Maximal L^1 Regularity for a Class of Parabolic Systems and Applications to Navier-Stokes Equations

by

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Abstract

This dissertation is devoted to the maximal L^1-in-time regularity for a class of linear parabolic systems with variable coefficients. This theory can be applied to investigate the global-in-time well-posedness and stability issues for density-dependent viscous fluids, even if the initial fluctuation of the density is large. The results in Chapter 3 and most of the results in Chapter 4 have been addressed in the author's papers [56] and [57], respectively.

The main result in Chapter 3 concerns the maximal L^1 regularity and asymptotic behavior for solutions to the inhomogeneous incompressible Navier-Stokes equations under a scaling-invariant smallness assumption on the initial velocity. We obtain a new global L^1-in-time estimate for the Lipschitz seminorm of the velocity field without any smallness assumption on the fluctuation of the initial density. In the derivation of this estimate, we study the maximal L^1 regularity for a linear Stokes system with variable coefficients. The analysis tools are a use of the semigroup generated by a generalized Stokes operator to characterize some Besov norms and a new gradient estimate for a class of second-order elliptic equations of divergence form.

In Chapter 4, we generalize the concrete maximal L^1 regularity result obtained in Chapter 3 and establish an abstract one for a class of Cauchy problems associated with composite operators. Then we apply this abstract theory to study maximal L^1 regularity for the Lamé system with rough variable coefficients. To lower the regularity of the coefficients, we work in the L^p (in space) framework. For this, we use a classical method to establish Gaussian bounds of the fundamental matrix of a generalized parabolic Lamé system with only bounded and measurable coefficients. As applications, we use a Lagrangian approach to study the global-in-time well-posedness of systems of compressible flows.

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Contents

Chapter 1

Introduction

1.1 Background and motivation

This dissertation is motivated by the study of the global well-posedness of the Cauchy problem for a class of hyperbolic-parabolic coupled systems modeling the motion of fluids. Probably the most famous example is the system of Navier-Stokes equations (see [43, 44]). In a fluid flow, the law of conservation of mass can be formulated mathematically using the continuity equation, given in differential form as

$$\partial_t \rho + \text{div}(\rho u) = 0, \text{ in } (0, \infty) \times \mathbb{R}^n,$$

where ρ is the density (mass per unit volume) and u is the flow velocity field. The law of conservation applied to momentum gives the momentum equation of the form

$$\rho(\partial_t u + u \cdot \nabla u) - \mathcal{A}u + \nabla P = 0, \text{ in } (0, \infty) \times \mathbb{R}^n, \tag{1.1}$$

where P is a scalar pressure and $\mathcal{A}$ is a dissipative operator. The fluid flow can be either incompressible or compressible. For example, the system modeling the motion of incompressible flows of mixing fluids with different densities reads

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla P = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

In the above system, we assume that the viscosity coefficient of the fluid is a constant normalized as 1. In the compressible case, we will consider two different systems. The first one is a system of pressureless flows, in which the pressure term in (1.1) is absent:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \rho(\partial_t u + u \cdot \nabla u) - \mathcal{A} u = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & \text{on } \mathbb{R}^n. \end{cases} \tag{1.3}$$

Note that (1.3) with $\mathcal{A} = \Delta$ can be viewed as a viscous regularization for the model of inviscid pressureless gases. The second one is the system of compressible Navier-Stokes equations for an ideal gas (see Section 4.6).

Throughout this dissertation, the initial density is always assumed to be bounded and bounded from below, namely,

$$m \le \rho_0(x) \le \frac{1}{m}, \;\; a.e.\ x \in \mathbb{R}^n \tag{1.4}$$

for some constant $m \in (0, 1]$.

Let us review some known results for the existence and uniqueness of solutions to (1.2). Global weak solutions with finite energy were first obtained by Kazhikhov [40] under the assumption that the initial density ρ_0 has a positive infimum. Several improvements can be found in [29, 43, 53]. The main estimate for weak solutions is the energy inequality

$$\|\sqrt{\rho} u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|\sqrt{\rho_0} u_0\|_{L^2}^2.$$

This estimate is far from enough to prove the uniqueness of weak solutions in 3-D. Ladyzhenskaya and Solonnikov [41] initiated the studies for unique solvability of (1.2) in a bounded domain with homogeneous Dirichlet boundary condition for u. In the last two decades, a large amount of work was devoted to the well-posedness of (1.2) under minimum regularity assumptions on the data. Firstly, Danchin [16] constructed a unique strong solution to (1.2) in the critical space $(L^\infty(\mathbb{R}^3) \cap \dot{B}^{3/2}_{2,\infty}(\mathbb{R}^3)) \times \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ in the case when the initial density is close to a constant. Later, many authors tried to improve Danchin's result to allow different Lebesgue indices of the critical spaces, or to remove the smallness assumption on the initial density (see [1–4, 12, 20, 58]). Secondly, it is interesting to lower the regularity of the density to allow discontinuity. A well-posedness result with only bounded density would demonstrate that the motion of a mixture of two incompressible fluids with different densities can be modeled by (1.2). One can see [13, 21, 37, 48] for the work towards this direction. Now one might ask whether a small initial velocity in some critical space and a discontinuous bounded initial density can generate a unique global-in-time solution to (1.2). To our knowledge, this question is not settled. Nevertheless, in his recent paper [59], Zhang established a global existence result in which ρ_0 merely satisfies (1.4) and $\|u_0\|_{\dot{B}^{1/2}_{2,1}}$ is small, yet the uniqueness is not known unless the initial velocity is more regular. Finally, we remark that our results will be based on the assumption (1.4). In case one is interested in the case when vacuum state is allowed, we refer to a recent paper [23] and the references therein.

The system of the form (1.3) has been studied by several authors. When $n = 1$ and $\mathcal{A} = \Delta$, Boudin [10] proved the existence of a global smooth solution to (1.3). Perepelitsa [50] considered (1.3) as a simplified model of compressible isentropic Navier-Stokes equations and he proved the global existence of a small energy weak solution with the density being a nonnegative bounded function throughout the half-space $\mathbb{R}^3_+$. Recently, Danchin et al. [24] formally derived the system (1.3), with $\mathcal{A}$ being the Laplacian or the Lamé operator, as a model of some collective behavior phenomena. They also proved the existence and uniqueness of a global solution with the initial density being only bounded and close to a constant in L^∞-norm.

In this dissertation, we are particularly interested in solving (1.2) and (1.3) using the Lagrangian method (see [17, 20]). The advantage is that one can convert the hyperbolic-parabolic coupled system into a parabolic system. Then the uniqueness and stability issues can be tackled in a relatively easy way compared to solving the system in Eulerian coordinates. In fact, one can prove the existence of solutions to the Lagrangian formulation by applying the contraction mapping theorem based on the maximal L^1-in-time regularity for the linear system.

In this framework, the heart of the matter is to get the estimate

$$\int_0^\infty \|\nabla u(t)\|_\infty \, dt \ll 1 \tag{1.5}$$

under a critical smallness condition on the initial velocity u_0, because this would imply the existence of a global-in-time coordinate transformation. Such an estimate is true if the initial density is close enough to a constant (see [19, 20]). To our knowledge, if the fluctuation of the density is not small, there is not much evidence in the literature that supports the validity of (1.5). For example, for (1.2) with $n = 3$, it was proved in [13, 48] that the quantity $\int_0^T \|\nabla u(t)\|_\infty \, dt$ has a polynomial growth in time provided that the initial velocity u_0 belongs to some Sobolev space $H^s(\mathbb{R}^3)$ with $s > \frac{1}{2}$, and that u_0 satisfies a scaling-invariant smallness condition. However, such growth could possibly be removed from the point of view of equivalent characterizations of norms. To see this, we suppose that u is the solution to the classical heat equation $\partial_t u - \Delta u = 0$ with initial value u_0. Then the estimate (1.5) dictates the smallest norm to be used to measure the smallness of the initial data because of the equivalence of norms

$$\int_0^\infty \|\nabla u(t)\|_\infty \, dt \simeq \|u_0\|_{\dot{B}^{-1}_{\infty,1}}.$$

The above equivalent characterization is classical and can be proved, for example, by applying [9, Lemma 2.4] and Lemma 2.13. In general, the space $\dot{B}^{-1}_{\infty,1}(\mathbb{R}^n)$ is too rough in order for the nonlinear system (1.2) (or (1.3)) to be well-posed in it. Then we have to replace it with smaller spaces $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$, $1 \le p < \infty$, so that there is still hope for (1.5) to be true.

It is worth noting that the scaling invariance of (1.2) and (1.3) also suggests the use of the $\dot{B}^{n/p-1}_{p,1}$-norms for the velocity. Let us take (1.2) for example. For any $\lambda > 0$, it is easy to see that (1.2) is invariant under the scaling

$$(\rho, u, P)(t,x) \rightsquigarrow (\rho_\lambda, u_\lambda, P_\lambda)(t,x) := (\rho, \lambda u, \lambda^2 P)(\lambda^2 t, \lambda x).$$

So the smallness condition on u_0 makes sense if it is measured by a norm which is invariant under the scaling $u_0(\cdot) \rightsquigarrow \lambda u_0(\lambda\cdot)$. And we do have $\|\lambda u_0(\lambda\cdot)\|_{\dot{B}^{n/p-1}_{p,1}} \simeq \|u_0\|_{\dot{B}^{n/p-1}_{p,1}}$. Nowadays, the spaces $\dot{B}^{n/q}_{q,1}(\mathbb{R}^n) \times \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ are called *critical spaces* for (1.2).

Without going into details, we are led to consider the linearized system of the Lagrangian formulation of (1.2) (or, (1.3)) that reads

$$\rho(x)\partial_t u - \mathcal{A}u = f. \tag{1.6}$$

Note that the coefficient ρ is now a time-independent function. In the incompressible case, the operator $\mathcal{A}$ in (3.7) is the so-called Stokes operator (see Chapter 3). But $\mathcal{A}$ is just what it used to be in the compressible case. Now the main task of this dissertation is to derive the estimate (1.5) for solutions u to (1.6) without requiring the fluctuation of the coefficient $\rho(x)$ to be small. To achieve this, we shall study the maximal L^1-in-time regularity for solutions to (1.6) in homogeneous type spaces.

In Chapter 2, we recall some necessary preliminaries. We deal with the incompressible system (1.2) in Chapter 3 and compressible pressureless system (1.3) in Chapter 4. Shortly after we completed the first draft of this dissertation, we realized that our method could also be applied to the global well-posedness of the heat-conductive compressible Navier-Stokes equations. We hence include Section 4.6.

1.2 Notations

Throughout this dissertation, the letter C denotes a harmless positive constant that may change from line to line, but whose meaning is clear from the context. The notation $a \lesssim b$ means $a \leq Cb$ for some C, and $a \simeq b$ means $a \lesssim b$ and $b \lesssim a$. For two quantities a, b, we denote by $a \vee b$ the bigger quantity and by $a \wedge b$ the smaller one. For $p \in [1, \infty]$, the conjugate index

p' is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $\| \cdot \|_p$ the Lebesgue L^p-norm. For a matrix A, A^T denotes its transpose. For a vector field $v = v(x)$, ∇v denotes the matrix $(\partial_{x_i} v_j)$ and $Dv = (\nabla v)^\mathsf{T}$.

Let $(X, \|\cdot\|)$ be a Banach space. $\mathscr{L}(X)$ denotes the space of all continuous linear operators on X and $\| \cdot \|_{\mathscr{L}(X)}$ denotes the operator norm. For $q \in [1, \infty]$, we may write $\| \cdot \|_{L^q_t(X)}$ for the norm of the space $L^q((0,t); X)$, and $\| \cdot \|_{L^q(X)}$ for the norm of $L^q(\mathbb{R}_+; X)$, where $\mathbb{R}_+ = (0, \infty)$. We denote operators on Banach spaces by "mathcal" letters (e.g., $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, etc.). For an operator $\mathcal{A}$, $D(\mathcal{A})$ and $R(\mathcal{A})$ denote the domain and range of $\mathcal{A}$, respectively.

Chapter 2

Preliminaries

2.1 Semigroups and abstract Cauchy problem

Let $(X, \|\cdot\|)$ be a real Banach space. We adopt the concept that a C_0 semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on X is called a bounded C_0 semigroup if $\|\mathcal{T}(t)\|_{\mathcal{L}(X)} \leq C < \infty$ for all $t \geq 0$, while it is called a contraction semigroup if $C = 1$.

Definition 2.1. $\{\mathcal{T}(t)\}_{t\geq 0}$ is called a bounded analytic semigroup on X if it is a bounded C_0 semigroup with generator $\mathcal{A}$ such that $\mathcal{T}(t)x \in D(\mathcal{A})$ for all $x \in X$ and $t > 0$, and

$$\sup_{t>0} \|t\mathcal{A}\mathcal{T}(t)x\| \leq C\|x\|, \ \ \forall x \in X. \tag{2.1}$$

Remark 2.2. In applications, one only needs to show (2.1) for x belonging to a dense subspace of X since $\mathcal{A}$ is closed.

Remark 2.3. In fact, (2.1) is also a real characterization of complex analyticity, see, for example, [32, Theorem 4.6], or [6, Theorem 3.7.19].

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. A linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is called dissipative on H if

$$\langle \mathcal{A}x, x\rangle \leq 0, \ \forall x \in D(\mathcal{A}).$$

We have the following well-known result:

Theorem 2.4. *Let $\mathcal{A}$ be a self-adjoint operator on H. Then $\mathcal{A}$ generates an analytic semigroup of contraction $\{e^{t\mathcal{A}}\}_{t\geq 0}$ if and only if $\mathcal{A}$ is dissipative. Moreover, $e^{t\mathcal{A}}$ is self-adjoint on H for every $t \geq 0$.*

For the complex version of Theorem 2.4, we refer to [6, Example 3.7.5] and [6, Corollary 3.3.9].

In applications, we will first apply Theorem 2.4 to construct a semigroup on L^2, and then extrapolate it to some other function spaces. However, it is usually not easy to identify the generator of the new semigroup. In this situation, we wish to identify the generator restricted on a dense subspace of its domain. Recall that a subspace Y of the domain $D(\mathcal{A})$ of a linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is called a core for $\mathcal{A}$ if Y is dense in $D(\mathcal{A})$ for the graph norm $\|x\|_{D(\mathcal{A})} := \|x\| + \|\mathcal{A}x\|$. In other words, Y is a core for $\mathcal{A}$ if and only if $\mathcal{A}$ is the closure of $\mathcal{A}|_Y$. The next result gives a useful sufficient condition for a subspace to be a core for the generator.

Lemma 2.5 (see [32, p. 53])**.** *Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ on X. If $Y \subset D(\mathcal{A})$ is a dense subspace of X and invariant under $\mathcal{T}(t)$ (i.e., $\mathcal{T}(t)Y \subset Y$), then Y is a core for $\mathcal{A}$.*

Next, we recall shortly how to use semigroups to solve abstract Cauchy problems. Suppose that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $e^{t\mathcal{A}}$ on a Banach space $(X, \|\cdot\|)$. We are concerned with the well-posedness of the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) - \mathcal{A}u(t) = f(t), \quad 0 < t \leq T, \\ u(0) = x. \end{cases} \tag{2.2}$$

We assume that $x \in X$ and the inhomogeneous term f only belongs to $L^1((0,T); X)$. Then (2.2) always has a unique *mild solution* $u \in C([0,T]; X)$ given by the formula

$$u(t) = e^{t\mathcal{A}}x + \int_0^t e^{(t-\tau)\mathcal{A}} f(\tau)\, d\tau.$$

A continuous function u is called a *strong solution* if $u \in W^{1,1}((0,T);X) \cap L^1((0,T);D(\mathcal{A}))$ satisfies (2.2) for a.e. $t \in (0,T)$. A strong solution is also a mild solution. Conversely, a mild solution with suitable regularity becomes a strong one.

Lemma 2.6 (see [49, Theorem 2.9])**.** *Let $u \in C([0,T];X)$ be a mild solution to* (2.2)*. If $u \in W^{1,1}((0,T),X)$, or $u \in L^1((0,T),D(\mathcal{A}))$, then u is a strong solution.*

2.2 Elliptic operators of divergence form

In this section, we recall some known results concerning elliptic equations of divergence form, heat kernels and Riesz transforms. Let $\mathcal{E}$ be the second order elliptic operator of divergence form formally defined by

$$\mathcal{E} = -\operatorname{div}(A\nabla), \tag{2.3}$$

where $A = A(x)$ is a real symmetric $n \times n$ matrix satisfying

$$mI_n \leq A(x) \leq \frac{1}{m} I_n, \quad a.e.\ x \in \mathbb{R}^n$$

for some constant $m \in (0,1]$.

Let $\dot{H}^1(\mathbb{R}^n)$ be the space of all distributions u such that $\nabla u \in L^2(\mathbb{R}^n)$, equipped with the inner product $\int \nabla u \cdot \nabla v \, dx$. We adopt the convention that two functions in $\dot{H}^1(\mathbb{R}^n)$ are identical if their difference is a constant. For any $f \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, as a consequence of the Lax-Milgram theorem, the elliptic equation

$$\mathcal{E}P = -\operatorname{div} f \tag{2.4}$$

has a unique weak solution $P \in \dot{H}^1(\mathbb{R}^n)$ satisfying $\|\nabla P\|_2 \leq \frac{1}{m}\|f\|_2$.

Let $D(\mathcal{E}) = \{u \in H^1(\mathbb{R}^n) | \mathcal{E}u \in L^2(\mathbb{R}^n)\}$. Then $-\mathcal{E} : D(\mathcal{E}) \subset L^2 \to L^2$ is a dissipative self-adjoint operator that generates an analytic semigroup of contraction $\{e^{-t\mathcal{E}}\}_{t\geq 0}$. The maximal accretive square root of $\mathcal{E}$ is given by the formula

$$\mathcal{E}^{1/2}u = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2\mathcal{E}}\mathcal{E}u\,dt, \quad u \in D(\mathcal{E}).$$

The above integral converges normally in L^2 since $\|e^{-t^2\mathcal{E}}\mathcal{E}u\|_2 \leq \|\mathcal{E}u\|_2 \wedge t^{-2}\|u\|_2$. The domain $D(\mathcal{E}^{1/2})$ of $\mathcal{E}^{1/2}$ coincides with $H^1(\mathbb{R}^n)$ and it holds that

$$m^{1/2}\|\nabla u\|_2 \leq \|\mathcal{E}^{1/2}u\|_2 \leq m^{-1/2}\|\nabla u\|_2, \quad \forall u \in H^1(\mathbb{R}^n).$$

Indeed, $\mathcal{E}^{1/2}$ extends to a continuous operator from $\dot{H}^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with a continuous inverse, denoted by $\mathcal{E}^{-1/2}$ for notational simplicity. Let $\mathcal{R} = \nabla\mathcal{E}^{-1/2}$ be the Riesz transform associated with $\mathcal{E}$. Then $\mathcal{R}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n;\mathbb{R}^n)$. Denote by $\mathcal{R}^*$ the adjoint of $\mathcal{R}$. Now for any $f \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, one can write the gradient of the solution to (3.19) as $\nabla P = \mathcal{R}\mathcal{R}^*f$. We refer the reader to the monograph [8] for more details in this paragraph[1].

The L^p theory of the square root problem for $\mathcal{E}$ is based on the famous Aronson-Nash estimates for the kernel of $e^{-t\mathcal{E}}$.

Lemma 2.7 (see, e.g., [7])**.** *The semigroup $e^{-t\mathcal{E}}$ acting on L^2 has a kernel $K_t(x,y)$ satisfying the Gaussian property. Precisely, there exist constants $\mu \in (0,1]$ and $C > 0$ depending only on m and n such that for all $t > 0$ and $x, y \in \mathbb{R}^n$,*

$$|K_t(x,y)| \leq Ct^{-n/2}\exp\left\{-\frac{|x-y|^2}{Ct}\right\},$$

and if in addition $2|h| \leq \sqrt{t} + |x-y|$,

$$\begin{aligned}|K_t(x+h,y) - K_t(x,y)| + |K_t(x,y+h) - K_t(x,y)| \\ \leq Ct^{-n/2}\left(\frac{|h|}{\sqrt{t}+|x-y|}\right)^\mu \exp\left\{-\frac{|x-y|^2}{Ct}\right\}.\end{aligned}$$

[1]The results stated here are much less general than those in [8], where the authors dealt with complex elliptic operators.

Thanks to this property, $e^{-t\mathcal{E}}$ extrapolates to a bounded analytic semigroup on $L^p(\mathbb{R}^n)$ for every $p \in (1,\infty)$ (see [47] and the references therein). We denote this semigroup by $e^{-t\mathcal{E}_p}$ and its generator by $-\mathcal{E}_p$. Then the square root $\mathcal{E}_p^{1/2}$ of $\mathcal{E}_p$ on $L^p(\mathbb{R}^n)$ is also well-defined. The following result can be found in [8, pp. 131-132].

Lemma 2.8. *$\mathcal{E}^{1/2}$ extends to a continuous operator from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which has a continuous inverse for $1 < p < 2+\varepsilon$ for some $\varepsilon > 0$. Moreover, the extension of $\mathcal{E}^{1/2}$ on $W^{1,p}(\mathbb{R}^n)$ coincides with $\mathcal{E}_p^{1/2}$ for $1 < p < 2+\varepsilon$.*

From now on, we do not distinguish between an operator and its continuous extension, and drop the p's for all operators associated with $\mathcal{E}_p$. The above lemma implies that the Riesz transform $\mathcal{R}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n;\mathbb{R}^n)$ for $1 < p < 2+\varepsilon$, and its adjoint $\mathcal{R}^*$ is bounded from $L^{p'}(\mathbb{R}^n;\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$. Moreover, all the bounds (i.e., operator norms) only depend on m, n and p.

2.3 Homogeneous Besov spaces

We collect some homogeneous Besov spaces relevant preliminaries. While there is a vast amount of literature on this topic, we will mainly refer to the book [9] because it focuses on applications of Fourier analysis to PDEs.

2.3.1 Definition and basic properties

Let χ, φ be two smooth radial functions valued in the interval [0,1], the support of χ be the closed ball $\overline{B_{4/3}}$, and the support of φ be the washer $\overline{B_{8/3}} \setminus B_{3/4}$. Moreover, $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, and $\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n$. Denote $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$, where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform $\mathcal{F}$. Then we define the homogeneous dyadic blocks $\dot{\Delta}_j$ and the low-frequency cutoff operators $\dot{S}_j$, respectively, by

$$\dot{\Delta}_j u = 2^{3j} \int_{\mathbb{R}^n} h(2^j y) u(x-y) dy, \text{ and } \dot{S}_j u = 2^{3j} \int_{\mathbb{R}^n} \widetilde{h}(2^j y) u(x-y) dy.$$

The frequency localization of the blocks $\dot{\Delta}_j$ and $\dot{S}_j$ leads to some fine properties. First, one has orthogonality due to the interaction of frequencies supported in disjoint regions. More precisely, for two tempered distributions $u, v \in \mathscr{S}'(\mathbb{R}^n)$, we have

$$\begin{aligned}
\dot\Delta_k \dot\Delta_j u =& 0 \ \text{ if } \ |k-j| \geq 2, \\
\dot\Delta_j (\dot S_{k-1} u \dot\Delta_k v) =& 0 \ \text{ if } \ |k-j| \geq 5, \\
\dot\Delta_j (\dot\Delta_k u \widetilde{\dot\Delta}_k v) =& 0 \ \text{ if } \ k \leq j-4, \ \text{ with } \ \widetilde{\dot\Delta}_k v := \sum_{|k'-k|\leq 1} \dot\Delta_{k'} v.
\end{aligned}$$

Second, we have the Poincare type inequalities (also called Bernstein's inequalities):

Lemma 2.9 (see [9, Lemmas 2.1-2.2])**.** *Let r and R be two constants satisfying $0 < r < R < \infty$. There exists a positive constant $C = C(r, R)$ such that for any $k \in \mathbb{N}$, any $\lambda > 0$, any smooth homogeneous function σ of degree $d \in \mathbb{N}$, any $1 \leq p \leq q \leq \infty$, and any function $u \in L^p(\mathbb{R}^n)$,*

$$\begin{aligned}
\operatorname{supp} \widehat{u} \subset \lambda B_R \Longrightarrow& \|D^k u\|_q := \sum_{|\alpha|=k} \|\partial^\alpha u\|_q \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_p, \\
\operatorname{supp} \widehat{u} \subset \lambda (B_R \setminus B_r) \Longrightarrow& C^{-k-1} \lambda^k \|u\|_p \leq \|D^k u\|_p \leq C^{k+1} \lambda^k \|u\|_p, \\
\operatorname{supp} \widehat{u} \subset \lambda (B_R \setminus B_r) \Longrightarrow& \|\sigma(D) u\|_q \leq C_{\sigma,d} \lambda^{m+n(\frac{1}{p}-\frac{1}{q})} \|u\|_p,
\end{aligned}$$

where $\widehat{u} = \mathcal{F} u$ and $\sigma(D) u$ is defined as $\mathcal{F}^{-1}(\sigma \widehat{u})$.

In most literature on the theory of function spaces, the homogeneous Besov spaces are defined in the ambient space of tempered distributions modulo polynomials (see, e.g., [55]). However, we wish to avoid this type of spaces when solving nonlinear PDEs. In this dissertation, we adopt the definitions of homogeneous spaces in [9, Section 2.3]. Let $\mathscr{S}'_h(\mathbb{R}^n)$ denote the space of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ that satisfy

$$u = \sum_{j \in \mathbb{Z}} \dot\Delta_j u \ \text{ in } \mathscr{S}'.$$

Note that $\mathscr{S}'_h$ is a large enough proper subspace of $\mathscr{S}'$. For example, if $b \in L^p$ with $p \in [1, \infty)$, then $b \in \mathscr{S}'_h$; and if $b \in L^\infty$, then $\nabla b \in \mathscr{S}'_h$.

Let us now recall the definition of homogeneous Besov spaces.

Definition 2.10. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of all the distributions u in $\mathscr{S}'_h(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| \left(2^{js} \|\dot{\Delta}_j u\|_p \right)_{j \in \mathbb{Z}} \right\|_{l^r} < \infty.$$

Remark 2.11. As an immediate consequence of the definition, we have that $u \in \dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a nonnegative sequence $\{c_{j,r}\}_{j \in \mathbb{Z}}$ such that $\|c_{j,r}\|_{l^r} \lesssim 1$ and $\|\dot{\Delta}_j u\|_p \lesssim c_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}}$ for every $j \in \mathbb{Z}$.

Let us collect some useful properties and inequalities for homogeneous Besov spaces.

Lemma 2.12 (see [9, Chapter 2])**.** (i) *For $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq r_1 \leq r_2 \leq \infty$ and $s \in \mathbb{R}$, $\dot{B}^s_{p_1,r_1}(\mathbb{R}^n)$ is continuously embedded in $\dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}(\mathbb{R}^n)$. $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$ is continuously embedded in $C_0(\mathbb{R}^n)$, the space of continuous functions that tend to zero at infinity.*

(ii) *Suppose that $(s_1, s_2, p, p_1, p_2, r) \in \mathbb{R}^2 \times [1,\infty]^4$, $\theta \in (0,1)$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then for any $u \in \mathscr{S}'_h(\mathbb{R}^n)$, we have*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p_2,r}}.$$

(iii) *If $(p,r) \in [1,\infty)^2$, then the space $\mathscr{S}_0(\mathbb{R}^n)$ of functions in $\mathscr{S}(\mathbb{R}^n)$ whose Fourier transforms are supported away from 0 is dense in $\dot{B}^s_{p,r}(\mathbb{R}^n)$.*

(iv) *$\dot{B}^s_{p,r}(\mathbb{R}^n)$ is a Banach space if (s,p,r) satisfies*

$$s < \frac{n}{p}, \;\; or \;\; s = \frac{n}{p} \;\; and \;\; r = 1. \tag{2.5}$$

(v) *Suppose that $(s,p,r) \in (0,\infty) \times [1,\infty]^2$ satisfies* (2.5)*. Assume that f is a smooth function on $\mathbb{R}$ which vanishes at 0. For any real-valued function $u \in L^\infty(\mathbb{R}^n) \cap \dot{B}^s_{p,r}(\mathbb{R}^n)$, the composite function $f \circ u$ belongs to the same space, and there exists a constant C depending on f' and $\|u\|_\infty$ such that*

$$\|f \circ u\|_{\dot{B}^s_{p,r}} \leq C \|u\|_{\dot{B}^s_{p,r}}.$$

Next, we recall Bony's paraproduct decomposition which can be used to define a product of two Besov functions. For two Besov functions u and v, we can formally write

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

where

$$\dot{T}_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v \quad \text{and} \quad \dot{R}(u,v) := \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v.$$

Sometimes it is sufficient to just estimate $\dot{T}'_v u := \dot{T}_v u + \dot{R}(u,v)$. We refer the reader to [9, section 2.6] for the convergence of the above series and the continuity of the paraproduct operators on homogeneous Besov spaces. In this dissertation, we will frequently use the following product laws:

$$\|uv\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}} \|v\|_{\dot{B}^{n/p}_{p,1}}, \ \text{ if } 1 \le p < \infty,$$
$$\|uv\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|u\|_{\dot{B}^{n/q}_{q,1}} \|v\|_{\dot{B}^{n/p-1}_{p,1}}, \ \text{ if } \frac{1}{p} - \frac{1}{q} \le \frac{1}{n} < \frac{1}{p} + \frac{1}{q},$$

and

$$\|uv\|_{\dot{B}^{n/p-2}_{p,1}} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}} \|v\|_{\dot{B}^{n/p-2}_{p,1}}, \ \text{ if } 1 \le p < n \text{ and } n \ge 3.$$

2.3.2 Characterizations of homogeneous Besov norms

Now we recall several equivalent characterizations of homogeneous Besov norms. The next lemma is well-known and can be implied by [9, Theorem 2.34].

Lemma 2.13. *Suppose that $s \in \mathbb{R}$ and $(p,q) \in [1,\infty]^2$. If $k > s/2$ and $k \ge 0$, we have*

$$\|f\|_{\dot{B}^{s,\Delta}_{p,q}} := \left\| t^{-s/2} \|(t\Delta)^k e^{t\Delta} f\|_p \right\|_{L^q(\mathbb{R}_+; \frac{dt}{t})} \simeq \|f\|_{\dot{B}^s_{p,q}}, \ \ \forall f \in \mathscr{S}'_h.$$

A similar result holds if we replace Δ by the Lamé operator. Here the Lamé operator $\mathcal{L}$ is defined by

$$\mathcal{L} := \mu\Delta + (\lambda + \mu)\nabla \operatorname{div} \tag{2.6}$$

with

$$\mu > 0, \text{ and } \nu := \lambda + 2\mu > 0. \tag{2.7}$$

Let us introduce the Hodge operator $\mathcal{Q} = -\nabla(-\Delta)^{-1} \operatorname{div}$ and let $\mathcal{P} = I - \mathcal{Q}$. The Lamé operator and the Laplacian can be expressed by each other, namely,

$$\mathcal{L} = (\mu\mathcal{P} + \nu\mathcal{Q})\Delta = \Delta(\mu\mathcal{P} + \nu\mathcal{Q}) \tag{2.8}$$

and

$$\Delta = \left(\frac{1}{\mu}\mathcal{P} + \frac{1}{\nu}\mathcal{Q}\right)\mathcal{L} = \mathcal{L}\left(\frac{1}{\mu}\mathcal{P} + \frac{1}{\nu}\mathcal{Q}\right). \tag{2.9}$$

So, for every $p \in (1, \infty)$ and $k \in \mathbb{N}$, we have

$$\|\mathcal{L}^k u\|_p \simeq \|\Delta^k u\|_p, \quad u \in W^{2k,p}(\mathbb{R}^n; \mathbb{R}^n). \tag{2.10}$$

Lemma 2.14. *Suppose that* $s \in \mathbb{R}$, $p \in (1, \infty)$ *and* $q \in [1, \infty]$. *If* $k > s/2$ *and* $k \geq 0$, *we have*

$$\|u\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} := \left\|t^{-s/2}\|(t\mathcal{L})^k e^{t\mathcal{L}} u\|_p\right\|_{L^q(\mathbb{R}_+; \frac{dt}{t})} \simeq \|u\|_{\dot{B}^s_{p,q}}, \quad \forall u \in L^p(\mathbb{R}^n; \mathbb{R}^n).$$

Proof. The lemma is a consequence of Lemma 2.13 along with the identities

$$e^{t\mathcal{L}} = e^{\mu t\Delta}\mathcal{P} + e^{\nu t\Delta}\mathcal{Q}$$

and

$$e^{t\Delta} = \mathcal{P}e^{\mu^{-1}t\mathcal{L}} + \mathcal{Q}e^{\nu^{-1}t\mathcal{L}}.$$

□

Next, we give two more equivalent characterizations of homogeneous Besov norms, one via the semigroup generated by the elliptic operator $\mathcal{E}$ defined by (2.3), the other via ball means of differences. Let us define

$$\|f\|_{\dot{B}^{s,-\mathcal{E}}_{p,q}} := \left\| t^{-s/2} \|(t\mathcal{E})^k e^{-t\mathcal{E}} f\|_p \right\|_{L^q\left(\mathbb{R}_+; \frac{dt}{t}\right)}$$

with $k > \frac{s}{2}$, and

$$\|f\|_{\dot{\Lambda}^s_{p,q}} := \left\| \left(\int_{\mathbb{R}^n} \fint_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{sp}} \, dy \, dx \right)^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}_+; \frac{dr}{r})},$$

where $\fint_{B(x,r)}$ denotes the integral mean over the ball $B(x,r)$ centered at x with radius r.

Lemma 2.15 (see [11, 36]). *Let μ be the Hölder index in Lemma 2.7. Suppose that $s \in (0, \mu)$, $p \in [1, \infty)$ and $q \in [1, \infty]$. If $k > s/2$, we have*

$$\|f\|_{\dot{B}^{s,-\mathcal{E}}_{p,q}} \simeq \|f\|_{\dot{\Lambda}^s_{p,q}}, \quad \forall f \in L^p(\mathbb{R}^n).$$

Proof. In [36], only the characterization of inhomogeneous norms was given. But the proof there can be easily adjusted to give the equivalence between homogeneous norms. □

Lemma 2.16. *Suppose that $s \in (0, 1)$, $p \in [1, \infty)$ and $q \in [1, \infty]$. It holds that*

$$\|f\|_{\dot{\Lambda}^s_{p,q}} \simeq \|f\|_{\dot{B}^s_{p,q}}, \ \ \forall f \in L^p(\mathbb{R}^n).$$

Proof. Note that the Hölder index in Lemma 2.7 becomes $\mu = 1$ if $\mathcal{E}$ is replaced by $-\Delta$. By Lemma 2.15, we have $\|f\|_{\dot{\Lambda}^s_{p,q}} \simeq \|f\|_{\dot{B}^{s,\Delta}_{p,q}}$ for any $s \in (0, 1)$. This together with Lemma 2.13 implies Lemma 2.16. □

2.4 Coordinate transformations

2.4.1 From Eulerian to Lagrangian coordinates

We start with the well-posedness issues of the ODE

$$\begin{cases} \frac{d}{dt}X(t,y) = u(t, X(t,y)), \quad t > 0 \\ X(0,y) = y \end{cases} \tag{2.11}$$

and the conservative continuity equation

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{R}^n \end{cases} \tag{2.12}$$

within the Cauchy-Lipschitz framework. Let us temporarily assume that u is C^1 vector field, namely,

$$u \in L^1([0,T]; C_b^1(\mathbb{R}^n; \mathbb{R}^n)).$$

The following result is well-known.

Theorem 2.17. *For any $y \in \mathbb{R}^n$, (2.11) has a unique solution $X(\cdot, y) = X_u(\cdot, y) \in W^{1,1}([0,T])$. For any $t \in [0,T]$, $X(t,\cdot)$ is a C^1 diffeomorphism over $\mathbb{R}^n$ satisfying*

$$\|\nabla X(t)\|_\infty \vee \|\nabla X^{-1}(t)\|_\infty \leq \exp\left(\|\nabla u\|_{L^1_t(L^\infty)}\right),$$

where $X^{-1}(t,\cdot)$ is the inverse of $X(t,\cdot)$. Let $J(t,y) = J_X(t,y)$ be the determinant of $DX(t,y)$. Then $J \in W^{1,1}([0,T]; C_b)$ and satisfies

$$\frac{d}{dt}J(t,y) = (\text{div}\, u)(t, X(t,y))J(t,y), \quad J(0,y) = 1.$$

Consequently,

$$\exp\left(-\|\operatorname{div} u\|_{L^1_t(L^\infty)}\right) \leq J(t,y) \leq \exp\left(\|\operatorname{div} u\|_{L^1_t(L^\infty)}\right).$$

In particular, if $\operatorname{div} u = 0$*, then* $J_X(t,y)$ *is identical to* 1.

Now we turn to the well-posedness of the continuity equation (2.12).

Definition 2.18. Let $\rho_0 \in L^\infty(\mathbb{R}^n)$. A bounded function ρ is called a weak solution to (2.12) if for any test function $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^n)$,

$$\int_0^T \int_{\mathbb{R}^n} \rho(\partial_t \varphi + u \cdot \nabla_x \varphi)\, dx\, dt + \int_{\mathbb{R}^n} \rho_0 \varphi(0,x)\, dx = 0,$$

and $\rho(t) \overset{*}{\rightharpoonup} \rho_0$ in $L^\infty(\mathbb{R}^n)$ as $t \to 0^+$.

Theorem 2.19 (see, e.g., [5, Proposition 2.1])**.** *For any* $\rho_0 \in L^\infty(\mathbb{R}^n)$*,* (2.12) *has a unique weak solution that is given by the formula*

$$\rho(t,x) = \frac{\rho_0(X^{-1}(t,x))}{J_X(t, X^{-1}(t,x))} = J_{X^{-1}}(t,x)\rho_0(X^{-1}(t,x)).$$

In particular, if $\operatorname{div} u = 0$*, we have* $\rho(t,x) = \rho_0(X^{-1}(t,x))$.

Next, we assume more regularity on u:

$$u \in C([0,T]; \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}^{n/p+1}_{p,1}(\mathbb{R}^n)) \tag{2.13}$$

for some $p \in [1,\infty)$. Then the corresponding trajectory X_u is also more regular. To see this, we need the following result which guarantees that Besov regularity of a function is preserved under changes of variables. To state the assumption of the lemma, let us recall that a function f is called a multiplier for $\dot{B}^s_{p,q}(\mathbb{R}^n)$ if f defines a continuous linear operator on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by pointwise multiplication. If f is a multiplier for $\dot{B}^s_{p,q}(\mathbb{R}^n)$, we write $f \in \mathscr{M}(\dot{B}^s_{p,q}(\mathbb{R}^n))$ and define the multiplier norm by

$$\|f\|_{\mathscr{M}(\dot{B}^s_{p,q}(\mathbb{R}^n))} := \sup_{\|\phi\|_{\dot{B}^s_{p,q}}=1} \|f\phi\|_{\dot{B}^s_{p,q}}.$$

Lemma 2.20 (see [22, Lemma 2.1.1])**.** *Let X be a C^1 diffeomorphism over $\mathbb{R}^n$. Let $p \in [1,\infty)$, $q \in [1,\infty]$ and $s \in (-n/p', n/p)$. Then the linear map $f \mapsto f \circ X$ is continuous on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ if one of the following conditions holds*

(i) *$s \in (0,1)$,*

(ii) *$s \in (-1,0)$ and $J_{X^{-1}} \in \mathscr{M}(\dot{B}^{-s}_{p',q'}(\mathbb{R}^n))$.*

Remark 2.21. The above lemma extends to $s = 0$ by interpolation, and to higher order regularities if we make stronger assumptions on X (again, see [22]).

By the above lemma and product laws in Besov spaces, one can prove the following

Lemma 2.22. *Let u satisfy* (2.13) *and $X = X_u$ solve* (2.11)*. Define $A = A_u = (DX_u)^{-1}$, $J = J_X = \det DX$, and $\mathscr{A} = \mathscr{A}_u =$ adj DX (the adjugate of DX, i.e., $\mathscr{A} = JA$). Then*

$$DX - I_n, J - I_n, A - I_n, \mathscr{A} - I_n \in C([0,T]; \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)).$$

Now for any scalar function $\phi = \phi(x)$ and any vector field $v = v(x)$, it is easy to see that

$$(\nabla\phi) \circ X = A^{\intercal}\nabla(\phi \circ X), \tag{2.14}$$

and

$$(\operatorname{div} v) \circ X = \operatorname{Tr}[AD(v \circ X)], \tag{2.15}$$

where $\operatorname{Tr}A$ denotes the trace of A. On the other hand, using an integration by part argument as in the appendix of [20], we also have

$$(\operatorname{div} v) \circ X = J^{-1} \operatorname{div}(\mathscr{A}(v \circ X)). \tag{2.16}$$

This along with (2.15) gives the following identity

$$\mathrm{Tr}[\mathscr{A} D(v\circ X)] = \mathrm{div}(\mathscr{A}(v\circ X)). \tag{2.17}$$

Applying (2.14) and (2.15), we see that

$$(\nabla \,\mathrm{div}\, v)\circ X = A^{\intercal}\nabla \mathrm{Tr}(AD(v\circ X)). \tag{2.18}$$

By writing $\Delta = \mathrm{div}\,\nabla$, we get from (2.14) and (2.16) that

$$(\Delta v)\circ X = J^{-1}\,\mathrm{div}(\mathscr{A} A^{\intercal}\nabla(v\circ X)). \tag{2.19}$$

Note that all the above equations hold even if u is not divergence free. But if $\mathrm{div}\, u = 0$, we have $J \equiv 1$, and thus, $A = \mathscr{A}$.

2.4.2 From Lagrangian to Eulerian coordinates

Throughout, we denote variables in Lagrangian coordinates by bold letters (e.g., $\boldsymbol{\rho}$, $\mathbf{u}$, $\mathbf{P}$, etc.). Given a velocity field $\mathbf{v}$ in Lagrangian coordinates, we define

$$X_{\mathbf{v}}(t,y) = y + \int_0^t \mathbf{v}(\tau, y)\, d\tau.$$

A smallness condition will be needed to recover the Eulerian velocity field v.

Lemma 2.23 (see also [20])**.** *Suppose that* $\mathbf{v} \in L^1([0,T]; C_b^1(\mathbb{R}^n;\mathbb{R}^n))$ *satisfies*

$$\|\nabla \mathbf{v}\|_{L^1_T(L^\infty)} \le \frac{1}{2}. \tag{2.20}$$

Then $X_{\mathbf{v}}(t,\cdot)$ is a C^1-diffeomorphism over $\mathbb{R}^n$ for every $t \in [0,T]$. Denote by $X_{\mathbf{v}}^{-1}(t,\cdot)$ the inverse of $X_{\mathbf{v}}(t,\cdot)$. It holds that

$$\|DX_{\mathbf{v}} - I_n\|_{L^\infty_T(L^\infty)} \leq \|D\mathbf{v}\|_{L^1_T(L^\infty)}, \tag{2.21}$$

$$\|DX_{\mathbf{v}}^{-1} - I_n\|_{L^\infty_T(L^\infty)} \leq 2\|D\mathbf{v}\|_{L^1_T(L^\infty)}. \tag{2.22}$$

Proof. The first inequality (2.21) is easily seen. Let $Y_{\mathbf{v}}(t,x)$ be the solution of the integral equation $Y_{\mathbf{v}}(t,x) = x - \int_0^t \mathbf{v}(\tau, Y_{\mathbf{v}}(t,x))\,d\tau$. Note that this equation is solvable under the assumption (2.20). Then it is not difficult to see that $Y_{\mathbf{v}}(t,\cdot)$ and $X_{\mathbf{v}}(t,\cdot)$ are inverses to each other. One can readily get $\|DY_{\mathbf{v}}\|_{L^\infty_T(L^\infty)} \leq 2$, which further implies (2.22). □

Next, we assume additionally that $\mathbf{v}$ satisfies

$$\mathbf{v} \in C([0,T]; \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)) \cap L^1((0,T); \dot{B}^{n/p+1}_{p,1}(\mathbb{R}^n)) \tag{2.23}$$

and

$$\|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})} \leq c_0, \tag{2.24}$$

so that (2.20) is fulfilled. The number c_0 may be chosen even smaller later. Define $A_{\mathbf{v}} = (DX_{\mathbf{v}})^{-1}$, $J_{X_{\mathbf{v}}} = \det DX_{\mathbf{v}}$, and $\mathscr{A}_{\mathbf{v}} = \operatorname{adj} DX_{\mathbf{v}}$. Then we have the following:

Lemma 2.24 (see [20]). *There exists a constant $C > 0$ such that*

$$\|DX_{\mathbf{v}} - I_n\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} + \|DX_{\mathbf{v}}^{-1} - I_n\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} \leq C\|D\mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})},$$
$$\|\partial_t X_{\mathbf{v}}^{-1}\|_{L^2_T(\dot{B}^{n/p}_{p,1})} \leq C\|\mathbf{v}\|_{L^2_T(\dot{B}^{n/p}_{p,1})}.$$

We conclude this section with some estimates that will be used to prove the existence and stability of solutions to the Lagrangian formulation of (1.2) or (1.3).

Lemma 2.25 (see [17, 20]). *Let* $\mathbf{v}, \mathbf{v}_1$ *and* $\mathbf{v}_2$ *be vector fields satisfying* (2.23) *and* (2.24). *Let* $\delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$. *Then we have*

$$\|A_{\mathbf{v}} - I_n\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} + \|\mathscr{A}_{\mathbf{v}} - I_n\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} \lesssim \|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})}, \tag{2.25}$$

$$\|J^{\pm 1}_{\mathbf{v}} - 1\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} \lesssim \|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})}, \tag{2.26}$$

$$\|A_{\mathbf{v}_1} - A_{\mathbf{v}_2}\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} + \|\mathscr{A}_{\mathbf{v}_1} - \mathscr{A}_{\mathbf{v}_2}\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} \lesssim \|\nabla \delta\mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})}, \tag{2.27}$$

$$\|J^{\pm 1}_{\mathbf{v}_1} - J^{\pm 1}_{\mathbf{v}_2}\|_{L^\infty_T(\dot{B}^{n/p}_{p,1})} \lesssim \|\nabla \delta\mathbf{v}\|_{L^1_T(\dot{B}^{n/p}_{p,1})}, \tag{2.28}$$

$$\|\partial_t \mathscr{A}_{\mathbf{v}}(t)\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|\nabla \mathbf{v}(t)\|_{\dot{B}^{n/p}_{p,1}}, \tag{2.29}$$

$$\|\partial_t \mathscr{A}_{\mathbf{v}}(t)\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|\mathbf{v}(t)\|_{\dot{B}^{n/p}_{p,1}}, \ \ if\ p < 2n, \tag{2.30}$$

$$\|\partial_t(\mathscr{A}_{\mathbf{v}_1} - \mathscr{A}_{\mathbf{v}_2})\|_{L^1_t(\dot{B}^{n/p}_{p,1})} \lesssim \|\nabla \delta\mathbf{v}\|_{L^1_t(\dot{B}^{n/p}_{p,1})}, \tag{2.31}$$

$$\|\partial_t(\mathscr{A}_{\mathbf{v}_1} - \mathscr{A}_{\mathbf{v}_2})\|_{L^2_t(\dot{B}^{n/p-1}_{p,1})} \lesssim \|\delta\mathbf{v}\|_{L^2_t(\dot{B}^{n/p}_{p,1})}, \ \ if\ p < 2n, . \tag{2.32}$$

Chapter 3

Incompressible Viscous Fluids

In this chapter, we mainly investigate the global existence and stability for solutions to (1.2) in three dimension.

3.1 Main results and strategy of the proof

Our first main theorem concerns the global existence and maximal L^1-in-time regularity estimates for solutions to (1.2) provided the initial velocity is small in the Besov space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. But we do not require the initial density to stay close to an equilibrium. First, let us be clear about what it means by a solution to (1.2). Let $\mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ be the space that consists of all divergence free vector fields whose components belong to $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$.

Definition 3.1. Let $T \in (0, \infty]$. Suppose that $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$ and $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. We say that $(\rho, u, \nabla P)$ is a *strong solution* to (1.2) if

$$\begin{cases} \rho - 1 \in C([0,T); \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)), \partial_t \rho \in L^2_{loc}([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)), \\ u \in C([0,T); \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)), \\ (\partial_t u, \Delta u, \nabla P) \in \Big(L^1_{loc}([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3))\Big)^3, \end{cases}$$

and $(\rho, u, \nabla P)$ satisfies (1.2) for a.e. $t \in (0, T)$.

Theorem 3.2. *Assume that the initial density ρ_0 satisfies* (1.4)*, $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$, and the initial velocity $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Then there exists some $T > 0$ such that* (1.2) *has a unique* *local-in-time solution* $(\rho, u, \nabla P)$ *with* ρ *verifying*

$$m \le \rho(t,x) \le \frac{1}{m}, \ \ \forall t \ge 0, \ a.e. \ x \in \mathbb{R}^3.$$

Moreover, there exists a positive constant ε_0 *depending on* m *and* $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$ *such that if* u_0 *satisfies*

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}} \le \varepsilon_0, \tag{3.1}$$

then the above solution exists globally in time and verifies

$$\|u\|_{L^\infty(\dot{B}^{1/2}_{2,1})} + \|\Delta u, \partial_t u, \nabla P\|_{L^1(\dot{B}^{1/2}_{2,1})} \le C_0 \|u_0\|_{\dot{B}^{1/2}_{2,1}}, \tag{3.2}$$

and

$$\|\rho - 1\|_{L^\infty(\dot{B}^{3/2}_{2,1})} \le C_1 \|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}, \tag{3.3}$$

where C_0 *is a constant depending on* m *and* $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$*, and* C_1 *is an absolute constant.*

Remark 3.3. The estimate of the $L^1(\mathbb{R}_+; \dot{B}^{1/2}_{2,1})$ norm in (3.2) is a maximal regularity type estimate. To our knowledge, this is the first such result concerning maximal L^1 regularity for density-dependent viscous fluids without any smallness assumption on the fluctuation of the initial density.

Remark 3.4. The regularity assumption on the initial density can be weakened to allow a slight discontinuity. But to simplify the exposition and avoid unpleasant technicalities, we do not pursue the optimal regularity assumption on ρ_0. Nevertheless, we do not know if our method can be improved to give maximal L^1 regularity estimates for weak solutions to (1.2) with merely measurable (or, piecewise constant) initial densities.

Remark 3.5. The estimate $\|u\|_{L^\infty(\dot{B}^{1/2}_{2,1})} \le C\|u_0\|_{\dot{B}^{1/2}_{2,1}}$ for weak solutions to (1.2) has been recently obtained by Zhang in [59], in which he also obtained some global L^2-in-time estimates.

The initial density in [59] is merely bounded and bounded from zero. But the uniqueness of weak solutions in critical spaces is not known, unless the initial velocity field has more regularity so that one can prove a local L^1-in-time estimate for the Lipschitz seminorm of the velocity field (see also [13,48]). However, the energy methods used in [13,48,59] are unlikely to give (3.2), even if the data is smooth.

As an application of (3.2), we prove a second result concerning the long time asymptotics for the globally-defined velocity constructed in Theorem 3.2.

Theorem 3.6. *Let* (ρ_0, u_0) *be the initial data in Theorem 3.2 that generates a global solution* $(\rho, u, \nabla P)$ *to* (1.2). *Then it holds that*

$$\lim_{t\to\infty} \|u(t)\|_{\dot{B}^{1/2}_{2,1}} = 0. \tag{3.4}$$

The proof of Theorem 3.6 relies on Theorem 3.2, so let us only elaborate the strategy for the proof of Theorem 3.2.

Step 1. Reformulating (1.2) **in Lagrangian coordinates.** The Lagrangian coordinate is a natural coordinate system at fluid motion, in which the observer follows an individual fluid parcel as it moves through space and time. It can be used to convert a free boundary problem into an equivalent problem in a fixed domain (see, e.g., [54]); or to convert a coupled hyperbolic-parabolic system into a merely parabolic system (see, e.g., [17,20,21]).

Let us introduce new unknowns

$$(\boldsymbol{\rho}, \mathbf{u}, \mathbf{P})(t, y) = (\rho, u, P)\big(t, X(t, y)\big).$$

In view of the continuity equation in (1.2), we have $\boldsymbol{\rho} \equiv \rho_0$. Using eqs. (2.14) to (2.17) and (2.19) and the chain rule, one can formally convert the system (1.2) into its Lagrangian formulation that reads

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \operatorname{div}(\mathscr{A}_\mathbf{u} \mathscr{A}_\mathbf{u}^\intercal \nabla \mathbf{u}) + \mathscr{A}_\mathbf{u}^\intercal \nabla \mathbf{P} = 0, \\ \operatorname{div}(\mathscr{A}_\mathbf{u} \mathbf{u}) = \operatorname{Tr}(\mathscr{A}_\mathbf{u} D\mathbf{u}) = 0, \\ \mathbf{u}|_{t=0} = u_0. \end{cases} \tag{3.5}$$

In this new system, we associate $\mathscr{A}_\mathbf{u}$ with the new velocity $\mathbf{u}$ so that the system is closed (i.e., determined). Precisely, we denote

$$\mathscr{A}_\mathbf{u} = \text{adj}\, DX_\mathbf{u}, \quad \text{with} \quad X_\mathbf{u}(t,y) = y + \int_0^t \mathbf{u}(\tau, y)\, d\tau. \tag{3.6}$$

Remark 3.7. One can write $A_\mathbf{u} := (DX_\mathbf{u}(t,y))^{-1}$ in place of $\mathscr{A}_\mathbf{u}$ in the "momentum" equation of (3.5). But as in the work of Solonnikov [54], one should use $\mathscr{A}_\mathbf{u}$ in the second equation. The reason is that, when linearizing (3.5) to seek existence, we will need the fact that (2.17) is an identity, whether u is divergence free or not. Of course, once the existence issue is settled, we can use either $\mathscr{A}_\mathbf{u}$ or $A_\mathbf{u}$ in (3.5). More precisely, the linearized system of (3.5) reads

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathbf{P} = \operatorname{div}((\mathscr{A}_\mathbf{v} \mathscr{A}_\mathbf{v}^\intercal - I)\nabla \mathbf{v}) + (I - \mathscr{A}_\mathbf{v}^\intercal)\nabla \mathbf{Q}, \\ \operatorname{div} \mathbf{u} = \operatorname{div}((I - \mathscr{A}_\mathbf{v})\mathbf{v}) = Tr((I - \mathscr{A}_\mathbf{v})D\mathbf{v}), \\ \mathbf{u}|_{t=0} = u_0. \end{cases} \tag{3.7}$$

To obtain *a priori* estimates for this system, we need to write $\operatorname{div} \mathbf{u}$ in two different ways. To conclude, we will solve (3.7) without assuming $\det DX_\mathbf{v} \equiv 1$.

Definition 3.8. Let $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$ and $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. We say that $(\mathbf{u}, \nabla \mathbf{P})$ is a *strong solution* to (3.5) if for some $T \in (0, \infty]$,

$$\begin{cases} \mathbf{u} \in C([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)), \\ \mathscr{A}_\mathbf{u} - I \in C([0,T); \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)), \\ (\partial_t \mathbf{u}, \Delta \mathbf{u}, \nabla \mathbf{P}) \in \left(L^1_{loc}([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)) \right)^3, \end{cases}$$

and $(\mathbf{u}, \nabla\mathbf{P})$ satisfies (3.5) for a.e. $t \in (0,T)$.

The justification of equivalence between Eulerian and Lagrangian can be found in [17,20], or one can show it using the preliminaries collected in Section 2.4. Let us now state a well-posedness result for (3.5).

Theorem 3.9. *Assume that the initial density ρ_0 satisfies* (1.4)*, $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$, and the initial velocity $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Then there exists some $T > 0$ such that* (3.5) *has a unique local solution $(\mathbf{u}, \nabla\mathbf{P})$ with $\mathbf{u}$ verifying*

$$\|\nabla\mathbf{u}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \le c_0 \tag{3.8}$$

for some positive constant c_0.

Moreover, there exists another constant ε_0 depending on m and $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$ so that if u_0 satisfies

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}} \le \varepsilon_0,$$

the local solution becomes globally in time and verifies

$$\|\mathbf{u}\|_{L^\infty(\mathbb{R}_+;\dot{B}^{1/2}_{2,1})} + \|\Delta\mathbf{u}, \partial_t\mathbf{u}, \nabla\mathbf{P}\|_{L^1(\mathbb{R}_+;\dot{B}^{1/2}_{2,1})} \le C_0\|u_0\|_{\dot{B}^{1/2}_{2,1}}$$

for some constant C_0 depending on m and $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$.

Step 2. Maximal regularity for Stokes system. The heart of the present paper is the well-posedness issue for the linearized system (3.7). To this end, we will mainly focus on the maximal L^1 regularity for the following linear Stokes-like system

$$\begin{cases} \rho(x)\partial_t u - \Delta u + \nabla P = f, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = \operatorname{div} R = g, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{3.9}$$

Here the coefficient ρ is a time-independent function that satisfies (1.4) and no other assumption is needed temporarily. The system is supplemented with the compatibility condition $\operatorname{div} u_0(x) = \operatorname{div} R(0,x)$. For simplicity, we will take $R(0,\cdot) = 0$, since this is the case in applications.

We are going to construct solutions to (3.9) using the theory of semigroups and abstract Cauchy problem. In doing so, let us introduce a new variable

$$v := u - \mathcal{Q}R = u + \nabla(-\Delta)^{-1}g,$$

where $\mathcal{Q} = -\nabla(-\Delta)^{-1}\operatorname{div}$ is the Hodge operator. Then the system (3.9) can be equivalently reformulated as

$$\begin{cases} \rho\partial_t v - \Delta v + \nabla(P-g) = f - \rho\mathcal{Q}\partial_t R, \\ \operatorname{div} v = 0, \\ v(0,x) = u_0(x). \end{cases} \tag{3.10}$$

We shall obtain maximal L^1 regularity estimates for the above system. Compared with the classical Stokes system, this is a challenging problem because the velocity and the pressure are strongly coupled in the presence of the density. Let us now explain how to achieve our goal. Denote $b = \rho^{-1}$ and let $\mathcal{E}_b = -\operatorname{div}(b\nabla)$. Applying $\operatorname{div} b$ to the first equation of (3.10) and using the second equation, we see that

$$\mathcal{E}_b(P-g) = -\operatorname{div}[b(\Delta v + f - \rho\mathcal{Q}\partial_t R)].$$

Next, we introduce the Hodge operator

$$\mathcal{Q}_b := -\nabla\mathcal{E}_b^{-1}\operatorname{div} b \tag{3.11}$$

associated with $\mathcal{E}_b$, and let

$$\mathcal{P}_b = I - \mathcal{Q}_b. \tag{3.12}$$

Then we can write

$$\nabla(P - g) = \mathcal{Q}_b(\Delta v + f - \rho\mathcal{Q}\partial_t R). \tag{3.13}$$

Plugging (3.13) in (3.10), we hence introduce a generalized Stokes operator

$$\mathcal{S} := b\mathcal{P}_b\Delta, \tag{3.14}$$

so (3.10) can be further reformulated as an abstract Cauchy problem

$$\begin{cases} \partial_t v - \mathcal{S}v = \tilde{f} := b\mathcal{P}_b(f - \rho\mathcal{Q}\partial_t R), \\ v(0,x) = u_0(x). \end{cases} \tag{3.15}$$

We will show that the Stokes operator $\mathcal{S}$ generates a semigroup $e^{t\mathcal{S}}$ on

$$\mathcal{P}L^2 := \{u \in L^2(\mathbb{R}^3, \mathbb{R}^3) |\, \operatorname{div} u = 0\}.$$

So by Duhamel's principle, we can formally write the solution v to (3.15) as

$$v(t) = e^{t\mathcal{S}}u_0 + \int_0^t e^{(t-\tau)\mathcal{S}}\tilde{f}(\tau)\, d\tau. \tag{3.16}$$

To obtain maximal regularity estimates for v, we will characterize some Besov norms for divergence free vector fields via the semigroup $e^{t\mathcal{S}}$. In fact, we are able to prove the following:

Theorem 3.10. *Assume that b satisfies* (1.4). *For any $s \in (0,2)$, any $q \in [1,\infty]$, and any*

$$u_0 \in \mathcal{P}H^2 := \{u \in H^2(\mathbb{R}^3, \mathbb{R}^3) |\, \operatorname{div} u = 0\},$$

we have

$$\|u_0\|_{\dot{B}^s_{2,q}} \simeq \left\| t^{-s/2} \| t\mathcal{S} e^{t\mathcal{S}} u_0 \|_2 \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \simeq \left\| t^{-s/2} \| t\Delta e^{t\mathcal{S}} u_0 \|_2 \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})}.$$

Remark 3.11. Note that we do not need any regularity for the coefficient b at this point.

Applying Theorem 3.10 (with $q = 1$) to (3.16) gives us an *a priori* maximal regularity estimate for v:

$$\|v\|_{L^\infty_T(\dot{B}^s_{2,1})} + \|\partial_t v, \mathcal{S}v\|_{L^1_T(\dot{B}^s_{2,1})} \lesssim \|u_0\|_{\dot{B}^s_{2,1}} + \|\tilde{f}\|_{L^1_T(\dot{B}^s_{2,1})}. \tag{3.17}$$

This gives good estimates for $(u, \nabla P)$ except for $\|u\|_{L^\infty_T(\dot{B}^s_{2,1})}$. If we apply (3.17) to bound $\|u\|_{L^\infty_T(\dot{B}^s_{2,1})}$ directly, we need to use $\|\mathcal{Q}R\|_{L^\infty_T(\dot{B}^s_{2,1})}$, which would cause trouble when proving local existence of large solutions to (3.5). To overcome this difficulty, we view ∇P in the first equation of (3.9) as a source term, and write

$$\partial_t u - b\Delta u = b(f - \nabla P). \tag{3.18}$$

Then the maximal L^1 regularity estimate for u can be obtained by equivalent characterizations of Besov norms via the semigroup $e^{tb\Delta}$.

Step 3. Elliptic estimates. It remains to bound the inhomogeneous term $\tilde{f}$ in (3.15). For this, we need to study the continuity of the operator $b\mathcal{P}_b$ on Besov spaces. In other words, we need to study the gradient estimates for solutions to the divergence form elliptic equation

$$-\mathcal{E}_b P = \operatorname{div} f. \tag{3.19}$$

But this is again a difficult problem for it is well-known that $\nabla \mathcal{E}_b^{-1} \operatorname{div}$ is not of Calderón-Zygmund type. In general, $\nabla \mathcal{E}_b^{-1} \operatorname{div}$ is not bounded on L^p for p not close enough to 2, even if the coefficient b is smooth (see [39]). In fact, in order to prove continuity of $\nabla \mathcal{E}_b^{-1} \operatorname{div}$ on homogeneous function spaces, one should treat it as a zeroth-order operator. This suggests that

b should be in some function spaces that have the same scaling as L^∞. So we once again need b to be in some "critical" spaces.

Our strategy is to use an iteration scheme to gain elliptic regularity. In the initial iteration, we prove an inequality of the form

$$\|\nabla \mathcal{E}_b^{-1} \operatorname{div} f\|_{\dot{B}^{s_0}_{p_0,r}} \leq C\|f\|_{\dot{B}^{s}_{p,r}},$$

in which a loss of regularity is allowed, but the scalings of both Besov spaces are the same, meaning that $s_0 - \frac{3}{p_0} = s - \frac{3}{p}$ with $s_0 < s$. The proof relies on Lemma 2.15, Lemma 2.16 and the boundedness of the Riesz transform $\nabla \mathcal{E}_b^{-1/2}$ on L^p, $1 < p \leq 2$ (see Section 2.2). In this step, we only require b to satisfy (1.4). But if b has more regularity (in critical spaces), the loss of regularity can be recovered via an iteration scheme. We are able to eventually prove that $\nabla \mathcal{E}_b^{-1} \operatorname{div}$ is bounded on some homogeneous Besov spaces including $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Such a result is nontrivial for $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ has the same scaling as $L^3(\mathbb{R}^3)$. Indeed, our method also applies to L^p elliptic gradient estimates for p slightly larger than the space dimensions.

Carrying out the details of the strategy, we can prove the following maximal L^1 regularity theorem for the Stokes system (3.9).

Theorem 3.12. *Let $T \in (0, \infty]$. Assume that $\rho = \rho(x)$ satisfies* (1.4)*, $\rho - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$ and $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Suppose that f, g, R satisfy*

$$R \in C_b([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)), \;\; (f, \nabla g, \partial_t R) \in \Big(L^1((0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3))\Big)^3,$$

$R(0) = 0$, and $\operatorname{div} R = g$*. Then the system* (3.9) *has a unique strong solution $(u, \nabla P)$ in the class*

$$u \in C_b([0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)), \; (\Delta u, \partial_t u, \nabla P) \in \Big(L^1((0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3))\Big)^3.$$

Moreover, there exists a constant C depending on m and $\|\rho - 1\|_{\dot{B}^{3/2}_{2,1}}$ such that

$$\|u\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})} + \|\partial_t u, \Delta u, \nabla P\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \leq C\|u_0\|_{\dot{B}^{1/2}_{2,1}} + C\|f, \partial_t R, \nabla g\|_{L^1_T(\dot{B}^{1/2}_{2,1})}. \tag{3.20}$$

The remainder of this chapter is organized as follows. Section 3.2 is devoted to the proof of maximal L^1 regularity for the Stokes system (3.9). To this end, we give equivalent characterizations of homogeneous Besov norms via the semigroup generated by the generalized Stokes operator, and prove elliptic gradient estimates. Then, in Section 3.3, we apply the results established in Section 3.2 to prove Theorem 3.2 and Theorem 3.9. The proof of Theorem 3.6 will be given in Section 3.4.

3.2 Maximal L^1 regularity for Stokes system

This section is devoted to the proof of Theorem 3.12, which is the heart of this chapter.

3.2.1 Stokes operator

We present some useful properties for the Stokes operator. Let $\rho = \rho(x)$ satisfy (1.4) and denote $b = \rho^{-1}$. In the sequel, we use the notations $L^2 = L^2(\mathbb{R}^3; \mathbb{R}^3)$, $\mathcal{P}L^2 = \{u \in L^2 | \operatorname{div} u = 0\}$, $\|\cdot\|$ the L^2 norm induced by the standard L^2 inner product $\langle \cdot, \cdot \rangle$, and $\|\cdot\|_\rho$ the weighted norm induced by the inner product

$$\langle u, v \rangle_\rho = \int_{\mathbb{R}^3} u(x) \cdot v(x) \rho(x)\, dx.$$

Let $\mathcal{Q}_b$, $\mathcal{P}_b$ and $\mathcal{S}$ be defined by (3.11), (3.12) and (3.14), respectively. It is well know that $\mathcal{Q}_b$ is bounded on L^2 and $\|\mathcal{Q}_b f\|_b \leq \|f\|_b$ for every $f \in L^2$. If $b \equiv 1$, we denote $\mathcal{P} = \mathcal{P}_1$ and $\mathcal{Q} = \mathcal{Q}_1$. Formally, we have $\operatorname{div} \mathcal{P} = 0$. It is for this reason we use $\mathcal{P}$ in front of a space of vectors to denote its subspace of divergence-free vectors.

Lemma 3.13. *With the above notations, we have*

(i) $b\mathcal{P}_b : L^2 \to \mathcal{P}L^2$ *is bounded and it holds that* $\mathcal{Q}(b\mathcal{P}_b) = b\mathcal{P}_b\mathcal{Q} \equiv 0$, *and* $\mathcal{P}(b\mathcal{P}_b) = b\mathcal{P}_b\mathcal{P} \equiv b\mathcal{P}_b$.

(ii) $b\mathcal{P}_b : \mathcal{P}L^2 \to \mathcal{P}L^2$ *is invertible with a continuous inverse* $\mathcal{P}\rho$. *Thus, it holds that*

$$\|b\mathcal{P}_b u\| \simeq \|u\|, \ \ \forall u \in \mathcal{P}L^2. \tag{3.21}$$

(iii) $b\mathcal{P}_b : L^2 \to L^2$ *is self-adjoint on* $(L^2, \langle\cdot,\cdot\rangle)$, *while* $b\mathcal{P}_b : \mathcal{P}L^2 \to \mathcal{P}L^2$ *is self-adjoint on both* $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle)$ *and* $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$.

(iv) $\mathcal{S} : \mathcal{P}H^2 \subset \mathcal{P}L^2 \to \mathcal{P}L^2$ *is a self-adjoint operator on* $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$.

The proof is straightforward and thus left to the reader.

Lemma 3.14. *The Stokes operator* $\mathcal{S} : \mathcal{P}H^2 \subset \mathcal{P}L^2 \to \mathcal{P}L^2$ *generates an analytic semigroup of contraction* $\{e^{t\mathcal{S}}\}_{t\geq 0}$ *on* $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$, *and* $e^{t\mathcal{S}}b\mathcal{P}_b$ *is self-adjoint on* $(L^2, \langle\cdot,\cdot\rangle)$ *for every* $t \geq 0$.

Proof. For $u \in \mathcal{P}H^2$, we have

$$\langle \mathcal{S}u, u\rangle_\rho = \int_{\mathbb{R}^n} \mathcal{P}_b \Delta u(x) \cdot u(x)\, dx = -\|\nabla u\|^2 \leq 0.$$

Since $\mathcal{S}$ is self-adjoint on $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$, so by Theorem 2.4, $\mathcal{S}$ generates an analytic semigroup of contraction $\{e^{t\mathcal{S}}\}_{t\geq 0}$ on $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$. Moreover, $e^{t\mathcal{S}}$ is self-adjoint on $(\mathcal{P}L^2, \langle\cdot,\cdot\rangle_\rho)$. Then we have for all $u, v \in L^2$ that

$$\langle e^{t\mathcal{S}} b\mathcal{P}_b u, v\rangle = \langle e^{t\mathcal{S}} b\mathcal{P}_b u, b\mathcal{P}_b v\rangle_\rho = \langle b\mathcal{P}_b u, e^{t\mathcal{S}} b\mathcal{P}_b v\rangle_\rho = \langle u, e^{t\mathcal{S}} b\mathcal{P}_b v\rangle.$$

This means that $e^{t\mathcal{S}}b\mathcal{P}_b$ is self-adjoint on $(L^2, \langle\cdot,\cdot\rangle)$. □

Proposition 3.15. *For any* $u_0 \in \mathcal{P}L^2$, *it holds that* $\lim_{t\to\infty} \|e^{t\mathcal{S}}u_0\| = 0$.

Proof. In view of Lemma 3.14 and (3.21), we apply Gagliardo–Nirenberg inequality to get

$$\|e^{t\mathcal{S}}u_0\|_p \leq C\|e^{t\mathcal{S}}u_0\|^{1-\theta}\|\mathcal{S}e^{t\mathcal{S}}u_0\|^\theta \leq Ct^{-\theta}\|u_0\|, \ \forall u_0 \in \mathcal{P}L^2,$$

where $\frac{1}{p} = \frac{1}{2} - \frac{2\theta}{3}$, $p \in (2, \infty]$, and $\theta \in (0, 1)$. Now for any $u_0 \in L^2$, since $b\mathcal{P}_b u_0 \in \mathcal{P}L^2$, we get $\|e^{t\mathcal{S}}b\mathcal{P}_b u_0\|_p \leq Ct^{-\theta}\|u_0\|$. By duality, we obtain $\|e^{t\mathcal{S}}b\mathcal{P}_b u_0\| \leq Ct^{-\theta}\|u_0\|_{p'}$ with $\frac{1}{p'} = \frac{1}{2} + \frac{2\theta}{3}$.

So for any $u_0 \in \mathcal{P}L^2 \cap L^{p'}$, we have

$$\|e^{t\mathcal{S}}u_0\| = \|e^{t\mathcal{S}}b\mathcal{P}_b(\rho u_0)\| \le Ct^{-\theta}\|u_0\|_{p'},$$

which implies $\lim_{t\to\infty} \|e^{t\mathcal{S}}u_0\| = 0$. By a density argument, the result still holds for $u_0 \in \mathcal{P}L^2$. This completes the proof. $\square$

3.2.2 Characterizations of Besov norms via semigroup $e^{t\mathcal{S}}$

In order to obtain maximal regularity estimates for the Stokes system, we use the semigroup $e^{t\mathcal{S}}$ to give equivalent characterizations of certain Besov norms for divergence-free vector fields. First, let us prove an easy but useful lemma.

Lemma 3.16. *Let $w(t,\tau)$ be a nonnegative weight function satisfying*

$$\sup_{\tau>0}\int_0^\infty w(t,\tau)\,\frac{dt}{t} + \sup_{t>0}\int_0^\infty w(t,\tau)\,\frac{d\tau}{\tau} \le C.$$

Then for any $q \in [1,\infty]$, we have

$$\left\|\int_0^\infty w(t,\tau)f(\tau)\frac{d\tau}{\tau}\right\|_{L^q(\mathbb{R}_+,\frac{dt}{t})} \le C\|f\|_{L^q(\mathbb{R}_+,\frac{dt}{t})}.$$

Proof. The cases $q=1$ and $q=\infty$ are trivial. Assume that $f \ge 0$. For $q \in (1,\infty)$, we apply Hölder's inequality to see that

$$\int_0^\infty w(t,\tau)f(\tau)\frac{d\tau}{\tau} \le \left(\int_0^\infty w(t,\tau)\frac{d\tau}{\tau}\right)^{1/q'}\left(\int_0^\infty w(t,\tau)f^q(\tau)\frac{d\tau}{\tau}\right)^{1/q}.$$

By the assumption of the lemma, we obtain

$$\left(\int_0^\infty w(t,\tau)f(\tau)\frac{d\tau}{\tau}\right)^q \le C\int_0^\infty w(t,\tau)f^q(\tau)\frac{d\tau}{\tau}.$$

Integrating both sides over $(0,\infty)$ with respect to $\frac{dt}{t}$, the result follows from the assumption again and the Fubini's theorem. $\square$

As in many other works (see, e.g., [11,36]) concerning characterizations of function spaces via semigroups, a fundamental ingredient is to obtain a sort of reproducing formulas associated with the semigroups. The following reproducing formula depends on the very special (and simple) structure of the operator $\mathcal{S}$.

Lemma 3.17. *For any $u_0 \in \mathcal{P}L^2$, it holds that*

$$u_0 = -\int_0^\infty \Delta e^{\tau\mathcal{S}} b\mathcal{P}_b u_0 \, d\tau := \lim_{\varepsilon\to 0^+} -\int_\varepsilon^{1/\varepsilon} \Delta e^{\tau\mathcal{S}} b\mathcal{P}_b u_0 \, d\tau, \tag{3.22}$$

where the limit converges in L^2.

Proof. Since $e^{t\mathcal{S}}$ is an analytic semigroup, the function $u(t) = e^{t\mathcal{S}}u_0$ is an classical solution to the equation $u'(t) = \mathcal{S}u(t)$. Integrating this equation in time from s to t, we get

$$u(t) - u(s) = \int_s^t \mathcal{S}u(\tau)\, d\tau.$$

Obviously, $u(s) \to u_0$ in L^2 as $s \to 0^+$. This together with Proposition 3.15 implies that

$$u_0 = -\int_0^\infty \mathcal{S}e^{\tau\mathcal{S}} u_0 \, d\tau.$$

Replacing u_0 by $b\mathcal{P}_b u_0$ and recalling the expression for $\mathcal{S}$, we have

$$b\mathcal{P}_b u_0 = -\int_0^\infty b\mathcal{P}_b \Delta e^{\tau\mathcal{S}} b\mathcal{P}_b u_0 \, d\tau.$$

Then the desired formula follows from the fact that $b\mathcal{P}_b$ is invertible on $\mathcal{P}L^2$. $\square$

Let us first consider characterizations of Besov norms with negative regularity.

Theorem 3.18. *Suppose that $s \in (0, 2)$ and $q \in [1, \infty]$. For any $u_0 \in \mathcal{P}L^2$, we have*

$$\big\| t^{s/2} \| e^{t\mathcal{S}} b\mathcal{P}_b u_0 \| \big\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \simeq \|u_0\|_{\dot{B}^{-s}_{2,q}}.$$

Proof. Let us first assume that $u_0 \in \dot{B}^{-s}_{2,q}$. We need the reproducing formula (by taking $b \equiv 1$ in (3.22))

$$u_0 = -\int_0^\infty \Delta e^{\tau\Delta} u_0 \, d\tau, \quad u_0 \in \mathcal{P}L^2.$$

Next, applying $e^{t\mathcal{S}} b\mathcal{P}_b$ to both sides of the above equation gives rise to

$$e^{t\mathcal{S}} b\mathcal{P}_b u_0 = -\int_0^\infty e^{t\mathcal{S}} b\mathcal{P}_b \Delta e^{\tau\Delta} u_0 \, d\tau.$$

We may estimate the L^2 norm of the integrand in two different ways: we get from Lemma 3.14 that

$$\|e^{t\mathcal{S}} b\mathcal{P}_b \Delta e^{\tau\Delta} u_0\| = \|\mathcal{S} e^{t\mathcal{S}} e^{\tau\Delta} u_0\| \le \frac{C}{t}\|e^{\tau\Delta} u_0\| \le \frac{C}{t}\|e^{\frac{\tau}{2}\Delta} u_0\|,$$

alternatively,

$$\|e^{t\mathcal{S}} b\mathcal{P}_b \Delta e^{\tau\Delta} u_0\| \le C\|\Delta e^{\tau\Delta} u_0\| \le \frac{C}{\tau}\|e^{\frac{\tau}{2}\Delta} u_0\|.$$

Hence, we get

$$\|e^{t\mathcal{S}} b\mathcal{P}_b u_0\| \le C\int_0^\infty \left(\frac{1}{t} \wedge \frac{1}{\tau}\right) \|e^{\tau\Delta} u_0\| \, d\tau.$$

Multiplying both sides by $t^{s/2}$, we write

$$t^{s/2}\|e^{t\mathcal{S}} b\mathcal{P}_b u_0\| \le C\int_0^\infty \left(\frac{t}{\tau}\right)^{s/2} \left(\frac{\tau}{t} \wedge 1\right) \|\tau^{s/2} e^{\tau\Delta} u_0\| \, \frac{d\tau}{\tau}.$$

Now if $s \in (0,2)$, it is easy to verify that

$$\sup_{t>0}\int_0^\infty \left(\frac{t}{\tau}\right)^{s/2} \left(\frac{\tau}{t} \wedge 1\right) \frac{d\tau}{\tau} + \sup_{\tau>0}\int_0^\infty \left(\frac{t}{\tau}\right)^{s/2} \left(\frac{\tau}{t} \wedge 1\right) \frac{dt}{t} \le C.$$

It follows form Lemma 3.16 and Lemma 2.13 that

$$\left\| t^{s/2} \| e^{t\mathcal{S}} b\mathcal{P}_b u_0 \| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \le C \left\| t^{s/2} \| e^{t\Delta} u_0 \| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \le C \|u_0\|_{\dot{B}^{-s}_{2,q}}.$$

For the reverse inequality, we apply $e^{t\Delta}$ to both sides of (3.22) to get

$$e^{t\Delta} u_0 = -\int_0^\infty e^{t\Delta} \Delta e^{\tau\mathcal{S}} b\mathcal{P}_b u_0 \, d\tau.$$

Again, in view of Lemma 3.14, a similar argument as before gives rise to

$$\|e^{t\Delta} u_0\| \le C \int_0^\infty \left(\frac{1}{t} \wedge \frac{1}{\tau} \right) \|e^{\tau\mathcal{S}} b\mathcal{P}_b u_0\| \, d\tau.$$

The rest of the steps are exactly the same as before. So the proof of the theorem is completed.

□

Now Theorem 3.10 is an immediate consequence of Theorem 3.18.

Proof of Theorem 3.10. Since $u_0 \in \mathcal{P}H^2$, the result immediately follows from Theorem 3.18 if we replace u_0 by $\Delta u_0 \in \mathcal{P}L^2$. □

With Theorem 3.10 in hand, we can extrapolate $e^{t\mathcal{S}}$ to a semigroup on Besov spaces without assuming any regularity on the coefficient b. To this end, let us first study some regularity estimates of $e^{t\mathcal{S}}$ on $\mathcal{P}\dot{B}^s_{2,q}$.

Proposition 3.19. *Suppose that $s \in (0,2)$, $q \in [1,\infty]$ and $k \in \mathbb{N} \cup \{0\}$. There exists a positive constant C such that for any $u_0 \in \mathcal{P}H^2$,*

$$\|(t\mathcal{S})^k e^{t\mathcal{S}} u_0\|_{\dot{B}^s_{2,q}} \le C \|u_0\|_{\dot{B}^s_{2,q}}, \quad \forall t \ge 0, \tag{3.23}$$

and

$$\left\| \|(t\mathcal{S})^{k+1} e^{t\mathcal{S}} u_0\|_{\dot{B}^s_{2,q}} \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \le C \|u_0\|_{\dot{B}^s_{2,q}}. \tag{3.24}$$

Proof. The first inequality follows immediately from Theorem 3.10 and the fact that $e^{t\mathcal{S}}$ is a bounded analytic semigroup on $\mathcal{P}L^2$.

For the second inequality, we only need to prove for $k = 0$ and $q < \infty$. Applying Theorem 3.10, Lemma 3.14 and Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty \|\tau\mathcal{S}e^{\tau\mathcal{S}}u_0\|_{\dot{B}^s_{2,q}}^q \frac{d\tau}{\tau} \simeq& \int_0^\infty \int_0^\infty \left(t^{-s/2}\|t\tau\mathcal{S}^2 e^{(t+\tau)\mathcal{S}}u_0\|\right)^q \frac{dt}{t}\frac{d\tau}{\tau} \\
=& \int_0^\infty \int_\tau^\infty (t-\tau)^{(1-s/2)q-1}\tau^{q-1}\|\mathcal{S}^2 e^{t\mathcal{S}}u_0\|^q \, dt\, d\tau \\
=& \int_0^\infty \|\mathcal{S}^2 e^{t\mathcal{S}}u_0\|^q \, dt \int_0^t (t-\tau)^{(1-s/2)q-1}\tau^{q-1}\, d\tau \\
\leq& C\int_0^\infty \left(t^{-s/2}\|t\mathcal{S}e^{t\mathcal{S}}u_0\|\right)^q \frac{dt}{t} \leq C\|u_0\|_{\dot{B}^s_{2,q}}^q.
\end{aligned}$$

This completes the proof. □

In the rest of this subsection, we assume that s and q satisfy

$$(s,q) \in (0,3/2)\times[1,\infty), \ \text{ or } s \in (0,3/2] \ \text{ and } \ q = 1. \tag{3.25}$$

Then $\mathcal{P}\dot{B}^s_{2,q}$ is a Banach space and $\mathcal{P}H^2$ is dense in $\mathcal{P}\dot{B}^s_{2,q}$. For each $t \geq 0$, the inequality (3.23) (with $k = 0$) guarantees that $e^{t\mathcal{S}}$ extends to a bounded operator on $\mathcal{P}\dot{B}^s_{2,q}$ with bounds uniform in t. We denote this extension by $\mathcal{T}(t) = \mathcal{T}_{s,q}(t)$. Then $\{\mathcal{T}(t)\}_{t\geq 0}$ is a bounded semigroup on $\mathcal{P}\dot{B}^s_{2,q}$. Also, it is easy to verify the strong continuity for $\mathcal{T}(t)$.

Proposition 3.20. *Suppose that (s,q) satisfies (3.25). Then $\mathcal{T}(t)$ is a bounded C_0 semigroup on $\mathcal{P}\dot{B}^s_{2,q}$.*

Proof. For $u_0 \in \mathcal{P}H^2$, the function $t \mapsto \mathcal{T}(t)u_0 = e^{t\mathcal{S}}u_0$ belongs to $C([0,\infty);\mathcal{P}H^2)$, thus $C([0,\infty);\mathcal{P}\dot{B}^s_{2,q})$. By a density argument, we get the strong continuity of $\mathcal{T}(t)$ on $\mathcal{P}\dot{B}^s_{2,q}$. □

Let us denote by $\mathcal{G} = \mathcal{G}_{s,q}$ the generator of $\mathcal{T}(t)$ on $\mathcal{P}\dot{B}^s_{2,q}$. In general, it is not easy to identify the domain of the generator of a semigroup. However, it would be easier to find a core for the generator.

Lemma 3.21. *Suppose that* (s,q) *satisfies* (3.25). *Define* $\mathscr{C} = \{u \in \mathcal{P}H^2 | \mathcal{S}u \in \mathcal{P}\dot{B}^s_{2,q}\}$. *Then* $\mathscr{C}$ *is a core for* $\mathcal{G}$, *and it holds that* $\mathcal{G}|_{\mathscr{C}} = \mathcal{S}|_{\mathscr{C}}$. *In other words,* $\mathcal{G}$ *is the closure of* $\mathcal{S} : \mathscr{C} \subset \mathcal{P}\dot{B}^s_{2,q} \to \mathcal{P}\dot{B}^s_{2,q}$.

Proof. Note that the domain $D(\mathcal{S}^2)$ of $\mathcal{S}^2$ is contained in $\mathscr{C}$. Since $D(\mathcal{S}^2)$ is dense in $D(\mathcal{S}) = \mathcal{P}H^2$ and $\mathcal{P}H^2$ is dense in $\mathcal{P}\dot{B}^s_{2,q}$, so $\mathscr{C}$ is also dense in $\mathcal{P}\dot{B}^s_{2,q}$. For every $u_0 \in \mathscr{C}$, since $\mathcal{T}(t)u_0 = e^{t\mathcal{S}}u_0$ is a classical solution to the equation $u'(t) = \mathcal{S}u(t)$, we have

$$\mathcal{T}(t)u_0 - u_0 = \int_0^t e^{\tau\mathcal{S}}\mathcal{S}u_0\, d\tau = \int_0^t \mathcal{T}(\tau)\mathcal{S}u_0\, d\tau.$$

So, we have

$$\frac{1}{t}(\mathcal{T}(t)u_0 - u_0) = \frac{1}{t}\int_0^t \mathcal{T}(\tau)\mathcal{S}u_0\, d\tau.$$

By strong continuity of $\mathcal{T}(t)$ on $\mathcal{P}\dot{B}^s_{2,q}$, the limit as $t \to 0^+$ on the right exists and equals to $\mathcal{S}u_0$. We thus infer that $\mathscr{C} \subset D(\mathcal{G})$ and $\mathcal{G}|_{\mathscr{C}} = \mathcal{S}|_{\mathscr{C}}$. Obviously, $\mathscr{C}$ is invariant under $\mathcal{T}(t)$. Thus, by Lemma 2.5, $\mathscr{C}$ is a core for $\mathcal{G}$. This completes the proof. □

Proposition 3.22. *Suppose that* (s,q) *satisfies* (3.25). *Then* $\mathcal{T}(t)$ *is a bounded analytic semigroup on* $\mathcal{P}\dot{B}^s_{2,q}$.

Proof. We know from the above lemma that $\mathscr{C}$ is dense in $\mathcal{P}\dot{B}^s_{2,q}$, and that $\mathcal{G}\mathcal{T}(t)u_0 = \mathcal{S}e^{t\mathcal{S}}u_0$ for $u_0 \in \mathscr{C}$. It then follows from (3.23) that $\|t\mathcal{G}\mathcal{T}(t)u_0\|_{\dot{B}^s_{2,q}} \le C\|u_0\|_{\dot{B}^s_{2,q}}$. This completes the proof. □

Remark 3.23. Now (3.24) actually holds for data in $\mathcal{P}\dot{B}^s_{2,q}$, that is,

$$\Big\| \|(t\mathcal{G})e^{t\mathcal{G}}u_0\|_{\dot{B}^s_{2,q}} \Big\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \le C\|u_0\|_{\dot{B}^s_{2,q}}, \ \ \forall u_0 \in \mathcal{P}\dot{B}^s_{2,q}.$$

In particular, choosing $q = 1$ gives

$$\|\mathcal{G}e^{t\mathcal{G}}u_0\|_{L^1(\mathbb{R}_+;\dot{B}^s_{2,1})} \le C\|u_0\|_{\dot{B}^s_{2,1}}, \ \ \forall u_0 \in \mathcal{P}\dot{B}^s_{2,1}. \tag{3.26}$$

We conclude this subsection with a maximal L^1 regularity result for the following abstract Cauchy problem

$$u'(t) - \mathcal{G}u(t) = f(t), \;\; u(0) = u_0. \tag{3.27}$$

Theorem 3.24. *Suppose that* $s \in (0, 3/2]$ *and* $T \in (0, \infty]$. *Let* $u_0 \in \mathcal{P}\dot{B}^s_{2,1}$ *and* $f \in L^1((0,T); \mathcal{P}\dot{B}^s_{2,1})$. *Then* (3.27) *has a unique strong solution* $u \in C_b([0,T); \mathcal{P}\dot{B}^s_{2,1})$. *Moreover, there exists a positive constant* $C = C(m, s)$ *such that*

$$\|u\|_{L^\infty_T(\dot{B}^s_{2,1})} + \|u', \mathcal{G}u\|_{L^1_T(\dot{B}^s_{2,1})} \leq C\|u_0\|_{\dot{B}^s_{2,1}} + C\|f\|_{L^1_T(\dot{B}^s_{2,1})}.$$

Proof. The homogeneous part $e^{t\mathcal{G}}u_0$ is a classical solution, and satisfies the estimates by Proposition 3.22 and (3.26). Denote the inhomogeneous part by $\mathcal{I}f(t) = \int_0^t e^{(t-\tau)\mathcal{G}} f(\tau)\, d\tau$. Since $e^{t\mathcal{G}}$ is uniformly bounded, we have $\|\mathcal{I}f\|_{L^\infty_T(\dot{B}^s_{2,1})} \leq C\|f\|_{L^1_T(\dot{B}^s_{2,1})}$. Using again (3.26) and Fubini's theorem, we have

$$\begin{aligned} \|\mathcal{G}\mathcal{I}f\|_{L^1_T(\dot{B}^s_{2,1})} &\leq \int_0^T \int_0^t \|\mathcal{G}e^{(t-\tau)\mathcal{G}} f(\tau)\|_{\dot{B}^s_{2,1}}\, d\tau\, dt \\ &= \int_0^T d\tau \int_\tau^T \|\mathcal{G}e^{(t-\tau)\mathcal{G}} f(\tau)\|_{\dot{B}^s_{2,1}}\, dt \leq C\|f\|_{L^1_T(\dot{B}^s_{2,1})}. \end{aligned}$$

So by Lemma 2.6, u is a strong solution to (3.27). The estimate for u' follows directly from the previous estimates and the equation (3.27). So the proof is completed. □

3.2.3 Elliptic estimates

So far we have not assumed any regularity on the coefficient b. In what follows, we shall prove that $b\mathcal{P}_b$ is bounded on some Besov spaces if b has suitable "critical" regularity. We allow a slight discontinuity for b and point out that it is of independent interest to study elliptic estimates with discontinuous coefficients. The continuity of $b\mathcal{P}_b$ will also help us identify the domain of $\mathcal{G}$.

In the sequel, $P \in \dot{H}^1(\mathbb{R}^3)$ is the weak solution to (3.19) with b satisfying (1.4), and μ is the Hölder index in Lemma 2.7. The main result in this subsection is the following:

Theorem 3.25. *Given any $(p,r) \in [2, \frac{3}{1-\mu}) \times [1,\infty]$, any $s \in (0, \frac{3}{p}+\mu-1)$. If b satisfies $\nabla b \in \dot{B}^{3/q-1}_{q,\infty}(\mathbb{R}^3)$ with $\frac{3}{q} > s \vee (1-\mu)$, there exists a constant C depending on m and $\|b\|_{\dot{B}^{3/q}_{q,\infty}}$ such that*

$$\|\nabla P\|_{\dot{B}^s_{p,r}} \le C\|f\|_{\dot{B}^s_{p,r}}$$

for all vectors f whose components belong to $L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,r}(\mathbb{R}^3)$.

The proof of Theorem 3.25 takes two steps. In the first step, we do not assume any regularity on the elliptic coefficients and allow a loss of regularity for ∇P.

Lemma 3.26. *Given any $(p,r) \in [2, \frac{3}{1-\mu}) \times [1,\infty]$, any $s \in (0, \frac{3}{p}+\mu-1)$. For any $-s_0 \in (1-\mu, \frac{3}{p}-s]$, let p_0 be defined by $s_0 - \frac{3}{p_0} = s - \frac{3}{p}$. Then there exists a constant $C>0$ such that*

$$\|\nabla P\|_{\dot{B}^{s_0}_{p_0,r}} \le C\|f\|_{\dot{B}^s_{p,r}}, \quad \forall f \in L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,r}(\mathbb{R}^3).$$

Proof. Note that the assumptions on (p,r,s) guarantee the existence of (p_0,s_0). By Lemma 2.12 (i), we may assume that $s_0 > -1$ and $p_0 < \infty$. First, by Lemma 2.15 and Lemma 2.16, we have

$$\begin{aligned}
\|\nabla P\|_{\dot{B}^{s_0}_{p_0,r}} \simeq& \|\mathcal{E}^{-1/2}\mathcal{R}^* f\|_{\dot{B}^{s_0+1}_{p_0,r}} \\
\simeq& \left\| t^{-\frac{1}{2}-\frac{s_0}{2}} \|(t\mathcal{E})^{\frac{1}{2}} e^{-t\mathcal{E}} \mathcal{E}^{-1/2}\mathcal{R}^* f\|_{p_0} \right\|_{L^r(\mathbb{R}_+, \frac{dt}{t})} \\
=& \left\| t^{-\frac{s_0}{2}} \|e^{-t\mathcal{E}} \mathcal{R}^* f\|_{p_0} \right\|_{L^r(\mathbb{R}_+, \frac{dt}{t})}.
\end{aligned}$$

Next, applying $e^{-t\mathcal{E}}\mathcal{R}^*$ to the reproducing formula

$$f(x) = \frac{1}{(k-1)!}\int_0^\infty (-\tau\Delta)^k e^{\tau\Delta} f(x)\,\frac{d\tau}{\tau} \ \text{ with } \ k > \frac{s}{2},$$

we write

$$e^{-t\mathcal{E}}\mathcal{R}^* f(x) = \frac{1}{(k-1)!}\int_0^\infty e^{-t\mathcal{E}}\mathcal{R}^* e^{\frac{\tau}{2}\Delta}(-\tau\Delta)^k e^{\frac{\tau}{2}\Delta} f(x)\,\frac{d\tau}{\tau}.$$

The $L^p - L^q$ estimates for heat semigroups and the boundedness of $\mathcal{R}^*$ from $L^p(\mathbb{R}^3,\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ $(2 \le p < \infty)$ then imply that

$$\|e^{-t\mathcal{E}}\mathcal{R}^* f\|_{p_0} \le C\int_0^\infty \left(\frac{1}{t}\wedge\frac{1}{\tau}\right)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{p_0})} \|(\tau\Delta)^k e^{\tau\Delta} f\|_p\,\frac{d\tau}{\tau}.$$

Thus,

$$t^{-\frac{s_0}{2}}\|e^{-t\mathcal{E}}\mathcal{R}^* f\|_{p_0} \le C\int_0^\infty \left(\frac{\tau}{t}\right)^{s_0/2}\left(\frac{\tau}{t}\wedge 1\right)^{\frac{s-s_0}{2}} \tau^{-s/2}\|(\tau\Delta)^k e^{\tau\Delta} f\|_p\,\frac{d\tau}{\tau}.$$

Note that s and $-s_0$ are positive numbers. One can readily check that

$$\sup_{\tau>0}\int_0^\infty \left(\frac{\tau}{t}\right)^{s_0/2}\left(\frac{\tau}{t}\wedge 1\right)^{\frac{s-s_0}{2}}\frac{dt}{t} + \sup_{t>0}\int_0^\infty \left(\frac{\tau}{t}\right)^{s_0/2}\left(\frac{\tau}{t}\wedge 1\right)^{\frac{s-s_0}{2}}\frac{d\tau}{\tau} \le C.$$

Finally, we apply Lemma 3.16 and Lemma 2.13 to conclude the proof. □

In the second step, if the coefficient b has suitable regularity, the loss of regularity of ∇P can be recovered using an iteration technique in the spirit of De Giorgi-Nash-Moser. To this end, we need the following commutator estimates that can help us gain regularity.

Lemma 3.27. *Suppose that* $r \in [1,\infty]$, $1 \le p_2 < p_1 \le \infty$, $q \in [1,\infty)$, $(s_1,s_2) \in \mathbb{R}^2$, $s_1 - \frac{3}{p_1} = s_2 - \frac{3}{p_2}$, *and*

$$\frac{1}{p_2} \le \frac{1}{p_1} + \frac{1}{q},\ \frac{1}{p_2} < \frac{1}{p_1} + \frac{1}{3},\ \frac{3}{q\vee p_2} > s_2,\ \ \frac{3}{q\vee p_1'} > -s_1.$$

If $a \in L^\infty(\mathbb{R}^3)$ *and* $\nabla a \in \dot{B}^{3/q-1}_{q,\infty}(\mathbb{R}^3)$, *then there exists a constant* C *such that*

$$\left\|\left(2^{js_2}\|[\dot{\Delta}_j, a]f\|_{p_2}\right)_j\right\|_{l^r_j} \le C\|a\|_{\dot{B}^{3/q}_{q,\infty}}\|f\|_{\dot{B}^{s_1}_{p_1,r}},$$

where $[\dot{\Delta}_j, a]f$ *denotes the commutator* $\dot{\Delta}_j(af) - a\dot{\Delta}_j f$.

Proof. This type of estimate is nowadays classical (see, e.g., [9, Section 2.10]). We give proof here for the sake of completeness.

By Bony's paraproduct, we split the commutator $[\dot{\Delta}_j, a]f$ into four terms:

$$[\dot{\Delta}_j, \dot{T}_a]f + \dot{\Delta}_j(\dot{T}_f a) + \dot{\Delta}_j(\dot{R}(a, f)) - \dot{T}'_{\dot{\Delta}_j f} a. \tag{3.28}$$

Many terms in the summation can be canceled out because of the frequency localization of the dyadic blocks. Specifically, the first term can be expressed as

$$[\dot{\Delta}_j, \dot{T}_a]f = \sum_{|j'-j|\le 4} 2^{nj} \int h(2^j y)\big(\dot{S}_{j'-1}a(x-y) - \dot{S}_{j'-1}a(x)\big)\dot{\Delta}_{j'} f(x-y)\, dy$$

Choosing p such that $\frac{1}{p_2} = \frac{1}{p} + \frac{1}{p_1}$, we use Hölder's inequality to see

$$\begin{aligned}\|[\dot{\Delta}_j, \dot{T}_a]f\|_{p_2} \lesssim & \sum_{|j'-j|\le 4} 2^{nj} \int |h(2^j y)| \|\dot{S}_{j'-1}a(\cdot - y) - \dot{S}_{j'-1}a(\cdot)\|_p \|\dot{\Delta}_{j'} f\|_{p_1}\, dy \\ \lesssim & 2^{-j} \sum_{|j'-j|\le 4} \|\nabla \dot{S}_{j'-1}a\|_p \|\dot{\Delta}_{j'} f\|_{p_1}.\end{aligned}$$

Noticing that $p \ge q$ and $1 - \frac{3}{p} > 0$, we use Lemma 2.9 to get

$$\begin{aligned}\|\nabla \dot{S}_{j'-1}a\|_p \lesssim & \sum_{k\le j'-2} 2^{3k(1/q-1/p)} \|\dot{\Delta}_k \nabla a\|_q \\ \lesssim & \sum_{k\le j'-2} 2^{k(1-3/p)} \|\nabla a\|_{\dot{B}^{3/q-1}_{q,\infty}} \lesssim 2^{j'(1-3/p)} \|a\|_{\dot{B}^{3/q}_{q,\infty}}.\end{aligned}$$

Consequently,

$$\|[\dot{\Delta}_j, \dot{T}_a]f\|_{p_2} \lesssim c_{j,r} 2^{-js_2} \|a\|_{\dot{B}^{3/q}_{q,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r}}. \tag{3.29}$$

For the second term in (3.28), if $q \geq p_2$, we redefine p by $\frac{1}{p_2} = \frac{1}{p} + \frac{1}{q}$. Again, applying Hölder's inequality and Lemma 2.9, we get

$$\|\dot{\Delta}_j(\dot{T}_f a)\|_{p_2} \lesssim \sum_{|j'-j|\leq 4} \|\dot{\Delta}_{j'} a\|_q \|\dot{S}_{j'-1} f\|_p \lesssim c_{j,r} 2^{-js_2} \|a\|_{\dot{B}^{3/q}_{q,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r}}, \tag{3.30}$$

where in the second inequality we need the fact that $p \geq p_1$ and $\frac{3}{q} > s_2$. In the case $q \leq p_2$, thanks to Lemma 2.12 (i), the same result stays true whenever $\frac{3}{p_2} > s_2$.

For the third term in (3.28), we first assume $\frac{1}{p_1} + \frac{1}{q} \leq 1$ and redefine p by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q}$. Using Lemma 2.9 and Hölder's inequality, and noticing that $p \leq p_2$ and $\frac{3}{q} + s_1 > 0$, we obtain

$$\|\dot{\Delta}_j(\dot{R}(a,f))\|_{p_2} \lesssim 2^{3j(\frac{1}{p} - \frac{1}{p_2})} \sum_{j' \geq j-3} \|\dot{\Delta}_{j'} a\|_q \|\widetilde{\dot{\Delta}}_{j'} f\|_{p_1} \lesssim c_{j,r} 2^{-js_2} \|a\|_{\dot{B}^{3/q}_{q,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r}}. \tag{3.31}$$

If $\frac{1}{p_1} + \frac{1}{q} \geq 1$, the result still holds provided that $\frac{3}{p_1'} + s_1 > 0$.

For the last term in (3.28), redefining p by $\frac{1}{p_2} = \frac{1}{q} + \frac{1}{p}$ if $q \geq p_2$, we see that

$$\|\dot{T}'_{\dot{\Delta}_j f} a\|_{p_2} \lesssim \sum_{j' \geq j-2} \|\dot{\Delta}_{j'} a\|_q \|\dot{\Delta}_j f\|_p \lesssim c_{j,r} 2^{-js_2} \|a\|_{\dot{B}^{3/q}_{q,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r}}, \tag{3.32}$$

where in the second inequality we need the condition $q < \infty$. If $p_2 \geq q$, the same result holds under the assumption that $p_2 < \infty$.

Putting (3.29)-(3.32) together finishes the proof. □

Remark 3.28. The technical assumption $a \in L^\infty(\mathbb{R}^3)$ is needed in order for the product af to be well-defined via paraproducts.

With the above commutator estimates at our disposal, we are now able to prove the following elliptic regularity which will be used for iteration.

Lemma 3.29. *Suppose that $r \in [1, \infty]$, $2 \leq p_2 < p_1 \leq \infty$, $q \in [1, \infty)$, $(s_1, s_2) \in \mathbb{R}^2$, $s_1 - \frac{3}{p_1} = s_2 - \frac{3}{p_2}$, and*

$$\frac{1}{p_2} \leq \frac{1}{p_1} + \frac{1}{q}, \ \frac{1}{p_2} < \frac{1}{p_1} + \frac{1}{3}, \ \frac{3}{q \vee p_2} > s_2, \ \ \frac{3}{q \vee p_1'} > -s_1.$$

Let b satisfy $\nabla b \in \dot{B}^{3/q-1}_{q,\infty}(\mathbb{R}^3)$. If in addition, $f \in L^2(\mathbb{R}^3) \cap \dot{B}^{s_2}_{p_2,r}(\mathbb{R}^3)$ and $\nabla P \in \dot{B}^{s_1}_{p_1,r}(\mathbb{R}^3)$, then there exists a constant $C > 0$ such that

$$\|\nabla P\|_{\dot{B}^{s_2}_{p_2,r}} \leq C\|f\|_{\dot{B}^{s_2}_{p_2,r}} + C\|b\|_{\dot{B}^{3/q}_{q,\infty}}\|\nabla P\|_{\dot{B}^{s_1}_{p_1,r}}.$$

Proof. Testing $v = \dot{\Delta}_j(|\dot{\Delta}_j P|^{p_2-2}\dot{\Delta}_j P) \in H^\infty(\mathbb{R}^3)$ in the equation

$$\int_{\mathbb{R}^3} b\nabla P \cdot \nabla v \, dx = \int_{\mathbb{R}^3} f \cdot \nabla v \, dx,$$

we have

$$\int_{\mathbb{R}^3} b|\dot{\Delta}_j \nabla P|^2 |\dot{\Delta}_j P|^{p_2-2} \, dx = \int_{\mathbb{R}^3} (\dot{\Delta}_j f - [\dot{\Delta}_j, b]\nabla P) \cdot \dot{\Delta}_j \nabla P |\dot{\Delta}_j P|^{p_2-2} \, dx.$$

Applying [15, Lemma A.5], Lemma 2.9 and Hölder's inequality, we get

$$\|\dot{\Delta}_j \nabla P\|_{p_2} \lesssim \|\dot{\Delta}_j f\|_{p_2} + \|[\dot{\Delta}_j, b]\nabla P\|_{p_2}.$$

Multiplying both sides by 2^{js_2} and taking l^r norm with respect to j, we obtain

$$\|\nabla P\|_{\dot{B}^{s_2}_{p_2,r}} \lesssim \|f\|_{\dot{B}^{s_2}_{p_2,r}} + \left\| \left(2^{js_2} \|[\dot{\Delta}_j, b]\nabla P\|_{p_2} \right)_j \right\|_{l^r_j}.$$

The desired result then follows from Lemma 3.27. □

We are now in a position to complete the proof of Theorem 3.25.

Proof of Theorem 3.25. We start with choosing s_0 and p_0. Since $(-\frac{3}{q}) \vee (s - \frac{3}{p}) < \mu - 1$, we can choose an s_0 between both sides of this inequality. Define p_0 by $s_0 - \frac{3}{p_0} = s - \frac{3}{p}$, then $p_0 \in (p, \infty)$. By Lemma 3.26, we have $\|\nabla P\|_{\dot{B}^{s_0}_{p_0,r}} \leq C\|f\|_{\dot{B}^{s}_{p,r}}$. Next, we shall choose $(p_1, s_1) \in [p, p_0) \times (s_0, s]$ that satisfies the assumptions in Lemma 3.29. It is not difficult to see that those assumptions can be reduced to $s_1 - \frac{3}{p_1} = s - \frac{3}{p}$ and

$$\frac{1}{p_1} \leq \frac{1}{p_0} + \frac{1}{q}, \ \frac{1}{p_1} < \frac{1}{p_0} + \frac{1}{3}.$$

If $(p_1, s_1) = (p, s)$ satisfies the above assumptions, we are done by using Lemma 3.29. Otherwise, we define p_1 by $\frac{1}{p_1} = \frac{1}{p_0} + \frac{1}{2(q\vee 3)}$, and get

$$\|\nabla P\|_{\dot{B}^{s_1}_{p_1,r}} \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,r}} + \|b\|_{\dot{B}^{3/q}_{q,\infty}} \|\nabla P\|_{\dot{B}^{s_0}_{p_0,r}} \lesssim \|f\|_{\dot{B}^{s}_{p,r}}.$$

The (p_k, s_k) is defined by $\frac{1}{p_k} = \frac{1}{p_0} + \frac{k}{2(q\vee 3)}$ and $s_k - \frac{3}{p_k} = s - \frac{3}{p}$. So the iteration scheme will end in a finite number of steps. This completes the proof. □

Remark 3.30. If we know a priori that the norm $\|\nabla P\|_{\dot{B}^{s}_{p,r}}$ is finite, then the iteration process is not needed. We apply Lemma 3.29 only once to get

$$\|\nabla P\|_{\dot{B}^{s}_{p,r}} \leq C\|f\|_{\dot{B}^{s}_{p,r}} + C\|b\|_{\dot{B}^{3/q}_{q,\infty}} \|\nabla P\|_{\dot{B}^{s_1}_{p_1,r}}$$

with some $s_1 \in (0, s)$. To complete the proof, we use Lemma 2.12 (ii) and Young's inequality. In this way, we can also remove the technical assumption that $q < \frac{3}{1-\mu}$.

Given Theorem 3.25, we are now able to identify $\mathcal{G}$ and its domain $D(\mathcal{G})$. In order to avoid some unpleasant technicalities, we will simply assume that $b - 1 \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$. Then by Lemma 2.12 (v), $\rho - 1 = (1-b)/b$ satisfies the same assumption.

Lemma 3.31. *Let* $0 < s < \frac{1}{2} + \mu$ *and* $1 \leq q < \frac{3}{s\vee(1-\mu)}$. *Assume that* $b - 1 \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$. *Then* $\mathcal{G} = \mathcal{G}_{s,1}$ *coincides with the operator* $\tilde{\mathcal{G}}$ *defined by*

$$\tilde{\mathcal{G}} = b\mathcal{P}_b\Delta : \mathcal{P}\dot{B}^{s}_{2,1} \cap \mathcal{P}\dot{B}^{s+2}_{2,1} \subset \mathcal{P}\dot{B}^{s}_{2,1} \to \mathcal{P}\dot{B}^{s}_{2,1}.$$

Proof. First, we get from Theorem 3.25 that $\nabla\mathcal{E}_b^{-1}\,\mathrm{div}$ extends to a continuous operator on $\dot{B}^{s}_{2,1}$. In view of product laws in Besov spaces and Lemma 3.13 (ii), $b\mathcal{P}_b$ is also continuous on $\dot{B}^{s}_{2,1}$, and the restriction of $b\mathcal{P}_b$ on $\mathcal{P}\dot{B}^{s}_{2,1}$ is invertible with a continuous inverse $\mathcal{P}\rho$. Based on this, it is not difficult to see that $\tilde{\mathcal{G}}$ is a closed operator. Note that the space $\mathscr{C}$ (defined in Lemma 3.21) coincides with the inhomogeneous space $\mathcal{P}B^{s+2}_{2,1}$, so it is dense in $D(\tilde{\mathcal{G}})$. This shows that $\tilde{\mathcal{G}}$ is the closure of $\mathcal{S} : \mathscr{C} \subset \mathcal{P}\dot{B}^{s}_{2,1} \to \mathcal{P}\dot{B}^{s}_{2,1}$. So we have $\mathcal{G} = \tilde{\mathcal{G}}$ as a consequence of Lemma 3.21. □

With a slight abuse of notation, we shall not distinguish between $\mathcal{S}$ and $\mathcal{G}$.

3.2.4 Proof of Theorem 3.12

We give a proof of Theorem 3.12 in this subsection. To further simplify the exposition, we assume that $b-1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$. First, let us go back to the maximal regularity for the abstract Cauchy problem

$$u'(t) - b\mathcal{P}_b\Delta u(t) = b\mathcal{P}_b f(t), \;\; u(0) = u_0. \tag{3.33}$$

As a consequence of Theorem 3.24, Theorem 3.25 and Lemma 3.31, we have:

Corollary 3.32. *Let $T \in (0,\infty]$. Assume that b satisfies* (1.4) *and $b-1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$. Let $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ and $f \in L^1((0,T);\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))$. Then* (3.33) *has a unique strong solution $u \in C_b([0,T);\mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))$. Moreover, there exists a constant C depending on m and $\|b-1\|_{\dot{B}^{3/2}_{2,1}}$ such that*

$$\|u\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})} + \|u',\Delta u\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \le C\|u_0\|_{\dot{B}^{1/2}_{2,1}} + C\|f\|_{L^1_T(\dot{B}^{1/2}_{2,1})}.$$

Let us now give the proof of the well-posedness part of Theorem 3.12.

Proof of the well-posedness part of Theorem 3.12. Note that $b = \rho^{-1}$ satisfies the same assumptions as ρ. By Corollary 3.32, we see that the following Cauchy problem

$$\partial_t v(t) - b\mathcal{P}_b\Delta v(t) = b\mathcal{P}_b(f - \rho\mathcal{Q}\partial_t R), \; v(0) = u_0$$

has a unique strong solution $v \in C_b([0,T);\mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))$ satisfying

$$\|v\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})} + \|\partial_t v,\Delta v\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \lesssim \|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|f,\partial_t R\|_{L^1_T(\dot{B}^{1/2}_{2,1})}. \tag{3.34}$$

Define $u = v + \mathcal{Q}R = v - \nabla(-\Delta)^{-1}g$ and $\nabla P = \mathcal{Q}(f - \rho\partial_t v - \rho\mathcal{Q}\partial_t R) + \nabla g$. One can readily check that $(u,\nabla P)$ is a strong solution to (3.9). □

By (3.34) and the construction of ∇P, we also have

$$\|\nabla P\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \lesssim \|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|f, \partial_t R, \nabla g\|_{L^1_T(\dot{B}^{1/2}_{2,1})}. \tag{3.35}$$

But if we apply (3.34) to bound u directly, we have to include the term $\|\mathcal{Q}R\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})}$ on the right side of (3.20). This would cause serious trouble for us to prove local existence of large solutions to (3.5). Thanks to (3.35), we can view ∇P in the first equation of (3.9) as a source term. So we choose to prove the maximal regularity for the solution u to the parabolic system (3.18). Having had success in establishing maximal L^1 regularity for (3.33) based on Theorem 3.10, we are going to obtain maximal L^1 regularity for the parabolic Cauchy problem

$$\partial_t u - b\Delta u = f, \;\; u(0) = u_0 \tag{3.36}$$

by characterizations of Besov norms via the semigroup $e^{tb\Delta}$. To simplify the exposition, we only prove what is needed for the proof of (3.20).

Lemma 3.33. *Suppose that b satisfies* (1.4)*, and b and b^{-1} belong to the multiplier space $\mathscr{M}(\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))$. Let $u_0 \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ and $f \in L^1((0,T); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3))$. Then* (3.36) *has a unique strong solution u satisfying*

$$\|u\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})} + \|\partial_t u, \Delta u\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \lesssim \|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|f\|_{L^1_T(\dot{B}^{1/2}_{2,1})}.$$

Proof. Since the proof is analogous to that of Theorem 3.24, we only outline the key steps.

First, $b\Delta : H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ generates a bounded analytic semigroup $e^{tb\Delta}$ whose kernel has a Gaussian upper bound (see [31, 46]). So we have $\lim_{t\to\infty} \|e^{tb\Delta} f\| = 0$ for every $f \in L^2(\mathbb{R}^3)$. Based on this, we can mimic the proof of Theorem 3.10 to get the equivalence of norms:

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}} \simeq \left\| t^{-1/4} \|tb\Delta e^{tb\Delta} u_0\| \right\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}, \;\; \forall u_0 \in H^2(\mathbb{R}^3).$$

This would imply that

$$\sup_{t>0}\|e^{tb\Delta}u_0\|_{\dot{B}^{1/2}_{2,1}}+\sup_{t>0}\|tb\Delta e^{tb\Delta}u_0\|_{\dot{B}^{1/2}_{2,1}}+\Big\|\|tb\Delta e^{tb\Delta}u_0\|_{\dot{B}^{1/2}_{2,1}}\Big\|_{L^1(\mathbb{R}_+,\frac{dt}{t})}\le C\|u_0\|_{\dot{B}^{1/2}_{2,1}},$$

for all $u_0\in H^2(\mathbb{R}^3)$. So $e^{tb\Delta}|_{H^2(\mathbb{R}^3)}$ extends to a bounded analytic semigroup $\mathcal{T}(t)$ on $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Denote by $\mathcal{G}$ the generator of $\mathcal{T}(t)$. Then $\mathscr{C}:=\{u_0\in H^2(\mathbb{R}^3)|b\Delta u_0\in\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\}$ is a core for $\mathcal{G}$, and $\mathcal{G}|_{\mathscr{C}}=b\Delta|_{\mathscr{C}}$. So far, all statements hold if b merely satisfies (1.4).

Next, we assume that both b and b^{-1} are multipliers of $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. Then $\mathcal{G}$ coincides with the operator

$$b\Delta:\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\cap\dot{B}^{5/2}_{2,1}(\mathbb{R}^3)\subset\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\to\dot{B}^{1/2}_{2,1}(\mathbb{R}^3).$$

Now that all preparation work is done, we mimic the proof of Theorem 3.24 to finish the proof of the present lemma. The details are left to the reader. □

We conclude this section by completing the proof of Theorem 3.12

Proof of Theorem 3.12. It remains to show (3.20). Applying Lemma 3.33 to (3.18), we have

$$\|u\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})}+\|\partial_t u,\Delta u\|_{L^1_T(\dot{B}^{1/2}_{2,1})}\lesssim\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|f\|_{L^1_T(\dot{B}^{1/2}_{2,1})}+\|\nabla P\|_{L^1_T(\dot{B}^{1/2}_{2,1})}.$$

This together with (3.35) gives (3.20). Thus, the proof of Theorem 3.12 is completed. □

3.3 Well-posedness of (1.2)

In this section, we prove the well-posedness of (3.5), then the well-posedness of (1.2) will follow. Let $E(T)$ denote the space of all pairs $(\mathbf{u},\nabla\mathbf{P})$ satisfying

$$\mathbf{u}\in C([0,T];\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)),\ (\partial_t\mathbf{u},\Delta\mathbf{u},\nabla\mathbf{P})\in\Big(L^1((0,T);\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))\Big)^3,$$

endowed with the norm

$$\|(\mathbf{u}, \nabla \mathbf{P})\|_{E(T)} = \|\mathbf{u}\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})} + \|\Delta \mathbf{u}, \partial_t \mathbf{u}, \nabla \mathbf{P}\|_{L^1_T(\dot{B}^{1/2}_{2,1})}.$$

Given Theorem 3.12, we can prove the well-posedness of (3.5) in $E(T)$ by using the contraction mapping theorem.

Theorem 3.34. *Assume that the initial density* ρ_0 *satisfies* (1.4) *and* $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$*, and the initial velocity* $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$*. Then there exists some* $T > 0$*, such that the system* (3.5) *has a unique strong solution* $(\mathbf{u}, \nabla \mathbf{P}) \in E(T)$*.*

Proof. We shall construct a contraction mapping on $E(T)$ by solving the linearized system (3.7). Let us denote the inhomogeneous terms in (3.7) by

$$\begin{cases} f(\mathbf{v}, \nabla \mathbf{Q}) = \operatorname{div}((\mathscr{A}_{\mathbf{v}}\mathscr{A}^T_{\mathbf{v}} - I)\nabla \mathbf{v}) + (I - \mathscr{A}^T_{\mathbf{v}})\nabla \mathbf{Q}, \\ R(\mathbf{v}) = (I - \mathscr{A}_{\mathbf{v}})\mathbf{v}, \\ g(\mathbf{v}) = \operatorname{Tr}((I - \mathscr{A}_{\mathbf{v}})D\mathbf{v}), \end{cases}$$

where $(\mathbf{v}, \nabla \mathbf{Q}) \in E(T)$. However, since no smallness is assumed on the initial data, one has to perform the contraction mapping theorem around a neighborhood of some reference, which here is chosen as the solution $(\mathbf{u}_L, \nabla \mathbf{P}_L)$ to the homogeneous linear system

$$\begin{cases} \rho_0 \partial_t \mathbf{u}_L - \Delta \mathbf{u}_L + \nabla \mathbf{P}_L = 0, \\ \operatorname{div} \mathbf{u}_L = 0, \\ \mathbf{u}_L(0, \cdot) = u_0. \end{cases} \tag{3.37}$$

In view of Theorem 3.12, we immediately have

$$\|(\mathbf{u}_L, \nabla \mathbf{P}_L)\|_{E(T)} \le C\|u_0\|_{\dot{B}^{1/2}_{2,1}}. \tag{3.38}$$

Hence, we see that

$$M(t) := \|\Delta \mathbf{u}_L, \partial_t \mathbf{u}_L, \nabla \mathbf{P}_L\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \to 0$$

as t tends to 0. We will solve (3.7) in a closed ball in $E(T)$ centered at $(\mathbf{u}_L, \nabla \mathbf{P}_L)$ with radius r, that is,

$$B_r(\mathbf{u}_L, \nabla \mathbf{P}_L) = \{(\mathbf{u}, \nabla \mathbf{P}) \in E(T) : \|(\bar{\mathbf{u}}, \nabla \bar{\mathbf{P}})\|_{E(T)} \leq r\},$$

where $(\bar{\mathbf{u}}, \bar{\mathbf{P}}) = (\mathbf{u} - \mathbf{u}_L, \mathbf{P} - \mathbf{P}_L)$. The numbers r and T will be chosen suitably small later.

Let us estimate the inhomogeneous terms first. For any $(\mathbf{v}, \nabla \mathbf{Q}) \in B_r(\mathbf{u}_L, \nabla \mathbf{P}_L)$, we denote $(\bar{\mathbf{v}}, \bar{\mathbf{Q}}) = (\mathbf{v} - \mathbf{u}_L, \mathbf{Q} - \mathbf{P}_L)$. Obviously, we have

$$\|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq \|\nabla \bar{\mathbf{v}}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} + \|\nabla \mathbf{u}_L\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq r + M(T).$$

So Lemma 2.24 and Lemma 2.25 are effective if we require

$$r + M(T) \leq c_0.$$

Then applying (2.25) and product laws in Besov spaces, we see that

$$\|f(\mathbf{v}, \nabla \mathbf{Q}), \nabla g(\mathbf{v})\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \lesssim \|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \|\Delta \mathbf{v}, \nabla \mathbf{Q}\|_{L^1_T(\dot{B}^{1/2}_{2,1})}. \tag{3.39}$$

While applying (2.25) and (2.30), we get

$$\begin{aligned} \|\partial_t R(\mathbf{v})\|_{L^1_T(\dot{B}^{1/2}_{2,1})} \lesssim & \int_0^T \|\partial_t \mathscr{A}_{\mathbf{v}}\|_{\dot{B}^{1/2}_{2,1}} \|\mathbf{v}\|_{\dot{B}^{3/2}_{2,1}} + \|I - \mathscr{A}_{\mathbf{v}}\|_{\dot{B}^{3/2}_{2,1}} \|\partial_t \mathbf{v}\|_{\dot{B}^{1/2}_{2,1}} \, dt \\ \lesssim & \|\mathbf{v}\|^2_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla \mathbf{v}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \|\partial_t \mathbf{v}\|_{L^1_T(\dot{B}^{1/2}_{2,1})}. \end{aligned} \tag{3.40}$$

So, by Theorem 3.12, the system (3.7) has a unique solution $(\mathbf{u}, \nabla \mathbf{P}) \in E(T)$. Subtracting (3.37) from (3.7), and then applying again Theorem 3.12 to the resulting system for $(\bar{\mathbf{u}}, \nabla \bar{\mathbf{P}})$, we obtain

$$\|(\bar{\mathbf{u}}, \nabla\bar{\mathbf{P}})\|_{E(T)} \lesssim \|f(\mathbf{v}, \nabla\mathbf{Q}), \partial_t R(\mathbf{v}), \nabla g(\mathbf{v})\|_{L^1_T(\dot{B}^{1/2}_{2,1})}.$$

Plugging (3.39) and (3.40) in the above estimate, and using Lemma 2.12 (ii) and (3.38), we arrive at

$$\begin{aligned}\|(\bar{\mathbf{u}}, \nabla\bar{\mathbf{P}})\|_{E(T)} \lesssim &\|\mathbf{v}\|^2_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla\mathbf{v}\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\|\Delta\mathbf{v}, \partial_t\mathbf{v}, \nabla\mathbf{Q}\|_{L^1_T(\dot{B}^{1/2}_{2,1})}\\ \le &C_0\|u_0\|_{\dot{B}^{1/2}_{2,1}} M(T) + C_0(r + M(T))^2,\end{aligned}$$

where C_0 depends on m and $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$. Now choosing T and r small enough so that

$$r \le \frac{1}{8C_0} \wedge \frac{c_0}{2} \ \text{ and } \ M(T) \le r \wedge \frac{r}{2C_0\|u_0\|_{\dot{B}^{1/2}_{2,1}}}, \tag{3.41}$$

we have $\|(\bar{\mathbf{u}}, \nabla\bar{\mathbf{P}})\|_{E(T)} \le r$. So the solution mapping $\mathcal{N}$ that assigns $(\mathbf{v}, \nabla\mathbf{Q})$ to $(\mathbf{u}, \nabla\mathbf{P})$ is a self-mapping on $B_r(\mathbf{u}_L, \nabla\mathbf{P}_L)$.

It remains to show that $\mathcal{N}$ is contractive. Let $(\mathbf{v}_i, \nabla\mathbf{Q}_i) \in B_r(\mathbf{u}_L, \nabla\mathbf{P}_L)$ and $(\mathbf{u}_i, \nabla\mathbf{P}_i) = \mathcal{N}(\mathbf{v}_i, \nabla\mathbf{Q}_i)$, $i = 1, 2$. In what follows, for two quantities q_1 and q_2, δq always denotes their difference $q_1 - q_2$. Then the system for $(\delta\mathbf{u}, \nabla\delta\mathbf{P})$ reads

$$\begin{cases}\rho_0\partial_t\delta\mathbf{u} - \Delta\delta\mathbf{u} + \nabla\delta\mathbf{P} = \delta f,\\ \operatorname{div}\delta\mathbf{u} = \operatorname{div}\delta R = \delta g,\\ \delta\mathbf{u}|_{t=0} = 0,\end{cases} \tag{3.42}$$

where $f_i = f(\mathbf{v}_i, \nabla\mathbf{Q}_i)$, $g_i = g(\mathbf{v}_i)$, and $R_i = R(\mathbf{v}_i)$ with $\mathscr{A}_i = \mathscr{A}_{\mathbf{v}_i}$.

We write $\delta f = (\delta f)_1 + (\delta f)_2$, where

$$\begin{aligned}(\delta f)_1 =& \operatorname{div}((\mathscr{A}_{\mathbf{v}_1}\mathscr{A}^T_{\mathbf{v}_1} - I)\nabla\delta\mathbf{v}) + (I - \mathscr{A}^T_{\mathbf{v}_1})\nabla\delta\mathbf{Q},\\ (\delta f)_2 =& -(\delta\mathscr{A})^T\nabla\mathbf{Q}_2 + \operatorname{div}\left[(\mathscr{A}_{\mathbf{v}_1}\mathscr{A}^T_{\mathbf{v}_1} - \mathscr{A}_{\mathbf{v}_2}\mathscr{A}^T_{\mathbf{v}_2})\nabla\mathbf{v}_2\right].\end{aligned}$$

Along the lines of deriving (3.39), we have

$$\|(\delta f)_1\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim \|\nabla \mathbf{v}_1\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\Delta\delta\mathbf{v}, \nabla\delta\mathbf{Q}\|_{L^1_T(\dot B^{1/2}_{2,1})}.$$

Applying (2.25), (2.27) and product laws in Besov spaces, we obtain

$$\|(\delta f)_2\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim \|\nabla \delta\mathbf{v}\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\Delta\mathbf{v}_2, \nabla\mathbf{Q}_2\|_{L^1_T(\dot B^{1/2}_{2,1})}.$$

Summing up the estimates, we have

$$\|\delta f\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim (M(T)+r)\|(\delta\mathbf{v}, \nabla\delta\mathbf{Q})\|_{E(T)}. \tag{3.43}$$

Note that $\partial_t\delta R = -\partial_t\mathscr{A}_{\mathbf{v}_1}\delta\mathbf{v} + (I-\mathscr{A}_{\mathbf{v}_1})\partial_t\delta\mathbf{v} - \partial_t(\delta\mathscr{A})\mathbf{v}_2 - \delta\mathscr{A}\,\partial_t\mathbf{v}_2$. Again, applying (2.25), (2.27), (2.29) and (2.32) gives

$$\begin{aligned}
\|\partial_t\mathscr{A}_{\mathbf{v}_1}\delta\mathbf{v}\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim&\|\nabla\mathbf{v}_1\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\delta\mathbf{v}\|_{L^\infty_T(\dot B^{1/2}_{2,1})},\\
\|(I-\mathscr{A}_{\mathbf{v}_1})\partial_t\delta\mathbf{v}\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim&\|\nabla\mathbf{v}_1\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\partial_t\delta\mathbf{v}\|_{L^1_T(\dot B^{1/2}_{2,1})},\\
\|\partial_t(\delta\mathscr{A})\mathbf{v}_2\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim&\|\delta\mathbf{v}\|_{L^2_T(\dot B^{3/2}_{p,1})}\|\mathbf{v}_2\|_{L^2_T(\dot B^{3/2}_{2,1})},\\
\|\delta\mathscr{A}\,\partial_t\mathbf{v}_2\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim&\|\nabla\delta\mathbf{v}\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\partial_t\mathbf{v}_2\|_{L^1_T(\dot B^{1/2}_{2,1})}.
\end{aligned}$$

Putting things together, and using (3.38) and interpolation inequality in Besov spaces, we arrive at

$$\|\partial_t\delta R\|_{L^1_T(\dot B^{1/2}_{2,1})} \lesssim (\|u_0\|_{\dot B^{1/2}_{2,1}}M^{1/2}(T) + M(T)+r)\|(\delta\mathbf{v}, \nabla\delta\mathbf{Q})\|_{E(T)}. \tag{3.44}$$

For the estimate of δg, we write $\delta g = \mathrm{Tr}((I-\mathscr{A}_{\mathbf{v}_1})D\delta\mathbf{v}) - \mathrm{Tr}(\delta\mathscr{A}D\mathbf{v}_2)$. We have

$$\begin{aligned}
\|\delta g\|_{L^1_T(\dot B^{3/2}_{2,1})} \lesssim&\|\nabla\mathbf{v}_1\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\nabla\delta\mathbf{v}\|_{L^1_T(\dot B^{3/2}_{2,1})} + \|\nabla\mathbf{v}_2\|_{L^1_T(\dot B^{3/2}_{2,1})}\|\nabla\delta\mathbf{v}\|_{L^1_T(\dot B^{3/2}_{2,1})}\\
\lesssim&(M(T)+r)\|(\delta\mathbf{v}, \nabla\delta\mathbf{Q})\|_{E(T)}.
\end{aligned} \tag{3.45}$$

Now summing up (3.43)–(3.45) and applying Theorem 3.12 to (3.42), we obtain

$$\begin{aligned}\|(\delta\mathbf{u},\nabla\delta\mathbf{P})\|_{E(T)} \lesssim & \|\delta f,\partial_t\delta R,\nabla\delta g\|_{L^1_T(\dot{B}^{1/2}_{2,1})}\\ \le & C_1(\|u_0\|_{\dot{B}^{1/2}_{2,1}}M^{1/2}(T)+M(T)+r)\|(\delta\mathbf{v},\nabla\delta\mathbf{Q})\|_{E(T)}\end{aligned}$$

with C_1 depending on m and $\|\rho_0-1\|_{\dot{B}^{3/2}_{2,1}}$. Taking (3.41) into consideration, we choose r and T so small that

$$r\le\frac{c_0}{2}\wedge\frac{1}{8C_0}\wedge\frac{1}{8C_1}\ \text{ and }\ M(T)\le r\wedge\frac{r}{2C_0\|u_0\|_{\dot{B}^{1/2}_{2,1}}}\wedge\Big(4C_1\|u_0\|_{\dot{B}^{1/2}_{2,1}}\Big)^{-2}.$$

Then $\mathcal{N}$ is a contraction mapping on $B_r(\mathbf{u}_L,\nabla\mathbf{P}_L)$. So it admits a unique fixed point $(\mathbf{u},\nabla\mathbf{P})$ in $B_r(\mathbf{u}_L,\nabla\mathbf{P}_L)$, which is a solution to (3.5) in $E(T)$. The proof of uniqueness in $E(T)$ is similar to the stability estimates. So the proof of the theorem is completed. □

Theorem 3.35. *Under the assumptions in Theorem 3.34, there exists a constant ε_0 depending on m and $\|\rho_0-1\|_{\dot{B}^{3/2}_{2,1}}$ such that if*

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}}\le\varepsilon_0,$$

then the local solution $(\mathbf{u},\nabla\mathbf{P})$ exists globally in time and verifies

$$\|(\mathbf{u},\nabla\mathbf{P})\|_{E(\infty)}:=\|\mathbf{u}\|_{L^\infty(\dot{B}^{1/2}_{2,1})}+\|\Delta\mathbf{u},\partial_t\mathbf{u},\nabla\mathbf{P}\|_{L^1(\dot{B}^{1/2}_{2,1})}\le C\|u_0\|_{\dot{B}^{1/2}_{2,1}}.$$

Proof. The proof is almost the same as that of Theorem 3.34. Let us just mention a few modifications. First, we should replace $E(T)$ by $E(\infty)$ that consists of all pairs $(\mathbf{u},\nabla\mathbf{P})$ satisfying

$$\mathbf{u}\in C_b([0,\infty);\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)),\ (\partial_t\mathbf{u},\Delta\mathbf{u},\nabla\mathbf{P})\in\Big(L^1(\mathbb{R}_+;\dot{B}^{1/2}_{2,1}(\mathbb{R}^3))\Big)^3.$$

Second, we replace the reference solution $(\mathbf{u}_L,\nabla\mathbf{P}_L)$ by $(0,0)$, and choose r as a small number depending on m and $\|\rho_0-1\|_{\dot{B}^{3/2}_{2,1}}$ and $\|u_0\|_{\dot{B}^{1/2}_{2,1}}\lesssim r$. The details are left to the reader. □

For completeness, let us now give the proof of Theorem 3.2 and Theorem 3.9.

Proof of Theorem 3.9. Theorem 3.34 and Theorem 3.35 constitute a proof of Theorem 3.9. $\square$

Proof of Theorem 3.2. First, by Theorem 3.9, we can solve (3.5) for a solution $(\mathbf{u}, \nabla \mathbf{P})$. Define

$$\rho(t,x) = \rho_0(X_{\mathbf{u}}^{-1}(t,x)) \text{ and } (u,P)(t,x) = (\mathbf{u},\mathbf{P})(t, X_{\mathbf{u}}^{-1}(t,x)).$$

By Lemma 2.24 and Lemma 2.20, $(\rho, u, \nabla P)$ has the regularity stated in Definition 3.1 and hence is a strong solution to (1.2).

Let $(\rho_i, u_i, \nabla P_i)$, $i = 1, 2$, be two strong solutions to (1.2) with the same initial value. By Lemma 2.22 and Lemma 2.20, the corresponding unknowns $(\mathbf{u}_i, \nabla \mathbf{P}_i)$ in Lagrangian coordinates are solutions to (3.5) with the same initial value. So it follows from the uniqueness part of Theorem 3.9 that $(\rho_1, u_1, P_1) = (\rho_2, u_2, P_2)$. $\square$

3.4 Long-time asymptotics

This section is devoted to the proof of Theorem 3.6. Besides the maximal regularity estimate (3.2), the proof also relies on a recent result in [59].

Lemma 3.36. *Assume that ρ_0 satisfies* (1.4) *and $\rho_0 - 1 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$, and $u_0 \in \mathcal{P}\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. There exists a constant ε_1 depending on m and $\|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}$ such that if u_0 satisfies*

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}} \le \varepsilon_1,$$

then (1.2) *has a global solution $(\rho, u, \nabla P)$ that verifies* (3.2)*,* (3.3) *and*

$$\|u\|_{L^\infty(\dot{B}^{1/2}_{2,1})} + \|\sqrt{t}(\partial_t u + u \cdot \nabla u)\|_{L^2(\dot{B}^{1/2}_{2,1})} \le C_2 \|u_0\|_{\dot{B}^{1/2}_{2,1}}, \tag{3.46}$$

where C_2 is a constant depending only on m.

Proof. Let ε_1 be so small that the Theorem 3.2 in the present paper and the Theorem 1.2 in [59] hold. We start with mollifying the data by defining

$$\rho_{0,N} = 1 + \sum_{|j|\le N} \dot\Delta_j(\rho_0 - 1) \ \text{ and } \ u_{0,N} = \sum_{|j|\le N} \dot\Delta_j u_0.$$

As in [59], the above data generates a global solution $(\rho_N, u_N, \nabla P_N)$ to (1.2) that satisfies the estimates in [59, Theorem 1.2]. On the other hand, in view of Theorem 3.2, $(\rho_N, u_N, \nabla P_N)$ also satisfies (3.2) and (3.3). The uniform estimates allow us to pass to a limit to obtain a global strong solution $(\rho, u, \nabla P)$ to (1.2), which is unique due to Theorem 3.2. Finally, we use the estimates in [59, Theorem 1.2] to get

$$\begin{aligned}&\|\sqrt{t}(\partial_t u + u\cdot\nabla u)\|_{L^2(\mathbb{R}_+;\dot B^{1/2}_{2,1})}\\ \lesssim&\|\sqrt{t}\partial_t u\|_{L^2(\mathbb{R}_+;\dot B^{1/2}_{2,1})} + \|\sqrt{t}u\|_{L^\infty(\mathbb{R}_+;\dot B^{3/2}_{2,1})}\|u\|_{L^2(\mathbb{R}_+;\dot B^{3/2}_{2,1})} \lesssim \|u_0\|_{\dot B^{1/2}_{2,1}}.\end{aligned}$$

This completes the proof of the lemma. □

We are now in a position to give the proof of Theorem 3.6. Our proof is motivated by [33,34] concerning asymptotics and stability for global solutions to the classical Navier-Stokes equations.

Proof of Theorem 3.6. Fix any $\varepsilon < \varepsilon_1$. We first split the initial velocity into two parts $u_0 = u_{0,h} + u_{0,l}$, where $u_{0,h} = \sum_{j\ge -N}\dot\Delta_j u_0$ is the high frequency part that belongs to the inhomogeneous Besov space $B^{1/2}_{2,1}(\mathbb{R}^3)$, while $u_{0,l}$ satisfies that

$$\|u_{0,l}\|_{\dot B^{1/2}_{2,1}} \le \varepsilon.$$

By Lemma 3.36, $(\rho_0, u_{0,l})$ generates a global solution $(\rho_l, u_l, \nabla P_l)$ to (1.2) that satisfies

$$\|u_l\|_{L^\infty(\mathbb{R}_+;\dot B^{1/2}_{2,1})} + \|\nabla u_l\|_{L^1(\mathbb{R}_+;\dot B^{3/2}_{2,1})} + \|\sqrt{t}(\partial_t u + u\cdot\nabla u)\|_{L^2(\mathbb{R}_+;\dot B^{1/2}_{2,1})} \lesssim \|u_{0,l}\|_{\dot B^{1/2}_{2,1}}, \quad (3.47)$$

and

$$\|\rho_l - 1\|_{L^\infty(\mathbb{R}_+;\dot{B}^{1/2}_{2,1})} \lesssim \|\rho_0 - 1\|_{\dot{B}^{3/2}_{2,1}}. \tag{3.48}$$

Let $(\rho_h, u_h, P_h) = (\rho - \rho_l, u - u_l, P - P_l)$. Then (ρ_h, u_h, P_h) satisfies the system

$$\begin{cases} \partial_t \rho_h + u \cdot \nabla \rho_h + u_h \cdot \nabla \rho_l = 0, \\ \rho(\partial_t u_h + u \cdot \nabla u_h) - \Delta u_h + \nabla P_h = -\rho u_h \cdot \nabla u_l - \rho_h(\partial_t u_l + u_l \cdot \nabla u_l), \\ \operatorname{div} u_h = 0, \\ (\rho_h, u_h)|_{t=0} = (0, u_{0,h}). \end{cases} \tag{3.49}$$

We shall use the energy method to derive an $L^4_t(\dot{B}^{1/2}_{2,1})$ estimate for u_h. Taking the L^2 inner product of the second equation in (3.49) with u_h, and using Hölder's inequality and Sobolev inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho} u_h\|_2^2 + \|\nabla u_h\|_2^2 &= -\int \rho(u_h \cdot \nabla u_l) \cdot u_h \, dx - \int \rho_h(\partial_t u_l + u_l \cdot \nabla u_l) \cdot u_h \, dx \\ &\lesssim \|\nabla u_l\|_\infty \|\sqrt{\rho} u_h\|_2^2 + \|\rho_h\|_2 \|\partial_t u_l + u_l \cdot \nabla u_l\|_3 \|\nabla u_h\|_2. \end{aligned}$$

To bound $\|\rho_h\|_2$, we get from the transport equation in (3.49) that

$$\|\rho_h(t)\|_2 \le \int_0^t \|u_h \cdot \nabla \rho_l\|_2 \, d\tau \lesssim \int_0^t \|\nabla u_h\|_2 \|\nabla \rho_l\|_3 \, d\tau \lesssim \sqrt{t} \|\nabla u_h\|_{L^2_t(L^2)} \|\nabla \rho_l\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}.$$

Putting things together and using (3.48), we have

$$\frac{d}{dt}\|\sqrt{\rho} u_h\|_2^2 + \|\nabla u_h\|_2^2 \lesssim \|\nabla u_l\|_\infty \|\sqrt{\rho} u_h\|_2^2 + \|\nabla u_h\|_{L^2_t(L^2)} \|\sqrt{t}(\partial_t u_l + u_l \cdot \nabla u_l)\|_3 \|\nabla u_h\|_2.$$

Integrating both sides of the above inequality over the time interval $[0, t]$, then using (3.47), we have

$$\|\sqrt{\rho} u_h\|_2^2(t) + \|\nabla u_h\|_{L^2_t(L^2)}^2 \lesssim \|\sqrt{\rho_0} u_{0,h}\|_2^2 + \int_0^t \|\nabla u_l\|_\infty \|\sqrt{\rho} u_h\|_2^2 \, d\tau + \varepsilon \|\nabla u_h\|_{L^2_t(L^2)}^2.$$

So, if ε is small enough, we arrive at

$$\|\sqrt{\rho}u_h\|_2^2(t) + \|\nabla u_h\|_{L_t^2(L^2)}^2 \lesssim \|\sqrt{\rho_0}u_{0,h}\|_2^2 + \int_0^t \|\nabla u_l\|_\infty \|\sqrt{\rho}u_h\|_2^2 \, d\tau.$$

Applying Gronwall's inequality and using (3.47) give us that

$$\|u_h\|_{L_t^\infty(L^2)} + \|\nabla u_h\|_{L_t^2(L^2)} \le C\|u_{0,h}\|_2 \exp\{C\|u_{0,l}\|_{\dot{B}_{2,1}^{1/2}}\}.$$

We interpolate to have

$$\|u_h\|_{L_t^4(\dot{B}_{2,1}^{1/2})} \le C\|u_{0,h}\|_2 \exp\{C\|u_{0,l}\|_{\dot{B}_{2,1}^{1/2}}\}.$$

This implies that there exists a positive number t_ε such that $\|u_h(t_\varepsilon)\|_{\dot{B}_{2,1}^{1/2}} \le \varepsilon$, and so $\|u(t_\varepsilon)\|_{\dot{B}_{2,1}^{1/2}} \lesssim \varepsilon$. Now we apply Lemma 3.36 to conclude that

$$\|u\|_{L^\infty((t_\varepsilon,\infty);\dot{B}_{2,1}^{1/2})} \lesssim \|u(t_\varepsilon)\|_{\dot{B}_{2,1}^{1/2}} \lesssim \varepsilon,$$

which implies (3.4) since ε is arbitrarily small. This completes the proof of Theorem 3.6. □

Chapter 4

Compressible Flows

4.1 Introduction

The main purpose of this chapter is to investigate the maximal L^1 regularity of (1.6) under the least regularity assumption on the density ρ. In Chapter 3, we were only able to work in the L^2 (in space) framework due to the presence of pressure. In this chapter, the operator $\mathcal{A}$ in (1.6) will be a local operator, and we will work in the general L^p (in space) framework. A practical benefit of doing so is that one can lower the regularity of the density (see [20]). For the analysis in Chapter 3 to adapt to the L^p framework, we need to make the extra effort to obtain pointwise bounds for the kernel of the semigroup generated by $\rho^{-1}\mathcal{A}$. Let us consider two concrete examples. For $\mathcal{A} = \Delta$ (the Laplacian), McIntosh and Nahmod [46] proved that the kernel of the L^2 semigroup $e^{t\rho^{-1}\Delta}$ generated by $\rho^{-1}\Delta$ satisfies Gaussian bounds (see also [31]). This guarantees that the semigroup $e^{t\rho^{-1}\Delta}$ extrapolates to a bounded analytic semigroup on L^p, $1 < p < \infty$. Note that the kernel of $e^{t\rho^{-1}\Delta}$ is essentially a scalar kernel. If $\mathcal{A}$ is the Lamé operator $\mathcal{L}$ defined by (2.6), however, (1.6) is a truly coupled system whose fundamental matrix does not necessarily satisfy Gaussian bounds. Nevertheless, we can prove the bounds for the fundamental matrix and its derivatives using a rather classical method if the dimensions of the Euclidean space ≤ 3. The tricks are due to Davies, one is to use Sobolev inequalities to bound L^∞-norm (see [27]), the other is a perturbation technique to obtain exponential decay (see [26]). In the spirit of [31, 46], once we obtain Gaussian upper bounds of the fundamental matrix (denoted by $K_t(x, y)$), we can easily get the $C^{1,\gamma}$ estimates for the kernel $K_t(x, y)\rho^{-1}(y)$.

Before we study the maximal regularity for (1.6), we will establish a maximal L^1-in-time regularity result for the abstract Cauchy problem

$$\begin{cases} u'(t) - \mathcal{S}u(t) = f(t), \\ u(0) = x \end{cases} \tag{4.1}$$

in homogeneous type spaces. Let us assume that $\mathcal{S}$ is an unbounded linear operator on a Banach space $(X, \|\cdot\|)$ that generates a bounded analytic semigroup $e^{t\mathcal{S}}$. Given (4.1) with $x = 0$, $\mathcal{S}$ is said to have maximal L^r-in-time regularity in X for $r \in [1, \infty]$, if for every $f \in L^r((0,\infty); X)$, (4.1) has a unique solution verifying

$$\|\mathcal{S}u\|_{L^r((0,\infty);X)} \leq C\|f\|_{L^r((0,\infty);X)}. \tag{4.2}$$

The maximal L^r regularity issue for $r \in (1, \infty)$ has been extensively studied in the literature. We refer to [25,28,30,35,51], amongst which [25] also covered the L^1 theory, but the global-in-time estimate (4.2) holds only if 0 belongs to the resolvent set $\rho(\mathcal{S})$ of $\mathcal{S}$ (i.e., $\mathcal{S}^{-1} \in \mathscr{L}(X)$). It goes without saying that such a condition is very demanding in many concrete examples. Recently, Ri and Farwig [52] established maximal L^1 regularity for $\mathcal{S}$ in inhomogeneous type spaces without assuming $0 \in \rho(\mathcal{S})$. Later, a similar result in the homogeneous space setting was proved by Danchin et al. [18]. The authors in [18] also nicely explained the importance of maximal L^1 regularity for parabolic systems in homogeneous spaces. Our work is more relevant to the one in [18]. But [18] did not cover maximal regularity in homogeneous spaces with negative regularity. For us, working in spaces with negative regularity can weaken the regularity of the density. Here we follow Chapter 3 closely. It turns out that the strategy of the proof of the concrete result in Chapter 3 works equally well for the abstract problem.

This chapter is organized as follows. In Section 4.2, we prove the $C^{1,\gamma}$ regularity for $K_t(x,y)b(y)$, where $K_t(x,y)$ is the matrix-valued heat kernel of $-b\mathcal{L}$ and $\mathcal{L}$ is the Lamé operator. We remark that the coefficient b is only bounded and bounded from below by a positive constant. In Section 4.3, we derive the maximal L^1 regularity for the abstract Cauchy problem (4.1) when $\mathcal{S}$ is a composition of bounded and unbounded operators. Then, in Section 4.4, we apply the abstract theory to study the maximal L^1 regularity for (1.6), where $\mathcal{A}$ is the Laplacian or the Lamé operator. Section 4.5 is devoted to the global-in-time well-posedness of the pressureless system (1.3). The method in Section 4.5 is also suitable for solving compressible Navier-Stokes equations, and the corresponding result will be reported in Section 4.6.

4.2 Bounds of fundamental matrix

Let ρ be a measurable function defined in $\mathbb{R}^n$ satisfying

$$m \leq \rho(x) \leq \frac{1}{m}, \quad \text{a.e. } x \in \mathbb{R}^n \tag{4.3}$$

for some $m \in (0, 1]$. Denote $b = \rho^{-1}$. The main results of this section, in the spirit of those in [31, 46], are the Gaussian bounds of the matrix-valued heat kernel of $-b\mathcal{L}$.

For notational convenience, we denote $L^2 = L^2(\mathbb{R}^n; \mathbb{R}^n)$, $H^2 = H^2(\mathbb{R}^n; \mathbb{R}^n)$. Let $\| \cdot \|$ be the L^2 norm induced by the standard L^2 inner product $\langle \cdot, \cdot \rangle$, and $\| \cdot \|_\rho$ the weighted norm induced by the inner product

$$\langle u, v \rangle_\rho = \int_{\mathbb{R}^n} u(x) \cdot v(x) \rho(x) \, dx.$$

Roughly, our method is a classical PDE method, and we will study various weighted estimates for the solutions to the parabolic Lamé system

$$\rho(x) \partial_t u - \mathcal{L} u = 0, \ \text{ in } (0, \infty) \times \mathbb{R}^n. \tag{4.4}$$

Before studying the variable coefficient problem, let us point out a basic fact about the Lamé operator $\mathcal{L}$. The assumption (2.7) guarantees the ellipticity of $-\mathcal{L}$, and we have

$$\|(-\mathcal{L})^{1/2} u\|^2 = \langle -\mathcal{L} u, u \rangle = \mu \|\nabla u\|^2 + (\mu + \lambda) \| \operatorname{div} u\|^2 \geq (\mu \wedge \nu) \|\nabla u\|^2 \tag{4.5}$$

for all vectors $u \in H^2$.

Lemma 4.1. *The operator $b\mathcal{L} : H^2 \subset L^2 \to L^2$ generates an analytic semigroup of contraction $\{e^{tb\mathcal{L}}\}_{t\geq 0}$ on $(L^2, \langle\cdot,\cdot\rangle_\rho)$, and $e^{tb\mathcal{L}}b$ is self-adjoint on $(L^2, \langle\cdot,\cdot\rangle)$ for every $t \geq 0$.*

Proof. First, it is readily to verify that $b\mathcal{L}$ is a self-adjoint operator on $(L^2, \langle\cdot,\cdot\rangle_\rho)$. In view of (4.5), we have $\langle b\mathcal{L}u, u\rangle_\rho \leq 0$ for all $u \in H^2$. So by Theorem 2.4, $b\mathcal{L}$ generates an analytic semigroup of contraction $\{e^{tb\mathcal{L}}\}_{t\geq 0}$ on $(L^2, \langle\cdot,\cdot\rangle_\rho)$. Since $e^{tb\mathcal{L}}$ is self-adjoint on $(L^2, \langle\cdot,\cdot\rangle_\rho)$, we have for all $u, v \in L^2$ that

$$\langle e^{tb\mathcal{L}}bu, v\rangle = \langle e^{tb\mathcal{L}}bu, bv\rangle_\rho = \langle bu, e^{tb\mathcal{L}}bv\rangle_\rho = \langle u, e^{tb\mathcal{L}}bv\rangle.$$

This means that $e^{tb\mathcal{L}}b$ is self-adjoint on $(L^2, \langle\cdot,\cdot\rangle)$. □

Lemma 4.2. *Let $n \in \{2, 3\}$. For every $t > 0$, the bounded operator $e^{tb\mathcal{L}}$ on L^2 admits a Schwartz kernel, denoted by $K_t(x, y)$, which is bounded and satisfies the pointwise bound*

$$|K_t(x, y)| \leq \frac{C}{t^{n/2}}$$

for some constant $C = C(m, \mu, \lambda)$.

Proof. Since $n \in \{2, 3\}$, we get from the Gagliardo-Nirenberg inequality and (2.10) that

$$\|u\|_\infty \leq C\|u\|^{1-n/4}\|\mathcal{L}u\|^{n/4} \;\; u \in H^2. \tag{4.6}$$

This along with the analyticity of $e^{tb\mathcal{L}}$ implies that

$$\|e^{tb\mathcal{L}}u_0\|_\infty \leq Ct^{-n/4}\|u_0\|, \;\; u_0 \in L^2.$$

Then $e^{tb\mathcal{L}}$ is also bounded from L^1 to L^2 due to the self-adjointness of $e^{tb\mathcal{L}}b$, and from L^1 to L^∞ due to the semigroup property. So the Schwartz kernel $K_t(x, y)$ of $e^{tb\mathcal{L}}$ is indeed bounded and satisfies the desired bound. This completes the proof. □

Next, we adopt the well-known Davies perturbation method (see [26]) to show Gaussian bounds for the kernel $S_t(x, y) := K_t(x, y)b(y)$.

The main theorem in this section is the following:

Theorem 4.3. *Let $n \in \{2,3\}$. For any $\gamma \in (0,1)$ and $t > 0$, each entry of $S_t(x,y)$ is a $C^{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n)$ function. More precisely, there exist constants $C_1 = C_1(m,\mu,\lambda)$ and $C_2 = C_2(m,\mu,\lambda,\gamma)$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,*

$$|S_t(x,y)| + \sqrt{t}|\nabla_x S_t(x,y)| \leq \frac{C_1}{t^{n/2}} \exp\left\{-\frac{|x-y|^2}{C_1 t}\right\},$$

$$|\nabla_x S_t(x+h,y) - \nabla_x S_t(x,y)| \leq \left(\frac{|h|}{\sqrt{t}}\right)^{\gamma} \frac{C_2}{t^{(n+1)/2}} \exp\left\{-\frac{|x-y|^2}{C_2 t}\right\}, \tag{4.7}$$

and

$$|\nabla_x S_t(x,y+h) - \nabla_x S_t(x,y)| \leq \left(\frac{|h|}{\sqrt{t}}\right)^{\gamma} \frac{C_2}{t^{(n+1)/2}} \exp\left\{-\frac{|x-y|^2}{C_2 t}\right\} \tag{4.8}$$

provided $2|h| \leq \sqrt{t}$.

Remark 4.4. In view of Lemma 4.1, we have $S_t(x,y) = S_t^{\intercal}(y,x)$. So the y-derivative also satisfies each of the bounds.

Let $\mathscr{W}$ denote the set of all bounded real-valued smooth functions ψ on $\mathbb{R}^n$ such that $\|\nabla\psi\|_\infty \leq 1$ and $\|\nabla^2\psi\|_\infty \leq 1$. Let $d(x,y) := \sup\{\psi(x) - \psi(y) | \psi \in \mathscr{W}\}$.

Lemma 4.5 (see [27, Lemma 4])**.** *There exists a positive constant $C = C(n)$ such that*

$$C^{-1}|x-y| \leq d(x,y) \leq C|x-y|$$

for all $x, y \in \mathbb{R}^n$.

Given $\alpha \in \mathbb{R}$ and $\psi \in \mathscr{W}$, define $\psi_\alpha(x) = \psi(\alpha x)$ and $\phi(x) = e^{\psi_\alpha(x)}$. The analysis is based on the key observation that

$$\begin{aligned}\langle -\phi^{-1}\mathcal{L}\phi v, u\rangle =& \mu \int (\alpha(\nabla\psi)_\alpha \otimes v + \nabla v) : (-\alpha(\nabla\psi)_\alpha \otimes u + \nabla u)\, dx \\ &+ (\mu+\lambda)\int (\alpha(\nabla\psi)_\alpha \cdot v + \operatorname{div} v)(-\alpha(\nabla\psi)_\alpha \cdot u + \operatorname{div} u)\, dx \end{aligned} \tag{4.9}$$

for any smooth vector fields u and v. In particular, if $u = v$, we have

$$\langle -\phi^{-1}\mathcal{L}\phi v, v\rangle \geq \|(-\mathcal{L})^{1/2} v\|^2 - C\alpha^2\|v\|^2. \tag{4.10}$$

In what follows, we divide the proof of Theorem 4.3 into three lemmas.

Lemma 4.6. *Let $n \in \{2,3\}$. There exists a constant $C = C(m,\mu,\lambda)$ such that for all $t > 0$ and $x, y \in \mathbb{R}^n$,*

$$|K_t(x,y)| \leq \frac{C}{t^{n/2}} \exp\left\{-\frac{|x-y|^2}{Ct}\right\}.$$

Proof. Denote $v = \phi^{-1}e^{tb\mathcal{L}}(\phi u_0)$, where $u_0 \in L^2$. Then v is a solution to the system

$$\begin{cases} \rho\partial_t v - \phi^{-1}\mathcal{L}\phi v = 0, \\ v(0) = u_0. \end{cases} \tag{4.11}$$

We start with the energy estimates for v. Taking inner product of (4.11) with v, then using (4.10), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_\rho^2 + \|(-\mathcal{L})^{1/2}v\|^2 \leq C\alpha^2\|v\|_\rho^2.$$

Applying Gronwall's inequality, we obtain

$$\|v(t)\|^2 + \int_0^t \|(-\mathcal{L})^{1/2}v\|^2\, d\tau \leq C\|u_0\|^2 e^{C\alpha^2 t}. \tag{4.12}$$

Differentiating (4.11) with respect to t, we get by a similar argument that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t v\|_\rho^2 + \|(-\mathcal{L})^{1/2}\partial_t v\|^2 \le C\alpha^2\|\partial_t v\|_\rho^2.$$

So the function $t \mapsto \|\partial_t v\|_\rho^2 e^{-C\alpha^2 t}$ is decreasing. Consequently, we have

$$\|\partial_t v\|_\rho^2 e^{-C\alpha^2 t} \le \frac{2}{t}\int_{t/2}^t \|\partial_t v\|_\rho^2 e^{-C\alpha^2\tau}\, d\tau. \tag{4.13}$$

Next, multiplying (4.11) by v_t and integrating in x, then using (4.9) and the Cauchy-Schwarz inequality, we get

$$\|\partial_t v\|_\rho^2 + \frac{1}{2}\frac{d}{dt}\|(-\mathcal{L})^{1/2}v\|^2 \le C\left(\alpha^2\|v\|\|\partial_t v\| + |\alpha|\|\nabla v\|\|\partial_t v\|\right).$$

The term $\|\partial_t v\|$ on the right side can be absorbed by $\|\partial_t v\|_\rho^2$ on the left side. This together with (4.12) and (4.5) gives

$$\|\partial_t v\|_\rho^2 + \frac{d}{dt}\|(-\mathcal{L})^{1/2}v\|^2 \le C\left(\alpha^4 e^{C\alpha^2 t}\|u_0\|^2 + \alpha^2\|(-\mathcal{L})^{1/2}v\|^2\right).$$

So,

$$\|\partial_t v\|_\rho^2 e^{-C\alpha^2 t} + \frac{d}{dt}(\|(-\mathcal{L})^{1/2}v\|^2 e^{-C\alpha^2 t}) \le C\alpha^4\|u_0\|^2. \tag{4.14}$$

Combining (4.12) and (4.14), we have

$$\int_{t/2}^t \|\partial_t v\|_\rho^2 e^{-C\alpha^2\tau}\, d\tau + \|(-\mathcal{L})^{1/2}v(t)\|^2 \le C\left(\frac{1}{t} + \alpha^4 t\right)\|u_0\|^2 e^{C\alpha^2 t},$$

which together with (4.13) further implies

$$\|\partial_t v(t)\| \le C\left(\alpha^2 + \frac{1}{t}\right)\|u_0\|e^{C\alpha^2 t} \le \frac{C}{t}\|u_0\|e^{C\alpha^2 t}. \tag{4.15}$$

The above estimate should imply the corresponding L^2 estimate of $\mathcal{L}v$. To see this, we get by a direct computation that

$$-\mathcal{L}v = -\rho\partial_t v + \mu(\alpha^2|\nabla\psi|_\alpha^2 v + 2\alpha(\nabla\psi)_\alpha \cdot \nabla v + \alpha^2(\Delta\psi)_\alpha v)$$
$$+ (\mu+\lambda)(\alpha \operatorname{div} v(\nabla\psi)_\alpha + \alpha\nabla v(\nabla\psi)_\alpha + \alpha^2(v\cdot(\nabla\psi)_\alpha)(\nabla\psi)_\alpha + \alpha^2(\nabla^2\psi)_\alpha v). \tag{4.16}$$

Then it is easy to see that

$$\|\mathcal{L}v\| \le C(\|\partial_t v\| + \alpha^2\|v\| + |\alpha|\|\nabla v\|).$$

The first order derivative can be handled by using the interpolation inequality

$$\|\nabla v\| \le C\|v\|^{1/2}\|\mathcal{L}v\|^{1/2}.$$

So,

$$\|\mathcal{L}v\| \le C(\|\partial_t v\| + \alpha^2\|v\|).$$

Substituting for $\|v\|$ and $\|\partial_t v\|$ by (4.12) and (4.15), respectively, we have

$$\|\mathcal{L}v(t)\| \le \frac{C}{t}\|u_0\|e^{C\alpha^2 t}.$$

Now using the Gagliardo-Nirenberg inequality (4.6), we obtain

$$\|v(t)\|_\infty \le \frac{C}{t^{n/4}}\|u_0\|e^{C\alpha^2 t}. \tag{4.17}$$

This means that the operator $\phi^{-1}e^{tb\mathcal{L}}\phi$ is bounded from L^2 to L^∞. A duality argument gives the bound from L^1 to L^2, that is,

$$\|v(t)\| \le \frac{C}{t^{n/4}}\|u_0\|_1 e^{C\alpha^2 t}. \tag{4.18}$$

While this along with the semigroup property of $\phi^{-1}e^{tb\mathcal{L}}\phi$ gives

$$\|v(t)\|_\infty \le \frac{C}{t^{n/2}}\|u_0\|_1 e^{C\alpha^2 t}. \tag{4.19}$$

Noticing that the kernel of $\phi^{-1}e^{tb\mathcal{L}}\phi$ is $K_t(x,y)e^{\psi(\alpha y)-\psi(\alpha x)}$, we get

$$|K_t(x,y)| \le \frac{C}{t^{n/2}}\exp\{C\alpha^2 t + \psi(\alpha x) - \psi(\alpha y)\}.$$

Replacing ψ by $-\psi$, we have

$$|K_t(x,y)| \le \frac{C}{t^{n/2}}\exp\{C\alpha^2 t - |\psi(\alpha x) - \psi(\alpha y)|\}.$$

It follows by optimizing with respect to $\psi \in \mathscr{W}$ and applying Lemma 4.5 that

$$|K_t(x,y)| \le \frac{C}{t^{n/2}}\exp\{C\alpha^2 t - C^{-1}|\alpha||x-y|\}.$$

Finally, minimizing the bound by choosing $\alpha = \frac{|x-y|}{2C^2 t}$ completes the proof. □

Lemma 4.7. *Let $n \in \{2,3\}$. There exists a constant $C = C(m,\mu,\lambda)$ such that for all $t > 0$ and $x, y \in \mathbb{R}^n$,*

$$|\nabla_x S_t(x,y)| \le \frac{C}{t^{(n+1)/2}}\exp\left\{-\frac{|x-y|^2}{Ct}\right\}.$$

Proof. Apparently, we only need to show the bound for $|\nabla_x K_t(x,y)|$. Denote $u = e^{tb\mathcal{L}}(\phi u_0)$ and $v = \phi^{-1}u$, where $u_0 \in L^2$. We need to bound the norm $\|\phi^{-1}\nabla u(t)\|_\infty$. To this end, let us first study the norm $\|\nabla v(t)\|_\infty$ since

$$\phi^{-1}\nabla u(t) = \nabla v + \alpha(\nabla\psi)_\alpha \otimes v. \tag{4.20}$$

By the equation (4.16), we see that

$$\|\mathcal{L}v\|_\infty \le C(\|\partial_t v\|_\infty + \alpha^2\|v\|_\infty + |\alpha|\|\nabla v\|_\infty). \tag{4.21}$$

Using Littlewood-Paley and (2.9), one can prove the interpolation inequality

$$\|\nabla v\|_\infty \le C\|v\|_\infty^{1/2}\|\mathcal{L}v\|_\infty^{1/2}. \tag{4.22}$$

Plugging (4.22) in (4.21), we easily get

$$\|\mathcal{L}v\|_\infty \le C(\|\partial_t v\|_\infty + \alpha^2\|v\|_\infty). \tag{4.23}$$

Then combining (4.20), (4.22) and (4.23), we arrive at

$$\|\phi^{-1}\nabla u(t)\|_\infty \le C\left(|\alpha|\|v(t)\|_\infty + \|v(t)\|_\infty^{1/2}\|\partial_t v(t)\|_\infty^{1/2}\right). \tag{4.24}$$

Next, in order to bound $\|\partial_t v(t)\|_\infty$, we observe that

$$\partial_t v(t) = \phi^{-1}e^{\frac{t}{2}b\mathcal{L}}\phi[\partial_t v(t/2)].$$

So, in view of (4.17), (4.15) and (4.18), we get

$$\|\partial_t v(t)\|_\infty \le \frac{C}{t^{n/4}}e^{C\alpha^2 t}\|\partial_t v(t/2)\| \le \frac{C}{t^{1+n/4}}e^{C\alpha^2 t}\|v(t/4)\| \le \frac{C}{t^{1+n/2}}e^{C\alpha^2 t}\|u_0\|_1. \tag{4.25}$$

Plugging the above in (4.24) and using (4.19), we have

$$\|\phi^{-1}\nabla u(t)\|_\infty \le C\left(\frac{|\alpha|}{t^{n/2}} + \frac{1}{t^{(n+1)/2}}\right)e^{C\alpha^2 t}\|u_0\|_1 \le \frac{C}{t^{(n+1)/2}}e^{C\alpha^2 t}\|u_0\|_1. \tag{4.26}$$

Thus,

$$|\nabla_x K_t(x,y)| \le \frac{C}{t^{(n+1)/2}}\exp\{C\alpha^2 t + \psi(\alpha x) - \psi(\alpha y)\}.$$

Again, we finish the proof by optimizing the bound with respect to $\psi \in \mathscr{W}$ and then $\alpha \in \mathbb{R}$. □

Remark 4.8. From (4.25), we also see that the kernel of $tb\mathcal{L}e^{tb\mathcal{L}}$ has a pointwise Gaussian upper bound. In particular, $tb\mathcal{L}e^{tb\mathcal{L}}$ extends to a bounded operator on L^p for every $t > 0$.

Lemma 4.9. *Let $n \in \{2,3\}$. For any $\gamma \in (0,1)$, there exists a constant $C = C(m,\mu,\lambda,\gamma)$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,* (4.7) *and* (4.8) *hold whenever $2|h| \le \sqrt{t}$.*

Proof. Let $u = e^{tb\mathcal{L}}u_0$. By Lemmas 4.1 and 4.6, we have

$$t^{n/4}\|\mathcal{L}u(t)\| + t^{n/2}\|\mathcal{L}u(t)\|_\infty \le \frac{C}{t}\|u_0\|_1.$$

For any $\gamma \in (0,1)$, let $q = \frac{n}{1-\gamma}$ and $\theta = \frac{2(1-\gamma)}{n}$. Then we use the embedding $\dot{W}^{1,q}(\mathbb{R}^n) \hookrightarrow \dot{C}^\gamma(\mathbb{R}^n)$ to get

$$\|\nabla u\|_{\dot{C}^\gamma} \le C\|\nabla^2 u\|_q \le C\|\mathcal{L}u\|_q \le C\|\mathcal{L}u\|_2^\theta \|\mathcal{L}u\|_\infty^{1-\theta} \le \frac{C}{t^{(n+1+\gamma)/2}}\|u_0\|_1.$$

Thus, we have for any $h \in \mathbb{R}^n$ that

$$|\nabla_x K_t(x+h,y) - \nabla_x K_t(x,y)| \le \frac{C}{t^{(n+1)/2}}\left(\frac{|h|}{\sqrt{t}}\right)^\gamma.$$

The exponential decay factor in (4.7) can be easily obtained by the observation that

$$\begin{aligned} &|\nabla_x K_t(x+h,y) - \nabla_x K_t(x,y)| \\ \le &(|\nabla_x K_t(x+h,y)| + |\nabla_x K_t(x,y)|)^{1-\beta}|\nabla_x K_t(x+h,y) - \nabla_x K_t(x,y)|^\beta \end{aligned}$$

for any $\beta \in (0,1)$. This proves (4.7).

To prove (4.8), we write

$$\int [\nabla_x S_t(x,y+h) - \nabla_x S_t(x,y)]u_0(y)\,dy = \nabla e^{tb\mathcal{L}}(b\delta_h u_0)$$

with $\delta_h u_0(x) = u_0(x-h) - u_0(x)$. Using Lemma 4.7, the right side can be estimated as follows

$$\|\nabla e^{tb\mathcal{L}}(b\delta_h u_0)\|_\infty \le \frac{C}{\sqrt{t}}\|e^{\frac{t}{2}b\mathcal{L}}(b\delta_h u_0)\|_\infty \le \frac{C|h|}{t^{1+n/2}}\|u_0\|_1.$$

The bound in (4.8) can be shown by a similar argument as the first part of the proof. This completes the proof of the lemma. □

For completeness, we conclude this section by finishing the proof of Theorem 4.3.

Proof of Theorem 4.3. Lemmas 4.6, 4.7 and 4.9 constitute the proof of Theorem 4.3. □

4.3 An abstract L^1 theory

In this section, we are concerned with the L^1-in-time theory for the abstract Cauchy problem (4.1), where $\mathcal{S}$ is a composition of bounded and unbounded operator. We follow Chapter 3 closely and we do not explicitly use the theory of interpolation spaces.

Let $(X, \|\cdot\|)$ be a Banach space. We temporarily just assume

Assumption 4.1. $\mathcal{S} : D(\mathcal{S}) \subset X \to X$ is an one-to-one operator that generates a bounded analytic semigroup $e^{t\mathcal{S}}$ on X.

Given $s \in (0, 2)$, we define

$$\|x\|_{\dot{B}^{s,\mathcal{S}}_{X,1}} := \|t^{-s/2}\|t\mathcal{S}e^{t\mathcal{S}}x\|\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}$$

and

$$\|x\|_{\dot{B}^{-s,\mathcal{S}}_{X,1}} := \|t^{s/2}\|e^{t\mathcal{S}}x\|\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}.$$

In view of Lemmas 2.13 and 2.14, the above notations make sense if we pretend that $\mathcal{S}$ is a second-order elliptic operator. For any $x \in D(\mathcal{S})$, since $\|t\mathcal{S}e^{t\mathcal{S}}x\| \lesssim \|x\| \wedge \|t\mathcal{S}x\|$, we easily see that

$$\|x\|_{\dot{B}^{s,\mathcal{S}}_{X,1}} \lesssim \|x\|_{D(\mathcal{S})} := \|x\| + \|\mathcal{S}x\|.$$

While for $x \in R(\mathcal{S})$, the range of $\mathcal{S}$, we have

$$\|x\|_{\dot{B}^{-s,\mathcal{S}}_{X,1}} = \|\mathcal{S}^{-1}x\|_{\dot{B}^{2-s,\mathcal{S}}_{X,1}} \lesssim \|x\|_{R(\mathcal{S})} := \|x\| + \|\mathcal{S}^{-1}x\|.$$

Definition 4.10. Let $s \in (0,2)$. Define $\dot{B}^{s,\mathcal{S}}_{X,1}$ as the completion of $(D(\mathcal{S}), \|\cdot\|_{\dot{B}^{s,\mathcal{S}}_{X,1}})$, and $\dot{B}^{-s,\mathcal{S}}_{X,1}$ as the completion of $(R(\mathcal{S}), \|\cdot\|_{\dot{B}^{-s,\mathcal{S}}_{X,1}})$.

The space $\dot{B}^{s,\mathcal{S}}_{X,1}$ can also be defined via interpolation (see, e.g., [35, Remark 2.4]), but we do not need this fact in this dissertation.

For notational convenience, we temporarily denote $\dot{B}^{\pm s,\mathcal{S}}_{X,1}$ by $\dot{B}^{\pm s}$. But we shall not use the abbreviated notations if the norms are associated with different operators.

Lemma 4.11. *For every $k \in \mathbb{N} \cup \{0\}$ and $s \in (0,2)$, there exists a constant C depending on s and k such that*

$$\sup_{t>0} \|(t\mathcal{S})^k e^{t\mathcal{S}} x\|_{\dot{B}^s} \le C\|x\|_{\dot{B}^s}, \ \ \forall x \in D(\mathcal{S}), \tag{4.27}$$

$$\sup_{t>0} \|(t\mathcal{S})^k e^{t\mathcal{S}} x\|_{\dot{B}^{-s}} \le C\|x\|_{\dot{B}^{-s}}, \ \ \forall x \in R(\mathcal{S}), \tag{4.28}$$

$$\big\|\|(t\mathcal{S})^{k+1} e^{t\mathcal{S}} x\|_{\dot{B}^s}\big\|_{L^1(\mathbb{R}_+,\frac{dt}{t})} \le C\|x\|_{\dot{B}^s}, \ \ x \in D(\mathcal{S}), \tag{4.29}$$

and

$$\big\|\|(t\mathcal{S})^{k+1} e^{t\mathcal{S}} x\|_{\dot{B}^{-s}}\big\|_{L^1(\mathbb{R}_+,\frac{dt}{t})} \le C\|x\|_{\dot{B}^{-s}}, \ \ x \in R(\mathcal{S}). \tag{4.30}$$

Proof. The first two inequalities follow immediately from the definitions of the norms and the analyticity of $e^{t\mathcal{S}}$.

The proofs for (4.29) and (4.30) are similar, so let us only prove (4.29). In view of (4.27), we only need to prove (4.29) for $k = 0$. Applying Fubini's theorem, we have

$$\begin{aligned}\int_0^\infty \|\tau \mathcal{S} e^{\tau\mathcal{S}} x\|_{\dot{B}^s} \frac{d\tau}{\tau} &= \int_0^\infty \int_0^\infty t^{-s/2} \|\mathcal{S}^2 e^{(t+\tau)\mathcal{S}} x\| \, dt \, d\tau \\ &= \int_0^\infty \int_\tau^\infty (t-\tau)^{-s/2} \|\mathcal{S}^2 e^{t\mathcal{S}} x\| \, dt \, d\tau \\ &= \int_0^\infty \|\mathcal{S}^2 e^{t\mathcal{S}} x\| \, dt \int_0^t (t-\tau)^{-s/2} \, d\tau \\ &= \frac{2}{2-s} \int_0^\infty t^{-s/2} \|t\mathcal{S}^2 e^{t\mathcal{S}} x\| \, dt.\end{aligned}$$

Finally, by the analyticity of $\mathcal{S}$, we end up with

$$\big\| \|(t\mathcal{S}) e^{t\mathcal{S}} x\|_{\dot{B}^s} \big\|_{L^1(\mathbb{R}_+, \frac{dt}{t})} \le C \int_0^\infty t^{-s/2} \|\mathcal{S} e^{t\mathcal{S}} x\| \, dt = C\|x\|_{\dot{B}^s}.$$

This completes the proof. □

The inequality (4.27) (with $k = 0$) guarantees that $e^{t\mathcal{S}}|_{D(\mathcal{S})}$ extends to a bounded operator on $\dot{B}^s$ with bounds uniform in t. Denote this extension by $\mathcal{T}_s(t)$. Then $\{\mathcal{T}_s(t)\}_{t\ge 0}$ is a bounded semigroup on $\dot{B}^s$. Similarly, (4.28) implies that $e^{t\mathcal{S}}$ also extrapolates to a bounded semigroup $\{\mathcal{T}_{-s}(t)\}_{t\ge 0}$ on $\dot{B}^{-s}$. In fact, both semigroups are strongly continuous.

Lemma 4.12. *$\{\mathcal{T}_s(t)\}_{t\ge 0}$ (resp., $\{\mathcal{T}_{-s}(t)\}_{t\ge 0}$) is a bounded C_0 semigroup on $\dot{B}^s$ (resp., $\dot{B}^{-s}$).*

Proof. For $x \in D(\mathcal{S})$, the function $t \mapsto \mathcal{T}_s(t)x = e^{t\mathcal{S}}x$ belongs to $C([0,\infty); D(\mathcal{S}))$, hence $C([0,\infty); \dot{B}^s)$ since $D(\mathcal{S}) \hookrightarrow \dot{B}^s$. One can easily get the strong continuity of $\mathcal{T}_s(t)$ on $\dot{B}^s$ by a density argument.

The strong continuity of $\mathcal{T}_{-s}(t)$ on $\dot{B}^{-s}$ can be proved analogously. □

Let us denote by $\mathcal{G}_s$ and $\mathcal{G}_{-s}$ the generators of $\mathcal{T}_s(t)$ and $\mathcal{T}_{-s}(t)$, respectively. In general, it is not easy to identify the domain of the generator of a semigroup. However, it would be easier to find a core for the generator.

Lemma 4.13. (i) *The domain $D(\mathcal{S}^2)$ of $\mathcal{S}^2$ is a core for $\mathcal{G}_s$, and it holds that $\mathcal{G}_s|_{D(\mathcal{S}^2)} = \mathcal{S}|_{D(\mathcal{S}^2)}$, that is, $\mathcal{G}_s$ is the closure of $\mathcal{S} : D(\mathcal{S}^2) \subset \dot{B}^s \to \dot{B}^s$.*

(ii) $\mathcal{G}_{-s}$ *is the closure of* $\mathcal{S} : D(\mathcal{S}) \cap R(\mathcal{S}) \subset \dot{B}^{-s} \to \dot{B}^{-s}$.

Proof. Note that $D(\mathcal{S}^2)$ is dense in $\dot{B}^s$ since $D(\mathcal{S}^2)$ is dense in $D(\mathcal{S})$ and $D(\mathcal{S})$ is dense in $\dot{B}^s$. For every $x \in D(\mathcal{S}^2)$, we have

$$\frac{1}{t}(\mathcal{T}_s(t)x - x) = \frac{1}{t}\int_0^t \mathcal{T}_s(\tau)\mathcal{S}x\, d\tau.$$

Letting $t \to 0^+$, the right side converges to $\mathcal{S}x$ in $D(\mathcal{S})$, thus, in $\dot{B}^s$. From this, we infer that $D(\mathcal{S}^2) \subset D(\mathcal{G}_s)$ and $\mathcal{G}_s|_{D(\mathcal{S}^2)} = \mathcal{S}|_{D(\mathcal{S}^2)}$. Obviously, $D(\mathcal{S}^2)$ is invariant under $\mathcal{T}_s(t)$. Thus, by Lemma 2.5, $D(\mathcal{S}^2)$ is a core for $\mathcal{G}_s$.

We prove the second part along the lines of the above proof. First, $D(\mathcal{S}) \cap R(\mathcal{S})$ is dense in $\dot{B}^{-s}$ since $D(\mathcal{S}) \cap R(\mathcal{S})$ is dense in $(R(\mathcal{S}), \|\cdot\|_{R(\mathcal{S})})$ and $R(\mathcal{S})$ is dense in $\dot{B}^{-s}$. Next, we can show that $D(\mathcal{S}) \cap R(\mathcal{S}) \subset D(\mathcal{G}_{-s})$ and $\mathcal{G}_{-s}|_{D(\mathcal{S})\cap R(\mathcal{S})} = \mathcal{S}|_{D(\mathcal{S})\cap R(\mathcal{S})}$. Moreover, since $D(\mathcal{S}) \cap R(\mathcal{S})$ is invariant under $\mathcal{T}_{-s}(t)$, so it is a core for $\mathcal{G}_{-s}$. This completes the proof. □

Lemma 4.14. $\{\mathcal{T}_s(t)\}_{t\geq 0}$ *(resp.,* $\{\mathcal{T}_{-s}(t)\}_{t\geq 0}$*) is a bounded analytic semigroup on* $\dot{B}^s$ *(resp.,* $\dot{B}^{-s}$*).*

Proof. We know from Lemma 4.13 that $D(\mathcal{S}^2)$ is dense in $\dot{B}^s$, and that $\mathcal{G}_s\mathcal{T}_s(t)x = \mathcal{S}e^{t\mathcal{S}}x$ for $x \in D(\mathcal{S}^2)$. It then follows from (4.27) that $\|t\mathcal{G}_s\mathcal{T}_s(t)x\|_{\dot{B}^s} \leq C\|x\|_{\dot{B}^s}$ for every $t > 0$. So $\mathcal{T}_s(t)$ is a bounded analytic semigroup. An analogous argument gives the analyticity of $\mathcal{T}_{-s}(t)$ on $\dot{B}^{-s}$. □

Remark 4.15. By Fatou's lemma, now (4.29) (resp., (4.30)) actually holds for data in $\dot{B}^s$ (resp., $\dot{B}^{-s}$). In particular, choosing $k = 0$, we have

$$\|\mathcal{G}_s e^{t\mathcal{G}_s}x\|_{L^1(\mathbb{R}_+,\dot{B}^s)} \leq C\|x\|_{\dot{B}^s}, \;\; \forall x \in \dot{B}^s \tag{4.31}$$

and

$$\|\mathcal{G}_{-s} e^{t\mathcal{G}_{-s}}x\|_{L^1(\mathbb{R}_+,\dot{B}^{-s})} \leq C\|x\|_{\dot{B}^{-s}}, \;\; \forall x \in \dot{B}^{-s}. \tag{4.32}$$

Next, we take advantage of Lemma 4.14, (4.31) and (4.32) to obtain the maximal L^1 regularity for the abstract Cauchy problems

$$u'(t) - \mathcal{G}_s u(t) = f(t), \ \ u(0) = x \tag{4.33}$$

and

$$u'(t) - \mathcal{G}_{-s} u(t) = f(t), \ \ u(0) = x. \tag{4.34}$$

Theorem 4.16. *Assume Assumption 4.1. Let $s \in (0,2)$ and $T \in (0,\infty]$. There exists a constant $C = C(s)$ such that*

(i) *For any $x \in \dot{B}^s$ and $f \in L^1((0,T);\dot{B}^s)$, the equation* (4.33) *has a unique strong solution $u \in C([0,T);\dot{B}^s)$ satisfying*

$$\|u\|_{L^\infty_T(\dot{B}^s)} + \|u', \mathcal{G}_s u\|_{L^1_T(\dot{B}^s)} \le C\|x\|_{\dot{B}^s} + C\|f\|_{L^1_T(\dot{B}^s)}.$$

(ii) *For any $x \in \dot{B}^{-s}$ and $f \in L^1((0,T);\dot{B}^{-s})$, the equation* (4.34) *has a unique strong solution $u \in C([0,T);\dot{B}^{-s})$ satisfying*

$$\|u\|_{L^\infty_T(\dot{B}^{-s})} + \|u', \mathcal{G}_{-s} u\|_{L^1_T(\dot{B}^{-s})} \le C\|x\|_{\dot{B}^{-s}} + C\|f\|_{L^1_T(\dot{B}^{-s})}.$$

Proof. Let us only give the proof of the first part. The homogeneous part $e^{t\mathcal{G}_s}x$ is a classical solution to the homogeneous equation, and satisfies the desired estimate by Lemma 4.14 and (4.31). Denote the inhomogeneous part by $\mathcal{I}f(t) = \int_0^t e^{(t-\tau)\mathcal{G}_s} f(\tau)\,d\tau$. Then it is easy to see that $\|\mathcal{I}f\|_{L^\infty_T(\dot{B}^s)} \lesssim \|f\|_{L^1_T(\dot{B}^s)}$. Using again (4.31) and Fubini's theorem, we have

$$\begin{aligned}
\|\mathcal{G}_s\mathcal{I}f\|_{L^1_T(\dot{B}^s)} &\le \int_0^T \int_0^t \|\mathcal{G}_s e^{(t-\tau)\mathcal{G}_s} f(\tau)\|_{\dot{B}^s}\,d\tau\,dt \\
&= \int_0^T d\tau \int_\tau^T \|\mathcal{G}_s e^{(t-\tau)\mathcal{G}_s} f(\tau)\|_{\dot{B}^s}\,dt \lesssim \|f\|_{L^1_T(\dot{B}^s)}.
\end{aligned}$$

So by Lemma 2.6, $u = e^{t\mathcal{G}_s}x + \mathcal{I}f(t)$ is a strong solution to (4.33). The estimate for u' follows directly by the previous estimates and the equation (4.33). So the proof is completed. □

Now we consider the abstract Cauchy problem (4.1) associated with a composite operator of the form $\mathcal{S} = \mathcal{B}\mathcal{A}$, where $\mathcal{B}$ is a bounded invertible operator and $\mathcal{A}$ is an unbounded operator. Our analysis relies on an abstract version of Theorem 3.10. Such a result has been demonstrated useful for studying density-dependent fluids.

We start with some assumptions.

Assumption 4.2. The linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ generates a bounded analytic semigroup $e^{t\mathcal{A}}$ satisfying $\lim_{t\to\infty} \|e^{t\mathcal{A}}x\| = 0$ for every $x \in X$.

Assumption 4.3. $\mathcal{B} \in \mathscr{L}(X)$ is invertible with an inverse $\mathcal{B}^{-1} \in \mathscr{L}(X)$.

Assumption 4.4. $\mathcal{S} = \mathcal{B}\mathcal{A} : D(\mathcal{A}) \subset X \to X$ generates a bounded analytic semigroup $e^{t\mathcal{S}}$ satisfying $\lim_{t\to\infty} \|e^{t\mathcal{S}}x\| = 0$ for every $x \in X$.

Lemma 4.17. *Under Assumptions 4.2-4.4, it holds for any* $(s, q) \in (0, 1) \times [1, \infty]$ *and* $x \in X$ *that*

$$\left\| t^s \|e^{t\mathcal{S}}x\| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \simeq \left\| t^s \|e^{t\mathcal{A}}\mathcal{B}^{-1}x\| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})}. \tag{4.35}$$

Consequently, we have for any $x \in D(\mathcal{A})$*,*

$$\left\| t^{-s} \|t\mathcal{S}e^{t\mathcal{S}}x\| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \simeq \left\| t^{-s} \|t\mathcal{A}e^{t\mathcal{A}}x\| \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})}. \tag{4.36}$$

Proof. By Assumption 4.2, we have for any $x \in X$ that

$$x = -\lim_{\varepsilon\to 0^+} \int_\varepsilon^{1/\varepsilon} \mathcal{A}e^{\tau\mathcal{A}}x \, d\tau,$$

where the limit converges in X. Replacing x by $\mathcal{B}^{-1}x$ gives

$$\mathcal{B}^{-1}x = -\int_0^\infty \mathcal{A}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x \, d\tau.$$

Applying $e^{t\mathcal{S}}\mathcal{B}$ to both sides of the above identity, we obtain

$$e^{t\mathcal{S}}x = -\int_0^\infty e^{t\mathcal{S}}\mathcal{B}\mathcal{A}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\,d\tau.$$

We can bound the integrand in two different ways:

$$\|e^{t\mathcal{S}}\mathcal{B}\mathcal{A}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\| = \|e^{t\mathcal{S}}\mathcal{S}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\| \lesssim \frac{1}{t}\|e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\| \lesssim \frac{1}{t}\|e^{\frac{\tau}{2}\mathcal{A}}\mathcal{B}^{-1}x\|,$$

or,

$$\|e^{t\mathcal{S}}\mathcal{B}\mathcal{A}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\| \lesssim \|\mathcal{A}e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\| \lesssim \frac{1}{\tau}\|e^{\frac{\tau}{2}\mathcal{A}}\mathcal{B}^{-1}x\|.$$

So we arrive at

$$\|e^{t\mathcal{S}}x\| \lesssim \int_0^\infty \frac{1}{t\vee\tau}\|e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\|\,d\tau.$$

Multiplying both sides by t^s, we get

$$t^s\|e^{t\mathcal{S}}x\| \lesssim \int_0^\infty \left(\frac{t}{\tau}\right)^s \left(1\wedge\frac{\tau}{t}\right)\tau^s\|e^{\tau\mathcal{A}}\mathcal{B}^{-1}x\|\,\frac{d\tau}{\tau}.$$

Since $s\in(0,1)$, it is easy to verify that

$$\sup_{t>0}\int_0^\infty \left(\frac{t}{\tau}\right)^s \left(\frac{\tau}{t}\wedge 1\right)\frac{d\tau}{\tau} + \sup_{\tau>0}\int_0^\infty \left(\frac{t}{\tau}\right)^s \left(\frac{\tau}{t}\wedge 1\right)\frac{dt}{t} \le C.$$

It then follows from Lemma 3.16 that

$$\left\|t^s\|e^{t\mathcal{S}}x\|\right\|_{L^q(\mathbb{R}_+,\frac{dt}{t})} \lesssim \left\|t^s\|e^{t\mathcal{A}}\mathcal{B}^{-1}x\|\right\|_{L^q(\mathbb{R}_+,\frac{dt}{t})}.$$

The reverse inequality can be proved in a similar way. By Assumption 4.4, we have for any $x \in X$ that

$$x = -\int_0^\infty \mathcal{B}\mathcal{A}e^{\tau\mathcal{S}}x\,d\tau.$$

This time we apply $e^{t\mathcal{A}}\mathcal{B}^{-1}$ to both sides of the above identity to get

$$e^{t\mathcal{A}}\mathcal{B}^{-1}x = -\int_0^\infty e^{t\mathcal{A}}\mathcal{A}e^{\tau\mathcal{S}}x\,d\tau.$$

So bounding the integrand in two different ways as before gives rise to

$$\|e^{t\mathcal{A}}\mathcal{B}^{-1}x\| \lesssim \int_0^\infty \frac{1}{t\vee\tau}\|e^{\tau\mathcal{S}}x\|\,d\tau.$$

This can further imply that

$$\left\|t^s\|e^{t\mathcal{A}}\mathcal{B}^{-1}x\|\right\|_{L^q(\mathbb{R}_+,\frac{dt}{t})} \lesssim \left\|t^s\|e^{t\mathcal{S}}x\|\right\|_{L^q(\mathbb{R}_+,\frac{dt}{t})}.$$

Thus, we have verified (4.35).

Finally, (4.36) follows by replacing x by $\mathcal{S}x$ in (4.35). □

We assume additionally that

Assumption 4.5. $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is one-to-one.

So $\mathcal{S}$ satisfies Assumption 4.1. Then the equivalence of norms implies the equivalence of spaces. More precisely, we get immediately from Lemma 4.17 that

Corollary 4.18. *Let $s \in (0,2)$. Under Assumptions 4.2-4.5, we have*

(i) $\dot{B}^{s,\mathcal{S}}_{X,1} = \dot{B}^{s,\mathcal{A}}_{X,1}$ *with equivalent norms.*

(ii) $\dot{B}^{-s,\mathcal{S}}_{X,1}$ *coincides with the completion of $R(\mathcal{S})$ with respect to the norm $\|\mathcal{B}^{-1}\cdot\|_{\dot{B}^{-s,\mathcal{A}}_{X,1}}$, where the spaces and norms associated with $\mathcal{A}$ are defined in an obvious way.*

It turns out that the operator $\mathcal{B}$ acting on $\dot{B}^{-s,\mathcal{A}}_{X,1}$ is meaningful. Indeed, (4.35) implies that $\mathcal{B}|_{R(\mathcal{A})}$ extends to a continuous operator, denoted by $\overline{\mathcal{B}}$, from $\dot{B}^{-s,\mathcal{A}}_{X,1}$ to $\dot{B}^{-s,\mathcal{S}}_{X,1}$; and that

$\mathcal{B}^{-1}|_{R(\mathcal{S})}$ extends to a continuous operator, denoted by $\overline{\mathcal{B}^{-1}}$, from $\dot{B}^{-s,\mathcal{S}}_{X,1}$ to $\dot{B}^{-s,\mathcal{A}}_{X,1}$. Obviously, $\overline{\mathcal{B}}$ is invertible and $\overline{\mathcal{B}}^{-1} = \overline{\mathcal{B}^{-1}}$. These facts can help us identify $\mathcal{G}_{-s}$ in the following

Lemma 4.19. *Assuming Assumptions 4.2-4.5, then the operator*

$$\mathcal{A} : D(\mathcal{A}) \cap R(\mathcal{S}) \subset \dot{B}^{-s,\mathcal{S}}_{X,1} \to \dot{B}^{-s,\mathcal{A}}_{X,1}$$

is closable. Moreover, we have $\mathcal{G}_{-s} = \overline{\mathcal{B}}\,\overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the closure of the above $\mathcal{A}$.

Proof. We see from Lemma 4.13 (ii) that $\mathcal{G}_{-s}$ is the closure of

$$\mathcal{B}\mathcal{A} : D(\mathcal{A}) \cap R(\mathcal{S}) \subset \dot{B}^{-s,\mathcal{S}}_{X,1} \to \dot{B}^{-s,\mathcal{S}}_{X,1}.$$

It follows that $\overline{\mathcal{A}} := \overline{\mathcal{B}}^{-1}\mathcal{G}_{-s}$ is the closure of

$$\mathcal{A} : D(\mathcal{A}) \cap R(\mathcal{S}) \subset \dot{B}^{-s,\mathcal{S}}_{X,1} \to \dot{B}^{-s,\mathcal{A}}_{X,1}.$$

This completes the proof. □

We conclude this section with the maximal L^1 regularity for the Cauchy problem

$$\overline{\mathcal{B}}^{-1}u'(t) - \overline{\mathcal{A}}u(t) = f(t), \;\; u(0) = x. \tag{4.37}$$

Theorem 4.20. *Let $s \in (0,2)$ and $T \in (0,\infty]$. Assuming Assumptions 4.2-4.5, if $x \in \dot{B}^{-s,\mathcal{S}}_{X,1}$ and $f \in L^1((0,T); \dot{B}^{-s,\mathcal{A}}_{X,1})$, then* (4.37) *has a unique strong solution u in the class*

$$u \in C([0,T); \dot{B}^{-s,\mathcal{S}}_{X,1}), \; u' \in L^1((0,T); \dot{B}^{-s,\mathcal{S}}_{X,1}), \; \overline{\mathcal{A}}u \in L^1((0,T); \dot{B}^{-s,\mathcal{A}}_{X,1}).$$

Moreover, it holds that

$$\|\overline{\mathcal{B}}^{-1}u\|_{L^\infty_T(\dot{B}^{-s,\mathcal{A}}_{X,1})} + \|\overline{\mathcal{B}}^{-1}u', \overline{\mathcal{A}}u\|_{L^1_T(\dot{B}^{-s,\mathcal{A}}_{X,1})} \le C\|\overline{\mathcal{B}}^{-1}x\|_{\dot{B}^{-s,\mathcal{A}}_{X,1}} + C\|f\|_{L^1_T(\dot{B}^{-s,\mathcal{A}}_{X,1})},$$

where C depends on s, $\|\mathcal{B}\|_{\mathscr{L}(X)}$ and $\|\mathcal{B}^{-1}\|_{\mathscr{L}(X)}$.

Proof. Note that $\overline{\mathcal{B}}f \in L^1((0,T); \dot{B}^{-s,\mathcal{S}}_{X,1})$. Thanks to the continuity of $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}^{-1}$, and Lemma 4.19, then Theorem 4.20 follows by applying Theorem 4.16 (ii) to the Cauchy problem

$$u'(t) - \overline{\mathcal{B}}\,\overline{\mathcal{A}}u(t) = \overline{\mathcal{B}}f(t), \;\; u(0) = x.$$

□

4.4 Concrete examples

In this section, we apply the abstract theory to two concrete examples. The linear system to be considered reads

$$\begin{cases} \rho\partial_t u - \mathcal{A}u = f, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0) = u_0, & \text{on } \mathbb{R}^n, \end{cases} \tag{4.38}$$

where the coefficient ρ is a time-independent function satisfying (4.3), and $\mathcal{A}$ is either the Laplacian Δ or the Lamé operator $\mathcal{L}$ defined by (2.6). We denote $b = \rho^{-1}$. From now on, we always assume

Assumption 4.6. $n \geq 2$ if $\mathcal{A} = \Delta$, or $n \in \{2, 3\}$ if $\mathcal{A} = \mathcal{L}$.

We choose $X = L^p = L^p(\mathbb{R}^n; \mathbb{R}^n)$ $(1 < p < \infty)$, $D(\mathcal{A}) = W^{2,p} = W^{2,p}(\mathbb{R}^n; \mathbb{R}^n)$, and $\mathcal{S} = b\mathcal{A}$. Obviously, Assumptions 4.3 and 4.5 are satisfied. That $\mathcal{A}$ satisfies Assumption 4.2 is a classical result (see, e.g., [6, Example 3.7.6]). That $b\Delta : W^{2,p} \subset L^p \to L^p$ satisfies Assumption 4.4 was essentially proved in [31, 46]. Analogously, we can use Lemma 4.1, Lemma 4.6 and Remark 4.8 to show that $b\mathcal{L}$ satisfies Assumption 4.4 as well.

Let us identify the spaces $\dot{B}^{\pm s,\mathcal{A}}_{X,1}$. Let $s \in (0, 2)$. We know from Lemmas 2.13 and 2.14 that the $\dot{B}^{-s,\mathcal{A}}_{X,1}$-norm is equivalent to the Besov $\dot{B}^{-s}_{p,1}$-norm. One can see from (2.8) and (2.9) that $R(\Delta) = R(\mathcal{L})$. It is however easy to see that $R(\Delta)$ is dense in $\dot{B}^{-s}_{p,1}$. So $\dot{B}^{-s,\mathcal{A}}_{X,1}$ is identified as $\dot{B}^{-s}_{p,1}$ for every $s \in (0, 2)$. To identify $\dot{B}^{s,\mathcal{A}}_{X,1}$, we assume additionally $s \leq \frac{n}{p}$ so that $\dot{B}^{s}_{p,1}$ is complete. Then applying Corollary 4.18 (i), Lemmas 2.13 and 2.14, and the obvious fact that $D(\mathcal{A}) = W^{2,p}$ is dense in $\dot{B}^{s}_{p,1}$, we get $\dot{B}^{s,\mathcal{S}}_{X,1} = \dot{B}^{s,\mathcal{A}}_{X,1} = \dot{B}^{s}_{p,1}$.

We now turn to the central problem of this section, that is, the maximal L^1 regularity for (4.38). In view of Theorem 4.16 (i) and Lemma 4.13 (i), the smooth solutions to (4.38) should satisfy the *a priori* estimate

$$\|u\|_{L^\infty_T(\dot{B}^s_{p,1})} + \|u', b\mathcal{A}u\|_{L^1_T(\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|bf\|_{L^1_T(\dot{B}^s_{p,1})}.$$

But if ρ merely satisfies (4.3), we can not handle the inhomogeneous term, nor can we obtain the estimate for $\|\mathcal{A}u\|_{L^1_T(\dot{B}^s_{p,1})}$. Solving (4.38) in Besov spaces with negative regularity seems to be a more promising way to lower the regularity of the density. In fact, from Theorem 4.20, the *a priori* estimate for smooth solutions becomes

$$\|\rho u\|_{L^\infty_T(\dot{B}^{-s}_{p,1})} + \|\rho u', \mathcal{A}u\|_{L^1_T(\dot{B}^{-s}_{p,1})} \lesssim \|\rho u_0\|_{\dot{B}^{-s}_{p,1}} + \|f\|_{L^1_T(\dot{B}^{-s}_{p,1})}.$$

Unfortunately, the above is not quite true if u is only a strong solution.

By Corollary 4.18 (ii), the space $\dot{B}^{-s,\mathcal{S}}_{X,1}$ agrees with the completion of $(R(\mathcal{S}), \|\rho\cdot\|_{\dot{B}^{-s}_{p,1}})$. Then the multiplication by ρ extends to a bounded operator, still denoted by ρ, from $\dot{B}^{-s,\mathcal{S}}_{X,1}$ to $\dot{B}^{-s}_{p,1}$ with a bounded inverse that coincides with the extension of the multiplication by b. By Lemma 4.19, the operator

$$\mathcal{A} : W^{2,p} \cap b\mathcal{A}(W^{2,p}) \subset \dot{B}^{-s,\mathcal{S}}_{X,1} \to \dot{B}^{-s}_{p,1}$$

is closable, and we denote its closure by $\overline{\mathcal{A}}$. Then One can directly interpret Theorem 4.20 as follows:

Corollary 4.21. *Let $s \in (0,2)$ and $T \in (0,\infty]$. If $u_0 \in \dot{B}^{-s,\mathcal{S}}_{X,1}$ and $f \in L^1((0,T); \dot{B}^{-s}_{p,1})$, then* (4.38) *has a unique strong solution u in the class*

$$u \in C([0,T); \dot{B}^{-s,\mathcal{S}}_{X,1}),\ \partial_t u \in L^1((0,T); \dot{B}^{-s,\mathcal{S}}_{X,1}),\ \overline{\mathcal{A}}u \in L^1((0,T); \dot{B}^{-s}_{p,1}).$$

Moreover, there exists some constant $C = C(s, m, \mu, \nu)$ such that

$$\|\rho u\|_{L^\infty_T(\dot{B}^{-s}_{p,1})} + \|\rho u', \overline{\mathcal{A}}u\|_{L^1_T(\dot{B}^{-s}_{p,1})} \leq C\|\rho u_0\|_{\dot{B}^{-s}_{p,1}} + C\|f\|_{L^1_T(\dot{B}^{-s}_{p,1})}.$$

Unfortunately, it is not clear whether $\|\nabla u\|_\infty$ can be bounded by $\|\overline{\mathcal{A}}u\|_{\dot{B}^{n/p-1}_{p,1}}$ for $n < p < \infty$. Note that an element in $\dot{B}^{-s,\mathcal{S}}_{X,1}$ might not even be a distribution. So Theorem 4.16 and Corollary 4.21 may be too abstract to be useful in applications if one insists to work in an L^1-in-time framework. For this, we require a little more regularity on the coefficients.

Lemma 4.22. (i) *Let $p \in (1, \infty)$ and $s \in (0, 2) \cap (0, \frac{n}{p}]$. Assume that $\rho, b \in \mathscr{M}(\dot{B}^s_{p,1})$. Then $\mathcal{G}_s$ coincides with the operator*

$$b\mathcal{A} : \dot{B}^s_{p,1} \cap \dot{B}^{2+s}_{p,1} \subset \dot{B}^s_{p,1} \to \dot{B}^s_{p,1}. \tag{4.39}$$

(ii) *Let $p \in (1, \infty)$ and $s \in (0, 2)$. Assume that $\rho, b \in \mathscr{M}(\dot{B}^{-s}_{p,1})$. Then the space $\dot{B}^{-s,\mathcal{S}}_{X,1}$ coincides with $\dot{B}^{-s}_{p,1}$, and the operator $\overline{\mathcal{A}}$ is given by*

$$\mathcal{A} : \dot{B}^{2-s}_{p,1} \cap \dot{B}^{-s}_{p,1} \subset \dot{B}^{-s}_{p,1} \to \dot{B}^{-s}_{p,1}. \tag{4.40}$$

Proof. (i) First, along the same lines of the proof of Lemma 4.13 (i), we can show that $\mathcal{G}_s$ is the closure of

$$\mathcal{S} : \{u \in D(\mathcal{S}) | \mathcal{S}u \in \dot{B}^s_{p,1}\} \subset \dot{B}^s_{p,1} \to \dot{B}^s_{p,1}. \tag{4.41}$$

Since $\rho, b \in \mathscr{M}(\dot{B}^s_{p,1})$, we can identify $\{u \in D(\mathcal{S}) | \mathcal{S}u \in \dot{B}^s_{p,1}\} = \{u \in W^{2,p} | b\mathcal{A}u \in \dot{B}^s_{p,1}\}$ as the inhomogeneous Besov space $B^{2+s}_{p,1} = L^p \cap \dot{B}^{2+s}_{p,1}$. On the other hand, it is easy to see that the operator $b\mathcal{A}$ defined in (4.39) is closed and is an extension of the operator $\mathcal{S}$ defined in (4.41). The desired result then follows from the fact that $B^{2+s}_{p,1}$ is dense in $\dot{B}^s_{p,1} \cap \dot{B}^{2+s}_{p,1}$.

(ii) Let us first refine several results in Section 4.3. Using (4.35) and the fact that $R(\mathcal{A}) = \mathcal{A}(W^{2,p})$ is dense in $\dot{B}^{-s}_{p,1}$, we can verify that $\dot{B}^{-s,\mathcal{S}}_{X,1}$ agrees with the completion of

$$\mathscr{D}_{-s} := \{u \in L^p | \|\rho u\|_{\dot{B}^{-s}_{p,1}} < \infty\}$$

with respect to the norm $\|\rho \cdot \|_{\dot{B}^{-s}_{p,1}}$. Then (4.28) holds for every $u \in \mathscr{D}_{-s}$, so $\mathcal{T}_{-s}(t)$ is the continuous extension of $e^{t\mathcal{S}}|_{\mathscr{D}_{-s}}$ to $\dot{B}^{-s,\mathcal{S}}_{X,1}$. From this, we can follow the same lines as the proof of Lemma 4.13 (ii) to show that $\mathcal{G}_{-s}$ is the closure of

$$b\mathcal{A} : W^{2,p} \cap \dot{B}^{-s,\mathcal{S}}_{X,1} \subset \dot{B}^{-s,\mathcal{S}}_{X,1} \to \dot{B}^{-s,\mathcal{S}}_{X,1}.$$

Now assuming $\rho, b \in \mathscr{M}(\dot{B}^{-s}_{p,1})$, it is easy to see that $\dot{B}^{-s,\mathcal{S}}_{X,1}$ coincides with $\dot{B}^{-s}_{p,1}$. So $\rho\mathcal{G}_{-s}$ is the closure of

$$\mathcal{A} : W^{2,p} \cap \dot{B}^{-s}_{p,1} \subset \dot{B}^{-s}_{p,1} \to \dot{B}^{-s}_{p,1}.$$

But it is not difficult to see that the closure of the above operator is the one defined by (4.40). This completes the proof. □

Finally, we obtain a concrete version of maximal L^1 regularity for (4.38).

Theorem 4.23. *Let $p \in (1, \infty)$, $s \in (0, 2)$ and $T \in (0, \infty]$. Let ρ satisfy* (4.3) *and $b = \rho^{-1}$.*

(i) *Assume that $s \leq \frac{n}{p}$ and $\rho, b \in \mathscr{M}(\dot{B}^{s}_{p,1})$. Then for $u_0 \in \dot{B}^{s}_{p,1}$ and $f \in L^1((0,T); \dot{B}^{s}_{p,1})$, the equation* (4.38) *has a unique strong solution $u \in C([0,T); \dot{B}^{s}_{p,1})$ satisfying*

$$\|u\|_{L^\infty_T(\dot{B}^{s}_{p,1})} + \|\partial_t u, b\mathcal{A}u\|_{L^1_T(\dot{B}^{s}_{p,1})} \leq C\|u_0\|_{\dot{B}^{s}_{p,1}} + C\|bf\|_{L^1_T(\dot{B}^{s}_{p,1})}$$

for some constant C depending on s, m, μ and ν.

(ii) *Assume $\rho, b \in \mathscr{M}(\dot{B}^{-s}_{p,1})$. If $u_0 \in \dot{B}^{-s}_{p,1}$ and $f \in L^1((0,T); \dot{B}^{-s}_{p,1})$, then* (4.38) *has a unique strong solution $u \in C([0,T); \dot{B}^{-s}_{p,1})$ satisfying*

$$\|\rho u\|_{L^\infty_T(\dot{B}^{-s}_{p,1})} + \|\rho\partial_t u, \mathcal{A}u\|_{L^1_T(\dot{B}^{-s}_{p,1})} \leq C\|\rho u_0\|_{\dot{B}^{-s}_{p,1}} + C\|f\|_{L^1_T(\dot{B}^{-s}_{p,1})} \tag{4.42}$$

for some constant C depending on s, m, μ and ν.

Proof. The first part follows from Theorem 4.16 (i), the equivalence between $\dot{B}^{s,\mathcal{S}}_{X,1}$ and $\dot{B}^{s}_{p,1}$, and Lemma 4.22 (i). The second part follows from Corollary 4.21 and Lemma 4.22 (ii). □

4.5 An application to pressureless flows

In this section, we study the global-in-time well-posedness for the pressureless flow

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \rho(\partial_t u + u\cdot\nabla u) - \mathcal{L}u = 0, & \text{in } (0,\infty)\times\mathbb{R}^n, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & \text{on } \mathbb{R}^n, \end{cases} \tag{4.43}$$

where $\mathcal{L}$ is the Lamé operator defined in (2.6) with coefficients satisfying (2.7). The structure of our proof is in the spirit of the one established in [20]. But the substantial progress we make is the removal of the smallness assumption on the fluctuation of the initial density.

In this section, we always assume that

Assumption 4.7. Let $n \in \{2,3\}$, $p \in (1,2n)\setminus\{n\}$, ρ_0 satisfy (4.3), $u_0 \in \dot{B}^{n/p-1}_{p,1} = (\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))^n$, and $\rho_0, \rho_0^{-1} \in \mathscr{M}(\dot{B}^{n/p-1}_{p,1})$.

Let us be clear about what it means by a solution to the system (4.43).

Definition 4.24. The unknown (ρ, u) is called a global-in-time solution to (4.43) if

$$\rho \in L^\infty(\mathbb{R}_+\times\mathbb{R}^n)\cap L^\infty(\mathbb{R}_+;\mathscr{M}(\dot{B}^{n/p-1}_{p,1})),$$
$$u \in C([0,\infty);\dot{B}^{n/p-1}_{p,1}),\ (\partial_t u, \mathcal{L}u) \in \left(L^1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\right)^2,$$

ρ is a weak solution to the continuity equation of (4.43) (see Definition 2.18), (ρ, u) satisfies the momentum equation of (4.43) for a.e. $t\in(0,\infty)$, $u(0)=u_0$, and $\rho(t) \overset{*}{\rightharpoonup} \rho_0$ in $L^\infty(\mathbb{R}^n)$ as $t\to 0^+$.

The main result in the section is the following

Theorem 4.25. *Assuming Assumption 4.7, there exists a positive constant c depending on $m, p, n, \mu, \nu, \|\rho_0\|_{\mathscr{M}(\dot{B}^{n/p-1}_{p,1})}$ and $\|\rho_0^{-1}\|_{\mathscr{M}(\dot{B}^{n/p-1}_{p,1})}$ such that if $\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c$, then* (4.43) *has a unique global-in-time solution.*

Remark 4.26. The above theorem holds without constraint on the dimensions if $\mathcal{L}$ is replaced by Δ.

Firstly, we shall convert (4.43) into its Lagrangian formulation. Let $X(t,y) = X_u(t,y)$ be the trajectory of the velocity field u. Recall the notations $A(t,y) = \big(D_y X(t,y)\big)^{-1}$, $J(t,y) = \det DX(t,y)$, and $\mathscr{A}(t,y) = \operatorname{adj} DX(t,y)$. Then introduce new unknowns in Lagrangian coordinates and define

$$(\boldsymbol{\rho}, \mathbf{u})(t,y) = (\rho, u)\big(t, X(t,y)\big). \tag{4.44}$$

The continuity equation in (4.43) has a unique weak solution $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ such that $J\boldsymbol{\rho} \equiv \rho_0$ (see Theorem 2.19). Using (2.18), (2.19) and the chain rule, one can formally convert the system (4.43) into its Lagrangian formulation that reads

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \mu \operatorname{div}(\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^{\intercal} \nabla \mathbf{u}) - (\mu + \lambda) \mathscr{A}_{\mathbf{u}}^{\intercal} \nabla \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}) = 0, \\ \mathbf{u}|_{t=0} = u_0, \end{cases} \tag{4.45}$$

where we associate $\mathscr{A}_{\mathbf{u}}$ and $A_{\mathbf{u}}$ with the new velocity $\mathbf{u}$, namely,

$$\mathscr{A}_{\mathbf{u}} = \operatorname{adj} DX_{\mathbf{u}}, \ \text{ and } \ A_{\mathbf{u}} = (DX_{\mathbf{u}}(t,y))^{-1}$$

with

$$X_{\mathbf{u}}(t,y) = y + \int_0^t \mathbf{u}(\tau, y)\, d\tau.$$

We shall prove the well-posedness of the highly nonlinear system (4.45) using the contraction mapping theorem. In order to apply the linear theory established in Theorem 4.23, we shall rewrite (4.45) as

$$\rho_0 \partial_t \mathbf{u} - \mathcal{L}\mathbf{u} = f(\mathbf{u}),$$

where

$$f(\mathbf{u}) = \mu \operatorname{div}((\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^{\intercal} - I_n)\nabla \mathbf{u}) + (\mu+\lambda)\{(\mathscr{A}_{\mathbf{u}}^{\intercal} - I_n)\nabla \mathrm{Tr}(A_{\mathbf{u}} D\mathbf{u}) + \nabla \mathrm{Tr}((A_{\mathbf{u}} - I_n) D\mathbf{u})\}.$$

To bound the above nonlinear term, we invoke (2.25) and the product laws in Besov spaces to get

$$\|f(\mathbf{v})\|_{L^1(\dot{B}^{n/p-1}_{p,1})} \lesssim \|\nabla \mathbf{v}\|^2_{L^1(\dot{B}^{n/p}_{p,1})} \tag{4.46}$$

whenever $\mathbf{v}$ satisfies (2.24).

Again, in view of Theorem 4.23, we shall perform the contraction mapping theorem in the Banach space E_p defined as

$$E_p := \Big\{\mathbf{u} \in C_b([0,\infty); \dot{B}^{n/p-1}_{p,1}) | \partial_t \mathbf{u} \in L^1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}), \mathbf{u} \in L^1(\mathbb{R}_+; \dot{B}^{n/p+1}_{p,1})\Big\}$$

endowed with the norm

$$\|\mathbf{u}\|_{E_p} := \|\mathbf{u}\|_{L^\infty(\dot{B}^{n/p-1}_{p,1})} + \|\partial_t \mathbf{u}, \mathcal{L}\mathbf{u}\|_{L^1(\dot{B}^{n/p-1}_{p,1})}.$$

Now we can prove the global-in-time well-posedness for (4.45).

Theorem 4.27. *Assuming Assumption 4.7, there exists a positive constant c depending on $m, p, n, \mu, \nu, \|\rho_0\|_{\mathscr{M}(\dot{B}^{n/p-1}_{p,1})}$ and $\|\rho_0^{-1}\|_{\mathscr{M}(\dot{B}^{n/p-1}_{p,1})}$ such that if $\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \le c$, then* (4.45) *has a unique global-in-time strong solution $\mathbf{u} \in E_p$ satisfying $\|\mathbf{u}\|_{E_p} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}}$.*

Proof. For $r > 0$, let $E_p(r)$ denote the closed ball in E_p centered at $u = 0$ with radius r. We shall construct a contraction mapping on $E_p(r)$ by solving the linearized system

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \mathcal{L}\mathbf{u} = f(\mathbf{v}), \\ \mathbf{u}|_{t=0} = u_0, \end{cases} \tag{4.47}$$

where the input $\mathbf{v} \in E_p(r)$. To bound the inhomogeneous term, we require r to be small so that

$$\|\nabla \mathbf{v}\|_{L^1(\dot{B}^{n/p}_{p,1})} \leq C\|\mathcal{L}\mathbf{v}\|_{L^1(\dot{B}^{n/p-1}_{p,1})} \leq C_1 r \leq c_0.$$

This then implies (4.46).

Now, applying Theorem 4.23, we can solve (4.47) for a strong solution $\mathbf{u} \in E_p$ satisfying

$$\|\mathbf{u}\|_{E_p} \leq C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + C\|f(\mathbf{v})\|_{L^1(\dot{B}^{n/p-1}_{p,1})} \leq C_2\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + C_2 r^2.$$

To ensure that the mapping $\mathbf{v} \mapsto \mathbf{u}$ is a self-map on $E_p(r)$, we need

$$r \leq \frac{c_0}{C_1} \wedge \frac{1}{2C_2}$$

and

$$\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq \frac{r}{2C_2}.$$

Next, we need to show the contraction property of the mapping $\mathbf{v} \mapsto \mathbf{u}$. Given $\mathbf{v}_1, \mathbf{v}_2 \in E_p(r)$, let $\mathbf{u}_1, \mathbf{u}_2 \in E_p(r)$ be the corresponding solutions to (4.47). As before, for two quantities q_1 and q_2, we always denote by δq their difference $q_1 - q_2$. Then applying Theorem 4.23 to the system satisfied by $\delta\mathbf{u}$, we obtain

$$\|\delta\mathbf{u}\|_{E_p} \leq C\|f(\mathbf{v}_1) - f(\mathbf{v}_2)\|_{L^1(\dot{B}^{n/p-1}_{p,1})}.$$

We write

$$
\begin{aligned}
f(\mathbf{v}_1) - f(\mathbf{v}_2) =& \mu \operatorname{div}((\mathscr{A}_1 A_1^\intercal - I)\nabla\delta\mathbf{v}) + \mu \operatorname{div}((\mathscr{A}_1 A_1^\intercal - \mathscr{A}_2 A_2^\intercal)\nabla\mathbf{v}_2) \\
& + (\mu+\lambda)(\mathscr{A}_1^\intercal - I)\nabla\mathrm{Tr}(A_1 D\delta\mathbf{v}) + (\mu+\lambda)(\mathscr{A}_1^\intercal - I)\nabla\mathrm{Tr}(\delta A D\mathbf{v}_2) \\
& + (\mu+\lambda)(\delta\mathscr{A})^\intercal\nabla\mathrm{Tr}(A_2 D\mathbf{v}_2) + (\mu+\lambda)\nabla\mathrm{Tr}((A_1 - I)D\delta\mathbf{v}) \\
& + (\mu+\lambda)\nabla\mathrm{Tr}(\delta A D\mathbf{v}_2),
\end{aligned}
$$

where $\mathscr{A}_i = \mathscr{A}_{\mathbf{v}_i}$ and $A_i = A_{\mathbf{v}_i}$, $i = 1, 2$. Applying (2.25), (2.27) and product laws in Besov spaces, we arrive at

$$
\|f(\mathbf{v}_1) - f(\mathbf{v}_2)\|_{L^1(\dot{B}^{n/p-1}_{p,1})} \leq C\|\nabla\mathbf{v}_1, \nabla\mathbf{v}_2\|_{L^1(\dot{B}^{n/p}_{p,1})}\|\nabla\delta\mathbf{v}\|_{L^1(\dot{B}^{n/p}_{p,1})}.
$$

We thus infer

$$
\|\delta\mathbf{u}\|_{E_p} \leq C\|\nabla\mathbf{v}_1, \nabla\mathbf{v}_2\|_{L^1(\dot{B}^{n/p}_{p,1})}\|\nabla\delta\mathbf{v}\|_{L^1(\dot{B}^{n/p}_{p,1})} \leq C_3 r\|\delta\mathbf{v}\|_{E_p},
$$

from which we see that $\|\delta\mathbf{u}\|_{E_p} \leq \frac{1}{2}\|\delta\mathbf{v}\|_{E_p}$ if $r \leq \frac{1}{2C_3}$.

Finally, we choose

$$
r = \frac{c_0}{C_1} \wedge \frac{1}{2C_2} \wedge \frac{1}{2C_3} \text{ and } c = \frac{r}{2C_2}.
$$

Then the mapping $\mathbf{v} \mapsto \mathbf{u}$ is a contraction on $E_p(r)$, thus, admits a unique fixed point $\mathbf{u} \in E_p(r)$, which is a solution to (4.45) in E_p. The proof of the uniqueness of strong solutions in E_p is similar to the proof of the contraction property of the mapping $\mathbf{v} \mapsto \mathbf{u}$. This completes the proof of the theorem. □

Remark 4.28. For $n < p < 2n$, in view of (4.42), one can prove the global well-posedness under the assumption that $\|\rho_0 u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c$ with c only depending on m, p, n, μ, ν.

Let us finish the proof of Theorem 4.25.

Proof of Theorem 4.25. The proof is similar to that of Theorem 3.2. □

4.6 An application to heat-conductive compressible Navier–Stokes

In this section, we study the global well-posedness of the three-dimensional heat-conductive compressible Navier-Stokes equations

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ \rho(\partial_t u + u\cdot\nabla u) - \mathcal{L}u + \nabla P = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ c_v\rho(\partial_t\theta + u\cdot\nabla\theta) - \kappa\Delta\theta = \dfrac{\mu}{2}|\nabla u + Du|^2 + \lambda(\operatorname{div} u)^2 - P\operatorname{div} u, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ (\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0), & \text{on } \mathbb{R}^3. \end{cases} \tag{4.48}$$

Compared with (4.43), a pressure term is added in the momentum equation of (4.48), and the temperature equation (i.e., the θ equation) is included. $\mathcal{L}$ is the Lamé operator defined in (2.6) with coefficients satisfying (2.7). c_v and κ are positive constants. For a perfect gas, a good approximation of the pressure P is provided by *Boyle's law*

$$P = P(\rho,\theta) = R\rho\theta,$$

where R is a constant.

There is a vast amount of literature that is devoted to the well-posedness of (4.48), but we will review a few of the results that are most relevant to ours. Note that (4.48) is invariant under the scaling

$$(\rho,u,\theta)(t,x) \rightsquigarrow (\rho,u,\theta)_\lambda(t,x) := (\rho,\lambda u,\lambda^2\theta)(\lambda^2 t,\lambda x).$$

Based on this scaling-invariance, Chikami and Danchin [14] proved the local well-posedness of the full compressible Navier-Stokes equations with variable coefficients in critical Besov spaces, but the global well-posedness was not covered in their paper. Global well-posedness of strong solutions close to the equilibrium $(\rho,u,\theta) = (1,0,1)$ in Sobolev spaces was first proved in [45]. This result was generalized in [38] to allow initial vacuum. The smallness assumption on the initial data in [38] is imposed as

$$\int \left(\frac{1}{2}\rho_0|u_0|^2 + R(\rho_0\log\rho_0 - \rho_0 + 1) + c_v\rho_0(\theta_0 - \log\theta_0 - 1)\right)\,dx \ll 1,$$

where ρ_0 and θ_0 are bounded and nonnegative functions. Note that this assumption implies that

$$\|\rho_0 - 1\|_2 + \|\theta_0 - 1\|_2 \ll 1.$$

Global well-posedness in the presence of a vacuum without smallness assumption on the fluctuation of ρ_0 was recently obtained by Li [42]. For more results concerning the well-posedness of (4.48), we refer the reader to the references in [42].

The main result in this section reads

Theorem 4.29. *Let* $p \in (1,3) \setminus \{\frac{3}{2}\}$. *Assume that* ρ_0 *satisfies* (1.4) *and* $(\rho_0 - 1, u_0, \theta_0) \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)\times\dot{B}^{3/p-2}_{p,1}(\mathbb{R}^3)$. *There exists a constant* ε_0 *depending on* $p, m, \mu, \lambda, c_v, \kappa, R$ *and* $\|\rho_0 - 1\|_{\dot{B}^{3/p}_{p,1}}$ *such that if*

$$\|u_0\|_{\dot{B}^{3/p-1}_{p,1}} + \|\theta_0\|_{\dot{B}^{3/p-2}_{p,1}} \le \varepsilon_0, \tag{4.49}$$

then (4.48) *has a unique solution* (ρ, u, θ) *satisfying*

$$\begin{aligned}\rho - 1 \in C_b([0,\infty);\dot{B}^{3/p}_{p,1}),\ u \in C_b([0,\infty);\dot{B}^{3/p-1}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{3/p+1}_{p,1}),\\ \theta \in C_b([0,\infty);\dot{B}^{3/p-2}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{3/p}_{p,1})\end{aligned}$$

and

$$\begin{aligned}&\|u\|_{L^\infty(\dot{B}^{3/p-1}_{p,1})} + \|\partial_t u, \Delta u\|_{L^1(\dot{B}^{3/p-1}_{p,1})} + \|\theta\|_{L^\infty(\dot{B}^{3/p-2}_{p,1})} + \|\partial_t\theta, \Delta\theta\|_{L^1(\dot{B}^{3/p-2}_{p,1})}\\ \le& C\left(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}} + \|\theta_0\|_{\dot{B}^{3/p-2}_{p,1}}\right).\end{aligned} \tag{4.50}$$

Remark 4.30. Compared with [42], we cannot allow an initial vacuum, but the coefficients μ and λ in our result satisfies (2.7) only, while the assumption $2\mu > \lambda$ was assumed in [42].

Remark 4.31. Roughly, the underlying reason why Theorem 4.29 holds is that $(\rho, u, \theta) = (\rho_0, 0, 0)$ is an equilibrium of (4.48). So one can expect global well-posedness of (4.48) under the assumption (4.49) but without smallness assumption on the fluctuation of ρ_0. It is for the same reason that our method does not work for the barotropic compressible Navier-Stokes equations because $(\rho, u) = (\rho_0, 0)$ is not an equilibrium.

Remark 4.32. If the initial data is smooth, it is quite standard to derive from (4.50) more time-independent estimates for some higher norms of the solution (ρ, u, θ) and decay estimates for u and θ.

Remark 4.33. It follows from (4.50) that $\|\nabla u\|_{L^1(L^\infty)} + \|\theta\|_{L^1(L^\infty)} \ll 1$. By the Cauchy-Lipschiz theorem, we infer that $\frac{|X(t,y_1)-X(t,y_2)|}{|y_1-y_2|} \approx 1$ for any $t \geq 0$ and $y_1, y_2 \in \mathbb{R}^n$. So the microscopic motion of the gas particles is quite stable despite the large density oscillation. This result is in agreement with physics intuition: the motion of the gas particles is inactive in a low-temperature environment. Nevertheless, the L^∞ norm of the initial temperature can be large.

In the rest of this section, we will just outline the proof of Theorem 4.29 by following the lines of the proof of Theorem 4.25.

Step 1. Lagrangian formulation of (4.48)**.** As in Section 4.5, we here introduce variables in Lagrangian coordinates:

$$(\boldsymbol{\rho}, \mathbf{u}, \boldsymbol{\theta})(t, y) = (\rho, u, \theta)\big(t, X(t, y)\big).$$

Then (4.48) can be reformulated as

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \mu \operatorname{div}(\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^{\mathsf{T}} \nabla \mathbf{u}) - (\mu + \lambda) \mathscr{A}_{\mathbf{u}}^{\mathsf{T}} \nabla \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}) + R \operatorname{div}(\rho_0 \boldsymbol{\theta} A_{\mathbf{u}}) = 0, \\ c_v \rho_0 \partial_t \boldsymbol{\theta} - \kappa \operatorname{div}(\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^{\mathsf{T}} \nabla \boldsymbol{\theta}) = \mu \operatorname{Tr}[(A_{\mathbf{u}}^{\mathsf{T}} \nabla \mathbf{u} + D\mathbf{u} A_{\mathbf{u}}) \mathscr{A}_{\mathbf{u}}^{\mathsf{T}} \nabla \mathbf{u}] \\ \qquad\qquad + \lambda \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}) \operatorname{Tr}(\mathscr{A}_{\mathbf{u}} D\mathbf{u}) - R \rho_0 \boldsymbol{\theta} \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}), \\ (\mathbf{u}, \boldsymbol{\theta})|_{t=0} = (u_0, \theta_0). \end{cases} \tag{4.51}$$

We shall rewrite the above system as

$$\begin{cases} \rho_0 \partial_t \mathbf{u} - \mathcal{L}\mathbf{u} + \nabla(R\rho_0 \boldsymbol{\theta}) = f(\mathbf{u}, \boldsymbol{\theta}), \\ c_v \rho_0 \partial_t \boldsymbol{\theta} - \kappa \Delta \boldsymbol{\theta} = g(\mathbf{u}, \boldsymbol{\theta}), \\ (\mathbf{u}, \boldsymbol{\theta})|_{t=0} = (u_0, \theta_0), \end{cases}$$

where

$$f(\mathbf{u}, \boldsymbol{\theta}) = \mu \operatorname{div}((\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^\intercal - I_n)\nabla \mathbf{u}) + (\mu + \lambda)(\mathscr{A}_{\mathbf{u}}^\intercal \nabla \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}) - \nabla \operatorname{div} \mathbf{u}) - R \operatorname{div}(\rho_0 \boldsymbol{\theta}(A_{\mathbf{u}} - I_n))$$

and

$$\begin{aligned} g(\mathbf{u}, \boldsymbol{\theta}) = &\kappa \operatorname{div}((\mathscr{A}_{\mathbf{u}} A_{\mathbf{u}}^\intercal - I_n)\nabla \boldsymbol{\theta}) + \mu \operatorname{Tr}[(A_{\mathbf{u}}^\intercal \nabla \mathbf{u} + D\mathbf{u} A_{\mathbf{u}})\mathscr{A}_{\mathbf{u}}^\intercal \nabla \mathbf{u}] \\ &+ \lambda \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}) \operatorname{Tr}(\mathscr{A}_{\mathbf{u}} D\mathbf{u}) - R\rho_0 \boldsymbol{\theta} \operatorname{Tr}(A_{\mathbf{u}} D\mathbf{u}). \end{aligned}$$

Step 2. Estimates for linear system. Next, we need the maximal L^1 regularity for the following linear system

$$\begin{cases} \rho_0 \partial_t u - \mathcal{L}u + \nabla(R\rho_0 \theta) = f, \\ c_v \rho_0 \partial_t \theta - \kappa \Delta \theta = g, \\ (u, \theta)|_{t=0} = (u_0, \theta_0). \end{cases} \tag{4.52}$$

As a consequence of Theorem 4.23, we have the following

Proposition 4.34. *Let* $p \in (1, 3) \setminus \{\frac{3}{2}\}$. *Assume that* ρ_0 *satisfies* (1.4) *and* $(\rho_0 - 1, u_0, \theta_0) \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p-2}_{p,1}(\mathbb{R}^3)$. *Then* (4.52) *has a unique solution* (u, θ) *satisfying*

$$u \in C_b([0, \infty); \dot{B}^{3/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{3/p+1}_{p,1}),\ \theta \in C_b([0, \infty); \dot{B}^{3/p-2}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{3/p}_{p,1})$$

and

$$
\begin{aligned}
&\|u\|_{L^\infty(\dot B^{3/p-1}_{p,1})}+\|\partial_t u,\Delta u\|_{L^1(\dot B^{3/p-1}_{p,1})}+\|\theta\|_{L^\infty(\dot B^{3/p-2}_{p,1})}+\|\partial_t\theta,\Delta\theta\|_{L^1(\dot B^{3/p-2}_{p,1})}\\
\le &C\left(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|\theta_0\|_{\dot B^{3/p-2}_{p,1}}+\|f\|_{L^1(\dot B^{3/p-1}_{p,1})}+\|g\|_{L^1(\dot B^{3/p-2}_{p,1})}\right),
\end{aligned}
$$

where C is a constant depending on $p, m, \mu, \lambda, c_v, \kappa, R$ and $\|\rho_0-1\|_{\dot B^{3/p}_{p,1}}$.

Remark 4.35. The assumption $p<3$ guarantees that both ρ_0 and ρ_0^{-1} are multipliers of $\dot B^s_{p,1}(\mathbb{R}^3)$ for $s=3/p, 3/p-1$ and $3/p-2$.

Step 3. Estimates of nonlinearities. According to the product laws in Besov spaces and Lemma 2.25, we have

$$
\|f(\mathbf{u},\boldsymbol{\theta})\|_{L^1(\dot B^{3/p-1}_{p,1})}\lesssim\|\nabla\mathbf{u}\|^2_{L^1(\dot B^{3/p}_{p,1})}+\|\nabla\mathbf{u}\|_{L^1(\dot B^{3/p}_{p,1})}\|\rho_0\boldsymbol{\theta}\|_{L^1(\dot B^{3/p}_{p,1})}, \tag{4.53}
$$

and

$$
\|g(\mathbf{u},\boldsymbol{\theta})\|_{L^1(\dot B^{3/p-2}_{p,1})}\lesssim\|\nabla\mathbf{u}\|_{L^1(\dot B^{3/p}_{p,1})}\left(\|\boldsymbol{\theta}\|_{L^1(\dot B^{3/p}_{p,1})}+\|\mathbf{u}\|_{L^\infty(\dot B^{3/p-1}_{p,1})}+\|\rho_0\boldsymbol{\theta}\|_{L^\infty(\dot B^{3/p-2}_{p,1})}\right), \tag{4.54}
$$

provided that $1\le p<3$ and $\mathbf{u}$ satisfies (2.23) and (2.24) with $n=3$. Let $(\mathbf{u}_i,\boldsymbol{\theta}_i)$, $i=1,2$, satisfy the same assumptions as $(\mathbf{u},\boldsymbol{\theta})$. Then we have

$$
\begin{aligned}
\|f(\mathbf{u}_1,\boldsymbol{\theta}_1)-f(\mathbf{u}_2,\boldsymbol{\theta}_2)\|_{L^1(\dot B^{3/p-1}_{p,1})}\lesssim&\|\nabla\mathbf{u}_1,\nabla\mathbf{u}_2,\rho_0\boldsymbol{\theta}_2\|_{L^1(\dot B^{3/p}_{p,1})}\|\nabla\delta\mathbf{u}\|_{L^1(\dot B^{3/p}_{p,1})}\\
&+\|\nabla\mathbf{u}_1\|_{L^1(\dot B^{3/p}_{p,1})}\|\rho_0\delta\boldsymbol{\theta}\|_{L^1(\dot B^{3/p}_{p,1})},
\end{aligned} \tag{4.55}
$$

and

$$
\begin{aligned}
&\|g(\mathbf{u}_1,\boldsymbol{\theta}_1)-g(\mathbf{u}_2,\boldsymbol{\theta}_2)\|_{L^1(\dot B^{3/p-2}_{p,1})}\\
\lesssim&\|\nabla\delta\mathbf{u}\|_{L^1(\dot B^{3/p}_{p,1})}\left(\|\boldsymbol{\theta}_2\|_{L^1(\dot B^{3/p}_{p,1})}+\|\mathbf{u}_1,\mathbf{u}_2\|_{L^\infty(\dot B^{3/p-1}_{p,1})}+\|\rho_0\boldsymbol{\theta}_2\|_{L^\infty(\dot B^{3/p-2}_{p,1})}\right)\\
&+\|\nabla\mathbf{u}_1\|_{L^1(\dot B^{3/p}_{p,1})}\left(\|\delta\boldsymbol{\theta}\|_{L^1(\dot B^{3/p}_{p,1})}+\|\rho_0\delta\boldsymbol{\theta}\|_{L^\infty(\dot B^{3/p-2}_{p,1})}\right).
\end{aligned} \tag{4.56}
$$

Step 4. Contraction mapping theorem. In view of Proposition 4.34 and the estimates (4.53)-(4.56), one can apply the contraction mapping theorem to prove the following

Theorem 4.36. *Let* $p \in (1,3) \setminus \{\frac{3}{2}\}$. *Assume that* ρ_0 *satisfies* (1.4) *and* $(\rho_0 - 1, u_0, \theta_0) \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p-2}_{p,1}(\mathbb{R}^3)$. *There exists a constant* ε_0 *depending on* $p, m, \mu, \lambda, c_v, \kappa, R$ *and* $\|\rho_0 - 1\|_{\dot{B}^{3/p}_{p,1}}$ *such that if*

$$\|u_0\|_{\dot{B}^{3/p-1}_{p,1}} + \|\theta_0\|_{\dot{B}^{3/p-2}_{p,1}} \le \varepsilon_0,$$

then (4.51) *has a unique solution* $(\mathbf{u}, \boldsymbol{\theta})$ *satisfying*

$$\mathbf{u} \in C_b([0,\infty); \dot{B}^{3/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{3/p+1}_{p,1}),\ \boldsymbol{\theta} \in C_b([0,\infty); \dot{B}^{3/p-2}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{3/p}_{p,1})$$

and

$$\begin{aligned}&\|\mathbf{u}\|_{L^\infty(\dot{B}^{3/p-1}_{p,1})} + \|\partial_t \mathbf{u}, \Delta \mathbf{u}\|_{L^1(\dot{B}^{3/p-1}_{p,1})} + \|\boldsymbol{\theta}\|_{L^\infty(\dot{B}^{3/p-2}_{p,1})} + \|\partial_t \boldsymbol{\theta}, \Delta \boldsymbol{\theta}\|_{L^1(\dot{B}^{3/p-2}_{p,1})}\\ \le &C\left(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}} + \|\theta_0\|_{\dot{B}^{3/p-2}_{p,1}}\right).\end{aligned}$$

Step 5. Back to Eulerian coordinates. Going back to the Eulerian coordinates, we finish the proof of Theorem 4.29.

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