The Intersection Problem for Maximum Packings of the Complete Graph with 4-Cycles

by

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Abstract

A maximum packing of K_n with 4-cycles may leave some edges of the graph unpacked. The size of the leave is determined by the congruence class of n modulo 8. When $n \equiv 1 \pmod 8$ the packing coincides with a 4-cycle system of K_n. Billington (J. Combin. Des. 1 (1993) 435-452) has solved the intersection problem for 4-cycle systems. In this dissertation we complete the solution for maximum packings with 4-cycles by solving the cases where $n \not\equiv 1 \pmod 8$.

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Chapter 1

Introduction

The objective of this dissertation is to solve a problem that involves packing the complete graph K_n with 4-cycles. If we label the vertices of K_6 as in Figure 1.1, we can pack the graph with a 4-cycle, a graph isomorphic to C_4, by selecting the 4-cycle (1245). We can pack a second 4-cycle by selecting (2356), since this cycle has no edges in common with the first cycle we have chosen. Since (1346) has no edges in common with either of the other two cycles, we can pack it as well.

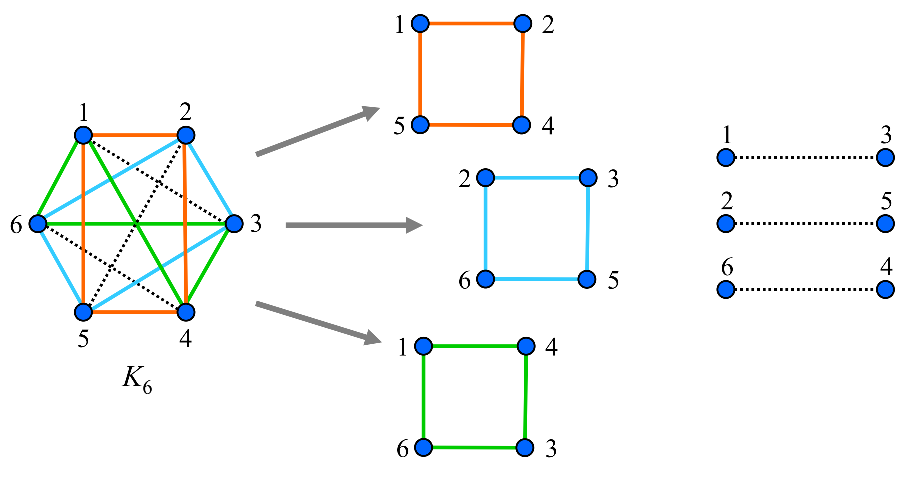


Figure 1.1: Packing with 4-cycles

A collection $\{H_1, H_2, \ldots, H_t\}$ of t edge-disjoint subgraphs of K_n, each isomorphic to C_4, is a *packing* of K_n with 4-cycles. If there is no such collection of cardinality $t+1$, then the collection is a *maximum packing*. For any packing which leaves some edges unpacked, the graph consisting of these edges and their endpoints is called the *leave* of the packing. The collection

$$P = \{(1245), (2356), (1436)\}$$

that we selected above is a packing of K_6 with 4-cycles. Its leave contains the three unpacked edges $\{1, 3\}, \{2, 5\}$, and $\{6, 4\}$. Since the graph has 15 edges, we cannot pack it with more than three edge-disjoint 4-cycles. This implies that P is a maximum packing.

If we look, we can find other ways to pack K_6 with 4-cycles. In Figure 1.2 the cycles of the packing P along with those of two other packings S and T are highlighted. Since each of these consists of three 4-cycles, each is a maximum packing. Let us compare these three

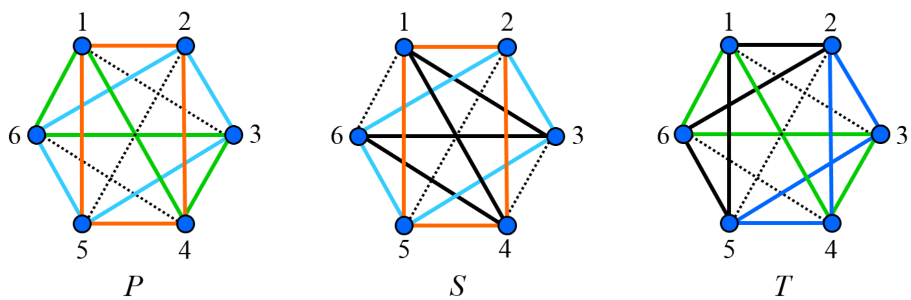


Figure 1.2: Some packings of K_6

maximum packings. They are

$$P = \{(1245), (2356), (1436)\}$$

$$S = \{(1245), (2356), (1364)\}$$

$$T = \{(1265), (2354), (1436)\}$$

We see that

$$P \cap S = \{(1245), (2356)\}, \quad P \cap T = \{(1436)\} \quad \text{and} \quad S \cap T = \varnothing.$$

This gives us a pair of maximum packings having two common cycles, a pair having one common cycle and another pair with no common cycles. Selecting P twice gives us a pair with three common cycles. Thus for $k = 0$, 1, 2 or 3, there is a pair of maximum packings of K_6 having k cycles in common. We have answered the following question.

> *For which integers k do there exists maximum packings P_1 and P_2 of K_6 such that $|P_1 \cap P_2| = k$?*

This is an instance of what is known as the *intersection problem.* The goal is to find the possible intersection sizes of two combinatorial structures based on the same set. These possible sizes are called *intersection values* or *intersection numbers.* In the problem above, the intersection values for maximum packings of K_6 with 4-cycles are $k = 0$, 1, 2 or 3. These values form the *intersection set* or *intersection spectrum* of the packing. For a maximum packing of K_n with 4-cycles, we let $I(n)$ denote the intersection set. So $I(6) = \{0, 1, 2, 3\}$.

Elizabeth Billington's [2] 1993 survey covers many instances of the intersection problem. One instance which will be of interest in this work involves a special type of packing. A *decomposition* of a graph G is a collection of its subgraphs such that each edge of G is in exactly one of the subgraphs. An *m-cycle system* of K_n is a decomposition of K_n where each subgraph of the decomposition is isomorphic to C_m. So an m-cycle system is a maximum packing with m-cycles that uses all edges of the graph, giving an empty leave. The intersection problem for this design is to find the values k for which there are two m-cycle systems of K_n having k common cycles. If $m = 3$, the design is equivalent to a Steiner triple system. Curt Lindner and Alex Rosa [3] solved this intersection problem in 1975. C.K. Fu [4] solved the $m = 5$ case in 1987, while Elizabeth Billington [1] solved the cases $m = 4, 6, 7, 8$ and 9 in 1993.

In the example above, we found all intersection values for maximum packings of K_6 with 4-cycles. We now generalize to K_n. If can we pack K_n with 4-cycles so that the leave is empty, then the packing is certainly a maximum packing. The packing is also a 4-cycle system of K_n. By a result of A. Kotzig [6], this is possible if and only if $n \equiv 1 \pmod 8$. For these values of n, we can get the intersection values from Billington's work. Let p_n denote the number of 4-cycles in a maximum packing of K_n. If two different 4-cycle systems of K_n have $p_n - 1$ cycles in common, then the four edges not included in these cycles must be the same in both systems. This implies then that the two systems have p_n cycles in common. We can conclude from this that the p_{n-1} is not an intersection value and thus $I(n) \subseteq \{0, 1, \ldots, p_n - 2, p_n\}$. Billington's result is that equality in fact holds. This gives a partial solution to the intersection problem for maximum packings of the complete graph with 4-cycles:

> *For which integers k do there exists maximum packings P_1 and P_2 of K_n with 4-cycles such that $|P_1 \cap P_2| = k$?*

The work of this dissertation to find a complete solution. This requires that we find the intersection spectrum $I(n)$ for all $n \not\equiv 1 (\text{mod } 8)$.

Chapter 2

Small Cases

To completely solve the intersection problem for maximum packings of K_n with 4-cycles, it is necessary to have at least one construction that can provide a solution for infinitely many values of n. Such constructions will come in later chapters. Here we address the cases where $n \leq 8$. The solutions we obtain will enable us to develop general constructions. Throughout this dissertation, figures depicting packings with 4-cycles will indicate leave edges with dotted line segments.

2.1 The Cases $n \leq 5$

For $n = 1, 2$ or 3, the graph K_n, which is shown in Figure 2.1, has less than 4 edges and therefore no 4-cycle. A maximum packing with 4-cycles has size $p_n = 0$. The intersection

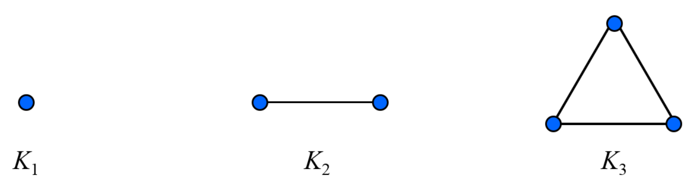


Figure 2.1: The complete graphs that have no 4-cycles

spectra for these three case are

$$I(1) = I(2) = I(3) = \{0\}$$

Now we consider K_4 which has six edges. A maximum packing contains a single cycle. From the two packings indicated in Figure 2.2, we see that is possible for two packings to be either the same or disjoint. Hence $I(4) = \{0, 1\}$.

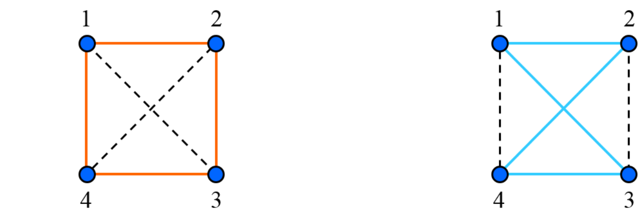


Figure 2.2: Packings $P = \{(1234)\}$ and $S = \{(1243)\}$

In the case of K_5, there are ten edges. Packing a 4-cycle as in Figure 2.3 leaves four edges incident with the same vertex, namely the vertex labelled 1. A second 4-cycle must pass through this vertex, but only a 3-cycle can be formed this way. Therefore the two packings shown in the figure are maximum packings. We conclude that $I(5) = \{0, 1\}$.

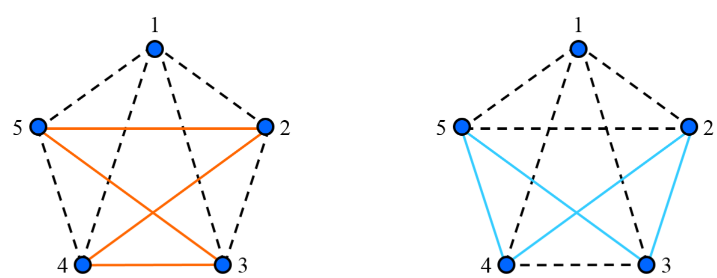


Figure 2.3: Packings $P = \{(5243)\}$ and $S = \{(5324)\}$

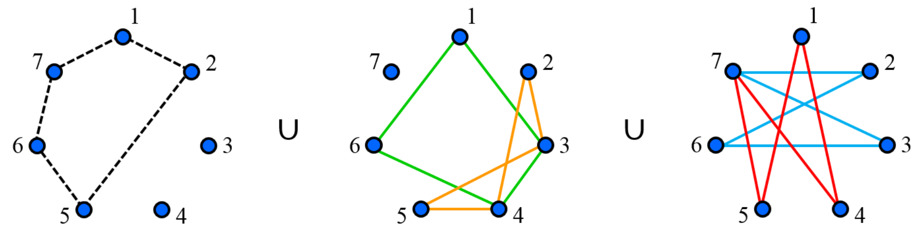
2.2 The Case $n = 7$

In the complete graph K_7 there are $7 \cdot 6/2 = 21$ edges. This implies that a packing must have a leave L where $|E(L)| \equiv 1 \pmod 4$. Each vertex of K_7 has degree 6 and a packing uses an even number of these edges. Therefore any vertex incident with leave edges must be incident with an even number of leave edges. This condition requires there to be at least two leave edges. Hence $|E(L)| \geq 5$. We now know that that any packing using 16 edges is a maximum one. In Figure 2.4 there are five such packings represented. Each consists of four 4-cycles and is distinct from the other others. Comparing these packings we obtain the following intersection values:

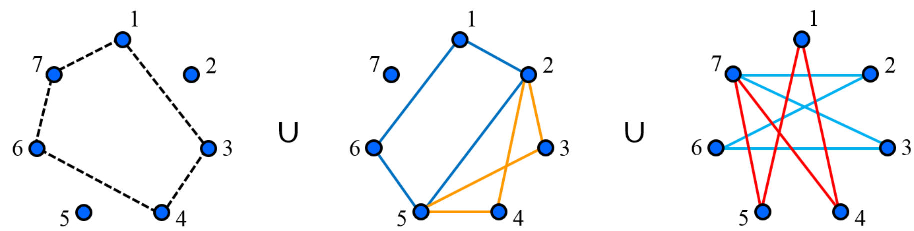
$$|P_1 \cap P_2| = |\{(2354), (1475), (7263)\}| = 3 \qquad |P_1 \cap P_3| = |\{(1346), (2354)\}| = 2$$

$$|P_2 \cap P_3| = |\{(2354)\}| = 1 \qquad |P_2 \cap P_4| = |\varnothing| = 0$$

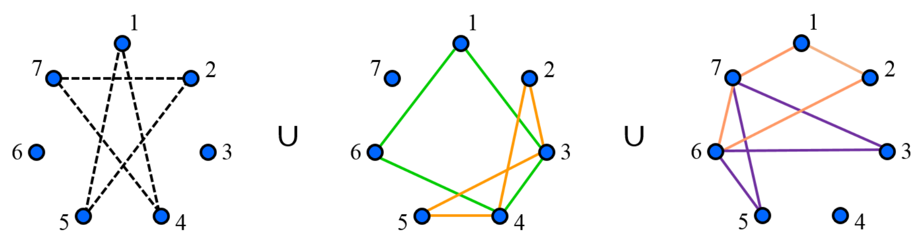
The spectrum is $I(7) = \{0, 1, 2, 3, 4\}$.



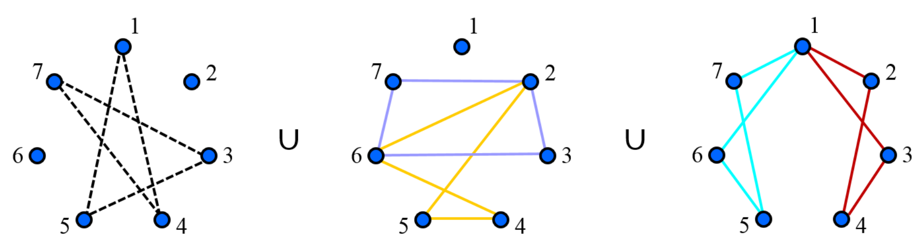
$P_1 = \{(1346), (2354), (1475), (7263)\}$



$P_2 = \{(1256), (2354), (1475), (7263)\}$



$P_3 = \{(1346), (2354), (7365), (7126)\}$



$P_4 = \{(7263), (6245), (1657), (1342)\}$

Figure 2.4: Maximum packings of K_7

2.3 The Cases $n = 6$ and $n = 8$

The examples in the introduction establish that $I(6) = \{0, 1, 2, 3\}$. Here we show how to arrive at this result by a more systematic method. The construction we demonstrate has the benefit that it readily extends to obtain $I(8)$. The graph K_6 contains a subgraph that is isomorphic to the complete bipartite graph $K_{2,4}$ while K_8 has a subgraph isomorpic to $K_{2,6}$. As we will show, each of these bipartite graphs can be decomposed into 4-cycles. We denote by $I(m, n)$, the intersection spectrum of a maximum packing of a complete bipartite graph $K_{m,n}$ into 4-cycles. This graph has mn edges. Suppose that it is possible to decompose $K_{m,n}$ into 4-cycles. Then a decomposition consists of $mn/4$ cycles. If two 4-cycle decompositions have $mn - 1$ cycles in common, then the remaining cycle most be the same in both decompositions. This implies that

$$I(m, n) \subseteq \{0, 1, \ldots, mn/4 - 2, mn/4\}$$

In this section we will use the spectra $I(2, 4)$ and $I(2, 6)$.

Lemma 2.1. $I(2, 4) = \{0, 2\}$.

Proof. Let $\{a, b\}$ and $\{1, 2, 3, 4\}$ be the independent sets of $K_{2,4}$, which has 8 edges. A 4-cycle decomposition must contain exactly 2 cycles. The intersection spectrum cannot contain 1. The sets

$$D_1 = \{(a, 1, b, 2), (a, 3, b, 4)\} \text{ and } D_2 = \{(a, 1, b, 3), (a, 2, b, 4)\}$$

are disjoint decompositions of the graph into 4-cycles. See Figure 2.5. □

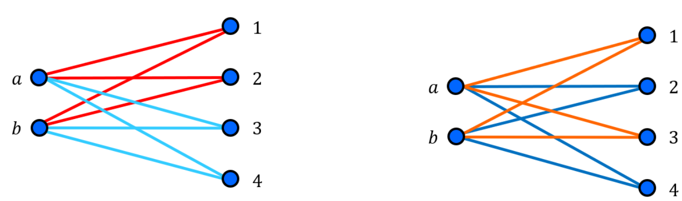


Figure 2.5: Disjoint decompositions of $K_{2,4}$

Lemma 2.2. $I(2,6) = \{0,1,3\}$.

Proof. Let $\{a,b\}$ and $\{1,2,3,4,5,6\}$ be the independent sets of $K_{2,6}$. There are 12 edges and so a 4-cycle decomposition has exactly 3 cycles. This implies that 2 is not in the spectrum. Each of the following sets is a 4-cycle decomposition of the graph. See Figure 2.6.

$$D_1 = \{(a,1,b,2),(a,3,b,4),(a,5,b,6)\}$$

$$D_2 = \{(a,1,b,2),(a,3,b,5),(a,4,b,6)\}$$

$$D_3 = \{(a,1,b,4),(a,3,b,5),(a,2,b,6)\}$$

Intersecting, we have $|D_1 \cap D_2| = 1$ and $|D_1 \cap D_3| = 0$. □

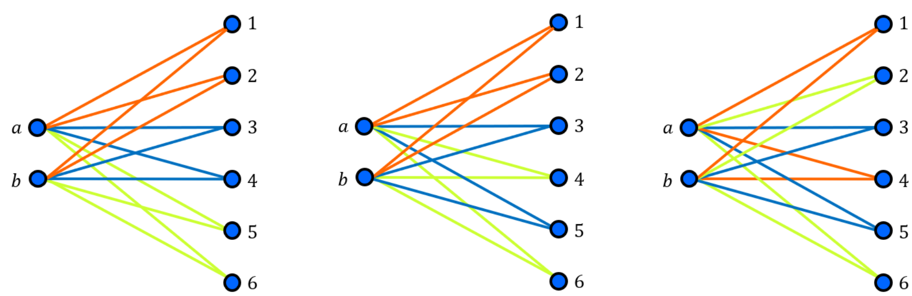


Figure 2.6: Three decompositions of $K_{2,6}$

The Bipartite Construction for K_6

Label the vertices of K_6 with the symbols $\{1,1',2,2',3,3'\}$. Select 4-cycles as follows.

1. Let B be a 4-cycle decomposition of $K_{2,4}$ with independent sets $\{1,1'\}$ and $\{2,2',3,3'\}$.

2. Let $C = \{(w,x,y,z)\}$ where (w,x,y,z) is a 4-cycle of K_4 with vertices $\{2,2',3,3'\}$.

Since the independent set $\{2,2',3,3'\}$ in Step 1 is used as the vertex set in Step 2, the cycle chosen in Step 2 has no edges in common with any cycle chosen in Step 1. Therefore $B \cup C$ consists of three edge-disjoint 4-cycles. Since K_6 has 15 edges, $B \cup C$ is a maximum packing.

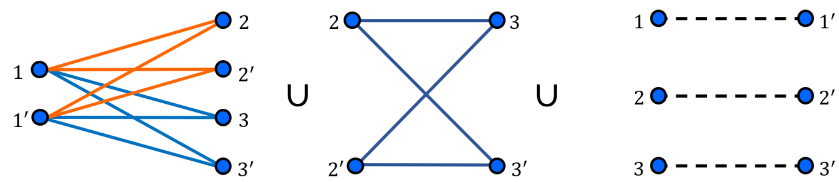


Figure 2.7: The Bipartite Construction for K_6

Figure 2.7 shows an example selection

$$B = \{(1,2,1',2'),(1,3,1',3)\} \quad C = \{(2,3,2',3')\}$$

that leaves the edges $\{1,1'\}$, $\{2,2'\}$ and $\{3,3'\}$ unpacked. The union of the three edge-disjoint subgraphs in the figure is K_6. To determine the spectrum for K_6, let $s \in \{0,2\}$ and let $t \in \{0,1\}$. By Lemma 2.1, we can reselect B so that there are s cycles in common with the first selection. Since $I(4) = \{0,1\}$, we may reselect C so that there are t cycles in common with the first choice. The spectrum therefore consists of the possible values of $s+t$. These values appear in Table 2.1. We see that $I(6) = \{0,1,2,3\}$.

$s \in \{0,2\}$	$t \in \{0,1\}$	$s+t$
0	0	0
0	1	1
2	0	2
2	1	3

Table 2.1: Intersection values $s+t$

The Bipartite Construction for K_8

Label the vertices of K_8 with the symbols $\{1,1',2,2',3,3',4,4'\}$. Select 4-cycles as follows.

1. Let A be a 4-cycle decomposition of $K_{2,6}$ with independent sets $\{1,1'\}$ and $\{2,2',3,3',4,4'\}$.

2. Remove the vertices 1 and $1'$. On the remaining graph, apply the Bipartite Construction for K_6 to select the sets B and C.

The set A consists of three 4-cycles. The vertices of the independent set $\{2, 2', 3, 3', 4, 4'\}$ in Step 1 remain after the vertex removal in Step 2. Consequently, the cycles chosen in Step 2 use different edges than those of the set A. This makes $A \cup B \cup C$ a collection of six edge-disjoint 4-cycles. Since every vertex of K_8 has degree seven and packed cycles must use two incident edges of a vertex, any packing must leave some edges. There are 28 edges in K_8, therefore any 4-cycle packing must contain less than seven cycles. This implies that $A \cup B \cup C$ is a maximum packing with 4-cycles.

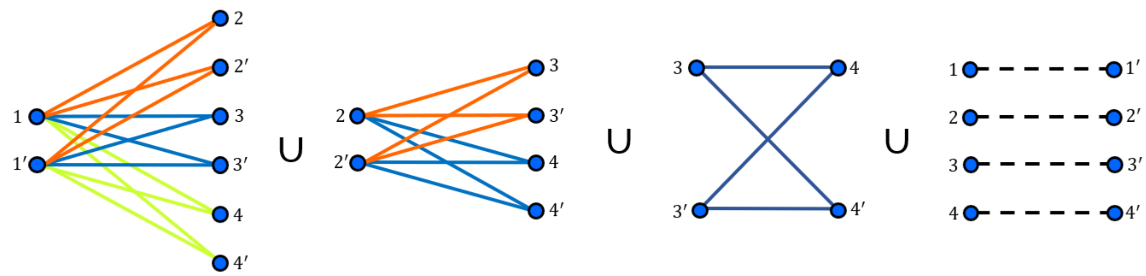


Figure 2.8: The Bipartite Construction for K_8

In Figure 2.8 we see the construction with the selection

$$A = \{(1, 2, 1', 2'), (1, 3, 1', 3'), (1, 4, 1', 4')\} \quad B = \{(2, 3, 2', 3), (2, 4, 2', 4')\} \quad C = \{(3, 4, 3', 4')\}$$

which leaves the edges $\{1, 1'\}$, $\{2, 2'\}$, $\{3, 3'\}$, and $\{4, 4'\}$. Turning our attention to the intersection spectrum for K_8, we consider how we may reselect the sets A, B and C. Let $p \in \{0, 1, 3\}$ and let $q \in \{0, 1, 2, 3\}$. By Lemma 2.2, we can reselect A so that there are p cycles in common with the first selection. Since $I(6) = \{0, 1, 2, 3\}$, we may reselect B and C so that $B \cup C$ has q cycles that also resulted from the original selection of B and C. Table 2.2 shows the possible intersection values $p + q$. By the table, $I(8) = \{0, 1, 2, 3, 4, 5, 6\}$.

$p \in \{0, 1, 3\}$	$q \in \{0, 1, 2, 3\}$	$p + q$
0	0	0
0	1	1
0	2	2
0	3	3
1	3	4
3	2	5
3	3	6

Table 2.2: Intersection values $p + q$

2.4 Summary

For $n \leq 8$, we have solved the intersection problem for maximum packings of K_n with 4-cycles by determining $I(n)$. Also we have found the the size p_n of maximum packing in each of these cases. Combining the results of this section gives the following.

Theorem 2.3. *For* $n = 1, 2, \ldots 8$, $I(n) = \{0, 1, \ldots, p_n\}$, *and*

$$p_n = \begin{cases} 0 & \text{if } n = 1, 2 \text{ or } 3 \\ 1 & \text{if } n = 4 \text{ or } 5 \\ 3 & \text{if } n = 6 \\ 4 & \text{if } n = 7 \\ 6 & \text{if } n = 8 \end{cases}$$

Chapter 3

Complete Bipartite Graphs and 4-cycle Systems

We will continue to use intersection spectra of complete bipartite graphs to solve larger cases of the intersection problem for maximum packings of K_n with 4-cycles. It will also be useful to incorporate Billington's[1] solution to the intersection problem for 4-cycle systems.

3.1 The Intersection Spectrum for $K_{m,n}$

We have seen that it is possible to decompose $K_{2,4}$ into 4-cycles. Furthermore, the decomposition obtained forms part of a maximum packing of K_6 with 4-cycles. Similarly, we used a 4-cycle decomposition of $K_{2,6}$ in the construction of a maximum packing of K_8. We would like to know which complete bipartite graphs decompose into 4-cycles. In their work on packing K_n with 4-cycles, J. Schönheim and A. Bialostocki's [5] showed that $K_{m,n}$ decomposes into 4-cycles if and only if m and n are even. We now turn our attention to $I(m,n)$, the intersection spectrum for a maximum packing of $K_{m,n}$ with 4-cycles. As we have noted in Section 2.3, whenever $K_{m,n}$ decomposes into 4-cycles

$$I(m,n) \subseteq \{0, 1, \ldots, mn/4 - 2, mn/4\}$$

In this section, we will prove that equality holds if $n \geq 4$.

Lemma 3.1. $I(2,8) = \{0, 1, 2, 4\}$.

Proof. Let $\{a_1, a_2\}$ and $\{0, 1, \ldots, 8\}$ be the independent sets of $K_{2,8}$. We define the following decompositions of the graph:

$$A_0 = \{(a_1, 1, a_2, 2), (a_1, 3, a_2, 4), (a_1, 5, a_2, 6), (a_1, 7, a_2, 8)\}$$

$$A_1 = \{(a_1, 1, a_2, 2), (a_1, 3, a_2, 5), (a_1, 4, a_2, 8), (a_1, 6, a_2, 7)\}$$

$$A_2 = \{(a_1, 1, a_2, 2), (a_1, 3, a_2, 4), (a_1, 5, a_2, 8), (a_1, 6, a_2, 7)\}$$

$$A_3 = \{(a_1, 1, a_2, 3), (a_1, 2, a_2, 4), (a_1, 5, a_2, 8), (a_1, 6, a_2, 7)\}$$

See Figure 3.1. Intersecting each decomposition with A_3, we have

$$|A_0 \cap A_3| = 0 \qquad |A_1 \cap A_3| = 1 \qquad |A_2 \cap A_3| = 2$$

□

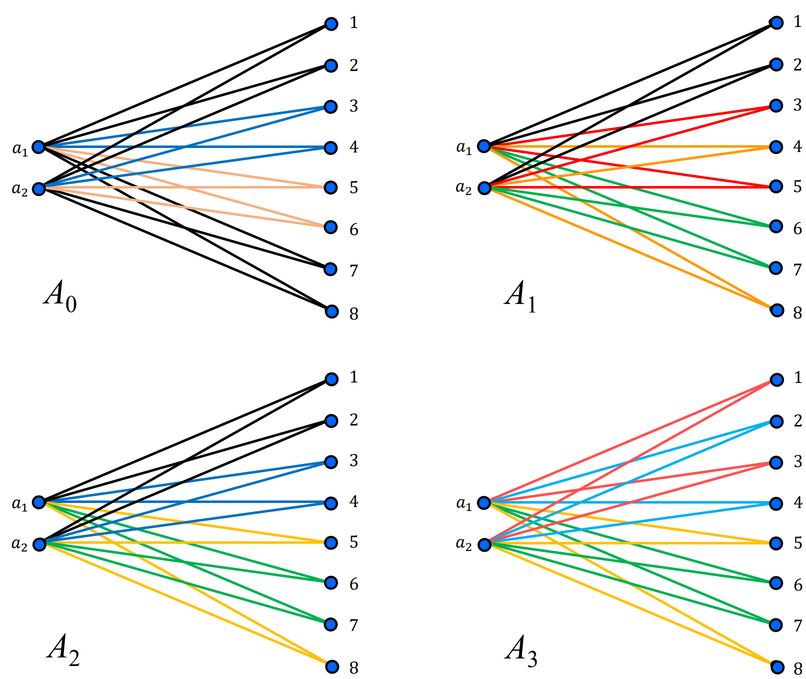


Figure 3.1: Maximum packings of $K_{2,8}$

Proposition 3.1. *Let $\delta_1, \delta_2, \ldots, \delta_d$ be positive integers greater than or equal to 3 and let $\delta = \sum_1^d \delta_i$. If $s \in \{0, 1, \ldots, \delta - 2, \delta\}$, then there exists non-negative integers $s_1, s_2, \ldots, s_d$ such that $s_i \in \{0, 1, \ldots, \delta_i - 2, \delta_i\}$ and $s = \sum_1^d s_i$.*

Proof. Let x be the smallest integer for which $s \leq k = \sum_1^x \delta_i$. Then one of the following is true.

1. $s = k$

2. $s = k - 1$

3. $s = k - r$ for some r with $1 < r < \delta_x$

For $i < x$ choose $s_i = \delta_i$ and for $i > x + 1$ choose $s_i = 0$. According to the cases, make the following choices for s_x and s_{x+1}.

	s_x	s_{x+1}
$s = k$	δ_x	0
$s = k - 1$	$\delta_x - 2$	1
$s = k - r$	$\delta_x - r$	0

□

Theorem 3.2. *Let n be even with $n \geq 4$. Then $I(2, n) = \{0, 1, \ldots, n/2 - 2, n/2\}$.*

Proof. Let the independent sets of $K_{2,n}$ be $\{x, y\}$ and V. Lemmas 2.1, 2.2 and 3.1 establish the cases $n = 4, 6$ and 8 respectively. Now we assume that $n \geq 10$. Then $n = 6d + c$ for some positive integer d, and $c = 0, 2$ or 4. Partition V into sets $V_1, V_2, \ldots, V_d$ of six vertices each and a set U with c vertices. For $i = 1, \ldots, d-1$, let A_i be a decomposition of $K_{2,6}$ into 4-cycles where the independent sets are $\{x, y\}$ and V_i. We take the following cases.

Case 1: $n \equiv 0 \pmod 6$

Here $c = 0$. Form a 4-cycle decomposition A_d of $K_{2,6}$ where the independent sets are $\{x, y\}$

and V_d. Then $\bigcup_{i=1}^{d} A_i$ is a decomposition of $K_{2,n}$ into 4-cycles. Let $s_i \in \{0, 1, 3\}$. By Lemma 2.2, we may reselect the set A_i so that there are s_i cycles in common with the original choice. This forms a new packing of $K_{2,n}$ that shares

$$s_1 + s_2 + \cdots + s_d$$

cycles with the original decomposition. By Proposition 3.1, it is possible to make the new selection so that the sum is any value in $\{0, 1, \ldots, 3d - 2, 3d\}$. Since $n = 6d$, $3d = n/2$.

Case 2: $n \equiv 2 \pmod 6$

The subgraph induced on the vertex set $\{x, y\} \cup V_d \cup U$ is isomorphic to $K_{2,8}$. Decompose this subgraph into a collection B of 4-cycles. This forms a packing $B \cup \left(\bigcup_{i=1}^{d-1} A_i\right)$. For $s_i \in \{0, 1, 3\}$, we may reselect A_i so that there are s_i common cycles. By Lemma 3.1 we may replace B so that there are t common cycles for any t in $\{0, 1, 2, 4\}$. In this way, we can repack to have

$$s_1 + s_2 + \cdots + s_{d-1} + t$$

common cycles. By Proposition 3.1, we can make the s_i's sum to any value in

$$\{0, 1, \ldots, 3(d - 1) - 2, 3(d - 1)\}$$

Consequently, the number of cycles common to the original packing can be any value in

$$\{0, 1, \ldots, 3(d - 1) + 2, 3(d - 1) + 4\}$$

Since $n = 6d + 2$, the upper value of the spectrum is $3(d - 1) + 4 = 3d + 1 = n/2$.

Case 3: $n \equiv 4 \pmod 6$

Select a 4-cycle decomposition A_d of $K_{2,6}$ where the independent sets are $\{x, y\}$ and V_d. The subgraph induced on the vertex set $\{x, y\} \cup U$ is isomorphic to $K_{2,4}$. Select a 4-cycle decomposition B of this subgraph to form a packing $B \cup \left(\bigcup_{i=1}^{d} A_i\right)$. For any $s_i \in \{0,1,3\}$ and $t \in \{0,2\}$, Lemmas 2.1 and 2.2 allow us to repack so that each A_i has s_i common cycles and B has t common cycles. The number of common cycles in the new packing is $s_1 + s_2 + \cdots + s_d + t$ and so the spectrum is $\{0,1,\ldots,3d,3d+2\}$. Here $n = 6d+4$ and the upper value is $3d+2 = n/2$. □

Theorem 3.3. *If m and n are even, then $I(m,n) = \{0,1,\ldots,mn/4-2,mn/4\}$.*

Proof. Let S and V be the independent sets of $K_{m,n}$ of sizes m and n respectively. Partition S into the sets $\{x_1,y_1\},\{x_2,y_2\},\ldots,\{x_{m/2},y_{m/2}\}$. For $i = 1,2,\ldots,m/2$, let A_i be a decomposition of $K_{2,n}$ into 4-cycles with independent sets $\{x_i,y_i\}$ and V. The packing $A = \bigcup_{i=1}^{m/2} A_i$ decomposes $K_{m,n}$ into 4-cycles. Since $|A_i| = n/2$,

$$|A| = \frac{m}{2} \cdot \frac{n}{2} = \frac{mn}{4}$$

By Theorem 3.2, we can reselect each set A_i so that s_i cycles are left unchanged for any $s_i \in \{1,2,\ldots,n/2-2,n/2\}$. Let $s \in \{1,2,\ldots,mn/4-2,mn/4\}$. Apply Proposition 3.1 with each $\delta_i = n/2$ so that $\delta = mn/4$. We conclude that it is possible to choose the $s_i's$ so that their sum is s. □

3.2 The Intersection Problem for 4-cycle Systems

The result of A. Kotzig's [6] that was mentioned in the introduction is the following statement.

> *If $n \equiv 1 \pmod{8k}$, then there is a partition of the edges of K_n into 4k-cycles, the condition being also necessary if k is a power of 2.*

Letting $k = 1$, we see that a 4-cycle system of K_n exists if and only if $n \equiv 1 \pmod 8$. In the introduction, we stated that Billington [1] has shown that $I(n) = \{0,1,\ldots,p_n-2,p_n\}$. A 4-cycle system of K_1 consists of 0 cycles. A system of K_9 has $(9 \cdot 8)/(2 \cdot 4) = 9$ cycles. It is possible to obtain Billington's solution to the intersection problem for 4-cycle systems using her solution for the K_9 case and Theorem 3.3.

Lemma 3.4. (Billington [1]) $I(9) = \{0, 1, \ldots, 7, 9\}$.

Proof. Take the group $\mathbb{Z}_9$ as the vertices of K_9. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the permutations on $\mathbb{Z}_9$ defined by $\alpha_1 = (0132)$, $\alpha_2 = (345)$, $\alpha_3 = (35)$, $\alpha_4 = (135)(246)$. Define the following 4-cycle systems on K_9:

$$D_0 = \{(0+i, 1+i, 5+i, 2+i) \mid i \in \mathbb{Z}_9\}$$

$$D_1 = \{0132, 0354, 0516, 0738, 1264, 1758, 2478, 2567, 3486\}$$

$$D_2 = D_1 \backslash \{0738, 1758\} \cup \{0758, 1738\}$$

$$D_3 = D_1 \backslash \{0738, 2478, 3486\} \cup \{0748, 2438, 3687\}$$

$$D_4 = D_1\alpha_1 \qquad D_5 = D_1\alpha_2 \qquad D_6 = D_1\alpha_3 \qquad D_7 = D_2\alpha_4$$

These are the systems

$$D_0 = \{0152, 1263, 2374, 3485, 4506, 5617, 6728, 7830, 8041\}$$

$$D_1 = \{0132, 0354, 0516, 0738, 1264, 1758, 2478, 2567, 3486\}$$

$$D_2 = \{0132, 0354, 0516, 0758, 1264, 1738, 2478, 2567, 3486\}$$

$$D_3 = \{0132, 0354, 0516, 0748, 1264, 1758, 2438, 2567, 3687\}$$

$$D_4 = \{0132, 1254, 1536, 1728, 3064, 3758, 0478, 0567, 2486\}$$

$$D_5 = \{0142, 0435, 0316, 0748, 1265, 1738, 2567, 2367, 4586\}$$

$$D_6 = \{0152, 0534, 0316, 0758, 1264, 1738, 2478, 2367, 5486\}$$

$$D_7 = \{0354, 0516, 0132, 3426, 4678, 4127, 5682, 0718, 3758\}$$

Table 3.1, shows how to obtain all possible intersection sizes. □

Intersection	Size
$D_0 \cap D_1 = \varnothing$	0
$D_1 \cap D_4 = \{0132\}$	1
$D_2 \cap D_5 = \{2567, 1738\}$	2
$D_2 \cap D_7 = \{0132, 0354, 0516\}$	3
$D_2 \cap D_6 = \{1264, 2478, 0758, 1738\}$	4
$D_2 \cap D_3 = \{0132, 0354, 0516, 1264, 2567\}$	5
$D_1 \cap D_3 = \{0132, 0354, 0516, 1264, 1758, 2567\}$	6
$D_1 \cap D_2 = \{0132, 0354, 0516, 1264, 2478, 2567, 3486\}$	7

Table 3.1: Intersections among 4-cycle systems of K_9

The solution to the intersection problem for 4-cycle systems is the following.

Theorem 3.5. (Billington [1]) *If* $n \equiv 1 \pmod 8$ *then* $I(n) = \{0, 1, \ldots, p_n - 2, p_n\}$.

Proof. We assume that $n \geq 17$. For some positive integer m, we have $n = 8m + 1$. Label the vertices of $K_n = K_{8m+1}$ with the elements

$$\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq 8\}$$

Pack the graph with 4-cycles as follows.

1. For $1 \leq i \leq m$, let D_i be a 4-cycle system of K_9 with vertices $\{\infty\} \cup \{(i, 1), (i, 2), \ldots, (i, 8)\}$.

2. For each pair $\{h, k\} \subseteq S = \{1, 2, \ldots, m\}$, let $B_{h,k}$ be a decomposition of $K_{8,8}$ into 4-cycles where the independent sets are $\{(h, 1), (h, 2), \ldots, (h, 8)\}$ and $\{(k, 1), (k, 2), \ldots, (k, 8)\}$.

Define the collections

$$D = \bigcup_{i=1}^{m} D_i \qquad\qquad B = \bigcup_{\{h,k\} \subseteq S} B_{h,k}$$

Then $D \cup B$ is a 4-cycle system of K_{8m+1}. Each set D_i contains 9 cycles and each set $B_{h,k}$ contains $8^2/4 = 16$ cycles. By Lemma 3.4, $I(9) = \{0, 1, \ldots, 7, 9\}$ and by Theorem 3.3,

$I(8,8) = \{0, 1, \ldots, 14, 16\}$. Define $b = |B| = 16\binom{m}{2}$ and let

$$s \in \{0, 1, \ldots, 9m-2, 9m\} \qquad\qquad t \in \{0, 1, \ldots, b-2, b\}$$

where $b = |B| = 16\binom{m}{2}$. Applying Proposition 3.1, we see that it is possible to reselect each packing D_i so that the collection D is reformed to have s cycles in common with the original collection. Similarly, we may reselect each packing $B_{h,k}$ to reform the collection B so that t common cycles are kept. With the appropriate choice of s and t, it is possible to form a new system where the number of cycles common to the original is any value in $\{0, 1, \ldots, 9m+b-2, 9m+b\} = \{0, 1, \ldots, p_n - 2, p_n\}$. □

Figure 3.2 depicts the 4-cycle system constructed in the proof. The number of cycles in the system is

$$\begin{aligned} p_n = |D \cup B| = |D| + |B| = 9m + 16\binom{m}{2} &= 9m + 8m(m-1) = 8m^2 + m \\ = m(8m+1) = \left(\frac{n-1}{8}\right) n = \frac{n(n-1)}{8} \end{aligned}$$

which agrees with the fact the there must be $|E(K_n)|/4$ cycles.

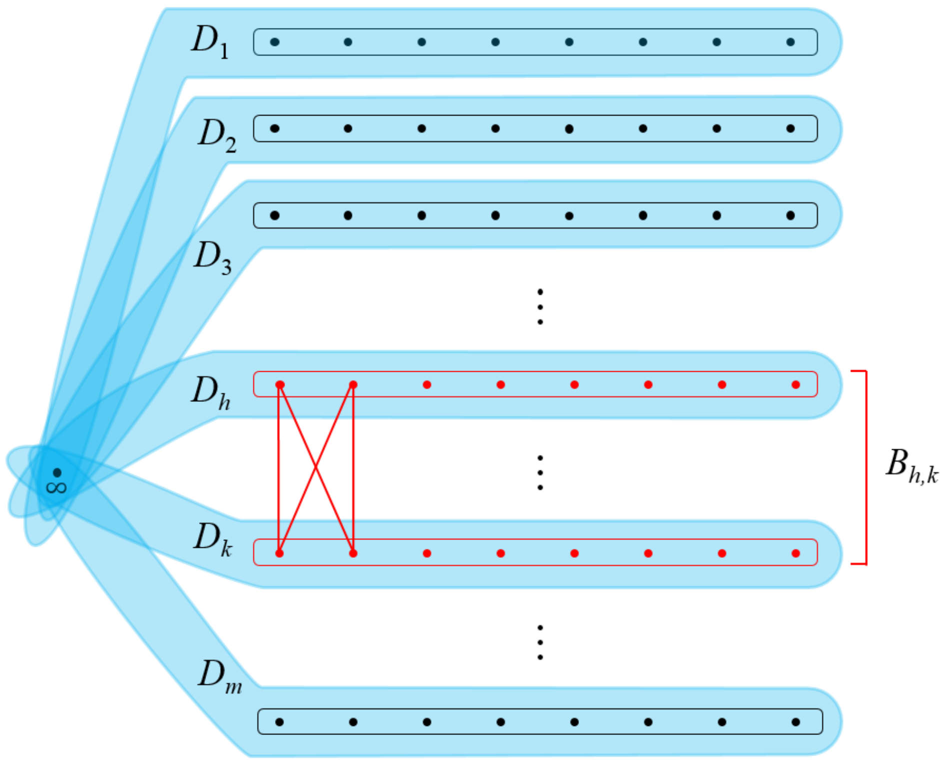


Figure 3.2: A 4-cycle system of K_{8m+1}

Chapter 4

General Constructions

4.1 A Maximum Packing of K_n with 4-cycles

Since a 4-cycle system of K_n exists if and only if $n \equiv 1 \pmod 8$, we cannot achieve perfect packing whenever $n \not\equiv 1 \pmod 8$. Any packing with 4-cycles will leave some edges of K_n unpacked. From Schönheim and Bialostocki's work [5], we can determine the possible leaves which are shown in Table 4.1. The leave graphs and our notation used to denote them are as follows.

Possible Leave Graphs	
C_3	3-cycle
C_5	5-cycle
C_6	6-cycle
F_n	1-factor of of K_n
L_B	Bow tie (two 3-cycles with a common vertex)
L_C	Two 3-cycles without a common vertex

The result from [5] that determines the leaves given in Table 4.1 is the following theorem which we state using the notation given above.

Theorem 4.1. (Schönheim and Bialostocki [5]) *If P is a maximum packing of K_n with 4-cycles with a leave L then*

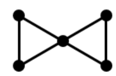
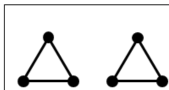
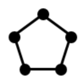
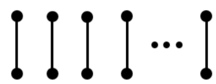
$n \equiv 1 \pmod 8$	$\emptyset$				
$n \equiv 3 \pmod 8$					
$n \equiv 5 \pmod 8$		or		or	
$n \equiv 7 \pmod 8$					
n even					

Table 4.1: Leaves of a maximum packing

(i)

$$|P| = \begin{cases} \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1 & \text{if } n \equiv 5 \text{ or } 7 \pmod 8 \\ \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor & \text{if } n \not\equiv 5 \text{ or } 7 \pmod 8 \end{cases}$$

(ii) *the 4-cycles can be chosen so that*

$$L \cong \begin{cases} C_3 & \text{if } n \equiv 3 \pmod 8 \\ L_B & \text{if } n \equiv 5 \pmod 8 \\ C_5 & \text{if } n \equiv 7 \pmod 8 \\ F_n & \text{if } n \text{ is even} \end{cases}$$

Except for the $n \equiv 5 \pmod 8$ case, these are the only possibilities. We confirm this claim for each case. First note that a packing with 4-cycles uses an even number of edges from each vertex. This implies that all vertices of L have an even degree if n is odd and all vertices of L have an odd degree if n is even. Using this observation and the theorem which gives us the number of edges in the leave we conclude the following.

- The $n \equiv 3 \pmod 8$ case:

  Since L has 3 edges and all degrees are even, the only possibility is that $L \cong K_3$.

- The $n \equiv 5 \pmod 8$ case:

  Here L has 6 edges with all degrees are even. This condition is satisfied if and only if L is isomorphic to any of C_6, L_B or L_C.

- The $n \equiv 7 \pmod 8$ case:

  There are 5 edges with all degrees even. Therefore $L \cong C_5$.

- The n even case:

  All vertices of K_n have odd degree and as a result, each is incident with at least one leave edge. By the theorem, L has $n/2$ edges. To satisfy both of these conditions, L must be a 1-factor of K_n.

We have established a corollary to Theorem 4.1.

Corollary 4.1.1. *If a maximum packing of K_n with 4-cycles has a leave L then,*

$$L \cong \begin{cases} K_3 \text{ if } n \equiv 3 \pmod 8 \\ C_5 \text{ if } n \equiv 7 \pmod 8 \\ F_n \text{ if } n \text{ is even} \end{cases}$$

and L is isomorphic to either $C_6, L_B,$ or L_C if $n \equiv 5 \pmod 8$.

We can express the size p_n of a maximum packing of K_n without the floor functions that appear in Theorem 4.1. We can do this by using the number of edges in the leave given in the theorem. Computing p_n we have the following.

$$n \equiv 1 \pmod 8 \quad \Longrightarrow \quad p_n = \frac{n(n-1)}{8}$$

$$n \equiv 3 \pmod 8 \quad \Longrightarrow \quad p_n = \frac{1}{4}\left[\frac{n(n-1)}{2} - 3\right] = \frac{n^2-n-6}{8} = \frac{(n+2)(n-3)}{8}$$

$$n \equiv 5 \pmod 8 \quad \Longrightarrow \quad p_n = \frac{1}{4}\left[\frac{n(n-1)}{2} - 6\right] = \frac{n^2-n-12}{8} = \frac{(n+3)(n-4)}{8}$$

$$n \equiv 7 \pmod 8 \quad \Longrightarrow \quad p_n = \frac{1}{4}\left[\frac{n(n-1)}{2} - 5\right] = \frac{n^2-n-10}{8}$$

$$n \text{ even} \quad \Longrightarrow \quad p_n = \frac{1}{4}\left[\frac{n(n-1)}{2} - \frac{n}{2}\right] = \frac{n^2-2n}{8} = \frac{n(n-2)}{8}$$

Table 4.2 summarizes these packing sizes.

	$n \equiv 1 \pmod 8$	$n \equiv 3 \pmod 8$	$n \equiv 5 \pmod 8$	$n \equiv 7 \pmod 8$	n even
p_n	$\frac{n(n-1)}{8}$	$\frac{(n+2)(n-3)}{8}$	$\frac{(n+3)(n-4)}{8}$	$\frac{n^2-n-10}{8}$	$\frac{n(n-2)}{8}$

Table 4.2: The number p_n of 4-cycles in a maximum packing of K_n

4.2 The $8m + 2r + 1$ Construction

Suppose that n is odd. Then $n \equiv 1, 3, 5$ or $7 \pmod 8$. In the $n \equiv 1 \pmod 8$ case, a maximum packing is a 4-cycle system. To construct a maximum packing for the other cases, we express n as $n = 8m + 2r + 1$, where $r = 1, 2$ or 3.

The $8m + 2r + 1$ Construction

Let $m \geq 1$, $r \in \{1, 2, 3\}$. Partition the vertices of $K_{8m+2r+1}$ into sets U and V where

$$|U| = 2r + 1 \qquad \text{and} \qquad |V| = 8m$$

Label the vertices of U so that

$$U = \begin{cases} \{\infty, 1, 2\} & \text{if } r = 1 \\ \{\infty, 1, 2, 3, 4\} & \text{if } r = 2 \\ \{\infty, 1, 2, 3, 4, 5, 6\} & \text{if } r = 3 \end{cases}$$

Pack the graph with 4-cycles as follows.

1. Let D be a 4-cycle system of K_{8m+1} where the vertex set is $\{\infty\} \cup V$.

2. Let A be a 4-cycle decomposition of $K_{2r,8m}$ on independent sets $U \setminus \{\infty\}$ and V.

3. Let P be a maximum packing of K_{2r+1} where the vertex set is U.

   (a) For K_3, choose $P = \varnothing$

   (b) For K_5, choose P so that the leave is a bow tie.

   (c) For K_7, choose any maximum packing.

The 4-cycles of $D \cup A \cup P$ include all the edges of the graph except for those forming the leave of P. Since $8m + 2r + 1 \equiv 2r + 1 \pmod{8}$, the leave of P has the same number of edges as that of a maximum packing of $K_{8m+2r+1}$ by Theorem 4.1. Thus $D \cup A \cup P$ is a maximum packing of $K_{8m+2r+1}$. See Figures 4.1, 4.2 and 4.3. The collections have sizes

$$|D| = p_{8m+1} \qquad |A| = 2r \cdot 8m/4 = 4rm \qquad |P| = p_{2r+1}$$

Enumerating cycles, we can obtain p_n where $n = 8m + 2r + 1$.

$r = 1: \quad n = 8m + 3 \quad \Longrightarrow \quad m = \frac{n-3}{8}$

$$p_n = p_{n-2} + 4m + p_3 = \frac{(n-2)(n-3)}{8} + \frac{4n-12}{8} + 0 = \frac{n^2-n-6}{8} = \frac{(n+2)(n-3)}{8}$$

$r = 2: \quad n = 8m + 5 \quad \Longrightarrow \quad m = \frac{n-5}{8}$

$$p_n = p_{n-4} + 8m + p_5 = \frac{(n-4)(n-5)}{8} + \frac{8n-40}{8} + 1 = \frac{n^2-n-12}{8} = \frac{(n+3)(n-4)}{8}$$

$r = 3: \quad n = 8m + 7 \quad \Longrightarrow \quad m = \frac{n-7}{8}$

$$p_n = p_{n-6} + 12m + p_7 = \frac{(n-6)(n-7)}{8} + \frac{12n-84}{8} + 4 = \frac{n^2-n-10}{8}$$

In each case the value for p_n is in agreement with Table 4.2.

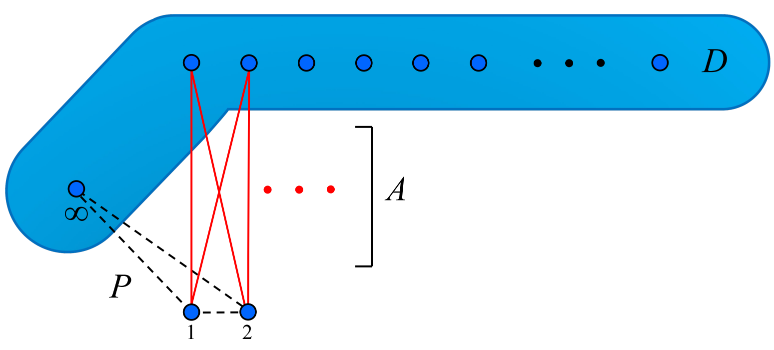


Figure 4.1: A maximum packing of K_{8m+3}

Theorem 4.2. *If* $n \equiv 3, 5$ or $7 \pmod 8$, *then* $I(n) = \{0, 1, \ldots, p_n\}$.

Proof. For $n \leq 8$, this is true by Theorem 2.3. For $n \geq 11$, apply the $8m+2r+1$ Construction

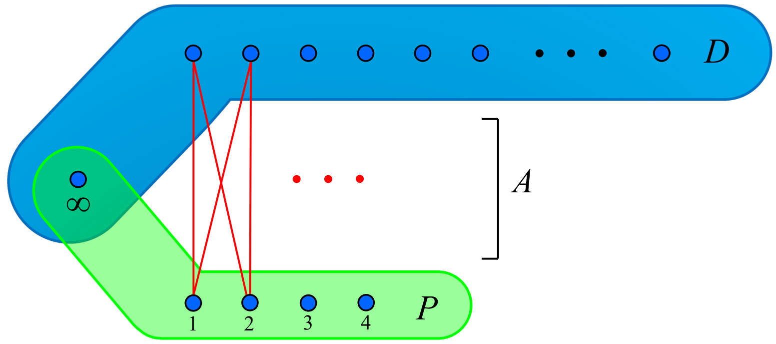


Figure 4.2: A maximum packing of K_{8m+5}

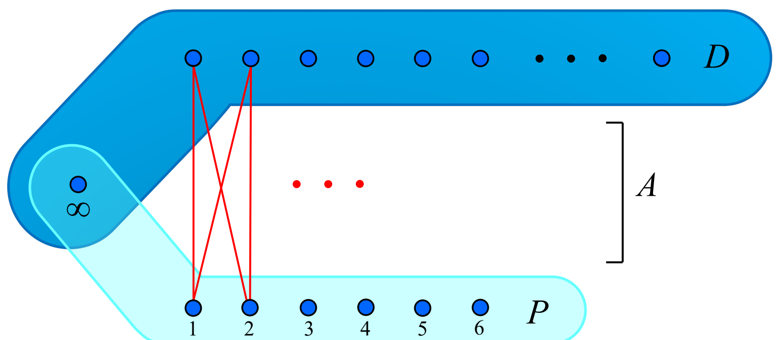


Figure 4.3: A maximum packing of K_{8m+7}

to K_n to form the packing $D \cup A \cup P$. Let

$$s \in \{0, 1, \ldots, p_{8m+1} - 2, p_{8m+1}\} \quad \text{and} \quad t \in \{0, 1, \ldots, 4rm - 2, 4rm\}$$

By Theorem 3.5 we can replace D with a 4-cycle system that has s cycles in common with D. By Theorem 3.3 we can reselect A so that there are t common cycles. Section 2.1 shows how to reselect P in the $r = 2$ case to obtain $v = 0$ or 1 common cycles. In the $r = 3$ case, we may obtain v common cycles for any v in $\{0, 1, 2, 3, 4\}$. This follows from Theorem 2.3. For these values of r, which correspond to $n \equiv 5$ or 7 (mod 8), we see that it is possible to obtain any intersection size from $\{0, 1, \ldots, p_n\}$ by appropriately choosing s, t and v.

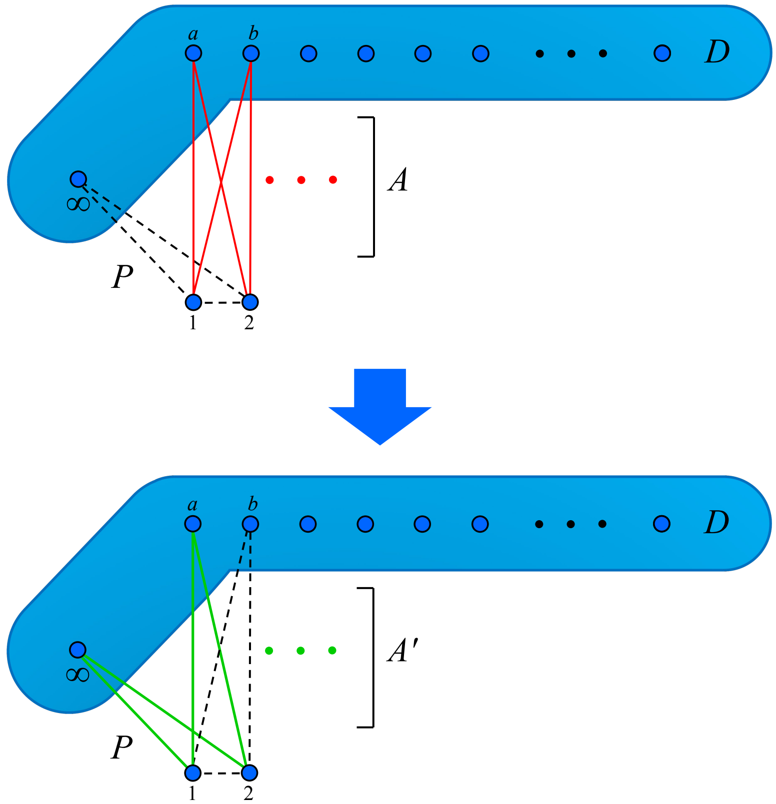


Figure 4.4: Modifying the packing of K_{8m+3} to obtain the intersection value $p_n - 1$

In the $n \equiv 3 \pmod 8$ case, where $P = \varnothing$, a re-selection of D and A can give any intersection size in $\{0, 1, \ldots, p_n - 2, p_n\}$. To get the missing value, we can adjust the packing by moving the leave, which is the 3-cycle $(\infty, 1, 2)$. Let $(1, a, 2, b)$ be one of the 4-cycles in A. Replace this cycle with $(1, a, 2, \infty)$, which does not include any edges from D. The packing is now identical except for this cycle and the leave, which is now $(1, 2, b)$. (See Figure 4.4.) This proves the theorem. □

4.3 The $8m$ and $8m + 2r$ Constructions

If n is even, then $n \equiv 0, 2, 4$ or $6 \pmod 8$. In this section we express n as $n = 8m + 2r$ where $r = 0, 1, 2$ or 3. We will first address the $n = 8m$ case and use the solution to solve the other cases.

The $8m$ Construction

Let $m \geq 1$. Label the vertices of K_{8m} with the set

$$\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq 8\}$$

Pack the graph with 4-cycles as follows.

1. For $1 \leq i \leq m$, let R_i be a maximum packing of K_8 with 4-cycles where the vertex set is $\{(i,1),(i,2),\ldots,(i,8)\}$.

2. For each pair $\{h,k\} \subseteq S = \{1,2,\ldots,m\}$, let $B_{h,k}$ be a decomposition of $K_{8,8}$ into 4-cycles where the independent sets are $\{(h,1),(h,2),\ldots,(h,8)\}$ and $\{(k,1),(k,2),\ldots,(k,8)\}$.

Define the collections

$$R = \bigcup_{i=1}^{m} R_i \qquad\qquad B = \bigcup_{\{h,k\}\subseteq S} B_{h,k}$$

Then $R \cup B$ is a maximum packing of K_{8m} with 4-cycles. See Figure 4.5. Each set R_i contains 6 cycles. (See Section 2.3) and each set $B_{h,k}$ contains $8^2/4 = 16$ cycles. If $n = 8m$, then the total number of cycles is

$$p_n = 6m + 16\binom{m}{2} = 6m + 16\,\frac{m(m-1)}{2} = \frac{16m^2 - 4m}{2} = \frac{4 \cdot 2m(8m-2)}{4 \cdot 2} = \frac{n(n-2)}{8}$$

in agreement with Table 4.2.

Theorem 4.3. *If* $n \equiv 0 \pmod 8$*, then* $I(n) = \{0,1,\ldots,p_n\}$

Proof. The $n = 8$ case is true by Theorem 2.3. We assume that $n \geq 16$ and apply the $8m$ Construction. Define $b = |B| = 16\binom{m}{2}$ and let

$$s \in \{0,1,\ldots,6m\} \qquad\qquad t \in \{0,1,\ldots,b-2,b\}$$

Since $I(8) = \{0,1,\ldots,p_8\}$ by Theorem 2.3, we can repack the 6 cycles of each set R_i

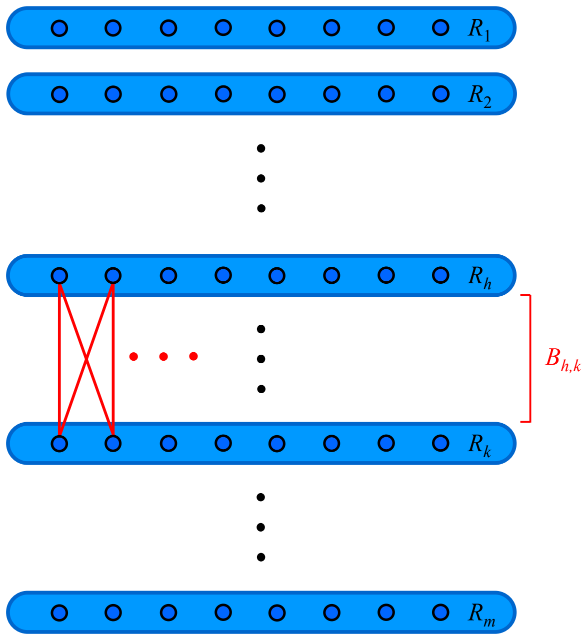


Figure 4.5: A maximum packing of K_{8m}

preserving any desired number of cycles. Therefore it is possible to reform the collection R to obtain a new collection with s common cycles. Since $I(8,8) = \{0, 1, \ldots, 14, 16\}$ by Theorem 3.3, we can apply Proposition 3.1, to reselect each packing $B_{h,k}$ so that collection B is altered but retains t of the original cycles. By choosing s and t in this way, it is possible to form a new maximum packing so that the number of common cycles is any value from $\{0, 1, \ldots, s + t = p_n\}$. □

The $8m + 2r$ Construction

Let $m \geq 2$, $r \in \{1, 2, 3\}$. Partition the vertices of K_{8m+2r} into sets U and V of sizes $2r$ and $8m$ respectively. Label the vertices of U so that

$$U = \begin{cases} \{1,2\} & \text{if } r = 1 \\ \{1,2,3,4\} & \text{if } r = 2 \\ \{1,2,3,4,5,6\} & \text{if } r = 3 \end{cases}$$

Pack the graph with 4-cycles as follows.

1. Let C be a maximum packing of K_{8m} where the vertex set V.

2. Let A be a 4-cycle decomposition of $K_{2r,8m}$ on independent sets U and V.

3. Let P be a maximum packing of K_{2r} where the vertex set is U.

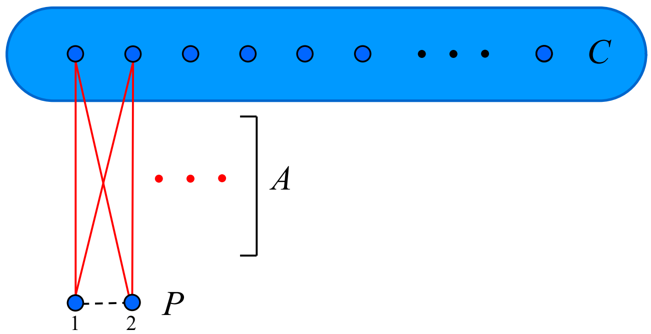


Figure 4.6: A maximum packing of K_{8m+2}

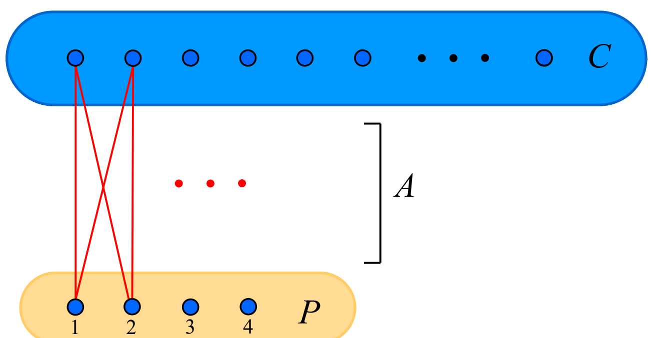


Figure 4.7: A maximum packing of K_{8m+4}

Step 1 is possible using the $8m$ Construction. The 4-cycles of $C \cup A \cup P$ include all the edges of the graph except for those of the leaves of the packings C and P. The first of these leaves is a 1-factor of K_{8m} with vertex set V, and the second is a 1-factor of K_{2r} with vertex set U. Together these leaves constitute a 1-factor of K_{2m+r} and therefore $C \cup A \cup P$ is a

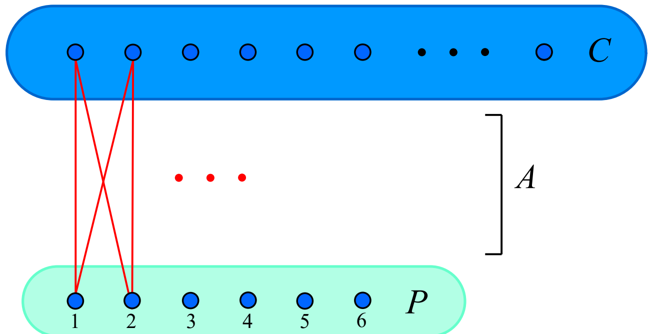


Figure 4.8: A maximum packing of K_{8m+6}

maximum packing of K_n. See Figures 4.6, 4.7 and 4.8. We have

$$|C| = p_{8m} \qquad |A| = 2r \cdot 8m/4 = 4rm \qquad |P| = p_{2r}$$

By Table 4.2, $p_n = n(n-2)/8$ hence

$$p_{8m} = \frac{8m(8m-2)}{8} = \frac{64m^2 - 16m}{8}$$

Enumerating cycles, we can obtain p_n where $n = 8m + 2r$.

$r = 1: \quad n = 8m + 2$

$$p_n = p_{8m} + 4m + p_2 = \frac{64m^2 - 16m}{8} + \frac{32m}{8} + 0 = \frac{64m^2 + 16m}{8}$$

$$= \frac{(8m+2)8m}{8} = \frac{n(n-2)}{8}$$

$r = 2: \quad n = 8m + 4$

$$p_n = p_{8m} + 8m + p_4 = \frac{64m^2 - 16m}{8} + \frac{64m}{8} + 1 = \frac{64m^2 + 48m + 8}{8}$$

$$= \frac{(8m+4)(8m+2)}{8} = \frac{n(n-2)}{8}$$

$r = 3: \quad n = 8m + 6$

$$p_n = p_{8m} + 12m + p_6 = \frac{64m^2 - 16m}{8} + \frac{96m}{8} + 3 = \frac{64m^2 + 80m + 24}{8}$$

$$= \frac{(8m+6)(8m+4)}{8} = \frac{n(n-2)}{8}$$

Theorem 4.4. *If* $n \equiv 2, 4$ or $6 \pmod 8$, *then* $I(n) = \{0, 1, \ldots, p_n\}$.

Proof. The cases $n = 2, 4$ and 6 are true by Theorem 2.3. Assume that $n \geq 10$. Applying the $8m + 2r$ construction gives a maximum packing $C \cup A \cup P$ of K_{8m+2r}. Let

$$s \in \{0, 1, \ldots, p_m\} \quad t \in \{0, 1, \ldots, 4rm - 2, 4rm\} \quad \text{and} \quad v \in \{0, 1, \ldots, p_{2r}\}$$

We can repack the cycles of C, A and P retaining s, t and v cycles respectively. This follows from Theorems 4.3, 3.3 and 2.3. With the right selection of (s, t, v) any desired intersection size is possible. □

Chapter 5

Summary

The intersection problem for maximum packings of the complete graph K_n with 4-cycles is to answer the following question.

> *For which integers k do there exists maximum packings P_1 and P_2 of K_n with 4-cycles such that $|P_1 \cap P_2| = k$?*

A complete solution to this intersection problem is one which specifies the intersection spectrum $I(n)$ for each value of n. Combining Theorems 2.3, 3.5, 4.2, 4.3 and 4.4 gives a complete solution.

Theorem 5.1. *The intersection spectrum $I(n)$, of maximum packings of K_n with 4-cycles is given by*

$$I(n) = \begin{cases} \{0, 1, \ldots, p_n - 2, p_n\} & \text{if } n \equiv 1 \pmod{8} \\ \{0, 1, \ldots, p_n\} & \text{otherwise} \end{cases}$$

where p_n is the number of cycles in the packing.

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