Slow Coloring Cyclic Permutation Graphs

by

Joan Mary Morris

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Approved by

Gregory J. Puleo, Chair, Assistant Professor of Mathematics
Peter Johnson, Alumni Professor of Mathematics
Jessica McDonald, Associate Professor of Mathematics

Abstract

The *slow coloring game* is played by two players, Lister and Painter, on a graph G. In round i, Lister marks a nonempty subset of $V(G)$, which we'll call M. By doing this he scores $|M|$ points. Painter responds by deleting a maximal independent subset of M. This process continues until all vertices are deleted. Lister aims to maximize the score, while Painter aims to minimize it. The best score that both players can guarantee is called the *slow coloring number* or *sum-color cost* of G, denoted $\mathring{s}(G)$.

Puleo and West [1] found that for an n-vertex tree T, $\mathring{s}(T) \leq \lfloor \frac{3n}{2} \rfloor$, and that the maximum is reached when T contains a spanning forest with vertices of degree 1 or 3. This implies that graphs with a perfect matching have a slow coloring number bounded by $\mathring{s}(G) \geq \frac{3n}{2}$. We find a stronger lower bound for cyclic permutation graphs. Given a cyclic permutation graph G_σ, $\sigma \in S_k$, we show $\mathring{s}(G_\sigma) \geq \frac{3n}{2} + 1$.

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Chapter 1

Introduction

This thesis studies the *slow coloring game* [2] which is played between two players, Lister and Painter, on a graph G. In each round of the game, Lister marks a nonempty subset of the graph, which we'll call M, and scores $|M|$ points. Painter then chooses a maximal independent subset of M to delete. This process continues until all the vertices are deleted. Lister seeks to maximize the score and Painter seeks to minimize it. The best score that each player can guarantee is called the slow coloring number, or sum-color cost of G, $\mathring{s}(G)$.

Slow coloring, also called online sum-paintability, is a recent problem that comes from a history of coloring parameters. These other variations of coloring can help us to better understand the slow-coloring game and sum-color cost. We postpone formal definitions of these concepts for Section 1.1 in favor of an informal discussion. A proper coloring of a graph, G, is an assignment of colors to the vertices of G such that adjacent vertices must get distinct colors. List coloring, introduced independently by Erdos-Rubin-Taylor [3] and Vizing [4, 5], gives the graph a list assignment L, such that each vertex v receives a list of $L(v)$ available colors. A graph is L-colorable if it has a proper vertex coloring using the colors from the lists assigned by $L(v)$. Choosability looks at the largest sizes of these lists: A graph is f-choosable, for a function $f : V(G) \to \mathbb{N}$, if it is L-colorable for every list assignment L such that $|L(v)| \geq f(v)$ for all v. It is k-choosable for an integer k if it is f-choosable when $f(v) = k$ for all v. Instead of looking at the least size of the lists, we can look at the least sum, or average, of the list sizes. And so we have sum-choosability, first defined and studied by Isaak [6]. The sum-choosability of G is $\chi_{SC}(G)$, the minimum $\sum(f(v))$ over all f such that G is f-choosable. [6]

We can take list coloring, and introduce an online factor– revealing the lists of vertices little by little. To model worst possible behavior, we can view this as a game with the same players and round structure described in relation to slow coloring. In round i, Lister marks a subset of M vertices of the graph. We can view this marking as revealing all of the vertices with color i in their lists. Painter then chooses an independent subset of M to receive color i. In comparison with slow coloring however, we score this game differently: Lister tries to maximize the amount of times that a vertex is chosen, thus revealing the entire hidden list before Painter colors it. We note here the connection to choosability, where we're concerned with the largest list on any vertex. For a function f determining the list sizes for the vertices, Lister wins if some vertex v is marked more than $f(v)$ times. Painter wins by coloring all the vertices before this happens. Thus, Painter wins the f-painting game by preventing a vertex v from being marked more than $f(v)$ times. If so, then the graph is f-paintable. A graph is k-paintable if it is f-paintable for the function $f(v) = k$ for all v, and the *paintability* of a graph is the least such k. Paintability was independently introduced by Schauz [7] and Zhu [8] Just as in choosability, we can study the least sum, or average, of this property– the sum-paintability of G, studied first by Carraher, Mahoney, Puleo, and West [9]. Denoted $\chi_{SP}(G)$, the sum-paintability of a graph G is the least value of $\sum(f(v))$ such that G is f-paintable.

Since paintability deals with how many times a vertex is marked, and the size of its list, we are not focused on specific colors the vertices get. Because of this, as first noted in [9], we can view paintability in the following way: Painter allots tokens to the vertices of G, according to a function$f(v)$, corresponding to the size of their lists. Every time Lister marks a vertex, a token is removed. When all the tokens of a vertex have been used, then the vertex has been marked $f(v)$ times. If a vertex is marked more than $f(v)$ times (having no tokens left to "pay"), Lister wins the game. In this fashion, sum-paintability is the least amount of total tokens used. If continue this concept and look at sum-paintability in an online progression, we arrive at slow coloring. In slow coloring, rather than assigning tokens beforehand according to f, Painter can distribute tokens to the vertices as the game progresses. This allows Painter to reserve tokens, and use them as needed, perhaps on especially difficult vertices. Thus, we can see that $\mathring{s}(G) \leq \chi_{SP}(G)$, since Painter can always play according to the function defined by

$\chi_{SP}(G)$. Here again, we can see that the specific color of the vertices marked does not affect the parameter, and for each round i, we can use a different color i. Thus deleting a vertex in round i, as we discussed earlier, is a model for assigning it color i.

Since slow coloring is such a new parameter, relatively little is known about it. It was first introduced by Mahoney, Puleo, and West in 2017 [2] where they provided a general upper and lower bound on $\mathring{s}$ according to the graph's number of vertices and its independence number. They also found results for specific cases, such as when the independence number is two, for n-vertex trees and complete bipartite graphs. In 2018, Gutowski, et al, [10] studied the property on several classes of sparse graphs including k-degenerate, acyclically k-colorable, planar, and outerplanar graphs.

Around the same time, Puleo and West [1] published results studying slow coloring on trees. They developed an algorithm to compute the slow coloring number for a tree and produced results characterizing n-vertex trees with the largest and smallest values. They proved two theorems in particular that will be useful for our results:

Theorem 1.1. *[1] For every n-vertex tree T,*

$$n + \sqrt{2n} \approx n + u_{n-1} = \mathring{s}(K_{1,n-1}) \leq \mathring{s}(T) \leq \mathring{s}(P_n) = \lfloor \frac{3n}{2} \rfloor$$

where $u_r = \max\{k : t_k \geq r\}$ for $t_k = \binom{k+1}{2}$, $k, r \in \mathbb{N}$.

Theorem 1.2. *[1] If T is an n-vertex forest, then $\mathring{s}(T) = \lfloor \frac{3n}{2} \rfloor$ (the maximum) if and only if T contains a spanning forest in which every vertex has degree 1 or 3, except for one vertex of degree 0 or 6 when n is odd.*

A natural corollary of this is that any graph with a perfect matching has one of these spanning forests as a subgraph. Since more edges would only push the sum-color cost higher, this becomes a lower bound for classes of graphs with a perfect matching. Cyclic permutation graphs, discussed in Chapter 2, are one such class of graphs. Created by two copies of a cycle of length k, with vertices joined together according to a permutation $\sigma \in S_k$, a perfect matching can be found from all the permutation edges. Also, since these graphs will always have an even number of vertices with $n = |V(G)| = 2k$, then for a cyclic permutation graph G_σ, we have $\mathring{s}(G_\sigma) \geq \frac{3n}{2}$.

In this paper, we use a Lister strategy to guarantee a higher bound for cyclic permutation graphs:

Theorem 1.3. *For any permutation $\sigma \in S_k$, the sum-color cost of the cyclic permutation graph G_σ is bounded by*

$$\mathring{s}(G_\sigma) \geq \frac{3n}{2} + 1$$

At the end of this paper, we include some results on the existence of disjoint cycles within the cyclic permutation graphs. Although they are not necessary for the proof of our main result, they are interesting and demonstrate proficiency in using probabilistic methods.

1.1 Definitions

Definition. A *graph*, G, is a set of elements, $V(G)$, called *vertices*, and a set of unordered pairs of these elements, $E(G)$, called *edges*.

Definition. A pair of vertices are said to be *adjacent* if they are endpoints of the same edge. A subset of vertices of a graph G is *independent* if none of those vertices are adjacent. The *independence number* $\alpha(G)$ of a graph is the size of the largest independent subset of G.

Definition. A *proper coloring* of a graph, G, is an assignment of colors to the vertices of G in which vertices that are adjacent to each other must get distinct colors. The *chromatic number* $\chi(G)$, is the least amount of colors that can be used to produce a proper coloring on a graph.

Definition. A *list assignment* for a graph, G, is a function L that assigns every vertex v a list of colors $L(v)$. A graph is *L-colorable* if it has a proper vertex coloring using the colors from the lists assigned by $L(v)$. A graph is *k-choosable* if it is L-colorable whenever $|L(v)| \geq k$ for all vertices of the graph. The *choosability*, or *list chromatic number*, of a graph G is $\chi_l(G)$: the least k such that G is k-choosable.

Definition. For a function $f : V(G) \to \mathbb{N}$, we define the f-painting game in the following way. In round i, Lister marks a subset M of vertices of G. Painter responds by deleting an independent subset of M. If Painter can ensure that no vertex v gets marked more than $f(v)$ times, then the graph is f-paintable. The *paintability* of a graph G is the smallest positive integer k such that G is f-paintable whenever $f(v) \geq k$. The *sum-paintability* of G, denoted $\chi_{SP}(G)$, is the least value of $\sum(f(v))$ such that G is f-paintable.

Definition. Following [11], we write S_k for the symmetric group on the elements $\{1, \ldots, k\}$. Given $\sigma \in S_k$, we define a *cyclic permutation graph* G_σ as the graph with vertex set $\{v_1, \ldots, v_k, w_1, \ldots w_k\}$ and the edge set

$$\{v_1v_2, , \ldots, , v_kv_1, w_1w_2, \ldots, w_kw_1\} \cup \{v_iw_{\sigma(i)} : 1 \leq i \leq k\}$$

. We call the subset of edges $\{v_iw_{\sigma(i)} : 1 \leq i \leq k\}$ *permutation edges*, and the complement of this subset *cycle edges.*

Definition. For two sets A, B, the *symmetric difference* of A and B is the set of elements that are in exactly one of A, B, denoted $A \oplus B := (A - B) \cup (B - A)$

Definition. For a family of probability spaces, (Ω_n, P_n), indexed by n, Property Q holds *asymptotically almost surely* as $n \to \infty$ if P(Property Q holds) $\to 1$ as $n \to \infty$.

Chapter 2

Cyclic Permutation Graphs

There are several known ways in which we can view groups as graphs, and vice versa. The most notable of these are Cayley Graphs, which are built from a group and its generating set.

Rather than building a graph out of a generating set, Chartand and Harary [12] introduced permutation graphs, which take two disjoint copies of a graph of size $|V(G)| = k$, and join the vertices of these copies together according to a permutation $\sigma \in S_k$. Ringeisen [13] considered the subset of these where G is a cycle of length k: cyclic permutation graphs. Since we're joining two cycles of length k, we note that $|V(G_\sigma)| = n = 2k$. We can also see that these graphs are 3-regular.

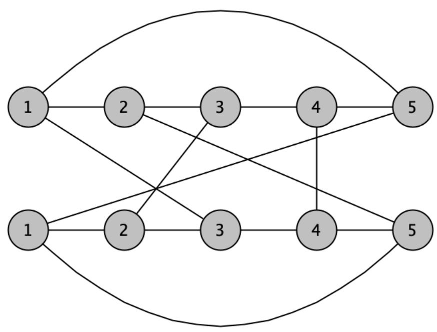

Figure 2.1: A cyclic permutation graph G_σ, with $\sigma = (1, 3, 2, 5) \in S_5$.

Lemma 2.1. *For $\sigma \in S_k$, $k \geq 3$, the cyclic permutation graph G_σ is 3-connected.*

Proof. Let G_σ be the cyclic permutation graph composed of two cycles of length k. We'll call the two cycle copies C_1 and C_2. Let $x, y \in V(G_\sigma)$. We delete $\{x, y\}$ and show that G_σ is still connected.

First consider if x and y belong to the same cycle copy, say C_1. To find a path between a vertex on C_2 and another vertex on C_2, you simply move along the cycle, since no vertices on that copy were deleted. Every remaining vertex in C_1 has a permutation edge connecting it to C_2, so we can get from any vertex in C_1 to any vertex in C_2. We can also get from any vertex in C_1 to any other vertex in C_1 by moving to C_2 via the permutation edge, moving along C_2, and then returning to C_1 using another permutation edges.

Next we consider if x and y belong to different cycles, say $x \in C_1$ and $y \in C_2$. By deleting $\{x, y\}$, we delete two permutation edges. However, since $k \geq 3$, at least one permutation edge, remains. We only deleted one vertex from each cycle, so both $C_1 - x$ and $C_2 - y$ are connected. Since these are connected, and joined by at least one permutation edge, there is a path between every pair of vertices in the remaining graph.

$\square$

There are three main observations we want to make concerning cyclic permutation graphs. The first is the above lemma: these graphs are 3-connected. The second is that a perfect matching can always be found on these graphs by taking all the permutation edges. And finally, since these graphs always have an even number of vertices, $n = |V(G_\sigma)| = 2k$, by Theorem 1.2, $\mathring{\mathrm{s}}(G_\sigma) \geq \frac{3n}{2}$.

Chapter 3

Main Result

Recall that $\sigma \in S_k$ and G_σ is the cyclic permutation graph on cycles of length k. Lister chooses the permutation pairs $i, \sigma(i)$ and $j, \sigma(j)$. Painter replies by deleting an independent subset of the marked vertices, D. We may assume, as observed by [2], that Painter deletes a maximal independent subset, so we may assume $|D| = 2$. To delete a maximal independent subset of Lister's choice, Painter deletes one each of the selected pairs. We call the graph after Painter's response $G_{\sigma^-} = G_\sigma - D$.

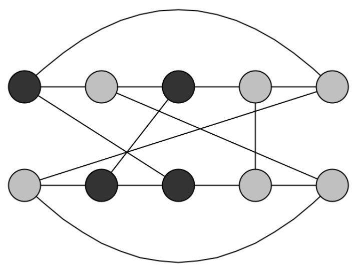

Figure 3.1: An example of Lister's choice on G_σ.

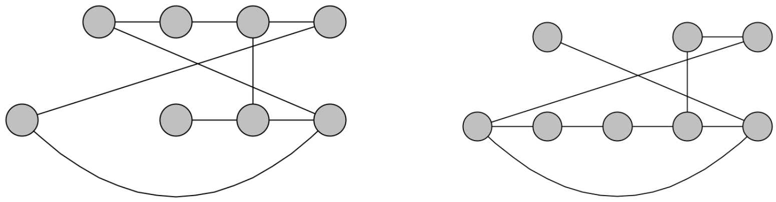

Figure 3.2: Examples of Painter's possible replies, creating G_{σ^-}.

For our main result, we rely heavily on the following Lemma.

Lemma 3.1. *For any Painter reply D with $|D| = 2$, the graph G_{σ^-} has a spanning forest, F, in which all vertices have degree* 1 *or* 3.

For now, we will assume this lemma, and prove our main result.

Theorem 3.2. *For any permutation $\sigma \in S_k$, the sum-color cost of the cyclic permutation graph G_σ is bounded by*

$$\mathring{s}(G_\sigma) \geq \frac{3n}{2} + 1$$

Proof. Lister starts by marking the permutation pairs $i, \sigma(i)$ and $j, \sigma(j)$. Let D be the independent subset deleted by Painter with, as noted before $|D| = 2$. We call the graph after deletion G_{σ^-}, which has $|V(G_{\sigma^-})| = n - 2$. By Lemma 3.1, G_{σ^-} has a spanning forest in which all vertices have degree 1 or 3. Thus by Theorem 1.2,

$$\mathring{s}(G_{\sigma^-}) \geq \frac{3}{2}(n-2) = \frac{3}{2}n - 3$$

Playing an optimal strategy on G_{σ^-}, the final score $\mathring{s}$ achieved by Lister against this reply is bounded from below by

$$\mathring{s}(G_\sigma) \geq \left(\frac{3}{2}n - 3\right) + 4 = \frac{3}{2}n + 1$$

Since this bound holds no matter which vertices Painter deletes in response to our Lister's choice, we conclude that

$$\mathring{s}(G_\sigma) \geq \frac{3}{2}n + 1.$$

□

We now prove the supporting Lemma.

Proof. Consider the vertices that were adjacent to the deleted vertices by permutation edges, call them β-vertices. Note that after deletion, there are two β-vertices. We construct F as follows: Let F_0 be the set of all permutation edges remaining in G_{σ^-}, and let P be a path between the β-vertices. By Lemma 2.1 at least one such path exists. We construct F_1 to be the symmetric difference of F_0 and $E(P)$:

$$F_1 = F_0 \oplus E(P).$$

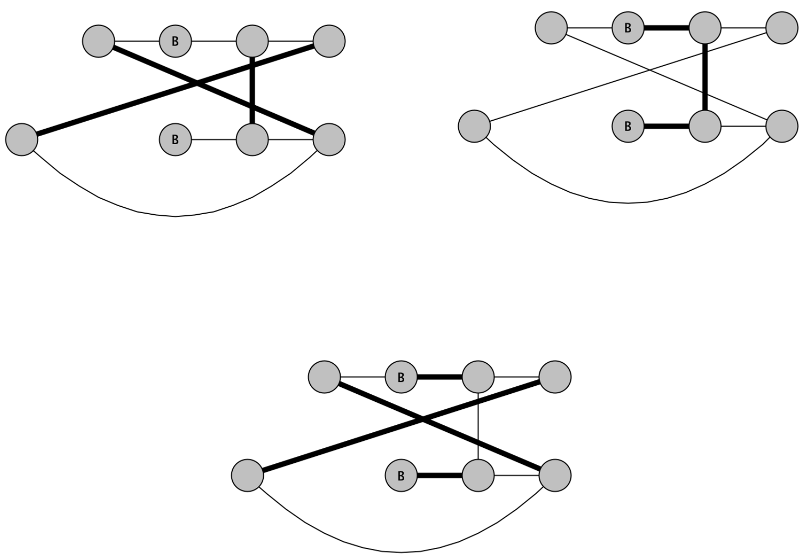

Figure 3.3: Construction of F_1. Top left: F_0, top right: β-path, bottom: F.

Note that, by the definition of symmetric difference,

$$d_{F_1}(v) = d_{F_0}(v) + d_P(v) \pmod 2.$$

We claim that the vertices of F_1 have degree 1 or 3. To see this, we'll consider each type of vertex. First, consider a β-vertex, v_β. Since β-vertices no longer have a permutation edge, $d_{F_0}(v_\beta) = 0$. Since they are an endpoint of the β-path, $d_P(v_\beta) = 1$. Thus $d_{F_1}(v_\beta) = 1 \pmod 2$.

Now consider all the remaining vertices. Let v be a vertex that is not a β-vertex. Since v retains its permutation edges in G_{σ^-}, $d_{F_0}(v) = 1$. Also, $d_P(v)$ is even: if the vertex is disjoint from the β-path, then $d_P(v) = 0$; if the vertex belongs to the β-path, then $d_P(v) = 2$. Thus for non-β-vertices, $d_{F_1}(v) = 1 \pmod 2$. Hence every vertex has odd degree in F_1. This implies that every vertex in F_1 has degree either 1 or 3, since G has maximum degree 3.

Finally, we remove cycles, should any exist. Let C be a cycle contained in F_1. Let $F = F_1 \oplus E(C) = F_1 - E(C)$. If F_1 contains more than one cycle, we can continue to take symmetric differences until all are removed. Each cycle can only contribute degree 2 to any vertex in F_1. Thus removing any cycle maintains the fact that all vertices have odd degree. And so G_{σ^-} contains a spanning forest of vertices of degree 1 or 3.

□

Chapter 4

Probabilistic Methods

Originally, we believed in order to prove our sum-color cost bound for cyclic permutation graphs, there would need to be two disjoint cycles (using permutation edges) within the graph– see the figure below. In this section, we use probabilistic methods to show that these disjoint cycles exist for large enough k. It turned out that our proof did not rely on these cycles, however, the result is interesting and demonstrates a mastery of these techniques.

For our results here, we rely heavily on Chebyshev's Inequality, as formulated in [14].

Lemma 4.1. *[14] (Chebyshev's Inequality) Let X be a random variable on a finite probability space. For any real $t > 0$,*

$$P(|X - E(X)| \geq t) \leq \frac{V(X)}{t^2}$$

Let G_σ be the cyclic permutation graph of $\sigma \in S_k$. Label the vertices of each cycle $\{1, \ldots, t, \ldots, k\}$ with $t = \frac{k}{2}$.

Theorem 4.2. *Let $k = 2t$, and let σ be a uniform random permutation from S_k. Asymptotically almost surely, there exist distinct $i, j \in \{1, ..., t\}$ such that $\sigma(i), \sigma(j) \in \{1, ..., t\}$, and there exist distinct $i', j' \in \{t+1, ..., k\}$ such that $\sigma(i'), \sigma(j') \in \{t+1, ..., k\}$, for t sufficiently large.*

Proof. We prove the first statement, and the second statement follows with a similar argument. Since the descriptions of these vertices rely solely on the permutation σ, we think of $i, j, \sigma(i), \sigma(j)$ as elements of this permutation. We call an element $i \in \{1, \ldots, t\}$ *good* if it satisfies the above, that is, if $\sigma(i) \in \{1, \ldots, t\}$.

Let A_i be the event that $i \in \{1, \ldots, t\}$ is good. Let X_i be its indicator random variable, i.e.

$$X_i = \begin{cases} 1 & \text{if } \sigma(i) \in \{1, \ldots, t\} \\ \\ 0 & \text{if } \sigma(i) \notin \{1, \ldots, t\} \end{cases}$$

Note that $P(\sigma(i) \in \{1, \ldots, t\}) = \frac{1}{2}$ and so $E(X_i) = \frac{1}{2}$. Let X be the number of good elements. Then $X = \sum_{i=1}^{t} X_i$ and

$$E(X) = E\left(\sum_{i=1}^{t} X_i\right) = \sum_{i=1}^{t} (E(X_i)) = \frac{t}{2}$$

We use Chebyshev's Inequality to show that almost surely there exist two good elements. To do this we first find the variance of X.

Since X_i is an indicator random variable, we have

$$V(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i) - E(X_i)^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

For the covariance we have

$$\begin{aligned} C(X_i, X_j) &= E(X_i X_j) - E(X_i)E(X_j) \\ &= \frac{1}{2}\left(\frac{t-1}{2t-1}\right) - \frac{1}{4} \\ &= \frac{-2}{16t-8} \end{aligned}$$

To see this, note that, $E(X_i X_j) = P(A_i \text{ and } A_j) = P(A_i)P(A_j|A_i)$ where $P(A_i) = \frac{t}{2t} = \frac{1}{2}$ and $P(A_j|A_i) = \frac{t-1}{2t-1}$.

Thus the variance of X is as follows:

$$
\begin{aligned}
V(X) &= \sum_i V(X_i) + \sum_{i \neq j} C(X_i X_j) \\
&= \frac{t}{4} + \left(\binom{t}{2}\right)\left(\frac{-2}{16t-8}\right) \\
&\leq \frac{t}{4} + \left(\frac{(t-1)^2}{2}\right)\left(\frac{-2}{16t-8}\right)
\end{aligned}
$$

We use Chebyshev's Inequality and consider $P(X \leq 1)$. $X \leq 1$ would require $|X - E(X)| \geq E(X) - 1$. Thus we have,

$$
\begin{aligned}
P(X \leq 1) &\leq P(|X - E(X)| \geq E(X) - 1) \\
&\leq \frac{V(X)}{(E(X)-1)^2} \\
&\leq \frac{\frac{t}{4} + \frac{(t-1)^2}{2}(\frac{-2}{16t-8})}{(\frac{t}{4}-1)^2} \\
&\leq \frac{t + (\frac{-2t^2+2t-1}{32t-16})}{\frac{t^2}{16} - \frac{t}{2} + 1} \\
&= \frac{480t^2 - 224t - 16}{32t^3 - 272t^2 + 640t - 256}
\end{aligned}
$$

So as $t \to \infty$, $P(X \leq 1) \to 0$ and almost surely there exist at least two good elements.

$\square$

Since two good elements exist within the permutation, we have that corresponding vertices within the graph, and so we achieve our disjoint cycles as described before.

Chapter 5

Future Interests

Since slow coloring is such a recently defined parameter, there are many interesting directions to follow in studying it. As discussed in [13], there are other types of permutation graphs: we can take two copies of any graph and join the copies according to a permutation [12]. Thus first point of interest would be seeing how to modify this result for other types of permutation graphs (or all permutation graphs).

This leads us to begin thinking about other types of algebraically generated graphs, such as Cayley graphs. The algebraic structures of Cayley graphs lend themselves to high connectivity, which suggests a higher sum-color cost. We can also look to expand on the probabilistic work we've done, seeing what structure and connectivity we can guarantee for a graph through probabilistic methods.

The final and most interesting future question is if our result can be directly generalized to graphs that have a perfect matching and are $(2k+1)$-connected. We see that our result extends to this idea that for $k = 1$, such graphs have sum-color cost, $\mathring{s}(G) \geq \frac{3n}{2} + k$. Therefore we conjecture the following.

Conjecture 5.1. *Let G be a graph with a perfect matching. If G is $(2k+1)$-connected, then*

$$\mathring{s}(G) \geq \frac{3n}{2} + k$$

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