The Application of Complex Numbers and Quaternions in Navigation

by

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Abstract

Complex numbers and quaternions have a wide array of applications. This paper begins by briefly going over the history of complex numbers and quaternions, bringing up little known facts on main discoverers. Then, we dive into their important properties in relation to the navigation applications we will go over. We will see their application in navigation can greatly minimize time, work and error in various fields of navigation.

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Chapter 1

The Imaginary Number i

1.1 Introduction

The imaginary number i is a fairly known idea today. Many cannot finish a secondary education without the understanding $i^2 = -1$. Why do we continue to call this result imaginary? Let's consider $\sqrt{4}$. We understand this is equivalent to 2 since $2^2 = 4$. Now, let's consider $\sqrt{-1}$. There is no number one can square to receive a negative result. Yet, while there is no real solution, $\sqrt{-1}$ has many real-world uses. One use is in many forms of navigation, as we will see throughout this paper. The use of this number will be seen to significantly improve the time and accuracy of calculations needed in navigation. Many who came across the result $\sqrt{-1}$ would respond as Simon Stevin of Bruges who stated, "There are enough legitimate things to work on without need to get busy on uncertain matter." But, there are a few mathematicians whose time to note and consider negative square roots allowed for it to become the essential number that it is today.

1.2 History

In 50 AD, Heron of Alexandria was studying frustrums. In his book, $Stereometria$, he questions how high a frustrum is given the side lengths of the base, the top square, and the edge connecting the two. He came up with the equation

$$h = \sqrt{c^2 - \frac{(a-b)^2}{2}}$$

In his first example, he uses $a = 10, b = 2, c = 9$ which yields the answer 7. He then tried another example $a = 28, b = 4$, and $c = 15$. In his book, he records the solution $\sqrt{63}$. But, we can see the true solution, given his equation, is

$$h = \sqrt{15^2 - \frac{(28-4)^2}{2}} = \sqrt{225 - 288} = \sqrt{-63}.$$

It is unsure if it was a deliberate error or simply a mistake while recording his answer, but this is the earliest known recorded work with a negative square root as a result. Unfortunately, there is no known evidence of Heron further pursuing this result.

In the *Ars Magna*, published in 1545, Cardan mentions a solution to the quadratic equation $x(10 - x) = 40$. He states, "Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ making 25 - (-15), whence the product is 40." This is how he receives credit as the mathematician to introduce complex numbers. He leaves his solution at that with a comment about the arithmetic being "as refined as it is useless." He too does not further pursue what was, at the time, a peculiar solution. However, he does stumble upon a similar situation in his book when explaining a method to solve cubic equations of the form

$$y^3 + py = q$$

He finds the solution to equations of this form with what is now known as Cardan's formula. While Cardan still receives most credit for this formula, he did not come up with this formula on his own. Niccolò Fontana, also known as Tartaglia, first introduces this formula in a contest. While the formula was leaked to Cardan, he promised to leave it out of his book. Evidently, he does not uphold this promise

$$x = a - b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

What he discerns with this formula is when $\left(\frac{q}{2}\right)^2$ is less than $\left(\frac{p}{3}\right)^3$, the result will be a negative square root. While he did make note of this, he does not resolve it. It isn't until Rafael Bombelli publishes his book *Algebra* in 1572 that we see some attempt to reason with these unexplained solutions. Consider the equation

$$x^3 = 15x + 4$$

Using Cardan's formula, or Cardan-Tartaglia's formula to be fair, let $p = -15$ and $q = 4$. Then,

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\frac{4^2}{4} - \frac{(-15)^3}{27}}} + \sqrt[3]{\frac{4}{2} - \sqrt{\frac{4^2}{4} - \frac{(-15)^3}{27}}}$$

reduces to

$$x = \sqrt[3]{2 + \sqrt{4 - \frac{-3375}{27}}} + \sqrt[3]{2 - \sqrt{4 - \frac{3375}{27}}}$$

resulting in

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

However, Bombelli knew there were real solutions to this equation: $4, -2+\sqrt{3}, -2-\sqrt{3}$. This is what one calls an "irreducible case," which is a situation in which Cardan's formula makes impossible solutions when real solutions exist. This problem lead him to the ingenious idea to express how the imaginary solution could result in the equation's real solution. He began by setting

$$\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1} \text{ and } \sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1}$$

where $a > 0$ and $b > 0$. Rearranging the expression

$$2 + \sqrt{-121} = (a + b\sqrt{-1})^3$$

$$= a^3 + 3a^2b\sqrt{-1} + 3ab^2(\sqrt{-1})^2 + b^3(\sqrt{-1})^3$$

$$= a(a^2 - 3b^2) + b(3a^2 - b^2)\sqrt{-1}$$

where $a(a^2 - 3b^2) = 2$ and $b(3a^2 - b^2) = 11$. Then $a = 2$ and $b = 1$ are the only solutions that will satisfy both conditions. This leads to

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$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

$$= \sqrt[3]{(2 + \sqrt{-1})^3} + \sqrt[3]{(2 - \sqrt{-1})^3}$$

$$= (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4$$

With this, Bombelli demonstrated that through the square root of negative numbers, real solutions could be achieved. He can also be noted as one of the first to give a name to the square root of negative numbers rather than just trying to solve their arithmetic as is. He called $\sqrt{-1}$ plus of minus and $-\sqrt{-1}$ minus of minus and noted their arithmetic as follows:

"Plus by plus of minus, makes plus of minus.

Minus by plus of minus, makes minus of minus.

Plus by minus of minus, makes minus of minus.

Minus by minus of minus, makes plus of minus.

Plus of minus by plus of minus, makes minus.

Plus of minus by minus of minus, makes plus.

Minus of minus by plus of minus, makes plus.

Minus of minus by minus of minus makes minus." [4]

After Bombelli's discoveries, a shift is seen in Mathematicians' attitude on imaginary numbers. In 1629, Albert Girard notices, while solving polynomials, that he can have "impossible" solutions. In one of his examples, he uses

$$x^4 + 3 = 4x$$

and states there are four factions: 0, 0, 4, and 3. The two remaining solutions have the property that their sum is -2 and their product is 3 which leads to the solutions

$$-1 \pm \sqrt{-2}$$

Girard then goes on to explain his belief in the value of this solution stating, "they are good for three things: for the certainty of the general rule, for being sure that there are no other solutions and for its utility."

Shortly after, in 1637, Rene Descartes introduces the terms real and imaginary in relation to the discussion of the square root of negative numbers. In his book, *La Geometrie*, he states, "neither the true roots nor the false are always real; sometimes they are, however, imaginary; namely, whereas we can always imagine as many roots for each equation as I have predicted, there is still not always a quantity which corresponds to each root so imagined. Thus, while we may think of the equation $x^3 - 6xx + 13x - 10 = 0$ as having three roots, yet there is just one real root, which is 2, and the other two, however, increased, diminished, or multiplied them as we just laid down, remain always imaginary."

In 1674, Gottfried Wilhelm von Leibniz wasn't satisfied with the work of Bombelli on Cardan-Taraglia's formula involving the irreducible case. This leads him to try and factor a polynomial with only real coefficients into irreducible factors with a degree of one or two. He then factors $x^4 + a^4$ into

$$(x^2 + a^2\sqrt{-1})(x^2 - a^2\sqrt{-1})$$

Leading to

$$(x - a\sqrt{-\sqrt{-1}})(x - a\sqrt{-\sqrt{-1}})(x + a\sqrt{\sqrt{-1}})(x - a\sqrt{\sqrt{-1}})$$

Although untrue, this leads him to believe that no nontrivial combination of linear factors can result in a real divisor. This is not the only work he did with imaginary numbers. He also solved

$$\sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$$

While this looks interchangeable with Bombelli's solutions involving his real root 4, Leibniz's contemporaries were still amazed with this calculation.

In 1748, 10 years after he had gone completely blind, Leonhard Euler publishes his book *Introductio in Analysin Infinitorum*. Throughout his book, he works with real and then imaginary solutions. His most notable work with imaginary numbers was when he decided to plug the imaginary number in for x in the equation $e^{(x\pi)}$. While he was the mathematician to introduce the letter i as a representation for the $\sqrt{-1}$, he had not come up with it yet. Instead, he left throughout his calculations $\sqrt{-1}$ as is. When he plugged in $\sqrt{-1}z$ in for x in the series for e^x, he knew

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2 \cdot 1} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4}...$$

substiting $\sqrt{-1}z$ results in

$$e^{\sqrt{-1}z} = 1 + \frac{(\sqrt{-1}z)}{1} + \frac{(\sqrt{-1}z)^2}{2 \cdot 1} + \frac{(\sqrt{-1}z)^3}{1 \cdot 2 \cdot 3} + \frac{(\sqrt{-1}z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(\sqrt{-1}z)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}...$$

and then simplifying with the knowledge $(\sqrt{-1})^2 = -1$

$$e^{\sqrt{-1}z} = 1 + \frac{(\sqrt{-1}z)}{1} - \frac{z^2}{2 \cdot 1} - \frac{\sqrt{-1}z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\sqrt{-1}z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}...$$

and then, separating terms without imaginary numbers from terms with imaginary numbers, we get

$$e^{\sqrt{-1}z} = \left(1 - \frac{z^2}{2 \cdot 1} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4}...\right) + \left(\frac{(z)}{1} - \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\right)\sqrt{-1}...$$

As Euler points out, the real part is the same as the expansion for $cos(x)$ and the imaginary part is the expansion for $sin(x)$. This leads him to what is now known as the "Euler's formula"

$$e^{\sqrt{-1}z} = cos(z) + \sqrt{-1}sin(z)$$

More on Euler and his famous formula will be presented in the next chapter. In a memoir presented in 1777, he, at-last, introduces the use of the letter i as a representation of the imaginary number $\sqrt{-1}$, but this was towards the end of his life, and so it was not used in many of his originally published books, instead using $\sqrt{-1}$, though many translated copies after 1777 will go ahead and make this replacement.

Chapter 2

Complex Numbers

2.1 Introduction

A complex number is defined as a number of the form $z = a + bi$ where a and b are real numbers and i is the solution to $\sqrt{-1}$. We see this form used throughout history in the previous section. But, the name for this form, known as a complex number, wasn't coined until Gauss. Figure 2.1, taken from Gauss' $Werke$, shows his explanation of the term.
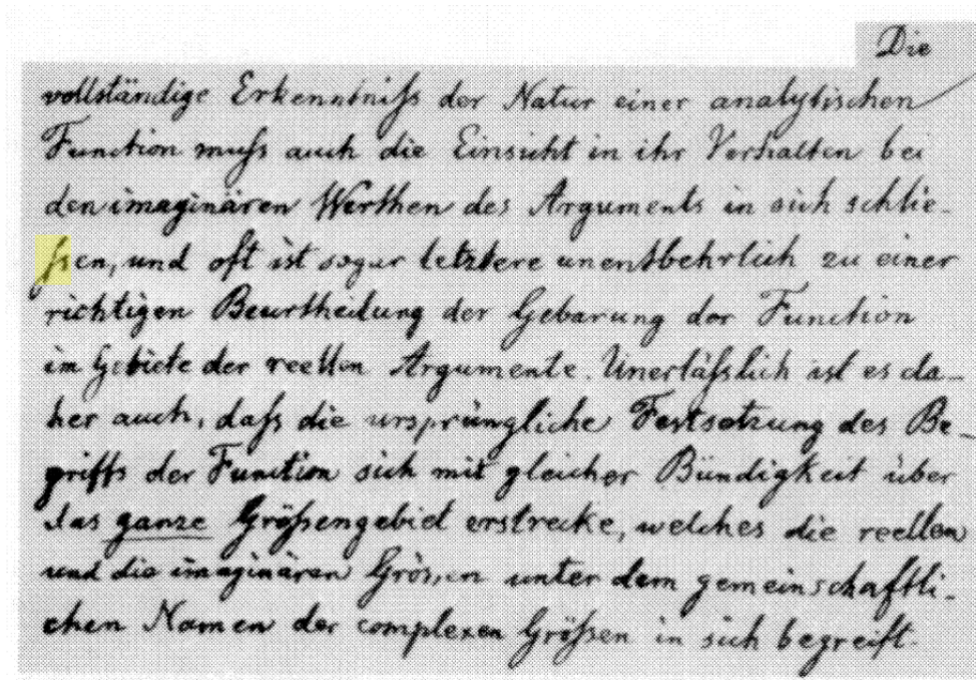

Figure 2.1: $Werke$: Volume 10 Part 1, page 405

This translates to

"Complete knowledge of the nature of an analytic function must also include insight into its behavior for imaginary values of the arguments. Often the latter is indispensable even for a proper appreciation of the behavior of the function for real arguments. It is therefore essential that the original determination of the function concept be broadened to a domain of magnitudes which includes both the real and the imaginary quantities, on an equal footing, under the single designation complex numbers." [13]

2.2 Properties of Complex Numbers

Complex numbers generally hold the same properties as real numbers, except for $i^2 = -1$. Consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$. Then,

For addition,

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i \tag{2.1}$$

For subtraction,

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

For multiplication,

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (bc + ad)i$$

To understand division let us first understand the complex conjugate of a complex number.

A complex conjugate, denoted as $\bar{z}$ of a complex number, z, is the same as the complex number given, except the imaginary part is the opposite sign of the complex number given. For example, consider the complex number $z = a + bi$, then the complex conjugate is

$$\bar{z} = a - bi$$

Now, to divide complex numbers, one would write the complex numbers in fraction form, then multiply the numerator and denominator of the fraction by the denominator's complex conjugate. For example, consider dividing $z_1 = a + bi$ by $z_2 = c + di$. Then,

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$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\left(\frac{c-di}{c-di}\right) = \frac{ac-adi+bci-bdi^2}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

For the norm, aka modulus, let $z = a + bi$, then the norm is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

2.3   Complex Plane

Poplar nicknames for the complex plane include the Gaussian plane or Argand plane, however, another mathemetacian by the name of Casper Wessel visualized the complex plane before either. In 1629, Albert Girard proposes the idea of putting negative numbers on a line. He also was one of the first people to recognize imaginary roots of an equation, but there is no written publications of him putting the two ideas together. Then, in a letter he wrote in 1673, John Wallis mentions the idea of giving a "negative square" a placement on a plane since, just as impossibly, negative lengths have a place on the line. This idea gets executed in 1799 when a cartographer, Caspar Wessel, published graphical representations using complex numbers prior to either Gauss or Argand. This will be seen in more detail in Section 2.4.1. Since he was not an academic, unfortunately, his work went unnoticed for quite some time.
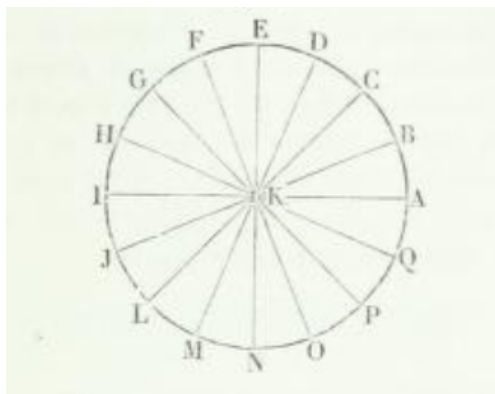

Figure 2.2: Figure 1 in Argand's 1806 Publication

In 1806, Argand anonymously publishes *Essai sur une maniere de representer les quantites imaginaires dans les constructions geometriques*. In it, under section 4 of his paper, he

includes a plane with a circle drawn in it centered at K, seen in Figure 2.2 [5]. He describes the figure as KA being a positive unit, KI being a negative unit, KE representing the value $\sqrt{-1}$, and KN representing the value $-\sqrt{-1}$. The pages in his paper fully describing this figure can be found in the Appendix.

Modern views of the complex plane parallel that of Wessel and Argand, with the only difference being in notation. We take a horizontal line to be the real axis with a vertical line perpendicular to that of the real line to be the imaginary line. We can use this plane to consider a complex number number $z = a + bi$ with a being the real value and b being the imaginary value, as seen in Figure 2.3 [16]. We can use this plane to visualize properties of complex numbers.
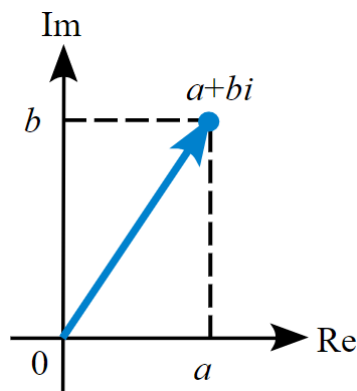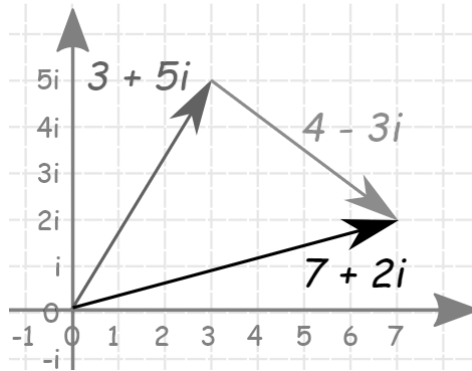

Figure 2.3: The Complex Plane

Let us now consider some examples of complex numbers and their graphical representation [12]

Let $z_1 = 3 + 5i$ and $z_2 = 4 - 3i$. Adding z_1 to z_2 using equation (2.1) results in,

$$3+5i+4-3i= 7+2i$$

The graphic representation of these values would be

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Now, let $z_1 = 1 + i$ and $z_2 = 3 + i$, multiplying z_1 to z_2 brings about

$$(1 + i)(3 + i) = 2 + 4i$$

Dividing 2+4i by 1+i

$$\frac{2+4i}{1+i} = \frac{(2)(1)+(4)(1)}{1^2+1^2} + \frac{(4)(1)-(2)(1)}{(1^2)+(1^2)}$$

$$= \frac{6}{2} + \frac{2}{2}$$

$$= 3 + i$$

The graphic representation of these values would be
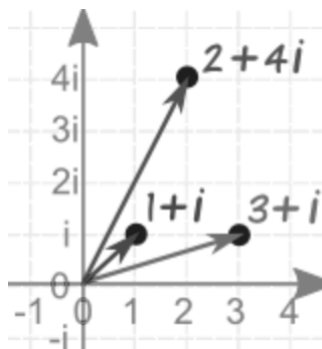
The complex plane can also be used to represent rotations. First, we will start with the famous Euler's identity and formula. Most textbooks or articles mentioning this identity and formula describe it with adjectives like "the most important" and "the most beautiful formulae." And while it is credited to Euler, there is no known writing of the exact following, but from his *Introductio in Analysin Infinitorum*, one can deduce

$$e^{\pi i} + 1 = 0 \tag{2.2}$$

and

$$e^{xi} = cos(x) + isin(x) \tag{2.3}$$

Rewriting (2.2), one can see a rotation of π results in -1. Looking at equation (2.3) and referring to our identities,

$$a = cos(\pi) \text{ and } b = sin(\pi),$$

giving us the coordinates $(-1, 0)$ as to be expected.

Now, consider we want to rotate this coordinate $90°$ counterclockwise. Converting this to radians, we want to rotate by $\frac{\pi}{2}$. Recall from Chapter 1 that $i^0 = 1$, $i^1 = i$, $1^2 = -1$ and $i^3 = -i$. Looking at Figure 2.4 [16], we see a rotation of 90 degrees is equivalent to multiplying a complex number by i. From this we can infer that a rotation of $90°$ from $(-1, 0)$ will move us to $(0, -i)$. We can check this by multiplying $(-1, 0)$, or -1, by i which does result in $-i$, or $(0, -i)$, as expected.
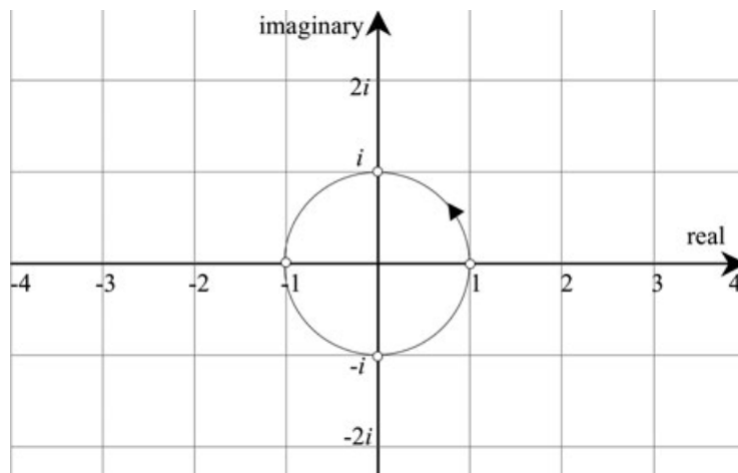

Figure 2.4: The Complex Plane with a Unit Circle

Complex Numbers can also be represented in what is known as polar form. Consider a complex number number $z = a + bi$ in the form

$$z = r\cos(\theta) + ir\sin(\theta) \tag{2.4}$$

where $a = r\cos(\theta)$ and $b = r\sin(\theta)$, r is the radius and is equivalent to $|z|$ and θ is the angle from the real axis to the line formed between the origin and the complex number z. The angle θ is also known as the argument of z, written as

$$arg(z)$$

where

$$\text{In Quadrant 1 } \theta = tan^{-1}\left(\frac{b}{a}\right) \text{ s.t. } a, b > 0$$
$$\text{In Quadrant 2 \& 3 } \theta = tan^{-1}\left(\frac{b}{a}\right) + \pi \text{ s.t. } a < 0$$
$$\text{In Quadrant 4 } \theta = tan^{-1}\left(\frac{b}{a}\right) + 2\pi \text{ s.t. } a > 0, b < 0$$

We can rewrite equation (2.4) in regard to angle since, by trigonemetric identities,

$$cos(\theta) = \frac{a}{r} \text{ and } sin(\theta) = \frac{b}{r}.$$

We can also rewrite (2.4) in exponential form using (2.3)

$$z = r\cos(\theta) + ir\sin(\theta) = r(cos(\theta) - isin(\theta))$$

$$z = r(e^{i\theta}) = re^{i\theta}$$

This is what we also call polar form. Let us explore the complex identities now in polar form. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

For the product,

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$

$$r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$r_1 r_2 [cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

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The product of z_1z_2 then has a norm

$$|z_1z_2| = r_1r_2$$

and the argument

$$arg(z_1z_2) = \theta_1 + \theta_2$$

For the quotient,

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}$$

$$= \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

$$= \frac{r_1}{r_2}[cos(\theta_1 - theta_2) + isin(\theta_1 - \theta_2)]$$

The norm of the quotient is then

$$\left|\frac{z_1}{z_1}\right| = \frac{r_1}{r_2}$$

and the argument is

$$arg(\tfrac{z_1}{z_2}) = \theta_1 - \theta_2$$

Referring back to our product and quotient example, let us first convert z_1, z_2 and z_3 into polar form

$$z_1 = 1 + i = 1.4142(cos(45°) + isin(45°) = 1.4142e^{i(45°)}$$
$$z_2 = 3 + i = 3.1623(cos(18.4349°) + isin(18.4349°) = 3.1623e^{i(18.4349°)}$$
$$z_3 = 2 + 4i = 4.4721(cos(63.4349°) + isin(63.4349°) = 4.4721e^{i(63.4349°)}$$

Let us multiply z_1 and z_2 using polar form. We will do this by looking at what would the norm and argument of the product be. The norm of the product is

$$norm|z_1z_2| = 1.41421 \times 3.16228 = 4.4721$$

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and the argument of the product is

$$arg(z_1 z_2) = 45° + 18.4349° = 63.4349°$$

This would lead to the value 4.4721. Let us see if this equivalates to the solution found multiplying in complex number form. We will do this by using the Pythagorean theorem $a^2 + b^2 = c^2$

$$c^2 = 2^2 + 4^2 = 20$$
$$c = \sqrt{20} \approx 4.4721$$

This does indeed. Now let us consider dividing z_3 by z_1 using polar form. We will do this by looking at the norm and argument of the difference. The norm of the difference is

$$norm\left|\frac{z_3}{z_1}\right| = \frac{4.4721}{1.4142} = 3.1623$$

and the argument of the difference is

$$arg\left(\frac{z_3}{z_1}\right) = 63.4349° - 45° = 18.4349°$$

This leads to the solution 3.1623. Let's compare this with the solution in complex number form by again using the pyathagorean theorem.

$$c^2 = 3^2 + 1^2 = 10$$
$$c = \sqrt{10} = 3.1623$$

Again, we see the solutions parallel one another.

2.4   Applications

2.4.1   Trigonometrical Surveying

When Caspar Wessel was working as a trigonometrical surveyor, he discovered one could use complex numbers to simplify the calculations needed. The job entailed determining the position of a triangular net of points through heavy trigonometric calculations. Since calculations

would have to be rounded as the solution found tended to contain long or infinite decimals, a series of corrections would be applied to reach a higher accuracy. These calculations found would then be compared to others' calculations to compensate for less accurate geographical measurements to obtain a more reliable map. Wessel believed complex numbers could simplify these trigonometrical calculations on the triangular nets.

As early at 1787, there is documentation of Wessel having the idea to use complex numbers to represent direction in a plane. To understand his illustration of a complex plane, one should first understand the use of the observatory in surveying. Astronomical knowledge was considered essential to surveying. So, it was considered important that the Round Tower Observatory, in the heart of Copenhagen, was modernised so it would equal principal observatories in Europe. This made the Observatory the origin of triangulation, making it important to determine it's position in order to coordinate with measurements in other countries. In his Trigonometrical Surverying Report of 1787, we see him take an arbitrary coordinate point, $(p, \sqrt{-1}m)$, from the Observatory coordinate system and connect it to f, whose line creates the tangent plane of the Observatory labeled $T(cosw + \sqrt{-1}w)$ [3].
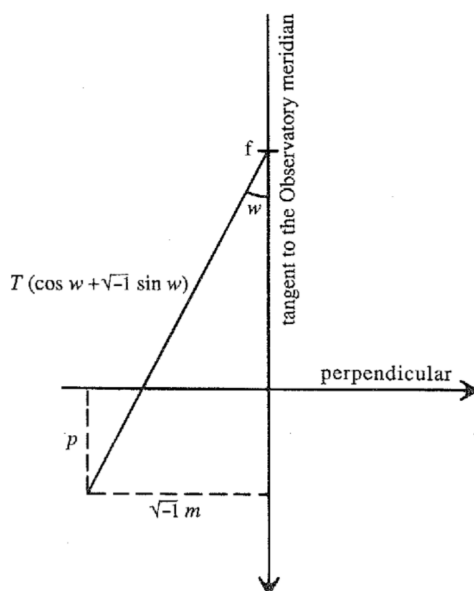

Figure 2.5: Wessel's Complex Plane from his Trigonometrical Surveying Report of 1787

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One can see the line labeled tangent to the Observatory meridian represents the real axis and the perpendicular represents the imaginary axis. Unfortunately, in this paper he does not go into detail about this figure, but this figure seems to parallel ideas we know about stereographic projection, which will be discussed in the next section. What we can take from this figure, along with his work in "Analytical Representation of Direction," is his understanding of how the complex plane would work and examples of how he may apply this knowledge to triangulation.

We can see from his "Analytical Representation of Direction," Wessel understood the complex identities. He was able to state how the complex plane could be drawn with two perpendicular lines, one real and one imaginary, and explained that if you consider 1 to be 0 degrees, -1 to be 180 degrees, ϵ to be 90 degrees and $-\epsilon$ to be 270 degrees, how taking two imaginary units would lead to -1, or $\epsilon \cdot \epsilon = -1$. From this information, and figure 2.5, one can allude to what an image of this description would be, as seen in figure 2.6 [3]. He also understood that, given a segment ab and another segment bc such that ab and bc share a common endpoint that ac could be considered the sum of the two segments. Then

$$ac = ab + bc$$
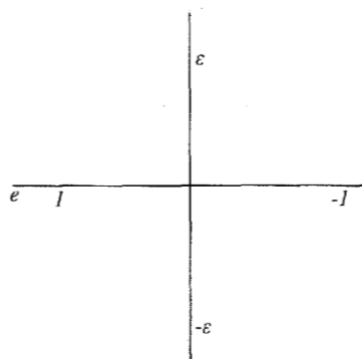
or, when $ca = -ac$.

$$0 = ab + bc + ca \tag{2.5}$$

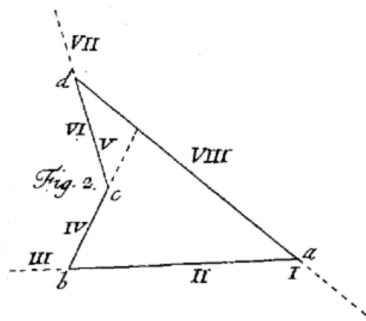Figure 2.6: Wessel's Complex Plane

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Figure 2.7: Wessel's Original Figure 2
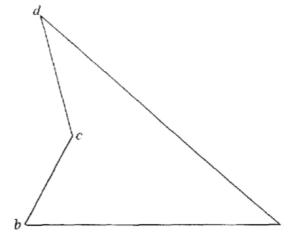
Figure 2.8: Wessel's Figure 2 Modified

He even explains that a line segment on his complex plane, $a + \epsilon b$ could be represented with

$$a = r cos v \text{ and } b = r sin v,$$

then,

$$a + \epsilon b = r(cos v + \epsilon sin v) \tag{2.6}$$

While we know when working with he imaginary unit, Wessel denoted it as ϵ, the following calculations will use the letter "i". And, when explaining how to find unknown elements in a polygon, he created a system of abbreviations using roman numerals, as can be seen in Figure 2.7, but to understand the math behind his calculations, I find it better to use letters as traditionally used in finding information on complex numbers and polygons, as seen in Figure 2.8. Let the inner polygon angles be represented by their capital letter. In other words, at point a, let the inner polygon angle be A. Let the length ab in figure (2.8) represent itself. Then,

$$bc \text{ deviates from } ab \text{ by } 180 - B$$

$$cd \text{ deviates from } ab \text{ by } (180 - B) + 180 + C, \text{ or by } 360 - (A + C)$$

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$$da \text{ deviates from } ab \text{ by } 180 + A$$

Using (2.5) and (2.6), the above can be rewritten as

$$|ab|+|bc|[cos(180-B)+isin(180-B)]+|cd|[cos(360-(B+C))+isin(360-(B+C))]$$

$$+|da|[cos(180 + A) + isin(180 + A)] = 0 \tag{2.7}$$

where $A + B + C + D = 360$

This formula is how Wessel would obtain unknown elements in a quadrangle during his surveying. An example of how he would apply (2.7) goes as follows:

Consider quadrangle (2.8) where angles A, B, and side $|cd|$ are unknown. and let us consider we are just looking for B. We must rewrite (2.7) so that $|cd|$ no longer is a coefficient to a complex number. This is done by multiplying $cos(B + c) + isin(B + C)$ to (2.7) resulting in:

$$|ab|[cos(B + C) + isin(B + C)] + |bc|[cos(180 + C) + isin(180 + C)] + |cd| +$$
$$|ad|[cos(180 - D) + isin(180 - D)] = 0$$

This works only if the imaginary part's coefficient is equal to 0, or:

$$|ab|sin(B + C) + |bc|sin(180 + C) + |ad|sin(180 - D) = 0$$

Now, the only unknown left would be B, making it easy to find.

2.4.2 Celestial Navigation

The complex plane can be used to simplify measurements found in celestial navigation. Celestial Navigation involves measuring altitudes of either the Sun, Moon, planets, or stars versus either a reference horizon or visible horizon. Then, a series of sight reduction corrections must

be used to come to a more accurate result of the position the user is at. These corrections can be simplified significantly with the use of the complex plane and stereographic projection.

Consider a sphere, like in Figure (2.9), with a North Pole, labeled N, and South Pole, labeled S, and lying it on the complex plane so that the South Pole intersects the complex plane at it's origin [14]. Now, if one chooses a point on the sphere, like p in (2.9), and creates a line intersecting the point p and the North pole, one will find a point the line intersects on the complex plane as well, labeled z_p in (2.9). The point found on the complex plane is the image of the point taken on the sphere. There is also a circle centered at p shown in (2.9). This circle maps to a circle on the complex plane centered at z_c rather than z_p except when p is either the North pole, N, or South pole, S. This is an example of operations carried out in stereographic projection.

Figure 2.9: Stereographic Projection Figure
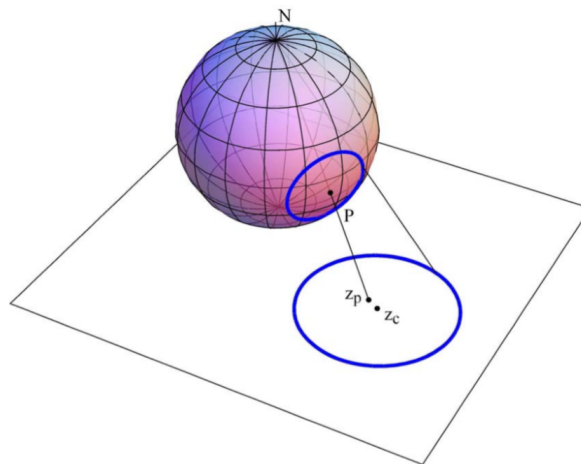
Now let us consider the p's coordinates to be latitude, L, and longitude λ, then the image, z_p, can be written as:

$$z_p = tan(\frac{\pi}{4} + \frac{L}{2})e^{i\lambda} \tag{2.8}$$

The South Pole, S, maps to zero on the complex plane, the longitude at 0 maps to the real axis, and as this longitude increases, the mapping follows in an anticlockwise direction. Note

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that when L = 90 degrees, $tan(\frac{\pi}{4} + \frac{L}{2})$ is undefined. When finding points closer to the North Pole, N, one should consider the complex plane to intersect the North Pole and use the function:

$$z_p = tan(\frac{\pi}{4} - \frac{L}{2})e^{-i\lambda} \tag{2.9}$$

One use of this complex equation is in computing the altitude and azimuth of a celestial body. This was essential for celestial navigators. To find the altitude and azimuth, the observer would need the latitude and longitude of their current location. They would then convert these coordinates to radians and input them into

$$a = e^{\frac{-i\lambda}{2}}, \tag{2.10}$$

$$b = tan(\frac{\pi}{4} + \frac{L}{2})e^{i(\frac{\lambda}{2}+\pi)} \tag{2.11}$$

and into equation (2.8) when working with values closer to the South Pole

$$z = tan(\frac{\pi}{4} + \frac{\delta}{2})e^{-i(GHA)} \tag{2.12}$$

or into equation (2.9) when working with values closer to the North Pole

$$z = tan(\frac{\pi}{4} - \frac{\delta}{2})e^{i(GHA)}$$

where $L = \delta$ and $\lambda = -GHA$

The values a,b and z are then used to perform a bilinear rotation using the equation

$$T(z) = \frac{ax + b}{-\bar{b}z + \bar{a}} \tag{2.13}$$

This will lead to a complex number

$$\omega = tan(\frac{\pi}{4} - \frac{h}{2})e^{iZ}$$

where h is the celestial body's altitude and Z is it's azimuth

An example of the application of this complex equation. Saint-Hilaire [14] takes an altitude sight of the star Vega at 8 o'clock on October 24, 1874. The longitude and hour angles were measured from the meridian of Paris and are:

$$L = 35°30'N, \ \lambda = 9°30'W$$

Next, we convert the values to radians and substitute these values into equations (2.10) and (2.11).

$$a = e^{\frac{-i\lambda}{2}}$$

$$= e^{-\frac{i0.1658062789}{2}}$$

$$= cos\left(\frac{-0.1658062789}{2}\right) - isin\left(\frac{-.1658062789}{2}\right)$$

$$= 0.99656550249 + 0.08280820749i$$

and

$$b = tan\left(\frac{\pi}{4} + \frac{L}{2}\right)e^{i\left(\frac{\lambda}{2} + \pi\right)}$$

$$b = tan\left(\frac{\pi}{4} + \frac{0.6195918845}{2}\right)e^{i\left(\frac{0.1658062789}{2} + \pi\right)}$$

$$= 1.941619979169439e^{i3.224495793039793}$$

$$b = -1.934951490 + 0.160782070i$$

For Equation (2.12), we will find the location of the star Vega at this time on this day using reference to Greenwich.

$$GHA = 62°16'00'', \delta = 38°40'13''$$

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First we will convert these values to radians for easier calculation.

$$GHA = 1.0867583476, \delta = 0.6749236699$$

Next, we will input these values into Equation (2.12)

$$z = tan(\tfrac{\pi}{4} + \tfrac{\delta}{2})e^{-i(GHA)}$$

$$z = tan(\tfrac{\pi}{4} + \tfrac{0.6749236699}{2})e^{-i(1.0867583476)}$$

$$z = 2.081111817929621e^{-i1.842040022620980}$$

$$z = 0.968460094 - 1.842040022i$$

Values a, b, and z are then applied to Equation (2.13)

$$T(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$$

$$T(z) = \frac{(0.996565502+0.082808207i)(0.968460094-1.842040022i)+(1.934951490+0.160782070i)}{(-1.934951490-0.160782070i)(0.968460094-1.842040022i)+(0.996565502-0.082808207i)}$$

$$T(z) = 0.134118059 - 0.355732135i = 0.380174967115913e^{-i1.210256048409223}$$

Then Z is,

$$Z = -1.210256049739083$$

Then, to find h, we rewrite our equation for ω

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$$T(z) = \omega = tan(\tfrac{\pi}{4} - \tfrac{h}{2})e^{iZ}$$

$$\frac{0.380174967115913e^{i1.210256048409223}}{e^{i1.210256048409223}} = \omega = tan(\tfrac{\pi}{4} - \tfrac{h}{2})$$

$$\tfrac{\pi}{4} - \tfrac{h}{2} = tan^{-1}(0.380174967427535)$$

$$h = (tan^{-1}(0.380174967427535) - \tfrac{\pi}{4}) * -2 = 0.844196544474261$$

Converting Z and h back to degrees we have:

$$Z = 69°20'33.2''W$$

$$h = 48°22'8''$$

Saint- Hilaire's findings yielded similar results with error due to less significant figures used

$$Z = 69°20'W$$

$$h = 48°22'15''$$

2.4.3   Global Positioning System

The Global Positioning System, or GPS, is a navigational system that uses satellite signals to fix the location of a radio receiver on or above the earth's surface. There are many uses of complex numbers in this intricate navigational system, but we will explore a use in GPS signals.

Using polar form of complex numbers, we get what is known as an "imaginary exponential" used in the study of GPS signals and receivers[11]

$$x(t) = Ae^{j(2\pi f_0 t + \theta)}$$

where A stands for amplitude, j represents $\sqrt{-1}$, f_0 for frequency, t for time, and θ for the phase.

Then, by using equation (2.4), one can get

$$Ae^{j(2\pi f_0 t + \theta)} = A(cos(2\pi f_0 t + \theta) + jsin(2\pi f_0 t + \theta)$$

By adding the conjugate,

$$Ae^{j(2\pi f_0 t + \theta)} + Ae^{-j(2\pi f_0 t + \theta)} = A(cos(2\pi f_0 t + \theta) + A(cos(2\pi f_0 t + \theta) + jsin(2\pi f_0 t + \theta) - jsin(2\pi f_0 t + \theta)$$

$$Ae^{j(2\pi f_0 t + \theta)} + Ae^{-j(2\pi f_0 t + \theta)} = 2Acos(2\pi f_0 t + \theta)$$

Rearranging,

$$Acos(2\pi f_0 t + \theta) = \tfrac{A}{2}(e^{j(2\pi f_0 t + \theta)} + e^{-j(2\pi f_0 t + \theta)})$$

$$= Re\{Ae^{j(2\pi f_0 t + \theta)}\}$$

When subtracting the conjugate,

$$Ae^{j(2\pi f_0 t + \theta)} - Ae^{-j(2\pi f_0 t + \theta)} = A(cos(2\pi f_0 t + \theta) - A(cos(2\pi f_0 t + \theta) + jsin(2\pi f_0 t + \theta) + jsin(2\pi f_0 t + \theta)$$

$$Ae^{j(2\pi f_0 t + \theta)} - Ae^{-j(2\pi f_0 t + \theta)} = 2Ajsin(2\pi f_0 t + \theta)$$

Rearranging,

$$Asin(2\pi f_0 t + \theta) = \tfrac{A}{2j}(e^{j(2\pi f_0 t + \theta)} - Ae^{-j(2\pi f_0 t + \theta)})$$

$$= Im\{Ae^{j(2\pi f_0 t + \theta)}\}$$

One use of this function in relation to GPS signals is it's involvement in ambiguity functions. An ambiguity function allows for satellite detection and estimation of received signal parameters, these parameters being the code phase offset and Doppler frequency. The code phase offset is the signal used for timing measurements, allowing to improve the accuracy of GPS. The Doppler effect is the "apparent change in the transmission frequency of the received signal and is experienced whenever there is any relative motion between the emitter and receiver of wave signals"[2]. The ambiguity function is

$$\tilde{R}(\triangle \tau, \triangle f_D) = \tfrac{1}{T_{CO}} \int_{T_{CO}}^{0} x(t - \tau)x(t - \hat{\tau})e^{j(2\pi \triangle f_D t + \triangle \theta)}dt$$

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Where j is the imaginary unit, T_CO is the coherent averaging time, x is code from a satellite, τ is the code phase offset, $\triangle f_D$ is the Doppler Error, $\triangle \tau$ is the code phase error, and $\triangle \theta$ is the phase of the signal such that

$$\triangle \theta = \delta \theta - \hat{\theta}$$

where $\delta \theta$ is the input phase and $\hat{\theta}$ is the receiver's best estimate of phase. To estimate the Doppler frequency and the code phase offset, one first needs to understand the equation

$$\tilde{S} = S_I(\triangle \tau, \triangle f_D, \triangle \theta) + j S_Q(\triangle \tau, \triangle f_D, \triangle \theta)$$

where

$$S_I(\triangle \tau, \triangle f_D, \triangle \theta) = \frac{\sqrt{C}D}{T_{CO}} \int_0^{T_{CO}} c(t - \tau) x(t - \hat{\tau}) \cos(2\pi \triangle f_D t + \triangle \theta) dt$$

and

$$S_Q(\triangle \tau, \triangle f_D, \triangle \theta) = \frac{\sqrt{C}D}{T_{CO}} \int_0^{T_{CO}} c(t - \tau) x(t - \hat{\tau}) \sin(2\pi \triangle f_D t + \triangle \theta) dt$$

Rewriting $\tilde{S}$ in exponential form we get

$$\tilde{S} = \sqrt{C}De^{j\triangle\theta \tilde{R}(\triangle \tau, \triangle f_D)}$$

The GPS Reciever takes the magnitude of this equation to remove the two "nuiance parameters" D, the data bit, and $\triangle \theta$, the carrier phase offset

$$\tilde{S} = S_1^2 + S_Q^2 = C|\tilde{R}(\triangle \tau, \triangle f_D)|^2$$

This can then be used to estimate the Doppler frequency and code phase offset.

Chapter 3

Quaternions

3.1   Introduction

An Irish Mathematician by the name of William Rowan Hamilton was so intrigued by how complex numbers worked in two-dimensional geometry that he wanted to discover "triplets" that would work the same way, but in 3D. Alas, he never could come up with a working model of this. Instead, he discovered the 4-D division algebra he named quaternions. He carved the equation $i^2 = j^2 = k^2 = ijk = -1$ into the Brougham Bridge where he had his eureka moment in 1843 [7].
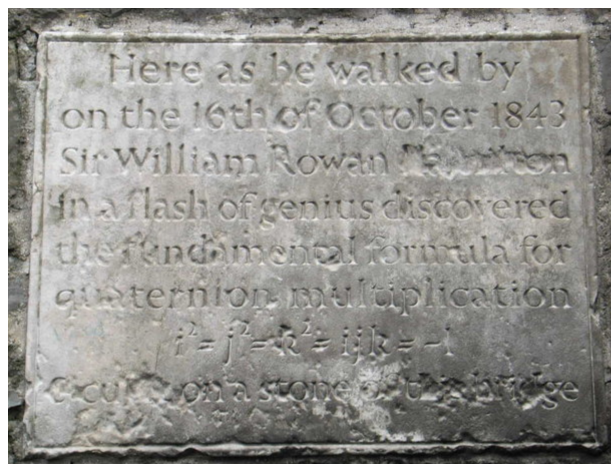
Quaternion Plaque on Brougham Bridge

We can easily read about his success with the quaternion, but there is little explanation on his failure with the creation of three dimensional numbers. According to a paper published on

the University of London website, Hamilton wanted his triples to work in a similar manner to complex numbers. The problem with this is he could not define the multiplication for triples, as he did for quaternions stated earlier. Let us take a 3 dimensional number $t = a + bi + cj$ and lets assume $i^2 = j^2 = ij = -1$ Then

$$ij = -1$$
$$iij = -1i$$
$$i^2 j = -i$$
$$-j = -i$$
$$j = i$$
$$jj = ij$$
$$j^2 = -1$$
$$ijj = -1j$$
$$ij^2 = -j$$
$$-i = -j$$
$$i = j$$

If i = j, then the 3 dimensional number t=a+bi+cj is the same as t=a+(b+c)i, which is just a complex number, losing that third value we need. There are numerous methods to define the initial value equivalencies, but after 8 years of trying to define it, I believe we can take William Howard's word that defining such a number would prove difficult if even possible.

It is important to note that while Hamilton is the first known person to define quaternions algebraically and give the system a name, a mathematician by the name of Benjamin Olinde Rodrigues actually understood and geometrically explained what we call quaternions before Hamilton. In 1840, he published a paper describing how one could represent one rotation and then another by one rotation about a third axes. He expressed this solution using a scalar and a 3D axis, which Hamilton didn't discover for a couple of years after his quaternion discovery. While Rodrigues' paper was published before Hamilton's discovery, the paper was known by few.

3.2 Properties

Quaternions can be compared to their 2D counterpart, the complex number. As we know, a complex number consists of a real part and imaginary part, denoted by $a + bi$. Quaternions consist of one real and three imaginary parts and can be denoted in three ways:

$$q = x + iy + jz + kw$$
$$q = s + v$$
$$q = [s, v]$$

where $i^2 = j^2 = k^2 = -1$, s, x, y, and $z \in \mathbb{R}$, and $v \in \mathbb{R}^3$

And while quaternions have a similar way of carrying out mathememathical operations, a curious fact to note is quaternions non-commutative properties. Unlike complex numbers, where $abi = bia$, changing the placement of a variable during multiplication with quaternions can cause a sign flip. For example,

Let's consider ijk=-1

$$iijk = i - 1$$
$$i^2 jk = -i$$
$$-jk = -i$$
$$jk = i$$

However, kj=-i as you can see below

$$ijkk = -1k$$
$$ijk^2 = -k$$
$$-ij = -k$$
$$ij = k$$
$$ijj = kj$$
$$ij^2 = kj$$
$$-i = kj$$

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Likewise j=ki, –j=ik , k=ij and –k=ji as seen below in the table below [15].

	-1	$-i$	$-j$	$-k$	1	i	j	k
-1	1	i	j	k	-1	$-i$	$-j$	$-k$
$-i$	i	-1	k	$-j$	$-i$	1	$-k$	j
$-j$	j	$-k$	-1	i	$-j$	k	1	$-i$
$-k$	k	j	$-i$	-1	$-k$	$-j$	i	1
1	-1	$-i$	$-j$	$-k$	1	i	j	k
i	$-i$	1	$-k$	j	i	-1	k	$-j$
j	$-j$	k	1	$-i$	j	$-k$	-1	i
k	$-k$	$-j$	i	1	k	j	$-i$	-1

Table 3.1: Quaternion Multiplication Table

While quaternions aren't commutative, they do hold associative properties.

$$i(jk) = (ij)k$$

$$i(i) = (k)k$$

$$-1 = -1$$

Other properties include:

The identity $q_e = 1$

The conjugate $q^* = x - yi - zw - wk$

The length of a quaternion (aka norm) $|q| = \sqrt{qq^*}$

The inverse of a quaternion $q^{-1} = \frac{q^*}{|q|^2}$

Quaternions are able to be added and multiplied similarily to complex numbers. Lets consider

$$z_1 = a_1 + ib_1 \text{ and } z_2 = a_2 + ib_2 \text{ where } z_1, z_2 \in C$$

And

$$q_1 = x_1 + iy_1 + jz_1 + kw_1 \text{ and } q_2 = x_2 + iy_2 + jz_2 + kw_2 \text{ where } q_1, q_2 \in H$$

Just as

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

We can add quaternions by combining like terms, for example

$$q_1 + q_2 = (x_1 + x_2) + i(y_1 + y_2) + j(z_1 + z_2) + k(w_1 + w_2)$$

Also, just as we can multiply complex numbers with the understanding that $i^2 = -1$, we can multiply quaternions with the understanding $i^2 = j^2 = k^2 = ijk = -1$. Manipulating this, as seen earlier, we can also understand $i = jk$, $j = ik$, and $k = ij$

Then, just as two complex numbers can be multiplied,

$$
\begin{aligned}
z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\
&= a_1 a_2 + i(a_1 b_2 + a_2 b_1) + i^2 b_1 b_2 \\
&= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)
\end{aligned}
$$

quaternions can similarly be multiplied,

$$q_1 \cdot q_2 = (x_1 + iy_1 + jz_1 + kw_1)(x_2 + iy_2 + jz_2 + kw_2) \tag{3.1}$$

$$
\begin{aligned}
= x_1 x_2 + i x_1 y_2 + j x_1 z_2 + k x_1 w_2 + i y_1 x_2 + i^2 y_1 y_2 + i j y_1 z_2 + i k y_1 w_2 + j z_1 x_2 + j i z_1 y_2 + \\
j^2 z_1 z_2 + j k z_1 w_2 + k w_1 x_2 + k i w_1 y_2 + k j w_1 z_2 + k^2 w_1 w_2
\end{aligned}
$$

$$
\begin{aligned}
= x_1 x_2 + i x_1 y_2 + j x_1 z_2 + k x_1 w_2 + i y_1 x_2 - y_1 y_2 + k y_1 z_2 - j y_1 w_2 + j z_1 x_2 - k z_1 y_2 - z_1 z_2 + \\
i z_1 w_2 + k w_1 x_2 + j w_1 y_2 - i w_1 z_2 - w_1 w_2
\end{aligned}
$$

$$
\begin{aligned}
q_1 \cdot q_2 = (x_1 x_2 - y_1 y_2 - z_1 z_2 - w_1 w_2) + i(x_1 y_2 + y_1 x_2 + z_1 w_2 - w_1 z_2) \\
+ j(x_1 z_2 - y_1 w_2 + z_1 x_2 + w_1 y_2) + k(x_1 w_2 + y_1 z_2 - z_1 y_2 + w_1 x_2)
\end{aligned}
$$

Now let's look at properties of inverse:

$$\frac{q^*}{|q|^2} = \frac{x-yi-zj-wk}{\sqrt{((x+yi+zj+wk)(x-yi-zj-wk))}}$$

$$= \frac{x-yi-zj-wk}{(x+yi+zj+wk)(x-yi-zj-wk)}$$

$$= \frac{x-yi-zj-wk}{x^2-xyi-xzj-xwk+xyi-y^2i^2-yzij-ywik+xzj-yzji-z^2j^2-zwjk+xwk-wyki-wzkj-w^2k^2}$$

$$= \frac{x-yi-zj-wk}{x^2-xyi-xzj-xwk+xyi+y^2-yzk-ywj+xzj+yzk+z^2-zwi+xwk+wyj+wzi+w^2}$$

$$= \frac{x-yi-zj-wk}{x^2+y^2+z^2+w^2}$$

To divide quaternions, we multiply one quaternion to the inverse of the quaternion to be divided.

Let $q_1 = a + bi + cj + dk$ and $q_2 = x + yi + zj + wk$, then

$$\frac{q_1}{q_2} = q_1 q_2^{-1} = a + bi + cj + dk \times \frac{x-yi-zj-wk}{x^2+y^2+z^2+w^2}$$

$$= \frac{(a+bi+cj+dk)(x-yi-zj-wk)}{x^2+y^2+z^2+w^2}$$

$$= \frac{ax-ayi-azj-awk+bxi-byi^2-bzji-bwik+cjx-cyji-czj^2-cwjk+dkx-dyki-dzkj-dwk^2}{x^2+y^2+z^2+w^2}$$

$$= \frac{ax-ayi-azj-awk+bxi+by-bzk-bwj+cxj-cyk+cz+cwi+dxk+dyj-dzi+dw}{x^2+y^2+z^2+w^2}$$

$$= \frac{(a+by+cz+dw)+(-ay+bx+cw-dz)i+(-az-bw+cx+dy)j+(-aw-bz-cy+dx)k}{x^2+y^2+z^2+w^2}$$

If we multiply a quaternion by it's own inverse, will the value be equivalent to multiplying the inverse by a quaternion?

Let $q = x + yi + zj + wk$

And as simplified above

$$\frac{q^*}{|q|^2} = \frac{x-yi-zj-wk}{x^2+y^2+z^2+w^2}$$

$$\frac{qq^*}{|q|^2} = \frac{(x+yi+zj+wk)(x-yi-zj-wk)}{x^2+y^2+z^2+w^2}$$

$$= \frac{x^2+y^2+z^2+w^2}{x^2+y^2+z^2+w^2} = 1$$

This shows that multiplying the inverse from the left or multiplying it from the right will yield the same result of 1. This means a quaternion multiplied by its multiplicative inverse is commutative, an unexpected fact given normally multiplying quaternions is not commutative.

Quaternions and inverse quaternions have the possibility of being zero divisors. This is in part by the norm of a quaternion, $|q| = \sqrt{qq^*}$. There is a theorem that states "The quaternion algebra $(\frac{(a,b)}{F})$ is a division algebra if and only if its norm form $N : (\frac{a,b}{F})F$ satisfies $N(q) = 0 \Rightarrow q = 0$, i.e. the norm form is anisotropic." By this theorem, we can consider the contrapositive. If the norm does not satisfy $N(q) = 0 \Rightarrow q = 0$, then the quaternion algebra is not a division algebra. This means there can be cases where the quaternion and it's inverse being multiplied is not equal to 0 and taking the norm will become 0, but note this means it will not be a division algebra.

3.3   Rotations

Quaternions have proved to be a great help in advancing computer games, robotics, and spaceflight to name a few. With it, programmers can avoid gimbal lock. A gimbal is defined as "a mechanism, typically consisting of rings pivoted at right angles, for keeping an instrument such as a compass or chronometer horizontal in a moving vessel or aircraft." The gimbals that would result in a gimbal lock are three ring gimbals in which each ring represents a different axis allowing for 3d rotations. A gimbal lock is the loss of a degree of freedom when two axes get into a parallel figuration causing all new rotations to only occur about the axis not

parallel. Figure 3.1 shows on the left an object where the three gimbals are in an independent configuration and therefore with three degrees of freedom [9]. On the right shows two of the axles having got into a parallel configuration, leaving the object only able to move around a vertical axis and therefore being gimbal locked [10].
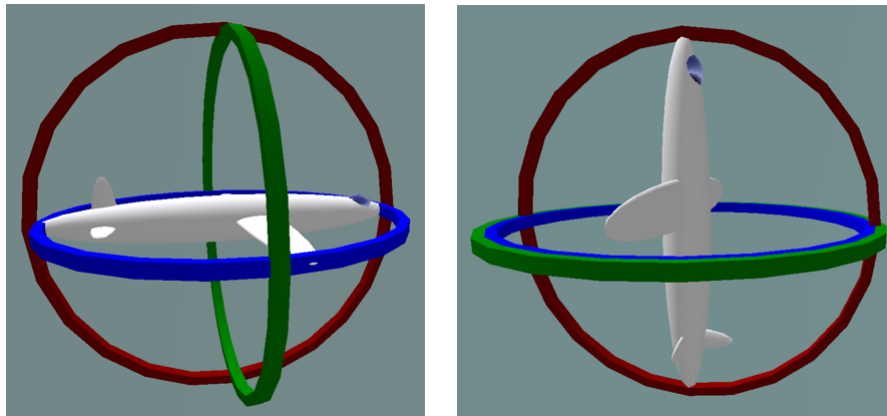

Figure 3.1: Gimbal Lock Example

The most notable occurance of gimbal lock would be Apollo 11. In a race against time, NASA scientists decide to use a three gimbal mechanism instead of four gimbals. According to Paul Fjeld, a writer of the Apollo Lunar Surface Journal, "If the middle (Z) gimbal was +-90 degrees roll, the inner gimbal axis would be coincident with the outer gimbal axis and you would loose a dimension. Near that point, in a closed stabilization loop, the torque motors could theoretically be commanded to flip the gimbal 180 degrees instantaneously. Instead, in the LM, the computer flashed a 'gimbal lock' warning at 70 degrees and froze the IMU at 85 degrees, flashing an unfriendly 'no att' light. Then it waited for a realignment." [6] Quaternions are a solution to this problem of gimbal lock by allowing one to take a single rotation equivalent to the set of three gimbal rotations that, if done in the wrong order, creates the problem of gimbal lock.

Unit Quaternions are able to represent any rotations in R^3 where

$$q = w + xi + yj + zk$$

and

$$w^2 + x^2 + y^2 + z^2 = 1$$

The real part of the equation, w, is thought of as a scalar part while the rest of the equation, $xi + yj + wk$, is thought of as an imaginary part, or the vector part. The i, j, and k axis are all perpendicular to each other and the real axis. To rotate an object in 3D, you would need a vector, a point p, and the degrees with which you are trying to rotate the object. You will then multiply the sine of half the angle to the vector and then add the cosine of the half the angle needed to create the quaternion that will represent the rotation of this object. This can be written as:

$$p = ai + bj + ck$$

and

$$q = cos(\frac{\theta}{2}) + sin(\frac{\theta}{2})(xi + yj + zk) \tag{3.2}$$

where $x^2 + y^2 + z^2 = 1$

Note that we must divide the angle to be rotated around in equation (3.2) by 2 to accurately implement the given angle. This is because we apply the quaternion twice, as will be seen next. To carry out the rotation you would multiply q to p and then take that product and multiply it to the inverse of q, also known as conjugating p by q. This can be written as:

$$qpq^{-1} \tag{3.3}$$

or

$$(cos(\tfrac{\theta}{2}) + sin(\tfrac{\theta}{2}))(xi + yj + zk) \cdot (ai + bj + ck) \cdot \frac{-(cos(\frac{\theta}{2})+sin(\frac{\theta}{2})(xi+yj+zk))}{|cos(\frac{\theta}{2})+sin(\frac{\theta}{2})(xi+yj+zk)|^2}$$

Multiplying two quaternions has an equivalent relationship to scaling one quaternion by the magnitude of the other and then applying a 4 dimensional rotation. This can be written as:

$$q_1 \cdot q_2 = (\tfrac{q_1}{||q_1||})||q_1|| \cdot q_2$$

While a quaternion, q, and the negative of that quaternion, or –q, represent the same rotation, the rest of the relationships between rotations and quaternions are unique. As we learned earlier, quaternion multiplication is not commutative. Representing it with rotations, you can see how quaternions are not commutative visually. For example, consider cube i in Figure 3.2. If the cube is rotated $45°$ around the x-axis and then $45°$ around the y-axis, it results in ii. If cube i is rotated $45°$ around the y-axis first, and then rotated around the x-axis by $90°$, it results in cube iii. As one can see, cubes ii and iii are in two different positions [5].
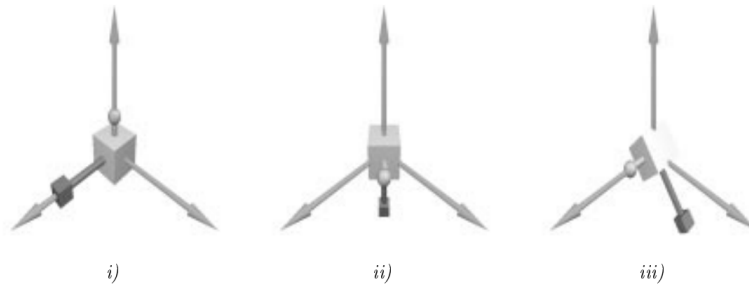

i)   *ii)*   *iii)*

Figure 3.2: Quaternion Rotation Example

3.4   Applications

3.4.1   Space Navigation

To properly navigate an aircraft, it is essential to understand the aircraft's rotations to properly translate the compass. To do this, it is essential to predict the aircraft's rotation movements

before they are executed. These directions can be expressed as vectors emerging from the cockpit, described in the aircraft's frame. One way to do this would be to use Euler angles, but this is very costly and the calculations are more involved and complex, creating an oppurtunity for greater error. A less expensive way to calculate rotations would be to use quaternions.

To rotate with quaternions, one needs the rotation quaternion

$$Q_n(\theta) = (cos(\theta/2), n sin(\theta/2))$$

where n is a given unit row vector and θ is a given angle.

A double multiplication will be applied using rotation quaternions:

$$(0, x') = Q_n(\theta)(0, x)Q_{-n}(\theta) \tag{3.4}$$

Where x is a given vector.

Let's consider an example in which a vector (2,0,0) needs to be rotated by 90 degrees about the y axis [8]. Using equation (3.4)

$$(0, x') = (cos(\tfrac{90}{2}), (0, 1, 0)sin(\tfrac{90}{2}))(0, (2, 0, 0))(cos(\tfrac{90}{2}), (0, -1, 0)sin(\tfrac{90}{2})$$

$$= (\tfrac{\sqrt{2}}{2}, (0, 1, 0)\tfrac{\sqrt{2}}{2})(0, (2, 0, 0))(\tfrac{\sqrt{2}}{2}, (0, -1, 0)\tfrac{\sqrt{2}}{2})$$

$$= \tfrac{1}{2}(1, 0, 1, 0)(0, 2, 0, 0)(1, 0, -1, 0)$$

$$= (1, 0, 1, 0)(0, 1, 0, 0)(1, 0, -1, 0)$$

Now applying equation (3.1)

$$= (0, 1, 0, -1)(1, 0, -1, 0)$$
$$(0, 0, 0, -2)$$

This idea can be implemented to determine the aircraft's axes so that a pilot may know information like where the compass direction will lie, where "up/down" is, and, if another aircraft is nearby, where it could be seen from their point of view.

Given vectors N, E, D, which stands for north, east, and down respectively, we will convert these values to to vectors x,y and z which describes the aircraft's axes. Let

$$N= \begin{bmatrix} -0.429 \\ 0.379 \\ 0.820 \end{bmatrix}, E= \begin{bmatrix} -0.663 \\ -0.749 \\ 0 \end{bmatrix}, D= \begin{bmatrix} 0.614 \\ -0.543 \\ 0.572 \end{bmatrix}$$

Now let $x_0 = N, y_0 = E, and z_0 = D$.

In this example, we want to rotate the aircraft axes around z_0 by 45 degrees, and then rotate about our new y_0, called y_1, by 20 degrees.

First, we will find x_1, y_1, and z_1 we will use equation (3.4)

For x_1

$$x_1 = (cos(45/2), z_0 sin(45/2)(0, x_0)(cos(45/2), -z_0 sin(45/2))$$

$$x_1 = (0.9239, 0.2350, -0.2078, 0.2189)(0, -0.4290, 0.3790, 0.8200)(0.9239, -0.2350, 0.2078, -0.2189)$$

$$x_1 = (0, -0.7716, -0.2615, 0.5797)$$

For y_1

$$y_1 = (cos(45/2), z_0 sin(45/2)(0, y_0)(cos(45/2), -z_0 sin(45/2))$$

$$y_1 = (0.9239, 0.2350, -0.2078, 0.2189)(0, -0.6630, -0.7490, 0)(0.9239, -0.2350, 0.2078, -0.2189)$$

$$y_1 = (0, -0.1660, -0.7978, -0.5798)$$

For z_1

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$$z_1 = (cos(45/2), z_0 sin(45/2)(0, z_0)(cos(45/2), -z_0 sin(45/2) = z_0$$

$$z_1 = (0, 0.614, -0.543, 0.572)$$

Note that rotating z around itself does not change it's location.

Now, we will rotate x_1, y_1, and z_1 around y_1 to get x, y, and z needed.

For x,

$$x = (cos(20/2), y_1 sin(20/2)(0, x_1)(cos(20/2), -y_1 sin(20/2)$$

$$x = (0.9848, -0.0288, -0.1244, -0.1212)(0, -0.7716, -0.2615, 0.5797)(0.9848, 0.0288, 0.1244, 0.1212)$$

$$x = (0, -0.9279, -0.0246, 0.3739)$$

For y,

$$y = (cos(20/2), y_1 sin(20/2)(0, y_1)(cos(20/2), -y_1 sin(20/2)$$

$$y = y_1$$

Note: rotating around y_1 around the y-axis does not change it's location

For z,

$$z = (cos(20/2), y_1 sin(20/2)(0, z_1)(cos(20/2), -y_1 sin(20/2)$$

$$z = (0.9848, -0.0288, -0.1244, -0.1212)(0, 0.6140, -0.5430, 0.5720)(0.9848, 0.0288, 0.1244, 0.1212)$$

$$z = (0, 0.3079, -0.6190, 0.7230)$$

Now we convert the quaternions back to vector form, resulting in

$$x = \begin{bmatrix} -0.9279 \\ -0.0246 \\ 0.3739 \end{bmatrix}, y = \begin{bmatrix} -0.1660 \\ -0.7978 \\ -0.5798 \end{bmatrix}, z = \begin{bmatrix} 0.3079 \\ -0.6190 \\ 0.7230 \end{bmatrix}$$

More detailed calculations done in Matlab can be found in Appendix A. This solution help to give a set of components required to allow another aircraft to have coordinates specific coordinates in the frame of our aircraft. To find the coordinates of the other aircraft within the frame of our aircraft, we take the dot product of our solution to a vector representing the other aircraft relative to us. This vector is

$$r_{them \leftarrow us} = \begin{bmatrix} -725 \\ -921 \\ 92 \end{bmatrix}$$

where r is in km.

3.4.2   3D Game Navigation

Similarly to Aerospace navigation, computer graphics rely on quaternions to rotate objects in 3D simulations, video games, robots and virtual reality. And, just as Apollo 11 had a problem with gimbal lock, computer games can also run into this problem, and avoid it with quaternions. In any of the above computer graphics situations, to properly navigate through, one needs smooth rotations. If one rotates a character in a 3D game simulation if the object isn't able to smoothly turn, ie when rotating the character to the left it goes straight to a 90 degree turn with no ability to turn it any degree between, the controller would not be able to properly navigate the character through the game. Interpolating quaternions allow for smooth rotations giving the

character the ability to turn with more precision. Normally one starts with a given vector. The spherical interpolant for a vector is

$$v = \frac{sin(1-t)\theta}{sin\theta} v_1 + \frac{sint\theta}{sin\theta} v_2$$

Simply replacing v_1 and v_2 with q_1 and q_2. the spherical interpolant for quaternions is:

$$q_s = \frac{sin(1-t)\theta}{sin\theta} q_1 + \frac{sint\theta}{sin\theta} q_2 \qquad (3.5)$$

where

$$q_1 = [s_1, x_1 i + y_1 j + z_1 k]$$
$$q_2 = [s_2, x_2 i + y_2 j + z_2 k]$$

and

$$\theta = cos^{-1}\left(\frac{q_1 \cdot q_2}{|q_1||q_2|}\right)$$

or, if working with unit-norm quaternions,

$$\theta = cos^{-1}(s_1 s_2 + x_1 x_2 + y_1 y_2 + z_1 z_2) \qquad (3.6)$$

Then, we will substitute q_s into equation (3.3) using the vector method below:

$$p' = \begin{bmatrix} 2(s^2 + x^2) - 1 & 2(xy - sz) & 2(xz + sy) \\ 2(xy + sz) & 2(s^2 + y^2) - 1 & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & 2(s^2 + z^2) - 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} \qquad (3.7)$$

where $[0, p'] = qpq^{-1}$

Let's consider a simple example [17]. Let $v_1 = (0, 1, 0)$ and $v_2 = (1, 0, 0)$ Consider two scenerios: a point at (0,1,1) that is rotated 90 degrees about v_1 and then a point at (0,1,1) that is rotated 90 degrees about the v_2.

$$q_1 = [cos\tfrac{1}{2}\theta, sin\tfrac{1}{2}\theta v_1]$$

$$q_1 = [cos(45), sin(45)j]$$

$$q_1 = [\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}j]$$

$$q_2 = [cos\tfrac{1}{2}\theta, sin\tfrac{1}{2}\theta v_2]$$

$$q_2 = [cos(45), sin(45)i]$$

$$q_2 = [\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}i]$$

Then, by (3.6)

$$\theta = cos^{-1}(\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}) + (0)(\tfrac{\sqrt{2}}{2}) + (\tfrac{\sqrt{2}}{2})(0) + (0)(0))$$

$$\theta = cos^{-1}(\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}) = 60°$$

Now, to find the mid-way point between two interpolated quaternions where t=0.5, one would use (3.5)

$$q_s = \frac{sin((1-0.5)(60))}{sin(60)}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}j] + \frac{sin((0.5)(60))}{sin(60)}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}i]$$

$$q_s = \frac{\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}j] + \frac{\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}i]$$

$$q_s = \frac{1}{\sqrt{3}}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}j] + \frac{1}{\sqrt{3}}[\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}i]$$

$$q_s = [\tfrac{\sqrt{2}}{\sqrt{3}}, \tfrac{1}{\sqrt{6}}i + \tfrac{1}{\sqrt{6}}j]$$

Then substituting into (3.7)

$$p' = \begin{bmatrix} 2((\tfrac{\sqrt{2}}{\sqrt{3}})^2 + (\tfrac{1}{\sqrt{6}})^2) - 1 & 2((\tfrac{1}{\sqrt{6}})(\tfrac{1}{\sqrt{6}}) - (\tfrac{\sqrt{2}}{\sqrt{3}})(0)) & 2((\tfrac{1}{\sqrt{6}})(0) + (\tfrac{\sqrt{2}}{\sqrt{3}})(\tfrac{1}{\sqrt{6}})) \\ 2((\tfrac{1}{\sqrt{6}})(\tfrac{1}{\sqrt{6}}) + (\tfrac{\sqrt{2}}{\sqrt{3}})(0)) & 2((\tfrac{\sqrt{2}}{\sqrt{3}})^2 + (\tfrac{1}{\sqrt{6}})^2) - 1 & 2((\tfrac{1}{\sqrt{6}})(0) - (\tfrac{\sqrt{2}}{\sqrt{3}})(\tfrac{1}{\sqrt{6}})) \\ 2((\tfrac{1}{\sqrt{6}})(0) - (\tfrac{\sqrt{2}}{\sqrt{3}})(\tfrac{1}{\sqrt{6}})) & 2((\tfrac{1}{\sqrt{6}})(0) + (\tfrac{\sqrt{2}}{\sqrt{3}})(\tfrac{1}{\sqrt{6}})) & 2((\tfrac{\sqrt{2}}{\sqrt{3}})^2 + (0)^2) - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the vector between our two scenarios at half-time would be at $[1,0,1]$ as can be seen in the figure below
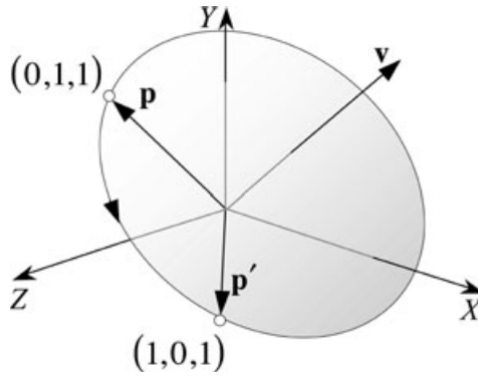


Figure 3.3: Quaternion Interpolation Example

The more points in time the vector is interpolated, the smoother, and more accurate, a turn will be. To get a visual idea of interpolation movements, let us begin by finding the quaternion product of q_1 and q_2.

Let us begin with q_1. Recall $q_1 = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}j]$. Then using Equation (3.7),

$$p'_1 = \begin{bmatrix} 2(\frac{\sqrt{2}}{2}^2 + 0^2) - 1 & 2(0 - \frac{\sqrt{2}}{2}(0)) & 2(0 + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2})) \\ 2((0)(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(0)) & 2((\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2) - 1 & 2((\frac{\sqrt{2}}{2})(0) - (\frac{\sqrt{2}}{2})(0)) \\ 2((0)(0) - (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2})) & 2((\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(0)) & 2((\frac{\sqrt{2}}{2})^2 + (0)^2) - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p_1' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

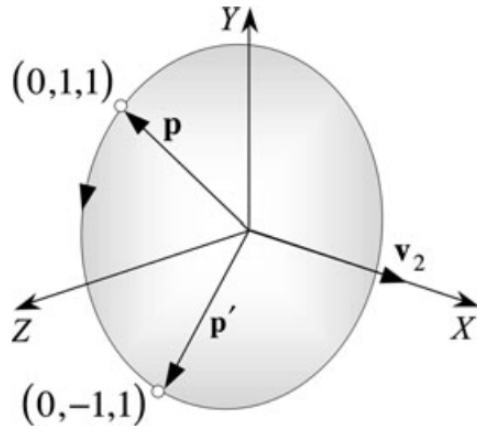$$p_1' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
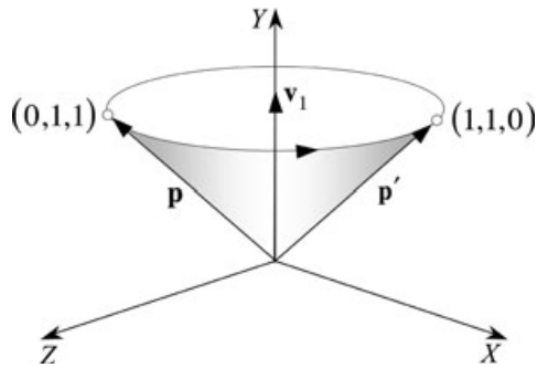
Then, with $q_2 = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}i]$ and using Equation (3.7),
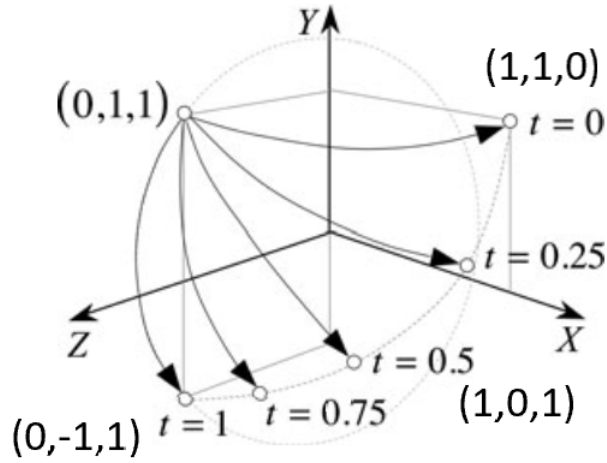
$$p_2' = \begin{bmatrix} 2((\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2) - 1 & 2((\frac{\sqrt{2}}{2})(0) - (\frac{\sqrt{2}}{2})(0)) & 2((\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(0)) \\ 2((\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(0)) & 2((\frac{\sqrt{2}}{2})^2 + (0)^2) - 1 & 2((0)(0) - (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2})) \\ 2((\frac{\sqrt{2}}{2})(0) - (\frac{\sqrt{2}}{2})(0)) & 2((0)(0) + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2})) & 2((\frac{\sqrt{2}}{2})^2 + (0)^2) - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2' = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

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Looking at p'_1, p'_2 and p' altogether, one can see below how the movements would be along a 3D space With p'_1 representing when t=0, p'_2 representing t=1, and p' representing t=0.5.

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Appendix A

```matlab
%Aerospace example

%finding y1. Note y=y1
c=cos(0.785398/2)
s=sin(0.785398/2)

z0= [0.614; -0.543; 0.572]
y0= [-0.663; -0.749; 0]

z0s= z0*s

qz0= [c ; z0s]'

qy0= [0; y0]'

qz0n=[c; -z0s]'

qprod1=quatmultiply(qz0,qy0)

qprod2=quatmultiply(qprod1,qz0n)

c2=cos(0.349066/2)
s2=sin(0.349066/2)

qy1=qprod2

y1= [-0.1660   -0.7162   -0.6978]


%finding x1

c=c
s=s

qz0= qz0
qz0n=qz0n

x0= [0 -0.429 0.379 0.820]

qx1prod=quatmultiply(qz0,x0)

qx1prod2=quatmultiply(qx1prod,qz0n)

x1=qx1prod2

%finding x

xqy1=[c2 s2*y1]

xqy1n=[c2 -s2*y1]
```

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```
qxprod=quatmultiply(xqy1,x1)

qxprod2=quatmultiply(qxprod,xqy1n)

x=qxprod2

% finding z Note: z1=z0

z0= [0 0.614 -0.543 0.572]

z1=z0

qzprod=quatmultiply(xqy1,z1)
qzprod2=quatmultiply(qzprod,xqy1n)

z=qzprod2
```

```
>> thesisaerospaceexample

c =

    0.9239


s =

    0.3827


z0 =

    0.6140
   -0.5430
    0.5720


y0 =

   -0.6630
   -0.7490
         0


z0s =

    0.2350
   -0.2078
    0.2189


qz0 =

    0.9239    0.2350   -0.2078    0.2189


qy0 =

         0   -0.6630   -0.7490         0


qz0n =

    0.9239   -0.2350    0.2078   -0.2189


qprod1 =
```

```
    0.0001   -0.4486   -0.8371   -0.3138
```

qprod2 =

```
   -0.0000   -0.1660   -0.7978   -0.5798
```

c2 =

```
    0.9848
```

s2 =

```
    0.1736
```

qy1 =

```
   -0.0000   -0.1660   -0.7978   -0.5798
```

y1 =

```
   -0.1660   -0.7162   -0.6978
```

c =

```
    0.9239
```

s =

```
    0.3827
```

qz0 =

```
    0.9239    0.2350   -0.2078    0.2189
```

qz0n =

```
    0.9239   -0.2350    0.2078   -0.2189
```

x0 =

```
     0    -0.4290    0.3790    0.8200


qx1prod =

   0.0001   -0.6497    0.0636    0.7575


qx1prod2 =

     0    -0.7716   -0.2615    0.5797


x1 =

     0    -0.7716   -0.2615    0.5797


xqy1 =

   0.9848   -0.0288   -0.1244   -0.1212


xqy1n =

   0.9848    0.0288    0.1244    0.1212


qxprod =

   0.0155   -0.8636   -0.1473    0.4825


qxprod2 =

   0.0000   -0.9279   -0.0246    0.3739


x =

   0.0000   -0.9279   -0.0246    0.3739


z0 =

     0     0.6140   -0.5430    0.5720


z1 =
```

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```
       0     0.6140    -0.5430     0.5720


qzprod =

    0.0195     0.4677    -0.5927     0.6553


qzprod2 =

    0.0000     0.3079    -0.6190     0.7230


z =

    0.0000     0.3079    -0.6190     0.7230

>>
```

différents, c'est-à-dire la quantité x qui satisfait à la proportion

$$+ 1 : + x :: + x : -1.$$

On est arrêté ici comme on l'a été en voulant continuer au delà de o la progression arithmétique décroissante, car on ne peut égaler x à aucun nombre positif ou négatif; mais, puisqu'on a trouvé plus haut que la quantité négative, imaginaire lorsque la numération était appliquée à de certaines espèces de grandeurs, devenait réelle lorsque l'on combinait d'une certaine manière l'idée de *grandeur absolue* avec l'idée de *direction*, ne serait-il pas possible d'obtenir le même succès relativement à la quantité dont il s'agit, quantité réputée imaginaire par l'impossibilité où l'on est de lui assigner une place dans l'échelle des quantités positives ou négatives?

En y réfléchissant, il a paru qu'on parviendrait à ce but si l'on pouvait trouver un genre de grandeurs auquel pût s'allier l'idée de direction, de manière que, étant adoptées deux directions opposées, l'une pour les valeurs positives, l'autre pour les valeurs négatives, il en existât une troisième telle, que la direction positive fût à celle dont il s'agit comme celle-ci est à la direction négative.

4. *Or*, si l'on prend un point fixe K (*fig.* 1) et qu'on adopte pour unité positive la ligne KA considérée comme ayant sa direction de K en A, ce qu'on pourra désigner par $\overline{\text{KA}}$, pour distinguer cette quantité de la ligne KA dans laquelle on ne considère ici que la grandeur absolue, l'unité négative sera $\overline{\text{KI}}$, le trait supérieur ayant la même destination que celui qui est placé sur $\overline{\text{KA}}$, et la condi-

tion à laquelle il s'agit de satisfaire sera remplie par la ligne KE, perpendiculaire aux précédentes et considérée

Fig. 1.
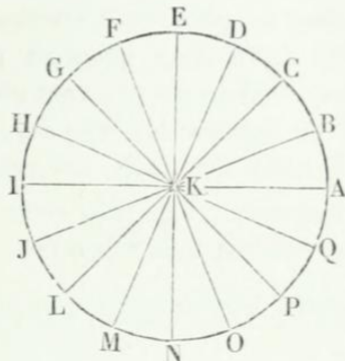
comme ayant sa direction de K en E, et qu'on exprimera également par $\overline{KE}$. En effet, la direction de $\overline{KA}$ est, à l'égard de la direction de $\overline{KE}$, ce que cette dernière est à l'égard de la direction de $\overline{KI}$. De plus, on voit que cette même condition est aussi bien remplie par $\overline{KN}$ que par $\overline{KE}$, ces deux dernières quantités étant entre elles comme $+1$ et -1, ainsi que cela doit être. Elles sont donc ce qu'on exprime ordinairement par $+\sqrt{-1}$, $-\sqrt{-1}$.

Par une marche analogue, on pourra insérer de nouvelles moyennes proportionnelles entre les quantités dont il vient d'être question. En effet, pour construire la moyenne proportionnelle entre $\overline{KA}$ et $\overline{KE}$, il faudra tirer la ligne CKL qui divise l'angle AKE en deux parties égales, et la moyenne cherchée sera $\overline{KC}$ ou $\overline{KL}$. La ligne GKP donnera également les moyennes entre $\overline{KE}$ et $\overline{KI}$ ou entre