Scattering Resonances for Three-dimensional Subwavelength Holes

by

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Abstract

This thesis aims to investigate scattering resonances and the field amplification at resonant frequencies for two different subwavelength structures: The first structure is a cavity with a closed bottom and width ε perforated in a slab of sound hard material. The second structure is a hollow hole with both sides open with an upper and lower aperture of width ε, embedded in a sound hard slab.

For both structures, we reformulate the boundary value problems by integral equations, and apply the asymptotic analysis and Gohberg-Sigal type theory to study the scattering resonances of the underlying differential operator. We prove the existence of scattering resonances, which are the set of complex-valued frequencies for the homogeneous problem with zero incident field, derive the asymptotic expansion of those resonances, and quantitatively analyze the field amplifications at the resonant frequencies for both cases.

It is shown that the complex-valued scattering resonances attain imaginary parts of order $O(\varepsilon^2)$ and the real part of order $O(1)$. We also show that, at the resonant frequencies, the field amplification inside the cavity(hole) and in the far field is of order $O(\frac{1}{\varepsilon^2})$. This is much stronger the enhancement order in the two-dimensional subwavelength hole of the same width, which attain order $O(\frac{1}{\varepsilon})$

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Chapter 1

Introduction

1.1 Light passing through subwavelength holes

Wave scattering by subwavelength apertures and holes has attracted a lot of attention in recent years due to its important applications in biological and chemical sensing [11, 13, 20, 21, 32, 33]. The so-called extraordinary optical transmission (EOT) through the holes provides a label-free and highly sensitive manner to detect biomolecular events efficiently. The EOT transmission anomaly is related to a variety of resonances of the underlying subwavelength structures [9,16,17,24–31,34]
.

The Extraordinary optical transmission (EOT) is an optical phenomenon in which light is passed through a structure perforated with a single or an array of different shapes of holes, transmits more light than expected and refers to the existence of resonances [21]. It was discovered by Thomos Ebbesen and his colleagues in [20] has induced extensive reasearch in plasmonic nanostructures and their applications. In the original experimental set up, an array of subwavelength holes were perforated in optically thick metallic film so that the optical waves would only transmit through the apertures. Generally a hole is perforated in a sheet of noble metal such as gold or silver with a prescribed shape and size.
The resonances induced by the subwavelength holes in metallic slabs can provoke extraordinary optical transmission EOT phenomenon [20].Through the experiment in [20] it was found out that these array of subwavelength holes transmit more light than that of a single hole with same area as the sum of all tiny holes. In physical context, to enhance the transmission through the periodic array of subwavelength hole,an aggregated response must occur.
As the study of EOT progressed these past years and the development of scientific apparatus, it is now possible to form the subwavelenth aperture with great precision to efficiently enhance the

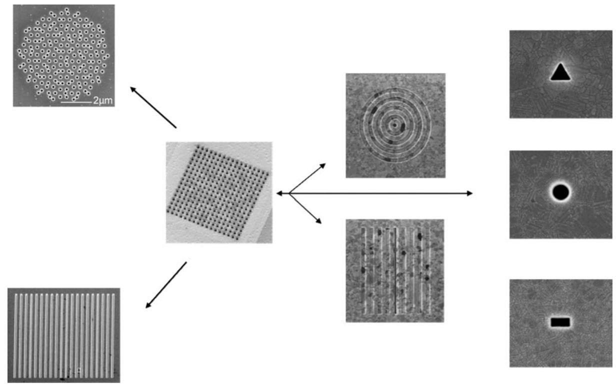

Figure 1.1: Various types of subwavelgth structure(left column) with different hole shapes (right column) [21].

wave field Figure 1.1. As depicted in Figure 1.1, EOT provides the anticipation of multitude of applications such as high transmission efficiency and local field amplification at wavelengths that can be manipulated through the geometry of the metal surface. Figure 1.3 demonstrates the field amplification through the nanohole depicted in the center of Figure 1.1

1.2 Scattering resonances

Scattering resonances emerge in different branches of physics, mathematics and engineering. Scattering resonances generalise characteristic values of the systems where energy can be scattered to infinity [35]. The mathematical formulation can be derived from the meromorphic continuation of the Green's function. The poles of the meromorphic continuation apprehend the physical information by associating the oscillation rate with the real part of the pole and decay rate by the imaginary part, which can be seen in Figure 1.4.

Let us consider the time-harmonic solution $v(x,t) = e^{-i\omega t}u(x)$ of the acoustic wave equation is $\frac{1}{c^2}\frac{\partial^2 v(x,t)}{\partial t^2} - \Delta v(x,t) = 0$.

Substituting into wave equation and simplifying gives,

$$\Delta u(x) + k^2 u(x) = 0$$

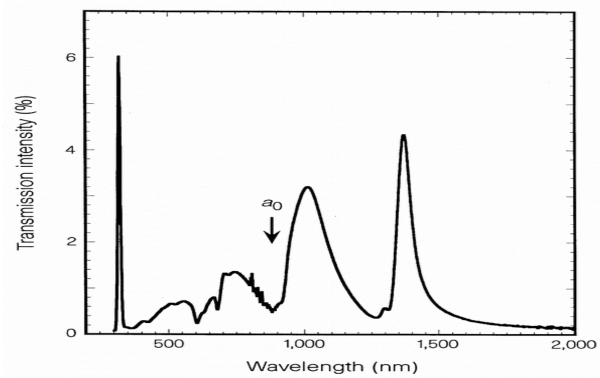

Figure 1.2: Extra ordinary optical transmission occurs at the wavelength near 1000 nm and 1500 nm through the square nanohole array sitting in the center of Figure 1.1. After [1]

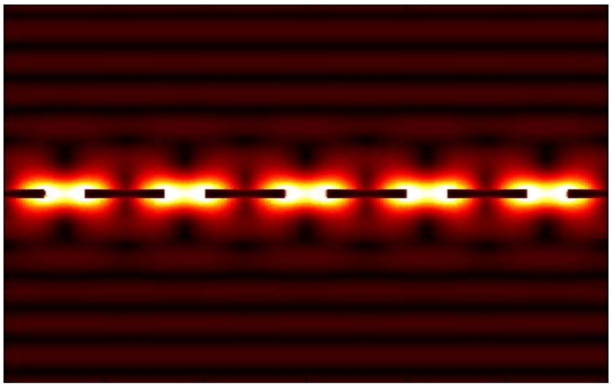

Figure 1.3: Cross sectional plot of strongly enhanced electric field inside the hole.

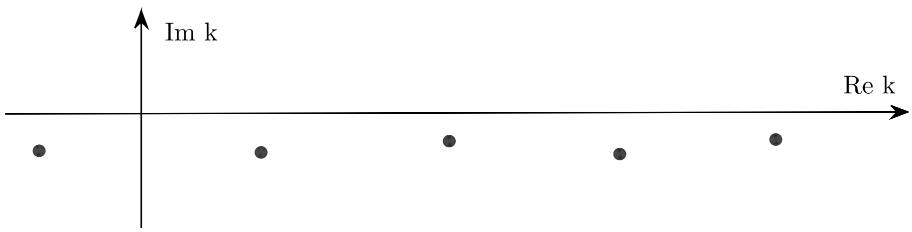

Figure 1.4: Representation of resonances with $Imk < 0$

where $k = \frac{\omega}{c}$ is wavenumber. The *scattering resonances* are the poles of the resolvent $R(k) := (\partial_x^2 + k^2)^{-1}$ when continued meromorphically to the whole complex plane. In a simple case, for the line, $\mathbb{R}$ assume that $R(k) := (\partial_x^2 + k^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator for $k \notin R$. Its integral represented is given as follow:

$$R(k)g(x) = \int_{\mathbb{R}} R(k, x, z)g(z)dz,$$

$$R(k, x, z) = \frac{i}{2k} e^{ik|x-z|}, \quad Imk > 0.$$

For the fixed value of x, z and with one pole at $k = 0$, the resolvent is a meromorphic function of k, such pole is called the scattering Resonance of ∂_x^2. To see its significance, we consider the wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big) w(t, x) = 0,$$
$$w(0, x) = 0, \quad \frac{\partial w(0, x)}{\partial t} = g(x),$$

where

$$w(t, x) = \frac{1}{2} \int_{x-t}^{x+t} g(z)dz.$$

satisfies the initial value problem If $g(x) = 0$ for $|x| > C$ then

$$w(t, x) = \frac{1}{2} \int_{\mathbb{R}} g(z)dz \quad \text{for} \quad t > |x| + C.$$

In context of resolvent $R(k, x, z)$, it can be seen as:

$$w(t, x) = -i \int_{\mathbb{R}} (Res_{k=0} R(k, x, z))g(z)dz, \quad \text{where} \quad Res(R(k, x, z)) = \frac{i}{2} \quad \text{at} \quad k = 0.$$

This equivalently says that the residue of R at the pole describes the long time behaviour of wave in (in t) in compact sets [35]. Consider wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x)\Big)w(t,x) = 0, \tag{1.2.1}$$

where V is the one dimensional compactly supported potential, the solution of (1.2.1) is given then in resonnace expansion instead of eigenfunction expansions. To explain this expansion, first consider $\Delta_V = \frac{d^2}{dx^2} + V$ on $[a,b]$ with the Dirichlet boundary condition. Then

$$\begin{cases} (\Delta_V - k^2)w = 0 \quad \text{on} \quad (a,b), \\ w(a) = w(b) = 0, \end{cases}$$

which has the set of solutions

$$(i\sqrt{-A_m}, u_m), \quad (k_j, w_j), \quad A_N < \dots A_1 < 0 < k_0^2 < k_1^2 < \dots \to \infty,$$

$$\int_a^b |w_j|^2 dx = \int_a^b |u_m|^2 dx = 1.$$

Then we can rewrite the wave equation (1.2.1) as

$$\begin{cases} (\frac{d^2}{dt^2} - \Delta_V)v = 0 \quad \mathbb{R} \times (a,b), \\ v(0,x) = v_0(x), \quad \frac{\partial v(0,x)}{\partial t} = v_1(x), \end{cases}$$

The solution of above system can be found by the eigenfunction expansion for $V \equiv 0$:

$$\begin{aligned} v(t,x) = \quad & \sum_{m=1}^{N} \cosh(t\sqrt{-A_m})a_m u_m(x) + \sum_{m=1}^{N} \frac{\sinh(t\sqrt{-A_m})}{\sqrt{-A_m}} b_m u_m(x) + \sum_{j=0}^{\infty} \cos(tk_j)c_j w_(x) \\ & + \sum_{j=0}^{\infty} \frac{\sin(tk_j)}{k_j} d_j w_j(x), \end{aligned} \tag{1.2.2}$$

where

$$a_m = \int_a^b v_0(x)\bar{u}_m(x)dx, \quad b_m = \int_a^b v_1(x)\bar{u}_m(x)dx,$$

$$c_j = \int_a^b v_0(x)\bar{w}_j(x)dx, \quad d_j = \int_a^b v_1(x)\bar{w}_j(x)dx.$$

The extension of (1.2.2), when $[a, b]$ is replaced by $\mathbb{R}$ is given explicitly by Theorem 2.9 in [35] or equivalently given by the following equation:

$$w(t, x) = \sum_{Imk_j > -M} e^{-ik_j t} w_j(x) + O(e^{-tB}), \quad x \in \mathbb{R} \tag{1.2.3}$$

The solution of the wave equation $(1.2.1)$ is calculated as the resonance expansion instead of the eigenfunction expansion and the negative complex numbers $Imk_j < 0$ represents the scattering resonances k_j,which are independent of the initial data.

1.3 Contribution of the dissertation

Significant progress has been made in the last several years on the quantitative analysis of the resonances as well as the induced enhanced transmission for the two-dimensional structures. The readers are referred to [9, 16, 17, 24–31, 34] for the detailed investigation of different resonant phenomena for several typical subwavelength structures. Other related mathematical studies on the subwavelength resonant wave scattering and their applications can be found in [4–7] and references therein.

In this thesis, we consider the three-dimensional problems and present quantitative analysis of scattering resonances for the acoustic wave scattering through subwavelength holes embedded in a sound hard slab. Two configurations are considered. The first one is a subwavelength cavity hole with a closed bottom, and the second one is a hollow subwavelgth hole throuh the slab.

In the cavity case, we reformulate the problem by a boundary integral equation then derive the asymptotic expansion of the Green's function inside the cavity and in the semi-infinite domain above the cavity aperture, followed by the asymptotic expansion of the boundary integral operators.

By using the simplified Gohberg-Sigal theory, we prove the existence of scattering resonances. We also study the wave field amplification when the frequency of the incident wave coincides with the real part of the complex-valued resonances. It is shown that the enhancement order is $O(1/\varepsilon^2)$. The study of scattering resonances through a hollow hole follow the same methodology but more complicated due to the opening of both upper and lower apertures. The scattering problem for the hole is then reformulated by a system of two boundary integral equations. The characteristic values of the integral operator can be decomposed as the union of the characteristic values of two scalar integral operator. We then perform the asymptotic expansion of the boundary integral operator and prove the existence of the scattering resonances, following the procedure in the cavity case. It is shown that the complex-valued scattering resonances attain imaginary parts of order $O(\varepsilon^2)$. We also analyze the field amplification at resonant frequencies and show that the enhancement is of order $O(1/\varepsilon^2)$ and it is much stronger than the enhancement in the two-dimensional hole, which attains an order of $O(1/\varepsilon)$ [26].

1.4 Outline

The rest of the dissertation is organized as follows. In Chapter 2, we preset preliminary concepts about fractional Sobolev spaces and single-layer and double-layer potentials. In Chapter 3, we investigate the configuration where a subwavelength cavity is embedded in a slab of sound hard medium. First we reformulate the scattering problem for the cavity by boundary integral equation, followed by the derivation of the asymptotic expansion of Green's functions and the scattering resonances, finishing chapter with the quantitative analysis of the field amplification at the resonant frequencies. In Chapter 4, following a parallel strategy as of Chapter 3, we derive the boundary integral representation of the the scattering problem for a subwavelength hole. Then we present the asymptotic expansions of the boundary integral operators and derive the asymptotic expansions of the scattering resonances. The quantitative analysis of the field enhancement at the even and odd resonant frequencies is then derived. Chapter 5 gives the proof of the invertibility of the integral operator K used in the quantitative analysis of resonances. Finally, Chapter 6 presents the conclusion of our work and plans for future work.

Chapter 2

Preliminaries

2.1 Sobolev spaces

Let D be convex and connected subset of $\mathbb{R}^n$, for any $s \in \mathbb{R}$ and for any $p \in [1, \infty)$. The Sobolev space $H^s(\mathbb{R}^n)$ is defined as follows

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n), (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n)\},$$

with norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \Big\| \mathcal{F}^{-1}\Big[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f\Big]\Big\|_{L^p(\mathbb{R}^n)}.$$

We also define

$$\tilde{H}^s(D) := \{u = V|_D \mid V \in H^s(\mathbb{R}^n) \quad and \quad suppV \subset \bar{D}\}.$$

The definition of Fractional Sobolev spaces is given as follows:

$$W^{s,p}(D) = H^s(D) := \Big\{ u \in L^p(D) : \frac{|u(x) - u(y)|^p}{|x - y|^{\frac{n}{p} + s}} \in L^p(D \times D) \Big\}, \tag{2.1.1}$$

or equivalently

$$H^s(D) := \{v = V|_D \mid V \in H^s(\mathbb{R}^n)\};$$

is Banach space with norm

$$\|u\|_{W^{s,p}(D)} := \Big(\int_D |u|^p + \int_D \int_D \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dxdy \Big)^{\frac{1}{p}}. \tag{2.1.2}$$

The space $\tilde{H}^s(D)$ is the dual of $H^{-s}(D)$ and the norm for $\tilde{H}^s(D)$ can be defined via duality [3].

The *dual space* of any topological vector space X is the space of all bounded linear functional on X.

2.2 Single-layer and double-layer potentials

We introduce single-layer and double-layer potentials in this section. The readers are referred to [19] for detailed discussions about the layer potential theory. Consider the Helmholtz equation

$$\Delta u + k^2 u = 0$$

for k a complex number such that $\mathrm{Im} k \geq 0$. Then the function

$$\Phi(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y, \tag{2.2.3}$$

is the fundamental solution to the Helmholtz equation with respect to x for any fixed y. Let the bounded set D be an open complement of an unbounded domain of class C^2. Let φ be an integrable function and $\Phi(x,y)$ is a function defined above in the domain D, then the integrals

$$u(x) := \int_{\partial D} \varphi(y)\Phi(x,y)ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D$$

and

$$v(x) := \int_{\partial D} \varphi(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D$$

are called the acoustic single-layer and acoustic double-layer potentials respectively with density φ. u and v satisfy the Sommerfeld radiation condition and they are the solutions to the Helmholtz equation in D and in $\mathbb{R}^3 \backslash \partial D$. The solution to Helmholtz can be represented by the combination of single-layer and double-layer potentials [10]. We refer readers to [19] for further details on single-layer and double-layer potentials, here we collect useful results to used in this thesis. [19].

Theorem 2.2.1 *([19], p. 35)For the twice continuosly differentiable surface ∂D there exists a positive constant L such that*

$$|\nu(y), x - y| \leq L||x - y|^2 \tag{2.2.4}$$

and

$$|\nu(x) - \nu(y)| \leq L|x - y| \tag{2.2.5}$$

for all $x, y \in \partial D$.

Theorem 2.2.2 *([19], p. 47) The double-layer potential v with density function φ can be continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with limiting values*

$$v_{\pm}(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \pm \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{2.2.6}$$

where

$$v_{\pm}(x) := \lim_{h \to +0} v(x \pm h\nu(x))$$

with improper integral.

Proof. ([19], p.47) First, we prove that the integral (2.2.6) is an improper integral, we consider the case when $k = 0$. From

$$\frac{\partial \Phi_0(x, y)}{\partial \nu(y)} = \frac{(\nu(y), x - y)}{4\pi |x - y|^3} \tag{2.2.7}$$

and Theorem 2.2.1,

we observe that

$$\Big|\frac{\partial \Phi_0(x, y)}{\partial \nu(y)}\Big| \leq \frac{L}{4\pi |x - y|}, \quad x, y \in \partial D, \quad x \neq y. \tag{2.2.8}$$

Hence $(2.2.6)$ exists as an improper integral.

Consider the double-layer potential

$$w(x) = \int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D \tag{2.2.9}$$

with density function $\varphi = 1$. From Gauss theorem, we see that

$$w(x) = \begin{cases} o, & x \in \mathbb{R}^3 \backslash \bar{D}, \\ 1, & x \in D. \end{cases} \tag{2.2.10}$$

By applying Gauss theorem again gives

$$\begin{aligned} \int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} ds(y) = & \lim_{r \to 0} \int_{H_{s,r}} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} ds(y) \\ = & \lim_{r \to 0} \frac{-1}{4\pi r^2} \int_{H_{s,r}} ds(y), \quad x \in \partial D \end{aligned}$$

where $H_{x,r}$ denotes that part of the surface of the sphere $\Omega_{x,r}$ of radius r and the center x that is contained in D and where ν denotes the exterior unit normal to this sphere. By Theorem $2.2.1$, it can be seen that

$$\int_{H_{s,r}} ds(y) = 2\pi r^2 + O(r^3)$$

uniformly on ∂D. Hence

$$\int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} ds(y) = -\frac{1}{2}, \quad x \in \partial D, \tag{2.2.11}$$

which concludes the case of constant density.

Let the parallel surfaces ∂D_h to ∂D by the representation

$$x = z + h\nu(z), \quad z \in \partial D$$

where h is the distance of ∂D_h from the generating surface ∂D. Let us choose a positive number h_0 such that the parallel surfaces defined above are well defined for all $|h| \leq h_0$ and define the set as follows $D_{h_0} := \{x = z + h\nu(z), \quad z \in \partial D, \quad |h| \leq h_0\}$. For the arbitrary continuous density in $D_{h_0} \backslash \partial D$, we first write v in the form

$$v(x) = \varphi(z)w(z) + u(x), \quad x = z + h\nu(z), \quad x \in D_{h_0} \backslash \partial D \tag{2.2.12}$$

where

$$u(x) = \int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} [\varphi(y) - \varphi(z)] ds(y). \tag{2.2.13}$$

To prove the theorem, we must show that u is continuous on D_{h_0}. By $(5.0.5)$ and Theorem 2.6 in [10], the integral above exists as an improper integral for $x \in \partial D$ and represents a continuous function on ∂D. This is enough to show that

$$\lim_{x \to z} u(x) = \lim u(z + h\nu(z)) = u(z), \quad z \in \partial D,$$

uniformly on ∂D.

Using Theorem $2.2.1$, we have the estimate

$$\begin{aligned} |x-y|^2 = & \quad |z-y|^2 + 2(z-y, x-z) + |x-z|^2 \\ \geq & \quad \frac{1}{2}\{|z-y|^2 + |x-z|^2\} \end{aligned}$$

provided h_0 is sufficiently small. Then

$$4\pi \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} = \frac{(\nu(y), z-y)}{|x-y|^3} + \frac{(\nu(y), x-z)}{|x-y|^3},$$

for $r < R$, by projecting onto the tangent plane give

$$\begin{aligned}\int_{S_{z,r}} \Big|\frac{\partial \Phi_0(x,y)}{\partial \nu(y)}\Big| ds(y) \leq \quad & C_1 \Big\{ \int_0^r d\rho + |x-z| \int_0^\infty \frac{\rho}{(\rho^2 + |x-z|^2)^{3/2}} d\rho \Big\} \\ = \quad & C_1(r+1) \\ \leq \quad & C_1(R+1) \end{aligned} \tag{2.2.14}$$

where C_1 denotes some constant depending on ∂D. From the mean value theorem, we see that

$$\Big|\frac{\partial \Phi_0(x,y)}{\partial \nu(y)} - \frac{\partial \Phi_0(z,y)}{\partial \nu(y)}\Big| \leq C_2 \frac{|x-z|}{|z-y|^3}$$

for $2|x-z| \leq |z-y|$ and therefore

$$\int_{\partial D \setminus S_{z,r}} \Big|\frac{\partial \Phi_0(x,y)}{\partial \nu(y)} - \frac{\partial \Phi_0(z,y)}{\partial \nu(y)}\Big| ds(y) \leq C_3 \frac{|x-z|}{r^3} \tag{2.2.15}$$

for some constants C_2 and C_3. By combining (2.2.14) and (2.2.15), we get

$$|u(x) - u(z)| \leq C \Big\{ \sup_{|y-z| \leq} r|\varphi(y) - \varphi(z)| + \frac{|x-z|}{r^3} \Big\} \tag{2.2.16}$$

for some constant C. Given $\varepsilon > 0$, we can choose $r > 0$ such that

$$|\varphi(y) - \varphi(z)| < \frac{\varepsilon}{2C}$$

for all $y, z \in \partial D$ with $|y-z| < r$, since φ is uniformly continuous on ∂D. Then by taking $\delta < \frac{\varepsilon}{2Cr^3}$, we see that

$$|u(x) - u(z)| < \varepsilon$$

for all $|x-z| < \delta$. □

Theorem 2.2.3 *([19], p.53) For single-layer potential u with continuous density function φ*

$$u(x) = \int_{\partial D} \varphi(y)\Phi(x,y)ds(y), \quad x \in \partial D, \tag{2.2.17}$$

we have

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y) \mp \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{2.2.18}$$

where

$$\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h\to+0} \nu(x) \cdot \mathrm{grad} u(x \pm h\nu(x))$$

is taken in context of uniform convergence on ∂D and where integrals exists as improper integrals [19].

Proof. Let v be the double-layer potential with φ density function. Let us consider $\varphi = 1$, then by the symmetry of the fundamental solution, we can write

$$(\nu(z)), gradu(x)) + v(x) = \int_{\partial D} (\nu(y) - \nu(z), grad_y \Phi(x,y))\varphi(y)ds(y),$$

where x is defined in (2.2.12). By applying Theorem 2.2.2 to the double-layer potential and Theorem 2.7 in [10], completes the proof.

Theorem 2.2.4 *([19], p.54) The double-layer potential v with continuous density function φ satisfies*

$$\lim_{h\to+0} \left\{ \frac{\partial v}{\partial \nu}(x + h\nu(x)) - \frac{\partial v}{\partial \nu}(x - h\nu(x)) \right\} = 0, \quad x \in \partial D. \tag{2.2.19}$$

We refer reader to [**?**] for the detailed proof of this theorem. □

Chapter 3

Scattering resonances for a three-dimensional subwavelength cavity

3.1 Problem formulation

In this chapter, we present the quantitative analysis of scattering resonances for the acoustic wave scattering by a subwavelength cavity and the wave field amplification at the resonant frequencies.

The cavity is perforated in a semi-infinite slab of sound hard material, and its geometry is presented in Figure 3.1. The slab occupies the domain $\{(x_1.x_2, x_3) \mid -\infty < x_3 < L\}$, and the cavity shown is a rectangular cuboid $C_\varepsilon := \{(x_1.x_2, x_3) \mid 0 < x_1 < \varepsilon, 0 < x_2 < \varepsilon, 0 < x_3 < L\}$. We consider the case when the width ε is much smaller than the height of the slab and wavelength of the incident wave λ that is $\varepsilon \ll L \sim \lambda$. Without loss of generality, in what follows we scale the geometry of the problem by assuming that the cavity deepness is $L = 1$. Let us denote the aperture of the cavity by Γ_ε, and semi-infinite domain above the slab by Ω_ε^+. The exterior domain is given by $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup C_\varepsilon \cup \Gamma_\varepsilon$.

We consider the scattering when the plane wave u^i is incident upon the structure, where $u^i = e^{ik(d(x \cdot x_0))}$ is the incident field, $(d_1, d_2, -d_3)$ is the incident direction with $d_3 > 0$, k is the wave number, and $x_0 = (0, 0, L)$. In the absence of cavity, the total field in the domain Ω_ε^+, consists of the incident field u^i and reflected field $u^r = e^{ik(d'(x \cdot x_0))}$ where $d' = (d_1, d_2, d_3)$.

In the presence of cavity C_ε, the total field u_ε, in the upper domain Ω^+ consists of u^i, u^r and the scattered field u_ε^s radiating from Γ_ε. In addition, the Neumann boundary condition $\partial_\nu u_\varepsilon = 0$ is imposed on $\partial\Omega_\varepsilon$ for the sound hard material, where ν is the unit outward normal pointing to Ω_ε. The scattered field u_ε^s satisfy the Sommerfeld radiation condition at the semi infinite domains. The total field satisfy the scattering problem:

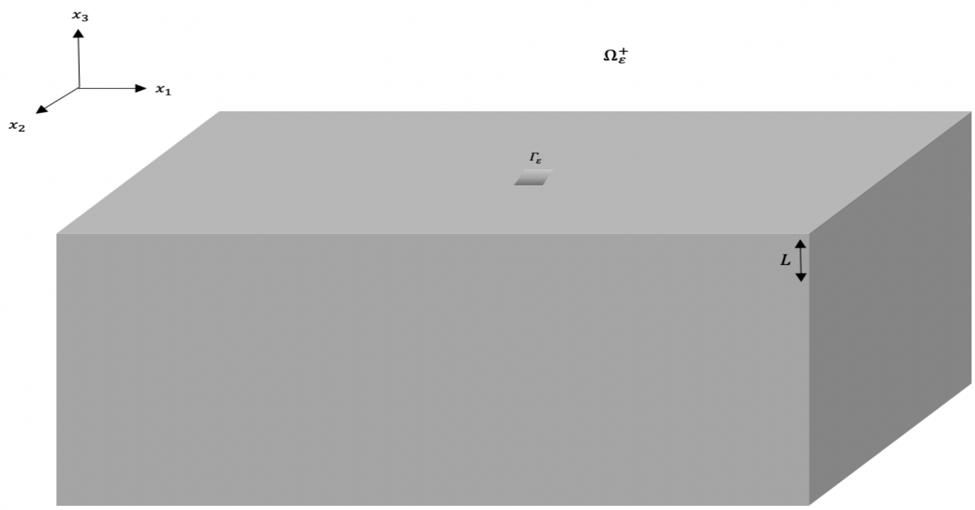

Figure 3.1: Geometry of the problem. The cavity C_ε has a cuboid shape with height L and width ε. The domains above the hard sound slab is denoted as Ω_ε^+ and the exterior domain $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \bar{C}_\varepsilon$ is denoted by Ω_ε. The aperture of the cavity is denoted by Γ_ε.

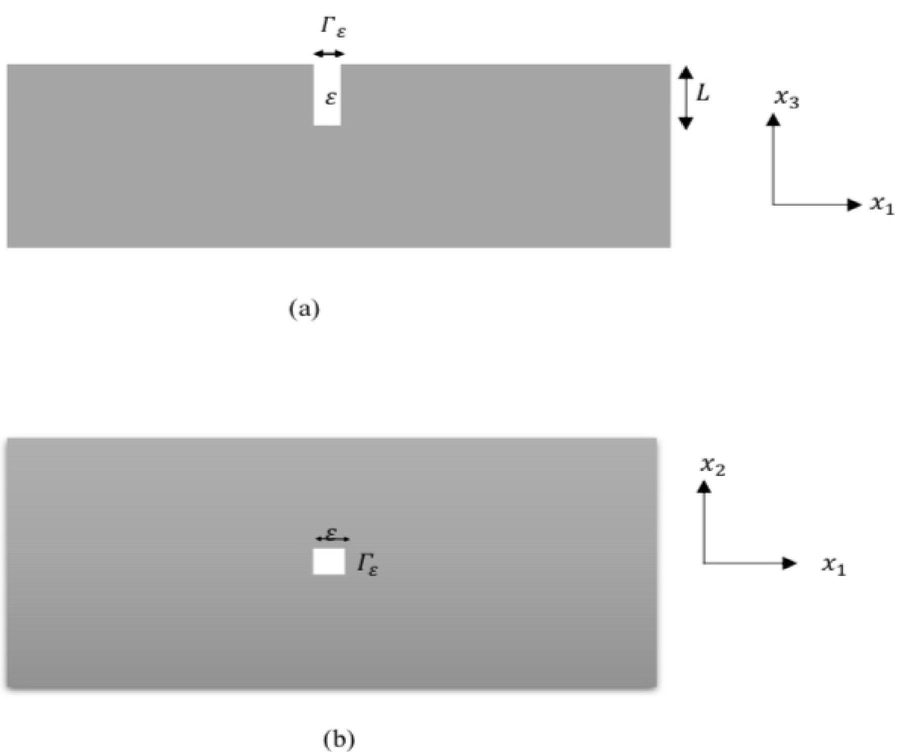

Figure 3.2: (a)(b): Vertical and horizontal cross section of the subwavelenth structure.

$$
\begin{aligned}
\Delta u_\varepsilon + k^2 u_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, && (3.1.1)\\
\frac{\partial u_\varepsilon}{\partial \nu} &= 0 \quad \text{on } \partial\Omega_\varepsilon, && (3.1.2)\\
u_\varepsilon &= u_\varepsilon^s + u^i + u^r, \quad \text{in } \Omega_\varepsilon^+ && (3.1.3)\\
\lim_{r\to\infty} r\Big(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\Big) &= 0, \;\; r = |x|. && (3.1.4)
\end{aligned}
$$

3.2 Boundary integral equation formulation

The scattering problem (3.1.1)–(3.1.4) can be formulated equivalently as a system of boundary-integral equations. The development in this section is standard. Let $g^{\mathrm{e}}(k;x,y)$ and $g_\varepsilon^{\mathrm{i}}(k;x,y)$ be the Green's functions for the Helmholtz equations with the Neumann boundary condition in Ω_ε^+ and C_ε respectively. They satisfy the following equations:

$$
\Delta g^{\mathrm{e}}(k;x,y) + k^2 g^{\mathrm{e}}(k;x,y) = \delta(x-y), \;\; x,y \in \Omega_\varepsilon^+,
$$

$$
\Delta g_\varepsilon^{\mathrm{i}}(k;x,y) + k^2 g_\varepsilon^{\mathrm{i}}(k;x,y) = \delta(x-y), \;\; x,y \in C_\varepsilon.
$$

In addition $\frac{\partial g^{\mathrm{e}}(k;x,y)}{\partial \nu_y} = 0$ for $y_3 = 1$ and $y_3 = 0$, and $\frac{\partial g^{\mathrm{i}}(k;x,y)}{\partial \nu_y} = 0$ on ∂C_ε

Green's function in the upper half domain Ω_ε^+ [15] is given by

$$
g^{\mathrm{e}}(k;x,y) = -\frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{ik|x'-y|}}{|x'-y|},
$$

where

$$
x' = (x_1, x_2, 2 - x_3) \quad \text{if } x,y \in \Omega_\varepsilon^+.
$$

The interior Green function $g_\varepsilon^{\mathrm{i}}(x,y)$ in the cavity C_ε with the Neumann boundary condition is:

$$g_\varepsilon^i(k;x,y) = \sum_{m,n,l=0}^{\infty} c_{mnl}\phi_{mnl}(x)\phi_{mnl}(y),$$

where $c_{mnl} = \frac{1}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2-(l\pi)^2}$, $\phi_{mnl} = \sqrt{\frac{\alpha_{mnl}}{\varepsilon}}\cos(\frac{m\pi x_1}{\varepsilon})\cos(\frac{n\pi x_2}{\varepsilon})\cos(l\pi x_3)$ and

$$a_{mnl} = \begin{cases} 1 & mnl \in Z_1, \\ 2 & mnl \in Z_2, \\ 4 & mnl \in Z_3, \\ 8 & mnl \in Z_4. \end{cases}$$

In the above $Z_1 = \{mnl \mid m = n = l = 0\}$,

$Z_2 = \{mnl \mid m = n = 0, l \geq 1 \quad \text{or} \quad n = l = 0, m \geq 1 \quad \text{or} \quad m = l = 0, n \geq 1\}$,

$Z_3 = \{mnl \mid m = 0, n \geq 1, l \geq 1 \quad \text{or} \quad n = 0, m \geq 1, l \geq 1 \quad \text{or} \quad l = 0, m \geq 1, n \geq 1\}$

and $Z_4 = \{mnl \mid m \geq 1, n \geq 1, l \geq 1\}$.

Using the second Green's identity in Ω_ε^+ and noting that $\dfrac{\partial u^i}{\partial \nu} + \dfrac{\partial u^r}{\partial \nu} = 0$ on $x_3 = 1$, we obtain

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon} g^{\mathrm{e}}(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \;\; x \in \Omega_\varepsilon^+. \tag{3.2.1}$$

By the continuity of single layer potential [23], we have

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu}ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \;\; x \in \Gamma_\varepsilon. \tag{3.2.2}$$

The solution inside the cavity can be expressed as

$$u_\varepsilon(x) = -\int_{\Gamma_\varepsilon} g_\varepsilon^{\mathrm{i}}(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y, \;\; x \in C_\varepsilon.$$

By imposing continuity of the solution along the gap aperture Γ_ε, we obtain the boundary integral equation as follows

$$\int_{\Gamma_\varepsilon} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x) = 0 \ \text{ on } \quad \Gamma_\varepsilon. \tag{3.2.3}$$

Proposition 3.2.1 *The scattering problem (*$3.1.1$*) is equivalent to the system of boundary integral equations (*$3.2.3$*).*

It is clear that$\frac{\partial u^\varepsilon}{\partial \nu}|_{\Gamma_\varepsilon} = \frac{\partial u_\varepsilon}{\partial y_3}(y_1, y_2, 1), \ \ , \ \ (u^i + u^r)|_{\Gamma_\varepsilon} = 2e^{ik(d_1x_1+d_2x_2)}$.

We re-scale the functions by introducing $X_1 = \frac{x_1}{\varepsilon}, X_2 = \frac{x_2}{\varepsilon}$ and $Y_1 = \frac{y_1}{\varepsilon}, Y_2 = \frac{y_2}{\varepsilon}$, and define the inequalities as follows:

$$\varphi_1(Y) := -\frac{\partial u_\varepsilon}{\partial y_3}(\varepsilon Y, 1); \quad f(X) := (u^i + u^r)(\varepsilon X, 1) = 2e^{ik\varepsilon X(d_1+d_2)};$$

$$\hat{G}^{\mathrm{e}}(X,Y) = -\frac{1}{2\pi}\frac{e^{ik\varepsilon|X-Y|}}{\varepsilon|X-Y|};$$

$$\hat{G}^{\mathrm{i}}_\varepsilon(X,Y) := g^{\mathrm{i}}_\varepsilon(k; \varepsilon X_1, \varepsilon X_2, 1; \varepsilon Y_1, \varepsilon Y_2, 1) = g^{\mathrm{i}}_\varepsilon(k; \varepsilon X_1, \varepsilon X_2, 0; \varepsilon Y_1, \varepsilon Y_2, 0)$$

$$= \sum_{m,n,l=0}^{\infty} \frac{c_{mnl}\alpha_{mnl}}{\varepsilon^2}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2).$$

By denoting $R_1 := (0,1)\times(0,1)$ call $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, we define the integral operators:

$$(S^e\varphi)(X) = \varepsilon\int_{R_1}\hat{G}^e_\varepsilon(X,Y)\varphi(Y)dY, \ \ X \in R_1; \tag{3.2.4}$$

$$(S^i\varphi)(X) = \varepsilon\int_{R_1}\hat{G}^i_\varepsilon(X,Y)\varphi(Y)dY, \ \ X \in R_1. \tag{3.2.5}$$

Proposition 3.2.2 *The integral equation* (3.2.3) *is equivalent to*

$$(S^{\mathrm{e}} + S^{\mathrm{i}})\varphi = \varepsilon^{-1} f. \tag{3.2.6}$$

3.3 Asymptotic expansion of the integral operators

First we introduce several notations below.

$$\begin{aligned}
\alpha(k,\varepsilon) &= -\frac{i\varepsilon k}{2\pi} + \frac{\cot k}{\varepsilon k}, \\
K_1(X,Y) &= -\frac{1}{2\pi\varepsilon |X-Y|}, \\
K_2(X,Y) &= -\frac{1}{\varepsilon} \sum_{m\geq 0, n\geq 0} \frac{2^j}{\pi\sqrt{m^2+n^2}} \cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2), \\
& \quad \text{where} \quad j=1 \quad \text{when} \quad m=0 \quad \text{or} \quad n=0 \quad \text{and} \quad j=2 \quad \text{when} \quad m,n\geq 1. \\
\kappa(X,Y) &= \varepsilon(K_1(X,Y) + K_2(X,Y)).
\end{aligned} \tag{3.3.7}$$

3.3.1 Asymptotic expansions of Green's functions

We present the asymptotic expansions of the Green's functions $\hat{G}^e_\varepsilon$ and $\hat{G}^i_\varepsilon$ in the following lemma.

Lemma 3.3.1 *If* $|k\varepsilon| \ll 1$, *and* $X, Y \in R_1$ *then*

$$\hat{G}^e_\varepsilon(X,Y) = \quad K_1(X,Y) - \frac{ik}{2\pi} + \kappa_{1,\varepsilon}(X,Y),$$

$$\hat{G}^i_\varepsilon(X,Y) = \quad \frac{\cot k}{k\varepsilon^2} + K_2(X,Y) + \kappa_{2,\varepsilon}(X,Y),$$

$$\text{where} \quad \kappa_{1,\varepsilon}(X,Y) \sim O(k^2\varepsilon), \quad \kappa_{2,\varepsilon}(X,Y) \sim O(k^2\varepsilon),$$

Proof. The asymptotic expansion of $\hat{G}^{\mathrm{e}}_{\varepsilon}(X,Y)$ is straightforward result of Taylor expansion.

$$\begin{aligned}
\hat{G}^{e}_{\varepsilon}(X,Y) &= -\frac{e^{ik\varepsilon|X-Y|}}{2\pi\varepsilon|X-Y|} \\
&= -\frac{1}{2\varepsilon\pi|X-Y|}\Big[1+ik\varepsilon|X-Y|+\frac{1}{2}(k\varepsilon)^2(X-Y)^2+O(k\varepsilon)^3\Big] \\
&= -\frac{1}{2\varepsilon\pi|X-Y|}-\frac{ik}{2\pi}+O(k^2\varepsilon).
\end{aligned}$$

Recall that

$$\hat{G}^{i}_{\varepsilon}(X,Y)=\frac{1}{\varepsilon^2}\sum_{m,n=0}^{\infty}\left(\sum_{l=0}^{\infty}c_{mnl}\alpha_{mnl}\right)\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2). \quad (3.3.8)$$

Let $A_{mn}=\sum_{l=0}^{\infty}c_{mnl}\alpha_{mnl}$, then

$$A_{00}(k)=\sum_{l=1}^{\infty}\frac{2}{k^2-(l\pi)^2}+\frac{1}{k^2}=\frac{\cot k}{k}.$$

$$\begin{aligned}
A_{m0}(k,\varepsilon) &= \sum_{l=1}^{\infty}\frac{4}{k^2-(m\pi/\varepsilon)^2-(l\pi)^2}+\frac{2}{k^2-(m\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2-k^2}}\coth\left(\sqrt{(m\pi/\varepsilon)^2-k^2}\right) \\
&= -\frac{2\varepsilon}{m\pi}-\frac{k^2\varepsilon^3}{m^3\pi^3}+O(\frac{\varepsilon^5}{m^5}), \quad m\geq 1.
\end{aligned}$$

$$
\begin{aligned}
A_{n0}(k,\varepsilon) = \quad & \sum_{l=1}^{\infty} \frac{4}{k^2-(n\pi/\varepsilon)^2-(l\pi)^2} + \frac{2}{k^2-(n\pi/\varepsilon)^2} \\
= \quad & -\frac{2}{\sqrt{(n\pi/\varepsilon)^2-k^2}} \coth\left(\sqrt{(n\pi/\varepsilon)^2-k^2}\right) \\
= \quad & -\frac{2\varepsilon}{n\pi} - \frac{k^2\varepsilon^3}{n^3\pi^3} + O(\frac{\varepsilon^5}{n^5}), \quad n \geq 1.
\end{aligned}
$$

For $m \geq 1, n \geq 1$,

$$
\begin{aligned}
A_{mn}(k,\varepsilon) = \quad & \sum_{l=1}^{\infty} \frac{8}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2-(l\pi)^2} + \frac{4}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2} \\
= \quad & -4\left(\sum_{l=1}^{\infty} \frac{2}{(m\pi/\varepsilon)^2+(n\pi/\varepsilon)^2+(l\pi)^2-k^2} + \frac{1}{(m\pi/\varepsilon)^2+(n\pi/\varepsilon)^2-k^2}\right) \\
= \quad & \frac{-4}{\sqrt{(m^2+n^2)\pi^2/\varepsilon^2-k^2}} \coth\left(\sqrt{(m^2+n^2)\pi^2/\varepsilon^2-k^2}\right) \\
= \quad & \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}}\left(1+\frac{-k^2\varepsilon^2}{\pi\sqrt{m^2+n^2}}\right)^{-\frac{1}{2}} = \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}} + O(k^2\varepsilon^3).
\end{aligned}
$$

Substituting these into $(3.3.8)$, we obtain

$$
\begin{aligned}
\hat{G}^i_\varepsilon(X,Y) = \quad & \frac{1}{\varepsilon^2}\frac{\cot k}{k} - \frac{1}{\varepsilon}\left(\sum_{m\geq 1}^{\infty} \frac{2}{m\pi}\cos(m\pi X_1)\cos(m\pi Y_1)\right) \\
& -\frac{1}{\varepsilon}\left(\sum_{n\geq 1}^{\infty} \frac{2}{n\pi}\cos(n\pi X_2)\cos(n\pi Y_2)\right) \\
& -\sum_{m\geq 1, n\geq 1} \frac{4}{\pi\varepsilon\sqrt{m^2+n^2}}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2) + O(k^2\varepsilon). \\
= \quad & \frac{1}{\varepsilon^2}\frac{\cot k}{k} + K_2(X,Y) + O(k^2\varepsilon).
\end{aligned}
$$

3.3.2 Decomposition of integral operators

Define the function spaces as follows:

$$V_1 = \tilde{H}^{-\frac{1}{2}}(R_1) := \{u = U|_{R_1} \mid U \in H^{-1/2}(\mathbf{R}) \text{ and } supp\, U \subset \bar{R}_1\} \quad \text{and} \quad V_2 = H^{\frac{1}{2}}(R_1).$$

We define the projection operator $P : V_1 \to V_2$ such that

$$P\varphi(X) = \langle \varphi, 1 \rangle 1,$$

where 1 is a function defined on R_1 and is equal to one therein. We denote by K and K_∞ the integral operators corresponding to the kernels $\kappa(X,Y)$ and $\kappa_\infty(X,Y)$ respectively. where $\kappa(X,Y)$ is defined $(3.3.7)$. Let $\kappa_\infty(X,Y) = \varepsilon(\kappa_{1,\varepsilon}(X,Y) + \kappa_{2,\varepsilon}(X,Y))$.

Lemma 3.3.2 *The operators $S^{\rm e} + S^{\rm i}$ admit the decomposition*

$$S^{\rm e} + S^{\rm i} = \alpha P + K + K_\infty.$$

Moreover, K_∞ is bounded from V_1 to V_2 with the operator norm $\|K_\infty\| \lesssim \varepsilon^2$ uniformly for bounded k's. The operator K is bounded from V_1 to V_2 with a bounded inverse.

Proof.

$$
\begin{aligned}
(S^{\mathrm{e}}+S^{\mathrm{i}})\varphi_1(X) = \quad & \varepsilon\int\Big[-\frac{1}{2\pi\varepsilon|X-Y|}-\frac{ik}{2\pi}-\frac{k^2\varepsilon|X-Y|}{4\pi}-O(k^3\varepsilon^2)+\frac{\cot k}{\varepsilon^2 k} \\
- \quad & \frac{1}{\varepsilon}\sum_{m\geq 1}^{\infty}\frac{2}{m\pi}\cos(m\pi X_1)\cos(m\pi Y_1)-\frac{1}{\varepsilon}\sum_{n\geq 1}^{\infty}\frac{2}{n\pi}\cos(n\pi X_2)\cos(n\pi Y_2) \\
- \quad & \sum_{m\geq 1,n\geq 1}\frac{1}{\varepsilon}\frac{4\varepsilon}{\pi\sqrt{m^2+n^2}}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2) \\
+ \quad & O(k^2\varepsilon)\Big]\varphi(Y)dY \\
= \quad & \varepsilon\int\Big[\frac{ik}{2\pi}+K_1(X,Y)+\kappa_{1,\varepsilon}(X,Y)+\frac{\cot k}{\varepsilon^2 k}+K_2(X,Y) \\
+ \quad & \kappa_{2,\varepsilon}(X,Y)\Big]\varphi(Y)dY. \\
= \quad & \alpha P\varphi+K\varphi+K_\infty\varphi.
\end{aligned}
$$

The operator $K : H^{-\frac{1}{2}}(R_1) \to H^{\frac{1}{2}}(R_1)$ is invertible and its proof is postponed to Chapter 5. □

3.4 Asymptotic expansion of the resonances

Lemma 3.4.1 *L is invertible for sufficiently small ε, and*

$$
\begin{aligned}
L &= K+K_\infty, \\
L^{-1}1 &= K^{-1}1+O(\varepsilon^2), \\
\langle L^{-1}1,1\rangle &= \gamma+O(\varepsilon^2), && (3.4.9)
\end{aligned}
$$

where $\gamma = \langle K^{-1}1,1\rangle_{L^2(R_1)}$.

Consider that

$$(S^e + S^i)\varphi = (B + L)\varphi = 0, \quad \text{where} \quad B = \alpha P, \quad L = K + K_\infty.$$

or equivalently

$$L^{-1}B\varphi + \varphi = 0. \tag{3.4.10}$$

Note that

$$B\varphi = \left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\right) P\varphi = \left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\right) \langle \varphi, 1\rangle 1.$$

It follows that

$$L^{-1}B\varphi \;\; = \;\; \left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\right) L^{-1}1\langle \varphi, 1\rangle.$$

Substituting it into (3.4.10) and taking inner product with the constant function 1 yields

$$\left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\right) \langle L^{-1}1, 1\rangle\langle \varphi, 1\rangle + \langle \varphi, 1\rangle = 0.$$

$$\Big(\Big(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\Big)\langle L^{-1}1, 1\rangle + 1\Big)\langle \varphi, 1\rangle = 0$$

We obtain the corresponding resonant condition

$$\lambda(k, \varepsilon) = 1 + \left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{\varepsilon k}\right) \langle L^{-1}1, 1\rangle.$$

Lemma 3.4.2 *The resonances of the scattering problem are the roots of the function* $\lambda(k, \varepsilon) = 0$.

Theorem 3.4.3 *The scattering resonances of* (3.1.1) *attains the following resonance expansion .*

$$k_n = \frac{n\pi}{2} + \frac{n\pi\varepsilon}{2\gamma} - \frac{i\pi n^2\varepsilon^2}{8} + O(\varepsilon^3) \quad n = 1, 3, 5, \ldots \tag{3.4.11}$$

where $\gamma = \langle K^{-1}1, 1\rangle$ *is defined in Lemma* 3.4.1

Proof. We aim to find the roots of

$$\lambda(k,\varepsilon) = 1 + \left(-\frac{ik\varepsilon}{2\pi} + \frac{\cot k}{k}\right)(\langle L^{-1}1, 1\rangle) = 0,$$

or equivalently,

$$q(k,\varepsilon) = \varepsilon\lambda(k,\varepsilon) = \varepsilon + \left[-\frac{ik\varepsilon^2}{2\pi} + \frac{\cot k}{k}\right](\gamma + r(k,\varepsilon)) = 0, \quad r(k,\varepsilon) \sim O(k^2\varepsilon^2).$$

Let

$$c(k) = \frac{\cot k}{k}.$$

The leading order term of $c(k)$ of $\lambda(k,\varepsilon)$ attains roots at $k_0 = n\pi/2$, for odd values of n. We see that $q(k,\varepsilon)$ is analytic for k in $\{z | arg z \neq \pi\}$. For finite solutions, we consider bounded domain for some fixed number $N \geq 0$

$$D_{\delta_o,\theta_o,N} = \{z \mid |z| \geq \delta\} \cup \{z \mid |z| \leq N, -(\pi - \theta_o) \leq arg z \leq (\pi - \theta_o)\}, \quad \delta > 0.$$

To derive the leading order asymptotic term of k_n,

We define

$$q_1(k,\varepsilon) = \varepsilon + \left(c(k) - \frac{i\varepsilon^2 k}{2\pi}\right)\gamma \tag{3.4.12}$$

Then by applying Taylor expansion we obtain

$$q_1(k,\varepsilon) = \varepsilon + \left[c^{'}(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0\right]\gamma \tag{3.4.13}$$

To find the value of $c'(k_0)$, first we calculate $c'(k)$ as follow

$$\begin{aligned} c'(k) = & \left(\frac{\cos k}{k \sin k}\right)' \\ = & \frac{-k\sin^2 k - \cos k(\sin k + k\cos k)}{(k\sin k)^2} \\ = & \frac{-k\sin^2 k - \cos k \sin k - k\cos^2 k}{(k\sin k)^2} \\ = & \frac{-k(\sin^2 k + \cos^2 k) - \cos k \sin k}{(k\sin k)^2} = \frac{-k - \cos k \sin k}{(k\sin k)^2}. \end{aligned}$$

Next to find the value of $c'(k_0) = \frac{n\pi}{2}$, substituting k_0 in the above derivative, we get

$$\begin{aligned} c'(k_0) = & \frac{-k_0 - \cos k_0 \sin k_0}{(k_0 \sin k_0)^2} \\ = & \frac{-k_0 - \cos k_0 \sin k_0}{k_0^2(1-\cos^2 k_0)} \\ = & \frac{-k_0 - \cos k_0 \sin k_0}{k_0^2(1-\cos k_0)(1+\cos k_0)} = -\frac{1}{k_0} \end{aligned}$$

by using the fact that $\sin k_0 = 1$ and $\cos k_0 = 0$. We see that q_1 has simple roots in $D_{\delta_o,\theta_o,N}$ which are close to k_0's. The expansion of the roots of q_1, denoted by $k_{n,1}$ in terms of ε can be evaluated as follows

$$\begin{aligned} \varepsilon + \left[c^{'}(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0\right]\gamma = 0, \\ c^{'}(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0 = -\frac{\varepsilon}{\gamma} \\ -\frac{1}{k_0}(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0 = -\frac{\varepsilon}{\gamma} \\ 1 + \frac{\varepsilon}{\gamma} - \frac{i\varepsilon^2}{2\pi}k_0 + O(\varepsilon^3) = -\frac{k}{k_0} \end{aligned}$$

Hence

$$k_{n,1} = k_0 + \frac{k_0\varepsilon}{\gamma} - \frac{ik_0^2\varepsilon^2}{2\pi} + O(\varepsilon^3)$$

or equivalently, for the odd value of n

$$k_{n,1} = \frac{n\pi}{2} + \frac{n\pi\varepsilon}{2\gamma} - \frac{i\pi n^2\varepsilon^2}{8} + O(\varepsilon^3).$$

To prove that $k_{n,1}$ is the leading order term of the asymptotic expansion of k_n, note that

$$q(k,\varepsilon) - q_1(k,\varepsilon) = (q_1(k,\varepsilon) + O(\varepsilon^2)) \cdot O(\varepsilon).$$

One may find a constant N such that

$$\mid q(k,\varepsilon) - q_1(k,\varepsilon) \mid < \mid q_1(k,\varepsilon) \mid$$

for all k such that $\mid k - k_{n,1} \mid = N\varepsilon^3$. Hence we obtain the expansion $(3.4.11)$ by the Rouche's theorem.

3.5 Quantitative analysis of the field enhancement at the resonant frequencies

Lemma 3.5.1 *If $n\varepsilon \ll 1$, then at the odd frequencies $k =$ Re k_n, we have*

$$q(k,\varepsilon) = -\frac{in\gamma\varepsilon^2}{4} + O(\varepsilon^3).$$

Proof. Let us consider $q(k,\varepsilon)$. First assume that

$$|k - \text{Re}k_n| \leq \varepsilon.$$

From the definition of q_1 and its expansion, it follows that

$$\begin{aligned} q(k,\varepsilon) = & \quad q_1(k,\varepsilon) + O(\varepsilon^3) \\ = & \quad q_1'(k_n)(k-k_n) + O(k-k_n) + O(\varepsilon^3) \\ = & \quad \gamma c'(k_0)\cdot(k-k_n) + O(\varepsilon^3) \\ = & \quad \frac{2\gamma}{n\pi}(k - \mathrm{Re}k_n - iImk_n) + O(\varepsilon^3). \end{aligned}$$

Since

$$Imk_n = Imk_{n,1} + O(\varepsilon^2) = -\frac{in^2\pi\varepsilon^2}{8} + O(\varepsilon^3).$$

$$q(k,\varepsilon) = -\frac{in\gamma\varepsilon^2}{2} + O(\varepsilon^3).$$

Lemma 3.5.2 *For the incidence direction $d = (d_1, d_2, -d_3)$, the following asymptotic expansion holds for the solution φ in V_1:*

$$\varphi = K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{q}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon^2) + O(k\varepsilon)^2\right] + O(k^2\varepsilon).$$

In addition

$$\langle\varphi, 1\rangle = \frac{1}{q}\Big(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\Big).$$

Proof. Let us consider

$$(S^e + S^i)\varphi = \varepsilon^{-1}\frac{f}{2}.$$

By the same calculations in Section 4, we have

$$\varphi = L^{-1}\frac{f}{2\varepsilon} - L^{-1}B\varphi, \tag{3.5.14}$$

or equivalently

$$\tilde{c}(k)\langle\varphi,1\rangle L^{-1}1+\varphi=L^{-1}\frac{f}{2\varepsilon}, \tag{3.5.15}$$

where $\tilde{c}(k)=\frac{1}{\varepsilon}\frac{\cot k}{k}-\frac{ik\varepsilon}{2\pi}$. Thus by taking inner product with 1 on both sides of (3.5.15), we get

$$\langle\varphi,1\rangle+\tilde{c}(k)\langle L^{-1}1,1\rangle\langle\varphi,1\rangle=\langle L^{-1}\frac{f}{2\varepsilon},1\rangle,$$

and it follows that

$$\langle L^{-1}\frac{f}{2\varepsilon},1\rangle = \quad \langle\varphi,1\rangle\Big(1+\tilde{c}(k)\langle L^{-1}1,1\rangle\Big),$$

$$\langle\varphi,1\rangle = \quad \Big(1+\tilde{c}(k)\langle L^{-1}1,1\rangle\Big)^{-1}\langle L^{-1}\frac{f}{2\varepsilon},1\rangle.$$

Since $\lambda(k,\varepsilon)=1+\left(\frac{ik\varepsilon}{2\pi}+\frac{\cot k}{\varepsilon k}\right)\langle L^{-1}1,1\rangle$, by substituting this value in the above equation, we obtain

$$\langle\varphi,1\rangle=\lambda^{-1}\langle L^{-1}\frac{f}{2\varepsilon},1\rangle. \tag{3.5.16}$$

Note that

$$L^{-1}\frac{f}{2\varepsilon}=\frac{1}{\varepsilon}\left(1+\frac{(d_1+d_2)}{2}O(k\varepsilon)\right)\left(K^{-1}1+O(k\varepsilon)^2\right).$$

substituting it into (3.5.16) yields

$$\langle\varphi,1\rangle=\frac{1}{q}\Big(\gamma+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\Big).$$

By substituting into $(3.5.14)$ and solving for φ

$$\begin{aligned}
\varphi = \quad & L^{-1}\frac{f}{2\varepsilon} - \tilde{c}(k)L^{-1}1\langle\varphi, 1\rangle \\
= \quad & L^{-1}\frac{f}{2\varepsilon} - \tilde{c}(k)\lambda^{-1}\langle L^{-1}\frac{f}{2\varepsilon}, 1\rangle \cdot L^{-1}1 \\
= \quad & L^{-1}\frac{f}{2\varepsilon} - \frac{\tilde{c}(k)\langle L^{-1}1, 1\rangle}{\langle L^{-1}1, 1\rangle\lambda}\langle L^{-1}\frac{f}{2\varepsilon}, 1\rangle \cdot L^{-1}1 \\
= \quad & L^{-1}\frac{f}{2\varepsilon} + \frac{1-(1+\tilde{c}(k))\langle L^{-1}1, 1\rangle}{\langle L^{-1}1, 1\rangle\lambda}\langle L^{-1}\frac{f}{2\varepsilon}, 1\rangle \cdot L^{-1}1,
\end{aligned}$$

we have

$$\varphi = \varepsilon^{-1}L^{-1}\frac{f}{2} + \frac{1-\lambda}{\langle L^{-1}1, 1\rangle\lambda}\langle L^{-1}\frac{f}{2\varepsilon}, 1\rangle \cdot L^{-1}1, \tag{3.5.17}$$

where

$$\begin{aligned}
L^{-1} = \quad & K^{-1}1 + O(k\varepsilon)^2, \\
\frac{f}{2\varepsilon} = \quad & \frac{1}{\varepsilon}\cdot 1 + \frac{(d_1+d_2)}{2}O(k) \quad \text{in} \quad V_2 \times V_2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\varepsilon\varphi = \quad & \left(1+\frac{(d_1+d_2)}{2}\cdot O(k\varepsilon)\right)K^{-1}1+O(k^2\varepsilon^2)\\
& +\frac{1-\lambda}{(\gamma+O(k\varepsilon)^2)\lambda}\left[\gamma+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\right][K^{-1}1+O(k\varepsilon)^2]\\
= \quad & K^{-1}1+\left(K^{-1}1\frac{(d_1+d_2)}{2}\cdot O(k\varepsilon)\right)\\
& +\frac{1-\lambda}{(\gamma+O(k\varepsilon)^2)\lambda}\left[\gamma K^{-1}1+K^{-1}1.1\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\right]+O(k^2\varepsilon^2)\\
= \quad & K^{-1}1+\left(K^{-1}1\frac{(d_1+d_2)}{2}\cdot O(k\varepsilon)\right)\\
& +\frac{\gamma(1-\lambda)}{(\gamma+O(k\varepsilon)^2)\lambda}\left[K^{-1}1+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\right]+O(k^2\varepsilon^2)\\
= \quad & K^{-1}1\frac{(d_1+d_2)}{2}O(k\varepsilon)+\frac{1}{\lambda}\left[K^{-1}1+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\right]+O(k\varepsilon)^2.
\end{aligned}
$$

Hence

$$
\varphi = K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right)+\frac{1}{q}\left[K^{-1}1+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\right]+O(k^2\varepsilon).
$$

□

Proposition 3.5.3 *At the resonant frequencies, solution to* $(3.2.6)$ *namely* $\varphi \sim O(1/\varepsilon^2) \quad in \quad V_1$ *and* $\langle \varphi_j, 1\rangle \sim O(1/\varepsilon^2), \quad j=1,2.$

It was observed in two dimensional case [26] that at resonant frequencies $\boldsymbol{\varphi} \sim O(1/\varepsilon) \quad in \quad (V_1\times V_1)$ and $\langle \varphi_i, 1\rangle \sim O(1/\varepsilon), \quad i=1,2.$

3.5.1 Field enhancement in the cavity

To investigate the field inside the cavity, note that u_ε satisfies the following boundary value problem

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in} \quad C_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_1} = 0 \quad \text{on} \quad x_1 = 0, x_1 = \varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_2} = 0 \quad \text{on} \quad x_2 = 0, x_2 = \varepsilon. \end{cases}$$

Then $u_\varepsilon(x)$ can be expanded as

$$\begin{aligned} u_\varepsilon(x) = \quad & a_{00}\cos(kx_3) + \sum_{m=0,n>0} \Big(a_{0n} \cos\frac{n\pi x_2}{\varepsilon} \exp\left(-k_{0n}x_3\right) \Big) \\ & + \sum_{m>0,n=0} \Big(a_{m0} \cos\frac{m\pi x_1}{\varepsilon} \exp\left(-k_{m0}x_3\right) \Big) \\ & + \sum_{m,n>0} \Big(a_{mn} \cos\frac{m\pi x_1}{\varepsilon} \cos\frac{n\pi x_2}{\varepsilon} \exp\left(-k_{mn}x_3\right) \Big), \end{aligned} \tag{3.5.18}$$

where $k_{mn} = \sqrt{(\frac{m\pi}{\varepsilon})^2 - (\frac{n\pi}{\varepsilon})^2 - k^2}$.

Lemma 3.5.4 *The following hold for the expansion coefficients in Equation (3.5.18):*

$$\begin{aligned} a_{00} = \quad & \frac{1}{k\sin k}\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right]\frac{1}{q}, \\ & \varepsilon\sqrt{m}|a_{m0}| \leq C, \quad \text{for} \quad m \geq 1, \quad \varepsilon\sqrt{n}|a_{0n}| \leq C, \quad \text{for} \quad n \geq 1 \\ & \sqrt{m+n}|a_{mn}| \leq C, \quad \text{for} \quad m,n \geq 1. \end{aligned}$$

where C is a positive constant independent of ε, k, m and n.

Proof Taking the derivative of the expansion (3.5.18) with respect to x_3

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial x_3} = \quad & -a_{00}k\sin(kx_3) + \sum_{m\geq 1}\Big(-a_{m0}k_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}x_3\right)\Big) \\ & + \sum_{n\geq 1}\Big(-a_{0n}k_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}x_3\right)\Big) \\ & + \sum_{m,n\geq 1}\Big(-a_{mn}k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}x_3\right)\Big).\end{aligned}$$

In particular, we have

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,1) = \quad & -a_{00}\sin(kx_3) + \sum_{m\geq 1}\left(-a_{m0}\exp\left(-k_{m0}\right)\right)k_{m0}\cos\frac{m\pi x_1}{\varepsilon} \\ & + \sum_{n\geq 1}\left(-a_{0n}\exp\left(-k_{0n}\right)\right)k_{0n}\cos\frac{n\pi x_2}{\varepsilon} \\ & + \sum_{m,n\geq 1}\left(-a_{mn}\exp\left(-k_{mn}\right)\right)k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}, \qquad (3.5.19)\end{aligned}$$

Integrating over the aperture gives

$$-a_{00}k\sin k = \frac{1}{\varepsilon^2}\int_{\Gamma_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,1)dx_1dx_2 = -\langle\varphi,1\rangle = -\frac{1}{q}\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).$$

We obtain the required formula for a_{00}.

For coefficient a_{m0} taking inner product of $\partial_{x_3}u_\varepsilon$ and $\cos\frac{m\pi x_1}{\varepsilon}$ and integrating over aperture yields.

For $m \geq 1$,

$$a_{m0}k_{m0}\exp\left(-k_{m0}\right) = \frac{2}{\varepsilon^2}\int_{\Gamma_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,1)\cos\frac{m\pi x_1}{\varepsilon}dx_1dx_2 = -2\int_{R_1}\varphi(X)\cos m\pi X dX,$$

$$a_{m0}k_{m0} = \frac{-2}{e^{-k_{m0}}}\left(\int_{R_1}\varphi(X)\cos(m\pi X)dX\right)$$

Note that $k_{m0} = O(\frac{m}{\varepsilon})$ for $m \geq 1$, and

$$\|\varphi\|_{V_1} \leq \frac{1}{\varepsilon^2}, \qquad \|\cos(m\pi X)\|_{V_2} \leq \sqrt{m}$$

The estimate for a_{m0} follows. A parallel calculation yields required results for a_{0n}. For coefficient a_{mn}, taking inner product of $(3.5.19)$ $\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}$ and integrating over aperture yields,

$$(a_{mn}\exp(-k_{mn}))k_{mn} = -\frac{4}{\varepsilon^2}\int_{\Gamma_\varepsilon}\partial_{x_3}u_\varepsilon(x_1,x_2,1)\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}ds$$

solving above equation for a_{mn} yields

$$a_{mn}k_{mn} = \quad \frac{-4}{e^{-2k_{mn}}}\left(e^{k_{mn}}\int_{R_1}\varphi(X)\cos(m\pi X_1)\cos(n\pi X_2)dX_1dX_2\right)$$

since $k_{mn} = O(\sqrt{\frac{m^2+n^2}{\varepsilon^2}})$ and $\|\varphi\|_{V_1} \leq \frac{1}{\varepsilon^2}$, the desired formula for a_{mn} follows immediately.

Theorem 3.5.5 *The wave field in the cavity $C_\varepsilon^{int} := \{x \in C_\varepsilon \mid x_3 \gg \varepsilon, 1 - x_3 \gg \varepsilon\}$ is given by*

$$u_\varepsilon(x) = \left(1/\varepsilon^2 + \frac{(d_1+d_2)}{2}O(1/\varepsilon) + O(1)\right)\frac{2i\cos(kx_3)}{nk\sin k} + O(\exp(-1/\varepsilon^2)),$$

at the resonant frequencies $k = Rek_n$.

Proof. Since from (3.5.18) and Lemma 3.5.4, we obtain

$$\begin{aligned} u_\varepsilon(x) = \quad & a_{00}\cos(kx_3) + \sum_{m=0,n>0}\Big(a_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}x_3\right)\Big) \\ & + \sum_{m>0,n=0}\Big(a_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}x_3\right)\Big) \\ & + \sum_{m,n>0}\Big(a_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}x_3\right)\Big). \end{aligned}$$

For $\varepsilon \ll 1$,

$$u_\varepsilon(x) = \left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\left[\frac{1}{q}\frac{\cos kx_3}{k\sin k}\right] + O(\exp(-1/\varepsilon^2)).$$

At the odd resonant frequencies $k = Rek_n$, $\frac{1}{q} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon))$, above equation takes form

$$\begin{aligned} u_\varepsilon(x) = \quad & [1 + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2]\Big[\Big(\frac{1}{\varepsilon^2}\frac{2i\cos(kx_3)}{nk\sin k}(1+O(\varepsilon))\Big)\Big] \\ + \quad & \exp(-1/\varepsilon^2) \\ = \quad & \Big(1/\varepsilon^2 + \frac{(d_1+d_2)}{2}O(1/\varepsilon) + O(1)\Big)\frac{2i\cos(kx_3)}{nk\sin k} + O(\exp(-1/\varepsilon^2)). \end{aligned}$$

□

3.5.2 Scattering enhancement in the far field

In the domian $\Omega_1^+ \backslash H_1^+$ above the cavity, where $H_1^+ := \{x | x - (0,0,1) \leq 1\}$. Recall that

$$u_s^\varepsilon(x) = \int_{\Gamma_\varepsilon} g^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y, \quad x \in \Omega^+$$

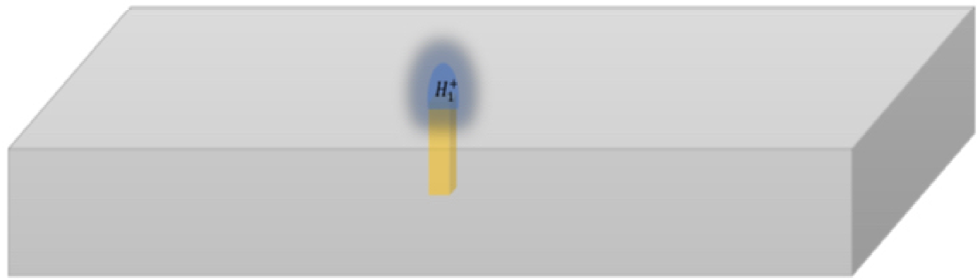

Figure 3.3: Depiction of far fields H_1^+.

and

$$\frac{\partial u_\varepsilon}{\partial \nu}(x_1, x_2, 1) = -\varphi_+(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}).$$

Therefore

$$\begin{aligned} u_s^\varepsilon(x) = & \quad -\int_{\Gamma_\varepsilon} g^e(x, (y_1, y_2, 1))\varphi_+(\frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon})dy_1 dy_2 \\ = & \quad -\varepsilon^2 \int_0^1 \int_0^1 g^e(x, \varepsilon Y_1, \varepsilon Y_2, 1)\varphi_+(Y_1, Y_2)dY_1 dY_2. \end{aligned}$$

Note that

$$g^e(x, \varepsilon Y_1, \varepsilon Y_2, 1) = g^e(x, (0,0,1))(1 + O(\varepsilon)) \quad x \in \Omega^+ \backslash H_1^+,$$

and

$$\langle \varphi, 1 \rangle_{L^2(R_1)} = \frac{1}{q}\left(\gamma + \frac{(d_1 + d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).$$

It follows that

$$u_s^\varepsilon(x) = -\varepsilon^2 g^e(x, (0,0,1))(1 + O(\varepsilon))\frac{1}{q}\left(\gamma + \frac{(d_1 + d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).$$

Since $\frac{1}{q} = \frac{i4}{n\gamma\varepsilon^2}(1 + O(\varepsilon))$ at resonant frequencies, the corresponding scattered field is

$$u_s^\varepsilon(x) = \frac{i4}{n} g^e(x, (0,0,1) + O(\varepsilon^2). \tag{3.5.20}$$

Chapter 4

Scattering resonances for a three-dimensional subwavelength hole

4.1 Problem formulation

The hole is bore through a sound hard material slab, and its geometry is shown in Figure 4.1. The slab occupied the domain $\{(x_1.x_2, x_3) \mid 0 < x_3 < L\}$, and the hole is a cuboid given by $C_\varepsilon := \{(x_1.x_2, x_3) \mid 0 < x_1 < \varepsilon, 0 < x_2 < \varepsilon, 0 < x_3 < L\}$. We consider the case when the length and width ε of the hole is much smaller than the thickness of the slab and the wavelength of the incident wave λ, i.e., $\varepsilon \ll L \sim \lambda$. Without loss of generality, in what follows we scale the geometry of the problem by assuming that the slab thickness $L = 1$. Let us denote the upper and lower aperture of the hole by Γ^+ and Γ^- respectively, and semi-infinite domains Ω^+ and Ω^- above and below the slab respectively. The exterior domain is given by $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup \bar{C}_\varepsilon$.

We consider the scattering when the plane wave u^i is incident upon the structure, where $u^i = e^{ik(d(x-x_0))}$ is the incident field. Here $(d_1, d_2, -d_3)$ is the incident direction with $d_3 > 0$, k is the wave number, and $x_0 = (0, 0, L)$. In the absence of hole, the total field in the domain Ω^+, consists of the incident field u^i and reflected field $u^r = e^{ik(d'(x-x_0))}$ where $d' = (d_1, d_2, d_3)$, while the field in the domain Ω^- is zero. In the presence of hole C_ε, the total field u_ε in the upper domain Ω^+ consists of u^i, u^r and the scattered field u^s_ε radiating from Γ^+. In the domain Ω^-_ε, u_ε only consists of the transmitted field through the lower aperture Γ^-. In addition, the Neumann boundary condition $\partial_\nu u_\varepsilon = 0$ is imposed on $\partial\Omega_\varepsilon$ for the sound hard material, where ν is the unit outward normal pointing to Ω_ε. Finally, the scattered field u^s_ε satisfies the Sommerfeld radiation condition at the semi-infinite domains [19]. In summary, the total field u_ε satisfies the following

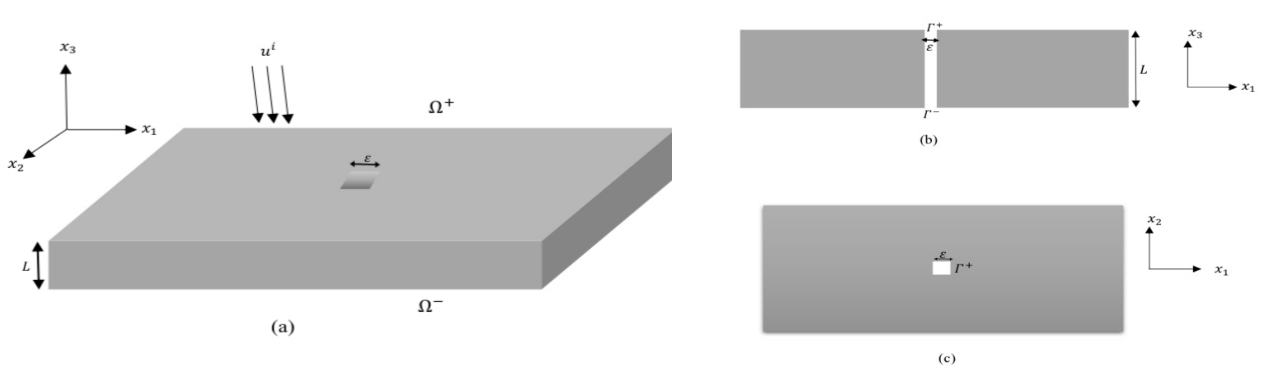

Figure 4.1: (a) Geometry of the problem. The hole C_ε has a cuboid shape with height L and width ε. The domains above and below the hard sound slab are denoted as Ω^+ and Ω^- respectively and the exterior domain $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup \bar{C}_\varepsilon$ is denoted by Ω_ε.(b)(c) The upper and lower aperture of the hole are denoted by Γ^+ and Γ^- respectively. (b)(c): Vertical and horizontal cross section of the subwavelenth structure.

scattering problem:

$$\Delta u_\varepsilon + k^2 u_\varepsilon \; = \; 0, \quad \text{in } \Omega_\varepsilon, \tag{4.1.1}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad \text{on } \partial\Omega_\varepsilon, \tag{4.1.2}$$

$$u_\varepsilon \; = \; u_\varepsilon^s + u^i + u^r, \quad \text{in } \Omega^+, \tag{4.1.3}$$

$$u_\varepsilon \; = \; u_\varepsilon^s, \quad \text{in } \Omega^-, \tag{4.1.4}$$

$$\lim_{r\to\infty} r\Big(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\Big) = 0, \;\; r = |x|. \tag{4.1.5}$$

For all complex wavenumbers k with $Imk \geq 0$, it can be shown that the above scattering problem has a unique solution. By analytic continuation, the resolvent $R(k) := (\Delta + k^2)^{-1}$ of the scattering problem (4.1.1)-(4.1.5) can be extended to the whole complex plane except at a countable number of poles. These poles are called the scattering resonances of the scattering problem. In this paper, we prove the existence of scattering resonances, derive the asymptotic expansions of those resonances, and present the quantitative analysis of the field amplification at the resonant frequencies. By reformulating the scattering problem (4.1.1)-(4.1.5) as the equivalent integral equation system, the resonances reduce to the characteristic values of the certain integral operators. We apply the asymptotic analysis of the integral operators and the simplified Gohberg-Sigal theory to obtain scattering resonances. It is shown that the complex-valued scattering resonances attain imaginary parts of order $O(\varepsilon^2)$. We also analyze the field amplification at resonant frequencies and show that the enhancement is of order $O(1/\varepsilon^2)$.

4.2 Boundary integral equation formulation

The scattering problem (4.1.1)-(4.1.5) can be formulated equivalently as a system of boundary-integral equations. The development in this section is standard, see for instance [7, 8, 23]. Let $g^{\mathrm{e}}(k;x,y)$ and $g^{\mathrm{i}}_{\varepsilon}(k;x,y)$ be the Green's functions for the Helmholtz equations with the Neumann boundary condition in Ω^+, Ω^- and C_ε respectively. They satisfy the following equations:

$$\Delta g^{\mathrm{e}}(k;x,y) + k^2 g^{\mathrm{e}}(k;x,y) = \delta(x-y), \;\; x,y \in \Omega^{\pm},$$

$$\Delta g^{\mathrm{i}}_{\varepsilon}(k;x,y) + k^2 g^{\mathrm{i}}_{\varepsilon}(k;x,y) = \delta(x-y), \;\; x,y \in C_\varepsilon.$$

In addition $\frac{\partial g^{\mathrm{e}}(k;x,y)}{\partial \nu_y} = 0$ for $y_3 = 1$ and $y_3 = 0$, and $\frac{\partial g^{\mathrm{i}}(k;x,y)}{\partial \nu_y} = 0$ on ∂C_ε.

The Green's function in $\Omega^{\pm}$ is given by

$$g^{\mathrm{e}}(k;x,y) = -\frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{ik|x'-y|}}{|x'-y|},$$

where

$$x' = \begin{cases} (x_1, x_2, 2-x_3) & \text{if } x,y \in \Omega^+, \\ (x_1, x_2, -x_3) & \text{if } x,y \in \Omega^-. \end{cases}$$

The interior Green function $g^{\mathrm{i}}_{\varepsilon}(x,y)$ in the hole C_ε with the Neumann boundary condition is

$$g^{i}_{\varepsilon}(k;x,y) = \sum_{m,n,l=0}^{\infty} c_{mnl}\varphi_{mnl}(x)\phi_{mnl}(y),$$

where $c_{mnl} = \frac{1}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2-(l\pi)^2}$, $\phi_{mnl} = \sqrt{\frac{\alpha_{mnl}}{\varepsilon}}\cos(\frac{m\pi x_1}{\varepsilon})\cos(\frac{n\pi x_2}{\varepsilon})\cos(l\pi x_3)$ and

$$a_{mnl} = \begin{cases} 1 & mnl \in Z_1, \\ 2 & mnl \in Z_2, \\ 4 & mnl \in Z_3, \\ 8 & mnl \in Z_4. \end{cases}$$

In the above $Z_1 = \{mnl \mid m = n = l = 0\}$, $Z_2 = \{mnl \mid m = n = 0, l \geq 1 \quad \text{or} \quad n = l = 0, m \geq 1 \quad \text{or} \quad m = l = 0, n \geq 1\}$, $Z_3 = \{mnl \mid m = 0, n \geq 1, l \geq 1 \quad \text{or} \quad n = 0, m \geq 1, l \geq 1 \quad \text{or} \quad l = 0, m \geq 1, n \geq 1\}$ and $Z_4 = \{mnl \mid m \geq 1, n \geq 1, l \geq 1\}$.

Using the second Green's identity in Ω^+ and noting that $\dfrac{\partial u^i}{\partial \nu} + \dfrac{\partial u^r}{\partial \nu} = 0$ on $x_3 = 1$, we obtain

$$u_\varepsilon(x) = \int_{\Gamma^+} g^{\mathrm{e}}(x,y)\frac{\partial u_\varepsilon}{\partial \nu} ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \;\; x \in \Omega^+. \tag{4.2.1}$$

By the continuity of single layer potential [23], we have

$$u_\varepsilon(x) = \int_{\Gamma^+} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu} ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \;\; x \in \Gamma^+. \tag{4.2.2}$$

Similarly,

$$u_\varepsilon(x) = \int_{\Gamma^-} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu} ds_y, \;\; x \in \Gamma^-. \tag{4.2.3}$$

The solution inside the hole can be expressed as

$$u_\varepsilon(x) = -\int_{\Gamma^+\cup\Gamma^-} g^{\mathrm{i}}_\varepsilon(x,y)\frac{\partial u_\varepsilon}{\partial \nu} ds_y, \;\; x \in C_\varepsilon.$$

Taking the limit when x approaches the hole apertures Γ^+ and Γ^-, there holds

$$u_\varepsilon(x) = -\int_{\Gamma^+\cup\Gamma^-} g^{\mathrm{i}}_\varepsilon(x,y)\frac{\partial u_\varepsilon}{\partial \nu} ds_y, \;\; x \in \Gamma^+ \cup \Gamma^-. \tag{4.2.4}$$

By imposing continuity of the solution along the hole apertures, we obtain the boundary integral equations as follows

$$\begin{cases} \int_{\Gamma^+} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu} ds_y + \int_{\Gamma^+\cup\Gamma^-} g^i_\varepsilon(x,y)\frac{\partial u_\varepsilon}{\partial \nu} ds_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x) = 0 \ \text{ on } \quad \Gamma^+, \\ \int_{\Gamma^-} -\Big(\frac{1}{2\pi}\Big)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu} ds_y + \int_{\Gamma^+\cup\Gamma^-} g^{\mathrm{i}}_\varepsilon(x,y)\frac{\partial u_\varepsilon}{\partial \nu} ds_y = 0 \ \text{ on } \quad \Gamma^-. \end{cases} \tag{4.2.5}$$

It is clear that $\frac{\partial u^\varepsilon}{\partial \nu}|_{\Gamma^+} = \frac{\partial u_\varepsilon}{\partial y_3}(y_1,y_2,1)$, $\frac{\partial u^\varepsilon}{\partial \nu}|_{\Gamma^-} = -\frac{\partial u_\varepsilon}{\partial y_3}(y_1,y_2,0)$, $(u^i+u^r)|_{\Gamma^+} = 2e^{ik(d_1x_1+d_2x_2)}$. We rescale the functions by introducing $X_1 = \frac{x_1}{\varepsilon}, X_2 = \frac{x_2}{\varepsilon}$ and $Y_1 = \frac{y_1}{\varepsilon}, Y_2 = \frac{y_2}{\varepsilon}$, and define the following quantities:

$$\varphi_1(Y) := -\frac{\partial u_\varepsilon}{\partial y_3}(\varepsilon Y, 1); \quad \varphi_2(Y) := \frac{\partial u_\varepsilon}{\partial y_3}(\varepsilon Y, 0); \quad f(X) := (u^i+u^r)(\varepsilon X, 1) = 2e^{ik\varepsilon X(d_1+d_2)};$$

$$G^{\mathrm{e}}(X,Y) = -\frac{1}{2\pi}\frac{e^{ik\varepsilon|X-Y|}}{\varepsilon|X-Y|};$$

$$G^{\mathrm{i}}_\varepsilon(X,Y) := g^{\mathrm{i}}_\varepsilon(k;\varepsilon X_1,\varepsilon X_2,1;\varepsilon Y_1,\varepsilon Y_2,1) = g^{\mathrm{i}}_\varepsilon(k;\varepsilon X_1,\varepsilon X_2,0;\varepsilon Y_1,\varepsilon Y_2,0)$$

$$= \sum_{m,n,l=0}^{\infty} \frac{c_{mnl}\alpha_{mnl}}{\varepsilon^2}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2);$$

$$\tilde{G}^{\mathrm{i}}_{\varepsilon(X,Y)} := g^{\mathrm{i}}_\varepsilon(k;\varepsilon X_1,\varepsilon X_2,1;\varepsilon Y_1,\varepsilon Y_2,0) = g^{\mathrm{i}}_\varepsilon(k;\varepsilon X_1,\varepsilon X_2,0;\varepsilon Y_1,\varepsilon Y_2,1)$$

$$= \sum_{m,n,l=0}^{\infty} (-1)^l\frac{c_{mnl}\alpha_{mnl}}{\varepsilon^2}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2).$$

Let $R_1 := (0,1)\times(0,1)$, $X = (X_1,X_2)$ and $Y = (Y_1,Y_2)$. For $X \in R_1$, we define the integral operators:

$$(Q^{\mathrm{e}}\varphi)(X) = \varepsilon\int_{R_1} G^e_\varepsilon(X,Y)\varphi(Y)dY; \tag{4.2.6}$$

$$(Q^{\mathrm{i}}\varphi)(X) = \varepsilon \int_{R_1} G^i_\varepsilon(X,Y)\varphi(Y)dY; \tag{4.2.7}$$

$$(\tilde{Q}^{\mathrm{i}}\varphi)(X) = \varepsilon \int_{R_1} \tilde{G}^i_\varepsilon(X,Y)\varphi(Y)dY. \tag{4.2.8}$$

By the change of variables, the following proposition follows.

Proposition 4.2.1 *The system of integral equations* (4.2.5) *is equivalent to the system* $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$*, in which*

$$\mathbb{Q} = \begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix}, \quad \boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \frac{f}{\varepsilon} \\ 0 \end{bmatrix}. \tag{4.2.9}$$

4.3 Asymptotic expansion of the integral operators

First we introduce several notations below:

$$\begin{aligned}
\beta(k,\varepsilon) &= -\frac{i\varepsilon k}{2\pi} + \frac{\cot k}{\varepsilon k}, \\
\tilde{\beta}(k,\varepsilon) &= \frac{1}{\varepsilon k \sin k}, \\
K_1(X,Y) &= -\frac{1}{2\pi\varepsilon|X-Y|}, \\
K_2(X,Y) &= -\frac{1}{\varepsilon}\sum_{m\geq 0, n\geq 0} \frac{2^j}{\pi\sqrt{m^2+n^2}} \cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2), \\
& \quad \text{where} \quad j=1 \quad \text{for} \quad m=0 \quad \text{or} \quad n=0 \quad \text{and} \quad j=2 \quad \text{for} \quad m,n\geq 1. \\
\kappa(X,Y) &= \varepsilon(K_1(X,Y)+K_2(X,Y)).
\end{aligned} \tag{4.3.10}$$

The asymptotic expansions of the kernels G^e_ε, G^i_ε, $\tilde{G}^i_\varepsilon$ are presented in the following lemma.

Lemma 4.3.1 *If $|k\varepsilon| \ll 1$ and $X, Y \in R_1$, then*

$$G^e_\varepsilon(X,Y) = -\frac{1}{2\varepsilon\pi|X-Y|} - \frac{ik}{2\pi} + \kappa_{1,\varepsilon}(X,Y),$$

$$G^i_\varepsilon(X,Y) = \frac{\cot k}{k\varepsilon^2} - K_2(X,Y) + \kappa_{2,\varepsilon}(X,Y),$$

$$\tilde{G}^i_\varepsilon(X,Y) = \frac{1}{\varepsilon^2}\frac{1}{k\sin k} + \tilde{\kappa}_\infty(X,Y),$$

where $\kappa_{1,\varepsilon}(X,Y) \sim O(k^2\varepsilon)$, $\kappa_{2,\varepsilon}(X,Y) \sim O(k^2\varepsilon)$, and $\tilde{\kappa}_\infty(X,Y) \sim O(\exp(-1/\varepsilon))$.

Proof. The asymptotic expansion of $G^e_\varepsilon(X,Y)$ is straightforward from the Taylor expansion:

$$\begin{aligned} G^e_\varepsilon(X,Y) &= -\frac{e^{ik\varepsilon|X-Y|}}{2\pi\varepsilon|X-Y|} \\ &= -\frac{1}{2\varepsilon\pi|X-Y|}\left[1 + ik\varepsilon|X-Y| + \frac{1}{2}(k\varepsilon)^2(X-Y)^2 + O(k\varepsilon)^3\right] \\ &= -\frac{1}{2\varepsilon\pi|X-Y|} - \frac{ik}{2\pi} + O(k^2\varepsilon). \end{aligned}$$

Recall that

$$G^i_\varepsilon(X,Y) = \frac{1}{\varepsilon^2}\sum_{m,n=0}^{\infty}\left(\sum_{l=0}^{\infty} c_{mnl}\alpha_{mnl}\right)\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2). \quad (4.3.11)$$

Let $C_{mn} = \sum_{l=0}^{\infty} c_{mnl}\alpha_{mnl}$, then using the formulas in [22], we have

$$C_{00}(k) = \sum_{l=1}^{\infty}\frac{2}{k^2-(l\pi)^2} + \frac{1}{k^2} = \frac{\cot k}{k}.$$

$$
\begin{aligned}
C_{m0}(k,\varepsilon) = & \sum_{l=1}^{\infty} \frac{4}{k^2 - (m\pi/\varepsilon)^2 - (l\pi)^2} + \frac{2}{k^2 - (m\pi/\varepsilon)^2} \\
= & -\frac{2}{\sqrt{(m\pi/\varepsilon)^2 - k^2}} \coth\left(\sqrt{(m\pi/\varepsilon)^2 - k^2}\right) \\
= & -\frac{2\varepsilon}{m\pi} - \frac{k^2\varepsilon^3}{m^3\pi^3} + O\Big(\frac{k^4\varepsilon^5}{m^5}\Big), \quad m \geq 1.
\end{aligned}
$$

$$
\begin{aligned}
C_{n0}(k,\varepsilon) = & \sum_{l=1}^{\infty} \frac{4}{k^2 - (n\pi/\varepsilon)^2 - (l\pi)^2} + \frac{2}{k^2 - (n\pi/\varepsilon)^2} \\
= & -\frac{2}{\sqrt{(n\pi/\varepsilon)^2 - k^2}} \coth\left(\sqrt{(n\pi/\varepsilon)^2 - k^2}\right) \\
= & -\frac{2\varepsilon}{n\pi} - \frac{k^2\varepsilon^3}{n^3\pi^3} + O\Big(\frac{k^4\varepsilon^5}{n^5}\Big), \quad n \geq 1.
\end{aligned}
$$

For $m \geq 1, n \geq 1$,

$$
\begin{aligned}
C_{mn}(k,\varepsilon) = & \sum_{l=1}^{\infty} \frac{8}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2 - (l\pi)^2} + \frac{4}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2} \\
= & -4\left(\sum_{l=1}^{\infty} \frac{2}{(m\pi/\varepsilon)^2 + (n\pi/\varepsilon)^2 + (l\pi)^2 - k^2} + \frac{1}{(m\pi/\varepsilon)^2 + (n\pi/\varepsilon)^2 - k^2}\right) \\
= & \frac{-4}{\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}} \coth\left(\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}\right) \\
= & \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}} \left(1 + \frac{-k^2\varepsilon^2}{\pi\sqrt{m^2+n^2}}\right)^{-\frac{1}{2}} = \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}} + O(k^2\varepsilon^3).
\end{aligned}
$$

Substituting these into $(4.3.11)$,we obtain

$$G^i_\varepsilon(X,Y) = \quad \frac{1}{\varepsilon^2}\frac{\cot k}{k} + K_2(X,Y) + O(k^2\varepsilon).$$

Similarly,

$$\tilde{G}^i_\varepsilon(X,Y) = \frac{1}{\varepsilon^2}\sum_{m,n=0}^{\infty}\left(\sum_{l=0}^{\infty}(-1)^l c_{mnl}\alpha_{mnl}\right)\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2). \tag{4.3.12}$$

Let $\tilde{C}_{mn} = \sum_{l=0}^{\infty}(-1)^l c_{mnl}\alpha_{mnl}$, then

$$\tilde{C}_{00} = \sum_{l=0}^{\infty}\frac{2(-1)^l}{k^2-(l\pi)^2} + \frac{1}{k^2} = \frac{1}{\sin k}.$$

$$\begin{aligned}
\tilde{C}_{m0} &= \sum_{l=1}^{\infty}\frac{4(-1)^l}{k^2-(m\pi/\varepsilon)^2-(l\pi)^2} + \frac{2}{k^2-(m\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2-k^2}\sinh\left(\sqrt{(m\pi/\varepsilon)^2-k^2}\right)} \\
&= O\left(\frac{\varepsilon}{m\pi}\exp(-\frac{m\pi}{\varepsilon})\right), \quad m \geq 1.
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_{0n} &= \sum_{l=1}^{\infty}\frac{4(-1)^l}{k^2-(n\pi/\varepsilon)^2-(l\pi)^2} + \frac{2}{k^2-(n\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(n\pi/\varepsilon)^2-k^2}\sinh\left(\sqrt{(n\pi/\varepsilon)^2-k^2}\right)} \\
&= O\left(\frac{\varepsilon}{n\pi}\exp(-n\pi/\varepsilon)\right), \quad n \geq 1.
\end{aligned}$$

For $m \geq 1, n \geq 1$,

$$\tilde{C}_{mn} = \sum_{l=1}^{\infty} \frac{8(-1)^l}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2 - (l\pi)^2} + \frac{4}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2}$$

$$= -\frac{4}{\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}\,\sinh\left(\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}\right)}.$$

Substituting into (4.3.12), we obtain

$$\tilde{G}^i_\varepsilon(X,Y) = \frac{1}{(k\sin k)\varepsilon^2} + O\left(\exp(-1/\varepsilon)\right).$$

$\square$

Define the function spaces

$$V_1 = \tilde{H}^{-\frac{1}{2}}(R_1) := \{u = U|_{R_1} \mid U \in H^{-1/2}(\mathbf{R}) \text{ and } supp\, U \subset \bar{R}_1\} \quad \text{and} \quad V_2 = H^{\frac{1}{2}}(R_1),$$

where $H^{\frac{1}{2}}(R_1)$ and $H^{-1/2}(\mathbf{R})$ are the standard Sobolev spaces [3]. We define a projection operator $P : V_1 \to V_2$ such that

$$P\varphi(X) = \langle \varphi, 1 \rangle 1,$$

where 1 is a function defined on R_1 and is equal to one therein. We denote by $K, K_\infty, \tilde{K}_\infty$ the integral operators corresponding to the kernels $\kappa(X,Y), \kappa_\infty(X,Y)$ and $\varepsilon\tilde{\kappa}_\infty(X,Y)$, respectively,where $\kappa(X,Y)$ is defined in (4.3.10), $\tilde{\kappa}_\infty(X,Y)$ is defined in Lemma 4.3.1, and $\kappa_\infty(X,Y) = \varepsilon(\kappa_{1,\varepsilon}(X,Y) + \kappa_{2,\varepsilon}(X,Y))$.

Lemma 4.3.2 *The operators $Q^{\mathrm{e}} + Q^{\mathrm{i}}$ and $\tilde{Q}^{\mathrm{i}}$ admit the decompositions*

$$Q^{\mathrm{e}} + Q^{\mathrm{i}} = \beta P + K + K_\infty, \quad \text{and} \quad \tilde{Q}^{\mathrm{i}} = \tilde{\beta} P + \tilde{K}_\infty.$$

Moreover,the operator $K : H^{-\frac{1}{2}}(R_1) \to H^{\frac{1}{2}}(R_1)$ is invertible, K_∞ and $\tilde{K}_\infty$ are bounded from V_1 to V_2 with the operator norms $\|K_\infty\| \lesssim \varepsilon^2$ and $\|\tilde{K}_\infty\| \lesssim \exp(-1/\varepsilon)$ uniformly for bounded k's respectively.

Proof. By using the definition of operators in $(4.2.6) - (4.2.8)$, and the decomposition in Lemma $4.3.1$, we have

$$
\begin{aligned}
(Q^{\mathrm{e}} + Q^{\mathrm{i}})\varphi(X) = \quad & \varepsilon \int \Big[-\frac{ik}{2\pi} + K_1(X,Y) + \kappa_{1,\varepsilon}(X,Y) + \frac{\cot k}{\varepsilon^2 k} \\
+ \quad & K_2(X,Y) + \kappa_{2,\varepsilon}(X,Y) \Big] \varphi(Y) dY. \\
= \quad & \beta P\varphi + K\varphi + K_\infty \varphi.
\end{aligned}
$$

The decomposition for $\tilde{Q}^i$ follows by similar calculations. The proof of the invertibility of K is postponed to Chapter 5. □

4.4 Asymptotic expansion of resonances

Note that the scattering problem $(4.1.1)$-$(4.1.5)$ and the system $(4.2.9)$ are equivalent. Thus the resonances of the scattering problem, which are the set of complex-valued frequencies for the homogeneous problem with zero incident field, are the characteristic frequencies k such that $\mathbb{Q}(k)\,\boldsymbol{\varphi} = 0$ attains non-trivial solutions in $(V_1)^2$.

Lemma 4.4.1 *Let $Q_+ = Q^{\mathrm{e}} + Q^{\mathrm{i}} + \tilde{Q}^{\mathrm{i}}$ and $Q_- = Q^{\mathrm{e}} + Q^{\mathrm{i}} - \tilde{Q}^{\mathrm{i}}$, then*

$$
\sigma(Q) = \sigma\big(Q_+\big) \,\cup\, \sigma\big(Q_-\big),
$$

where $\sigma(Q)$, $\sigma(Q_+)$ and $\sigma(Q_-)$ denote the sets of characteristic frequencies k of Q, Q_+ and Q_-, respectively.

Proof. Decomposing function space $(V_1)^2$ as $(V_1)^2 = V_{\text{even}} \oplus V_{\text{odd}}$, where $V_{\text{even}} = \{ [\varphi_+, \varphi_+]^T ; \varphi_+ \in V_1\}$ and $V_{\text{odd}} = \{ [\varphi_-, -\varphi_-]^T ; \varphi_- \in V_1\}$ are invariant subspaces for Q. Thus $\sigma(Q) = \sigma(Q|_{V_{\text{even}}}) \cup \sigma(Q|_{V_{\text{odd}}})$. By observing that

$$\begin{bmatrix} Q^{\text{e}} + Q^{\text{i}} & \tilde{Q}^{\text{i}} \\ \tilde{Q}^{\text{i}} & Q^{\text{e}} + Q^{\text{i}} \end{bmatrix} \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix} = \begin{bmatrix} Q_+\varphi_+ \\ Q_+\varphi_+ \end{bmatrix},$$

it follows that $\sigma(Q|_{V_{\text{even}}}) = \sigma\big(Q_+\big)$, and similarly $\sigma(Q|_{V_{\text{odd}}}) = \sigma\big(Q_-\big)$. □

By the virtue of Lemma 4.3.2, we have

$$\begin{aligned} Q_\pm = & \quad Q^{\text{e}} + Q^{\text{i}} \pm \tilde{Q}^{\text{i}} \\ = & \quad (\beta \pm \tilde{\beta})P + K + K_\infty \pm \tilde{K}_\infty =: P_\pm + L_\pm, \end{aligned}$$

where $P_\pm = (\beta \pm \tilde{\beta})P$ and $L_\pm = K + K_\infty \pm \tilde{K}_\infty$. Furthermore, the following lemma holds.

Lemma 4.4.2 *$L_\pm$ is invertible for sufficiently small ε, and there holds*

$$L_\pm = K + K_\infty \pm \tilde{K}_\infty,$$

$$L_\pm^{-1}1 = K^{-1}1 + O(\varepsilon^2),$$

$$\langle L_\pm^{-1}1, 1\rangle = \gamma + O(\varepsilon^2), \tag{4.4.13}$$

where $\gamma := \langle K^{-1}1, 1\rangle_{L^2(R_1)}$.

We first solve

$$Q_+\varphi = (P_+ + L_+)\varphi = 0,$$

which is equivalent to

$$L_+^{-1}P_+\varphi + \varphi = 0. \tag{4.4.14}$$

Note that

$$
\begin{aligned}
P_+\varphi &= \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) P\varphi \\
&= \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle \varphi, 1\rangle 1,
\end{aligned}
$$

it follows that

$$
L_+^{-1}P_+\varphi = \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) L_+^{-1}1\langle \varphi, 1\rangle.
$$

Substituting it into (4.4.14) and taking inner product with the constant function 1 yields

$$
\left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_+^{-1}1, 1\rangle\langle \varphi, 1\rangle + \langle \varphi, 1\rangle = 0.
$$

We obtain the corresponding resonance condition

$$
\theta_+(k,\varepsilon) := 1 + \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_+^{-1}1, 1\rangle = 0.
$$

Similar calculations for the equation

$$
Q_-\varphi(X) = (P_- + L_-)\varphi(X) = 0
$$

yields the second resonance condition

$$
\theta_-(k,\varepsilon) := 1 + \left(-\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_-^{-1}1, 1\rangle = 0.
$$

Lemma 4.4.3 *The resonances of the scattering problem are the roots of the functions* $\theta_\pm(k,\varepsilon) = 0$.

Theorem 4.4.4 *The scattering resonances of (4.1.1) attain the following resonance expansions:*

$$k_n = n\pi + \frac{2n\pi}{\gamma}\varepsilon - in^2\pi\varepsilon^2 + O(\varepsilon^3), \quad n = 1, 2, 3, \ldots \quad and \quad n\varepsilon \ll 1, \tag{4.4.15}$$

where $\gamma = \langle K^{-1}1, 1\rangle$ *is defined in Lemma* 4.4.2.

Proof. We solve for the roots of

$$\theta_+(k,\varepsilon) = 1 + \left(\frac{1}{\varepsilon}\left(\frac{1}{k\sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon}{2\pi}\right)\langle L_+^{-1}1, 1\rangle = 0,$$

or equivalently

$$p_+(k,\varepsilon) = \varepsilon\theta_+ = \varepsilon + \left[\left(\frac{1}{k\sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon^2}{2\pi}\right](\gamma + r(k,\varepsilon)) = 0, \quad r(k,\varepsilon) \sim O(k^2\varepsilon^2).$$

Let

$$b(k) = \frac{1}{k\sin k} + \frac{\cot k}{k}.$$

The leading order term $b(k)$ of $\theta_+(k,\varepsilon)$ attains roots $k_0 = n\pi$ for odd integers n. To this end, we consider the domain for some fixed number $C \geq 0$

$$W_{\delta_o,\theta_o,C} = \{z \mid |z| \geq \delta\} \cup \{z : \ |z| \ \leq C, -(\pi - \theta_o) \leq argz \leq (\pi - \theta_o)\}, \quad \delta > 0.$$

To derive the leading order asymptotic term of k_n,

define

$$p_{+,1}(k,\varepsilon) = \varepsilon + \left(b(k) - \frac{i\varepsilon^2 k}{2\pi}\right)\gamma. \tag{4.4.16}$$

Then

$$p_{+,1}(k,\varepsilon) = \varepsilon + \left[b'(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0\right]\gamma. \tag{4.4.17}$$

To find the value of $b'(k_0)$, first we calculate $b'(k)$ as follow

$$\begin{aligned}
b'(k) = \quad & \Big(\frac{1+\cos k}{k\sin k}\Big)' \\
= \quad & \frac{-k\sin^2 k - (1+\cos k)(\sin k + k\cos k)}{(k\sin k)^2} \\
= \quad & \frac{-k\sin^2 k - \sin k - \cos k\sin k - k\cos k - k\cos^2 k}{(k\sin k)^2} \\
= \quad & \frac{-k(\sin^2 k + \cos^2 k) - \cos k\sin k - \sin k - k\cos k}{(k\sin k)^2} \\
= \quad & \frac{-k(1+\cos k) - \sin k(\cos k + 1)}{(k\sin k)^2} \\
= \quad & -\frac{(k+\sin k)(1+\cos k)}{k^2(1+\cos k)(1-\cos k)} = -\frac{k+\sin k}{k^2(1-\cos k)}.
\end{aligned}$$

Next to find the value of $b'(k_0) = n\pi$, substituting k_0 in the above derivative, we get

$$b'(k_0) = \quad -\frac{k_0 + \sin k_0}{k_0^2(1-\cos k_0)} = -\frac{1}{2k_0}$$

by using the fact that $\sin k_0 = 0$ and $\cos k_0 = -1$. We see that $p_{+,1}$ has simple roots in $W_{\delta_o,\theta_o,C}$ which are close to k_0's. By expanding the roots $k_{n,1}$ of $p_{+,1}$ in terms of ε, we obtain

$$\begin{aligned}
\varepsilon + \left[b^{'}(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0\right]\gamma = 0 \\
b^{'}(k_0)(k-k_0) + O(k-k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0 = -\frac{\varepsilon}{\gamma} \\
-\frac{1}{2k_0}(k-k_0) - \frac{i\varepsilon^2}{2\pi}k_0 + O(\varepsilon^3) - \frac{\varepsilon}{\gamma},
\end{aligned}$$

at $k_0 = n\pi$, the expansion is given as follows

$$k_{n,1} = n\pi + \frac{2n\pi}{\gamma}\varepsilon - in^2\pi\varepsilon^2 + O(\varepsilon^3).$$

To prove that $k_{n,1}$ is the leading order term of the asymptotic expansion of k_n, note that

$$p_+(k,\varepsilon) - p_{+,1}(k,\varepsilon) = (p_{+,1}(k,\varepsilon) - \varepsilon^2) \cdot O(\varepsilon).$$

One can find a constant $M > 0$ such that

$$|p_+(k,\varepsilon) - p_{+,1}(k,\varepsilon)| < |p_{+,1}(k,\varepsilon)|$$

for all k such that $|k - k_{n,1,1}| = M\varepsilon^3$. Hence we obtain the expansion $(4.4.15)$ for odd integers n by the Rouches's theorem [2]. By similar calculations for

$$\theta_-(k,\varepsilon) = 1 + \left(-\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right)\langle L_-^{-1}1, 1\rangle = 0.$$

we obtain $(4.4.15)$ for even n. □

4.5 Quantitative analysis of the field enhancement at the resonant frequencies

To investigate the field amplification, we first study the solution of $(4.2.9)$.

4.5.1 Solution of the system $(4.2.9)$

Decompose the system $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$ as

$$\mathbb{Q}\boldsymbol{\varphi}_{\text{even}} = \mathbf{f}_{\text{even}} \quad \text{and} \quad \mathbb{Q}\boldsymbol{\varphi}_{\text{odd}} = \mathbf{f}_{\text{odd}},$$

where $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\text{even}} + \boldsymbol{\varphi}_{\text{odd}}$ and $\mathbf{f} = \mathbf{f}_{\text{even}} + \mathbf{f}_{\text{odd}}$, with

$$\boldsymbol{\varphi}_{\text{even}} = \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix}, \quad \boldsymbol{\varphi}_{\text{odd}} = \begin{bmatrix} \varphi_- \\ -\varphi_- \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{\text{even}} = \begin{bmatrix} \frac{f}{2\varepsilon} \\ \frac{f}{2\varepsilon} \end{bmatrix}, \quad \mathbf{f}_{\text{odd}} = \begin{bmatrix} \frac{f}{2\varepsilon} \\ -\frac{f}{2\varepsilon} \end{bmatrix}.$$

$\mathbb{Q}\boldsymbol{\varphi}_{\text{even}} = \mathbf{f}_{\text{even}}$ implies that

$$\begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix} \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \frac{f}{2} \\ \frac{f}{2} \end{bmatrix},$$

which is equivalent to solving

$$Q_+\varphi_+ = \varepsilon^{-1}\frac{f}{2}, \quad \text{where} \quad Q_+ = Q^{\mathrm{e}} + Q^{\mathrm{i}} + \tilde{Q}^{\mathrm{i}}.$$

$\mathbb{Q}\boldsymbol{\varphi}_{\text{odd}} = \mathbf{f}_{\text{odd}}$ implies that

$$\begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix} \begin{bmatrix} \varphi_- \\ -\varphi_- \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \frac{f}{2} \\ -\frac{f}{2} \end{bmatrix},$$

which is equivalent to solving $Q_-\varphi_- = \varepsilon^{-1}\frac{f}{2}$, where $Q_- = Q^{\mathrm{e}} + Q^{\mathrm{i}} - \tilde{Q}^{\mathrm{i}}$.

Recall that

$$p_\pm(k,\varepsilon) = \varepsilon\theta_+ = \varepsilon + \left[\left(\pm\frac{1}{k\sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon^2}{2\pi}\right](\gamma + r(k,\varepsilon)), \quad r(k,\varepsilon) \sim O(k^2\varepsilon^2).$$

Lemma 4.5.1 *The following asymptotic expansion holds for the solutions φ_+ and φ_- in V_1:*

$$\varphi_\pm \;=\; K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{p_\pm}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon),$$

where $d = (d_1, d_2, -d_3)$ is the incident direction. In addition,

$$\langle\varphi_\pm, 1\rangle = \frac{1}{p_\pm}\Big(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\Big). \tag{4.5.18}$$

Proof. Let us consider $Q_+\varphi_+ = \varepsilon^{-1}\frac{f}{2}$. By the same calculations in Section 4, we have

$$L_+^{-1}P_+\varphi_+ + \varphi_+ = L_+^{-1}\frac{f}{2\varepsilon}, \tag{4.5.19}$$

or equivalently,

$$\tilde{b}(k)L_+^{-1}1\langle\varphi_+,1\rangle+\varphi_+=L_+^{-1}\frac{f}{2\varepsilon}, \tag{4.5.20}$$

where $\tilde{b}(k)=\frac{1}{\varepsilon}\left(\frac{1}{k\sin k}+\frac{\cot k}{k}\right)-\frac{ik\varepsilon}{2\pi}$. Thus by taking inner product with 1 on both sides of (4.5.20), we get

$$\langle\varphi_+,1\rangle+\tilde{b}(k)\langle L_+^{-1}1,1\rangle\langle\varphi_+,1\rangle=\langle L_+^{-1}\frac{f}{2\varepsilon},1\rangle,$$

and it follows that

$$\langle\varphi_+,1\rangle=\theta_+^{-1}\langle L_+^{-1}\frac{f}{2\varepsilon},1\rangle. \tag{4.5.21}$$

Note that

$$L_+^{-1}\frac{f}{2\varepsilon}=\frac{1}{\varepsilon}\left(1+\frac{(d_1+d_2)}{2}O(k\varepsilon)\right)\left(K^{-1}1+O(k\varepsilon)^2\right).$$

Substituting it into (4.5.21) yields

$$\langle\varphi_+,1\rangle=\frac{1}{p_+}\Big(\gamma+\frac{(d_1+d_2)}{2}O(k\varepsilon)+O(k\varepsilon)^2\Big).$$

By substituting the above into (4.5.20),

$$\begin{aligned}\varphi_+= &\quad L_+^{-1}\frac{f}{2\varepsilon}-\tilde{b}(k)L_+^{-1}1\langle\varphi_+,1\rangle\\ = &\quad L_+^{-1}\frac{f}{2\varepsilon}-\tilde{b}(k)\theta_+^{-1}\langle L_+^{-1}\frac{f}{2\varepsilon},1\rangle L_+^{-1}1\\ = &\quad L_+^{-1}\frac{f}{2\varepsilon}-\frac{\tilde{b}(k)\langle L_+^{-1}1,1\rangle}{\langle L_+^{-1}1,1\rangle\theta_+}\langle L_+^{-1}\frac{f}{2\varepsilon},1\rangle L_+^{-1}1\\ = &\quad L_+^{-1}\frac{f}{2\varepsilon}+\frac{1-(1+\tilde{b}(k))\langle L_+^{-1}1,1\rangle}{\langle L_+^{-1}1,1\rangle\theta_+}\langle L_+^{-1}\frac{f}{2\varepsilon},1\rangle L_+^{-1}1,\end{aligned}$$

by substituting the value of θ_+, we obtain

$$\varphi_+ = \varepsilon^{-1} L_+^{-1} \frac{f}{2} + \frac{1-\theta_+}{\langle L_+^{-1} 1, 1\rangle \theta_+} \langle L_+^{-1} \frac{f}{2\varepsilon}, 1\rangle \cdot L_+^{-1} 1, \tag{4.5.22}$$

where

$$\begin{aligned} L_+^{-1} = & \quad K^{-1}1 + O(k\varepsilon)^2, \\ \frac{f}{2\varepsilon} = & \quad \frac{1}{\varepsilon} \cdot 1 + \frac{(d_1+d_2)}{2} O(k) \quad \text{in} \quad V_2 \times V_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon\varphi_+ = & \quad \left(1 + \frac{(d_1+d_2)}{2} \cdot O(k\varepsilon)\right) K^{-1}1 + O(k^2\varepsilon^2) \\ & + \frac{1-\theta_+}{(\gamma + O(k\varepsilon)^2)\theta_+} \left[\gamma + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right] [K^{-1}1 + O(k\varepsilon)^2] \\ = & \quad K^{-1}1 + \left(K^{-1}1 \frac{(d_1+d_2)}{2} \cdot O(k\varepsilon)\right) \\ & + \frac{1-\theta_+}{(\gamma + O(k\varepsilon)^2)\theta_+} \left[\gamma K^{-1}1 + K^{-1}1.1 \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon^2) \\ = & \quad K^{-1}1 + \left(K^{-1}1 \frac{(d_1+d_2)}{2} \cdot O(k\varepsilon)\right) \\ & + \frac{\gamma(1-\theta_+)}{(\gamma + O(k\varepsilon)^2)\theta_+} \left[K^{-1}1 + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon^2) \\ = & \quad \frac{(d_1+d_2)}{2} O(k\varepsilon) K^{-1}1 + \frac{1}{\theta_+} \left[K^{-1}1 + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k\varepsilon)^2. \end{aligned}$$

$$\varphi_+ = K^{-1}1 \left(\frac{(d_1+d_2)}{2} O(k)\right) + \frac{1}{p_+} \left[K^{-1}1 + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon).$$

Similarly,

$$\varphi_- = K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{p_-}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon).$$

□

Corollary 4.5.2 *Let* $\boldsymbol{\varphi} = [\varphi_1, \varphi_2]^T$ *be the solution of the system* $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$*, then* $\boldsymbol{\varphi} = [\varphi_+ + \varphi_-,\ \varphi_+ - \varphi_-]^T$*, where* $\varphi_\pm$ *are defined in Lemma 4.5.1. Furthermore,*

$$\langle \varphi_1, 1\rangle = \langle \varphi_+ + \varphi_-, 1\rangle = \left(\frac{1}{p_+} + \frac{1}{p_-}\right)\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right],$$

$$\langle \varphi_2, 1\rangle = \langle \varphi_+ - \varphi_-, 1\rangle = \left(\frac{1}{p_+} - \frac{1}{p_-}\right)\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right].$$

It was observed in the two dimensional case [26] that at resonant frequencies $\boldsymbol{\varphi} \sim O(1/\varepsilon) \quad in \quad (V_1 \times V_1)$ and $\langle \varphi_i, 1\rangle \sim O(1/\varepsilon), \quad i = 1, 2.$

Lemma 4.5.3 *If* $n\varepsilon \ll 1$*, then*

$$p_+(k,\varepsilon) = -\frac{in\gamma}{2}\varepsilon^2 + O(\varepsilon^3), \quad p_-(k,\varepsilon) = \Big(\frac{\cos k - 1}{k\sin k}\Big)\gamma + O(\varepsilon)$$

and

$$p_+(k,\varepsilon) = \Big(\frac{\cos k + 1}{k\sin k}\Big)\gamma + O(\varepsilon), \quad p_-(k,\varepsilon) = -\frac{in\gamma}{2}\varepsilon^2 + O(\varepsilon^3),$$

at the odd and even resonant frequencies $k = Rek_n$*, respectively.*

Proof. Let us consider $p_+(k,\varepsilon)$. First assume that $|k - \text{Re}k_n| \le \varepsilon$ with odd integers n. From the definition of $p_{+,1}$ and its expansion, it follows that

$$\begin{aligned} p_+(k,\varepsilon) = & \quad p_{+,1}(k,\varepsilon) + O(\varepsilon^3) \\ = & \quad p_1'(k_n)(k-k_n) + O(k-k_n)^2 + O(\varepsilon^3) \\ = & \quad \gamma b'(k_0)\cdot(k-k_n) + O(\varepsilon^3) \\ = & \quad \frac{\gamma}{2n\pi}(k - \text{Re}k_n - i\text{Im}k_n) + O(\varepsilon^3). \end{aligned}$$

Since

$$\text{Im}k_n = \text{Im}k_{n,1} + O(\varepsilon^3) = -n^2 i\pi\varepsilon^2 + O(\varepsilon^3),$$

we obtain

$$p_+(k,\varepsilon) = -\frac{in\varepsilon^2\gamma}{2} + O(\varepsilon^3).$$

To derive the expression for $p_-(k,\varepsilon)$ at $k = Rek_n$ for odd n, recall that

$$p_-(k,\varepsilon) = p_{-,1}(k,\varepsilon) + O(\varepsilon^2) = \varepsilon + \Big(c(k) - \frac{ik\varepsilon^2}{2\pi}\Big)\gamma + O(\varepsilon^2).$$

$c(k) = \frac{\cot k}{k} - \frac{1}{k\sin k}$ is well defined for $k = \text{Re}k_n$, hence

$$p_-(k,\varepsilon) = \Big(\frac{\cos k - 1}{k\sin k}\Big)\gamma + O(\varepsilon).$$

The calculations for $p_\pm(k,\varepsilon)$ at the even resonant frequencies follow similarly. $\square$

Proposition 4.5.4 *There hold $\varphi_1,\varphi_2 \sim O(1/\varepsilon^2)$ in V_1, and $\langle\varphi_1,1\rangle,\langle\varphi_2,1\rangle \sim O(1/\varepsilon^2)$ at the even and odd resonant frequencies $k = Rek_n$.*

It was observed in the two dimensional case [26] that at resonant frequencies $\boldsymbol{\varphi} \sim O(1/\varepsilon) \quad in \quad (V_1\times V_1)$ and $\langle\varphi_i,1\rangle \sim O(1/\varepsilon), \quad i=1,2.$

4.5.2 Field enhancement in the hole

To investigate the field inside the hole, note that u_ε satisfies the following boundary value problem

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in} \quad C_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_1} = 0, \quad \text{on} \quad x_1 = 0, x_1 = \varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_2} = 0, \quad \text{on} \quad x_2 = 0, x_2 = \varepsilon. \end{cases}$$

Then $u_\varepsilon(x)$ can be expanded as

$$\begin{aligned} u_\varepsilon(x) = \quad & a_{00}\cos(kx_3) + b_{00}\cos k(1-x_3) \\ & + \sum_{m\geq 1}\Big(a_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}x_3\right) + b_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}(1-x_3)\right)\Big) \\ & + \sum_{n\geq 1}\Big(a_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}x_3\right) + b_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}(1-x_3)\right)\Big) \\ & + \sum_{m,n\geq 1}\Big(a_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}x_3\right) \\ & + b_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}(1-x_3)\right)\Big) \end{aligned} \tag{4.5.23}$$

where $k_{mn} = \sqrt{(\frac{m\pi}{\varepsilon})^2 - (\frac{n\pi}{\varepsilon})^2 - k^2}$.

Lemma 4.5.5 *The following hold for the expansion coefficients in (4.5.23):*

$$a_{00} = \frac{1}{k \sin k}\left[\gamma + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right]\left(\frac{1}{p_+} + \frac{1}{p_-}\right),$$

$$b_{00} = \frac{1}{k \sin k}\left[\gamma + \frac{(d_1+d_2)}{2} O(k\varepsilon) + O(k\varepsilon)^2\right]\left(\frac{1}{p_+} - \frac{1}{p_-}\right),$$

$$\varepsilon\sqrt{m}|a_{m0}| \le C, \quad \varepsilon\sqrt{m}|b_{m0}| \le C \quad for \quad m \ge 1,$$

$$\varepsilon\sqrt{n}|a_{0n}| \le C, \quad \varepsilon\sqrt{n}|b_{0n}| \le C \quad for \quad n \ge 1$$

$$\sqrt{m+n}|a_{mn}| \le C, \quad \sqrt{m+n}|b_{mn}| \le C \quad for \quad m, n \ge 1,$$

where C is a positive constant independent of ε, k, m and n.

Proof. Taking the derivative of the expansion $(4.5.23)$ with respect to x_3 gives

$$\frac{\partial u_\varepsilon}{\partial x_3} = -a_{00}\sin(kx_3) + b_{00}\sin k(1-x_3)$$

$$+\sum_{m\ge1}\Big(-a_{m0}k_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}x_3\right) + b_{m0}k_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}(1-x_3)\right)\Big)$$

$$+\sum_{n\ge1}\Big(-a_{0n}k_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}x_3\right) + b_{0n}k_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}(1-x_3)\right)\Big)$$

$$+\sum_{m,n\ge1}\Big(-a_{mn}k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}x_3\right)$$

$$+b_{mn}k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}(1-x_3)\right)\Big).$$

Evaluating at $\Gamma_\pm$ gives

$$\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,1) = \quad -a_{00}\sin(kx_3) + \sum_{m\geq 1}\left(-a_{m0}\exp\left(-k_{m0}\right) + b_{m0}\right)k_{m0}\cos\frac{m\pi x_1}{\varepsilon}$$

$$+\sum_{n\geq 1}\left(-a_{0n}\exp\left(-k_{0n}\right) + b_{0n}\right)k_{0n}\cos\frac{n\pi x_2}{\varepsilon}$$

$$+\sum_{m,n\geq 1}\left(-a_{mn}\exp\left(-k_{mn}\right) + b_{mn}\right)k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}, \qquad (4.5.24)$$

$$\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,0) = \quad b_{00}\sin(kx_3) + \sum_{m\geq 1}\left(-a_{m0} + b_{m0}\exp\left(-k_{m0}\right)\right)k_{m0}\cos\frac{m\pi x_1}{\varepsilon}$$

$$+\sum_{n\geq 1}\left(-a_{0n} + b_{0n}\exp\left(-k_{0n}\right)\right)k_{0n}\cos\frac{n\pi x_2}{\varepsilon}$$

$$+\sum_{m,n\geq 1}\left(-a_{mn} + b_{mn}\exp\left(-k_{mn}\right)\right)k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}. \qquad (4.5.25)$$

Integrating over the hole apertures and using Corollary 4.5.2 leads to

$$-a_{00}k\sin k = \quad \frac{1}{\varepsilon^2}\int_{\Gamma+}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,1)dx_1dx_2 = -\int_{R_1}\varphi_1(X)dX$$

$$= \quad -\left(\frac{1}{p_+}+\frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right),$$

$$b_{00}k\sin k = \quad \frac{1}{\varepsilon^2}\int_{\Gamma-}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,0)dx_1dx_2 = \int_{R_1}\varphi_2(X)dX$$

$$= \quad \left(\frac{1}{p_+}-\frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).$$

We obtain the desired formulas for a_{00} and b_{00}. For coefficients a_{m0}, b_{m0}, taking inner product of eqs $(4.5.24)$ and $(4.5.25)$ with $\cos\frac{m\pi x_1}{\varepsilon}$ and integrating over aperture yields

$$a_{m0}k_{m0} = \frac{-2}{1-e^{-2k_{m0}}}\left(e^{-k_{m0}}\int_{R_1}\varphi_1(X)\cos(m\pi X)dX + \int_{R_1}\varphi_2(X)\cos(m\pi X)dX\right),$$

$$b_{m0}k_{m0} = \frac{-2}{1-e^{-2k_{m0}}}\left(\int_{R_1}\varphi_1(X)\cos(m\pi X)dX + e^{-k_{m0}}\int_{R_1}\varphi_2(X)\cos(m\pi X)dX\right).$$

Note that $k_{m0} = O(\frac{m}{\varepsilon})$ for $m \geq 1$, and

$$\|\varphi_1\|_{V_1} \leq \frac{1}{\varepsilon^2}, \quad \|\varphi_2\|_{V_1} \leq \frac{1}{\varepsilon^2}, \quad \|\cos(m\pi X)\|_{V_2} \leq \sqrt{m}.$$

The estimate for a_{m0} and b_{m0} follows. A parallel calculation can also be applied for a_{0n} and b_{0n}.

For coefficients a_{mn}, b_{mn} taking inner product of $(4.5.24)$ and $(4.5.25)$ with $\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}$ and integrating over the aperture yields

$$\begin{aligned} a_{mn}k_{mn} = \quad & \frac{-4}{1-e^{-2k_{mn}}}\Big(e^{-k_{mn}}\int_{R_1}\varphi_1(X)\cos(m\pi X_1)\cos(n\pi X_2)dX_1dX_2 \\ & + \int_{R_1}\varphi_2(X)\cos(m\pi X_1)\cos(n\pi X_2)dX_1dX_2\Big), \\ b_{mn}k_{mn} = \quad & \frac{-4}{1-e^{-2k_{mn}}}\Big(\int_{R_1}\varphi_1(X)\cos(m\pi X_1)\cos(n\pi X_2)dX_1 \\ & + e^{-k_{mn}}\int_{R_1}\varphi_2(X)\cos(m\pi X_1)\cos(n\pi X_2)dX_1dX_2\Big). \end{aligned}$$

Since $k_{mn} = O(\sqrt{\frac{m^2+n^2}{\varepsilon^2}})$ and $\|\varphi_1\|_{V_1} \leq \frac{1}{\varepsilon^2}, \quad \|\varphi_2\|_{V_1} \leq \frac{1}{\varepsilon^2}$, the desired formulas for a_{mn} and b_{mn} follow. □

Theorem 4.5.6 *The wave field in the hole $C_\varepsilon^{int} := \{x \in C_\varepsilon \mid x_3 \gg \varepsilon, 1 - x_3 \gg \varepsilon\}$ is given by*

$$u_\varepsilon(x) = \left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)} + \frac{\sin(k(x_3-1/2))}{\sin(k/2)} +O(\exp(-1/\varepsilon^2)),$$

$$u_\varepsilon(x) = -\left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\sin(k(x_3-1/2))}{nk\cos(k/2)} + \frac{\cos(k(x_3-1/2))}{k\sin(k/2)} +O(\exp(-1/\varepsilon^2))$$

at the odd and even resonant frequencies $k = Rek_n$ respectively.

Proof. From Lemma 4.5.5,

$$\begin{aligned} u_\varepsilon(x) = \quad & a_{00}\cos(kx_3) + b_{00}\cos k(1-x_3) \\ & + \sum_{m\geq 1}\left(a_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}x_3\right) + b_{m0}\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_{m0}(1-x_3)\right)\right) \\ & + \sum_{n\geq 1}\left(a_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}x_3\right) + b_{0n}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{0n}(1-x_3)\right)\right) \\ & + \sum_{m,n\geq 1}\left(a_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}x_3\right)\right. \\ & \left. +b_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp\left(-k_{mn}(1-x_3)\right)\right). \end{aligned}$$

For $\varepsilon \ll 1$,

$$u_\varepsilon(x) = \left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\left[\left(\frac{1}{p_+}+\frac{1}{p_-}\right)\frac{\cos kx_3}{k\sin k} + \left(\frac{1}{p_+}-\frac{1}{p_-}\right)\frac{\cos k(1-x_3)}{k\sin k}\right]$$
$$+O(\exp(-1/\varepsilon^2))$$

$$= \left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\Big[\frac{1}{p_+}\Big(\frac{\cos kx_3 + \cos k(1-x_3)}{k\sin k}\Big)$$

$$+\frac{1}{p_-}\Big(\frac{\cos kx_3 - \cos k(1-x_3)}{k\sin k}\Big)\Big] + O(\exp(-1/\varepsilon^2)$$

$$= \left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\Big[\frac{1}{p_+}\Big(\frac{2\cos(kx_3-k-kx_3)\cos(kx_3-k+kx_3)}{k\sin k}\Big)$$

$$+\frac{1}{p_-}\Big(\frac{-2\sin(kx_3-k-kx_3)\sin(kx_3-k+kx_3)}{k\sin k}\Big)\Big] + O(\exp(-1/\varepsilon^2)$$

$$= 2\left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\Big[\frac{1}{p_+}\frac{\cos(k/2)\cos(k(x_3-1/2))}{k\sin k}$$

$$-\frac{1}{p_-}\frac{\sin(k/2)\sin(k(x_3-1/2))}{k\sin k}\Big] + O(\exp(-1/\varepsilon^2)).$$

At the odd resonant frequencies $k = Rek_n$, it follows from Lemma 4.5.3 that $\frac{1}{p_+} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon))$ and $\frac{1}{p_-} = \frac{k\sin k}{(\cos k - 1)\gamma}(1+O(\varepsilon))$. Therefore,

$$u_\varepsilon(x) = 2\left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\Big[\frac{2i}{\gamma n\varepsilon^2}\frac{\cos(k/2)\cos\left(k(x_3-1/2)\right)}{k\sin k}(1+O(\varepsilon))$$

$$-\frac{k\sin k}{(\cos k-1)\gamma}\frac{\sin\left(k/2\right)\sin\left(k(x_3-1/2)\right)}{k\sin k}(1+O(\varepsilon))\Big] + O(\exp(-1/\varepsilon^2))$$

$$= [1 + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2]\Big[\Big(\frac{1}{\varepsilon^2}\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)}(1+O(\varepsilon))$$

$$+\frac{\sin(k(x_3-1/2))}{k\sin(k/2)}(1+O(\varepsilon)\Big)\Big] + \exp(-1/\varepsilon^2)$$

$$= \left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)} + \frac{\sin(k(x_3-1/2))}{\sin(k/2)}$$
$$+O(\exp(-1/\varepsilon^2)).$$

Similarly, at the even resonant frequencies, $\frac{1}{p_+} = \frac{k\sin k}{(\cos k+1)\gamma}(1+O(\varepsilon))$ and $\frac{1}{p_-} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon))$. We obtain

$$u_\varepsilon(x) = -\left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\sin(k(x_3-1/2))}{nk\cos(k/2)}$$

$$+\frac{\cos(k(x_3-1/2))}{k\sin(k/2)} + O(\exp(-1/\varepsilon^2)).$$

□

Therefore, scattering field amplification is of order $O(1/\varepsilon^2)$ in the hole, as the leading order term is of order $O(1/\varepsilon^2)$.

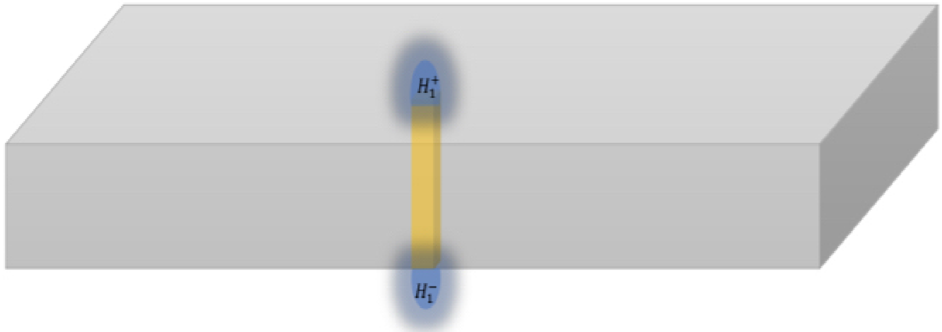

Figure 4.2: Depiction of upper and lower far fields H_1^+, and H_1^- respectively

4.5.3 Scattering enhancement in the far field

Consider the domain $\Omega^+ \backslash H_1^+$ above the hole, where $H_1^+ := \{x | x - (0,0,1) \leq 1\}$. Recall that

$$u_s^\varepsilon(x) = \int_{\Gamma^+} g^e(x,y) \frac{\partial u_\varepsilon}{\partial \nu} ds_y, \quad x \in \Omega^+,$$

and

$$\frac{\partial u_\varepsilon}{\partial \nu}(x_1, x_2, 1) = -\varphi_1(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}).$$

Therefore,

$$\begin{aligned} u_s^\varepsilon(x) = & \quad -\int_{\Gamma^+} g^e(x,(y_1,y_2,1))\varphi_1(\frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon}) dy_1 dy_2 \\ = & \quad -\varepsilon^2 \int_0^1 \int_0^1 g^e(x, \varepsilon Y_1, \varepsilon Y_2, 1)\varphi_1(Y_1, Y_2) dY_1 dY_2. \end{aligned}$$

Note that

$$g^e(x, \varepsilon Y_1, \varepsilon Y_2, 1) = g^e(x,(0,0,1))(1 + O(\varepsilon)), \quad x \in \Omega^+ \backslash H_1^+.$$

and

$$\langle \varphi_1, 1 \rangle_{L^2(R_1)} = \left(\frac{1}{p_+} + \frac{1}{p_-} \right) \left(\gamma + \frac{(d_1 + d_2)}{2} O(\varepsilon) + O(\varepsilon)^2 \right).$$

It follows that

$$u_s^\varepsilon(x) = -\varepsilon^2 g^e(x, (0,0,1))(1 + O(\varepsilon)) \left(\frac{1}{p_+} + \frac{1}{p_-} \right) \left(\gamma + \frac{(d_1 + d_2)}{2} O(\varepsilon) + O(\varepsilon)^2 \right).$$

From Lemma 4.5.3,

$$\frac{1}{p_+} = \frac{2i}{\gamma n \varepsilon^2}(1 + O(\varepsilon)) \quad \text{and} \quad \frac{1}{p_-} = \frac{2i}{\gamma n \varepsilon^2}(1 + O(\varepsilon))$$

when n is even and odd respectively. The corresponding scattered field is

$$u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,1) + O(\varepsilon), \quad u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,1) + O(\varepsilon).$$

Similarly, the scattered field in the domain $H_1^- := \{x | x - (0,0,0) \leq 1\}$ is

$$u_s^\varepsilon(x) = -\varepsilon^2 g^e(x, (0,0,0))(1 + O(\varepsilon)) \left(\frac{1}{p_+} - \frac{1}{p_-} \right) \left(\gamma + \frac{(d_1 + d_2)}{2} O(\varepsilon) + O(\varepsilon)^2 \right).$$

It follows that

$$u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,0) + O(\varepsilon), \quad u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,0) + O(\varepsilon),$$

at the odd and even resonant frequencies, respectively.

Chapter 5

The invertibility of the operator K

Recall the integral operator

$$K\varphi(X) = \int_{R_1} \kappa(X,Y)\varphi(Y)dY, \quad \text{for} \quad X \in R_1 \quad \varphi \in H^{-\frac{1}{2}}(R_1). \tag{5.0.1}$$

Where $\kappa(X,Y)$ is given by $(4.3.10)$. We consider the integral equation $K\varphi = f$, where $f \in H^{\frac{1}{2}}(R_1)$. We extend the argument in [16] to show that K is invertible from $H^{-\frac{1}{2}}(R_1)$ to $H^{\frac{1}{2}}(R_1)$.

Let us consider the domain depicted in Figures 5.1 and 5.2 as $\tilde{\Omega} = \tilde{\Omega}_e \cup \tilde{\Omega}_i$, where $\tilde{\Omega}_e = \mathbb{R}^3_+$, and $\tilde{\Omega}_i = (0,1)^2 \times \mathbb{R}_-$. Let $\tilde{\Omega}_\varepsilon = (0,1)^2 \times (0,\varepsilon)$ be the bounded domain inside the hole with the upper and lower boundary given by $\tilde{\Gamma} = (0,1)^2 \times \{0\}$ and $\tilde{\Gamma}_\varepsilon = (0,1)^2 \times \{-\varepsilon\}$, respectively. Let $u^{\pm}(X) = \lim_{t\to\pm 0} u(X + (t,0,0))$ for $x \in \tilde{\Gamma}$, and $u^{\mp}(X) = \lim_{t\to\pm 0} u(X + (t,0,0))$ on $X \in \tilde{\Gamma}_\varepsilon$. $[u]_{\tilde{\Gamma}}$ represents the jump $u^+(X) - u^-(X)$ for $X \in \tilde{\Gamma}$. The solution of the integral equation $K\varphi = f$

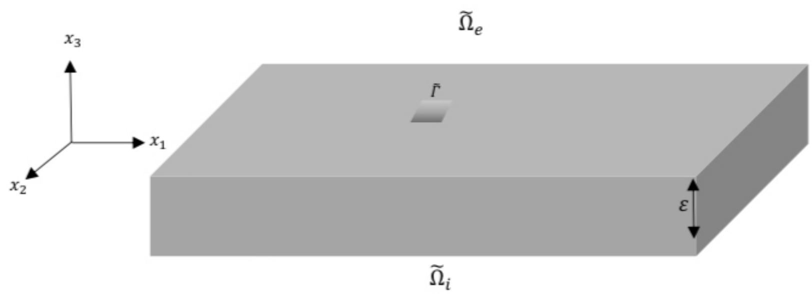

Figure 5.1: (a) Geometry of the problem. The domains above and below the sound hard slab are denoted as $\tilde{\Omega}_e$ and $\tilde{\Omega}_i$. The exterior domain $\tilde{\Omega} = \tilde{\Omega}_e \cup \tilde{\Omega}_i$. The aperture of the hole are denoted by $\tilde{\Gamma}$.

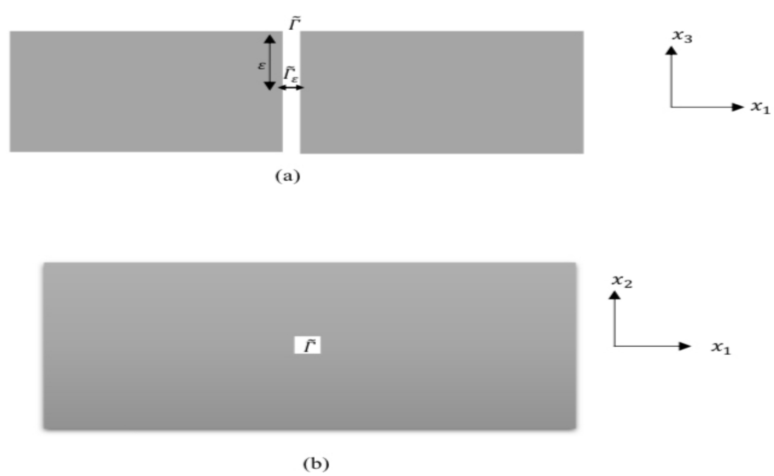

Figure 5.2: (a)(b): Vertical and horizontal cross section of the subwavelenth structure.

is related to the following transmission problem:

$$(A)\begin{cases} \Delta u(X) \;=\; 0, \quad \text{in } \tilde{\Omega}, \\ \frac{\partial u(X)}{\partial \nu} \;=\; 0, \quad \text{on } \partial\tilde{\Omega}, \quad \int_{\tilde{\Gamma}} u^-(X) ds_X \;=\; 0, \quad [u]_{\tilde{\Gamma}} \;=\; f(X), \quad \Big[\dfrac{\partial u(X)}{\partial X_3}\Big]_{\tilde{\Gamma}} = 0, \\ u(X) - X_3 \displaystyle\int_{\tilde{\Gamma}} \dfrac{\partial u(X)}{\partial X_3} ds_X \;=\; o(1), \quad X_3 \to -\infty \quad in \quad \tilde{\Omega}_i, \\ |\Delta(u(X) - X_3 \displaystyle\int_{\tilde{\Gamma}} \dfrac{\partial u(X)}{\partial X_3} ds_X)| \;=\; o(1), \quad X_3 \to -\infty \quad in \quad \tilde{\Omega}_i, \\ u(X) - \frac{1}{\pi|X|} \displaystyle\int_{\tilde{\Gamma}} \dfrac{\partial u(X)}{\partial X_3} ds_X \;=\; O\Big(\dfrac{1}{|X|^2}\Big), \quad |X| \to \infty \quad in \quad \tilde{\Omega}_e, \\ |\Delta u(X) \cdot \frac{X}{|X|} + \frac{1}{\pi|X|^2} \displaystyle\int_{\tilde{\Gamma}} \dfrac{\partial u(X)}{\partial X_3} ds_X \;=\; O\Big(\dfrac{1}{|X|^3}\Big), \quad |X| \to \infty \quad in \quad \tilde{\Omega}_e. \end{cases}$$

(A) can be reformulated in the bounded domain $\tilde{\Omega}_\varepsilon$. To this end, we introduce the Green's function for the exterior domain $\tilde{\Omega}_e$:

$$\begin{cases} \Delta \tilde{G}_{\mathrm{e}}(X,Y) = \delta(X-Y), \quad \text{in} \quad \tilde{\Omega}_{\mathrm{e}}, \\ \frac{\partial \tilde{G}_{\mathrm{e}}(X,Y)}{\partial \nu} \;=\; 0, \quad \text{on } \partial\tilde{\Omega}_e, \\ \tilde{G}_e(X,Y) + \frac{1}{\pi|X|} \;=\; O(\frac{1}{|X|^2}), \quad |X| \to \infty, \\ |\Delta_X \tilde{G}_{\mathrm{e}}(X,Y) \cdot \frac{X}{|X|} + \frac{1}{\pi|X|^2} \;=\; O(\frac{1}{|X|^3}), \quad |X| \to \infty. \end{cases}$$

The method of images shows that $\tilde{G}_e(X,Y) = -\frac{1}{4\pi}\frac{1}{|X-Y|} - \frac{1}{4\pi}\frac{1}{|X-Y'|}$, where $Y' = (Y_1, Y_2, -Y_3)$. The Green's function in $\tilde{\Omega}_i$ satisfies

$$\begin{cases} \Delta \tilde{G}_i(X,Y) = \delta(X-Y) \quad \text{in } \tilde{\Omega}_i, \\ \frac{\partial \tilde{G}_i(X,Y)}{\partial \nu} = 0, \quad \text{on } \partial\tilde{\Omega}_i, \\ \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y) ds_X = 0, \\ \tilde{G}_i(X,Y) = o(1), \quad |\Delta_X \tilde{G}_i(X,Y)| = o(1) \quad \text{as } X_3 \to \infty. \end{cases}$$

It can be shown that

$$\begin{aligned} \tilde{G}_{\rm i}(X,Y) = \quad & -\sum_{m\geq 0, n\geq 0} \frac{2^j}{\pi\sqrt{m^2+n^2}} \left(e^{-\frac{\pi}{2}\sqrt{m^2+n^2}|X_3+Y_3|} + e^{-\frac{\pi}{2}\sqrt{m^2+n^2}|X_3-Y_3|}\right) \\ & . \cos(m\pi X_1)\cos(m\pi Y_1)\cos(n\pi X_2)\cos(n\pi Y_2), \\ & \text{where} \quad j=0 \quad \text{for} \quad m=0 \quad \text{or} \quad n=0 \quad \text{and} \quad j=1 \quad \text{for} \quad m,n \geq 1. \end{aligned}$$

We define two integral operators $\Theta : H^{-\frac{1}{2}}(\tilde{\Gamma}) \to H^{\frac{1}{2}}(\tilde{\Gamma})$ and $\Theta_\varepsilon : H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$:

$$\begin{aligned} \Theta\varphi(X) &= \int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\varphi(Y) ds_Y, \\ \Theta_\varepsilon\varphi(X) &= \int_{\tilde{\Gamma}} \tilde{G}_i(X+(0,\varepsilon,0), Y+(0,\varepsilon,0))\varphi(Y) ds_Y. \end{aligned}$$

Here Θ_ε does not depend on ε and $\Theta_\varepsilon 1(X) = \int_{\tilde\Gamma} \tilde G_i(X,Y)ds_Y$. The bounded value problem in $\tilde\Omega_\varepsilon$ is formulated as follows:

$$(B)\begin{cases} \Delta u(X) = 0, \quad \text{in } \tilde\Omega_\varepsilon, \\ \displaystyle\int_{\tilde\Gamma} u(X)ds_X = 0, \\ \frac{\partial u(X)}{\partial \nu} = 0, \quad \text{on} \quad X_1 = \{0,1\}, X_2 = \{0,1\}, \\ \Theta_\varepsilon(\frac{\partial u(X)}{\partial X_3}) + \varepsilon \displaystyle\int_{\tilde\Gamma} \frac{\partial u(X)}{\partial X_3} ds_X = u(X), \quad \text{on} \quad \tilde\Gamma_\varepsilon, \\ -\Theta(\frac{\partial u(X)}{\partial X_3}) = u(X) + f(X), \quad \text{on } \tilde\Gamma. \end{cases}$$

As shown below, (A) and (B) are equivalent.

5.0.1 Equivalence of the well-posedness of a boundary value problem and the invertibility of K

Lemma 5.0.1 *The following two statements are equivalent:*
(1) K is invertible from $H^{-\frac{1}{2}}(\tilde\Gamma) \to H^{\frac{1}{2}}(\tilde\Gamma)$,
(2) For any function $f \in H^{\frac{1}{2}}(\tilde\Gamma)$, there exists a unique solution to (A).

Proof. If (1) holds, given $f \in H^{\frac{1}{2}}(\tilde\Gamma)$, let $\varphi_f(X) \in H^{-\frac{1}{2}}(\tilde\Gamma)$ be a unique solution to $K\varphi_f(X) = f(x)$. Define u_f in $\tilde\Omega$ by

$$u_f(X) = \begin{cases} -\displaystyle\int_{\tilde\Gamma} \tilde G_e(X,Y)\varphi_f(Y)ds_Y, \quad X \in \tilde\Omega_e, \\ X_3 \displaystyle\int_{\tilde\Gamma} \varphi_f(X)ds_X + \int_{\tilde\Gamma} \tilde G_i(X,Y)\varphi_f(Y)ds_Y, \quad X \in \tilde\Omega_i. \end{cases} \tag{5.0.2}$$

The function $u_f(X)$ is the solution to (A). To prove the uniqueness of the solution, let w_f be a solution to (A), with $[w_f] = f$ on $\tilde{\Gamma}$. Applying the Green's formula in $\tilde{\Omega}$ as

$$\int_{\tilde{\Omega}_e} \Delta \tilde{G}_e(X,Y) w_f(Y) - \Delta w_f(Y) \tilde{G}_e(X,Y) dY = \int_{\partial \tilde{\Omega}_e} \frac{\partial \tilde{G}_e(X,Y)}{\partial \nu} w_f(Y) - \frac{\partial w_f(Y)}{\partial \nu} \tilde{G}_e(X,Y) ds_Y,$$

we have

$$w_f(X) = -\int_{\tilde{\Gamma}} \tilde{G}_e(X,Y) \frac{\partial w_f(Y)}{\partial Y_3} ds_Y, \quad X \in \tilde{\Omega}_e. \tag{5.0.3}$$

Similarly apply Green's formula in $\tilde{\Omega}_i$ as follow

$$\begin{aligned} w_f(X) &= \int_{\partial \tilde{\Omega}_i} \frac{\partial \tilde{G}_i(X,Y)}{\partial \nu} w_f(Y) - \frac{\partial w_f(Y)}{\partial \nu} \tilde{G}_i(X,Y) ds_Y \\ &= -\int_{\partial \tilde{\Omega}_i} \frac{\partial w_f(Y)}{\partial \nu} \tilde{G}_i(X,Y) ds_Y \\ &= \int_{\tilde{\Gamma}} \frac{\partial w_f(Y)}{\partial x_3} \tilde{G}_i(X,Y) ds_Y - \int_{\tilde{\Gamma}_{-\infty}} \frac{\partial w_f(Y)}{\partial x_3} \tilde{G}_i(X,Y) ds_Y \\ &= \int_{\tilde{\Gamma}} \frac{\partial w_f(Y)}{\partial x_3} \tilde{G}_i(X,Y) ds_Y - \int_{\tilde{\Gamma}_{-\infty}} \tilde{G}_i(X,Y)(o(1) + \int_{\tilde{\Gamma}} \frac{\partial w_f(X)}{\partial x_3} ds_X) ds_Y, \end{aligned}$$

we obtain

$$w_f(X) = \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y) \frac{\partial w_f(Y)}{\partial Y_3} ds_Y + X_3 \int_{\tilde{\Gamma}} \frac{\partial w_f(X)}{\partial X_3} ds_X \quad X \in \tilde{\Omega}_i. \tag{5.0.4}$$

Taking trace of $w_f(X)$ on both sides of the boundary $\tilde{\Gamma}$ for $X = (X_1, X_2, 0) \in \tilde{\Gamma}$, we obtain

$$\begin{aligned} f(X) &= \int_0^1 \int_0^1 \Big(\tilde{G}_i(X_1, X_2, 0; Y_1, Y_2, 0) + \tilde{G}_e(X_1, X_2, 0; Y_1, Y_2, 0) \Big) \frac{\partial w_f(Y_1, Y_2, 0)}{\partial Y_3} dY_1 dY_2 \\ &= K \Big[\frac{\partial w_f}{\partial X_3} \Big]. \end{aligned}$$

We infer from (1) that $\frac{\partial w_f}{\partial X_3} = \varphi_f(Y)$. By (5.0.3) and (5.0.4), it follows that $u_f = w_f$ in $\tilde{\Omega}$, which proves the uniqueness of the solution to (A).

Assume that (2) holds. Then from the above, we see that the solution w_f to (A) satisfies (5.0.3) and (5.0.4), and consequently

$$K\varphi = f \tag{5.0.5}$$

has at least one solution $\varphi(Y) = \frac{\partial w_f}{\partial X_3} \in H^{-\frac{1}{2}}(\tilde{\Gamma})$. For $f = 0$, let φ_0 be the corresponding solution of $(5.0.5)$ and construct a solution u_0 to (A) by $(5.0.2)$. Hence by (2), $u_0 \equiv 0$ implies $\varphi_0 = \frac{\partial u_0}{\partial \nu} = 0$. Thus the solution to (5.0.5) is unique.

5.0.2 The equivalence of problem (A) and problem (B)

Lemma 5.0.2 *Problem (A) attains unique solution iff problem (B) has a unique solution.*

Proof. Let u_f be the solution to (A). Applying the Green's formula in $\hat{\Omega} = (0,1)^2 \times (-\varepsilon, -\infty)$ as follow

$$\begin{aligned} u_f(X) &= \int_{\partial\hat{\Omega}} \frac{\partial \tilde{G}_i(X,Y)}{\partial \nu} u_f(Y) ds_Y - \int_{\partial\tilde{\Omega}} \tilde{G}_i(X,Y) \frac{\partial u_f(Y)}{\partial \nu} ds_Y \\ &= -\int_{\partial\hat{\Omega}} \frac{\partial u_f(Y)}{\partial \nu} \tilde{G}_i(X,Y) ds_Y \\ &= -\int_{\tilde{\Gamma}\cup\tilde{\Gamma}_\varepsilon\cup\tilde{\Gamma}_\infty} \frac{\partial u_f(Y)}{\partial \nu} \tilde{G}_i(X,Y) ds_Y \\ &= \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u_f(Y)}{\partial Y_3} \tilde{G}_i(X+(0,\varepsilon), Y+(0,\varepsilon)) ds_Y + \int_{\tilde{\Gamma}_\infty} \tilde{G}_i(X,Y) \Big(o(1) + \int_{\tilde{\Gamma}} \frac{\partial u_f(X)}{\partial X_3} ds_X \Big) ds_Y \end{aligned}$$

we obtain

$$u_f(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(X+(0,\varepsilon,0), Y+(0,\varepsilon,0)) \frac{\partial u_f(Y)}{\partial Y_3} ds_Y + X_3 \int_{\tilde{\Gamma}} \frac{\partial u_f(Y)}{\partial X_3} ds_Y, \quad X \in \hat{\Omega}. \tag{5.0.6}$$

Taking the trace of (5.0.3) on $\tilde{\Gamma}$

$$u_f^+(X) = -\int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\frac{\partial u_f(Y)}{\partial x_3} ds_Y, \quad X \in \tilde{\Gamma}$$

,

or equivalently

$$f(X) = -\Theta(\frac{\partial u_f}{\partial X_3}) - u_f^-(X).$$

Similarly by taking trace of (5.0.4) on $\tilde{\Gamma}_\varepsilon$, we obtain

$$u_f^+(X) = \varepsilon\int_{\tilde{\Gamma}} \frac{\partial u_f(X)}{\partial x_3} ds_X + \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(Y+(0,\varepsilon), X+(0,\varepsilon))\frac{\partial u_f(Y)}{\partial x_3} ds_Y \quad X \in \hat{\Gamma}_\varepsilon,$$

or equivalently

$$u_f^+(X) = \Theta_\varepsilon \frac{\partial u_f}{\partial X_3}(X) + \varepsilon\int_{\tilde{\Gamma}} \frac{\partial u_f(Y)}{\partial X_3} ds_X.$$

Thus u_f is also a solution to (B).

Let $u(X)$ be the solution to (B), using Green's formula, (5.0.2) and (5.0.6), u can be extended continuously to $\tilde{\Omega}$. We claim that such extension is unique. Assume that there are two solutions u_1 and u_2 of (A) that coincides in $\tilde{\Omega}_\varepsilon$. Let $w = u_1 - u_2$ be the solution of the following system

$$\begin{cases} \Delta w(X) = 0, \quad \text{in} \quad \tilde{\Omega}\backslash\bar{\tilde{\Omega}}_\varepsilon, \\ \frac{\partial w(X)}{\partial \nu} = 0, \quad \partial\tilde{\Omega}_e \cup \{X_1 = \{0,1\}, X_2 = \{0,1\}\} \times (-\varepsilon, -\infty), \\ w(X) = 0, \quad \text{on} \quad \tilde{\Gamma} \cup \tilde{\Gamma}_\varepsilon, \\ w(X) = o(1) \quad |\nabla w(X)| = o(1), \qquad X_3 \to -\infty, \\ w(X) = O(\frac{1}{X^2}), \quad |\nabla w(X)| \cdot \frac{X}{|X|} = O(\frac{1}{|X|^3}), \qquad |X| \to \infty. \end{cases}$$

Let C_R^+ be the upper half sphere of radius R and center $(0,0,0)$ in $\tilde{\Omega}_e$. let $S_R^+ = \partial S_R^+ \cap \tilde{\Omega}_e$. We multiply Δw by $\bar{w}(X)$ and integrate by parts over C_R^+ to obtain

$$\int_{C_R^+} |\nabla w|^2 dX = \int_{S_R^+} \nabla w(X) \cdot \frac{X}{|X|} ds_X = O\Big(\frac{1}{R^3}\Big) \quad \text{as} \quad R \to +\infty.$$

Hence $w(X)$ is constant in C_R^+,therefore $w(X)$ is constant on $\tilde{\Omega}_e$. Since $w(X) = 0$ on $\tilde{\Gamma}$. we conclude that $w(X) = 0$ on $\tilde{\Omega}_e$. Let $P > \varepsilon$ be a positive constant. We multiply $\Delta w(X)$ by $\bar{w}(X)$ and integrating by parts over $(0,1)^2 \times (-\varepsilon, -P) = \tilde{\Omega}_P$ to acquire

$$\int_{\tilde{\Omega}_P} |\nabla w|^2 dX = \int_{\tilde{\Gamma}_P} \partial_{X_3} w(X_1, X_2, -P) \bar{w}(X_1, X_2, -P) dX_1 dX_2 = o(1), \quad \text{as} \quad P \to \infty.$$

Thus $w(X)$ is constant in $\hat{\Omega}$. Since $w(X) = 0$ on $\tilde{\Gamma}_\varepsilon$, we deduce that $w(X) \equiv 0$ on $\hat{\Omega}$ which proves the uniqueness. □

5.0.3 Well-possedness of problem (B)

Define the function spaces:

$$H_0^{-\frac{1}{2}}(\tilde{\Gamma}) := \{\varphi(X) \in H^{-\frac{1}{2}}(\tilde{\Gamma}) : \int_{\tilde{\Gamma}} \varphi(X) ds_X = 0\},$$

$$H_0^{\frac{1}{2}}(\tilde{\Gamma}) := \{\phi(X) \in H^{\frac{1}{2}}(\tilde{\Gamma}) : \int_{\tilde{\Gamma}} \phi(X) ds_X = 0\}.$$

Lemma 5.0.3 *The operator Θ has a bounded inverse from $H^{\frac{1}{2}}(\tilde{\Gamma})$ to $H^{-\frac{1}{2}}(\tilde{\Gamma})$. In addition, the following inequality holds*

$$Re(\langle \Theta\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0, \quad \forall \varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma}).$$

Proof. First we show that $Re(\langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0$ for any $\varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma})$. Assume $\varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma})$ such that $\langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}} \neq 0$. The function $w_\varphi(X) := \int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\varphi(Y)ds_Y$ is a solution to

$$\begin{cases} \Delta w_\varphi(X) = 0, \quad in \quad \tilde{\Omega}_e, \\ \\ \frac{\partial w_\varphi(X)}{\partial \nu} = 0, \quad \text{on} \quad \partial\tilde{\Omega}_e \backslash \tilde{\Gamma}, \\ \\ \frac{\partial w_\varphi(X)}{\partial \nu} = \varphi(X), \quad \text{on} \quad \tilde{\Gamma}, \\ \\ w_\varphi(X) = O(\frac{1}{|X|^2}), \quad |\nabla w_\varphi(X)| \cdot \frac{X}{|X|} = O(\frac{1}{|X|^3}), \qquad |X| \to \infty. \end{cases}$$

Let C_R^+ be the upper half sphere of radius R and centered at $(0,0,0)$ in $\tilde{\Omega}_e$, and $S_R^+ = \partial C_R^+ \cap \tilde{\Omega}_e$. Multiplying $\Delta \bar{w}_\varphi$ with w_φ and integrating by parts over C_R^+ yields

$$\begin{aligned} \int_{C_R^+} |\nabla w_\varphi|^2 dX = & \int_{S_R^+} \nabla \bar{w}_\varphi(X) \cdot \frac{X}{|X|} w_\varphi(X) ds_X + \langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}} \\ = & \langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}} + O\Big(\frac{1}{R^3}\Big), \quad R \to +\infty. \end{aligned} \tag{5.0.7}$$

Hence $Re(\langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0$. Since $\Theta : H^{-\frac{1}{2}}(\tilde{\Gamma}) \to H^{\frac{1}{2}}(\tilde{\Gamma})$ is a compact operator, to show invertibility it is sufficient to prove the injectivity of Θ. Let $\varphi \in H_0^{-\frac{1}{2}}(\tilde{\Gamma})$ such that $\langle \Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}} = 0$ and by substituting it in (5.0.7), we see that $w_\varphi(X)$ is constant in $\tilde{\Omega}_e$. Since $w_\varphi(X) = O\Big(\frac{1}{|X|}\Big)$ large $|X|$, so $w_\varphi(X) \equiv 0$ on $\tilde{\Omega}_e$. By taking its normal derivative on $\tilde{\Gamma}$, we conclude that $\varphi \equiv 0$, which proves the claim and hence Θ is injective in $H_0^{-\frac{1}{2}}(\tilde{\Gamma})$. □

Lemma 5.0.4 *The operator Θ_ε is invertible from $H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$ to $H_0^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$. In addition, the following inequality holds :*

$$Re(\langle \Theta_\varepsilon\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0, \quad \forall \varphi \in H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon).$$

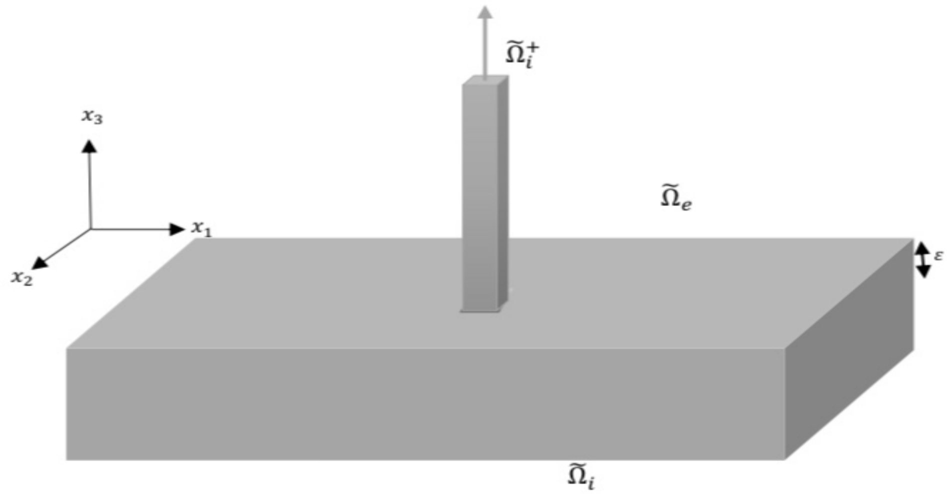

Figure 5.3: Geometry of the extended domain $\tilde{\Omega}_i^+$

Proof. Since Θ_ε is a compact operator, to prove the invertibility of the operator $\Theta_\varepsilon : H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H_0^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$ amounts to proving its injectivity. Recall that

$$\Theta_\varepsilon \varphi(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(Y + (0, \varepsilon, 0), X + (0, \varepsilon, 0))\varphi(Y) ds_Y = \int_{\tilde{\Gamma}} \tilde{G}_i(Y, X)\varphi(Y) ds_Y.$$

Define the single layer potential

$$u(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(Y, X)\varphi(Y) ds_Y \quad X \in \tilde{\Omega}_i \backslash \tilde{\Gamma}_\varepsilon.$$

Let us define $\tilde{\Omega}_i^+ = (0,1)^2 \times \mathbb{R}_+$ to be an extension of $\tilde{\Omega}_i$ to $+\infty$, can be seen in Figure 5.3. We then multiply $\Delta \bar{u}$ by u and integrate over $\tilde{\Omega}_i^+$ and $\tilde{\Omega}_i$ respectively to obtain

$$\int_{\tilde{\Omega}_i^+} |\nabla u|^2(X) dX = \int_{\partial \tilde{\Omega}_i^+} \partial_\nu \bar{u}_+(X) u(X) ds_X = -\int_{\tilde{\Gamma}} \partial_{X_3} \bar{u}_+(X) u(X) ds_X.$$

$$\int_{\tilde{\Omega}_i} |\nabla u|^2 dX = \int_{\partial \tilde{\Omega}_i} \partial_\nu \bar{u}_+(X) u(X) ds_X = \int_{\tilde{\Gamma}} \partial_{X_3} \bar{u}_-(X) u(X) ds_X.$$

Combining these results, we get

$$\int_{\tilde{\Omega}_i^+ \cup \tilde{\Omega}_i} |\nabla u|^2 dX = \int_{\tilde{\Gamma}} \left(\partial_{X_3}\bar{u}_-(X) - \partial_{X_3}\bar{u}_+(X)\right) u(X) ds_X.$$

Since

$$\partial_{X_3}\bar{u}_-(X) - \partial_{X_3}\bar{u}_+(X) = \bar{\varphi}(X),$$

there holds

$$\int_{\tilde{\Omega}_i} |\nabla u|^2 dX = \int_{\tilde{\Gamma}} \Theta_\varepsilon \varphi(X)\bar{\varphi}(X) ds_X \geq 0.$$

If $\langle \Theta_\varepsilon \varphi, \bar{\varphi}\rangle = 0$, it follows that $\nabla u = 0$ in $\tilde{\Omega}_i$ and $\mathbb{R}^3 \backslash \bar{\tilde{\Omega}}_i$ respectively. Hence $\varphi = \partial_\nu u_-(X) - \partial_\nu u_+(X) = 0$. □

To derive the variational formulation, we introduce the function space $V = \{w \in H^1(\tilde{\Omega}_\varepsilon) : \int_{\tilde{\Gamma}} w(X) ds_X = 0\}$ for any test function $w \in V$, we multiply it by (B) and integrate by parts over $\tilde{\Omega}_\varepsilon$ to obtain

$$\int_{\tilde{\Omega}_\varepsilon} \nabla u(X) \nabla \bar{w}(X) dX = \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) ds_X - \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) ds_X. \tag{5.0.8}$$

The integral $\int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) ds_X$ in $(5.0.8)$ can be understood as the dual product $\overline{\langle w, \frac{\partial u(X)}{\partial X_3}\rangle}_{\frac{1}{2},-\frac{1}{2}}$. $\forall w|_{\tilde{\Gamma}} \in H^{\frac{1}{2}}(\tilde{\Gamma})$, $(\Theta^{-1}w)(X)$ is well defined in $H^{-\frac{1}{2}}(\tilde{\Gamma})$. Since the Green's function $\tilde{G}_e$ is symmetric, we can write $\overline{\langle w, \frac{\partial u(X)}{\partial X_3}\rangle}_{\frac{1}{2},-\frac{1}{2}} = \langle \Theta(\frac{\partial u}{\partial X_3}), \Theta^{-1} w\rangle_{\frac{1}{2},-\frac{1}{2}}$. Let u be the solution to (B), then

$$\int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) ds_X = -\int_{\tilde{\Gamma}} u(X)(\Theta^{-1}\bar{w})(X) ds_X + \int_{\tilde{\Gamma}} u(X)(\Theta^{-1}f)(X) ds_X. \tag{5.0.9}$$

For the integral $\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) ds_X$ in $(5.0.8)$, we observe that the solution to (B) satisfies $\Theta_\varepsilon(\frac{\partial u}{\partial X_3})(X) + \varepsilon\langle 1, \frac{\partial u}{\partial X_3}\rangle = u(X)$ on $\tilde{\Gamma}_\varepsilon$. Integrating over $\tilde{\Gamma}_\varepsilon$ yields

$$4\varepsilon\langle 1, \frac{\partial u}{\partial X_3}\rangle = \int_{\tilde{\Gamma}_\varepsilon} u(X) dX. \tag{5.0.10}$$

Let $t(w) = \frac{1}{4}\int_{\tilde{\Gamma}_\varepsilon} w ds_X$, then

$$\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3}\bar{w}(X)ds_X = \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3}(\bar{w} - t(\bar{w}))ds_X + t(\bar{w})\langle 1, \frac{\partial u}{\partial X_3}\rangle.$$

We deduce by the invertibility of $\Theta_\varepsilon \ : \ H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$, and the symmetry that

$$\begin{aligned}\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3}\bar{w}(X))ds_X = & \int_{\tilde{\Gamma}_\varepsilon} \Theta_\varepsilon\Big(\frac{\partial u}{\partial X_3}\Big)\Theta_\varepsilon^{-1}(\bar{w} - t(\bar{w})ds_X + t(\bar{w})\langle 1, \frac{\partial u}{\partial X_3}\rangle. \\ = & \int_{\tilde{\Gamma}_\varepsilon} (u - t(u))\Theta_\varepsilon^{-1}(\bar{w} - t(\bar{w}))ds_X + \frac{1}{\varepsilon}t(u)t(w). \end{aligned} \quad (5.0.11)$$

Therefore, by virtue of $(5.0.8) - (5.0.11)$, we define the bilinear form $a(u, w)$ and the functional $F(w)$ as follows:

$$a(u, w) = \int_{\tilde{\Omega}_\varepsilon} \nabla u \nabla \bar{w} dX - \int_{\tilde{\Gamma}} u(\Theta^{-1}\bar{w})ds_X + \int_{\tilde{\Gamma}_\varepsilon} (u - t(u))\Theta_\varepsilon^{-1}(\bar{w} - t(\bar{w}))ds_X + \varepsilon^{-1}t(u)t(\bar{w}), \quad (5.0.12)$$

and

$$F(w) = \int_{\tilde{\Gamma}} (\Theta^{-1}f)\bar{w}ds_X, \quad (5.0.13)$$

so that $(5.0.8)$ reduces to

$$a(u, w) = F(w).$$

Theorem 5.0.5 *(1) $F(w)$ is a bounded linear functional from V to $\mathbb{C}$. The bilinear form $a(u, w)$ is bounded and coercive on $V \times V$: There exists constants $C_1 > 0$ and $C_2 > 0$, such that*

$$|a(u, w)| \le C_1\|u\|_1\|w\|_1, \quad \forall u, w \in V,$$

$$Re(a(u, u)) \ge C_2\|u\|_1^2, \quad \forall u \in V.$$

(2) For any $f \in H^{\frac{1}{2}}(R_1)$, there exists a unique solution to (B).

Proof. First to show that $F(w)$ is bounded from V to $\mathbb{C}$. It follows from Lemma $5.0.3$ that $\Theta^{-1} : H^{\frac{1}{2}}(\tilde{\Gamma}) \to H^{-\frac{1}{2}}(\tilde{\Gamma})$ is bounded operator. Since the operator on $\tilde{\Gamma}$ is continuous operator from V to $H^{\frac{1}{2}}(\tilde{\Gamma})$, $F(w)$ is continuous linear form from V to $\mathbb{C}$. The bilinear form $a(u, w)$ can easily seen to be bounded by using the trace theorem and the boundedness of $\Theta_{\varepsilon}^{-1}$ and Θ. Finally, for a fixed $u \in V$ as

$$a(u, u) = \int_{\Omega_{\varepsilon}} |\nabla u^2| dX + \int_{\hat{\Gamma}} u(\Theta^{-1}\bar{u}) ds_X + \int_{\hat{\Gamma}_{\varepsilon}} (u - t(u))\Theta_{\varepsilon}^{-1}(\bar{u} - t(\bar{u})) ds_X + \varepsilon^{-1}|t(u)|^2.$$

The coercivity of the bilinear form is a direct result of Poincare-Friedrichs inequality. Finally, from the Lax-Milgram theorem, (B) attains a unique solution $u \in H^{\frac{1}{2}}(\tilde{\Omega}_{\varepsilon})$. □

Chapter 6

Conclusion and future work

To recapitulate, we have investigated the acoustic wave field amplification for the scattering through three dimensional subwavelength structures namely a cavity and a hole. Mathematical theory required for the asymptotic expansion of resonances was studied. We derived the asymptotic expansion of the integral operator and the simplified Gohberg-Sigal theory to obtain scattering resonances. Furthermore, we quantitatively analyzed the field amplification at the even and odd resonant frequencies and show that the enhancement is of order $O(1/\varepsilon^2)$. It was shown that the complex-valued scattering resonances attained imaginary parts of order $O(\varepsilon^2)$.

In future, we plan to work on scattering resonances for periodic structures and the scattering resonances for the electromagnetic wave.

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