)| \le \psi'(0) = \mu e^{\nu M}.$$

We can therefore assert the following boundary gradient estimate on general domains.

Theorem 4.16 [11, p.337]. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $Q_\varepsilon u = 0$ in Ω and $u = \phi$ on $\partial\Omega$. Suppose that Ω satisfies a uniform exterior sphere condition and $\phi \in C^2(\overline{\Omega})$. Then we have*

$$\sup_{\partial\Omega} |Du| \le C$$

where $C = C(n, M, \mu(M), \Phi, \delta)$, $M = \sup_{\Omega} |u|$, $\Phi = |\phi|_{2:\Omega}$ and δ is the radius of the exterior spheres.

This estimate is used to approximate boundary gradient $\sup_{\partial\Omega} |Du|$ in terms of $\sup_{\Omega} |u|$. The constant C does not depend on ε.

4.10 Interior Gradient Bounds

In this section, we are mainly concerned with the derivation of apriori estimates for the gradients of solutions of quasilinear elliptic equations. That is Stage 3 in the Apriori Estimates Program. Let $u \in C^2(\Omega)$ satisfy the divergence form equation

$$Q_\varepsilon u = \text{div } \mathbf{A}_\varepsilon(x, Du) + B_\varepsilon(x, Du) = 0,$$

where the vector functions $\mathbf{A}_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}^n)$ and $B_\varepsilon \in C^0(\overline{\Omega} \times \mathbb{R}^n)$ are defined in (4.16) and (4.17) in the domain Ω.

It was shown in Section 4.8 that the derivatives $D_k u$ is a solution of the linear elliptic equation

$$Lu := D_i(\bar{a}^{ij} D_j u + f_k^i) = 0, \tag{4.33}$$

where

$$\bar{a}^{ij}(x) = D_{p_j} \mathbf{A}_\varepsilon^i(x, Du(x)),$$

$$f_k^i(x) = \delta_k \mathbf{A}_\varepsilon^i(x, Du(x)) + \delta_k^i B_\varepsilon(x, Du(x)).$$

We compute $D_{p_j}\mathbf{A}_\varepsilon^i(x,p)$ for $\mathbf{A}_\varepsilon$ given in (4.16) as follows:

$$D_{p_j}\mathbf{A}_\varepsilon^i(x,p) = \begin{cases} \dfrac{\delta_{ij}(P-|p|^2+\varepsilon|p|) + (1-\dfrac{\varepsilon}{2|p|})p_ip_j}{(P-|p|^2+\varepsilon|p|)^{\frac{3}{2}}} & |p|^2 \le P, \\ \dfrac{\delta_{ij}\Big((|p|^2-P+\varepsilon|p|)\cdot\varepsilon|p|\Big) + \Big(\dfrac{\varepsilon P}{|p|} - \dfrac{\varepsilon^2}{2}\Big)p_ip_j}{\varepsilon^2|p|^2(|p|^2-P+\varepsilon|p|)^{\frac{1}{2}}} & |p|^2 > P. \end{cases}$$

In the proof of Lemma 4.4, we show that

$$\min(\lambda_1,\lambda_n)\cdot|\xi|^2 \le \Big[\delta_{ij}(P-|p|^2+\varepsilon|p|) + (1-\frac{\varepsilon}{2|p|})p_ip_j\Big]\xi_i\xi_j, \tag{4.34}$$

where $\lambda_1 = P-|p|^2+\varepsilon|p|$ and $\lambda_n = P+\dfrac{1}{2}\varepsilon|p|$. Similarly,

$$\min(\lambda_1',\lambda_n')\cdot|\xi|^2 \le \Big[\delta_{ij}\Big((|p|^2-P+\varepsilon|p|)\cdot\varepsilon|p|\Big) + \Big(\frac{\varepsilon P}{|p|}-\frac{\varepsilon^2}{2}\Big)p_ip_j\Big]\xi_i\xi_j, \tag{4.35}$$

where $\lambda_1' = \Big(|p|^2-P+\varepsilon|p|\Big)\cdot\varepsilon|p|$ and $\lambda_n' = \varepsilon|p|^3+\dfrac{1}{2}\varepsilon^2|p|^2$.

Combining (4.34) and (4.35), we obtain

$$\bar{a}^{ij}(x,p)\xi_i\xi_j = D_{p_j}\mathbf{A}_\varepsilon^i(x,p)\xi_i\xi_j \ge \begin{cases} \dfrac{\min(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}|\xi|^2 & |p|^2\le P, \\ \dfrac{\min(\lambda_1',\lambda_n')}{\lambda_1'^{\frac{1}{2}}(2\lambda_n')^{\frac{3}{4}}}|\xi|^2 & |p|^2 > P, \end{cases}$$

which implies that there exists a positive constant ν on $\mathbb{R}$ such that

$$\bar{a}^{ij}(x,p)\xi_i\xi_j = D_{p_j}\mathbf{A}^i(x,p)\xi_i\xi_j \ge \nu|\xi|^2, \tag{4.36}$$

for all $\xi \in \mathbb{R}^n$, $(x,p) \in \Omega \times \mathbb{R}^n$ and ν is computed explicitly as follows

$$\begin{aligned}\nu &= \min\left\{\frac{\min(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\min(\lambda_1',\lambda_n')}{\lambda_1'^{\frac{1}{2}}(2\lambda_n')^{\frac{3}{4}}}\right\} \\ &= C\min\left\{\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt[4]{\lambda_n'}}, \frac{1}{\sqrt[4]{\lambda_1'}}\right\} \\ &= C\min\left\{\frac{1}{\sqrt{\lambda_n}}, \frac{1}{\sqrt[4]{\lambda_n'}}\right\}.\end{aligned}$$

where C is a constant. Taking the limit $\varepsilon \to 0$, we have

$$\nu = \inf_{\Omega} \frac{C}{\sqrt{P}},$$

which is a constant independent of ε, provided that P is bounded away from zero.

A global gradient bound for Q_ε can then be derived under the additional structure conditions:

$$D_x\mathbf{A}_\varepsilon, \;\; B_\varepsilon = o(|p|) \quad \text{as } |p| \to \infty. \tag{4.37}$$

These conditions are verified by computing $D_x\mathbf{A}_\varepsilon$ and recalling B_ε:

$$D_x\mathbf{A}_\varepsilon(x,p) = \begin{cases} \dfrac{-p_i \cdot D_xP}{2(P-|p|^2+\varepsilon|p|)^{\frac{3}{2}}} & |p|^2 \le P, \\[2ex] \dfrac{-p_i \cdot D_xP}{2\varepsilon|p|(|p|^2-P+\varepsilon|p|)^{\frac{1}{2}}} & |p|^2 > P, \end{cases}$$

and

$$B_\varepsilon(x,p) = \begin{cases} \dfrac{1}{2}\dfrac{\sigma q}{P-|p|^2+\varepsilon|p|}, & |p|^2 \le P, \\[2ex] \dfrac{1}{2}\dfrac{\sigma q}{\varepsilon|p|}, & |p|^2 > P, \end{cases}$$

We set $M = \sup_\Omega |u|$, $M_1 = \sup_\Omega |Du|$ and apply the maximum principle to the equation (4.21) in the domain $\tilde{\Omega} = \{x \in \Omega \mid |Du| > \dfrac{M_1}{2\sqrt{n}}\}$. Thus we obtain

$$\sup_{\tilde{\Omega}} |D_k u| \le \sup_{\partial\tilde{\Omega}} |D_k u| + C\|f_k^i\|_q$$

for $q > n$, $k = 1, \dots n$, where $C = C(n, \nu(M), q, |\Omega|)$. Taking $q = \infty$ and using conditions (4.37), we therefore have

$$\sup_{\Omega} |Du| \leq C(\sup_{\partial\Omega} |Du| + \sigma(M_1))$$

where $\sigma(t) = o(t)$ as $t \to \infty$. Consequently, an apriori estimate for M_1 follows.

Theorem 4.17 [11, p.374]. *Let $u \in C^2(\Omega)$ satisfy $Q_\varepsilon u = 0$ in the bounded domain Ω and the structure conditions (4.36) and (4.37) are satisfied. Then we have the estimate*

$$\sup_{\Omega} |Du| \leq C(1 + \sup_{\partial\Omega} |Du|)$$

where C depends on n, $\nu(M)$ and the quantities in (4.37).

The global gradient bound for solutions of the divergence structure, Theorem 4.17, is due to [26]. Instead of pursuing global bound further, at this stage, we now turn to a consideration of interior gradient estimates for uniformly elliptic equations. The estimates follow the standard theory of elliptic equations. We present a simplified treatment of interior gradient bounds taking into account the fact that the operator $Q_\varepsilon(x, z, p)$ does not depend on z, but only on x and p.

We first observe that the operator Q_ε is uniformly elliptic in Ω in the sense that

$$|D_p \mathbf{A}_\varepsilon(x, p)| \leq \mu \tag{4.38}$$

for all $(x, p) \in \Omega \times \mathbb{R}^n$, where

$$\mu = \max\left\{\frac{\max(\lambda_1, \lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\max(\lambda_1', \lambda_n')}{\lambda_1'^{\frac{1}{2}}(2\lambda_n - 2P)^{\frac{3}{4}}}\right\}$$

is a positive constant.

Conditions (4.36) and (4.38) imply respectively the inequalities

$$\begin{aligned} p \cdot \mathbf{A}_\varepsilon(x, p) - p \cdot \mathbf{A}_\varepsilon(x, 0) &\geq \nu \cdot |p|^2, \\ |\mathbf{A}_\varepsilon^i(x, p) - \mathbf{A}_\varepsilon^i(x, 0)| &\leq \mu \cdot (1 + |p|), \end{aligned} \tag{4.39}$$

Finally we take, in place of (4.37), the more general condition

$$g(x,p) = |D_x\mathbf{A}_\varepsilon| + |B_\varepsilon| \le \mu(1+|p|)^2 \tag{4.40}$$

for all $(x,p) \in \Omega \times \mathbb{R}^n$. Conditions (4.36), (4.38), (4.40) can be regarded as natural for divergence structure operators. The derivation of an apriori interior gradient bound under these conditions is accomplished in three stages:

(i) *Reduction to an L^p estimate.* We replace the function ζ in (4.21) by $\zeta D_k u$ and sum the resulting equations over k to get

$$\int_\Omega (\bar{a}^{ij} D_{jk}u + f_k^i) \cdot D_i(\zeta D_k u)dx = 0$$

Setting $v = |Du|^2$, we obtain

$$\int_\Omega \zeta \bar{a}^{ij} D_{ik}u D_{jk}u \; dx + \int_\Omega (\frac{1}{2}\bar{a}^{ij} D_j v + D_k u f_k^i) D_i\zeta \; dx + \int_\Omega \zeta f_k^i D_{ik}u \; dx = 0$$

Hence by Young's inequality, we have

$$\int_\Omega (\bar{a}^{ij} D_j v + 2D_k u f_k^i) D_i\zeta \; dx \le \frac{1}{2}\int_\Omega \zeta \lambda^{-1}(f_k^i)^2 \; dx \tag{4.41}$$

The following interior estimate for weak subsolutions of linear equations is applicable to inequality (4.41).

Theorem 4.18 [11, p.194]. *Let L be a linear operator satisfying conditions* (4.23), (4.24)*, and suppose that $f^i \in L^q(\Omega),\ i = 1, \dots, n$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ subsolution (supersolution) of equation $Lu = D_i f$ in Ω, we have, for any ball $B_{2R}(y) \subset \Omega$ and $p > 1$*

$$\sup_{B_R(y)} u(-u) \le C(R^{-n/p}\|u^+(u^-)\|_{L^2(B_{2R}(y))} + k(R))$$

where $C = C(n, \Lambda/\lambda, q, p)$.

Applying this theorem to $v = |Du|^2$ and operator L defined in (4.33) and using (4.40), we have for any ball $B_{2R} = B_{2R}(y) \subset \Omega$ and $q > n$, the estimate

$$\sup_{B_R(y)} v \leq C\{R^{-n/2}\|v\|_{L^2(B_{2R})} + \|(1+|Du|)^4)\|_{L^q(B_{2R})}\}$$

where $C = C(n, \nu(M), \mu(M), q, \operatorname{diam}\Omega)$, $M = \sup\limits_{B_{2R}(y)} |u|$. Consequently for sufficiently large p we have

$$\sup_{B_R(y)} v \leq C(n, \nu(M), \mu(M), \operatorname{diam}\Omega, R^{-n} \int_{B_{2R}(y)} v^p \, dx). \tag{4.42}$$

(ii) *Reduction to a Hölder estimate.* We utilize the weak form of equation

$$\int_\Omega (\bar{a}^{ij} D_j w + f_k^i(x)) \cdot D_i \zeta dx = 0 \qquad \forall \zeta \in C_0^1(\Omega)$$

that is

$$Q_\varepsilon(u, \phi) = \int_\Omega (\mathbf{A}_\varepsilon^i(x, Du) D_i \phi - B_\varepsilon(x, Du)\phi) \, dx = 0 \qquad \forall \phi \in C_0^1(\Omega) \tag{4.43}$$

From (4.39), we obtain that the function $\mathbf{A}_\varepsilon$ satisfies

$$\begin{aligned} &|\mathbf{A}_\varepsilon(x, p)| \leq \mu_1 \cdot (1 + |p|) \\ &p \cdot \mathbf{A}_\varepsilon(x, p) \geq \nu_1 \cdot |p|^2 - \mu_1 \end{aligned} \tag{4.44}$$

for all $(x, p) \in \Omega \times \mathbb{R}^n$, where

$$\mu_1 = \sup\{\mu, |\mathbf{A}(x, 0)|, p \cdot \mathbf{A}(x, 0)\} \quad \text{and} \quad \nu_1 = \nu$$

are both positive constants.

Next, we substitute into (4.43) the test function

$$\phi = \eta^2[u - u(y)].$$

where $\eta \in C_0^1(B_{2R})$, $B_{2R} = B_{2R}(y) \subset \Omega$, to get

$$\int_\Omega \mathbf{A} \cdot \eta^2 Du\, dx = -\int_\Omega \mathbf{A} \cdot 2\eta D\eta (u(x) - u(y))\, dx + \int_\Omega \mathbf{B}\eta^2 (u(x) - u(y))\, dx.$$

Using (4.40) and (4.44) we thus obtain

$$\begin{aligned}
\nu_1 \int_\Omega \eta^2 |Du|^2\, dx &\le \mu_1 \int_\Omega \eta^2\, dx + \mu \int_\Omega \eta^2 |u(x) - u(y)|(1 + |Du|)^2\, dx \\
&\quad + 2\mu_1 \int_\Omega |\eta D\eta|\, |u(x) - u(y)|(1 + |Du|)\, dx \\
&\le \mu_1 \int_\Omega \eta^2\, dx + 3\mu\omega(R) \int_\Omega \eta^2 (1 + |Du|^2)\, dx \\
&\quad + \mu_1 \omega(R) \int_\Omega |D\eta|^2\, dx + 3\mu_1 \omega(R) \int_\Omega \eta^2 (1 + |Du|^2)\, dx \\
&\le \mu_1 \int_\Omega \eta^2\, dx + 4(\mu + \mu_1)\omega(R) \int_\Omega \eta^2 (1 + |Du|^2)\, dx \\
&\quad + \mu_1 \omega(R) \int_\Omega |D\eta|^2\, dx,
\end{aligned}$$

where $\omega(R) = \sup\limits_{B_{2R}} |u(x) - u(y)|$. Hence if R is chosen small enough to ensure that

$$\omega(R) \le \frac{\nu_1}{8(\mu + \mu_1)},$$

we have

$$\int_\Omega \eta^2 |Du|^2\, dx \le C \int_\Omega (\eta^2 + \omega(R)|D\eta|^2)\, dx, \tag{4.45}$$

where $C = C(\mu, \mu_1, \nu_1)$. Replacing the function η in (4.45) by

$$\eta v^{(\beta+1)/2}, \quad \beta > 0,$$

where $v = |Du|^2$, we obtain the estimate

$$\begin{aligned}\int_\Omega \eta^2 |Du|^{2\beta+4}\,dx &\le C\int_\Omega \Big[\eta^2|Du|^{2\beta+2} + \omega(R)|D\eta\cdot v^{(\beta+1)/2} + \frac{\beta+1}{2}\eta v^{(\beta-1)/2}Dv|^2\Big]\,dx\\ &\le C\int_\Omega \Big[|Du|^{2(\beta+1)}(\eta^2+\omega(R)|D\eta|^2) + \frac{(\beta+1)^2}{4}\omega(R)\eta^2 v^{\beta-1}|Dv|^2\\ &\quad + \omega(R)|D\eta\cdot v^{(\beta+1)/2}(\beta+1)\eta v^{(\beta-1)/2}Dv|\Big]\,dx.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}&\int_\Omega \omega(R)|D\eta\cdot v^{(\beta+1)/2}(\beta+1)\eta v^{(\beta-1)/2}Dv|\,dx\\ &\qquad\le \frac{1}{2}\int_\Omega \omega(R)|D\eta|^2|Du|^{2\beta+2} + (\beta+1)^2\omega(R)\eta^2 v^{\beta-1}|Dv|^2\,dx.\end{aligned}$$

We thus obtain

$$\begin{aligned}&\int_\Omega \eta^2|Du|^{2\beta+4}\,dx\\ &\quad\le C\int_\Omega \Big[|Du|^{2(\beta+1)}(\eta^2+\omega(R)|D\eta|^2) + (\beta+1)^2\omega(R)\eta^2 v^{\beta-1}|Dv|^2\Big]\,dx. \qquad (4.46)\end{aligned}$$

To estimate the last term on the right hand side of (4.46), we choose in (4.41)

$$\zeta = \eta^2 v^\beta$$

that is,

$$\int_\Omega (\bar a^{ij}D_j v + 2D_k u f_k^i)(2\eta D_i\eta v^\beta + \eta^2\beta v^{\beta-1}D_i v)\,dx \le \frac{1}{2}\int_\Omega \eta^2 v^\beta \lambda^{-1}(f_k^i)^2\,dx$$

or, equivalently,

$$\begin{aligned}\int_\Omega \bar a^{ij}\eta^2\beta v^{\beta-1}D_i v D_j v\,dx &\le \int_\Omega \Big(-2\bar a^{ij}D_j v\cdot\eta D\eta v^\beta - 4\eta D\eta v^\beta\cdot D_k u f_k^i\\ &\quad - 2\eta^2\beta v^{\beta-1}D_i v\cdot D_k u f_k^i + \frac{1}{2}\eta^2 v^\beta\lambda^{-1}(f_k^i)^2\Big)\,dx.\end{aligned}$$

Using conditions (4.36), (4.38) and (4.40), we obtain

$$\begin{aligned}\beta\nu\int_\Omega \eta^2 v^{\beta-1}|Dv|^2\,dx &\le \int_\Omega \Big(2\mu\eta v^\beta|D\eta||Dv| + 4\eta v^\beta|D\eta||Du|\mu(1+|Du|)^2 \\ &\quad + 2\eta^2\beta v^{\beta-1}|Dv||Du|\mu(1+|Du|)^2 + \frac{1}{2}\eta^2 v^\beta\lambda^{-1}\mu^2(1+|Du|)^4\Big)\,dx \\ &\le C\int_\Omega \Big(\eta v^\beta|D\eta||Dv| + (1+|Du|)^3(\eta v^\beta|D\eta| + \beta\eta^2 v^{\beta-1}|Dv|) \\ &\quad + \eta^2(1+|Du|)^4 v^\beta\Big)\,dx, \end{aligned} \tag{4.47}$$

where $C = C(\mu,\nu)$. Hence, by Young's inequality, we have

$$\int_\Omega \eta v^\beta|D\eta||Dv|\,dx \le \frac{1}{2}\int_\Omega (\eta^2 v^{\beta-1}|Dv|^2 + v^{\beta+1}|D\eta|^2)\,dx \tag{4.48}$$

and

$$\begin{aligned}&\int_\Omega (1+|Du|)^3(\eta v^\beta|D\eta| + \beta\eta^2 v^{\beta-1}|Dv|)\,dx \\ &\qquad \le \frac{1}{2}\int_\Omega (1+|Du|)^3(v^\beta\eta^2 + v^\beta|D\eta|^2 + \beta\eta^2 v^{\beta-1}|Dv|^2 + \beta\eta^2 v^{\beta-1})\,dx. \end{aligned} \tag{4.49}$$

Putting together (4.47), (4.48) and (4.49), we thus obtain

$$\int_\Omega \eta^2 v^{\beta-1}|Dv|^2\,dx \le C\int_\Omega (1+|Du|)^{2\beta+2}[\eta^2(1+|Du|)^2 + |D\eta|^2]\,dx, \tag{4.50}$$

where $C = C(\mu,\nu,\beta)$. Consequently, by substituting sufficiently small $\omega(R)$ in (4.37) and using (4.46), we arrive at

$$\int_\Omega \eta^2(1+|Du|)^{2\beta+4}\,dx \le C\int_\Omega (\eta^2(1+|Du|)^{2\beta+2} + |D\eta|^2(1+|Du|)^{2\beta+2})\,dx,$$

where $C = C(\mu,\nu,\beta)$. Replacing η by $\eta^{\beta+2}$ and using Young's inequality, we then obtain

$$\int_\Omega [\eta(1+|Du|)]^{2\beta+4}\,dx \le C,$$

where $C = C(\mu, \nu, \mu_1, \nu_1, \beta, \sup |D\eta|)$. In particular, if $\eta \equiv 1$ on $B_R(y)$ and $|D\eta| \leq \dfrac{2}{R}$, it follows that, for $p \geq 1$,

$$\int_{B_R(y)} (1 + |Du|)^p \, dx \leq C, \tag{4.51}$$

where $C = C(\mu, \nu, \mu_1, \nu_1, p, R^{-1})$. Combining the estimates (4.42) and (4.51), we obtain for any ball $B_0 = B_{R_0}(y) \subset \Omega$ the estimate

$$|Du(y)| \leq C \tag{4.52}$$

where $C = C(n, \mu, \nu, \mu_1, \nu_1, \alpha, [u]_{\alpha,y})$, and

$$[u]_{\alpha,y} = \sup_{B_0} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \qquad 0 < \alpha < 1.$$

that is, the derivation of an interior gradient bound is reduced to the existence of an interior Hölder Estimate for u.

(iii) *A Hölder estimate for weak solutions of equation* (4.40). We write inequalities (4.44) together with the condition on B_ε in (4.40) in the form

$$\begin{aligned} &|\mathbf{A}_\varepsilon(x,p)| \leq a_0|p| + \chi, \\ &p \cdot \mathbf{A}_\varepsilon(x,p) \geq \nu_0 |p|^2 - \chi^2, \\ &|B_\varepsilon(x,p)| \leq b_0 |p|^2 + \chi^2, \end{aligned} \tag{4.53}$$

where

$$a_0 = \mu_1, \quad \nu_0 = \nu_1, \quad b_0 = \mu, \quad \chi = \sup_\Omega (\mu_1, \sqrt{\mu_1}).$$

are positive constants. The estimates of Theorems 4.19, 4.20 and 4.21 for uniformly elliptic divergence structure equations were substantially developed by [20] and [21].

Theorem 4.19 [11, p.379]. *Let $u \in C^1(\Omega)$ be a weak solution of $Q_\varepsilon u = 0$ in the domain Ω and suppose that Q_ε satisfies the structure conditions* (4.53). *Then for any ball $B_0 = B_{R_0}(y) \subset \Omega$*

and $R \leq R_0$, we have the estimate

$$\operatorname*{osc}_{B_R(y)} u \leq C(1 + R_0^{-\alpha} M_0) R^{\alpha},$$

where $C = C(n, a_0, b_0, \nu_0, \chi, R_0, M_0)$ and $\alpha = \alpha(n, a_0, b_0, \nu_0, M_0)$ are positive constants and $M_0 = \sup_{B_0} |u|$.

Combining Theorem 4.19 with estimate (4.52), we arrive at the following interior gradient estimate.

Theorem 4.20 [11, p.379]. *Let $u \in C^2(\Omega)$ satisfy $Q_\varepsilon u = 0$ in the domain Ω and the structure conditions (4.37), (4.38) and (4.40) are fulfilled. Then we have the estimate*

$$|Du(y)| \leq C,$$

for any $y \in \Omega$, where $C = C\Big(n, \mu, \nu, \sup_{\Omega} |\mathbf{A}(x,p)|, \dfrac{M_0}{d}\Big)$, and $M_0 = \sup_{B_d(y)} |u|$ and $d = \operatorname{dist}(y, \partial\Omega)$.

We conclude from the interior estimate and the boundary Lipschitz estimate the following global estimate.

Theorem 4.21 [11, p.380]. *Let $u \in C^2(\Omega) \cap C^0\bar{\Omega}$ satisfy $Q_\varepsilon u = 0$ in the bounded domain Ω and the structure conditions (4.37), (4.38) and (4.40) are fulfilled. Assume also that Ω satisfies a uniform exterior sphere condition that $u = \phi$ on $\partial\Omega$ for $\phi \in C^2(\bar{\Omega})$. Then we have the estimate*

$$\sup_{\Omega} |Du| \leq C$$

where $C = C(n, \mu, \nu, \sup_{\Omega} |\mathbf{A}(x,p)|, \partial\Omega, |\phi|_{2:\Omega})$.

This theorem provides the estimation of $\sup_{\Omega} |Du|$ in terms of $\sup_{\partial\Omega} |Du|$ and $\sup_{\Omega} |u|$. We note that the constant C is not guaranteed to be independent of ε.

4.11 Existence for Regularized Problems

In this section, we assemble the results from previous sections for the regularized equation in divergence form:

$$Q_\varepsilon u = \operatorname{div} \frac{Du}{\sqrt{\xi_\varepsilon(x,|Du|)}} + \frac{1}{2}\frac{\sigma q}{\zeta_\varepsilon(x,|Du|)} = 0,$$

where ξ_ε and ζ_ε are given by (4.10) and (4.11) respectively.

Recall taking the limit as $\varepsilon \to 0$ in the coefficients ξ_ε and ζ_ε for the case $|p|^2 < P$, we recover the ill-posed operator Q defined in $|Du|^2 < P$:

$$Qu = \operatorname{div} \frac{Du}{\sqrt{P - |Du|^2}} + \frac{1}{2}\frac{\sigma q}{P - |Du|^2} = 0.$$

Based on Theorem 4.11, Theorem 4.15, Theorem 4.16 and Theorem 4.21, we obtain the following estimate for the regularized operator:

Theorem 4.22. *Let $u_\varepsilon \in C^2(\overline{\Omega})$ satisfy $Q_\varepsilon u_\varepsilon = 0$ in Ω and $u_\varepsilon = \phi$ on $\partial\Omega$ where $\phi \in C^2(\overline{\Omega})$. Suppose that Ω satisfies a uniform exterior sphere condition, we have the estimate*

$$[Du_\varepsilon]_{\beta;\Omega} \le C_1 \sup_{\partial\Omega} |u_\varepsilon| + C_2 = C_1 \sup_{\partial\Omega}|\phi| + C_2$$

where

$$C_1 = C_1(n,\nu,\mu_1,\mu_2,\mu_3,\mu_4,\Omega), \quad C_2 = C_2(n, \sup_\Omega |\mathbf{A}(x,p)|, |\phi|_{2:\Omega}, \delta, \Omega),$$

$$\nu = \min\left\{\frac{\min(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\min(\lambda_1',\lambda_n')}{\lambda_1'^{\frac{1}{2}}(2\lambda_n')^{\frac{3}{4}}}\right\}, \quad \mu_1 = \max\left\{\frac{\max(\lambda_1,\lambda_n)}{\lambda_1^{\frac{3}{2}}}, \frac{\max(\lambda_1',\lambda_n')}{\lambda_1'^{\frac{1}{2}}(2\lambda_n - 2P)^{\frac{3}{4}}}\right\},$$

$$\mu_2 = \sup_{x\in\Omega}\left\{\sqrt{P},\ \frac{2}{\sqrt{P}} + \frac{|DP(x)|}{\sqrt{P(x)^3}} + \frac{q(x)}{2\sqrt{P(x)^3}}\right\},$$

$$\mu_3 = \sup_{x\in\Omega}\frac{|DP(x)|}{2P(x)} \quad \textit{and} \quad \mu_4 = \sup_{x\in\Omega}\left\{\frac{|DP(x)|}{2\sqrt{P(x)}} + \frac{q(x)}{2\sqrt{P(x)}}\right\}.$$

Taking the limit of as $\varepsilon \to 0$, we obtain that ν is bounded which is illustrated in Section 4.10, and that μ_2, μ_3 and μ_4 are independent of ε. We cannot guarantee that μ_1 is bounded as

$\varepsilon \to 0$ to establish the bound for C_1 defined in Theorem 4.22, but μ_1 is bounded for a fixed ε as in the following theorem.

Theorem 4.23. *Suppose that Ω satisfies a uniform exterior sphere condition and let $\varepsilon > 0$. Suppose that $u_\varepsilon \in C^2(\overline{\Omega})$ satisfies $Q_\varepsilon u_\varepsilon = 0$ in Ω and $u_\varepsilon = \phi$ on $\partial\Omega$ where $\phi \in C^2(\overline{\Omega})$. Then we have*

$$||u_\varepsilon||_{C^{1,\beta}(\bar{\Omega})} \leq M,$$

where $M = M(n, \Omega, P, q, \phi, \varepsilon)$.

Combing Theorem 4.3 and Theorem 4.23, we obtain the following existence theorem for the regularized problems.

Theorem 4.24. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ and satisfies a uniform exterior sphere condition and $\partial\Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}$. Then it follows that the Dirichlet problems, $Q_\varepsilon u_\varepsilon = 0$ in Ω, $u_\varepsilon = \phi$ on $\partial\Omega$ is solvable in $C^{2,\alpha}(\bar{\Omega})$.*

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Appendices

Appendix A

Dini's formulas for half-plane and the ball

First we derive the Dini's formula for the half-plane [5]. Let Ω be a bounded domain in $\mathbb{R}^2$. Suppose that the function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then the following Riemann-Green formula holds,

$$u(x) = -\int_{\Omega} G(y,x)\Delta u(y)\, dy + \int_{\partial\Omega} \Big(G(y,x)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial G(y,x)}{\partial \nu}\Big)\, dS(y), \qquad \text{(A.1)}$$

where $G(y,x) = \Phi(x-y) + \tilde{g}(y)$, $x \in \Omega$, $y \in \bar{\Omega}$, $x \neq y$, $\tilde{g}(y)$ is an arbitrary harmonic function in Ω and

$$\Phi(x-y) = -\frac{1}{2\pi}\log(|x-y|).$$

Now take Ω to be the upper half-plane $\mathbb{R}^2_+$. For $x = (x_1, x_2)$, define the reflection of x by $x^\star = (x_1, -x_2)$. Then the function

$$G(y,x) = -\frac{1}{2\pi}\Big[\log(|x-y|) + \log(|x^\star - y|)\Big]$$

is the Green's function for the Neumann problem on the half-plane.

The outward normal ν to $\partial\mathbb{R}^2_+$ is $\nu = (0,-1)$ and the normal derivative of G is $\dfrac{\partial G}{\partial \nu} = -\dfrac{\partial G}{\partial y_2} = 0$. Clearly, $G(y,x) = -\dfrac{1}{\pi}\log(|x-y|)$ for $y = (y_1, 0)$.

Therefore $(A.1)$ gives a representation of solution of the problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2_+, \qquad \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial\mathbb{R}^2_+,$$

in the form

$$u(x)=\int_{\partial\mathbb{R}^2_+}G(y,x)h(y)dS(y)=-\frac{1}{\pi}\int_{\mathbb{R}}\log(|x-y|)h(y_1)dy_1. \tag{A.2}$$

This is Dini's formula for the half-plane.

Next we consider the Dini's formula for the ball [10] [13]. Suppose that v is a solution of Laplace's equation $\Delta v=0$ with Neumann boundary condition on a ball $B_r(0)$ with the center at the origin and radius r, that is

$$\begin{cases}\Delta v=0 & \text{in } B_r(0),\\ \dfrac{\partial v}{\partial \nu}=h & \text{on } \partial B_r(0).\end{cases}$$

Using polar coordinates (r,θ) rather than rectangular coordinates and Fourier series, we construct a harmonic function v on $B_r(0)$ satisfying the Neumann condition.

Suppose the function $h(r,\theta)$ has the Fourier series expansion

$$h(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\theta)+\sum_{n=1}^{\infty}b_n\sin(n\theta),$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi}h(r,\theta)d\theta=0,$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}h(r,\theta)\cos(n\theta)d\theta \quad n\geq 1,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}h(r,\theta)\sin(n\theta)d\theta \quad n\geq 1.$$

Let $z=\rho e^{i\theta}$ be an arbitrary complex number where $0\leq\rho\leq r$ and $0\leq\theta\leq 2\pi$, then $z^n=\rho^n e^{in\theta}=\rho^n\cos(n\theta)+i\rho^n\sin(n\theta)$. Since the real and imaginary parts of an analytic function are harmonic, $\rho^n\cos(n\theta)$ and $\rho^n\sin(n\theta)$ are harmonic on $\mathbb{R}^2$.

It can be shown that

$$v(\rho,\theta)=v_0+\sum_{n=1}^{\infty}\alpha_n\rho^n\cos(n\theta)+\sum_{n=1}^{\infty}\beta_n\rho^n\sin(n\theta)$$

is a harmonic function.

Moreover,

$$\frac{\partial}{\partial \nu}v(r,\theta) = \frac{\partial}{\partial \rho}v(r,\theta) = \sum_{n=1}^{\infty} n\alpha_n r^{n-1}\cos(n\theta) + \sum_{n=1}^{\infty} n\beta_n r^{n-1}\sin(n\theta).$$

Choosing $\alpha_n = \dfrac{a_n}{nr^{n-1}}$, and $\beta_n = \dfrac{b_n}{nr^{n-1}}$, it appears that Neumann boundary condition $\dfrac{\partial v}{\partial \nu} = h$ is satisfied. Then harmonic function v can be rewritten as

$$v(\rho,\theta) = v_0 + \sum_{n=1}^{\infty} \frac{\rho^n}{nr^{n-1}}[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

Since

$$\begin{aligned} a_n\cos(n\theta) + b_n\sin(n\theta) &= \frac{1}{\pi}\int_0^{2\pi}\cos(n\phi)\cos(n\theta)h(r,\phi)d\phi \\ &+ \frac{1}{\pi}\int_0^{2\pi}\sin(n\phi)\sin(n\theta)h(r,\phi)d\phi \\ &= \frac{1}{\pi}\int_0^{2\pi}\cos[n(\phi-\theta)]h(r,\phi)d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}[e^{in(\phi-\theta)} + e^{-in(\phi-\theta)}]h(r,\phi)d\phi \end{aligned}$$

and it implies that

$$\begin{aligned} v(\rho,\theta) &= v_0 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\rho^n}{nr^{n-1}}\int_0^{2\pi}[e^{in(\phi-\theta)} + e^{-in(\phi-\theta)}]h(r,\phi)d\phi \\ &= v_0 + \frac{r}{2\pi}\int_0^{2\pi}\Big[\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\rho e^{in(\phi-\theta)}}{r}\Big)^n + \sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\rho e^{-in(\phi-\theta)}}{r}\Big)^n\Big]h(r,\phi)d\phi \end{aligned}$$

Using the fact that $\log(1-x) = -\sum_{n=1}^{\infty}\dfrac{x^n}{n}$, we have

$$\begin{aligned} v(\rho,\theta) &= v_0 + \frac{r}{2\pi}\int_0^{2\pi}\log\Big(\frac{r^2}{r^2 - 2\rho r\cos(\phi-\theta) + \rho^2}\Big)h(r,\phi)d\phi \\ &= v_0 - \frac{r}{2\pi}\int_0^{2\pi}\log(r^2 - 2\rho r\cos(\phi-\theta) + \rho^2)h(r,\phi)d\phi \qquad \text{(A.3)} \end{aligned}$$

This is Dini's formula for the ball in polar coordinates .This formula in rectangular coordinates takes the form

$$v(x) = v_0 - \frac{r}{2\pi} \int_{\partial B_1(0)} \log(|x-y|)h(y)\, dS(y). \tag{A.4}$$

where $x = (x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$ is an arbitrary point in the disk $(0 \leq \rho \leq r)$ and $y = (y_1, y_2) = (r\cos t, r\sin t)$ is a point on the circle.

Appendix B

Periodic Cubic Spline Interpolation

This is a variation of the cubic spline interpolation introduced in [1]. Suppose we are given data values (t_i, z_i), $i = 0, 1, \ldots n$, where

$$-\pi = t_0 < t_1 < \cdots < t_{n-1} < t_n = \pi, \quad \text{and} \quad z_i = f(t_i)$$

for some function f that may not be explicitly available and we are looking for a spline function $v(x)$ that satisfies $v(t_i) = z_i$ for all $i = 0, 1, \ldots, n$.

Suppose $s_i(t)$ is a cubic polynomial interpolation piece in each of the subintervals $[t_i, t_{i+1}]$. Then $s_i(t)$ can be patched together to form a continuous global spline $v(t)$ which satisfied $v(t) = s_i(t)$, $t_i \leq t \leq t_{i+1}$, $i = 0, 1, \cdots n-1$, where

$$s_i(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3, \tag{B.1a}$$

$$s_i'(t) = b_i + 2c_i(t - t_i) + 3d_i(t - t_i)^2, \tag{B.1b}$$

$$s_i''(t) = 2c_i + 6d_i(t - t_i). \tag{B.1c}$$

Moreover, we supplement the conditions to be satisfied by any cubic spline

$$s_i(t_i) = f(t_i), \quad i = 0, \ldots n-1, \tag{B.2a}$$

$$s_i(t_{i+1}) = f(t_{i+1}), \quad i = 0, \ldots n-1, \tag{B.2b}$$

$$s_i'(t_{i+1}) = s_{i+1}'(t_{i+1}), \quad i = 0, \ldots n-2, \tag{B.2c}$$

$$s_i''(t_{i+1}) = s_{i+1}''(t_{i+1}), \quad i = 0, \dots n-2, \tag{B.2d}$$

with periodic boundary conditions $s_0'(t_0) = s_{n-1}'(t_n)$ and $s_0''(t_0) = s_{n-1}''(t_n)$.

The conditions $(B.2a)$ immediately determine

$$a_i = f(t_i), \quad i = 0, \dots, n-1.$$

Denote $h_i = t_{i+1} - t_i$, $i = 0, \dots, n-1$, then conditions $(B.2b)$ give

$$a_i + h_i b_i + h_i^2 c_i + h_i^3 d_i = f(t_{i+1}).$$

Plugging in the values of a_i and dividing by h_i, we have

$$b_i + h_i c_i + h_i^2 d_i = \frac{f(t_{i+1}) - f(t_i)}{h} = f[t_i, t_{i+1}], \quad i = 0, \dots, n-1. \tag{B.3}$$

From $(B.1b)$ and $(B.2c)$, we get the condition

$$b_i + 2h_i c_i + 3h_i^2 d_i = b_{i+1}, \quad i = 0, \dots, n-2.$$

Likewise, from $(B.1c)$ and $(B.2d)$, we get

$$c_i + 3h_i d_i = c_{i+1}, \quad i = 0, \dots, n-2.$$

Note that we can extend the above two conditions to the case $i = n-1$ by defining $b_n = v'(t_n) = s_0'(t_0)$ and $c_n = \frac{1}{2}v''(t_n) = \frac{1}{2}s_0''(t_0)$.

We can obtain

$$d_i = \frac{c_{i+1} - c_i}{3h_i}, \quad i = 0, \dots, n-1.$$

and then yields

$$b_i = f[t_i, t_{i+1}] - \frac{h_i}{3}(2c_i + c_{i+1}) \quad i = 0, \dots, n-1.$$

Eliminating b_i and d_i and rearranging the terms, equation $(B.3)$ reads

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3(f[t_i, t_{i+1}] - f[t_{i-1}, t_i]), \quad i = 0, \ldots, n-1. \tag{B.4}$$

The periodic condition $s''_0(t_0) = s''_{n-1}(t_n)$ implies that

$$c_0 = c_{n-1} + 3h_{n-1}d_{n-1} = c_{n-1} + 3h_{n-1}(\frac{c_n - c_{n-1}}{3h_{n-1}}) = c_n.$$

The other periodic condition $s'_0(t_0) = s'_{n-1}(t_n)$ gives us

$$b_0 = b_{n-1} + 2h_{n-1}c_{n-1} + 3h_{n-1}^2 d_{n-1}$$

and is equivalent to

$$2c_0(h_{n-1} + h_0) + h_0 c_1 + h_{n-1}c_{n-1} = 3(f[t_0, t_1] - f[t_{n-1}, t_n]). \tag{B.5}$$

We write equations $(B.4)$ and $(B.5)$ in matrix form to obtain

$$Q \cdot c = \phi,$$

where

$$Q = \begin{pmatrix} 2(h_{n-1}+h_0) & h_0 & & & & h_{n-1} \\ h_0 & 2(h_0+h_1) & h1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ h_{n-1} & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix},$$

$c = (c_0, c_1, \cdots, c_{n-2}, c_{n-1})^T$ and $\psi = (\phi_0, \phi_1, \cdots, \phi_{n-2}, \phi_{n-1})^T$,and where ψ_i is a shorthand for

$$\psi_0 = 3(f[t_0, t_1] - f[t_{n-1}, t_n]),$$

$$\psi_i = 3(f[t_i, t_{i+1}] - f[t_{i-1}, t_i]), \quad i = 1, \ldots, n-1.$$

Appendix C

Legendre Transfrom

The general Legendre transform is outlined in [9]. Here we apply the transform to the following special case.

Equation (4.2) in $\mathbb{R}^2$ can be rewritten as

$$\Big(P - u_{x_1}^2 - u_{x_2}^2\Big)u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + \Big(P - u_{x_1}^2 - u_{x_2}^2\Big)u_{x_2x_2} \\ - \frac{1}{2}P_{x_1}u_{x_1} - \frac{1}{2}P_{x_2}u_{x_2} + \frac{q}{2}\sigma\sqrt{P - u_{x_1}^2 - u_{x_2}^2} = 0. \quad \text{(C.1)}$$

Assume that, in region $\Omega \subset \mathbb{R}^2$, we can invert the relations

$$p_1 = u_{x_1}(x_1, x_2), \quad p_2 = u_{x_2}(x_1, x_2),$$

to solve for

$$x_1 = x_1(p_1, p_2), \quad x_2 = x_2(p_1, p_2).$$

By Hodograph transform, we get

$$\begin{cases} u_{x_1x_1} = Jx_{2,p_2}, \;\; u_{x_1x_2} = -Jx_{1,p_2} \\ u_{x_2x_1} = -Jx_{2,p_1}, \;\; u_{x_2x_2} = Jx_{1,p_1} \end{cases}$$

as long as $J = \dfrac{\partial(u_{x_1}, u_{x_2})}{\partial(x_1, x_2)} = \det D^2 u = u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2 \neq 0.$

Define

$$v(p) = \mathbf{x} \cdot p - u(\mathbf{x}(p)),$$

where $\mathbf{x} = (x_1, x_2)$ and $p = (p_1, p_2)$. We then arrive at

$$v_{p_1} = x_1 + p_1 x_{1,p_1} + p_2 x_{2,p_1} - u_{x_1} x_{1,p_1} - u_{x_2} x_{2,p_1} = x_1,$$

$$v_{p_2} = p_1 x_{1,p_2} + x_2 + p_2 x_{2,p_2} - u_{x_1} x_{1,p_2} - u_{x_2} x_{2,p_2} = x_2,$$

that is,

$$x_{2,p_2} = v_{p_2 p_2}, \quad x_{1,p_2} = v_{p_1 p_2} \text{ and } x_{1,p_1} = v_{p_1 p_1}.$$

We discover that

$$\begin{cases} u_{x_1 x_1} = J v_{p_2 p_2} \\ u_{x_1 x_2} = -J v_{p_1 p_2} \\ u_{x_2 x_2} = J v_{p_1 p_1} \end{cases}$$

Upon substituting into the equation (4.2), we derive for v the equation

$$\begin{aligned} \Big(P - p_1^2 - p_2^2\Big) J v_{p_2 p_2} - 2 p_1 p_2 J v_{p_1 p_2} + \Big(P - p_1^2 - p_2^2\Big) J v_{p_1 p_1} \\ - \frac{1}{2} P_{v_{p_1}} p_1 - \frac{1}{2} P_{v_{p_2}} p_2 + \frac{q}{2} \sigma \sqrt{P - p_1^2 - p_2^2} = 0. \quad \text{(C.2)} \end{aligned}$$

The transformed equation $(C.2)$ is non-linear in the new variables p_1, p_2. The power of the Legendre transform is to obtain a linear equation from a non-linear one. This technique does not seem to attain a simple equation out of equation $(C.1)$. For this reason, we do not pursue this direction any further.