Global existence, persistence, spreading speeds, and forced waves in chemotaxis models

by

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Abstract

Chemotaxis models are widely used to describe the movements of biological species or living organisms in response to certain chemicals in their environments. This dissertation is devoted to the study of various dynamical aspects of a parabolic-elliptic chemotaxis model in shifting environments and a parabolic-parabolic chemotaxis model with logistic source on the whole space.

Concerning parabolic-elliptic chemotaxis models in shifting environments, we study persistence, spreading speeds and existence of forced waves in two different shifting environments. In particular, in the case favorable environment and unfavorable environment are separated, we prove that if the shifting speed of the environment is large, the biological species with compactly supported initial distribution will die out in the long run; if the shifting speed of the environment is not large, the species will persist and spread along the shifting habitat at a fixed asymptotic spreading speed. We also prove that there is a forced wave with speed c which coincides with the shifting speed of the environment connecting two points provided that c is large. In the case favorable environment is surrounded by unfavorable environment, we show that if the generalized principle eigenvalue of the linearized system at the trivial solution is positive, the species will persist surrounding the good habitat; if the generalized principle eigenvalue is negative and the degradation rate of the chemical substance is large, the species will become extinct in the habitat. We also show that there is a forced wave connecting $(0, 0)$ and $(0, 0)$ with the speed agreeing to the shifting speed of the environment provided that the chemotactic sensitivity is sufficiently small and the generalized principle eigenvalue is positive. Some numerical simulations are also carried out in both cases. The simulations indicate the existence of forced wave solutions in some parameter regions which are not covered in the theoretical results, induce several problems to be further studied, and also provide some illustration of the theoretical results.

Regarding parabolic-parabolic chemotaxis models with logistic source on the whole space, we first prove the local existence and uniqueness of classical solutions for given initial functions. We then prove the global existence and boundedness of classical solutions for given initial functions under the assumption that the logistic damping is large relative to the product of the chemotactic sensitivity and the production rate of the chemical substance. Next, we study the asymptotic behavior of the global classical solutions with strictly positive initial functions. We show that under further conditions on parameters, the nonnegative constant solution is globally stable in some sense. Finally, we investigate the spreading speeds of global classical solutions with nonempty compact supported initial functions and front like initial functions. We prove that the spreading speed of such global classical solutions of the parabolic-parabolic chemotaxis model with logistic source is the same as that of Fisher-KPP equation under the same assumption of the existence of global classical solutions. Note that when there is no chemotaxis, the parabolic-parabolic chemotaxis model reduces to famous Fisher-KPP equation. Hence, under the same assumption of the existence of global classical solutions, the chemotaxis neither speeds up nor slows down the spatial spreading in the Fisher-KPP equation. As a by-product of spreading speeds, we show that persistence phenomena occurs, that is, any globally defined bounded classical solution with strictly positive initial function is bounded below by a positive constant independent of its initial function when time is large.

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Chapter 1

Introduction

Chemotaxis is the directed movement of cells and organisms in response to chemical gradients which can be found in various biological phenomena ranging from microscopic phenomena to macroscopic phenomena such as cancer growth, embryo development, immune system response, wound healing, neuron migration, finding the location for food, avoiding predators, attracting mates, population dynamics, gravitational collapse, etc [11]. There are two types of chemotaxis, positive chemotaxis and negative chemotaxis. Positive chemotaxis occurs if the movement is toward a higher concentration of the chemical substance. Conversely, negative chemotaxis occurs if the movement is in the opposite direction. The chemical substances that lead to positive chemotaxis are called chemoattractants and those leading to negative chemotaxis are called chemorepellents.

Chemotaxis has been attracting increasing attention of biologists, ecologists, mathematicians, and so on due to its important role in a wide range of biological phenomena such as the aforementioned process. Mathematical models for chemotaxis date to the pioneering works of Keller and Segel in the 1970s to describe the aggregation of the slime mold *Dyctyostelium discoideum* [25, 26]. Since then, many mathematical models have been established for the chemotaxis process. Such mathematical models are called chemotaxis models which are also known for Keller-Segel models. The reader is referred to [3, 18, 19, 44] and the references therein for some detailed introduction into the mathematics and applications of various chemotaxis models. For the recent developments on chemotaxis models, we refer the reader to a survey paper [2].

This dissertation is devoted to the study of various dynamical aspects of parabolic-elliptic chemotaxis models in shifting environments and parabolic-parabolic chemotaxis models with logistic source on the whole space.

1.1 Parabolic-elliptic chemotaxis models in shifting environments

In reality, the living environments of species may be shifted due to climate change, in particular, the global warming or the worsening of the environment resulting from industrialization which lead to the shifting or translating of the habitat ranges [47]. It is of both biological and mathematical interests to study chemotaxis models in shifting environments.

One of the main objective of this dissertation is to investigate the persistence, spreading speeds and forced waves of the following parabolic-elliptic chemotaxis model in shifting environments,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(r(x-ct) - bu), \quad x \in \mathbb{R},\ t > 0, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where χ, b, λ, and μ are positive constants, $c \in \mathbb{R}$. In (1.1), $u(t,x)$ and $v(t,x)$ denote the population densities of some biological species and chemical substance at time t and location x, respectively; The term Δu describes the movement of the biological species from the places with higher population density to the places with lower population density following random walk; the term $\chi \nabla \cdot (u \nabla v)$ characterizes the influence of chemical substance, that is, the biological species moves from the places with lower concentration of chemical substance to the places with higher concentration of chemical substance when χ is positive measuring the sensitivity effect on the biological species. $c \neq 0$ is the shifting speed of the environment and $r(\cdot)$ is a sign changing function modeling the growth rate of the biological species. b is the logistic damping or the self-limitation rate of the biological species. The second equation indicates that the chemical substance diffuses via random walk very quickly and is produced over time by the biological species at a rate of μ. The chemical substance degrades at a rate of λ.

The shifting environments are incorporated into the sign changing function $r(\cdot)$. In particular, we will consider the following two different shifting environments:

Case 1. *Favorable and unfavorable habitats are separated in the sense that* $r(x)$ *is Hölder continuous, the limits* $r(\pm\infty) := \lim_{x\to\pm\infty} r(x)$ *exist and are finite, and* $r(-\infty) < 0 < r(\infty)$*,* $r(-\infty) \le r(x) \le r(\infty)$*,* $\forall\, x \in \mathbb{R}$*.*

Case 2. *Favorable habitat is surrounded by unfavorable habitat in the sense that* $r(x)$ *is Hölder continuous,* $\sup_{x\in\mathbb{R}} r(x) > 0$*, the limits* $r(\pm\infty) := \lim_{x\to\pm\infty} r(x)$ *exist and are finite, and* $r(\pm\infty) < 0$*,* $\min\{r(\infty), r(-\infty)\} \le r(x)$*,* $\forall\, x \in \mathbb{R}$*.*

In **Case 1**, $r(x-ct)$ divides the spatial domain into two regions: the region with good-quality habitat suitable for growth $\{x \in \mathbb{R}\colon r(x-ct) > 0\}$ and the region with poor-quality habitat unsuitable for growth $\{x \in \mathbb{R}\colon r(x-ct) < 0\}$. The edge of the habitat suitable for species growth is shifting at a speed c. In **Case 2**, $r(x-ct)$ still divides the spatial domain into two regions: one favorable for growth $\{x \in \mathbb{R}\colon r(x-ct) > 0\}$ and one unfavorable for growth $\{x \in \mathbb{R}\colon r(x-ct) < 0\}$. The favorable habitat is bounded and surrounded by the unfavorable habitat. The favorable habitat is shifting at a speed c.

Among interesting population dynamical problems in (1.1) are whether the species with compactly supported initial distribution can persist; whether the system (1.1) has so called forced wave solutions; whether it spreads into larger and larger regions, if so, how fast it spreads. A positive solution $(u(t,x), v(t,x))$ of (1.1) is called a *forced wave solution* if it is defined for all $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ for some one variable functions $\phi(\cdot)$ and $\psi(\cdot)$. Biologically, forced waves describe the propagation of species as a wave with a fixed shape and a fixed speed which coincides with the shifting speed of the environment. There are many studies of (1.1) towards these dynamical problems in the case without chemotaxis ($\chi = 0$), that is,

$$u_t = \Delta u + u(r(x-ct) - bu), \quad x \in \mathbb{R}, \tag{1.2}$$

as well as various variants of (1.2).

For example, in **Case 1**, Li et al. [34] studied (1.2). They proved that if $c > c^* = 2\sqrt{r(\infty)}$, then the species with compactly supported initial distribution will become extinct in the habitat, and if $0 < c < c^*$, the species will persist and spread along the shifting habitat at the asymptotic spreading speed c^*. Recently, Hu and Zou [22] demonstrated that in the case $b = 1$, for any given speed $c > 0$ of the shifting habitat edge, (1.2) admits a nondecreasing traveling wave solution $u(t, x) = \phi(x - ct)$ connecting 0 and $r(\infty)$ (i.e. $\phi(-\infty) = 0$ and $\phi(\infty) = r(\infty)$) with the speed c agreeing to the habitat shifting speed, which accounts for an extinction wave. Very recently, Wang and Zhao [64] obtained the uniqueness of the forced wave of (1.2) by using the sliding technique and established the global exponential stability of the forced wave via the monotone semiflows approach combined with the method of super- and subsolutions (see [64, Theorem 2.3]).

In **Case 2**, Berestycki et al. [4] proposed to use the following reaction-diffusion equation with a forced speed $c > 0$ to study the influence of climate change on the population dynamics of biological species:

$$u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}. \tag{1.3}$$

A typical f considered in [4] is

$$f(x,u) = \begin{cases} au(1 - \frac{u}{K}), & 0 \le x \le L, \\ -ru, & x < 0 \text{ and } x > L \end{cases}$$

for some positives constants a, r, K, L. They first considered this special case and derived an explicit condition for the persistence of species by gluing phase portraits. Then they established a strict qualitative dichotomy for a large class of models by the rigorous PDE methods. More precisely, they showed that if ζ_∞, defined to be the generalized principal eigenvalue of the operator $u \to u_{xx} + cu_x + f_u(x, 0)u$ on $\mathbb{R}$, is less than or equal to zero, then (1.3) has no forced wave solution and every positive solution of (1.3) converges to zero as $t \to \infty$, uniformly in $x \in \mathbb{R}$. If $\zeta_\infty > 0$, (1.3) has a unique forced wave solution and every nontrivial positive solution of (1.3) converges to this unique forced wave solution as $t \to \infty$, uniformly in $x \in \mathbb{R}$. For other related works on persistence, spreading speeds and forced waves under different climate change for certain reaction-diffusion equations, nonlocal dispersal equations, lattice differential equations, as well as integro-difference equations, we refer the readers to [5, 12, 21, 31, 32, 33, 35, 48, 62, 64] and the references therein.

In this dissertation, we study the spatial spreading dynamics of (1.1) with the presence of the chemotaxis and shifting environments and obtain the persistence criteria, spreading speeds and existence of forced wave solutions theoretically and numerically in both cases. Let $r^* = \sup_{x\in\mathbb{R}} r(x)$, $c^* = 2\sqrt{r^*}$ and $\zeta_\infty(r(\cdot), c)$ be the generalized principle eigenvalue of the operator $u \to u_{xx} + cu_x + r(x)u$. Among others, we prove the following for (1.1).

(1) Persistence and spreading speeds in Case 1. Suppose that $r(x)$ is as in **Case 1**, $b > \chi\mu$, $b > \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$, and $u_0(x)$ is nonnegative, uniformly continuous, bounded and has a nonempty compact support.

- **(i)** If $c > c^*$, then the species will die out in the habitat.
- **(ii)** If $-c^* \le c < c^*$, then the species will persist and spread along the shifting habitat into larger and larger regions at the asymptotic spreading speed c^*.
- **(iii)** If $c < -c^*$, then the species will persist and spread at the asymptotic spreading speed c^* (see Theorem 2.1 for details).

(2) Persistence and extinction in Case 2. Suppose that $r(x)$ is as in **Case 2**, $b > \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$, and $u_0(x)$ is nonnegative, uniformly continuous, bounded and has a nonempty compact support.

- **(i)** If $|c| > c^*$, then the species will die out in the habitat.
- **(ii)** If $\zeta_\infty(r(\cdot), c) < 0$ and $\lambda \ge \frac{(\sqrt{8r^*+c^2}+|c|)^2}{4}$, then the species will also die out in the habitat.
- **(iii)** If $\zeta_\infty(r(\cdot), c) > 0$, then the species will persist surrounding the good habitat (see Theorem 2.3 for details).

(3) Existence of forced wave solutions in Case 1. Suppose that $r(x)$ is as in **Case 1**, $b > 2\chi\mu$ and $c > \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} - 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$, then there is a forced wave solution connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ (see Theorem 2.4 for details).

(4) Existence of forced wave solutions in Case 2. Suppose that $r(x)$ is as in **Case 2**, $b \geq \frac{3}{2}\chi\mu$ and $\zeta_\infty(r(\cdot), c) > 0$. Then there exists a number $\chi_0 = \chi_0(r(\cdot), c) > 0$ such that for any $0 < \chi < \chi_0$, there is a forced wave solution connecting $(0,0)$ and $(0,0)$ (see Theorem 2.5 for details).

We also do some numerical simulations for the existence of forced wave solutions to see whether forced wave solutions still exist when the conditions in the theoretical results are not satisfied (see section 2.6 for details).

1.2 Parabolic-parabolic chemotaxis models with logistic source on $\mathbb{R}^N$

Another main objective of this dissertation is to study several dynamical aspects of the following parabolic-parabolic chemotaxis model with logistic source on $\mathbb{R}^N$:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \mathbb{R}^N, \ t > 0 \\ v_t = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N, \ t > 0, \end{cases} \tag{1.4}$$

where χ, a, b, λ and μ are positive constants. System (1.4) also describes the evolution of a biological species "u" in response to a chemical substance "v" in a large living environment. In comparison with (1.1), the second equation in (1.4) indicates that the chemical substance diffuses via random walk with a finite diffusion rate.

Among the central dynamical problems are the existence of nonnegative solutions of (1.4) which are globally defined in time or blow up at a finite time and the asymptotic behavior of global classical solutions such as persistence, spreading speeds and convergence as time goes to infinity. These dynamical problems have been well investigated for (1.4) in the case without chemotaxis ($\chi = 0$). Observe that, in the absence of chemotaxis (i.e. $\chi = 0$), (1.4) reduces to the following reaction-diffusion equation

$$u_t = \Delta u + u(a - bu), \quad x \in \mathbb{R}^N. \tag{1.5}$$

Due to the pioneering works of Fisher [13] and Kolmogorov, Petrowsky, Piskunov [27] on traveling wave solutions and take-over properties of (1.5), (1.5) is also referred to as the Fisher-KPP equation. It is known that the dynamics of (1.5) is completely determined by the logistic term $u(a-bu)$. More precisely, for any given nonnegative initial function u_0, (1.5) has a unique global classical solution. It is known that $u \equiv \frac{a}{b}$ is the unique positive steady-state solution of (1.5) and for any given strictly positive initial distribution $u_0(\cdot)$, the solution $u(t,x;u_0)$ of (1.5) with $u(0,x;u_0) = u_0(x)$ converges to $\frac{a}{b}$. It is also known that equation (1.5) has traveling wave solutions $u(t,x) = \phi(x\cdot\xi - ct))$ ($\xi \in S^{N-1}$) connecting $\frac{a}{b}$ and 0 ($\phi(-\infty) = \frac{a}{b}, \phi(\infty) = 0$) of all speeds $c \ge 2\sqrt{a}$ and has no such traveling wave solutions of slower speeds. For any given bounded initial function u_0 with nonempty compact support, the following is well known,

$$\lim_{t\to\infty} \sup_{|x|\ge ct} u(t,x;u_0) = 0 \quad \forall\, c > 2\sqrt{a}$$

and

$$\lim_{t\to\infty} \sup_{|x|\le ct} |u(t,x;u_0) - \frac{a}{b}| = 0 \quad \forall\, 0 < c < 2\sqrt{a}.$$

In literature, the number $c_0^* := 2\sqrt{a}$ is called the *spreading speed* for (1.5) which was first introduced by Aronson and Weinberger [1]. Biologically, spreading speeds can be understood as the asymptotic rate at which a species, initially introduced in a bounded range, expands its spatial range as time evolves, while traveling waves describe the propagation of species at certain direction as a wave with a fixed shape and a fixed speed.

Consider chemotaxis models. Considerable progress has been made in the analysis of various chemotaxis models towards these central dynamical problems on bounded domains. For example, consider the following counterpart of (1.4) on a bounded domain with Neumann boundary condition,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + u(a-bu), \quad x\in\Omega,\ t>0 \\ v_t = \Delta v - \lambda v + \mu u, \quad x\in\Omega,\ t>0 \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x\in\partial\Omega. \end{cases} \tag{1.6}$$

Chemotaxis model (1.6) is the so-called minimal model when $a \equiv b \equiv 0$. It is known that finite time blow up may occur for the minimal model. For example, when Ω is a ball in $\mathbb{R}^N$ with $N \geq 3$, then for all $M > 0$, there exists positive initial data $(u_0, v_0) \in C(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ with $\int_\Omega u_0 = M$ such that the corresponding solution blows up in finite time (see [69]). It is shown in [70] that, when Ω is a convex bounded domain with smooth boundary and $\frac{b}{\chi}$ is sufficiently large, for any choice of suitably regular nonnegative initial data (u_0, v_0) such that $u_0 \not\equiv 0$, (1.6) possesses a uniquely determined global classical solution and that the constant solution $(\frac{a}{b}, \frac{\mu a}{\lambda b})$ is asymptotically stable in the sense that

$$\lim_{t \to \infty} \left[\|u(t, \cdot; u_0, v_0) - \frac{a}{b}\|_{L^\infty(\Omega)} + \|v(t, \cdot; u_0, v_0) - \frac{\mu a}{\lambda b}\|_{L^\infty(\Omega)} \right] = 0.$$

Hence finite time blow-up phenomena in (1.6) can be suppressed to some extent by the logistic source. The particular requirement on the convexity of the bounded domain Ω was later removed in [23] and [73]. However, when b is not large relative to χ, numerical evidence shows that even in the spatially one-dimensional setting solutions may exhibit chaotic behavior (see [45]). Also a phenomenon suggested by the numerical simulations in [45] consists in the ability of (1.6) to enforce asymptotic smallness of the cell population density, undistinguishable from extinction, in large spatial regions (see e.g. Fig. 7(d) in [45]). In [60], the authors proved that any such extinction phenomenon must be localized in space, and that the population as a whole always persists, which is called persistence of mass in [60]. Recently, Issa and Shen [23] proved the pointwise persistence phenomena, that is, any globally defined positive solution is bounded below by a positive constant independent of its initial data, which implies that the cell population may become very small at some time and some location, but it persists at any location eventually. For other related works on (1.6), we refer the readers to [20, 30, 36, 39, 42, 43, 65] and references therein.

When the second equation of (1.6) is replaced by $0 = \Delta v - \lambda v + \mu u$, $x \in \Omega$, it becomes

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \Omega, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.7}$$

The dynamics of (1.7) has been studied in many research papers and very rich dynamical scenarios have been observed. For example, when $a \equiv b \equiv 0$, finite time blow-up may occur in (1.7) if either $N = 2$ and the total initial population mass is large enough, or $N \geq 3$ (see [17, 24, 37, 38], etc.). When a and b are positive constants, if either $N \leq 2$ or $b > \frac{N-2}{N}\chi$, then for any nonnegative initial data $u_0 \in C(\bar{\Omega})$, (1.7) possesses a unique bounded global classical solution $(u(t, x; u_0), v(t, x; u_0))$ with $u(0, x; u_0) = u_0(x)$, and hence the finite time blow-up phenomena in (1.7) is suppressed to some extent. Moreover, if $b > 2\chi$, then $(\frac{a}{b}, \frac{\mu a}{\lambda b})$ is the unique positive steady-state solution of (1.7), and for any nonnegative initial distribution $u_0 \in C(\bar{\Omega})$ ($u_0(x) \not\equiv 0$),

$$\lim_{t \to \infty} \left[\|u(t, \cdot; u_0) - \frac{a}{b}\|_{L^\infty(\Omega)} + \|v(t, \cdot; u_0) - \frac{\mu a}{\lambda b}\|_{L^\infty(\Omega)} \right] = 0$$

But if $b < 2\chi$, there may be more than one positive steady-state solutions of (1.7) (see [61]). For other studies of parabolic-elliptic chemotaxis models on bounded domains, we refer the readers to [9, 14, 29, 63, 66, 68, 71, 74] and the references therein.

There are also some studies of chemotaxis models on the whole space. For example, consider (1.4) when $a = b = 0$. It is possible for a non-negative solution in $\mathbb{R}^N$ ($N \geq 2$) to blow up in finite time (see [8]). It was shown in [40] that the unique solution exists globally in time and bounded under some conditions for initial data. Moreover, every bounded solution decays to 0 as $t \to \infty$ and behaves like the heat kernel with the self-similarity (see [41] for the asymptotic profiles of bounded solution in the case $N = 1$).

When the second equation in (1.4) being replaced by $0 = \Delta v - \lambda v + \mu u$, it becomes

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \mathbb{R}^N, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N. \end{cases} \tag{1.8}$$

Some studies of (1.8) are also carried out. For example, in the case of $a = b = 0$, it is known that finite time blow-up occurs if either $N = 2$ and the total initial population mass is large enough, or $N \geq 3$ (see [3], [10] and references therein). When a and b are positive constants, it is shown for the case $\lambda = \mu = 1$ in [49] that if $b > \chi$, then there exists a unique bounded global classical solution for any nonnegative uniformly continuous and bounded initial function u_0, and that if $b > 2\chi$, then for any strictly positive initial $\inf_{x \in \mathbb{R}^N} u_0(x) > 0$, the unique global classical solution $(u(t, x; u_0), v(t, x; u_0))$ with $u(0, 0; u_0) = u_0(x)$ converges to constant solution $(\frac{a}{b}, \frac{a}{b})$ as time goes to infinity. The spreading speeds and traveling wave solutions of (1.8) are studied in [49, 50, 54]. Among others, it is proved that if $b > \chi\mu$ and $b \geq \big(1 + \frac{1}{2} \frac{(\sqrt{a} - \sqrt{\lambda})_+}{(\sqrt{a} + \sqrt{\lambda})}\big)\chi\mu$, $c_0^* := 2\sqrt{a}$ is the spreading speed of the solutions of (1.8) with nonnegative continuous initial function u_0 with nonempty compact support, that is,

$$\lim_{t \to \infty} \sup_{|x| \geq ct} u(t, x; u_0) = 0 \quad \forall\, c > c_0^*$$

and

$$\lim_{t \to \infty} \inf_{|x| \leq ct} u(t, x; u_0) > 0 \quad \forall\, 0 < c < c_0^*.$$

It is also proved that, if $b > 2\chi\mu$ and $\lambda \geq a$, then $2\sqrt{a}$ is the minimal speed of the traveling wave solutions of (1.8) connecting $(0, 0)$ and $(\frac{a}{b}, \frac{\mu}{\lambda}\frac{a}{b})$, that is, for any $c \geq 2\sqrt{a}$, (1.8) has a traveling wave solution connecting $(0, 0)$ and $(\frac{a}{b}, \frac{\mu}{\lambda}\frac{a}{b})$ with speed c, and (1.8) has no such traveling wave solutions with speed less than $2\sqrt{a}$. In particular, if $\lambda \geq a$ and $b > \chi\mu$, or $\lambda < a$ and $b \geq \big(1 + \frac{1}{2} \frac{(\sqrt{a} - \sqrt{\lambda})}{(\sqrt{a} + \sqrt{\lambda})}\big)\chi\mu$, then the chemotaxis neither speeds up nor slows down the spatial spreading in the Fisher-KPP equation (1.5). For the persistence of globally defined classical solution with strictly positive initial function, we refer the readers to [52].

It is interesting to investigate the influence of chemotaxis on the spreading dynamics of (1.4). The authors of [51], [53] studied the existence of traveling wave solutions of (1.4). Among others, it is proved that if $b > 2\chi\mu$ and $1 \geq \frac{1}{2}(1-\frac{\lambda}{a})_+$, then for every $c \geq 2\sqrt{a}$, (1.4) has a traveling wave solution $(u,v)(t,x) = (U^c(x\cdot\xi - ct), V^c(x\cdot\xi - ct))$ ($\xi \in S^{N-1}$) connecting the two constant steady states $(0,0)$ and $(\frac{a}{b}, \frac{\mu}{\lambda}\frac{a}{b})$, and there is no such solutions with speed c less than $2\sqrt{a}$, which shows that (1.4) has a minimal wave speed $c_0^* = 2\sqrt{a}$, which is independent of the chemotaxis. Except this, there is little study of dynamics of (1.4).

In this dissertation, we also investigate the local and global existence of classical solutions with given nonnegative initial functions, asymptotic behavior of global classical solutions of (1.4) with strictly positive initial functions, spreading speeds of global classical solutions of (1.4) with compactly supported or front-like initial functions. Among others, we prove the following.

(1) Global existence of classical solutions. Suppose that $b > \frac{N\mu\chi}{4}$. Then for every nonnegative, bounded, uniformly continuous function u_0 and nonnegative, bounded, uniformly continuous differentiable function v_0, (1.4) has a unique bounded global classical solution (see Theorem 3.2 for details).

(2) Convergence. There exists $K = k(a,\lambda,N) > \frac{N}{4}$ such that if $b > K\chi\mu$ and $\lambda \geq \frac{a}{2}$, then the unique bounded global classical solution $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ of (1.4) with $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$, $v_0 \geq 0$ converges to $(\frac{a}{b}, \frac{\mu a}{\lambda b})$ uniformly in $x \in \mathbb{R}^N$ as $t \to \infty$ exponentially (see Theorem 3.3 for details).

(3) Spreading speeds. If $b > \frac{N\mu\chi}{4}$, $2\sqrt{a}$ is the spreading speed of (1.4) with nonempty compact supported initial functions, which implies that the chemotaxis neither speeds up nor slows down the spatial spreading in the Fisher-KPP equation (1.5) (see Theorem 3.4 for details).

The rest of the dissertation is organized as follows. In Chapter 2, we study persistence, spreading speeds and the existence of forced waves of (1.1). In Chapter 3, we explore the dynamical issues of (1.4) including the existence and boundedness of global classical solutions, asymptotic behavior of global classical solutions with strictly positive initial functions and spreading speeds. We end the dissertation with concluding remarks and future works.

Chapter 2

Persistence, spreading speeds and forced waves of parabolic-elliptic chemotaxis models in shifting environments

In this chapter, we study spatial spreading dynamics of chemotaxis model (1.1) with the presence of the chemotaxis and shifting environments in one-dimensional setting. In particular, we identify the circumstances under which persistence or extinction occurs, and in the case that persistence occurs, we study the existence of forced wave solutions of (1.1) both theoretically and numerically.

2.1 Notations, Assumptions, Definitions and Main results

2.1.1 Notations, Assumptions and Definitions

In order to state our main results, we first introduce some notations, assumptions and definitions. Let

$$C^b_{\rm unif}(\mathbb{R}) = \{u \in C(\mathbb{R}) \mid \ u \text{ is uniformly continuous and bounded on } \mathbb{R}\}.$$

For every $u \in C^b_{\rm unif}(\mathbb{R})$, we let $\|u\|_\infty := \sup_{x\in\mathbb{R}} |u(x)|$. For each given $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $u_0(x) \geq 0$, we denote by $(u(t, x; u_0), v(t, x; u_0))$ the classical solution of (1.1) satisfying $u(0, x; u_0) = u_0(x)$ for every $x \in \mathbb{R}$. Note that, by the comparison principle for parabolic equations, for every nonnegative initial function $u_0 \in C^b_{\rm unif}(\mathbb{R})$, it always holds that $u(t, x; u_0) \geq 0$ and $v(t, x; u_0) \geq 0$ whenever $(u(t, x; u_0), v(t, x; u_0))$ is defined. We shall only focus on nonnegative classical solutions of (1.1) since both functions $u(t, x)$ and $v(t, x)$ represent density functions.

The following proposition states the existence and uniqueness of classical solutions of (1.1) with given initial functions.

Proposition 2.1. *Suppose that $r(x)$ is globally Hölder continuous and bounded. For every nonnegative initial function $u_0 \in C^b_{\rm unif}(\mathbb{R})$, there is a unique maximal time $T_{max} > 0$, such that $(u(t, x; u_0), v(t, x; u_0))$ is defined for every $x \in \mathbb{R}$ and $0 \le t < T_{max}$. Moreover if $\chi\mu < b$ then $T_{max} = \infty$ and the solution is globally bounded.*

The above proposition can be proved by similar arguments as those in ([49, Theorem 1.1 and Theorem 1.5]).

Throughout this chapter, we assume that $r(x)$ is as in **Case 1** or **Case 2**. We put

$$r_* = \inf_{x\in\mathbb{R}} r(x), \quad r^* = \sup_{x\in\mathbb{R}} r(x), \quad c^* = 2\sqrt{r^*}.$$

Note that, in **Case 1**, $r_* = r(-\infty)$ and $r^* = r(+\infty)$, and in **Case 2**, $r_* = \min\{r(-\infty), r(\infty)\}$ and $r^* = \max_{x\in\mathbb{R}} r(x)$.

Let $\zeta_L(r(\cdot), c)$ be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} \phi_{xx} + c\phi_x + r(x)\phi = \zeta\phi, \quad -L < x < L \\ \phi(-L) = \phi(L) = 0. \end{cases} \tag{2.1}$$

Note that $\zeta_L(r(\cdot), c)$ is increasing as L increases (See [6, Proposition 4.2]). We also have $\zeta_L(r_1(\cdot), c) \le \zeta_L(r_2(\cdot), c)$ if $r_1(\cdot) \le r_2(\cdot)$. In particular,

$$\zeta_L(r(\cdot), c) \le \zeta_L(r^*, c) < r^*. \tag{2.2}$$

Let $\zeta_\infty(r(\cdot), c) = \lim_{L\to\infty} \zeta_L(r(\cdot), c)$. By (2.2),

$$\zeta_\infty(r(\cdot), c) \le r^*.$$

For convenience, we make the following standing assumptions.

(H1) $b > \chi\mu$ and $b \ge \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$.

(H2) $b > 2\chi\mu$ and $c > \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} - 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$.

(H3) $b \geq \frac{3}{2}\chi\mu$ and $\zeta_\infty(r(\cdot), c) > 0$.

Note that $b \geq \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ implies $b \geq \chi\mu$; that $b > \chi\mu$ and $\lambda \geq r^*$ imply $b \geq \big(1+\frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ (and hence **(H1)**); and that $b \geq \frac{3}{2}\chi\mu$ implies $b \geq \big(1+\frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ (and hence **(H1)**). Biologically, $\lambda \geq r^*$ means that the degradation rate of the chemo-attractant is greater than or equal to the supremum of the intrinsic growth rate of the biological species over the whole space, and the condition $b \geq \big(1+\frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ indicates that the chemotaxis sensitivity is small relative to the self-limitation rate of the biological species. Also note that $\zeta_\infty(r(\cdot), c) > 0$ implies that $-2\sqrt{r^*} < c < 2\sqrt{r^*}$.

A positive solution $(u(t,x), v(t,x))$ of (1.1) is called a *forced wave solution* if it is defined for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ for some one variable functions $\phi(\cdot)$ and $\psi(\cdot)$. It is clear that $(u, v) = (\phi(x), \psi(x))$ is a stationary solution of

$$\begin{cases} u_t = u_{xx} + cu_x - (\chi u v_x)_x + u(r(x) - bu), \quad x \in \mathbb{R} \\ 0 = v_{xx} - \lambda v + \mu u, \quad x \in \mathbb{R}. \end{cases} \tag{2.3}$$

We say that a positive *forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *of* (1.1) *connects* (u_+^*, v_+^*) *and* (u_-^*, v_-^*) if $(\phi(\pm\infty), \psi(\pm\infty)) = (u_\pm^*, v_\pm^*)$.

2.1.2 Main results

The main results are from our works [56] and [57]. We first state the results on the persistence and spreading speeds of the species with $r(x)$ being as in **Case 1** or **Case 2**.

Theorem 2.1. *Suppose that* $r(x)$ *is as in* **Case 1**, **(H1)** *holds, and* $u_0(x) \in C^b_{\rm unif}(\mathbb{R})$ *is nonnegative, bounded and has a nonempty compact support.*

(1) If $c > c^*$, *then*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x;u_0) = 0$$

.

(2) If $-c^* \leq c < c^*$*, then for any* $0 < \varepsilon < \frac{c^*-c}{2}$*, there hold*

$$\lim_{t\to\infty} \sup_{x\leq(c-\varepsilon)t} u(t,x;u_0) = \lim_{t\to\infty} \sup_{x\geq(c^*+\varepsilon)t} u(t,x;u_0) = 0,$$

and

$$\liminf_{t\to\infty} \inf_{(c+\varepsilon)t\leq x\leq(c^*-\varepsilon)t} u(t,x;u_0) > 0.$$

Moreover, if $b > 2\chi\mu$*, then*

$$\lim_{t\to\infty} \sup_{(c+\varepsilon)t\leq x\leq(c^*-\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0.$$

(3) If $c < -c^*$*, then for any* $0 < \varepsilon < c^*$*, there hold*

$$\lim_{t\to\infty} \sup_{x\leq(-c^*-\varepsilon)t} u(t,x;u_0) = \lim_{t\to\infty} \sup_{x\geq(c^*+\varepsilon)t} u(t,x;u_0) = 0,$$

and

$$\liminf_{t\to\infty} \inf_{(-c^*+\varepsilon)t\leq x\leq(c^*-\varepsilon)t} u(t,x;u_0) > 0.$$

Moreover, if $b > 2\chi\mu$*, then*

$$\lim_{t\to\infty} \sup_{(-c^*+\varepsilon)t\leq x\leq(c^*-\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0.$$

Theorem 2.2. *Suppose that* $r(x)$ *is as in* **Cases 1**, **(H1)** *holds, and* $u_0(x) \in C^b_{\rm unif}(\mathbb{R})$ *is non-negative, bounded, and* $u_0(x) = 0$ *for* $x \ll -1$ *and* $\liminf_{x\to\infty} u_0(x) > 0$.

(1) If $c \geq -c^*$*, then for any* $\varepsilon > 0$*, there hold*

$$\lim_{t\to\infty} \sup_{x\leq(c-\varepsilon)t} u(t,x;u_0) = 0,$$

and

$$\liminf_{t\to\infty} \inf_{x\geq(c+\varepsilon)t} u(t,x;u_0) > 0.$$

Moreover, if $b > 2\chi\mu$, then

$$\lim_{t\to\infty} \sup_{x\geq(c+\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0.$$

(2) If $c < -c^*$, then for any $\varepsilon > 0$, there hold

$$\lim_{t\to\infty} \sup_{x\leq(-c^*-\varepsilon)t} u(t,x;u_0) = 0,$$

and

$$\liminf_{t\to\infty} \inf_{x\geq(-c^*+\varepsilon)t} u(t,x;u_0) > 0.$$

Moreover, if $b > 2\chi\mu$, then

$$\lim_{t\to\infty} \sup_{x\geq(-c^*+\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0.$$

Theorem 2.3. *Suppose that $r(x)$ is as in* **Case 2**, **(H1)** *holds, and $u_0(x) \in C^b_{\rm unif}(\mathbb{R})$ is nonnegative, bounded and has a nonempty compact support.*

(1) If $|c| > c^*$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x;u_0) = 0.$$

(2) If $\zeta_\infty(r(\cdot),c) < 0$ and $\lambda \geq \lambda^* := \frac{(\sqrt{8r^*+c^2}+|c|)^2}{4}$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x;u_0) = 0.$$

(3) If $|c| < c^*$, then

$$\lim_{t\to\infty} \sup_{|x-ct|\geq c' t} u(t,x;u_0) = 0 \quad \forall\ c' > 0.$$

If, additionally, $\zeta_\infty(r(\cdot),c) > 0$, then

$$\liminf_{t\to\infty} \inf_{|x-ct|\leq L} u(t,x;u_0) > 0 \quad \forall\ L > 0.$$

Remark 2.1. *(1) Suppose that $r(x)$ is as in* **Case 1** *and the species initially lives in a region with $-M \leq x \leq M$ for some $M \in \mathbb{R}$. Theorem 2.1 (1) shows that if $c > c^*$, then the species will become extinct in the habitat. Theorem 2.1 (2) shows that if $-c^* \leq c < c^*$, then the species will persist and spread along the shifting habitat into larger and larger region at the asymptotic spreading speed c^*.Theorem 2.1 (3) shows that if $c < -c^*$, then the species will persist and spread at the asymptotic spreading speed c^*. When $\chi = 0$,* **(H1)** *becomes $b > 0$. Hence Theorem 2.1 (1) recovers [34, Theorem 2.1], and Theorem 2.1 (2) recovers [34, Theorem 2.2].*

(2) Suppose that $r(x)$ is as in **Case 1** *and the species initially lives in a region with $x \geq M$ for some $M \in \mathbb{R}$. Theorem 2.2 (1) shows that for any given $c \geq -c^*$, the species will persist and spread along the shifting habitat at the asymptotic spreading speed c. Theorem 2.2 (2) shows that for any given $c < -c^*$, the species will persist and spread at the asymptotic spreading speed c^*.*

(3) It is sufficient to assume that $b > \chi\mu$ for $\lim_{t\to\infty} \sup_{x \leq (c-\varepsilon)t} u(t,x;u_0) = 0$ in Theorem 2.1 (2). It is not necessary to assume that $b \geq \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ in Theorem 2.1 (2) for proving $\liminf_{t\to\infty} \inf_{(c+\varepsilon)t \leq x \leq (c^*-\varepsilon)t} u(t,x;u_0) > 0$ and in Theorem 2.1 (3) for proving $\liminf_{t\to\infty} \inf_{(-c^*+\varepsilon)t \leq x \leq (c^*-\varepsilon)t} u(t,x;u_0) > 0$. In Theorem 2.2, the condition $b \geq \big(1+\frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ is only needed in proving $\lim_{t\to\infty} \sup_{x \leq (-c^*-\varepsilon)t} u(t,x;u_0) = 0$.

(4) Suppose that $r(x)$ is as in **Case 2** *and the species initially lives in a region with $-M \leq x \leq M$ for some $M \in \mathbb{R}$. Theorem 2.3 (1) shows that if $|c| > c^*$, then the species will die out in the long run. If $\zeta_\infty(r(\cdot), c) < 0$ and the degradation rate λ of the chemo-attractant is greater than or equal to $\lambda^* = \frac{(\sqrt{8r^*+c^2}+|c|)^2}{4}$, then the species will also die out in the long run. If $\zeta_\infty(r(\cdot), c) > 0$, then the species will persist surrounding the good habitat. When $\chi = 0$,* **(H1)** *becomes $b > 0$. Hence Theorem 2.3 (2) and (3) recovers [4, Theorem 4.11]. The assumption $\lambda \geq \lambda^*$ indicates that the degradation rate of the chemo-attractant is large relative to the speed of the shifting environment.*

(5) In Theorem 2.3, the condition $b \geq \big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu$ is only needed in (1).

(6) *It is not easy to prove the persistence and spreading speeds of solutions of* (1.1). *Several new techniques are developed to prove the results stated in the above theorems. These techniques can also be applied to the case* $\chi = 0$.

Next, we state our results on the existence of forced wave solutions with $r(x)$ being as in **Case 1** or **Case 2**.

Theorem 2.4. *Suppose that* $r(x)$ *is as in* **Case 1**, *and* **(H2)** *holds. Then there is a forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *connecting* $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ *and* $(0,0)$.

Theorem 2.5. *Suppose that* $r(x)$ *is as in* **Case 2**, *and* **(H3)** *holds. Then there exists a number* $\chi_0 = \chi_0(r(\cdot), c) > 0$ *such that for any* $0 < \chi < \chi_0$, *there is a forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *connecting* $(0,0)$ *and* $(0,0)$, *that is,* $\phi(x) > 0$ *for all* $x \in \mathbb{R}$ *and* $\phi(\pm\infty) = 0$.

Remark 2.2. (1) *When* $\chi = 0$, **(H2)** *becomes* $c > -2\sqrt{r^*}$. *Hence Theorem 2.4 recovers [22, Theorem 1.1].*

(2) *In* **Case 1**, *for any given* $c > -2\sqrt{r^*}$, *there is* $\chi_0(c) > 0$ *such that for any* $0 < \chi < \chi_0(c)$, **(H1)** *holds. Then by Theorem 2.4, for any* $0 < \chi < \chi_0(c)$, *there is a forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *connecting* $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ *and* $(0,0)$.

(3) *When* $\chi = 0$, *Theorem 2.5 recovers [4, Theorem 4.8].*

(4) *Thanks to the presence of chemotaxis, the comparison principle for parabolic equations cannot be applied directly to* (1.1) *or* (2.3), *and the techniques used in the study of* (1.2) *and* (1.3) *are difficult to be applied to the study of* (1.1). *We use Schauder's fixed point theorem together with sub- and super-solutions to study the existence of forced wave solutions of* (1.1). *We point out that the construction of some appropriate sub-solutions is highly nontrivial in both cases.*

Observe that the conditions in Theorem 2.4 (resp. in Theorem 2.5) are sufficient conditions for the existence of forced wave solutions. To see whether (1.1) still has forced wave solutions when these sufficient conditions are not satisfied, some numerical simulations are carried out (see section 2.5 for details).

The rest of the chapter is organized as follows. In section 2.2, we present some preliminary lemmas to be used in the proofs of the main results. In section 2.3, we study persistence and spreading speeds of (1.1) with $r(x)$ being as in **Case 1** or **Case 2** and prove Theorems 2.1, 2.2 and 2.3. In section 2.4, we study the existence of forced wave solutions of (1.1) with $r(x)$ being as in **Case 1** or **Case 2** and prove Theorems 2.4 and 2.5. In section 2.5, we present some numerical simulations for the existence of forced wave solutions of (1.1) with $r(x)$ being as in **Case 1** or **Case 2**.

2.2 Preliminary lemmas

In this section, we present some preliminary lemmas to be used in the proofs of the main theorems in later sections.

Note that, by the second equation in (1.1),

$$\Delta v = \lambda v - \mu u.$$

Hence the first equation in (1.1) can be written as

$$u_t = \Delta u - \chi \nabla u \cdot \nabla v + u\big(r(x-ct) - \chi\lambda v - (b-\chi\mu)u\big), \quad x \in \mathbb{R}.$$

By the comparison principle for parabolic equations, if $b > \chi\mu$, then for any $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $u_0 \geq 0$,

$$0 \leq u(t,x;u_0) \leq \max\{\|u_0\|_\infty, \frac{r^*}{b-\chi\mu}\} \quad \forall\ t \geq 0,\ x \in \mathbb{R}.$$

Lemma 2.1. *Assume $b > \chi\mu$. For every $R \gg 1$, there are $C_R \gg 1$ and $\varepsilon_R > 0$ such that for any $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $u_0 \geq 0$, any $x_0 \in \mathbb{R}$, and any $t \geq 0$, we have*

$$\|\chi v_x(t,\cdot;u_0)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} + \|\chi\lambda v(t,\cdot;u_0)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C_R\|u(t,\cdot;u_0)\|_{L^\infty(B_R(x_0))} + \varepsilon_R M$$

with $\lim_{R\to\infty}\varepsilon_R=0$, *where* $M:=\max\{\|u_0\|_\infty,\frac{r^*}{b-\chi\mu}\}$.

Proof. It follows from the arguments of [54, Lemma 2.5]. □

Lemma 2.2. *Assume* $b>\chi\mu$. *For every* $p>1$, $t_0>0$, $s_0\ge 0$, $R>0$, *and* $u_0\in C^b_{\rm unif}(\mathbb{R})$ *with* $u_0\ge 0$, *there is* $C_{t_0,s_0,R,M,p}$ *such that if* $s\in[0,s_0],\ t\ge t_0,\ |x-y|\le R$, *then*

$$u(t,x;u_0)\le C_{t_0,s_0,R,M,p}[u(t+s,y;u_0)]^{\frac{1}{p}}(M+1), \tag{2.4}$$

where $M:=\max\{\|u_0\|_\infty,\frac{r^*}{b-\chi\mu}\}$.

Proof. The lemma can be proved by slightly modified arguments of [15, Lemma 2.2].

In fact, first, fix $t_0>0, s_0\ge 0, R>0, p\in(1,\infty)$ and $t\ge t_0>0$. Let $\delta=\min\{\frac{t_0}{2},1\}$ and

$$A=bM+\sup_{t>0}\big(\|\,|\chi\nabla v(t,\cdot;u_0)|\,\|_\infty+\|\nabla\cdot\chi\nabla v(t,\cdot;u_0)\|_\infty\big).$$

Note that $\|u(t,\cdot;u_0)\|_\infty\le M$ for all $t\ge 0$. Let $\bar u(s',x)$ be the solution of

$$\begin{cases}\bar u_{s'}+\chi\nabla v(s',\cdot;u_0)\cdot\nabla\bar u=\Delta\bar u \quad \text{in } (t-\delta,\infty)\times\mathbb{R}\\ \bar u(t-\delta,\cdot)=u(t-\delta,\cdot;u_0)\quad \text{in } \mathbb{R}.\end{cases}$$

By the comparison principle for parabolic equations, we have

$$0\le\bar u(s',x)\le\|u(t-\delta,\cdot;u_0)\|_\infty \quad \text{in} \quad (t-\delta,\infty)\times\mathbb{R}.$$

Next, let

$$u^-(s',x)=\min\{1,\|u(t-\delta,\cdot;u_0)\|^{-1}_\infty\}e^{(r_*-A)(s'-t+\delta)}\bar u(s',x)$$

and

$$u^+(s',x)=e^{(r^*+A)(s'-t+\delta)}\bar u(s',x).$$

Notice that

$$
\begin{aligned}
A &> bM + \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&> b\|u(t-\delta, \cdot; u_0)\|_\infty + \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&\geq b\bar{u}(s', x) + \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&\geq bu^-(s', x) + \nabla \cdot \chi \nabla v(s', \cdot; u_0) \ \ \forall\, s' > t-\delta,\ x \in \mathbb{R}.
\end{aligned}
$$

By a straightforward computation, we have

$$
\begin{aligned}
& u^-_{s'} + \chi \nabla v(s', \cdot; u_0) \cdot \nabla u^- + u^- \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&= (r_* - A)u^- + \Delta u^- + u^- \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&\leq r(x - cs')u^- + \Delta u^- + u^-(\nabla \cdot \chi \nabla v(s', \cdot; u_0) - A) \\
&\leq \Delta u^- + u^-(r(x-cs') - bu^-) \ \ \forall\, s' > t-\delta, x \in \mathbb{R}
\end{aligned}
$$

and

$$
\begin{aligned}
& u^+_{s'} + \chi \nabla v(s', \cdot; u_0) \cdot \nabla u^+ + u^+ \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&= (r^* + A)u^+ + \Delta u^+ + u^+ \nabla \cdot \chi \nabla v(s', \cdot; u_0) \\
&\geq \Delta u^+ + u^+(r(x-cs') - bu^+) \ \ \forall\, s' > t-\delta, x \in \mathbb{R}.
\end{aligned}
$$

Hence, $u^-(s', x)$ and $u^+(s', x)$ are, respectively, a sub- and super-solution of the following equation for $(s', x) \in (t-\delta, \infty) \times \mathbb{R}$,

$$
u_{s'} = \Delta u - \chi \nabla v(s', x; u_0) \cdot \nabla u - \chi u \nabla \cdot \nabla v(s', x; u_0) + u(r(x-cs') - bu), \quad x \in \mathbb{R}.
$$

Furthermore,

$$
u^-(t-\delta, \cdot) \leq u(t-\delta, \cdot, u_0) = u^+(t-\delta, \cdot).
$$

Then by the comparison principle for parabolic equations, we have that, for all $(s', x) \in (t - \delta, \infty) \times \mathbb{R}$,

$$\min\{1, \|u(t-\delta, \cdot; u_0)\|_\infty^{-1}\} e^{(r_* - A)(s' - t + \delta)} \bar{u}(s', x) \le u(s', x; u_0) \\ \le e^{(r^* + A)(s' - t + \delta)} \bar{u}(s', x). \qquad (2.5)$$

Now by (2.5),

$$u(t, x; u_0) \le e^{(r^* + A)\delta} \bar{u}(t, x) \quad \forall\, x \in \mathbb{R}.$$

By the arguments of [15, Lemma 2.2], there is $C_{t_0, s_0, R, M, p}$ such that if $s \in [0, s_0],\ t \ge t_0,\ |x - y| \le R$, then (2.4) holds. The lemma thus follows. □

By Lemma 2.1 and Lemma 2.2 with $p > 1$, $s_0 = 0$ and $t_0 = 1$, we have

$$|\chi v_x(t, x; u_0)| + \chi\lambda v(t, x; u_0) \le C_{R,p}\big(u(t, x; u_0)\big)^{\frac{1}{p}} + \varepsilon_R M \\ = C_{R,p}\big(u(t, x; u_0)\big)^{\frac{1}{p}} + \varepsilon_R M \quad \forall\, t \ge 1,\ x \in \mathbb{R}, \qquad (2.6)$$

where $C_{R,p} = C_R \cdot C_{1,0,R,M,p} \cdot (M+1) (> 0)$.

Lemma 2.3. *Assume that $b > \chi\mu$. Let $c_\kappa = \frac{\kappa^2 + r^*}{\kappa}$ with $0 < \kappa \le \sqrt{r^*}$ satisfying*

$$\frac{(\kappa - \sqrt{\lambda})_+}{(\kappa + \sqrt{\lambda})} \le \frac{2(b - \chi\mu)}{\chi\mu}.$$

The following hold.

(i) For any $u_0 \ge 0$ with nonempty compact support and any $M \gg \frac{r^*}{b - \chi\mu}$ satisfying

$$\max\{u_0(x), u_0(-x)\} \le U^+(x) := \min\{M, M e^{-\kappa x}\}, \quad \forall\, x \in \mathbb{R},$$

there holds

$$u(t, x; u_0) \le M e^{-\kappa(|x| - c_\kappa t)}, \quad \forall\, x \in \mathbb{R},\ t \ge 0.$$

(ii) For any $u_0 \in C^b_{\rm unif}(\mathbb{R})$, $u_0(x) \ge 0$, and any $M \gg \frac{r^*}{b-\chi\mu}$ satisfying

$$u_0(x) \le U^+(x) := \min\{M, Me^{\kappa x}\}, \quad \forall\, x \in \mathbb{R},$$

there holds

$$u(t,x;u_0) \le Me^{\kappa(x+c_\kappa t)}, \quad \forall\, x \in \mathbb{R},\ t \ge 0.$$

Proof. It follows from the arguments of [54, Lemma 2.3]. □

Consider

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + u(r^* - bu), \quad x \in \mathbb{R} \\ 0 = v_{xx} - \lambda v + \mu u, \quad x \in \mathbb{R}. \end{cases} \tag{2.7}$$

Lemma 2.4. *Assume that $b > 2\chi\mu$.*

(1) For any $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u_0(x) > 0$,

$$\lim_{t\to\infty} \|u(t,\cdot;u_0) - \frac{r^*}{b}\|_\infty = 0,$$

where $(u(t,x;u_0), v(t,x;u_0))$ is the solution of (2.7) with $u(0,x;u_0) = u_0(x)$.

(2) If $(u^*(t,x), v^*(t,x))$ is an entire solution of (2.7) satisfying that

$$0 < \inf_{t\in\mathbb{R},x\in\mathbb{R}} u^*(t,x) \le \sup_{t\in\mathbb{R},x\in\mathbb{R}} u^*(t,x) < \infty,$$

then

$$(u^*(t,x), v^*(t,x)) \equiv \Big(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b}\Big).$$

Proof. (1) It follows from [49, Theorem 1.8].

(2) It follows from the arguments in [49, Theorem 1.8]. □

For every $u \in C^b_{\rm unif}(\mathbb{R})$, let

$$\Psi(x;u) = \mu \int_0^\infty \int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|y-x|^2}{4s}}}{\sqrt{4\pi s}} u(y)dyds. \tag{2.8}$$

It is well known that $\Psi(x;u) \in C^2_{\rm unif}(\mathbb{R})$ and solves the elliptic equation

$$\frac{d^2}{dx^2}\Psi(x;u) - \lambda\Psi(x;u) + \mu u = 0.$$

Lemma 2.5.

$$\Psi(x;u) = \frac{\mu}{2\sqrt{\lambda}}\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}u(y)dy \tag{2.9}$$

and

$$\frac{d}{dx}\Psi(x;u) = -\frac{\mu}{2}e^{-\sqrt{\lambda}x}\int_{-\infty}^{x} e^{\sqrt{\lambda}y}u(y)dy + \frac{\mu}{2}e^{\sqrt{\lambda}x}\int_{x}^{\infty} e^{-\sqrt{\lambda}y}u(y)dy. \tag{2.10}$$

Proof. The lemma is proved in [54, Lemma 2.1]. □

Lemma 2.6. *For every $u \in C^b_{\rm unif}(\mathbb{R})$, $u(x) \geq 0$, it holds that*

$$\left|\frac{d}{dx}\Psi(x;u)\right| \leq \sqrt{\lambda}\Psi(x;u), \quad \forall\, x \in \mathbb{R}.$$

Proof. The lemma is proved in [54, Lemma 2.2]. □

Lemma 2.7. *Suppose that $b > \chi\mu$. For every $u \in C^b_{\rm unif}(\mathbb{R})$, $0 \leq u(x) \leq \frac{r^*}{b-\chi\mu}$, it holds that*

$$\Psi(x;u) \leq \frac{\mu r^*}{\lambda(b-\chi\mu)} \quad \text{and} \quad \frac{d}{dx}\Psi(x;u) \leq \frac{\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)}.$$

Proof. It follows from a direct calculation. □

2.3 Persistence and spreading speeds

In this section, we study persistence and spreading speeds of (1.1) with $r(x)$ being as in **Case 1** or **Case 2**.

2.3.1 Case 1

In this subsection, we study persistence and spreading speeds of (1.1) with $r(x)$ being as in **Case 1** and prove Theorems 2.1 and 2.2. Throughout this subsection, we assume that **(H1)** holds and $r(x)$ is as in **Case 1**.

We first prove two lemmas.

Observe that for any given $\bar{c}$, let $u(t,x) = \tilde{u}(t, x - \bar{c}t)$ and $v(t,x) = \tilde{v}(t, x - \bar{c}t)$. Then (1.1) becomes

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u} + \bar{c}\nabla\tilde{u} - \nabla\cdot(\chi\tilde{u}\nabla\tilde{v}) + \tilde{u}(r(x-(c-\bar{c})t) - b\tilde{u}), \quad x \in \mathbb{R} \\ 0 = \Delta\tilde{v} - \lambda\tilde{v} + \mu\tilde{u}, \quad x \in \mathbb{R}. \end{cases} \tag{2.11}$$

In the following, $(u(t,x;u_0), v(t,x;u_0))$ denotes the solution of (1.1) with $u(0,x;u_0) = u_0(x)$, and $(\tilde{u}(t,x;u_0), \tilde{v}(t,x;u_0))$ denotes the solution of (2.11) with $\tilde{u}(0,x;u_0) = u_0(x)$.

For any given $0 < \epsilon < 2\sqrt{r^*}$, fix $\bar{r} < r^*$ such that

$$4\bar{r} - \bar{c}^2 \geq \epsilon\sqrt{r^*} \quad \forall \; -2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon. \tag{2.12}$$

Let

$$l = \frac{2\pi}{(\epsilon\sqrt{r^*})^{\frac{1}{2}}} \tag{2.13}$$

and

$$\lambda(\bar{c},\bar{r}) = \frac{4\bar{r} - \bar{c}^2 - \frac{\pi^2}{l^2}}{4}. \tag{2.14}$$

Then $\lambda(\bar{c},\bar{r}) \geq \frac{3\epsilon\sqrt{r^*}}{16} > 0$ for any $-2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon$.

Lemma 2.8. *For given $0 < \epsilon < 2\sqrt{r^*}$, let $\bar{r}$ and l be as in* (2.12) *and* (2.13). *Then for any $-2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon$, $\lambda(\bar{c},\bar{r})$ which is defined as in* (2.14) *is the principal eigenvalue of*

$$\begin{cases} \phi_{xx} + \bar{c}\phi_x + \bar{r}\phi = \lambda\phi, \quad -l < x < l \\ \phi(-l) = \phi(l) = 0, \end{cases}$$

and $\phi(x;\bar{c},\bar{r}) = e^{-\frac{\bar{c}}{2}x}\cos\frac{\pi}{2l}x$ is a corresponding positive eigenfunction.

Proof. It follows from direct calculations. $\square$

Consider

$$\begin{cases} u_t = u_{xx} + \bar{c}u_x - A(t,x)u_x + u(r^* - 2\varepsilon_R M - C_{R,p}u^{\frac{1}{p}} - (\bar{b} - \chi\mu)u), \quad -l < x < l \\ u(t,-l) = u(t,l) = 0, \end{cases} \tag{2.15}$$

where $R \gg 1$ is such that $r^* - 2\varepsilon_R M > \bar{r}$, and $A(t,x)$ is globally Hölder continuous in $t \in \mathbb{R}$ and $x \in [-l,l]$ with Hölder exponent $0 < \alpha < 1$ and $\|A(\cdot,\cdot)\|_\infty < \infty$.

Lemma 2.9. *For given* $0 < \epsilon < 2\sqrt{r^*}$*, let* $\bar{r}$ *and* l *be as in* (2.12) *and* (2.13)*. There is* $\eta > 0$ *such that for any* $A(\cdot,\cdot)$ *with* $\|A(\cdot,\cdot)\|_\infty < \eta$*, any* $\bar{c} \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon]$*, and any* $\bar{b} \in (\chi\mu, \infty)$*,* (2.15) *has a unique positive bounded entire solution* $u(t,x;\bar{c},\bar{b},A)$ *satisfying that*

$$\inf_{-l+\delta \le x \le l-\delta, t \in \mathbb{R}, \bar{c} \in [-2\sqrt{r^*}+\epsilon, 2\sqrt{r^*}-\epsilon]} u(t,x;\bar{c},\bar{b},A) > 0 \quad \forall\, 0 < \delta < l. \tag{2.16}$$

Proof. First, consider

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + \bar{c}\tilde{u}_x - A(t,x)\tilde{u}_x + \bar{r}\tilde{u}, \quad -l < x < l \\ \tilde{u}(t,-l) = \tilde{u}(t,l) = 0. \end{cases} \tag{2.17}$$

For given $t_0 \in \mathbb{R}$, let $\tilde{u}(t,x;t_0,\phi(\cdot;\bar{c},\bar{r}),A)$ be the solution of (2.17) with $\tilde{u}(t_0,x;t_0,\phi(\cdot;\bar{c},\bar{r}),A) = \phi(x;\bar{c},\bar{r})$. By [16, Theorem 3.4.1],

$$\lim_{\|A(\cdot,\cdot)\|_\infty \to 0} \|\tilde{u}(t,\cdot;t_0,\phi(\cdot;\bar{c},\bar{r}),A) - e^{\lambda(\bar{c},\bar{r})(t-t_0)}\phi(\cdot;\bar{c},\bar{r})\|_{C^1([-l,l])} = 0$$

uniformly in $t \in [t_0, t_0+1]$, $t_0 \in \mathbb{R}$, and $\bar{c} \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon]$. This implies that there is $\eta > 0$ such that for any $A(\cdot,\cdot)$ with $\|A(\cdot,\cdot)\|_\infty < \eta$, and any $\bar{c} \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon]$,

$$\tilde{u}(t_0+1,x;t_0,\phi(\cdot;\bar{c},\bar{r}),A) \ge e^{\lambda(\bar{c},\bar{r})/2}\phi(x;\bar{c},\bar{r}) \;\; \forall -l \le x \le l, \;\; t_0 \in \mathbb{R}. \tag{2.18}$$

Next, suppose that $\|A(\cdot,\cdot)\|_\infty < \eta$. Let $u(t,x;t_0,\sigma\phi(\cdot;\bar c,\bar r))$ be the solution of (2.15) with $u(t_0,x;t_0,\sigma\phi(\cdot;\bar c,\bar r)) = \sigma\phi(x;\bar c,\bar r)$. Note that

$$\lim_{\sigma\to 0}\sup_{t\in[t_0,t_0+1],-l\le x\le l} u(t,x;t_0,\sigma\phi(\cdot;\bar c,\bar r)) = 0$$

uniformly in $t_0\in\mathbb{R}$ and in $\bar c\in[-2\sqrt{r^*}+\epsilon, 2\sqrt{r^*}-\epsilon]$. Hence there is $\sigma_0>0$ such that for any $0<\sigma\le\sigma_0$, and $\bar c\in[-2\sqrt{r^*}+\epsilon, 2\sqrt{r^*}-\epsilon]$,

$$r^*-2\varepsilon_R M-C_{R,p}u^{\frac{1}{p}}(t,x;t_0,\sigma\phi)-(\bar b-\chi\mu)u(t,x;t_0,\sigma\phi) > \bar r\ \forall\, t\in[t_0,t_0+1],\ -l\le x\le l,\ t_0\in\mathbb{R}.$$

This together with the comparison principle for parabolic equations implies that for $0<\sigma\le\sigma_0$ and $\bar c\in[-2\sqrt{r^*}+\epsilon, 2\sqrt{r^*}-\epsilon]$,

$$u(t,x;t_0,\sigma\phi(\cdot;\bar c,\bar r)) \ge \sigma\tilde u(t,x;t_0,\phi,A)\ \ \forall\, t\in[t_0,t_0+1],\ -l\le x\le l,\ t_0\in\mathbb{R}.$$

Then by (2.18), we have

$$u(t_0+1,x;t_0,\sigma\phi(\cdot;\bar c,\bar r)) \ge \sigma e^{\lambda(\bar c,\bar r)/2}\phi(x;\bar c,\bar r)\ \ \forall -l\le x\le l,\ t_0\in\mathbb{R} \qquad (2.19)$$

for any $0<\sigma\le\sigma_0$ and $\bar c\in[-2\sqrt{r^*}+\epsilon, 2\sqrt{r^*}-\epsilon]$.

Now, by (2.19) and the comparison principle for parabolic equations, we have

$$u(k,x;-n,\sigma\phi(\cdot;\bar c,\bar r)) > \sigma\phi(x;\bar c,\bar r)\quad \forall\ k\ge -n+1,\ -l<x<l \qquad (2.20)$$

and then

$$u(k,x;-(n+1),\sigma\phi(\cdot;\bar c,\bar r)) > u(k,x;-n,\sigma\phi(\cdot;\bar c,\bar r))\quad \forall\ k\ge -n+1,\ -l<x<l. \quad (2.21)$$

Let $u_n(t,x)=u(t,x;-n,\sigma\phi(\cdot;\bar c,\bar r))$. Then $\lim_{n\to\infty}u_n(t,x)$ exists and $u(t,x)=\lim_{n\to\infty}u_n(t,x)$ is a solution of (2.15). By (2.20) and (2.21), $u(t,x)$ is a positive bounded entire solution of (2.15) satisfying (2.16).

Finally, we prove that when $\|A(\cdot,\cdot)\|_\infty < \eta$, (2.15) has a unique positive bounded entire solution satisfying (2.16). Suppose that $u(t,x)$, $v(t,x)$ are two positive bounded entire solutions of (2.15) satisfying (2.16). By Hopf's lemma,

$$u_x(t,-l) > 0,\ u_x(t,l) < 0,\ v_x(t,-l) > 0,\ v_x(t,l) < 0 \quad \forall\, t \in \mathbb{R}.$$

This implies that for any $t \in \mathbb{R}$, the following set is not empty,

$$\{\gamma > 1 \,|\, \frac{1}{\gamma} u(t,x) \le v(t,x) \le \gamma u(t,x) \quad \forall\ -l < x < l\}.$$

Hence we can define

$$\rho(u(t,\cdot), v(t,\cdot)) = \inf\{\ln\gamma \,|\, \frac{1}{\gamma} u(t,x) \le v(t,x) \le \gamma u(t,x) \quad \forall\ -l < x < l\}.$$

To prove the uniqueness of positive entire solutions satisfying (2.16), it then suffices to prove $\rho(u(t,\cdot), v(t,\cdot)) \equiv 0$.

Fix $t_0 \in \mathbb{R}$. Suppose that $\gamma > 1$ is such that

$$\frac{1}{\gamma} u(t_0,x) \le v(t_0,x) \le \gamma u(t_0,x) \quad \forall\ -l < x < l.$$

By the comparison principle for parabolic equations, we have

$$\frac{1}{\gamma} u(t,x;t_0,u(t_0,\cdot)) < v(t,x;t_0,v(t_0,\cdot)) < \gamma u(t,x;t_0,u(t_0,\cdot)) \quad \forall\ t > t_0,\ -l < x < l.$$

This together with Hopf's lemma implies that there is $1 < \gamma(t) < \gamma$ such that

$$\frac{1}{\gamma(t)} u(t,x;t_0,u(t_0,\cdot)) < v(t,x;t_0,v(t_0,\cdot)) < \gamma(t) u(t,x;t_0,u(t_0,\cdot)) \quad \forall\ t > t_0,\ -l < x < l.$$

Hence if $\rho(u(t_0,\cdot), v(t_0,\cdot))) \ne 0$ for some $t_0 \in \mathbb{R}$, then $\rho(u(t,\cdot), v(t,\cdot))$ is strictly decreasing as t increases.

Assume that $\rho(u(t,\cdot),v(t,\cdot)) \not\equiv 0$. Let $\rho^* = \lim_{t\to-\infty}\rho(u(t,\cdot),v(t,\cdot))$. Then $\rho^* > 0$. Choose a sequence $t_n \to -\infty$. Without loss of generality, we may assume that

$$A(t_n+t,x) \to A^*(t,x), \quad u(t_n+t,x) \to u^*(t,x), \quad v(t_n+t,x) \to v^*(t,x)$$

as $n \to \infty$ uniformly in $x \in [-l,l]$ and locally uniformly in $t \in \mathbb{R}$. We then have that $u^*(t,x)$ and $v^*(t,x)$ are positive solutions of (2.15) with $A(t,x)$ being replaced by $A^*(t,x)$ and satisfy (2.16). Moreover,

$$\rho(u^*(t,\cdot),v^*(t,\cdot)) = \rho^* > 0 \quad \forall\, t \in \mathbb{R}.$$

But by the arguments in the above, $\rho(u^*(t,\cdot),v^*(t,\cdot))$ is strictly decreasing as t increases, which is a contradiction.

Therefore, $\rho(u(t,\cdot),v(t,\cdot)) \equiv 0$ and $u(t,x) \equiv v(t,x)$. □

Next, we prove Theorem 2.1.

Proof of Theorem 2.1. (1) Suppose $c > c^* = 2\sqrt{r^*}$. Choose $\bar{c}$ and $0 < \kappa \le \sqrt{r^*}$ such that

$$\frac{(\kappa-\sqrt{\lambda})_+}{(\kappa+\sqrt{\lambda})} \le \frac{2(b-\chi\mu)}{\chi\mu}$$

and

$$c^* = 2\sqrt{r^*} < c_\kappa < \bar{c} < c.$$

By Lemma 2.3(i),

$$\lim_{t\to\infty}\sup_{|x|\ge\bar{c}t} u(t,x;u_0) = 0.$$

We claim that $\lim_{t\to\infty}\sup_{x\in\mathbb{R}} u(t,x;u_0) = 0$. For otherwise, there are $\delta_0 > 0$, $t_n \to \infty$ and $x_n \in (-\bar{c}t_n, \bar{c}t_n)$ such that $u(t_n,x_n;u_0) \ge \delta_0$ for all $n \ge 1$.

Let $u_n(t,x) = u(t+t_n, x+x_n; u_0)$ and $v_n(t,x) = v(t+t_n, x+x_n; u_0)$. Note that $x_n - ct_n \to -\infty$ as $n \to \infty$. Without loss of generality, we may assume that there is $(u^*(t,x), v^*(t,x))$ such that

$$\lim_{n\to\infty}(u_n(t,x),v_n(t,x)) = (u^*(t,x),v^*(t,x))$$

locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$, and $(u^*(t, x), v^*(t, x))$ satisfies

$$\begin{cases} u_t^* = \Delta u^* - \nabla \cdot (\chi u^* \nabla v^*) + u^*(r(-\infty) - bu^*), \quad t \in \mathbb{R},\ x \in \mathbb{R} \\ 0 = \Delta v^* - \lambda v^* + \mu u^*, \quad t \in \mathbb{R},\ x \in \mathbb{R}. \end{cases} \tag{2.22}$$

By $r(-\infty) < 0$, it can be proved that $u^*(t, x) \equiv 0$, which contradicts to

$$u^*(0, 0) = \lim_{n \to \infty} u(t_n, x_n; u_0) \geq \delta_0.$$

Therefore, $\lim_{t \to \infty} \sup_{x \in \mathbb{R}} u(t, x; u_0) = 0$.

(2) Suppose $-c^* \leq c < c^* = 2\sqrt{r^*}$. We first prove

$$\lim_{t \to \infty} \sup_{x \leq (c-\varepsilon)t} u(t, x; u_0) = 0.$$

Assume that the result does not hold. Then there are constants $\delta_0 > 0$, $\varepsilon_0 > 0$, and a sequence $\{(t_n, x_n)\}_{n \in \mathbb{N}}$, $t_n \to \infty$, $x_n \in (-\infty, (c - \varepsilon_0)t_n]$ such that $u(t_n, x_n; u_0) \geq \delta_0$ for all $n \geq 1$.

Let $u_n(t, x) = u(t + t_n, x + x_n; u_0)$ and $v_n(t, x) = v(t + t_n, x + x_n; u_0)$. Note that $x_n - ct_n \to -\infty$ as $n \to \infty$. Similarly, without loss of generality, we may assume that there is $(u^*(t, x), v^*(t, x))$ such that

$$\lim_{n \to \infty} (u_n(t, x), v_n(t, x)) = (u^*(t, x), v^*(t, x))$$

locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$, and $(u^*(t, x), v^*(t, x))$ satisfies (2.22). Again, by $r(-\infty) < 0$, it can be proved that $u^*(t, x) \equiv 0$, which contradicts to

$$u^*(0, 0) = \lim_{n \to \infty} u(t_n, x_n; u_0) \geq \delta_0.$$

Therefore, $\lim_{t \to \infty} \sup_{x \leq (c-\varepsilon)t} u(t, x; u_0) = 0$.

Next, we prove

$$\lim_{t \to \infty} \sup_{x \geq (c^*+\varepsilon)t} u(t, x; u_0) = 0.$$

For any $\varepsilon > 0$, let $\kappa = \sqrt{r^*}$, then $c_\kappa = 2\sqrt{r^*} < c^* + \varepsilon$. Since $0 < \chi\mu < b$, $\big(1 + \frac{1}{2}\frac{(\sqrt{r^*}-\sqrt{\lambda})_+}{(\sqrt{r^*}+\sqrt{\lambda})}\big)\chi\mu \le b$, by Lemma 2.3 (i), we have

$$\sup_{x\ge(c^*+\varepsilon)t} u(t,x;u_0) \le Me^{-\kappa(c^*+\varepsilon-c_\kappa)t} \to 0 \quad \text{as} \quad t\to\infty.$$

Therefore, $\lim_{t\to\infty}\sup_{x\ge(c^*+\varepsilon)t} u(t,x;u_0) = 0$.

We now prove

$$\liminf_{t\to\infty} \inf_{(c+\varepsilon)t\le x\le(c^*-\varepsilon)t} u(t,x;u_0) > 0. \tag{2.23}$$

To this end, for any $0 < \varepsilon < \frac{c^*-c}{2}$, let $\bar r$ and l be as in (2.12) and (2.13). Fix a $\bar c$ satisfying $-c^* + \varepsilon \le c+\varepsilon \le \bar c \le c^* - \varepsilon$ and set $M = \max\{\|u_0\|_\infty, \frac{r^*}{b-\chi\mu}\}$. Let $\lambda(\bar c, \bar r)$ be as in (2.14). By (2.6), for any $R \gg 1, p > 1$, $(\tilde u(t,x;u_0), \tilde v(t,x;u_0))$ satisfies

$$\tilde u_t \ge \tilde u_{xx} + \bar c\tilde u_x - \chi\tilde v_x\tilde u_x + \tilde u(r(x-(c-\bar c)t) - \varepsilon_R M - C_{R,p}\tilde u^{\frac{1}{p}} - (b-\chi\mu)\tilde u), \quad t\ge 1,\ x\in\mathbb{R}.$$

Let $p = 2$ and η be as in Lemma 2.9. Choose $R \gg 1$ such that $\varepsilon_R M < \frac{\eta}{4}$,

$$|\chi\tilde v_x| \le C_R\sqrt{\tilde u(t,x;u_0)} + \frac{\eta}{4}, \quad t\ge 1,\ x\in\mathbb{R}.$$

Define

$$A(t,x) = \begin{cases} \frac{\chi\tilde v_x(1,x;u_0)}{\max\{1,|\chi\tilde v_x(1,x;u_0)|\eta^{-1}\}}, & \text{if } t<1,\ x\in\mathbb{R} \\ \frac{\chi\tilde v_x(t,x;u_0)}{\max\{1,|\chi\tilde v_x(t,x;u_0)|\eta^{-1}\}}, & \text{if } t\ge 1,\ x\in\mathbb{R}. \end{cases}$$

Note that $\tilde v_x(t,x;u_0)$ is globally Hölder continuous in $t \ge 1$ and $x \in \mathbb{R}$. We then have that $A(t,x)$ is globally Hölder continuous in $t\in\mathbb{R}$ and $x\in\mathbb{R}$. It is clear that $\|A(\cdot,\cdot)\|_\infty < \eta$.

Let $T > 1$ be such that $r(x-(c-\bar c)t) \ge r^* - \varepsilon_R M$ for $x \ge -l$ and $t \ge T$. Choose $\bar b > b$ and also $\bar b \gg 1$ such that

$$C_R\sqrt{\frac{r^*}{\bar b-\chi\mu}} + \frac{r^*}{\bar b-\chi\mu} < \frac{\eta}{4},$$

and $u(T,x;\bar c,\bar b,A) < \frac{r^*}{\bar b-\chi\mu} < \tilde u(T,x;u_0)$ for $-l \le x \le l$ and $\bar c \in [c+\varepsilon, c^*-\varepsilon]$, where $u(t,x;\bar c,\bar b,A)$ is the unique positive bounded entire solution of (2.15).

Fix such $\bar{b}$. We first claim that for any $c+\varepsilon \le \bar{c} \le c^*-\varepsilon$,

$$\tilde{u}(t,x;u_0) \ge u(t,x;\bar{c},\bar{b},A) \quad \forall\, t \ge T,\ -l \le x \le l. \tag{2.24}$$

Suppose, by contradiction that (2.24) does not hold. Then there are $c+\varepsilon \le \bar{c} \le c^*-\varepsilon$ and $t_{\inf} \in [T,\infty)$ satisfying

$$t_{\inf} := \inf\{t \in (T,\infty)\,|\,\exists x_t \in \mathbb{R}, \text{satisfying} \quad u(t,x_t;\bar{c},\bar{b},A) > \tilde{u}(t,x_t;u_0), |x_t| \le l\}.$$

Note that

$$u(T,x;\bar{c},\bar{b},A) < \tilde{u}(T,x;u_0) \quad \forall -l \le x \le l$$

Hence

$$t_{\inf} > T.$$

Moreover, note that $u(t,-l;\bar{c},\bar{b},A) = u(t,l;\bar{c},\bar{b},A) = 0$ for any $t \in \mathbb{R}$, there is $x_{\inf} \in \mathbb{R}$ such that $|x_{\inf}| < l$,

$$\frac{r^*}{\bar{b}-\chi\mu} > u(t_{\inf},x_{\inf};\bar{c},\bar{b},A) = \tilde{u}(t_{\inf},x_{\inf};u_0), \tag{2.25}$$

and

$$u(t,x;\bar{c},\bar{b},A) < \tilde{u}(t,x;u_0), \qquad |x| \le l, \quad T \le t < t_{\inf}.$$

Hence there is $0 < \delta \ll 1$ such that $[t_{\inf}-\delta, t_{\inf}] \times [x_{\inf}-\delta, x_{\inf}+\delta] \subset \{(t,y)\,|\,|y| < l\}$ and

$$A(t,x) = \chi\tilde{v}_x(t,x;u_0), \quad \forall\, t_{\inf}-\delta \le t \le t_{\inf}, \quad x_{\inf}-\delta \le x \le x_{\inf}+\delta.$$

Note that

$$u(t_{\inf}-\delta,x;\bar{c},\bar{b},A) < \tilde{u}(t_{\inf}-\delta,x;u_0) \quad \forall\, x \in [x_{\inf}-\delta, x_{\inf}+\delta]$$

and

$$u(t,x_{\inf}\pm\delta;\bar{c},\bar{b},A) \le \tilde{u}(t,x_{\inf}\pm\delta;u_0) \quad \forall\, t_{\inf}-\delta \le t \le t_{\inf}.$$

Thus, by the comparison principle for parabolic equations, we have

$$u(t,x;\bar c,\bar b,A) < \tilde u(t,x;u_0) \quad \forall\, t_{\rm inf}-\delta < t \le t_{\rm inf}, \quad x_{\rm inf}-\delta < x < x_{\rm inf}+\delta.$$

In particular,

$$u(t_{\rm inf},x_{\rm inf};\bar c,\bar b,A) < \tilde u(t_{\rm inf},x_{\rm inf};u_0).$$

Which contradicts to (2.25).

By Lemma 2.9,

$$\inf_{-l+\delta\le x\le l-\delta, c+\varepsilon\le \bar c\le c^*-\varepsilon, t\ge 1} u(t,x;\bar c,\bar b,A) > 0, \quad \forall\ 0<\delta<l.$$

This together with (2.24) implies that

$$\liminf_{t\to\infty} \inf_{-l+\delta\le x\le l-\delta, c+\varepsilon\le \bar c\le c^*-\varepsilon} \tilde u(t,x;u_0) > 0, \quad \forall\, 0<\delta<l.$$

Hence

$$\liminf_{t\to\infty} \inf_{-l+\bar c t+\delta\le x\le l+\bar c t-\delta, c+\varepsilon\le \bar c\le c^*-\varepsilon} u(t,x;u_0) > 0, \quad \forall\ 0<\delta<l.$$

It then follows that

$$\liminf_{t\to\infty} \inf_{-l+(c+\varepsilon)t+\delta\le x\le l+(c^*-\varepsilon)t-\delta} u(t,x;u_0) > 0, \quad \forall\, 0<\delta<l.$$

and

$$\liminf_{t\to\infty} \inf_{(c+\varepsilon)t\le x\le (c^*-\varepsilon)t} u(t,x;u_0) > 0.$$

Finally, suppose that $2\chi\mu < b$. We prove

$$\lim_{t\to\infty} \sup_{(c+\varepsilon)t\le x\le (c^*-\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0. \tag{2.26}$$

Suppose by contraction that the result does not hold. Then there are constants $0 < \varepsilon < \frac{c^*-c}{2}$, $\delta > 0$, and a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$, $t_n(c+\varepsilon) \le x_n \le t_n(c^*-\varepsilon)$, and

$$|u(t_n, x_n; u_0) - \frac{r^*}{b}| \ge \delta \quad \forall\, n \ge 1. \tag{2.27}$$

For every $n \ge 1$, define $(u_n(t,x), v_n(t,x)) = (u(t+t_n, x+x_n; u_0), v(t+t_n, x+x_n; u_0))$. By a priori estimates for parabolic equations, without loss of generality, we may suppose that $(u_n(t,x), v_n(t,x)) \to (u^*(t,x), v^*(t,x))$ locally uniformly in $C^{1,2}(\mathbb{R}\times\mathbb{R})$. Furthermore, $(u^*(t,x), v^*(t,x))$ is an entire solution of (2.7).

Choose $0 < \tilde{\varepsilon} < \varepsilon < \frac{c^*-c}{2}$. For every $x \in \mathbb{R}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} x + x_n &\le x + t_n(c^* - \varepsilon) \\ &= (c^* - \tilde{\varepsilon})(t_n + t) - (\varepsilon - \tilde{\varepsilon})(t_n - \frac{x - (c^* - \tilde{\varepsilon})t}{\varepsilon - \tilde{\varepsilon}}) \\ &\le (t + t_n)(c^* - \tilde{\varepsilon}) \end{aligned}$$

whenever $t_n \ge \frac{\|x\| + (c^*-\tilde{\varepsilon})|t|}{\varepsilon - \tilde{\varepsilon}}$. On the other hand, For every $x \in \mathbb{R}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} x + x_n &\ge x + t_n(c + \varepsilon) \\ &= (c + \tilde{\varepsilon})(t_n + t) + (\varepsilon - \tilde{\varepsilon})(t_n - \frac{(c + \tilde{\varepsilon})t - x}{\varepsilon - \tilde{\varepsilon}}) \\ &\ge (t + t_n)(c + \tilde{\varepsilon}) \end{aligned}$$

whenever $t_n \ge \frac{\|x\| + (c+\tilde{\varepsilon})|t|}{\varepsilon - \tilde{\varepsilon}}$. Thus, it follows that

$$(t + t_n)(c + \tilde{\varepsilon}) \le x + x_n \le (t + t_n)(c^* - \tilde{\varepsilon})$$

whenever $t_n \ge \max\{\frac{\|x\| + (c^*-\tilde{\varepsilon})|t|}{\varepsilon - \tilde{\varepsilon}}, \frac{\|x\| + (c+\tilde{\varepsilon})|t|}{\varepsilon - \tilde{\varepsilon}}\}$. Note that

$$u^*(t,x) = \lim_{n\to\infty} u(t + t_n, x + x_n; u_0) \ge \liminf_{s\to\infty} \inf_{s(c+\tilde{\varepsilon}) \le y \le s(c^*-\tilde{\varepsilon})} u(s, y; u_0) > 0$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}$. Hence $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}} u^*(t,x) > 0$. By Lemma 2.4 (ii), we must have $u^*(t,x) = \frac{r^*}{b}$ for every $(t,x) \in \mathbb{R}\times\mathbb{R}$. In particular, $u^*(0,0) = \frac{r^*}{b}$, which contradicts to (2.27).

(3) First, let $\kappa = \sqrt{r^*}$, for any $\bar{c}$ satisfies

$$c^* = 2\sqrt{r^*} = c_\kappa < \bar{c} < |c|.$$

By Lemma 2.3 (i),

$$\lim_{t\to\infty} \sup_{|x|\geq \bar{c}t} u(t,x;u_0) = 0.$$

This implies that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \sup_{|x|\geq (c^*+\varepsilon)t} u(t,x;u_0) = 0.$$

Next, for any $0 < \varepsilon < c^*$, let $\bar{r}$ and l be as in (2.12) and (2.13). Fix a $\bar{c}$ satisfying $-c^* + \varepsilon \leq \bar{c} \leq c^* - \varepsilon$. Let $\lambda(\bar{c},\bar{r})$ be as in (2.14). By the similar arguments as those in the proof of (2.23), it can be proved that

$$\liminf_{t\to\infty} \inf_{-l+\bar{c}t+\delta\leq x\leq l+\bar{c}t-\delta, -c^*+\varepsilon\leq\bar{c}\leq c^*-\varepsilon} u(t,x;u_0) > 0, \quad \forall\, 0<\delta<l.$$

Hence

$$\liminf_{t\to\infty} \inf_{-l+(-c^*+\varepsilon)t+\delta\leq x\leq l+(c^*-\varepsilon)t-\delta} u(t,x;u_0) > 0, \quad \forall\, 0<\delta<l$$

and then

$$\liminf_{t\to\infty} \inf_{(-c^*+\varepsilon)t\leq x\leq (c^*-\varepsilon)t} u(t,x;u_0) > 0.$$

Moreover, using the similar arguments as those in the proof of (2.26), we can prove if $2\chi\mu < b$, then

$$\lim_{t\to\infty} \sup_{(-c^*+\varepsilon)t\leq x\leq (c^*-\varepsilon)t} |u(t,x;u_0) - \frac{r^*}{b}| = 0.$$

□

Now we prove Theorem 2.2.

Proof of Theorem 2.2. (1) Suppose $c \geq -c^* = -2\sqrt{r^*}$. By the same arguments as those in Theorem 2.1 (2), it can be proved that

$$\lim_{t\to\infty} \sup_{x\leq(c-\varepsilon)t} u(t,x;u_0) = 0.$$

Next, we prove that

$$\liminf_{t\to\infty} \inf_{x\geq(c+\varepsilon)t} u(t,x;u_0) > 0 \quad \forall\, \varepsilon > 0. \tag{2.28}$$

To this end, for any $\tilde{\varepsilon} > 0$, let $u(t,x;u_0) = \tilde{u}(t,x-(c+\tilde{\varepsilon})t;u_0), v(t,x;u_0) = \tilde{v}(t,x-(c+\tilde{\varepsilon})t;u_0)$ in (1.1) and set $M = \max\{\|u_0\|_\infty, \frac{r^*}{b-\chi\mu}\}$. By (2.6), it follows that, for any $R \gg 1, p > 1$, $(\tilde{u}(t,x;u_0), \tilde{v}(t,x;u_0))$ satisfies

$$\tilde{u}_t \geq \tilde{u}_{xx}+(c+\tilde{\varepsilon})\tilde{u}_x-\chi\tilde{v}_x\tilde{u}_x+\tilde{u}(r(x+\tilde{\varepsilon}t)-\varepsilon_R M-C_{R,p}\tilde{u}^{\frac{1}{p}}-(b-\chi\mu)\tilde{u}),\ t\geq 1,\ x\in\mathbb{R}. \tag{2.29}$$

Let $p = 2$. Choose $R \gg 1$, $0 < \xi \ll 1$ and $0 < \epsilon \ll \min\{1, 2\sqrt{r^*}\}$ such that $\varepsilon_R M < \frac{\xi}{4}$,

$$|\chi\tilde{v}_x| \leq C_R\sqrt{\tilde{u}(t,x;u_0)} + \frac{\xi}{4}, \quad t\geq 1,\ x\in\mathbb{R},$$

and

$$-c^* + \epsilon = -2\sqrt{r^*} + \epsilon \leq c + \tilde{\varepsilon} - \xi.$$

Define

$$B(t,x) = \frac{\chi\tilde{v}_x}{\max\{1, |\chi\tilde{v}_x|\xi^{-1}\}}, \quad t\geq 1,\ x\in\mathbb{R}.$$

From this point, the remaining part of the proof is completed in four steps.

Step 1. *In this step we construct some sub-solution for* (2.29).

First, choose $0 < \xi_1 \ll 1$ satisfying

$$C_R\sqrt{\xi_1} + \xi_1 < \frac{\xi}{4}.$$

Next, let l be chosen as in (2.13). Let $T > 1$ be such that $r(x + \tilde{\varepsilon}t) \geq r^* - \varepsilon_R M$ for $x \geq -l$ and $t \geq T$. Let $\underline{u}_1(\cdot) \in C^b_{\rm unif}([-l, \infty)) \setminus \{0\}$ be such that

$$\underline{u}_1(-l) = 0, \quad \underline{u}_1^{'}(x) > 0, \quad \text{and} \quad \underline{u}_1(x) < \tilde{u}(T, x; u_0), \quad \forall\, x \geq -l$$

Choose $-2\sqrt{r^*} + \epsilon \leq \bar{c} \leq \min\{2\sqrt{r^*} - \epsilon, (c + \tilde{\varepsilon}) - \xi\}$. Let $\underline{u}(t, x)$ be the solution of

$$\begin{cases} \underline{u}_t = \underline{u}_{xx} + \bar{c}\underline{u}_x + \underline{u}(r^* - 2\varepsilon_R M - C_{R,p}\underline{u}^{\frac{1}{p}} - (\frac{r^*}{\xi_1} + (b - \chi\mu))\underline{u}), \quad t > T,\ x > -l \\ \underline{u}(t, -l) = 0 \\ \underline{u}(T, x) = \frac{\xi_1}{M + \xi_1}\underline{u}_1(x). \end{cases} \tag{2.30}$$

Note that $\underline{u}(t, x) \equiv \xi_1$ is a super-solution of (2.30) and $\|\underline{u}(T, \cdot)\|_\infty < \xi_1$. Thus, by the comparison principle for parabolic equations that

$$\underline{u}(t, x) < \xi_1, \quad \forall\, t \geq T,\ x \geq -l.$$

Since $\underline{u}_1^{'}(x) > 0$, we have $\underline{u}_x(t, x) > 0$ for any $t \geq T$, $x \geq -l$. Note that $|B(t, x)| < \xi$ for all $t \geq 1$, $x \in \mathbb{R}$. Thus $\underline{u}(t, x)$ satisfies

$$\begin{aligned} \underline{u}_t =& \underline{u}_{xx} + \bar{c}\underline{u}_x + \underline{u}(r^* - 2\varepsilon_R M - C_{R,p}\underline{u}^{\frac{1}{p}} - (\frac{r^*}{\xi_1} + (b - \chi\mu))\underline{u}) \\ \leq& \underline{u}_{xx} + (c + \tilde{\varepsilon} - B(t, x))\underline{u}_x + \underline{u}(r^* - 2\varepsilon_R M - C_{R,p}\underline{u}^{\frac{1}{p}} - (b - \chi\mu)\underline{u}) \quad \forall\, t > T,\ x > -l. \end{aligned}$$

Step 2. *In this step, we show that*

$$\liminf_{t\to\infty} \inf_{x \geq -l+\delta} \underline{u}(t, x) > 0, \quad \forall\, \delta > 0. \tag{2.31}$$

Choose $\bar{b} > \frac{r^*}{\xi_1} + b$ and also $\bar{b} \gg 1$ such that $u(T{+}1, x; \bar{c}, \bar{b}) < \underline{u}(T{+}1, x)$ for $-l \leq x \leq l$, where $u(t, x; \bar{c}, \bar{b})$ is the unique positive entire solution of (2.15) with $A(t, x) = 0$. Fix such $\bar{b}$.

It follows from the comparison principle for parabolic equations that

$$u(t,x;\bar{c},\bar{b}) < \underline{u}(t,x), \quad t > T{+}1,\ -l < x < l.$$

Repeating the same procedure, by induction, we get

$$u(t,x-kl;\bar{c},\bar{b}) < \underline{u}(t,x), \quad t > T{+}1,\ (k-1)l < x < (k+1)l,\ k = 0,1,2,\cdots$$

There exists $\delta_0 > 0$, such that

$$\inf_{(k-1)l+\delta \le x \le (k+1)l-\delta, t\in\mathbb{R}} u(t,x-kl;\bar{c},\bar{b}) > \delta_0,$$

for any $0 < \delta < l,\ k = 0,1,2,\cdots$

Therefore, we have

$$\liminf_{t\to\infty} \inf_{x\ge -l+\delta} \underline{u}(t,x) > 0, \quad \forall\ \delta > 0.$$

Step 3. *In this step we show that*

$$\underline{u}(t,x) \le \tilde{u}(t,x;u_0), \quad \forall\ x \ge -l,\ t \ge T. \tag{2.32}$$

First, note that

$$\underline{u}(t,-l) = 0 < \tilde{u}(t,-l;u_0) \quad \forall\ t \ge T$$

and

$$\underline{u}(T,x) < \tilde{u}(T,x;u_0) \quad \forall\, x \ge -l.$$

Note also that

$$\tilde{u}_\infty(T) := \liminf_{x\to\infty} \tilde{u}(T,x;u_0) > \underline{u}_\infty(T) := \lim_{x\to\infty} \underline{u}(T,x).$$

For given $t \geq T$, let

$$\tilde{u}_\infty(t) = \liminf_{x\to\infty} \tilde{u}(t, x; u_0), \quad \underline{u}_\infty(t) = \lim_{x\to\infty} \underline{u}(t, x).$$

We claim that

$$\tilde{u}_\infty(t) > \underline{u}_\infty(t) \quad \forall\, t > T.$$

In fact, for any given $t_0 > T$, there is $x_n \to \infty$ such that

$$\tilde{u}(t_0, x_n; u_0) \to \tilde{u}_\infty(t_0), \quad \underline{u}(t_0, x_n) \to \underline{u}_\infty(t_0)$$

as $n \to \infty$. Without loss of generality, we may assume that

$$\tilde{u}(t, x + x_n; u_0) \to \tilde{u}^*(t, x), \quad \underline{u}(t, x + x_n) \to \underline{u}^*(t)$$

as $n \to \infty$ locally uniformly in $(t, x) \in (T, \infty) \times \mathbb{R}$. By $\tilde{u}_\infty(T) > \underline{u}_\infty(T)$ and the comparison principle for parabolic equations, we have

$$\tilde{u}^*(t, x) > \underline{u}^*(t) \quad \forall\, t > T, \quad x \in \mathbb{R}.$$

In particular, we have

$$\tilde{u}_\infty(t_0) = \tilde{u}^*(t_0, 0) > \underline{u}_\infty(t_0).$$

Hence the claim holds true.

Next, assume that there are $t > T$ and $x > -l$ such that $\tilde{u}(t, x; u_0) < \underline{u}(t, x)$. Then there is $t_{\inf} > T$ such that

$$\tilde{u}(t, x; u_0) > \underline{u}(t, x) \quad \forall\, T \leq t < t_{\inf},\ x \geq -l$$

and

$$\inf_{x\geq -l} \big(\tilde{u}(t_{\inf}, x; u_0) - \underline{u}(t_{\inf}, x)\big) = 0.$$

By the above claim, there is $x_{\inf} \in (-l, \infty)$ such that

$$\tilde{u}(t_{\inf}, x_{\inf}; u_0) = \underline{u}(t_{\inf}, x_{\inf}).$$

Then the similar arguments as those in the proof of (2.24), we have

$$\tilde{u}(t_{\inf}, x_{\inf}; u_0) > \underline{u}(t_{\inf}, x_{\inf}),$$

which is a contradiction. Hence (2.32) holds.

Step 4. *In this step, we prove* (2.28).

By (2.31) and (2.32), we deduce that

$$\liminf_{t\to\infty} \inf_{x\geq -l+\delta} \tilde{u}(t, x; u_0) > 0, \quad \forall\, \delta > 0.$$

Since $u(t, x; u_0) = \tilde{u}(t, x - (c + \tilde{\varepsilon})t)$, we have

$$\liminf_{t\to\infty} \inf_{x\geq -l+(c+\tilde{\varepsilon})t+\delta} u(t, x; u_0) > 0 \quad \forall\, \tilde{\varepsilon} > 0, \quad \forall\, \delta > 0.$$

Hence,

$$\liminf_{t\to\infty} \inf_{x\geq (c+\varepsilon)t} u(t, x; u_0) > 0 \quad \forall\, \varepsilon > 0.$$

Finally, we prove that

$$\lim_{t\to\infty} \sup_{x\geq (c+\varepsilon)t} |u(t, x; u_0) - \frac{r^*}{b}| = 0.$$

It can be proved by similar arguments as those in the proof of (2.26).

(2) Suppose $c < -c^* = -2\sqrt{r^*}$. First, let $\kappa = \sqrt{r^*}$, for any $\bar{c}$ satisfies

$$c < \bar{c} < -c^* = -2\sqrt{r^*} = -c_\kappa.$$

By Lemma 2.3 (ii),

$$\lim_{t\to\infty}\sup_{x\le \bar{c}t} u(t,x;u_0)=0.$$

This implies that for any $\varepsilon>0$,

$$\lim_{t\to\infty}\sup_{x\le(-c^*-\varepsilon)t} u(t,x;u_0)=0.$$

Next, we prove

$$\liminf_{t\to\infty}\inf_{x\ge(-c^*+\varepsilon)t} u(t,x;u_0)>0\quad \forall\,\varepsilon>0. \tag{2.33}$$

It suffices to prove that for any $0<\varepsilon<2\sqrt{r^*}$, (2.33) holds.

Let $0<\varepsilon<2\sqrt{r^*}$ be given , let $u(t,x;u_0)=\tilde{u}(t,x-(-c^*+\varepsilon)t;u_0), v(t,x;u_0)=\tilde{v}(t,x-(-c^*+\varepsilon)t;u_0)$ in (1.1) and set $M=\max\{\|u_0\|_\infty,\frac{r^*}{b-\chi\mu}\}$. By (2.6), it follows that, for any $R\gg 1, p>1$, $(\tilde{u}(t,x;u_0),\tilde{v}(t,x;u_0))$ satisfies

$$\tilde{u}_t\ge\tilde{u}_{xx}+(-c^*+\varepsilon)\tilde{u}_x-\chi\tilde{v}_x\tilde{u}_x+\tilde{u}(r(x+(-c^*+\varepsilon-c)t)-\varepsilon_R M-C_{R,p}\tilde{u}^{\frac{1}{p}}-(b-\chi\mu)\tilde{u}),\ t\ge 1,\ x\in\mathbb{R}.$$

Choose $0<\xi\ll\min\{1,\frac{\varepsilon}{2}\}$. Fix a $\bar{c}$ satisfying $-c^*+\frac{\varepsilon}{2}\le\bar{c}\le -c^*+\varepsilon-\xi$. By the similar arguments as those in the proof of (2.28), it can be proved that

$$\liminf_{t\to\infty}\inf_{x\ge -l+\delta}\tilde{u}(t,x;u_0)>0,\quad\forall\,\delta>0.$$

Since $u(t,x;u_0)=\tilde{u}(t,x-(-c^*+\varepsilon)t)$, we have

$$\liminf_{t\to\infty}\inf_{x\ge -l+(-c^*+\varepsilon)t+\delta} u(t,x;u_0)>0,\quad\forall\,\delta>0.$$

Hence,

$$\liminf_{t\to\infty}\inf_{x\ge(-c^*+\varepsilon)t} u(t,x;u_0)>0.$$

Finally, we prove

$$\lim_{t\to\infty}\sup_{x\ge(-c^*+\varepsilon)t}|u(t,x;u_0)-\frac{r^*}{b}|=0.$$

It can be proved using similar arguments as those in the proof of (2.26). □

2.3.2 Case 2

In this subsection, we study persistence and extinction of solutions of (1.1) with $r(x)$ being as in **Case 2**, and prove Theorem 2.3. Throughout this subsection, we assume that **(H1)** holds and $r(x)$ is as in **Case 2**.

We first prove some lemmas.

Consider

$$\begin{cases} u_t = u_{xx} + cu_x - A(t,x)u_x + u(r(x) - \varepsilon_R M - C_{R,p}u^{\frac{1}{p}} - (\bar{b} - \chi\mu)u), \ -L < x < L \\ u(t,-L) = u(t,L) = 0. \end{cases} \tag{2.34}$$

Observe that $\varepsilon_R \to 0$ as $R \to \infty$ and

$$\zeta_L(r(\cdot) - 2\varepsilon_R M, c) = \zeta_L(r(\cdot), c) - 2\varepsilon_R M,$$

where $\zeta_L(\cdot, c)$ is defined as in (2.1). Hence

$$\lim_{R\to\infty} \zeta_L(r(\cdot) - 2\varepsilon_R M, c) = \zeta_L(r(\cdot), c).$$

Lemma 2.10. *Suppose that $\zeta_\infty(r(\cdot), c) > 0$. Then there are $L^* > 0$, $R^* > 0$, and $\epsilon^* > 0$ such that for any $L \ge L^*$ and $R \ge R^*$, $\|A(\cdot,\cdot)\|_\infty < \epsilon^*$,* (2.34) *has a unique positive bounded entire solution $u^*(t,x;L,R,A(\cdot,\cdot))$ with*

$$\inf_{t\in\mathbb{R},|x|\le L-\delta} u^*(t,x;L,R,A(\cdot,\cdot)) > 0 \quad \forall\ 0 < \delta < L. \tag{2.35}$$

Proof. First of all, there are $L^* > 0$ and $R^* > 0$ such that

$$\zeta_L(r(\cdot) - 2\varepsilon_R M, c) > 0 \quad \forall\ L \ge L^*,\ R \ge R^*.$$

It then follows from similar arguments as those in Lemma 2.9 that there is $\epsilon^* > 0$ such that for any $A(\cdot,\cdot)$ with $\|A\|_\infty \le \epsilon^*$, (2.34) has a unique positive bounded entire solution $u^*(t,x;L,R,A(\cdot,\cdot))$ satisfying (2.35). □

Let $\phi_L(x)$ be the positive principal eigenfunction of (2.1) corresponding to the principal eigenvalue $\zeta_L(r(\cdot),c)$ with $\phi_L(0) = 1$. By a priori estimates and Harnack's inequality for elliptic equations, there exist $L_n \to \infty$ and $\phi_\infty(x) > 0$ such that

$$\lim_{n\to\infty} \phi_{L_n}(x) = \phi_\infty(x)$$

locally uniformly, and

$$(\phi_\infty)_{xx} + c(\phi_\infty)_x + r(x)\phi_\infty = \zeta_\infty(r(\cdot),c)\phi_\infty, \quad x \in \mathbb{R}. \tag{2.36}$$

Lemma 2.11.

$$|\frac{d}{dx}\phi_\infty(x)| \le \frac{\sqrt{8r^* + c^2} + |c|}{2}\phi_\infty(x) \quad \forall\, x \in \mathbb{R}.$$

Proof. It follows from [53, Lemma 2.1, Lemma 2.2]. □

We now prove Theorem 2.3.

Proof of Theorem 2.3. (1) If $c > c^*$, using the same arguments as those in the proof of Theorem 2.1 (1), we can prove $\lim_{t\to\infty} u(t,x;u_0) = 0$ uniformly for $x \in \mathbb{R}$.

If $c < -c^*$, let $\kappa = \sqrt{r^*}$, for any $\bar{c}$ satisfies $c < \bar{c} < -c^* = -2\sqrt{r^*} = -c_\kappa$. By Lemma 2.3 (ii), $\lim_{t\to\infty} \sup_{x\le \bar{c}t} u(t,x;u_0) = 0$. We claim that $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x;u_0) = 0$.

For otherwise, there are $\delta_0 > 0$, $t_n \to \infty$ and $x_n \in (\bar{c}t_n, \infty)$ such that

$$u(t_n, x_n; u_0) \ge \delta_0 \quad \forall\, n \ge 1.$$

Let $u_n(t,x) = u(t+t_n, x+x_n; u_0)$ and $v_n(t,x) = v(t+t_n, x+x_n; u_0)$. Note that $x_n - ct_n \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that there is $(u^*(t,x), v^*(t,x))$ such that

$$\lim_{n\to\infty}(u_n(t,x),v_n(t,x))=(u^*(t,x),v^*(t,x))$$

locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$, and $(u^*(t,x),v^*(t,x))$ satisfies

$$\begin{cases} u^*_t=\Delta u^*-\nabla\cdot(\chi u^*\nabla v^*)+u^*(r(\infty)-bu^*), \quad t\in\mathbb{R},\ x\in\mathbb{R}\\ 0=\Delta v^*-\lambda v^*+\mu u^*, \quad t\in\mathbb{R},\ x\in\mathbb{R}. \end{cases} \tag{2.37}$$

By $r(\infty)<0$, it can be proved that $u^*(t,x)\equiv 0$, which contradicts to $u^*(0,0)\geq\delta_0$. Therefore, $\lim_{t\to\infty}\sup_{x\in\mathbb{R}}u(t,x;u_0)=0$. □

Proof of Theorem 2.3. (2) First, fix u_0 with nonempty compact support. Let M be such that $M\geq\max\{\frac{r^*}{b-\chi\mu},\sup_{x\in\mathbb{R}}u_0(x)\}$. Note that

$$u(t,x;u_0)\leq M \quad \forall\, t\geq 0,\ x\in\mathbb{R}.$$

Next, let $\phi_\infty(x)$ be as in Lemma 2.11. Without loss of generality, we may suppose that $u_0(x)\leq\phi_\infty(x)$ for any $x\in\mathbb{R}$.

Let $u_\infty(t,x)=e^{\zeta_\infty(r(\cdot),c)t}\phi_\infty(x)$. Then $u_\infty(t,x)$ satisfies the following parabolic equation

$$(u_\infty)_t\geq(u_\infty)_{xx}+c(u_\infty)_x+(r(x)-(b-\chi\mu)u_\infty)u_\infty,\ \forall\, t>0, x\in\mathbb{R}. \tag{2.38}$$

Hence, if $\lambda\geq\lambda^*:=\frac{(\sqrt{8r^*+c^2}+|c|)^2}{4}$, then by Lemma 2.6 and Lemma 2.11, we get

$$\begin{aligned}
&-\chi(u_\infty)_x\Psi_x(x;u(t,\cdot;u_0))-\chi\lambda\Psi(x;u(t,\cdot;u_0))u_\infty\\
&\leq\chi\sqrt{\lambda}\Psi(|(u_\infty)_x|-\sqrt{\lambda}u_\infty)\\
&=\chi\sqrt{\lambda}\Psi e^{\zeta_\infty(r(\cdot),c)t}\big(|(\phi_\infty)_x|-\sqrt{\lambda}\phi_\infty\big)\\
&\leq\chi\sqrt{\lambda}\Psi e^{\zeta_\infty(r(\cdot),c)t}\big(\sqrt{\lambda^*}-\sqrt{\lambda}\big)\phi_\infty\\
&\leq 0,\ \ \forall\, x\in\mathbb{R}.
\end{aligned} \tag{2.39}$$

By (2.38), we have

$$
\begin{aligned}
(u_\infty)_t \geq &(u_\infty)_{xx} + c(u_\infty)_x - \chi(u_\infty)_x\Psi_x(x+ct;u(t,\cdot;u_0)) \\
&+ (r(x) - \chi\lambda\Psi(x+ct;u(t,\cdot;u_0)) - (b-\chi\mu)u_\infty)u_\infty, \ \forall x \in \mathbb{R}.
\end{aligned}
$$

Let $\tilde{u}(t,x;u_0) = u(t,x+ct;u_0)$, then $\tilde{u}(t,x;u_0)$ satisfies

$$
\tilde{u}_t = \tilde{u}_{xx} + c\tilde{u}_x - \chi\tilde{u}_x\Psi_x(x+ct;u(t,\cdot;u_0)) + (r(x) - \chi\lambda\Psi(x+ct;u(t,\cdot;u_0)) - (b-\chi\mu)\tilde{u})\tilde{u}, \ \forall\, t>0, x\in\mathbb{R}.
$$

By the comparison principle for parabolic equations, we have

$$
\tilde{u}(t,x;u_0) \leq u_\infty(t,x) = e^{\zeta_\infty(r(\cdot),c)t}\phi_\infty(x), \quad \forall\ t \geq 0,\ x\in\mathbb{R}.
$$

Since $\phi_\infty(x)$ is bounded on any compact set and $\zeta_\infty(r(\cdot),c) < 0$, we then have $\lim_{t\to\infty}\tilde{u}(t,x;u_0) = 0$ locally uniformly in $x\in\mathbb{R}$.

We now prove that $\lim_{t\to\infty}\tilde{u}(t,x;u_0) = 0$ uniformly in $x\in\mathbb{R}$. Assume by contradiction that this is not true. Then there is $\epsilon_0 > 0$, $t_n\to\infty$, and $|x_n|\to\infty$ such that

$$
\tilde{u}(t_n,x_n;u_0) \geq \epsilon_0.
$$

Without loss of generality, we assume that $x_n \to \infty$, and $\lim_{n\to\infty}\tilde{u}(t+t_n,x+x_n;u_0) = U^*(t,x)$, $\lim_{n\to\infty}\Psi(x+x_n+c(t+t_n);u(t+t_n,\cdot;u_0)) = \Psi^*(t,x)$ locally uniformly. Then

$$
U^*_t = U^*_{xx} + cU^*_x - \chi\Psi^*_xU^*_x + U^*(r(\infty) - \chi\lambda\Psi^* - (b-\chi\mu)U^*), \quad t\in\mathbb{R},\ x\in\mathbb{R}.
$$

Note that $U^*(t,x)$ is bounded and nonnegative and $r(\infty) < 0$. We must have

$$
U^*(t,x) \equiv 0,
$$

which contradicts to $U^*(0,0) \geq \epsilon_0$. Therefore, $\lim_{t\to\infty}\tilde{u}(t,x;u_0) = 0$ uniformly in $x\in\mathbb{R}$, which implies that $\lim_{t\to\infty}u(t,x;u_0) = 0$ uniformly in $x\in\mathbb{R}$. □

Proof of Theorem 2.3. (3) Suppose $|c| < c^*$. We first prove

$$\lim_{t\to\infty} \sup_{|x-ct|\ge c' t} u(t,x;u_0) = 0 \quad \forall\, c^{'} > 0.$$

Assume that the result does not hold. Then there are constants $\delta_0 > 0$, and a sequence $\{(t_n, x_n)\}_{n\in\mathbb{N}}$, $t_n \to \infty$, $|x_n - ct_n| \ge c^{'} t_n$ such that $u(t_n, x_n; u_0) \ge \delta_0$ for any $n \ge 1$.

Let $u_n(t,x) = u(t+t_n, x+x_n; u_0)$ and $v_n(t,x) = v(t+t_n, x+x_n; u_0)$. Note that $|x_n - ct_n| \to \infty$ as $n \to \infty$. Thus, either $x_n - ct_n \to \infty$ as $n \to \infty$ or $x_n - ct_n \to -\infty$ as $n \to \infty$.

In the case $x_n - ct_n \to \infty$ as $n \to \infty$. Following similar arguments as those in the proof of Theorem 2.3 (1), we can get a contradiction.

In the case $x_n - ct_n \to -\infty$ as $n \to \infty$. Also using similar arguments as those in the proof of Theorem 2.3 (1) and the fact $r(-\infty) < 0$, we also can get a contradiction.

Next, we prove that, if $\zeta_\infty(r(\cdot), c) > 0$, then

$$\liminf_{t\to\infty} \inf_{|x-ct|\le L} u(t,x;u_0) > 0 \quad \forall\, L > 0.$$

To this end, let $u(t,x;u_0) = \tilde{u}(t, x-ct; u_0), v(t,x;u_0) = \tilde{v}(t, x-ct; u_0)$ in (1.1) and set $M = \max\{\|u_0\|_\infty, \frac{r^*}{b-\chi\mu}\}$. By (2.6), it follows that, for any $R \gg 1, p > 1$, $(\tilde{u}(t,x;u_0), \tilde{v}(t,x;u_0))$ satisfies

$$\tilde{u}_t \ge \tilde{u}_{xx} + c\tilde{u}_x - \chi\tilde{v}_x\tilde{u}_x + \tilde{u}(r(x) - \varepsilon_R M - C_{R,p}\tilde{u}^{\frac{1}{p}} - (b-\chi\mu)\tilde{u}), \quad t \ge 1,\ x \in \mathbb{R}. \quad (2.40)$$

Let $p = 2$ and ϵ^* be as in Lemma 2.10. By the similar arguments as those in the proof of (2.23), it can be proved that

$$u^*(t,x;L,R,A(\cdot,\cdot)) \le \tilde{u}(t,x;u_0) \quad \forall\, t \ge 1,\ -L \le x \le L. \quad (2.41)$$

By Lemma 2.10, there are $L^* > 0$, and $R^* > 0$ such that for any $L \geq L^*$, $R \geq R^*$, and $\|A(\cdot,\cdot)\|_\infty < \epsilon^*$,

$$\inf_{t\in\mathbb{R},|x|\leq L-\delta} u^*(t,x;L,R,A(\cdot,\cdot)) > 0 \quad \forall\ 0<\delta<L.$$

This together with (2.41) implies that

$$\liminf_{t\to\infty} \inf_{|x|\leq L-\delta} \tilde{u}(t,x;u_0) > 0, \quad \forall\ 0<\delta<L.$$

Since $u(t,x;u_0) = \tilde{u}(t,x-ct;u_0)$, we then have

$$\liminf_{t\to\infty} \inf_{|x-ct|\leq L-\delta} u(t,x;u_0) > 0, \quad \forall\ 0<\delta<L,$$

which implies that

$$\liminf_{t\to\infty} \inf_{|x-ct|\leq L} u(t,x;u_0) > 0 \quad \forall\ L>0.$$

□

2.4 Forced wave solutions

In this section, we study the existence of forced wave solutions of (1.1) with $r(x)$ being as in **Case 1** or **Case 2**.

2.4.1 Case 1

In this subsection, we study the existence of forced wave solutions of (1.1) with $r(x)$ being as in **Case 1** and prove Theorem 2.4. Throughout this subsection, we assume that $r(x)$ is as in **Case 1**.

We first present some lemmas. Suppose that $b > \chi\mu$. Fix r_1 with $r(-\infty) < r_1 < 0$. Let x_1 be given satisfying that $r(x) \leq r_1$ for any $x \leq x_1$. Let θ_1 be the positive root of the equation

$\theta^2 + c\theta + r_1 = 0$. Define

$$U_1^+(x) = \min\{\frac{r^*}{b-\chi\mu}, \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\}, \tag{2.42}$$

and consider the set

$$\mathcal{E}_1^+ = \{u \in C^b_{\rm unif}(\mathbb{R}) \ : \ 0 \le u(x) \le U_1^+(x), \ \forall \, x \in \mathbb{R}\}. \tag{2.43}$$

For every $u \in \mathcal{E}_1^+$, consider the operator

$$\mathcal{A}_u(U)(x) = U_{xx}(x) + (c - \chi\Psi_x(x;u))U_x(x) + (r(x) - \chi\lambda\Psi(x;u) - (b-\chi\mu)U(x))U(x), \tag{2.44}$$

where $\Psi(x;u)$ is given by (2.8).

Lemma 2.12. *Suppose that* $b \ge \frac{3}{2}\chi\mu$. *For every* $u \in \mathcal{E}_1^+$, *it holds that* $\mathcal{A}_u(\frac{r^*}{b-\chi\mu})(x) \le 0$ *for* $x \in \mathbb{R}$ *and* $\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\theta_1(\cdot - x_1)})(x) \le 0$ *for* $x \in (-\infty, x_1)$.

Proof. Let $u \in \mathcal{E}_1^+$ be given. First, we have

$$\mathcal{A}_u(\frac{r^*}{b-\chi\mu})(x) = \frac{r^*}{b-\chi\mu}(r(x) - \chi\lambda\Psi(x;u) - r^*) \le 0 \quad \forall \, x \in \mathbb{R}.$$

Next, for $x \in (-\infty, x_1)$, we have $r(x) \le r_1$, and hence

$$\begin{aligned}
&\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\theta_1(\cdot - x_1)})(x) \\
&= \theta_1^2\frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)} + (c - \chi\Psi_x(x;u))\theta_1\frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)} \\
&+ \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}(r(x) - \chi\lambda\Psi(x;u) - (b-\chi\mu)\frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}) \\
&= \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\left(\theta_1^2 + c\theta_1 + r(x) - \chi\theta_1\Psi_x(x;u) - \chi\lambda\Psi(x;u) - r^*e^{\theta_1(x-x_1)}\right) \\
&\le \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\left(\theta_1^2 + c\theta_1 + r_1 - \chi\theta_1\Psi_x(x;u) - \chi\lambda\Psi(x;u) - r^*e^{\theta_1(x-x_1)}\right) \\
&= \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\left(-\chi\theta_1\Psi_x(x;u) - \chi\lambda\Psi(x;u) - r^*e^{\theta_1(x-x_1)}\right).
\end{aligned} \tag{2.45}$$

It then follows from Lemma 2.5 and (2.45) that

$$\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\theta_1(\cdot - x_1)})(x) \le \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\left(\frac{\chi\mu}{2}(\theta_1-\sqrt{\lambda})e^{-\sqrt{\lambda}x}\int_{-\infty}^{x} e^{\sqrt{\lambda}y}u(y)dy - r^*e^{\theta_1(x-x_1)}\right).$$

If $\theta_1 \le \sqrt{\lambda}$, we then have

$$\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\theta_1(\cdot - x_1)})(x) \le 0 \quad \forall x \in (-\infty, x_1).$$

If $\theta_1 > \sqrt{\lambda}$, we then have

$$\begin{aligned}
&\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\theta_1(\cdot - x_1)})(x)\\
&\le \frac{r^*}{b-\chi\mu}e^{\theta_1(x-x_1)}\left(\frac{\chi\mu r^*}{2(b-\chi\mu)}(\theta_1-\sqrt{\lambda})e^{-\sqrt{\lambda}x}\int_{-\infty}^{x} e^{\sqrt{\lambda}y}e^{\theta_1(y-x_1)}dy - r^*e^{\theta_1(x-x_1)}\right)\\
&= \frac{r^{*2}}{b-\chi\mu}e^{2\theta_1(x-x_1)}\left(\frac{\chi\mu(\theta_1-\sqrt{\lambda})}{2(b-\chi\mu)(\theta_1+\sqrt{\lambda})}-1\right)\\
&\le 0 \quad \forall x < x_1.
\end{aligned}$$

The lemma thus follows. □

Suppose $b > 2\chi\mu$. For any $0 < \varepsilon \ll 1$, define an ignition nonlinearity by

$$f_\varepsilon(u) = \begin{cases} u\left(r^* - \varepsilon - \frac{\chi\mu r^*}{b-\chi\mu} - (b-\chi\mu)u\right), & \text{if } u \ge 0,\\ 0 & \text{if } -\varepsilon \le u < 0.\end{cases}$$

Consider the equation

$$u_t = u_{xx} + f_\varepsilon(u), \quad x \in \mathbb{R}. \tag{2.46}$$

Equation (2.46) has a decreasing traveling wave solution $\phi_\varepsilon(x-\tilde{c}_\varepsilon t)$ connecting $\frac{(r^*-\varepsilon)(b-\chi\mu)-\chi\mu r^*}{(b-\chi\mu)^2}$ and $-\varepsilon$ with speed $0 < \tilde{c}_\varepsilon < 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$ and $\lim_{\varepsilon\to 0^+}\tilde{c}_\varepsilon = 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$ (see [7]), that is,

$(\phi_\varepsilon, \tilde{c}_\varepsilon)$ satisfies

$$\begin{cases} -\tilde{c}_\varepsilon \phi_\varepsilon^{'} = \phi_\varepsilon^{''} + f_\varepsilon(\phi_\varepsilon), \\ \phi_\varepsilon(-\infty) = \frac{(r^*-\varepsilon)(b-\chi\mu)-\chi\mu r^*}{(b-\chi\mu)^2}, \ \phi_\varepsilon(\infty) = -\varepsilon, \ \phi_\varepsilon^{'} < 0. \end{cases} \tag{2.47}$$

Let $\psi_\varepsilon(x) = \phi_\varepsilon(-x)$ for any $x \in \mathbb{R}$. It then follows from (2.47) that

$$\begin{cases} \tilde{c}_\varepsilon \psi_\varepsilon^{'} = \psi_\varepsilon^{''} + f_\varepsilon(\psi_\varepsilon), \\ \psi_\varepsilon(\infty) = \frac{(r^*-\varepsilon)(b-\chi\mu)-\chi\mu r^*}{(b-\chi\mu)^2}, \ \psi_\varepsilon(-\infty) = -\varepsilon, \ \psi_\varepsilon^{'} > 0. \end{cases} \tag{2.48}$$

Without loss of generality, we can assume that $\psi_\varepsilon(x_0) = 0$, $r(x) \geq r^* - \varepsilon$ if $x > x_0$, and $x_0 > x_1$. This can be realized by some appropriate translation of $\psi_\varepsilon(x)$ if necessary.

Lemma 2.13. *Suppose that* **(H2)** *holds. For every* $u \in \mathcal{E}_1^+$ *and* $0 < \varepsilon \ll 1$, $U_1^-(x) = \max\{\psi_\varepsilon(x), 0\}$ *satisfies that* $\mathcal{A}_u(U_1^-(\cdot))(x) \geq 0$ *for any* $x \neq x_0$. *Moreover,* $U_1^-(x) < U_1^+(x)$ *for all* $x \in \mathbb{R}$.

Proof. Let $\delta = c - \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} + 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$. By **(H2)**, $\delta > 0$. Since $\lim_{\varepsilon\to 0^+} \tilde{c}_\varepsilon = 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$, it then follows that for $0 < \varepsilon \ll 1$, we have $\tilde{c}_\varepsilon > 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}} - \frac{\delta}{2}$.

Fix such ε. For every $u \in \mathcal{E}_1^+$. If $x > x_0$, $U_1^-(x) = \psi_\varepsilon(x) > 0$ and $U_{1x}^-(x) > 0$. By Lemma 2.7 and (2.48), we have

$$\begin{aligned}
\mathcal{A}_u(\psi_\varepsilon(\cdot))(x) &= \psi_\varepsilon^{''} + (c - \chi\Psi_x(x;u))\psi_\varepsilon^{'} + \psi_\varepsilon(r(x) - \chi\lambda\Psi(x;u) - (b-\chi\mu)\psi_\varepsilon) \\
&\geq \psi_\varepsilon^{''} - \tilde{c}_\varepsilon\psi_\varepsilon^{'} + (c - \chi\Psi_x(x;u) + \tilde{c}_\varepsilon)\psi_\varepsilon^{'} + \psi_\varepsilon(r^* - \varepsilon - \chi\lambda\Psi(x;u) - (b-\chi\mu)\psi_\varepsilon) \\
&\geq \psi_\varepsilon^{''} - \tilde{c}_\varepsilon\psi_\varepsilon^{'} + (c - \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} + 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}} - \frac{\delta}{2})\psi_\varepsilon^{'} \\
&\quad + \psi_\varepsilon(r^* - \varepsilon - \frac{\chi\mu r^*}{b-\chi\mu} - (b-\chi\mu)\psi_\varepsilon) \\
&= (c - \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} + 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}} - \frac{\delta}{2})\psi_\varepsilon^{'} \\
&\geq 0.
\end{aligned}$$

If $x < x_0$, $U_1^-(x) = 0$. Then $\mathcal{A}_u(U_1^-)(x) = 0$.

Since $x_1 < x_0$, it is clear that $U_1^-(x) < U_1^+(x)$ for all $x \in \mathbb{R}$. The lemma is thus proved. $\square$

Let

$$\mathcal{E}_1 = \{u \in C_{\rm unif}^b(\mathbb{R}) \, : \, U_1^-(x) \le u(x) \le U_1^+(x), \, \forall \, x \in \mathbb{R}\}.$$

For any $u \in \mathcal{E}_1$, let $U(t, x; u)$ be the solution of the following parabolic equation

$$\begin{cases} U_t = \mathcal{A}_u(U), \quad t > 0, \; x \in \mathbb{R} \\ U(0, x; u) = U_1^+(x). \end{cases} \tag{2.49}$$

Lemma 2.14. *Suppose that* **(H2)** *holds. For any* $u \in \mathcal{E}_1$*,* $U_1^*(x; u) = \lim_{t\to\infty} U(t, x; u)$ *exists and satisfies the elliptic equation*

$$0 = U_{xx} + (c - \chi\Psi_x(x; u))U_x + (r(x) - \chi\lambda\Psi(x; u) - (b - \chi\mu)U)U \quad \forall \; x \in \mathbb{R}. \tag{2.50}$$

Moreover, $U_1^*(\cdot; u) \in \mathcal{E}_1$*.*

Proof. First, thanks to Lemma 2.12, it follows from the comparison principle for parabolic equations that

$$U(t_2, x; u) \le U(t_1, x; u) \le U_1^+(x), \quad \forall \; x \in \mathbb{R}, \; 0 < t_1 < t_2, \; u \in \mathcal{E}_1.$$

Thus the function

$$U_1^*(x; u) = \lim_{t\to\infty} U(t, x; u), \quad \forall \; u \in \mathcal{E}_1 \tag{2.51}$$

is well defined. Moreover, by a priori estimates for parabolic equations, it is not difficult to see that $U_1^*(\cdot; u) \in \mathcal{E}_1^+$ and $U_1^*(x; u)$ satisfies (2.50).

Next, it follows from Lemma 2.13 and the comparison principle for parabolic equations that

$$U_1^-(x) \le U(t, x; u), \qquad \forall \; x \in \mathbb{R}, \; t > 0, \; u \in \mathcal{E}_1. \tag{2.52}$$

Hence,

$$U_1^-(x) \le U_1^*(x;u), \quad \forall\, x \in \mathbb{R},\ \forall\, u \in \mathcal{E}_1. \tag{2.53}$$

Therefore, $U_1^*(\cdot;u) \in \mathcal{E}_1$. The lemma is thus proved. $\square$

Lemma 2.15. *Suppose that* **(H2)** *holds. For any* $u \in \mathcal{E}_1$*, suppose that* $U_{1*}(x;u)$ *is also a solution of* (2.50) *in* $\mathcal{E}_1$*. Then*

$$\lim_{x\to\infty} \frac{U_{1*}(x;u)}{U_1^*(x;u)} = 1.$$

Proof. First of all, by Lemma 2.7, $\sup_{x\in\mathbb{R}} |\Psi(x;u)| < \infty$ and $\sup_{x\in\mathbb{R}} |\frac{\partial}{\partial x}\Psi(x;u)| < \infty$. By $\Psi_{xx}(x;u) = \lambda\Psi(x;u) - \mu u$, we have $\sup_{x\in\mathbb{R}} |\Psi_{xx}(x;u)| < \infty$. This implies that for any $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$, there is $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} \Psi(x+x_{n_k};u)$ and $\lim_{k\to\infty} \Psi_x(x+x_{n_k};u)$ exist locally uniformly on $\mathbb{R}$.

Next, note that $\frac{U_{1*}(x;u)}{U_1^*(x;u)} \le 1$ for all $x \in \mathbb{R}$. It then suffices to prove $\liminf_{x\to\infty} \frac{U_{1*}(x;u)}{U_1^*(x;u)} \ge 1$. Assume by contraction that $\liminf_{x\to\infty} \frac{U_{1*}(x;u)}{U_1^*(x;u)} < 1$. Then there are $0 < \delta < 1$ and $x_n \to \infty$ such that $\frac{U_{1*}(x_n;u)}{U_1^*(x_n;u)} \le 1-\delta$ for $n = 1, 2, \cdots$. Let

$$U_{n,1*}(x;u) = U_{1*}(x+x_n;u), \quad U_{n,1}^*(x;u) = U_1^*(x+x_n;u), \quad \Psi_n(x;u) = \Psi(x+x_n;u).$$

Without loss of generality, we assume that there are $\underline{U}_*(x;u)$, $\overline{U}^*(x;u)$, and $\Psi^*(x;u)$ such that

$$\lim_{n\to\infty} U_{n,1*}(x;u) = \underline{U}_*(x;u), \quad \lim_{n\to\infty} U_{n,1}^*(x;u) = \overline{U}^*(x;u), \text{ and } \lim_{n\to\infty} \Psi_n(x;u) = \Psi^*(x;u)$$

locally uniformly on $\mathbb{R}$. This implies that both $\underline{U}_*(x;u)$ and $\overline{U}^*(x;u)$ are solutions of

$$0 = U_{xx} + (c - \chi\Psi_x^*(x;u))U_x + (r^* - \chi\lambda\Psi^*(x;u) - (b-\chi\mu)U)U.$$

We now claim that $\underline{U}_*(x;u) \equiv \overline{U}^*(x;u)$. Indeed, note that

$$0 < \inf_{x\in\mathbb{R}} \underline{U}_*(x;u) \le \inf_{x\in\mathbb{R}} \overline{U}^*(x;u), \quad \text{and} \quad \sup_{x\in\mathbb{R}} \underline{U}_*(x;u) \le \sup_{x\in\mathbb{R}} \overline{U}^*(x;u) < \infty.$$

This implies that the following set is not empty,

$$\{\gamma \geq 1 \,|\, \frac{1}{\gamma}\overline{U}^*(x;u) \leq \underline{U}_*(x;u) \leq \gamma\overline{U}^*(x;u) \quad \forall\, x \in \mathbb{R}\}.$$

Hence we can define

$$\rho(\underline{U}_*, \overline{U}^*) = \inf\{\ln\gamma \,|\, \frac{1}{\gamma}\overline{U}^*(x;u) \leq \underline{U}_*(x;u) \leq \gamma\overline{U}^*(x;u) \quad \forall\, x \in \mathbb{R}\}.$$

Note that $\rho(\underline{U}_*, \overline{U}^*)$ is the so called *part metric* between $\underline{U}_*$ and $\overline{U}^*$. Assume that $\rho(\underline{U}_*, \overline{U}^*) > 0$. Then by the arguments of [28, Proposition 3.4], there is $\delta_0 > 0$ such that

$$\rho(\underline{U}_*, \overline{U}^*) \leq \rho(\underline{U}_*, \overline{U}^*) - \delta_0,$$

which is a contradiction. Hence $\rho(\underline{U}_*, \overline{U}^*) = 0$, and then $\underline{U}_*(x;u) = \overline{U}^*(x;u)$ for all $x \in \mathbb{R}$. But, by the assumption, $\underline{U}_*(0;u) \neq \overline{U}^*(0;u)$, which is a contradiction. The lemma thus follows. □

Lemma 2.16. *Suppose that* **(H2)** *holds. For any* $u \in \mathcal{E}_1$*,* $U_1^*(x;u)$ *is the unique positive solution of* (2.50) *in* $\mathcal{E}_1$.

Proof. Suppose that $U_{1*}(x;u)$ is any positive solution of (2.50) in $\mathcal{E}_1$. It suffices to prove that $U_{1*}(x;u) \equiv U_1^*(x;u)$. For any $\epsilon > 0$, let

$$K_\epsilon = \{k \geq 1 \,|\, kU_{1*}(x;u) \geq U_1^*(x;u) - \epsilon \quad \forall\, x \in \mathbb{R}\}.$$

By Lemma 2.15 and the fact $\lim_{x\to-\infty} U_{1*}(x;u) = \lim_{x\to-\infty} U_1^*(x;u) = 0$, $K_\epsilon \neq \emptyset$. Let

$$k_\epsilon = \inf K_\epsilon.$$

Then $k_\epsilon \geq 1$ and

$$k_\epsilon U_{1*}(x;u) \geq U_1^*(x;u) - \epsilon \quad \forall\, x \in \mathbb{R}. \tag{2.54}$$

For any $0 < \epsilon_1 < \epsilon_2$, since $k_{\epsilon_1} U_{1*}(x;u) \geq U_1^*(x;u) - \epsilon_1 > U_1^*(x;u) - \epsilon_2$ for any $x \in \mathbb{R}$, it then follows that $k_{\epsilon_1} \geq k_{\epsilon_2}$. Thus, k_ϵ is nonincreasing in $\epsilon > 0$. If $k_\epsilon = 1$ for any $\epsilon > 0$, clearly, we have that $U_1^*(x;u) \equiv U_{1*}(x;u)$.

Assume that there exists $\epsilon_0 > 0$ such that $k_{\epsilon_0} > 1$. Then $k_\epsilon \geq k_{\epsilon_0} > 1$ for any $0 < \epsilon \leq \epsilon_0$. For any given $0 < \epsilon \leq \epsilon_0$, since $k_\epsilon > 1$, there exists $\delta > 0$, such that $k_\epsilon - \delta > 1$. By lemma 2.15, we have that for such given $\epsilon > 0$, $\frac{U_{1*}(x;u)}{U_1^*(x;u)} \geq 1 - \frac{\epsilon}{U_1^*(x;u)}$ for $x \gg 1$. Hence,

$$(k_\epsilon - \delta) U_{1*}(x;u) \geq U_1^*(x;u) - \epsilon \quad x \gg 1. \tag{2.55}$$

Since $\lim_{x \to -\infty} \frac{U_1^*(x;u) - \epsilon}{U_{1*}(x;u)} = -\infty$, it is then clear that

$$(k_\epsilon - \delta) U_{1*}(x;u) \geq U_1^*(x;u) - \epsilon \quad x \ll -1. \tag{2.56}$$

It then follows from (2.55), (2.56) and the definition of k_ϵ that there is $x_\epsilon \in \mathbb{R}$ such that

$$k_\epsilon U_{1*}(x_\epsilon;u) = U_1^*(x_\epsilon;u) - \epsilon. \tag{2.57}$$

By Lemma 2.15, x_ϵ is bounded above.

We claim that x_ϵ is bounded from below. In fact, at x_ϵ, we have

$$\partial_{xx}(k_\epsilon U_{1*}(x_\epsilon;u) - U_1^*(x_\epsilon;u)) \geq 0 \quad \text{and} \quad \partial_x(k_\epsilon U_{1*}(x_\epsilon;u) - U_1^*(x_\epsilon;u)) = 0.$$

Hence,

$$\begin{aligned} 0 &\geq k_\epsilon U_{1*}(x_\epsilon)(r(x_\epsilon) - \chi\lambda\Psi(x_\epsilon;u) - (b-\chi\mu)U_{1*}(x_\epsilon)) - U_1^*(x_\epsilon)(r(x_\epsilon) - \chi\lambda\Psi(x_\epsilon;u) - (b-\chi\mu)U_1^*(x_\epsilon)) \\ &\geq k_\epsilon U_{1*}(x_\epsilon)(r(x_\epsilon) - \chi\lambda\Psi(x_\epsilon;u) - (b-\chi\mu)U_1^*(x_\epsilon)) - U_1^*(x_\epsilon)(r(x_\epsilon) - \chi\lambda\Psi(x_\epsilon;u) - (b-\chi\mu)U_1^*(x_\epsilon)) \\ &= -\epsilon(r(x_\epsilon) - \chi\lambda\Psi(x_\epsilon;u) - (b-\chi\mu)U_1^*(x_\epsilon)). \end{aligned}$$

This implies that $r(x_\epsilon) \geq 0$ and hence x_ϵ is bounded from below.

Therefore, x_ϵ is bounded both from below and above. By (2.57), $k_\epsilon = \frac{U_1^*(x_\epsilon;u)-\epsilon}{U_{1*}(x_\epsilon;u)}$. Hence k_ϵ is bounded, and there is $\epsilon_n \to 0$ such that $x_{\epsilon_n} \to x^*$ and $k_{\epsilon_n} \to k^*(\geq k_{\epsilon_0} > 1)$ as $n \to \infty$. This together with (2.57) implies that $k^* U_{1*}(x^*;u) = U_1^*(x^*;u)$. By (2.54), $k^* U_{1*}(x;u) \geq U_1^*(x;u)$ for all $x \in \mathbb{R}$. Since $k^* > 1$, by Lemma 2.15, $k^* U_{1*}(x;u) \not\equiv U_1^*(x;u)$. Then by the comparison principle for parabolic equations, we must have $U_1^*(x;u) < k^* U_{1*}(x;u)$ for all $x \in \mathbb{R}$, which is a contraction. Therefore, $k_\epsilon = 1$ for $0 < \epsilon \ll 1$ and then $U_{1*}(x;u) \equiv U_1^*(x;u)$. □

We now prove Theorem 2.4.

Proof of Theorem 2.4. Consider the mapping $U_1^*(\cdot;\cdot) : \mathcal{E}_1 \ni u \mapsto U_1^*(x;u) \in \mathcal{E}_1$ as defined by (2.51). It follows from the arguments of the proof of [50, Theorem 3.1] and the Lemma 2.16 that this function is continuous and compact in the compact open topology. Hence it has a fixed point u^* by the Schauder's fixed point Theorem. Taking $v^*(x) = \Psi(x;u^*)$, we have from (2.50), that $(u(t,x), v(t,x)) = (u^*(x-ct), v^*(x-ct))$ is an entire solution of (1.1). Moreover, since $U_1^-(x) \leq u^*(x) \leq U_1^+(x)$, it follows that $\lim_{x\to-\infty} u^*(x) = 0$.

In the following we show that

$$\lim_{x\to\infty} u^*(x) = \frac{r^*}{b}. \tag{2.58}$$

Suppose on the contrary that this is false. Then, there is a constant $\delta > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to \infty$ and $|u^*(x_n) - \frac{r^*}{b}| \geq \delta$ for any $n \geq 1$. Consider the sequence of functions

$$u^n(t,x) = u(t, x+x_n) \quad \text{and} \quad v^n(t,x) = v(t, x+x_n).$$

By a priori estimate for parabolic equations, without loss of generality, we may assume that there is $(u^{**}(t,x), v^{**}(t,x)) \in C^{1,2}(\mathbb{R}\times\mathbb{R})$ such that $(u^n, v^n)(t,x) \to (u^{**}(t,x), v^{**}(t,x))$ locally uniformly in $C^{1,2}(\mathbb{R}\times\mathbb{R})$ as $n \to \infty$. Furthermore, the function $(u^{**}(t,x), v^{**}(t,x))$ is an entire solution of (2.7). Note that

$$u^{**}(t,x) = \lim_{n\to\infty} u^n(t,x) \geq \lim_{n\to\infty} U_1^-(x + x_n - ct) = U_1^-(\infty) > 0, \quad \forall\, x \in \mathbb{R},\ t \in \mathbb{R}.$$

So $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}} u^{**}(t,x) > 0$. Therefore, since $\chi\mu < \frac{b}{2}$, it follows from Lemma 2.4 (2) that $u^{**}(t,x) = \frac{r^*}{b}$ for every $(t,x) \in \mathbb{R}\times\mathbb{R}$. In particular, $\frac{r^*}{b} = u^{**}(0,0) = \lim_{n\to\infty} u^n(0,0) = \lim_{n\to\infty} u(0,x_n) = \lim_{n\to\infty} u^*(x_n)$, which is a contradiction. Therefore, (2.58) must hold. $\square$

2.4.2 Case 2

In this subsection, we study the existence of forced wave solutions of (1.1) with $r(x)$ being as in **Case 2** and prove Theorem 2.5. Throughout this subsection, we assume that $r(x)$ is as in **Case 2** and **(H3)** holds.

We first present some lemmas.

Fix a $\bar{r}$ with $\max\{r(-\infty), r(\infty)\} < \bar{r} < 0$. Choose $\bar{x}$ such that the inequality $r(x) < \bar{r}$ holds for all $x < \bar{x}$. Let $\bar{\theta}$ be the positive solution of $\bar{\theta}^2 + c\bar{\theta} + \bar{r} = 0$. Choose $\tilde{x}$ such that the inequality $r(x) < \bar{r}$ holds for all $x > \tilde{x}$. Let $\tilde{\theta}$ be the positive solution of $\tilde{\theta}^2 - c\tilde{\theta} + \bar{r} = 0$.

Define

$$U_2^+(x) = \begin{cases} \frac{r^*}{b-\chi\mu} e^{\bar{\theta}(x-\bar{x})} & \text{if } x < \bar{x}, \\ \frac{r^*}{b-\chi\mu} & \text{if } \bar{x} \le x \le \tilde{x}, \\ \frac{r^*}{b-\chi\mu} e^{-\tilde{\theta}(x-\tilde{x})} & \text{if } x > \tilde{x}, \end{cases}$$

and consider the set

$$\mathcal{E}_2^+ = \{u \in C^b_{\rm unif}(\mathbb{R}) \ : \ 0 \le u(x) \le U_2^+(x), \ \forall\, x \in \mathbb{R}\}.$$

For every $u \in \mathcal{E}_2^+$, consider the operator

$$\mathcal{A}_u(U)(x) = U_{xx}(x) + (c - \chi\Psi_x(x;u))U_x(x) + (r(x) - \chi\lambda\Psi(x;u) - (b-\chi\mu)U(x))U(x),$$

where $\Psi(x;u)$ is given by (2.8).

Lemma 2.17. *Suppose that $b \ge \frac{3\chi\mu}{2}$. For every $u \in \mathcal{E}_2^+$, it holds that $\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{\bar{\theta}(\cdot-\bar{x})})(x) \le 0$ for $x \in (-\infty,\bar{x})$, $\mathcal{A}_u(\frac{r^*}{b-\chi\mu})(x) \le 0$ for $x \in \mathbb{R}$ and $\mathcal{A}_u(\frac{r^*}{b-\chi\mu}e^{-\tilde{\theta}(\cdot-\tilde{x})})(x) \le 0$ for $x \in (\tilde{x},\infty)$.*

Proof. It can be proved by the similar arguments as those used in the proof of Lemma 2.12. $\square$

Consider

$$\begin{cases} u_t = u_{xx} + cu_x - A(t,x)u_x + u(r(x) - B(t,x) - (\bar{b} - \chi\mu)u), \quad -L < x < L \\ u(t,-L) = u(t,L) = 0, \end{cases} \tag{2.59}$$

where both $A(t,x)$ and $B(t,x)$ are globally Hölder continuous in $t \in \mathbb{R}$ and $x \in [-L, L]$ with Hölder exponent $0 < \alpha < 1$ and $\|A(\cdot,\cdot)\|_\infty < \infty$, $\|B(\cdot,\cdot)\|_\infty < \infty$.

Lemma 2.18. *Suppose that* **(H3)** *holds. Then there are $L^* > 0$ and $\eta = \eta(r(\cdot), c) > 0$ such that for any $L \geq L^*$, any $A(\cdot,\cdot)$, $B(\cdot,\cdot)$ with $\|A(\cdot,\cdot)\|_\infty < \eta$, $\|B(\cdot,\cdot)\|_\infty < \eta$, and any $\bar{b} > \chi\mu$,* (2.59) *has a unique positive bounded entire solution $u^*(t,x;\bar{b},A(\cdot,\cdot),B(\cdot,\cdot))$ with*

$$\inf_{t\in\mathbb{R}, -L+\delta\leq x\leq L-\delta, \|A(\cdot,\cdot)\|_\infty<\eta, \|B(\cdot,\cdot)\|_\infty<\eta} u^*(t,x;\bar{b},A(\cdot,\cdot),B(\cdot,\cdot)) > 0 \quad \forall\, 0 < \delta < L.$$

Proof. It can be proved by the similar arguments as those used in the proof of Lemma 2.9. □

Let $L = L^*$ and η be fixed. For every $u \in \mathcal{E}_2^+$, let $A(t,x) = \chi\Psi_x(x;u)$, $B(t,x) = \chi\lambda\Psi(x;u)$. By Lemma 2.7, both $\Psi_x(x;u)$ and $\Psi_{xx}(x;u)$ are bounded for any $u \in \mathcal{E}_2^+$, we then have both $A(t,x)$ and $B(t,x)$ are globally Hölder continuous in $x \in [-L, L]$.

In the following, we assume that

$$0 < \chi < \chi_0 = \chi_0(r(\cdot), c) := \min\{\frac{2\sqrt{\lambda}\eta b}{\mu r^* + 2\sqrt{\lambda}\eta\mu}, \frac{b\eta}{\mu r^* + \mu\eta}\}. \tag{2.60}$$

Then by Lemma 2.7, we have

$$\|A(\cdot,\cdot)\|_\infty \leq \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} < \eta \quad \text{and} \quad \|B(\cdot,\cdot)\|_\infty \leq \frac{\chi\mu r^*}{b-\chi\mu} < \eta.$$

It then follows from Lemma 2.18 that (2.59) with $A(t,x) = \chi\Psi_x(x;u)$ and $B(t,x) = \chi\lambda\Psi(x;u)$ has a unique positive bounded entire solution $u^*(t,x;\bar{b},u) := u^*(t,x;\bar{b},\chi\Psi_x(\cdot;u),\chi\lambda\Psi(\cdot;u))$ with

$$\inf_{t\in\mathbb{R}, -L+\delta\leq x\leq L-\delta} u^*(t,x;\bar{b},u) > 0 \quad \forall\, 0 < \delta < L.$$

Note that, by the comparison principle for parabolic equations,

$$u^*(t,x;\bar{b},u) \le \frac{r^*}{\bar{b}-\chi\mu} \quad \forall\, t \in \mathbb{R},\ -L \le x \le L.$$

Fix $\bar{b} \gg b$ such that $u^*(t,x;\bar{b},u) < U_2^+(x)$ for any $-L \le x \le L$, any $t \in \mathbb{R}$, any $u \in \mathcal{E}_2^+$. By the proof of Lemma 2.9, we have that

$$\inf_{t\in\mathbb{R},-L+\delta\le x\le L-\delta,u\in\mathcal{E}_2^+} u^*(t,x;\bar{b},u) > 0 \quad \forall\ 0<\delta<L. \tag{2.61}$$

Define

$$U_2^-(x) = \begin{cases} \inf_{t\in\mathbb{R},u\in\mathcal{E}_2^+} u^*(t,x;\bar{b},u) & \text{if } \ -L<x<L, \\ 0 & \text{if } x \ge L,\ x \le -L \end{cases}$$

Then, $U_2^-(x) \not\equiv 0$ and $\inf_{-L+\delta\le x\le L-\delta} U_2^-(x) > 0$, and $U_2^-(x) < U_2^+(x)$ for any $x \in \mathbb{R}$.

Lemma 2.19. *For any $u \in \mathcal{E}_2^+$,*

$$U_2^-(x) < U(t,x;u) \quad \forall\ x\in\mathbb{R},\ t>0,$$

where $U(t,x;u)$ is the solution of the following parabolic equation

$$\begin{cases} U_t = \mathcal{A}_u(U), & t>0,\ x\in\mathbb{R} \\ U(0,x;u) = U_2^+(x),\ x\in\mathbb{R}. \end{cases} \tag{2.62}$$

Proof. Observe that

$$u^*_{xx} + (c-\chi\Psi_x(x;u))u^*_x + (r(x)-\chi\lambda\Psi(x;u)-(b-\chi\mu)u^*)u^* - u^*_t = (\bar{b}-b)u^{*2} > 0$$

for any $-L<x<L$. Hence, by the comparison principle for parabolic equations, we have

$$u^*(t,x;\bar{b},u) < U(t,x;u), \qquad \forall\ -L<x<L,\ t>0,\ u\in\mathcal{E}_2^+.$$

This implies that

$$\inf_{t\in\mathbb{R},u\in\mathcal{E}_2^+} u^*(t,x;\bar{b},u) < U(t,x;u), \quad \forall\ -L<x<L,\ t>0,\ u\in\mathcal{E}_2^+.$$

The lemma then follows. □

Note that, by Lemma 2.17 and the comparison principle for parabolic equations,

$$U(t_2,x;u) \le U(t_1,x;u) \le U_2^+(x), \quad \forall\ x\in\mathbb{R},\ 0<t_1<t_2,\ u\in\mathcal{E}_2^+.$$

Thus the function

$$U_2^*(x;u) = \lim_{t\to\infty} U(t,x;u), \quad \forall\, u\in\mathcal{E}_2^+ \tag{2.63}$$

is well defined, and

$$U_2^-(x) \le U_2^*(x;u) \le U_2^+(x), \quad \forall\ x\in\mathbb{R},\ u\in\mathcal{E}_2^+. \tag{2.64}$$

Let

$$\mathcal{E}_2 = \{u\in C^b_{\rm unif}(\mathbb{R})\ :\ U_2^-(x)\le u(x)\le U_2^+(x),\ \forall\, x\in\mathbb{R}\}.$$

For any $u\in\mathcal{E}_2$, it follows from (2.64) that $U_2^*(\cdot;u)\in\mathcal{E}_2$. Moreover, by a priori estimates for parabolic equation, we have that $U_2^*(x;u)$ satisfies

$$0 = U_{xx} + (c-\chi\Psi_x(x;u))U_x + (r(x)-\chi\lambda\Psi(x;u)-(b-\chi\mu)U)U \quad \forall\ x\in\mathbb{R}. \tag{2.65}$$

Since $U_2^-(x)\ge 0$ for any $x\in\mathbb{R}$ and $U_2^-(x)\not\equiv 0$ for any $x\in\mathbb{R}$, it follows from the comparison principle for parabolic equations that

$$U_2^*(x;u) > 0 \quad \forall\ x\in\mathbb{R},\ u\in\mathcal{E}_2^+. \tag{2.66}$$

Lemma 2.20. *For any given $u\in\mathcal{E}_2$, (2.65) has a unique solution $U_2^*(\cdot;u)\in\mathcal{E}_2$.*

Proof. Let $U_1(x;u), U_2(x;u)$ be two solutions of (2.65) in $\mathcal{E}_2$. Note that $U_i(x;u) > 0$ for $x \in \mathbb{R}$ and every $i = 1, 2$. For any $\epsilon > 0$, let

$$K_\epsilon = \{k \geq 1 \,|\, kU_2(x;u) \geq U_1(x;u) - \epsilon \quad \forall\, x \in \mathbb{R}\}.$$

K_ϵ is not empty because $\lim_{x\to\pm\infty} \frac{U_1(x;u)-\epsilon}{U_2(x;u)} = -\infty$. Let $k_\epsilon = \inf K_\epsilon$. Following the similar arguments as those used in the proof of Lemma 2.16, we have $k_\epsilon = 1$ for $0 < \epsilon \ll 1$. Therefore, $U_2(x;u) \geq U_1(x,u)$ for any $x \in \mathbb{R}$. Similarly, we have $U_1(x;u) \geq U_2(x,u)$ for any $x \in \mathbb{R}$. The lemma thus follows. □

We now prove Theorem 2.5.

Proof of Theorem 2.5. Consider the mapping $U_2^*(\cdot;\cdot) : \mathcal{E}_2 \ni u \mapsto U_2^*(x;u) \in \mathcal{E}_2$ as defined by (2.63). It follows from the arguments of [50, Theorem 3.1] and Lemma 2.20 that this function is continuous and compact in the compact open topology. Hence it has a fixed point u^* by the Schauder's fixed point Theorem. Taking $v^*(x) = \Psi(x;u^*)$, we have from (2.65), that $(u(t,x), v(t,x)) = (u^*(x-ct), v^*(x-ct))$ is an entire solution of (1.1). Moreover, by (2.66), $u^*(x) > 0$ for all $x \in \mathbb{R}$. Since $U_2^-(x) \leq u^*(x) \leq U_2^+(x)$, it follows that $\lim_{x\to\pm\infty} u^*(x) = 0$. The theorem is thus proved. □

2.5 Numerical Simulations

In this section, we present some numerical simulations by the finite difference method with $r(x)$ being as in **Case 1** or **Case 2**. All the numerical simulations were conducted using programming software MATLAB. It should be pointed out that the authors in [72] provided some numerical study for the vanishing and spreading dynamics of chemotaxis systems with logistic source and a free boundary by the finite difference method.

2.5.1 Case 1

In this subsection, we present some numerical simulations for the existence of forced wave solutions with $r(x)$ being as in **Case 1**. To this end, we use the finite difference method to numerically compute the solution of

$$\begin{cases} u_t = u_{xx} + cu_x - (\chi u v_x)_x + u(r(x) - bu), \quad -L < x < L \\ 0 = v_{xx} - \lambda v + \mu u, \quad -L < x < L \\ u(0,x) = u_0(x), \quad -L \le x \le L \\ u(t,-L) = v(t,-L) = 0 \\ \frac{\partial u}{\partial x}(t,L) = \frac{\partial v}{\partial x}(t,L) = 0 \end{cases} \tag{2.67}$$

for reasonable large $L > 1$, where

$$u_0(x) = \begin{cases} 0 & \text{if } x \le -1, \\ \frac{r^*}{2b}x + \frac{r^*}{2b} & \text{if } -1 < x < 1, \\ \frac{r^*}{b} & \text{if } x \ge 1. \end{cases}$$

Let $(u_L(t,x;u_0), v_L(t,x;u_0))$ be the solution of (2.67) with $u_L(0,x;u_0) = u_0(x)$. Observe that if $(u_L(x), v_L(x)) := \lim_{t\to\infty}(u_L(t,x;u_0), v_L(t,x;u_0))$ exists, then $(u_L(x), v_L(x))$ is a stationary solution of (2.67). If $(u_\infty(x), v_\infty(x)) := \lim_{L\to\infty}(u_L(x), v_L(x))$ exists, and $u_\infty(-\infty) = 0$ and $u_\infty(\infty) = \frac{r^*}{b}$, then $(u_\infty(x), v_\infty(x))$ is a forced wave solution of (1.1) connecting $(\frac{r^*}{b}, \frac{\mu r^*}{\lambda b})$ and $(0,0)$. We compute the numerical solution of (2.67) on a reasonable large time interval $[0,T]$ for some choices of L. The detail of numerical scheme can be found in [57].

We fix $\chi = 0.1$, $\mu = 1$, $\lambda = 0.05$, and choose $r(x)$ to be

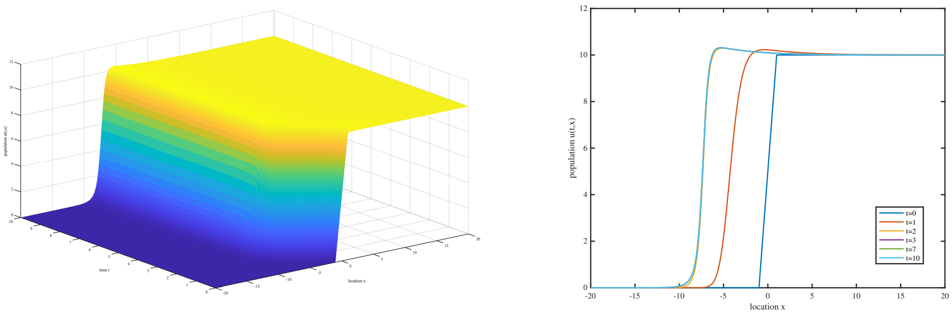
$$r(x) = \begin{cases} -1 & \text{if } x \le -8, \\ 11x + 87 & \text{if } -8 < x < -7, \\ 10 & \text{if } x \ge -7. \end{cases}$$

For this choice of $r(x)$, $r^* = 10$ and $-c^* = -2\sqrt{r^*} \approx -6.325$. We do four numerical experiments for different values of b and c. In these four numerical experiments, we use the same space step size $h = 0.1$ and the same time step size $\tau = 0.002$.

Numerical Experiment 1. Let $b = 1$ and $c = 1$. In this case, $c > \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} - 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$ becomes $c > \frac{5}{9\sqrt{0.05}} - 2\sqrt{\frac{8}{0.9}} \approx -3.478$. So for these choices of b and c, the assumption **(H1)** holds.

We compute the numerical solution of (2.67) with $L = 15, 20, 25, 30$, and 40 on the time interval $[0, 10]$. For all the choices of L, we observe that the numerical solution of (2.67) changes very little after $t = 3$, which indicates that the numerical solution converges to a stationary solution of (2.67) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 10$ changes very little and $u(10, L)$ is very close to $\frac{r^*}{b} = 10$ as L increases, which indicates the stationary solution of (2.67) converges to a stationary solution of (2.3) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ or a forced wave solution of (1.1) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ as $L \to \infty$, whose existence is proved in Theorem 2.4. Hence the numerical results for the choices $b = 1$ and $c = 1$ match the theoretical results.

We demonstrate the numerical solutions of (2.67) for the cases $L = 20$ in Figure 2.1. Figure 2.1(a) is the surface plot of the numerical solution of the system (2.67) on the interval $[-20, 20]$ as time evolves. Figure 2.1(b) is the profile of the numerical solution at time $t = 0, 1, 2, 3, 7, 10$.

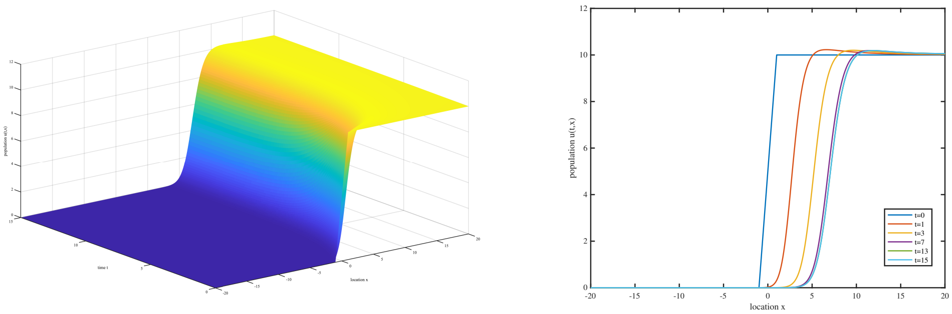


(a) (b)

Figure 2.1: **(a)** Evolution of numerical solution of (2.67) on the interval $[-20, 20]$ with $b = 1$ and $c = 1$. **(b)** numerical solution of (2.67) on the interval $[-20, 20]$ at time $t = 0, 1, 2, 3, 7, 10$ with $b = 1$ and $c = 1$.

Numerical Experiment 2. Let $b = 1$, $c = -6$. For these choices of b and c, b and c satisfy $b > 2\chi\mu$ and $c > -c^*$, but the assumption $c > \frac{\chi\mu r^*}{2\sqrt{\lambda}(b-\chi\mu)} - 2\sqrt{\frac{r^*(b-2\chi\mu)}{b-\chi\mu}}$ does not hold. We compute the numerical solution of (2.67) with $L = 15, 20, 25, 30$, and 40 on the time interval

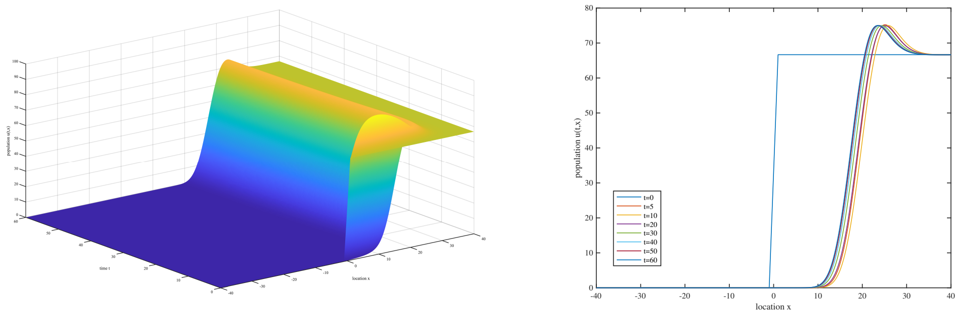
$[0, 15]$. For all the choices of L, we observe that the numerical solution of (2.67) changes very little after $t = 7$, which indicates that the numerical solution converges to a stationary solution of (2.67) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 15$ changes very little and $u(15, L)$ is very close to $\frac{r^*}{b} = 10$ as L increases, which indicates the stationary solution of (2.67) converges to a stationary solution of (2.3) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ or a forced wave solution of (1.1) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ as $L \to \infty$. The numerical results indicate that when $c > -c^*$ and $b > 2\chi\mu$, (1.1) also has a forced wave solution. We demonstrate the numerical solutions of (2.67) for the case $L = 20$ in Figure 2.2.



(a) (b)

Figure 2.2: **(a)** Evolution of numerical solution of (2.67) on the interval $[-20, 20]$ with $b = 1$ and $c = -6$. **(b)** numerical solution of (2.67) on the interval $[-20, 20]$ at time $t = 0, 1, 3, 7, 13, 15$ with $b = 1$ and $c = -6$.

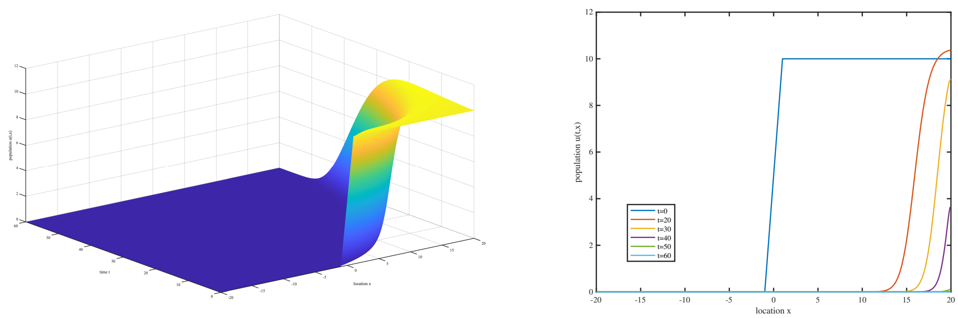
Numerical Experiment 3. Let $b = 0.15$, $c = -6$. For these choices of b and c, b and c satisfy $b > \chi\mu$ and $c > -c^*$. We compute the numerical solution of (2.67) with $L = 35, 40, 45, 50$, and 60 on the time interval $[0, 60]$. For all the choices of L, we observe that the numerical solution of (2.67) changes very little after $t = 50$, which indicates that the numerical solution converges to a stationary solution of (2.67) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 60$ changes very little and $u(60, L)$ is very close to $\frac{r^*}{b} = \frac{10}{0.15} \approx 66.67$ as L increases, which indicates the stationary solution of (2.67) converges to a stationary solution of (2.3) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ or a forced wave solution of (1.1) connecting $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ and $(0, 0)$ as $L \to \infty$. The numerical results indicate that when $c > -c^*$ and $b > \chi\mu$, (1.1) also has a forced wave solution. We demonstrate the numerical solutions of (2.67) for the case $L = 40$ in Figure 2.3.



(a) (b)

Figure 2.3: **(a)** Evolution of numerical solution of (2.67) on the interval $[-40, 40]$ with $b = 0.15$ and $c = -6$. **(b)** numerical solution of (2.67) on the interval $[-40, 40]$ at time $t = 0, 5, 10, 20, 30, 40, 50, 60$ with $b = 0.15$ and $c = -6$.

Numerical Experiment 4. Let $b = 1$, $c = -6.5$. For these choices of b and c, b and c satisfy $b > 2\chi\mu$ and $c < -c^*$. We compute the numerical solution of (2.67) with $L = 15$ on the time interval $[0, 50]$, with $L = 20$ on the time interval $[0, 60]$, with $L = 25$ on the time interval $[0, 70]$, with $L = 30$ on the time interval $[0, 90]$, and with $L = 40$ on the time interval $[0, 140]$. For all the choices of L, we observe that the numerical solution of (2.67) becomes very small after certain time, which indicates that the numerical solution converges to zero as $t \to \infty$, and also indicates that (2.3) has no positive stationary solutions or (1.1) has no forced wave solutions in the case that $c < -c^*$. We demonstrate the numerical solutions of (2.67) for the case $L = 20$ in Figure 2.4.



(a) (b)

Figure 2.4: **(a)** Evolution of numerical solution of (2.67) on the interval $[-20, 20]$ with $b = 1$ and $c = -6.5$. **(b)** numerical solution of (2.67) on the interval $[-20, 20]$ at time $t = 0, 20, 30, 40, 50, 60$ with $b = 1$ and $c = -6.5$.

Remark 2.3. *(1) The numerical simulations above illustrate our Theorem 2.4 and also shows that the assumptions in Theorem 2.4 can be weakened. Based on these numerical simulations, we conjecture that if* $b > \chi\mu$, *and* $c > -2\sqrt{r^*}$, *there is a forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *connecting* $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ *and* $(0,0)$, *that is,* $\phi(\infty) = \frac{r^*}{b}$ *and* $\phi(-\infty) = 0$. *If* $b > \chi\mu$ *and* $c < -2\sqrt{r^*}$, *there is no forced wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ *connecting* $(\frac{r^*}{b}, \frac{\mu}{\lambda}\frac{r^*}{b})$ *and* $(0,0)$, *that is,* $\phi(\infty) = \frac{r^*}{b}$ *and* $\phi(-\infty) = 0$.

(2) In the above four numerical experiments, we used the same space step size $h = 0.1$ *and the same time step size* $\tau = 0.002$. *They satisfy the numerical stable condition* $\frac{\tau}{h^2} < \frac{1}{2}$. *We do not give the accuracy analysis of the simulations in this paper. To see the reliability of the numerical results, we also tried different values of* h *and* τ *to simulate the existence of forced wave solutions. For the above four experiments, let* $h = 0.1$ *be fixed, choose* $\tau = 0.001, 0.002, 0.004$ *respectively, the graphs we got do not have a big difference. Fix* $h = 0.05$, *let* $\tau = 0.001, 0.0005, 0.00025$ *respectively, the graphs we got also do not have a big difference.*

(3) We also tried to use different initial conditions to simulate the forced wave. For example, let

$$u_0(x) = \begin{cases} 0 & \text{if } x < -1, \\ x+1 & \text{if } -1 \le x \le 1, \\ 2 & \text{if } x > 1. \end{cases}$$

We see similar dynamical scenarios. We then conjecture that the forced wave solution of (1.1) *is unique and stable in certain parameter regions.*

2.5.2 Case 2

In this subsection, we study the numerical simulations of the forced wave solutions in Case 2 by the finite difference method. To this end, we consider

$$\begin{cases} u_t = u_{xx} + cu_x - (\chi u v_x)_x + u(r(x) - bu), \quad -L < x < L \\ 0 = v_{xx} - \lambda v + \mu u, \quad -L < x < L \\ u(0,x) = u_0(x), \quad -L \le x \le L, \\ u(t,-L) = v(t,-L) = 0 \\ u(t,L) = v(t,L) = 0 \end{cases} \tag{2.68}$$

for reasonable large $L > 1$, where

$$u_0(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ (x+1)(1-x) & \text{if } -1 \le x \le 1. \end{cases}$$

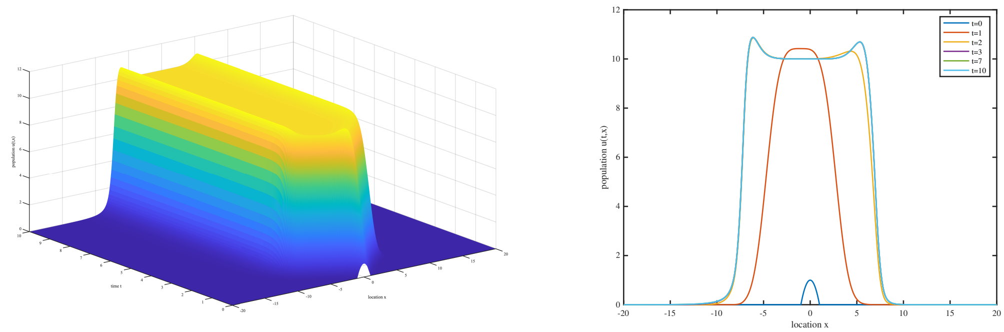
Choose $\mu = 1$, $\lambda = 1$ and

$$r(x) = \begin{cases} -1 & \text{if } |x| \ge 8, \\ 11x + 87 & \text{if } -8 < x < -7, \\ 10 & \text{if } -7 \le x \le 7, \\ -11x + 87 & \text{if } 7 < x < 8. \end{cases}$$

For this choice of $r(x)$, $r^* = 10$ and $c^* := 2\sqrt{r^*} \approx 6.325$. We do three numerical experiments for different values of b, c and χ. In these three numerical experiments, we use the same space step size $h = 0.1$ and the same time step size $\tau = 0.002$.

Numerical Experiment 1. Choose $c = 1$, then $\zeta_{-7}(r(\cdot), c) = \frac{40-1-\frac{\pi^2}{49}}{4} > 0$. Since $\zeta_L(r(\cdot), c) > \zeta_{-7}(r(\cdot), c)$ for $L > -7$, we have $\zeta_\infty(r(\cdot), c) \ge \zeta_{-7}(r(\cdot), c) > 0$. Choose $b = 1$ and $\chi = 0.6$. Then $b \ge \frac{3\chi\mu}{2}$.

We compute the numerical solution of (2.68) with $L = 15, 20, 25, 30$, and 40 on the time interval $[0, 10]$. In all the cases, we observe that the numerical solution changes very little after $t = 3$ and stays away from 0 on some fixed interval, which indicates that the numerical solution converges to a positive stationary solution of (2.68) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 10$ changes very little as L increases, which indicates that the stationary solution of (2.68) converges as $L \to \infty$ to a stationary solution of (2.3) or a forced wave solution of (1.1) connecting $(0, 0)$ and $(0, 0)$. We demonstrate the numerical solutions of (2.68) for the cases $L = 20$ in Figure 2.5.

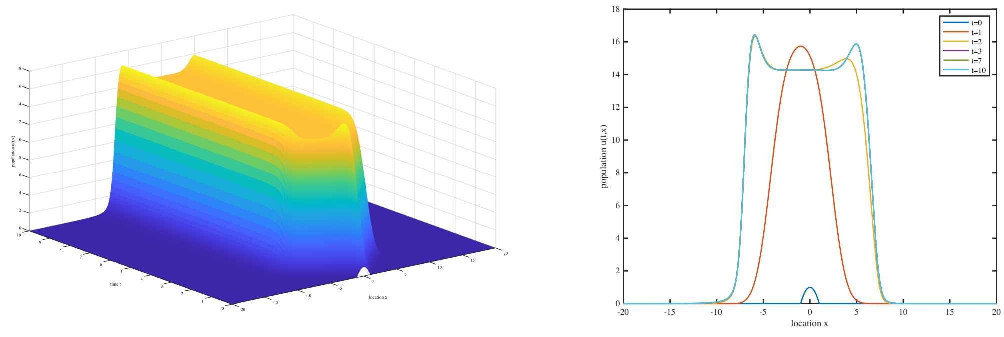


(a) (b)

Figure 2.5: **(a)** Evolution of numerical solution of (2.68) on the interval $[-20, 20]$ with $c = 1$, $b = 1$ and $\chi = 0.6$. **(b)** numerical solution of (2.68) on the interval $[-20, 20]$ at time $t = 0, 1, 2, 3, 7, 10$ with $c = 1$, $b = 1$ and $\chi = 0.6$.

Numerical Experiment 2. Choose $c = 1$ (then $\zeta_\infty(r(\cdot), c) > 0$). Choose $b = 0.7$ and $\chi = 0.6$ (then $\chi\mu < b < \frac{3\chi\mu}{2}$).

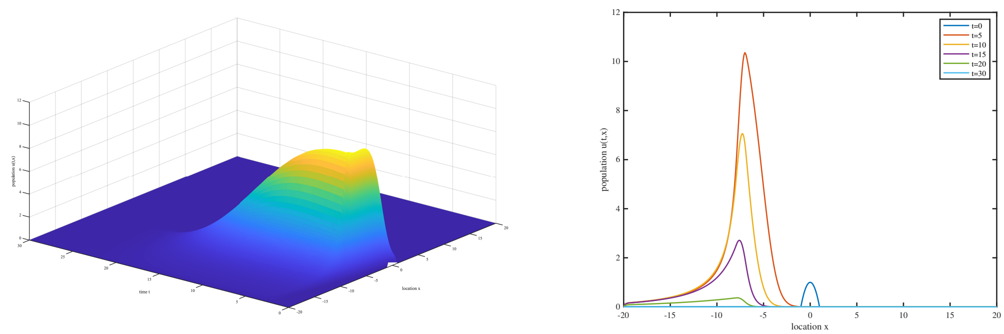
We compute the numerical solution of (2.68) with $L = 15, 20, 25, 30$, and 40 on the time interval $[0, 10]$. In all the cases, we observe that the numerical solution changes very little after $t = 3$ and stays away from 0 on some fixed interval, which indicates that the numerical solution converges to a positive stationary solution of (2.68) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 10$ changes very little as L increases, which indicates that the stationary solution of (2.68) converges as $L \to \infty$ to a stationary solution of (2.3) or a forced wave solution of (1.1) connecting $(0, 0)$ and $(0, 0)$. We demonstrate the numerical solutions of (2.68) for the cases $L = 20$ in Figure 2.6.



(a) (b)

Figure 2.6: **(a)** Evolution of numerical solution of (2.68) on the interval $[-20, 20]$ with $c = 1$, $b = 0.7$ and $\chi = 0.6$. **(b)** numerical solution of (2.68) on the interval $[-20, 20]$ at time $t = 0, 1, 2, 3, 7, 10$ with $c = 1$, $b = 0.7$ and $\chi = 0.6$.

Numerical Experiment 3. Let $c = 6.5$ (hence $c > c^*$). Let $b = 1$ and $\chi = 0.6$ (hence $b > \frac{3\chi\mu}{2}$). We compute the numerical solution of (2.68) with $L = 15, 20, 25, 30, 40$ on the time interval $[0, 30]$. For all the choices of L, we observe that the numerical solution of (2.68) becomes very small after $t = 20$, which indicates that the numerical solution converges to zero as $t \to \infty$, and also indicates that (2.3) has no positive stationary solutions or (1.1) has no forced wave solutions in the case that $c > c^*$ and $b > \frac{3}{2}\chi\mu$ which matches the theoretical result Theorem 2.3 (1). We demonstrate the numerical solutions of (2.68) for the case $L = 20$ in Figure 2.7.



(a) (b)

Figure 2.7: **(a)** Evolution of numerical solution of (2.68) on the interval $[-20, 20]$ with $c = 6.5$, $b = 1$ and $\chi = 0.6$. **(b)** numerical solution of (2.68) on the interval $[-20, 20]$ at time $t = 0, 5, 10, 15, 20, 30$ with $c = 6.5$, $b = 1$ and $\chi = 0.6$.

Similarly, if $c = -6.5$, $b = 1$ and $\chi = 0.6$, we observe that the numerical solution of (2.68) becomes very small after certain time, which indicates that the numerical solution converges to zero as $t \to \infty$, and also indicates that (2.3) has no positive stationary solutions or (1.1) has no forced wave solutions in the case that $c < -c^*$ and $b > \frac{3}{2}\chi\mu$ which matches the theoretical result Theorem 2.3 (1).

Remark 2.4. *(1) The numerical simulations above supports our Theorem 2.5 and also tells us that the assumptions in Theorem 2.5 may be weakened. Based on these numerical simulations, we conjecture that if $b > \chi\mu$ and $\zeta_\infty(r(\cdot), c) > 0$, there is a forced wave solution $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ connecting $(0,0)$ and $(0,0)$, that is, $\phi(x) > 0$ for all $x \in \mathbb{R}$ and $\phi(\pm\infty) = 0$. If $b > \chi\mu$ and $|c| > c^*$, there is no forced wave solution $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ connecting $(0,0)$ and $(0,0)$, that is, $\phi(x) > 0$ for all $x \in \mathbb{R}$ and $\phi(\pm\infty) = 0$.*

(2) In these three numerical simulations, we used the same space step size $h = 0.1$ and the same time step size $\tau = 0.002$, which satisfy the numerical stable condition $\frac{\tau}{h^2} < \frac{1}{2}$. Again, we do not give the accuracy analysis of the simulations in this paper. To see the reliability of the numerical results, we also tried different values of h and τ. For example, let $h = 0.1$ be fixed, let $\tau = 0.001, 0.002, 0.004$ respectively; let $h = 0.2$ be fixed, let $\tau = 0.01, 0.005, 0.0025$ respectively; let $h = 0.05$ be fixed, let $\tau = 0.001, 0.0005, 0.00025$ respectively. All the graphs we got do not change much.

(3) We also tried to use different initial conditions to simulate the existence of forced wave solutions. For example, let

$$u_0(x) = \begin{cases} 0 & \text{if } x \le -1, \\ sin(x+1) & \text{if } -1 < x < \pi - 1, \\ 0 & \text{if } x \ge \pi - 1. \end{cases}$$

We see similar dynamical scenarios. We then also conjecture that the forced wave solution of (1.1) is unique and stable in certain parameter regions.

Chapter 3

Global existence, asymptotic behavior and spreading speeds of parabolic-parabolic chemotaxis models with logistic source on $\mathbb{R}^N$

This chapter is devoted to the study of the asymptotic dynamics of the parabolic-parabolic chemotaxis model (1.4) with logistic source on $\mathbb{R}^N$. We first investigate the local existence and uniqueness of classical solutions with given initial functions. We then study the global existence and boundedness of classical solutions with given initial functions. Under the conditions that global classical solutions exist and some other further conditions, we study the asymptotic behavior of global classical solutions with strictly positive initial functions. Finally, we explore the spreading speeds of global classical solutions with compact supported initial functions and front like initial functions. As a by-product of spreading speeds, we get the persistence of global classical solutions with strictly positive initial functions (see Theorem 3.7).

3.1 Notations and statements of the main results

To state our main results, we first introduce some notations. Let

$$X_1 = C^b_{\text{unif}}(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) \,|\, u(x) \text{ is uniformly continuous in } x \in \mathbb{R}^N \text{ and } \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\}$$

equipped with the norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^N} |u(x)|$, and

$$X_2 = C^{b,1}_{\text{unif}} = \{u \in C^b_{\text{unif}}(\mathbb{R}^N) \,|\, \partial_{x_i} u \in C^b_{\text{unif}}(\mathbb{R}^N),\ i = 1, 2, \cdots, N\}$$

equipped with the norm $\|u\|_{C^{b,1}_{\text{unif}}} = \|u\|_\infty + \sum_{i=1}^N \|\partial_{x_i} u\|_\infty$ and

$$C^{b,2}_{\text{unif}} = \{u \in C^{b,1}_{\text{unif}}(\mathbb{R}^N) \,|\, \partial_{x_i x_j} u \in C^b_{\text{unif}}(\mathbb{R}^N),\ i, j = 1, 2, \cdots, N\}.$$

Let

$$X_1^+ = \{u \in X_1 \,|\, u \geq 0\}, \quad X_2^+ = \{v \in X_2 \,|\, v \geq 0\}.$$

For given $0 < \nu < 1$, let

$$C^{b,\nu}_{\rm unif}(\mathbb{R}^N) = \{u \in C^b_{\rm unif}(\mathbb{R}^N) \,|\, \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\nu} < \infty\}$$

with the norm $\|u\|_{\infty,\nu} = \sup_{x\in\mathbb{R}^N} |u(x)| + \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\nu}$.
For $0 < \theta < 1$, let

$$\begin{aligned} & C^\theta((t_1,t_2), C^{b,\nu}_{\rm unif}(\mathbb{R}^N)) \\ & = \{u(\cdot) \in C((t_1,t_2), C^{b,\nu}_{\rm unif}(\mathbb{R}^N)) \,|\, u(t) \text{ is locally Hölder continuous with exponent } \theta\}. \end{aligned}$$

We call $(u(x,t), v(x,t))$ a *classical solution* of (1.4) on $[0,T)$ if $u, v \in C(\mathbb{R}^N \times [0,T)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ and satisfies (1.4) for $(x,t) \in \mathbb{R}^N \times (0,T)$ in the classical sense. A classical solution $(u(x,t), v(x,t))$ of (1.4) on $[0,T)$ is called *non-negative* if $u(x,t) \geq 0$ and $v(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times [0,T)$. A *global classical solution* of (1.4) is a classical solution on $[0,\infty)$. Note that, due to the biological interpretations, only non-negative classical solutions will be of interest.

The main results are from our works [58] and [59]. We first state the result on the local existence and uniqueness of classical solution with initial function $(u_0, v_0) \in C^b_{\rm unif}(\mathbb{R}^N) \times C^{b,1}_{\rm unif}(\mathbb{R}^N)$.

Theorem 3.1. *For any $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $v_0 \in C^{b,1}_{\rm unif}(\mathbb{R}^N)$ with $u_0 \geq 0$, $v_0 \geq 0$, there exists $T_{\max} := T_{\max}(u_0, v_0) \in (0,\infty]$ such that* (1.4) *has a unique non-negative classical solution $(u(x,t;u_0,v_0),\ v(x,t;u_0,v_0))$ on $[0, T_{\max})$ satisfying that $\lim_{t\to 0^+} u(\cdot,t;u_0,v_0) = u_0$ in the $C^b_{\rm unif}(\mathbb{R}^N)$-norm and $\lim_{t\to 0^+} v(\cdot,t;u_0,v_0) = v_0$ in the $C^{b,1}_{\rm unif}(\mathbb{R}^N)$-norm,*

$$u(\cdot,\cdot;u_0,v_0) \in C([0,T_{\max}), C^b_{\rm unif}(\mathbb{R}^N)) \cap C^1((0,T_{\max}), C^b_{\rm unif}(\mathbb{R}^N)), \tag{3.1}$$

$$v(\cdot,\cdot;u_0,v_0) \in C([0,T_{\max}), C^{b,1}_{\rm unif}(\mathbb{R}^N)) \cap C^1((0,T_{\max}), C^{b,1}_{\rm unif}(\mathbb{R}^N)), \tag{3.2}$$

$$u(\cdot,\cdot;u_0,v_0),\ \partial_{x_i}u(\cdot,\cdot;u_0,v_0),\ \partial^2_{x_ix_j}u(\cdot,\cdot;u_0,v_0),\ \partial_t u(\cdot,\cdot;u_0,v_0)\in C^\theta((0,T_{\max}),C^{b,\nu}_{\rm unif}(\mathbb{R}^N)),\tag{3.3}$$

$$v(\cdot,\cdot;u_0,v_0),\ \partial_{x_i}v(\cdot,\cdot;u_0,v_0),\ \partial^2_{x_ix_j}v(\cdot,\cdot;u_0,v_0),\ \partial_t v(\cdot,\cdot;u_0,v_0)\in C^\theta((0,T_{\max}),C^{b,\nu}_{\rm unif}(\mathbb{R}^N))\tag{3.4}$$

for all $i,j=1,2,\cdots,N$, $0<\theta\ll 1$, *and* $0<\nu\ll 1$. *Moreover, if* $T_{\max}<\infty$, *then* $\lim_{t\to T_{\max}}\big(\|u(\cdot,t;u_0,v_0)\|_\infty+\|v(\cdot,t;u_0,v_0)\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)}\big)=\infty$.

We then state the result on the global existence and boundedness of the classical solution with initial function $(u_0,v_0)\in C^b_{\rm unif}(\mathbb{R}^N)\times C^{b,1}_{\rm unif}(\mathbb{R}^N)$.

Theorem 3.2. *Suppose that* $b>\frac{N\mu\chi}{4}$. *Then for every* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$, $v_0\in C^{b,1}_{\rm unif}(\mathbb{R}^N)$ *with* $u_0\geq 0$, $v_0\geq 0$, (1.4) *has a unique bounded global classical solution* $(u(x,t;u_0,v_0),$ $v(x,t;u_0,v_0))$ *and*

$$\limsup_{t\to\infty}\|u(\cdot,t;u_0,v_0)\|_\infty\leq\frac{(2\lambda+a)^2}{2\lambda(4b-N\mu\chi)}.\tag{3.5}$$

Moreover, if $\lambda\geq\frac{a}{2}$, *then*

$$\limsup_{t\to\infty}\|u(\cdot,t;u_0,v_0)\|_\infty\leq\frac{4a}{4b-N\mu\chi}.\tag{3.6}$$

We next state the result on the asymptotic behavior of the global classical solution with strictly positive initial function.

Theorem 3.3. *There exists* $K=k(a,\lambda,N)>\frac{N}{4}$ *such that if* $b>K\chi\mu$ *and* $\lambda\geq\frac{a}{2}$, *then the unique bounded global classical solution* $(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$ *of* (1.4) *with* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$, $v_0\in C^{b,1}_{\rm unif}(\mathbb{R}^N)$ *and* $\inf_{x\in\mathbb{R}^N}u_0(x)>0$, $v_0\geq 0$, *satisfies that*

$$\|u(\cdot,t;u_0,v_0)-\frac{a}{b}\|_\infty+\|v(\cdot,t;u_0,v_0)-\frac{\mu a}{\lambda b}\|_\infty\to 0 \text{ as } t\to\infty \text{ exponentially.}\tag{3.7}$$

Remark 3.1. *(1) Theorem 3.1 is on the local existence of a unique classical solution with nonnegative initial function* $(u_0,v_0)\in C^b_{\rm unif}(\mathbb{R}^N)\times C^{b,1}_{\rm unif}(\mathbb{R}^N)$. *We point out the local existence of a unique classical solution with* (u_0,v_0) *in some other spaces can also be proved. For example, following the similar arguments used in the proof of Theorem*

3.1, the local existence of a unique classical solution with nonnegative initial function $(u_0, v_0) \in L^p(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ for $p > N$ and $p \geq 2$ can be proved.

(2) As it is mentioned in the above, consider chemotaxis model (1.6) *on convex bounded domain with Neumann boundary condition and $\frac{b}{\chi}$ being sufficiently large, Winkler [70] proved the global existence of classical solution for every nonnegative initial function $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ and the global asymptotic stability of the constant solution $(\frac{a}{b}, \frac{\mu a}{\lambda b})$. Theorem 3.2 and Theorem 3.3 stated in the above extend the results in [70] on the global existence and global asymptotical stability of the constant solution for parabolic-parabolic chemotaxis systems on bounded domains to the whole space. Biologically, the conditions $b > \frac{N\chi\mu}{4}$ and $b > K\chi\mu$ in Theorems 3.2 and 3.3 indicate that the logistic damping b is large relative to the product of the chemotaxis sensitivity χ and the production rate μ at which the biological species produces the chemical substance. The condition $\lambda \geq \frac{a}{2}$ in Theorem 3.3 indicates that the degradation rate of the chemical substance is large relative to the intrinsic growth rate of the biological species.*

(3) Theorem 3.3 does not give an explicit expression on K, but it has the following property. According to the proof of Theorem 3.3, $K = \frac{N}{4\theta_0}$, where $\theta_0 \in (0, 1)$ is the largest number such that

$$\frac{2C_2\theta_0}{(1-\theta_0)^2 a} \leq \frac{1}{6} \quad \text{and} \quad \frac{8C\lambda^{-\frac{1}{2}}a^{\frac{1}{2}}\pi\theta_0}{N(1-\theta_0)} \leq \frac{1}{12} \tag{3.8}$$

hold simultaneously with $C_2 = \max\{C\lambda^{\gamma-\beta-\frac{3}{2}}a^{\beta+\frac{3}{2}}\sqrt{\pi}N^{-2} + C\lambda^{\gamma-\beta-1}a^{\beta+1}N^{-1}, a\}$, C here as well as in (3.8) *is a generic constant and β and γ are such that $\gamma \in (1, \frac{3}{2})$ and $\gamma - 1 < \beta < \frac{1}{2}$. It can then be verified directly that for fixed a and N, K is bounded in $\lambda \geq \frac{a}{2}$ and $K \to \frac{N}{0.28}$ as $\lambda \to \infty$.*

To state our main results on the spreading speeds of (1.4). We make the following notations.

For given $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, let $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$. Let

$$S^{N-1} = \{x \in \mathbb{R}^N \,|\, |x| = 1\}.$$

For $x=(x_1,x_2,\cdots,x_N)\in\mathbb{R}^N$, $y=(y_1,y_2,\cdots,y_N)\in\mathbb{R}^N$, define $x\cdot y=\sum_{i=1}^N x_iy_i$.

Let

$$C_{cp}^{+}=\{u\in X_1^{+}\mid \ \mathrm{supp}(u)\ \text{ is non-empty and compact}\},$$

and

$$C_{cp}^{+,1}=\{v\in X_2^{+}\mid \ \mathrm{supp}(v)\ \text{ is non-empty and compact}\}.$$

For any given $\xi\in S^{N-1}$, we define

$$C_{fl}^{+}(\xi)=\{u\in X_1^{+}\ \mid\ \liminf_{x\cdot\xi\to-\infty}u(x)>0,\ u(x)=0 \text{ for } x\in\mathbb{R}^N \text{ with } x\cdot\xi\gg 1\},$$

$$C_{fl}^{+,1}(\xi)=\{v\in X_2^{+}\ \mid\ \liminf_{x\cdot\xi\to-\infty}v(x)>0,\ v(x)=0 \text{ for } x\in\mathbb{R}^N \text{ with } x\cdot\xi\gg 1\},$$

$$C^{+}(\xi)=\{u\in X_1^{+}\mid \inf_{|x\cdot\xi|<r}u(x)>0 \text{ for some } r>0,\ u(x)=0 \text{ for } x\in\mathbb{R}^N \text{ with } |x\cdot\xi|\gg 1\},$$

and

$$C^{+,1}(\xi)=\{v\in X_2^{+}\mid \inf_{|x\cdot\xi|<r}v(x)>0 \text{ for some } r>0,\ v(x)=0 \text{ for } x\in\mathbb{R}^N \text{ with } |x\cdot\xi|\gg 1\}.$$

The main results on the spreading speeds are stated in the following theorems.

Theorem 3.4. *Suppose that $b>\frac{N\mu\chi}{4}$. For any $(u_0,v_0)\in C_{cp}^{+}\times C_{cp}^{+,1}$, the following hold.*

(1) For any $0<\epsilon<2\sqrt{a}$,

$$\liminf_{t\to\infty}\inf_{|x|\le(2\sqrt{a}-\epsilon)t}u(x,t;u_0,v_0)>0,$$

and

$$\liminf_{t\to\infty}\inf_{|x|\le(2\sqrt{a}-\epsilon)t}v(x,t;u_0,v_0)>0.$$

(2) For any $\varepsilon>0$,

$$\lim_{t\to\infty}\sup_{|x|\ge(2\sqrt{a}+\epsilon)t}u(x,t;u_0,v_0)=0,$$

and

$$\lim_{t\to\infty} \sup_{|x|\geq(2\sqrt{a}+\epsilon)t} v(x,t;u_0,v_0)=0.$$

Theorem 3.5. *Suppose that* $b>\frac{N\mu\chi}{4}$*. For any given* $\xi\in S^{N-1}$*,* $(u_0,v_0)\in C^+_{fl}(\xi)\times C^{+,1}_{fl}(\xi)$*, the following hold.*

(1) For any $0<\epsilon<2\sqrt{a}$*,*

$$\liminf_{t\to\infty} \inf_{x\cdot\xi\leq(2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0)>0,$$

and

$$\liminf_{t\to\infty} \inf_{x\cdot\xi\leq(2\sqrt{a}-\epsilon)t} v(x,t;u_0,v_0)>0.$$

(2) For any $\epsilon>0$*,*

$$\lim_{t\to\infty} \sup_{x\cdot\xi\geq(2\sqrt{a}+\epsilon)t} u(x,t;u_0,v_0)=0,$$

and

$$\lim_{t\to\infty} \sup_{x\cdot\xi\geq(2\sqrt{a}+\epsilon)t} v(x,t;u_0,v_0)=0.$$

Theorem 3.6. *Suppose that* $b>\frac{N\mu\chi}{4}$*. For any given* $\xi\in S^{N-1}$*,* $(u_0,v_0)\in C^+(\xi)\times C^{+,1}(\xi)$*, the following hold.*

(1) For any $0<\epsilon<2\sqrt{a}$*,*

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq(2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0)>0,$$

and

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq(2\sqrt{a}-\epsilon)t} v(x,t;u_0,v_0)>0.$$

(2) For any $\epsilon>0$*,*

$$\lim_{t\to\infty} \sup_{|x\cdot\xi|\geq(2\sqrt{a}+\epsilon)t} u(x,t;u_0,v_0)=0,$$

and

$$\lim_{t\to\infty} \sup_{|x\cdot\xi|\geq(2\sqrt{a}+\epsilon)t} v(x,t;u_0,v_0) = 0.$$

Remark 3.2. *(1) As it is recalled in the above, in the absence of chemotaxis (i.e. $\chi = 0$), $2\sqrt{a}$ is the spreading speed of* (1.5). *Theorems 3.4, 3.5, and 3.6 provide some new approach to prove that $2\sqrt{a}$ is the spreading speed of the Fisher-KPP equation* (1.5). *The new approach can also be applied to the study of the spreading speeds of general Fisher-KPP equation with time and space dependence.*

(2) Assume $b > \frac{N\mu\chi}{4}$. Theorem 3.4 (1), Theorem 3.5 (1) and Theorem 3.6 (1) show that the chemotaxis does not slow down the spreading speed in the Fisher-KPP equation (1.5). *Theorem 3.4 (2), Theorem 3.5 (2) and Theorem 3.6 (2) show that the chemotaxis does not speed up the spreading speed in the Fisher-KPP equation* (1.5). *Hence, when $b > \frac{N\mu\chi}{4}$, the chemotaxis neither speeds up nor slows down the spreading speed in the Fisher-KPP equation* (1.5). *Biologically, the condition $b > \frac{N\mu\chi}{4}$ means that the logistic damping is large relative to the product of the chemotaxis sensitivity and the production rate of the chemical substance.*

The following theorem is on the persistence of strictly positive solutions.

Theorem 3.7. *Suppose that $b > \frac{N\mu\chi}{4}$, then there exist $m > 0$ and $M > 0$ such that for any $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $v_0 \in C^{b,1}_{\rm unif}(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} u_0 > 0$ and $v_0 \geq 0$, there is $T(u_0,v_0)$ such that*

$$m \leq u(x,t;u_0,v_0) \leq M \quad \forall\, x \in \mathbb{R}^N,\ t \geq T(u_0,v_0).$$

The rest of the chapter is organized as follows. In section 3.2, we present some preliminary materials that will be needed in the proofs of our main results. In section 3.3, we study the local existence and uniqueness of the classical solutions of (1.4) with given initial functions and prove Theorem 3.1. In section 3.4, we study the global existence and boundedness of the classical solutions of (1.4) with given initial functions and prove Theorem 3.2. In section 3.5, we discuss the asymptotic behavior of global classical solutions with strictly positive initial functions and prove Theorem 3.3. In section 3.6, we investigate the lower bounds of spreading speeds of (1.4) and prove Theorems 3.4 (1), 3.5 (1) 3.6 (1), and Theorem 3.7. In section 3.7, we study the upper bounds of spreading speeds of (1.4) and prove Theorems 3.4 (2), 3.5 (2) and 3.6 (2).

3.2 Preliminaries

In this section, we present several lemmas which will be used often in the later sections. The reader is referred to [16], [46] for the details.

Throughout this paper, $\{e^{t(\Delta-\sigma I)}\}_{t>0}$, where $\sigma > 0$, denotes the analytic semigroup generated by $\Delta - \sigma I$ on $X := C^b_{\rm unif}(\mathbb{R}^N)$, unless specified otherwise. Then we have

$$\|e^{t(\Delta-\sigma I)}u\|_\infty \le e^{-\sigma t}\|u\|_\infty, \tag{3.9}$$

$$\|\nabla e^{t(\Delta-\sigma I)}u\|_\infty \le C_N t^{-\frac{1}{2}} e^{-\sigma t}\|u\|_\infty, \tag{3.10}$$

$$\|(\sigma I-\Delta)^\alpha e^{t(\Delta-\sigma I)}u\|_\infty \le C_\alpha t^{-\alpha} e^{-\sigma t}\|u\|_\infty \tag{3.11}$$

for every $t > 0$ and $\alpha \ge 0$. In fact, (3.9) and (3.10) follow directly from the following equation,

$$(e^{t(\Delta-\sigma I)}u)(x) = \int_{\mathbb{R}^N} e^{-\sigma t}\frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}} u(y)dy$$

for every $u \in C^b_{\rm unif}(\mathbb{R}^N)$, $t > 0$, $x \in \mathbb{R}^N$. (3.11) is a result of the combination of Theorem 1.4.3 in [16] and (3.9).

Lemma 3.1. *For every $t > 0$, the operator $e^{t(\Delta-\sigma I)}\nabla\cdot$ has a unique bounded extension on $\big(C^b_{\rm unif}(\mathbb{R}^N)\big)^N$ satisfying*

$$\|e^{t(\Delta-\sigma I)}\nabla\cdot u\|_\infty \le \frac{N}{\sqrt{\pi}} t^{-\frac{1}{2}} e^{-\sigma t}\|u\|_\infty \quad \forall\, u \in \big(C^b_{\rm unif}(\mathbb{R}^N)\big)^N,\ \forall\, t > 0.$$

Proof. It follows from [49, Lemma 3.2]. □

Note that $\mathrm{Dom}(\Delta - \sigma I) = C^{b,2}_{\rm unif}(\mathbb{R}^N)$. Let $X^\alpha = \mathrm{Dom}((\sigma I - \Delta)^\alpha)$ be the fractional power space of $\sigma I - \Delta$ on X ($\alpha \in [0, 1]$) equipped with graph norm $\|u\|_{X^\alpha} = \|(\sigma I - \Delta)^\alpha u\|_X$.

We have the following continuous imbedding

$$X^\alpha \hookrightarrow C^\nu \quad \text{if} \quad 0 \le \nu < 2\alpha \tag{3.12}$$

(see [16, Exercise 9, page 40]). Furthermore, there is a constant C_α such that

$$\|(e^{t(\Delta-\sigma I)} - I)u\|_X \le C_\alpha t^\alpha \|u\|_{X^\alpha} \quad \text{for all } u \in X^\alpha. \tag{3.13}$$

Inequality (3.13) comes from [16, Theorem 1.4.3]. Note that $X^0 = X$ and $X^1 = \mathrm{Dom}(\sigma I - \Delta)$.

We end this section by stating an important result that will be used in the proof of the local existence and uniqueness of classical solutions.

Lemma 3.2. *([16, Exercise 4^*, page 190]) Assume that a_1, a_2, α, β are non-negative constants, with $0 \le \alpha,\ \beta < 1$, and $0 < T < \infty$. There exists a constant $M(a_2, \beta, T) < \infty$ so that for any integrable function $u : [0,\ T] \to \mathbb{R}$ satisfying that*

$$0 \le u(t) \le a_1 t^{-\alpha} + a_2 \int_0^t (t-s)^{-\beta} u(s) ds$$

for a.e t in $[0,\ T]$, we have

$$0 \le u(t) \ \le \ \frac{a_1 M}{1-\alpha} t^{-\alpha}, \ \ \textit{a.e. on} \ \ 0 < t < T.$$

3.3 Local existence and uniqueness of classical solutions

In this section, we investigate the local existence and uniqueness of classical solutions of (1.4) with given initial functions and prove Theorem 3.1. Throughout this section, unless specified otherwise, C denotes a generic constant independent of u, v and may be different at different places. The main tools for the proof of this theorem are based on the contraction mapping theorem and the existence of classical solutions for linear parabolic equations with Hölder continuous coefficients. Throughout this subsection, $X_1 = C^b_{\rm unif}(\mathbb{R}^N)$, $X_2 = C^{b,1}_{\rm unif}(\mathbb{R}^N)$, and X_i^α is the fractional power space of $\lambda I - \Delta$ acting on X_i, $i = 1, 2$ ($\alpha \in (0,1)$).

Proof of Theorem 3.1. It can be proved by properly modifying arguments of [49, Theorem 1.1]. For self-completeness, we provide the outline of the proof.

(i) Existence of a mild solution. We first prove the existence of a mild solution of (1.4) with given initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $v_0 \in C^{b,1}_{\rm unif}(\mathbb{R}^N)$, that is, the existence of $(u(t), v(t))$ satisfying

$$\begin{cases} u(t) = e^{t(\Delta-\lambda I)}u_0 - \chi\int_0^t e^{(t-s)(\Delta-\lambda I)}\nabla\cdot(u(s)\nabla v(s))ds \\ \qquad\qquad + \int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)(a+\lambda-bu(s))ds \\ v(t) = e^{t(\Delta-\lambda I)}v_0 + \mu\int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)ds. \end{cases} \tag{3.14}$$

To this end, let $X = X_1\times X_2$. Fix $(u_0, v_0)\in X$. For every $T > 0$ and $R > 0$ satisfying $\|u_0\|_\infty \le R$ and $\|v_0\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)} \le R$, let

$$\mathcal{S}_{R,T} := \Big\{(u,v)\in C([0,T], C^b_{\rm unif}(\mathbb{R}^N))\times C([0,T], C^{b,1}_{\rm unif}(\mathbb{R}^N)) \mid \|u\|_\infty \le R \text{ and } \|v\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)}\le R\Big\}.$$

Note that $\mathcal{S}_{R,T}$ is a closed subset of the Banach space $C([0,T], C^b_{\rm unif}(\mathbb{R}^N))\times C([0,T], C^{b,1}_{\rm unif}(\mathbb{R}^N))$ with the norm $\|(u,v)\|_{\mathcal{S}_{R,T}} = \sup_{0\le t\le T}\|u(t)\|_\infty + \sup_{0\le t\le T}\|v\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)}$.

First, it is not difficult to prove that for any $(u,v)\in\mathcal{S}_{R,T}$ and $t\in[0,T]$, $\Phi(u,v)(t) = (\Phi_1(u,v)(t), \Phi_2(u,v)(t))$ is well defined in X, where

$$\Phi_1(u,v)(t) = e^{t(\Delta-\lambda I)}u_0 - \chi\int_0^t e^{(t-s)(\Delta-\lambda I)}\nabla\cdot(u(s)\nabla v(s))ds + \int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)(a+\lambda-bu(s))ds,$$

and

$$\Phi_2(u,v)(t) = e^{t(\Delta-\lambda I)}v_0 + \mu\int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)ds.$$

Second, we claim that *For every $(u,v)\in\mathcal{S}_{R,T}$, choose $0<\beta<\frac{1}{4}$ and $\frac{1}{2}<\gamma<1$ such that $\gamma+2\beta<1$. Then the function $(0,T]\ni t\to\Phi(u,v)(t)\in X^\beta$ is locally Hölder continuous, and Φ maps $\mathcal{S}_{R,T}$ into $C([0,T], C^b_{\rm unif}(\mathbb{R}^N))\times C([0,T], C^{b,1}_{\rm unif}(\mathbb{R}^N))$.*

Indeed, first, by the similar arguments to those in Claim 2 of [49, Theorem 1.1], the function $(0,T]\ni t\to\Phi_1(u,v)(t)\in X_1^\beta$ is locally Hölder continuous.

Next, observe that

$$\Phi_2(u,v)(t) = \underbrace{e^{t(\Delta-\lambda I)}v_0}_{J_0(t)} + \mu \underbrace{\int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)ds}_{J_1(t)}$$

For every $t > 0$, it is clear that $J_0(t) = e^{t(\Delta-\lambda I)}v_0 \in X_2^\beta$ because the semigroup $\{e^{t(\Delta-\lambda I)}\}_{t\geq 0}$ is analytic. Furthermore, since $X_1^\gamma \hookrightarrow C^1$, we have that

$$\begin{aligned}
\|J_1(t)\|_{X_2^\beta} &\leq \int_0^t \|(\lambda I - \Delta)^\beta e^{(t-s)(\Delta-\lambda I)}u(s)\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)}ds \\
&\leq C\int_0^t \|(\lambda I - \Delta)^{\gamma+\beta} e^{(t-s)(\Delta-\lambda I)}u(s)\|_\infty ds \\
&\leq C\int_0^t (t-s)^{-\gamma-\beta}e^{-\lambda(t-s)}\|u(s)\|_\infty ds \leq CR.
\end{aligned}$$

Since the operator $(\lambda I - \Delta)^\beta$ is closed, we have $J_1(t) \in X_2^\beta$. Hence, $\Phi_2(u,v)(t) \in X_2^\beta$ for every $t > 0$. Therefore, $\Phi(u,v)(t) \in X^\beta$ for every $t > 0$.

By (3.13), we have

$$\begin{aligned}
\|J_0(t+h) - J_0(t)\|_{X_2^\beta} &= \|(e^{h(\Delta-\lambda I)} - I)e^{t(\Delta-\lambda I)}v_0\|_{X_2^\beta} \leq Ch^\beta\|(\lambda I - \Delta)^\beta e^{t(\Delta-\lambda I)}v_0\|_{X_2^\beta} \\
&\leq Ch^\beta\|(\lambda I - \Delta)^{2\beta} e^{t(\Delta-\lambda I)}v_0\|_{X_1^\gamma} \leq CRt^{-2\beta-\gamma}h^\beta.
\end{aligned}$$

Hence, $(0,T] \ni t \to J_0(t) \in X_2^\beta$ is locally Hölder continuous. We also have

$$\begin{aligned}
&\|J_1(t+h) - J_1(t)\|_{X_2^\beta} \\
&\leq \int_0^t \|(e^{h(\Delta-\lambda I)} - I)e^{(t-s)(\Delta-\lambda I)}u(s)\|_{X_2^\beta}ds + \int_t^{t+h} \|e^{(t+h-s)(\Delta-\lambda I)}u(s)\|_{X_2^\beta}ds \\
&\leq Ch^\beta\int_0^t \|(\lambda I - \Delta)^{2\beta}e^{(t-s)(\Delta-\lambda I)}u(s)\|_{X_2}ds + \int_t^{t+h} \|(\lambda I - \Delta)^\beta e^{(t+h-s)(\Delta-\lambda I)}u(s)\|_{X_2}ds \\
&\leq Ch^\beta\int_0^t (t-s)^{-2\beta-\gamma}e^{-\lambda(t-s)}\|u(s)\|_\infty ds + C\int_t^{t+h} (t+h-s)^{-\beta-\gamma}e^{-\lambda(t+h-s)}\|u(s)\|_\infty ds \\
&\leq CR(h^\beta + h^{1-\gamma-\beta}).
\end{aligned}$$

Hence, $(0, T] \ni t \to \Phi_2(u, v)(t) \in X_2^\beta$ is locally Hölder continuous. Thus, $(0, T] \ni t \to \Phi(u, v)(t) \in X^\beta$ is locally Hölder continuous. It is clear that $t \to \Phi(u, v)(t) \in X$ is continuous in t at $t = 0$. The claim thus follows.

Third, it can be proved without much difficulty that for any $R > \max\{\|u_0\|_\infty, \|v_0\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)}\}$, there exists $T := T(R)$ such that Φ maps $\mathcal{S}_{R,T}$ into itself. Furthermore, Φ is a contraction map for T small and hence has a fixed point $(u(\cdot), v(\cdot)) \in \mathcal{S}_{R,T}$. This implies that (1.4) has a mild solution $(u(\cdot), v(\cdot))$ on a small time interval. By the standard extension arguments, this small interval can be extended to a maximal interval. That is, there is $T_{\max} \in (0, \infty]$ such that (1.4) has a mild solution $(u(\cdot), v(\cdot))$ on $[0, T_{\max})$ and if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}} \big(\|u(\cdot, t; u_0, v_0)\|_\infty + \|v(\cdot, t; u_0, v_0)\|_{C^{b,1}_{\rm unif}(\mathbb{R}^N)} \big) = \infty.$$

Moreover, for every $0 < \beta < \frac{1}{4}$, $\frac{1}{2} < \gamma < 1$ such that $\gamma + 2\beta < 1$, the function $(0, T_{\max}) \ni t \mapsto (u(\cdot), v(\cdot)) \in X^\beta$ is locally Hölder continuous.

(ii) Regularity and non-negativity. We next prove that the mild solution $(u(\cdot), v(\cdot))$ of (1.4) on $[0, T_{\max})$ obtained in (i) is a non-negative classical solution of (1.4) on $[0, T_{\max})$ and satisfies (3.1), (3.2), (3.3) and (3.4).

In fact, it follows from the claim in **(i)** and the fact X_1^β is continuously embedded into $C^{b,\nu}_{\rm unif}(\mathbb{R}^N)$ for $0 < \nu \ll 1$ that the mappings $t \to u(\cdot, t) := u(t)(\cdot) \in C^{b,\nu}_{\rm unif}(\mathbb{R}^N)$, $t \mapsto v(\cdot, t) := v(t)(\cdot) \in C^{b,\nu}_{\rm unif}(\mathbb{R}^N)$ are locally Hölder continuous in X for $t \in (0, T_{\max})$. By [16, Lemma 3.3.2], $v(x, t)$ is a classical solution of

$$v_t = (\Delta - \lambda I)v + \mu u(x, t), \quad x \in \mathbb{R}^N,\ 0 < t < T_{\max},$$

and

$$t \mapsto v_t(\cdot, t) \in C^{b,\nu}_{\rm unif}(\mathbb{R}^N), \ \ t \mapsto \frac{\partial v(\cdot, t)}{\partial x_i} \in C^{b,\nu}_{\rm unif}(\mathbb{R}^N), \ \ t \mapsto \frac{\partial^2 v(\cdot, t)}{\partial x_i \partial x_j} \in C^{b,\nu}_{\rm unif}(\mathbb{R}^N)$$

are also locally Hölder continuous in $t \in (0, T_{\max})$. Then by the similar arguments to those in the proof of [49, Theorem 1.1], $(u(x, t), v(x, t))$ is a classical solution of (1.4) on $(0, T_{\max})$

satisfying (3.1), (3.2), (3.3) and (3.4). Moreover, since $u_0 \geq 0$ and $v_0 \geq 0$, by comparison principle for parabolic equations, we get $u(x, t; u_0, v_0) \geq 0$ and $v(x, t; u_0, v_0) \geq 0$ for all $x \in \mathbb{R}$, $0 \leq t < T_{\max}$.

(iii) Uniqueness. We now prove that for given $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$, $v_0 \in C^{b,1}_{\text{unif}}(\mathbb{R}^N)$, (1.4) has a unique classical solution $(u(\cdot, \cdot; u_0, v_0), v(\cdot, \cdot; u_0, v_0))$ satisfying (3.1), (3.2), (3.3) and (3.4).

Any classical solution of (1.4) satisfying the properties of Theorem 3.1 clearly satisfies the integral equation (3.14). Suppose that for given $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$, $v_0 \in C^{b,1}_{\text{unif}}(\mathbb{R}^N)$ with $u_0 \geq 0$, $v_0 \geq 0$, $(u_1(x, t; u_0, v_0), v_1(x, t; u_0, v_0))$ and $(u_2(x, t; u_0, v_0), v_2(x, t; u_0, v_0))$ are two classical solutions of (1.4) on $\mathbb{R}^N \times [0, T_{\max})$ satisfying the properties of Theorem 3.1. Let $0 < T < T_{\max}$ be fixed. Thus $\sup_{0 \leq t \leq T}(\|u_1(\cdot, t; u_0, v_0)\|_\infty + \|u_2(\cdot, t; u_0, v_0)\|_\infty) < \infty$ and $\sup_{0 \leq t \leq T}(\|v_1(\cdot, t; u_0, v_0)\|_{C^{b,1}_{\text{unif}}(\mathbb{R}^N)} + \|v_2(\cdot, t; u_0, v_0)\|_{C^{b,1}_{\text{unif}}(\mathbb{R}^N)}) < \infty$. Let $u_i(t) = u_i(\cdot, t; u_0, v_0)$ and $v_i(t) = v_i(\cdot, t; u_0, v_0)$ $(i = 1, 2)$. For every $t \in [0, T]$, we have that

$$\begin{aligned}
\|u_1(t) - u_2(t)\|_\infty \leq\ & \chi C \sup_{0 \leq \tau \leq T} (\|\nabla v_1(\tau)\|_\infty) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda(t-s)} \|u_1(s) - u_2(s)\|_\infty ds \\
& + \chi C \sup_{0 \leq \tau \leq T} (\|u_2(\tau)\|_\infty) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda(t-s)} \|\nabla(v_2(s) - v_1(s))\|_\infty ds \\
& + (a + \lambda + b \sup_{0 \leq \tau \leq T} (\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty)) \int_0^t e^{-\lambda(t-s)} \|u_1(s) - u_2(s)\|_\infty ds,
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla(v_1(t) - v_2(t))\|_\infty &\leq \mu \int_0^t \|\nabla e^{(t-s)(\Delta - \lambda I)}(u_1(s) - u_2(s))\|_\infty ds \\
&\leq \mu C \int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda(t-s)} \|u_1(s) - u_2(s)\|_\infty ds.
\end{aligned}$$

Let $u(t) = u_1(t) - u_2(t)$, $v(t) = v_1(t) - v_2(t)$. We then have

$$\|u(t)\|_\infty + \|\nabla v(t)\|_\infty \leq M \int_0^t (t-s)^{-\frac{1}{2}} \left(\|u(s)\|_\infty + \|\nabla v(s)\|_\infty\right) ds,$$

where

$$M = \chi C \sup_{0\le\tau\le T}(\|\nabla v_1(\tau)\|_\infty) + \chi C \sup_{0\le\tau\le T}(\|u_2(\tau)\|_\infty) + \left(a+\lambda+b\sup_{0\le\tau\le T}(\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty)\right)\sqrt{T} + \mu C < \infty.$$

By Lemma 3.2, we get $\|u(t)\|_\infty \equiv 0$. Thus, $u_1(t) \equiv u_2(t)$ for all $0 \le t \le T$. Since $v(t) = \mu\int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)ds$, then $v(t) \equiv 0$. Hence, $v_1(t) \equiv v_2(t)$ for all $0 \le t \le T$. Since $T < T_{\max}$ was arbitrary chosen, then $u_1(t) \equiv u_2(t)$, $v_1(t) \equiv v_2(t)$ for all $0 \le t < T_{\max}$. The theorem is thus proved. □

3.4 Global existence and boundedness of classical solutions

This section is devoted to the study of the global existence and boundedness of classical solutions of (1.4) with given initial functions and prove Theorem 3.2. Again, throughout this section, unless specified otherwise, C denotes a generic constant independent of u, v and may be different at different places.

Proof of Theorem 3.2. Assume $b > \frac{N\mu\chi}{4}$. It can be proved by properly modifying the arguments in [70, Lemma 3.1].

First, we have

$$\frac{1}{2}\frac{d}{dt}|\nabla v|^2 = \sum_{i=1}^N v_{x_i}(v_t)_{x_i}.$$

From the second equation of (1.4), we have

$$\frac{1}{2}\frac{d}{dt}|\nabla v|^2 = \sum_{i=1}^N v_{x_i}(\Delta v - \lambda v + \mu u)_{x_i} = \nabla v\cdot\nabla(\Delta v) - \lambda|\nabla v|^2 + \mu\nabla v\cdot\nabla u. \tag{3.15}$$

Note that $\nabla v\cdot\nabla(\Delta v) = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$, (3.15) becomes

$$\frac{\chi}{2\mu}\frac{d}{dt}|\nabla v|^2 = \frac{\chi}{2\mu}\Delta|\nabla v|^2 - \frac{\chi}{\mu}\left|D^2 v\right|^2 - \frac{\chi\lambda}{\mu}|\nabla v|^2 + \chi\nabla v\cdot\nabla u. \tag{3.16}$$

Next, by the first equation of (1.4) and (3.16), we get

$$\frac{d}{dt}\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] = \Delta\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - \frac{\chi}{\mu}\left|D^2 v\right|^2 - \frac{\chi\lambda}{\mu}\left|\nabla v\right|^2 - \chi u\Delta v + u(a-bu). \quad (3.17)$$

By Young's inequality, we have

$$|u\Delta v| \le \frac{N\mu}{4}u^2 + \frac{1}{\mu}\left|D^2 v\right|^2.$$

Combining this with (3.17), we have

$$\begin{aligned}\frac{d}{dt}\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] &\le \Delta\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - \frac{\chi}{\mu}\left|D^2 v\right|^2 - \frac{\chi\lambda}{\mu}\left|\nabla v\right|^2 + \chi\left|u\Delta v\right| + u(a-bu)\\ &\le \Delta\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - \frac{\chi\lambda}{\mu}\left|\nabla v\right|^2 + \frac{N\mu\chi}{4}u^2 + u(a-bu)\\ &= \Delta\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - 2\lambda\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - (b-\frac{N\mu\chi}{4})\big(u-\frac{2(2\lambda+a)}{4b-N\mu\chi}\big)^2\\ &\quad + (b-\frac{N\mu\chi}{4})\frac{4(2\lambda+a)^2}{(4b-N\mu\chi)^2}.\end{aligned}$$

Since $b > \frac{N\mu\chi}{4}$, then for $0 < t < T_{max}$, we have

$$\frac{d}{dt}\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] \le \Delta\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] - 2\lambda\big[u+\frac{\chi}{2\mu}\left|\nabla v\right|^2\big] + \frac{(2\lambda+a)^2}{(4b-N\mu\chi)}. \quad (3.18)$$

By the comparison principle for parabolic equations, we have

$$u+\frac{\chi}{2\mu}\left|\nabla v\right|^2 \le \max\{\|u_0\|_\infty + \frac{\chi}{2\mu}\|\nabla v_0\|_\infty^2, \frac{(2\lambda+a)^2}{2\lambda(4b-N\mu\chi)}\} \quad \forall\ 0\le t < T_{max}, x\in\mathbb{R}^N.$$

Let $M = \max\{\|u_0\|_\infty + \frac{\chi}{2\mu}\|\nabla v_0\|_\infty^2, \frac{(2\lambda+a)^2}{2\lambda(4b-N\mu\chi)}\}$, then

$$u(x,t;u_0,v_0) \le M \quad \forall\ 0\le t < T_{max}, x\in\mathbb{R}^N, \quad (3.19)$$

and

$$|\nabla v(x,t;u_0,v_0)| \le \sqrt{\frac{2\mu M}{\chi}} \quad \forall\ 0\le t < T_{max}, x\in\mathbb{R}^N. \quad (3.20)$$

From the second equation of (1.4), by the variation of constant formula,

$$v(\cdot,t;u_0,v_0) = e^{t(\Delta-\lambda I)}v_0 + \mu\int_0^t e^{(t-s)(\Delta-\lambda I)}u(s)ds.$$

Thus,

$$\begin{aligned}\|v(\cdot,t;u_0,v_0)\|_\infty &\le e^{-\lambda t}\|v_0\|_\infty + \mu\int_0^t e^{-\lambda(t-s)}\|u(s)\|_\infty ds\\ &\le \|v_0\|_\infty + \frac{\mu M}{\lambda} \quad \forall\ 0\le t<T_{\max},\ x\in\mathbb{R}^N. \end{aligned} \tag{3.21}$$

In view of (3.19), (3.20) and (3.21), we obtain that $\limsup_{t\to T_{\max}}\|u(\cdot,t;u_0,v_0)\|_\infty$ is finite and that $\limsup_{t\to T_{\max}}\|v(\cdot,t;u_0,v_0)\|_{C^{b,1}_{\mathrm{unif}}(\mathbb{R}^N)}$ is also finite. Therefore, it follows from the blow-up criterion that $T_{\max}=\infty$ and the solution $(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$ is bounded for $(x,t)\in\mathbb{R}^N\times(0,\infty)$. By (3.18) and the comparison principle for parabolic equations,

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N} u(\cdot,t;u_0,v_0) \le \frac{(2\lambda+a)^2}{2\lambda(4b-N\mu\chi)} \tag{3.22}$$

and

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N} |\nabla v(\cdot,t;u_0,v_0)| \le \sqrt{\frac{\mu(2\lambda+a)^2}{\chi\lambda(4b-N\mu\chi)}}. \tag{3.23}$$

Finally, we prove (3.6). To this end, let

$$U(x,t) = u(x,t;u_0,v_0) - \frac{a}{b} \quad \text{and} \quad V(x,t) = v(x,t;u_0,v_0) - \frac{\mu}{\lambda}\frac{a}{b}.$$

Then (U,V) solves

$$\begin{cases} U_t = \Delta U - \chi\nabla\cdot(u(x,t;u_0,v_0)\nabla V) - aU - bU^2, \quad x\in\mathbb{R}^N,\ t>0,\\ V_t = \Delta V - \lambda V + \mu U, \quad x\in\mathbb{R}^N,\ t>0,\\ U(x,0) = u_0(x) - \frac{a}{b}, \quad x\in\mathbb{R}^N,\\ V(x,0) = v_0(x) - \frac{\mu}{\lambda}\frac{a}{b}, \quad x\in\mathbb{R}^N. \end{cases} \tag{3.24}$$

By (3.24) and the similar arguments as those used at the beginning of the proof, we have

$$\frac{d}{dt}\big[U+\frac{\chi}{2\mu}\left|\nabla V\right|^2\big]=\Delta\big[U+\frac{\chi}{2\mu}\left|\nabla V\right|^2\big]-\chi u\Delta V-aU-bU^2-\frac{\chi}{\mu}\left|D^2V\right|^2-\frac{\chi\lambda}{\mu}\left|\nabla V\right|^2.$$

It follows from Young's inequality and $U=u-\frac{a}{b}$ that

$$|u\Delta V|\leq\frac{N\mu}{4}u^2+\frac{1}{\mu}\left|D^2V\right|^2=\frac{N\mu}{4}U^2+\frac{N\mu a}{2b}U+\frac{a^2N\mu}{4b^2}+\frac{1}{\mu}\left|D^2V\right|^2.$$

Combining these and $b>\frac{N\chi\mu}{4}$, and $\lambda\geq\frac{a}{2}$, we have

$$\frac{d}{dt}\big[U+\frac{\chi}{2\mu}\left|\nabla V\right|^2\big]\leq\Delta\big[U+\frac{\chi}{2\mu}\left|\nabla V\right|^2\big]-a\big(U+\frac{\chi}{2\mu}\left|\nabla V\right|^2\big)+\frac{a^2N\mu\chi}{b(4b-N\mu\chi)}.$$

This together with the comparison principle for parabolic equations and $4b>N\chi\mu$ implies

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N}\big(U(\cdot,t)+\frac{\chi}{2\mu}\left|\nabla V(\cdot,t)\right|^2\big)\leq\frac{aN\mu\chi}{b(4b-N\mu\chi)}.$$

We then have

$$\limsup_{t\to\infty}\|U_+(\cdot,t)\|_\infty\leq\frac{aN\mu\chi}{b(4b-N\mu\chi)}.\tag{3.25}$$

and

$$\limsup_{t\to\infty}\|u(\cdot,t;u_0,v_0)\|_\infty\leq\frac{a}{b}+\frac{aN\mu\chi}{b(4b-N\mu\chi)}=\frac{4a}{4b-N\mu\chi}.$$

Hence (3.6) holds and the theorem is thus proved. □

3.5 Asymptotic behavior of global classical solutions

In this section, we discuss the asymptotic behavior of global bounded classical solutions of (1.4) with strictly positive initial functions and prove Theorem 3.3. The proof of Theorem 3.3 can be done by following the ideas given in [70]. Throughout this section, We assume that $\inf_{x\in\mathbb{R}^N}u_0(x)>0$ for $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ and $v_0(x)\geq0$ for $v_0\in C^{b,1}_{\rm unif}(\mathbb{R}^N)$. We also assume that $b>\frac{N\chi\mu}{4}$ and $\lambda\geq\frac{a}{2}$, and $\theta=\frac{N\mu\chi}{4b}$. Hence $0<\theta<1$. We denote by

$(u(x,t;u_0,v_0), v(x,t;u_0,v_0))$ the global bounded classical solution of (1.4) associated with initial function (u_0, v_0). Again, throughout this section, unless specified otherwise, C denotes a generic constant independent of u, v and may be different at different places.

Recall that $U(x,t) = u(x,t;u_0,v_0) - \frac{a}{b}$, $V(x,t) = v(x,t;u_0,v_0) - \frac{\mu}{\lambda}\frac{a}{b}$. Then (U, V) solves (3.24). We first present some lemmas on the estimates of U and V. The first lemma provides an estimate of $\|\nabla V(\cdot,t)\|_\infty$.

Lemma 3.3. *There exists $C_0 = C_0(a, \mu, \lambda) > 0$ such that*

$$\limsup_{t\to\infty} \|\nabla V(\cdot,t)\|_\infty \le \frac{C_0}{b(1-\theta)}.$$

Proof. By (3.6), we can fix a sufficiently large t_1 such that

$$\|u(\cdot,t;u_0,v_0)\|_\infty \le \frac{2a}{b(1-\theta)} \quad \forall\ t > t_1. \tag{3.26}$$

By the variation of constant formula, we have that

$$v(\cdot,t;u_0,v_0) = e^{(t-t_1)(\Delta-\lambda I)} v(\cdot,t_1;u_0,v_0) + \mu \int_{t_1}^t e^{(t-s)(\Delta-\lambda I)} u(\cdot,s;u_0,v_0) ds \quad \forall\ t \ge t_1.$$

Note that $\nabla V(x,t) = \nabla v(x,t;u_0,v_0)$. Thus, we have

$$\begin{aligned}\|\nabla V(\cdot,t)\|_\infty \le & \|\nabla e^{(t-t_1)(\Delta-\lambda I)} v(\cdot,t_1;u_0,v_0)\|_\infty \\ & + \mu \int_{t_1}^t \|\nabla e^{(t-s)(\Delta-\lambda I)} u(\cdot,s;u_0,v_0)\|_\infty ds \quad \forall\ t \ge t_1.\end{aligned} \tag{3.27}$$

By (3.10) and (3.26), we have

$$\|\nabla e^{(t-t_1)(\Delta-\lambda I)} v(\cdot,t_1;u_0,v_0)\|_\infty \le C(t-t_1)^{-\frac{1}{2}} e^{-\lambda(t-t_1)} \|v(\cdot,t_1;u_0,v_0)\|_\infty \quad \forall\ t \ge t_1, \tag{3.28}$$

and

$$\mu \int_{t_1}^{t} \|\nabla e^{(t-s)(\Delta-\lambda I)} u(\cdot,s;u_0,v_0)\|_\infty ds \le \mu C \int_{t_1}^{t} (t-s)^{-\frac{1}{2}} e^{-\lambda(t-s)} \|u(\cdot,s;u_0,v_0)\|_\infty ds$$
$$\le \frac{2a\mu C\sqrt{\pi}}{b(1-\theta)\sqrt{\lambda}} \quad \forall\, t \ge t_1. \tag{3.29}$$

The lemma with $C_0 = C_0(a,\mu,\lambda) = \frac{2a\mu C\sqrt{\pi}}{\sqrt{\lambda}}$ then follows from (3.27), (3.28) and (3.29). □

The second lemma provides an estimate of $\|\Delta V(\cdot,t)\|_\infty$.

Lemma 3.4. *There exists $C_1 = C_1(\lambda,a,N) > 0$ such that the following holds*

$$\limsup_{t\to\infty} \|\Delta V(\cdot,t)\|_\infty \le \frac{\mu C_1}{b(1-\theta)^2}.$$

Proof. Fix β and γ such that $\gamma \in (1,\frac{3}{2})$ and $\gamma - 1 < \beta < \frac{1}{2}$. We first prove that there exists $\tilde{C}_1 = \tilde{C}_1(\lambda,a,N) > 0$ such that

$$\limsup_{t\to\infty} \|A^\beta U(\cdot,t)\|_\infty \le \frac{\tilde{C}_1}{b(1-\theta)^2}, \tag{3.30}$$

where $A^\beta = (\lambda I - \Delta)^\beta$. To this end, let $t_1 > 0$ be such that (3.26) holds, then

$$\|U(\cdot,t)\|_\infty \le \|U(\cdot,t) - \frac{a}{b}\|_\infty + \frac{a}{b} \le \frac{2a}{b(1-\theta)} + \frac{a}{b} \le \frac{3a}{b(1-\theta)} \quad \forall\ t > t_1. \tag{3.31}$$

By lemma 3.3, we can fix $t_2 \ge t_1$ sufficiently large such that

$$\|\nabla v(\cdot,t;u_0,v_0)\|_\infty = \|\nabla V(\cdot,t)\|_\infty \le \frac{2C_0}{b(1-\theta)} \quad \forall\ t > t_2. \tag{3.32}$$

It follows from the variation of constant formula that

$$U(\cdot,t) = \underbrace{e^{(t-t_2)(\Delta-aI)}U(\cdot,t_2)}_{I_1} - \underbrace{\chi\int_{t_2}^{t} e^{(t-s)(\Delta-aI)}\nabla\cdot(u(\cdot,s;u_0,v_0)\nabla V(\cdot,s))ds}_{I_2}$$
$$\underbrace{-\,b\int_{t_2}^{t} e^{(t-s)(\Delta-aI)}U(\cdot,s)^2 ds}_{I_3}.$$

By (3.11), Lemma 3.1, (3.26), (3.31) and (3.32), we have

$$\|A^\beta I_1\|_\infty = \|A^\beta e^{(t-t_2)(\Delta-aI)}U(\cdot,t_2)\|_\infty \le C(t-t_2)^{-\beta}e^{-a(t-t_2)}\|U(\cdot,t_2)\|_\infty \quad \forall\ t\ge t_2,$$

$$\begin{aligned}\|A^\beta I_2\|_\infty &\le \chi\int_{t_2}^{t}\|A^\beta e^{(t-s)(\Delta-aI)}\nabla\cdot(u(\cdot,s)\nabla V(\cdot,s))ds\|_\infty ds\\ &\le \chi C\int_{t_2}^{t}(t-s)^{-\beta-\frac{1}{2}}e^{-a(t-s)}\|u(\cdot,s)\|_\infty\|\nabla V(\cdot,s)\|_\infty ds\\ &\le \chi C a^{\beta+\frac{1}{2}}\frac{4C_0}{b^2(1-\theta)^2}\quad \forall\ t\ge t_2,\end{aligned}$$

and

$$\begin{aligned}\|A^\beta I_3\|_\infty &\le b\int_{t_2}^{t}\|A^\beta e^{(t-s)(\Delta-aI)}U(\cdot,s)^2\|_\infty ds\\ &\le bC\int_{t_2}^{t}(t-s)^{-\beta}e^{-a(t-s)}\|U(\cdot,s)\|_\infty^2 ds\\ &\le \frac{9C}{b(1-\theta)^2}a^{\beta+1}\quad \forall\ t\ge t_2.\end{aligned}$$

We then obtain that

$$\|A^\beta U(\cdot,t)\|_\infty \le C(t-t_2)^{-\beta}e^{-a(t-t_2)}\|U(\cdot,t_2)\|_\infty + \chi C a^{\beta+\frac{1}{2}}\frac{4C_0}{b^2(1-\theta)^2} + \frac{9C}{b(1-\theta)^2}a^{\beta+1}.$$

This together with $\frac{\chi}{b} = \frac{4\theta}{N\mu}$ and $0<\theta<1$ implies (3.30) with $\tilde{C}_1 = \tilde{C}_1(\lambda,a,N) = \frac{32Ca^{\beta+\frac{3}{2}}\sqrt{\pi}}{N\sqrt{\lambda}} + 9Ca^{\beta+1}$.

Next, by (3.30), we can fix $t_3 \geq t_2$ sufficiently large such that

$$\|A^\beta U(\cdot,t)\|_\infty \leq \frac{2\tilde{C}_1}{b(1-\theta)^2} \quad \forall\ t > t_3. \tag{3.33}$$

By the variation of constant formula, we have

$$V(\cdot,t) = e^{(t-t_3)(\Delta-\lambda I)}V(\cdot,t_3) + \mu\int_{t_3}^t e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)ds \quad \forall\ t > t_3.$$

Note that

$$\|\Delta V(\cdot,t)\|_\infty \leq C\|A^\gamma V(\cdot,t)\|_\infty.$$

By (3.11),

$$\|A^\gamma e^{(t-t_3)(\Delta-\lambda I)}V(\cdot,t_3)\|_\infty \leq C_\gamma(t-t_3)^{-\gamma}e^{-\lambda(t-t_3)}\|V(\cdot,t_3)\|_\infty \to 0 \quad \text{as} \quad t\to\infty.$$

By (3.11) and (3.33), we have

$$\begin{aligned}
\mu\int_{t_3}^t \|A^\gamma e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)\|_\infty ds &= \mu\int_{t_3}^t \|A^{\gamma-\beta}e^{(t-s)(\Delta-\lambda I)}A^\beta U(\cdot,s)\|_\infty ds \\
&\leq \mu C\frac{2\tilde{C}_1}{b(1-\theta)^2}\int_{t_3}^t (t-s)^{-(\gamma-\beta)}e^{-\lambda(t-s)}ds \\
&= \mu C\frac{2\tilde{C}_1}{b(1-\theta)^2}\lambda^{\gamma-\beta-1} \quad \forall t \geq t_3.
\end{aligned}$$

It then follows that the Lemma holds with $C_1 = C_1(\lambda,a,N) = 2C\tilde{C}_1\lambda^{\gamma-\beta-1}$. □

Observe that, by (3.31),

$$\limsup_{t\to\infty}\|U(\cdot,t)\|_\infty \leq \frac{3a}{b(1-\theta)}.$$

In the following lemma, we provide a better estimate of $\|U(\cdot,t)\|_\infty$.

Lemma 3.5. *There exists $C_2 = C_2(\lambda,a,N) > 0$ such that*

$$\limsup_{t\to\infty}\|U(\cdot,t)\|_\infty \leq \frac{C_2\theta}{b(1-\theta)^2}.$$

Proof. We first prove that there exists $\tilde{C}_2 = \tilde{C}_2(\lambda, a, N) > 0$ such that

$$\liminf_{t\to\infty}\big(\inf_{x\in\mathbb{R}^N} u(x,t;u_0,v_0)\big) \geq \frac{a}{b} - \frac{\tilde{C}_2\theta}{b(1-\theta)^2}. \tag{3.34}$$

By Lemma 3.4, we can choose t_4 large enough such that

$$\|\Delta v(\cdot,t;u_0,v_0)\|_\infty = \|\Delta V(\cdot,t)\|_\infty \leq \frac{2\mu C_1}{b(1-\theta)^2} \quad \forall\ t \geq t_4.$$

Therefore,

$$u_t \geq \Delta u - \chi\nabla v\cdot\nabla u + u(a - \frac{2\mu C_1\chi}{b(1-\theta)^2}) - bu^2 \quad \forall\ x\in\mathbb{R},\ t > t_4.$$

By the comparison principle for parabolic equations, we have

$$\liminf_{t\to\infty}\big(\inf_{x\in\mathbb{R}^N} u(x,t;u_0,v_0)\big) \geq \frac{a - \frac{2\mu C_1\chi}{b(1-\theta)^2}}{b}.$$

This together with $\frac{\chi}{b} = \frac{4\theta}{N\mu}$ implies that (3.34) holds with $\tilde{C}_2 = \tilde{C}_2(\lambda, a, N) = \frac{8C_1}{N}$.

Next, by (3.34),

$$\limsup_{t\to\infty}\|U_-(\cdot,t)\|_\infty \leq \frac{\tilde{C}_2\theta}{b(1-\theta)^2}.$$

By (3.25),

$$\limsup_{t\to\infty}\|U_+(\cdot,t)\|_\infty \leq \frac{a\theta}{b(1-\theta)}.$$

Thus, the Lemma holds if we let $C_2 = C_2(\lambda, a, N) = \max\{\tilde{C}_2, a\}$. □

We now prove Theorem 3.3.

Proof of Theorem 3.3. First of all, let $\theta_0 \in (0,1)$ be defined

$$\theta_0 = \sup\{\theta\in(0,1)\,|\,\frac{2C_2\theta}{(1-\theta)^2 a} \leq \frac{1}{6} \quad\text{and}\quad \frac{8C\lambda^{-\frac{1}{2}}a^{\frac{1}{2}}\pi\theta}{N(1-\theta)} \leq \frac{1}{12}\}.$$

Then

$$\frac{2C_2\theta_0}{(1-\theta_0)^2 a} \leq \frac{1}{6} \quad\text{and}\quad \frac{8C\lambda^{-\frac{1}{2}}a^{\frac{1}{2}}\pi\theta_0}{N(1-\theta_0)} \leq \frac{1}{12}. \tag{3.35}$$

Let $K = \frac{N}{4\theta_0} > \frac{N}{4}$. We prove that for any $b > K\chi\mu$, there are $C > 0$ and $\alpha > 0$ such that

$$\|U(\cdot,t)\|_\infty \le Ce^{-\alpha t} \quad \forall\ t > 0. \tag{3.36}$$

To this end, first fix $b > K\chi\mu$. Note that $\theta = \frac{N\mu\chi}{4b} < \theta_0$. Then by (3.35), there is $0 < \alpha < \min\{\lambda, a\}$ such that

$$\frac{2C_2\theta}{(1-\theta)^2(a-\alpha)} \le \frac{1}{6} \quad \text{and} \quad \frac{8aC(\lambda-\alpha)^{-\frac{1}{2}}(a-\alpha)^{-\frac{1}{2}}\pi\theta}{N(1-\theta)} \le \frac{1}{12}. \tag{3.37}$$

Fix $0 < \alpha < \min\{\lambda, a\}$ such that (3.37) holds and fix $B > 0$ large enough such that

$$\frac{2C_2\theta}{b(1-\theta)^2} \le \frac{B}{6} \quad \text{and} \quad \frac{16aC_0(a-\alpha)^{-\frac{1}{2}}\sqrt{\pi}\theta}{N\mu b(1-\theta)^2} \le \frac{B}{12}. \tag{3.38}$$

By Lemma 3.5, there exists $t_0 \ge t_4 (\ge t_3 \ge t_2 \ge t_1)$ such that

$$\|U(\cdot,t)\|_\infty \le \frac{2C_2\theta}{b(1-\theta)^2} \quad \forall\ t \ge t_0. \tag{3.39}$$

Consider the set

$$S = \{T_0 \ge t_0 \,|\, \|U(\cdot,t)\|_\infty \le Be^{-\alpha(t-t_0)},\ \forall\ t \in [t_0, T_0]\}.$$

By (3.38) and (3.39), we have $\|U(\cdot,t_0)\|_\infty \le \frac{B}{6}$. Thus, S is not empty and $T := \sup S \in (t_0, \infty]$ is well-defined. Hence, to prove (3.36), it is sufficient to prove that

$$T = \infty. \tag{3.40}$$

Next, by the variation of constant formula,

$$\|\nabla V(\cdot,t)\|_\infty = \|\nabla e^{(t-t_0)(\Delta-\lambda I)}V(\cdot,t_0) + \mu\int_{t_0}^t \nabla e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)ds\|_\infty \quad \forall\ t \ge t_0.$$

By (3.9), (3.32), we have

$$\begin{aligned}\|\nabla e^{(t-t_0)(\Delta-\lambda I)}V(\cdot,t_0)\|_\infty &= \|e^{(t-t_0)(\Delta-\lambda I)}\nabla V(\cdot,t_0)\|_\infty \le e^{-\lambda(t-t_0)}\|\nabla V(\cdot,t_0)\|_\infty \\ &\le e^{-\lambda(t-t_0)}\frac{2C_0}{b(1-\theta)} \le \frac{2C_0}{b(1-\theta)}e^{-\alpha(t-t_0)} \quad \forall\, t \ge t_0. \end{aligned} \tag{3.41}$$

Furthermore, (3.10) along with the definition of T gives us that

$$\begin{aligned} &\mu\int_{t_0}^t \|\nabla e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)\|_\infty ds \\ &\le \mu C\int_{t_0}^t (t-s)^{-\frac{1}{2}}e^{-\lambda(t-s)}\|U(\cdot,s)\|_\infty ds \\ &\le \mu BC\lambda^{-\frac{1}{2}}\Big(\int_0^{\lambda(t-t_0)}\sigma^{-\frac{1}{2}}e^{-(1-\frac{\alpha}{\lambda})\sigma}d\sigma\Big)e^{-\alpha(t-t_0)} \\ &\le \mu BC\lambda^{-\frac{1}{2}}(1-\frac{\alpha}{\lambda})^{-\frac{1}{2}}\sqrt{\pi}e^{-\alpha(t-t_0)} \quad \forall\ t\in(t_0,T). \end{aligned} \tag{3.42}$$

Combing (3.41) and (3.42), we get that

$$\|\nabla V(\cdot,t)\|_\infty \le \Big\{\frac{2C_0}{b(1-\theta)} + \mu BC\lambda^{-\frac{1}{2}}(1-\frac{\alpha}{\lambda})^{-\frac{1}{2}}\sqrt{\pi}\Big\}e^{-\alpha(t-t_0)} \quad \forall\ t\in(t_0,T). \tag{3.43}$$

By the variation of constant formula again, we have

$$\begin{aligned}\|U(\cdot,t)\|_\infty \le \|e^{(t-t_0)(\Delta-aI)}U(\cdot,t_0)\|_\infty &+ \chi\int_{t_0}^t \|e^{(t-s)(\Delta-aI)}\nabla\cdot(u(\cdot,s;u_0,v_0)\nabla V(\cdot,s))\|_\infty ds \\ &+ b\int_{t_0}^t \|e^{(t-s)(\Delta-aI)}U^2(\cdot,s)\|_\infty ds \quad \forall\ t>t_0.\end{aligned}$$

It follows from (3.38) and (3.39) that

$$\begin{aligned}\|e^{(t-t_0)(\Delta-aI)}U(\cdot,t_0)\|_\infty &\le e^{-a(t-t_0)}\|U(\cdot,t_0)\|_\infty \le e^{-a(t-t_0)}\frac{2C_2\theta}{b(1-\theta)^2} \\ &\le e^{-\alpha(t-t_0)}\frac{2C_2\theta}{b(1-\theta)^2} \le \frac{B}{6}e^{-\alpha(t-t_0)} \quad \forall\ t>t_0. \end{aligned}\tag{3.44}$$

By Lemma 3.1, (3.26), (3.37), (3.38), (3.43), and $\frac{\chi}{b}=\frac{4\theta}{N\mu}$, we have

$$\begin{aligned}
&\chi\int_{t_0}^{t}\|e^{(t-s)(\Delta-aI)}\nabla\cdot(u(\cdot,s;u_0,v_0)\nabla V(\cdot,s))\|_\infty ds\\
&\le \chi C\int_{t_0}^{t}e^{-a(t-s)}(t-s)^{-\frac{1}{2}}\frac{2a}{b(1-\theta)}\{\frac{2C_0}{b(1-\theta)}+\mu BC\lambda^{-\frac{1}{2}}(1-\frac{\alpha}{\lambda})^{-\frac{1}{2}}\sqrt{\pi}\}e^{-\alpha(s-t_0)}ds\\
&\le\{\frac{16\theta aC_0}{bN\mu(1-\theta)^2}+\frac{8\theta aBC(\lambda-\alpha)^{-\frac{1}{2}}\sqrt{\pi}}{N(1-\theta)}\}(a-\alpha)^{-\frac{1}{2}}\sqrt{\pi}e^{-\alpha(t-t_0)}\\
&\le\frac{B}{6}e^{-\alpha(t-t_0)}\quad\forall\ t\in(t_0,T).
\end{aligned}\tag{3.45}$$

By (3.37), (3.39), and the definition of T, we have

$$\begin{aligned}
b\int_{t_0}^{t}\|e^{(t-s)(\Delta-aI)}U^2(\cdot,s))\|_\infty ds&\le b\int_{t_0}^{t}e^{-a(t-s)}\|U(\cdot,s)\|_\infty\|U(\cdot,s)\|_\infty ds\\
&\le b\int_{t_0}^{t}e^{-a(t-s)}\frac{2C_2\theta}{b(1-\theta)^2}Be^{-\alpha(s-t_0)}ds\\
&\le\frac{2C_2\theta}{(1-\theta)^2}\cdot B\cdot\frac{1}{a-\alpha}\cdot e^{-\alpha(t-t_0)}\\
&\le\frac{1}{6}Be^{-\alpha(t-t_0)}\quad\forall\ t\in(t_0,T).
\end{aligned}\tag{3.46}$$

Combing (3.44), (3.45) and (3.46), we can obtain that

$$\|U(\cdot,t)\|_\infty\le 3\cdot\frac{1}{6}Be^{-\alpha(t-t_0)}=\frac{B}{2}e^{-\alpha(t-t_0)}\quad\forall\ t\in(t_0,T),$$

which together with the continuity of U implies that T cannot be finite. This shows (3.40), and (3.36) then follows.

We now prove that there is $C>0$ such that

$$\|V(\cdot,t)\|_\infty\le Ce^{-\alpha t}\quad\forall t>0.\tag{3.47}$$

By variation of constants formula associated with the second equation in (3.24), we get that

$$V(\cdot,t)=e^{t(\Delta-\lambda I)}\big(v_0-\frac{\mu}{\lambda}\frac{a}{b}\big)+\mu\int_0^t e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)ds\quad\forall\ t>0.$$

By (3.36), we have

$$\begin{aligned}\|V(\cdot,t)\|_\infty &\le \|e^{t(\Delta-\lambda I)}\big(v_0-\frac{\mu}{\lambda}\frac{a}{b}\big)\|_\infty+\mu\int_0^t \|e^{(t-s)(\Delta-\lambda I)}U(\cdot,s)\|_\infty ds\\ &\le e^{-\lambda t}\|v_0-\frac{\mu}{\lambda}\frac{a}{b}\|_\infty + C\mu\int_0^t e^{-\lambda(t-s)}e^{-\alpha s}ds\\ &= e^{-\lambda t}\|v_0-\frac{\mu}{\lambda}\frac{a}{b}\|_\infty+\frac{C\mu}{\lambda-\alpha}(e^{-\alpha t}-e^{-\lambda t})\quad \forall\ t>0.\end{aligned}$$

(3.47) then follows, and (3.36) and (3.47) establish (3.7). □

3.6 Lower bounds of spreading speeds

In this section, we investigate lower bounds of spreading speeds of global classical solutions of (1.4) with different initial functions and prove Theorems 3.4 (1), 3.5 (1) and 3.6 (1), and Theorem 3.7. Throughout this section, we assume that $b>\frac{N\mu\chi}{4}$.

We first prove some lemmas.

For any given $\xi\in S^{N-1}$ and $c\in\mathbb{R}$, let $\tilde u(x,t)=u(x+ct\xi,t)$ and $\tilde v(x,t)=v(x+ct\xi,t)$. Then (1.4) becomes

$$\begin{cases}\tilde u_t=\Delta\tilde u+c\xi\cdot\nabla\tilde u-\chi\nabla\cdot(\tilde u\nabla\tilde v)+\tilde u(a-b\tilde u) \quad x\in\mathbb{R}^N,\\ \tilde v_t=\Delta\tilde v+c\xi\cdot\nabla\tilde v-\lambda\tilde v+\mu\tilde u,\quad x\in\mathbb{R}^N.\end{cases}\tag{3.48}$$

In the following, $(\tilde u(x,t;\xi,c,u_0,v_0),\tilde v(x,t;\xi,c,u_0,v_0))$ denotes the classical solution of (3.48) with $\tilde u(x,0;\xi,c,u_0,v_0)=u_0\in X_1^+$ and $\tilde v(x,0;\xi,c,u_0,v_0))=v_0\in X_2^+$.

For any given $0<\epsilon<2\sqrt{a}$, fix $0<\bar a<a$ such that

$$4\bar a-c^2\ge\epsilon\sqrt{a}\quad\forall -2\sqrt{a}+\epsilon\le c\le 2\sqrt{a}-\epsilon.\tag{3.49}$$

Let

$$l=\frac{2\pi\sqrt{N}}{(\epsilon\sqrt{a})^{\frac12}}\tag{3.50}$$

and

$$\zeta(c,\bar a) = \frac{4\bar a - c^2 - \frac{N\pi^2}{l^2}}{4}. \tag{3.51}$$

Then $\zeta(c,\bar a) \ge \frac{3\epsilon\sqrt{a}}{16} > 0$ for any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$. Let

$$D_l = \{x \in \mathbb{R}^N \mid |x_i| < l \text{ for } i = 1, 2, \cdots N\}.$$

For every $x \in \mathbb{R}^N$, and $r > 0$, we define

$$B_r(x) := \{y \in \mathbb{R}^N \,|\, |y - x| < r\}.$$

Lemma 3.6. *For any given $0 < \epsilon < 2\sqrt{a}$, let $\bar a$ and l be as in* (3.49) *and* (3.50)*. Then for any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$ and $\xi \in S^{N-1}$, $\zeta(c,\bar a)$ which is defined as in* (3.51) *is the principal eigenvalue of*

$$\begin{cases} \Delta\phi + c\xi\cdot\nabla\phi + \bar a\phi = \zeta\phi, \quad x \in D_l \\ \phi(x) = 0, \quad x \in \partial D_l, \end{cases} \tag{3.52}$$

and $\phi(x;\xi,c,\bar a) = e^{-\frac{c}{2}\xi\cdot x}\prod_{i=1}^N \cos\frac{\pi}{2l}x_i$ is a corresponding positive eigenfunction.

Proof. It follows from direct calculations. □

Lemma 3.7. *There are $M > 0$, $M_1 > 0$, and $0 < \theta < \frac{1}{2}$ such that for any $(u_0, v_0) \in X_1^+ \times X_2^+$, there is $T_0(u_0,v_0) > 1$ such that for any $c \in \mathbb{R}$, any $\xi \in S^{N-1}$, it holds that*

$$\begin{cases} \|\tilde u(\cdot,t;\xi,c,u_0,v_0)\|_\infty \le M \quad \forall\, t \ge T_0(u_0,v_0) \\ \|\tilde v(\cdot,t;\xi,c,u_0,v_0)\|_\infty \le M \quad \forall\, t \ge T_0(u_0,v_0) \\ \|\nabla\tilde v(\cdot,t;\xi,c,u_0,v_0)\|_\infty \le M \quad \forall\, t \ge T_0(u_0,v_0) \\ \|\Delta\tilde v(\cdot,t;\xi,c,u_0,v_0)\|_\infty \le M \quad \forall\, t \ge T_0(u_0,v_0) \end{cases} \tag{3.53}$$

and

$$\sup_{t,s\ge T_0(u_0,v_0)+1, t\ne s} \frac{\|\nabla\tilde v(\cdot,t;c,u_0,v_0) - \nabla\tilde v(\cdot,s;c,u_0,v_0)\|_\infty}{|t-s|^\theta} \le MM_1. \tag{3.54}$$

Proof. It suffices to prove (3.53) holds with $\tilde{u}(\cdot,t;\xi,c,u_0,v_0)$, $\tilde{v}(\cdot,t;\xi,c,u_0,v_0)$, $\nabla\tilde{v}(\cdot,t;\xi,c,u_0,v_0)$, and $\Delta\tilde{v}(\cdot,t;\xi,c,u_0,v_0)$ being replaced by $u(\cdot,t;u_0,v_0)$, $v(\cdot,t;u_0,v_0)$, $\nabla v(\cdot,t;u_0,v_0)$, $\Delta v(\cdot,t;u_0,v_0)$ respectively, where$(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$ is the classical solution of (1.4) with $u(x,0;u_0,v_0)=u_0$ and $v(x,0;u_0,v_0))=v_0$.

First, we obtain the upper bound for $\|u(\cdot,t;u_0,v_0)\|_\infty$ and $\|\nabla v(\cdot,t;u_0,v_0)\|_\infty$. It follows from (3.22) and (3.23) that there exists $T_1=T_1(u_0,v_0)$ such that

$$\|u(\cdot,t;u_0,v_0)\|_\infty \le \frac{(2\lambda+a)^2}{\lambda(4b-N\mu\chi)} \quad \forall\ t\ge T_1 \tag{3.55}$$

and

$$\|\nabla v(\cdot,t;u_0,v_0)\|_\infty \le 2\sqrt{\frac{\mu(2\lambda+a)^2}{\lambda\chi(4b-N\mu\chi)}} \quad \forall\ t\ge T_1. \tag{3.56}$$

Next, we obtain the upper bound for $\|v(\cdot,t;u_0,v_0)\|_\infty$. By the variation of constant formula, we have that

$$v(\cdot,t;u_0,v_0)=e^{(t-T_1)(\Delta-\lambda I)}v(\cdot,T_1;u_0,v_0)+\mu\int_{T_1}^t e^{(t-s)(\Delta-\lambda I)}u(\cdot,s;u_0,v_0)ds \quad \forall\ t\ge T_1.$$

Then by (3.9) and (3.55),

$$\begin{aligned}\|v(\cdot,t;u_0,v_0)\|_\infty &\le e^{-\lambda(t-T_1)}\|v(\cdot,T_1;u_0,v_0)\|_\infty+\mu\int_{T_1}^t e^{-\lambda(t-s)}\|u(\cdot,s;u_0,v_0)\|_\infty ds\\ &\le e^{-\lambda(t-T_1)}\|v(\cdot,T_1;u_0,v_0)\|_\infty+\frac{\mu(2\lambda+a)^2}{\lambda^2(4b-N\mu\chi)} \quad \forall\ t\ge T_1.\end{aligned}$$

This implies that

$$\limsup_{t\to\infty}\|v(\cdot,t;u_0,v_0)\|_\infty \le \frac{\mu(2\lambda+a)^2}{\lambda^2(4b-N\mu\chi)}.$$

It thus follows that there exists $T_2=T_2(u_0,v_0)>T_1$ such that

$$\|v(\cdot,t;u_0,v_0)\|_\infty \le \frac{2\mu(2\lambda+a)^2}{\lambda^2(4b-N\mu\chi)} \quad \forall\ t\ge T_2. \tag{3.57}$$

Now, we obtain the upper bound for$\|\Delta v(\cdot,t;u_0,v_0)\|_\infty$. Similar arguments to those used in Lemma 3.4 yield that

$$\limsup_{t\to\infty}\|\Delta v(\cdot,t;u_0,v_0)\|_\infty\le \mu C\lambda^{\gamma-\beta-1}\Big(\chi\lambda^{\beta-\frac{1}{2}}\frac{(2\lambda+a)^2}{\lambda(4b-N\mu\chi)}\sqrt{\frac{\mu(2\lambda+a)^2}{\lambda\chi(4b-N\mu\chi)}}+\big((a+\lambda)\frac{(2\lambda+a)^2}{\lambda(4b-N\mu\chi)}+b\frac{(2\lambda+a)^4}{\lambda^2(4b-N\mu\chi)^2}\big)\lambda^{\beta-1}\Big). \quad (3.58)$$

By (3.55), (3.56), (3.57), and (3.58), there are $M>0$ and $T_0(u_0,v_0)>1$ such that (3.53) holds. Finally, (3.54) follows form the arguments in the Claim of Theorem 3.1. □

In the following, $M>0$ is as in Lemma 3.7, and for given $\epsilon>0$, $l>0$ is as in (3.50). For given $\eta>0$, let $T=T(\eta)\ge 1$ be such that

$$e^{-\lambda T}M\le\eta, \quad (3.59)$$

and $L=L(\eta)\ge l$ be such that $B_L(0)\supset D_l$ and

$$\max\{\int_{\mathbb{R}^N\setminus B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)}e^{-|z|^2}dz,\int_{\mathbb{R}^N\setminus B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)}|z|e^{-|z|^2}dz\}\le\eta. \quad (3.60)$$

Lemma 3.8. *For any given $0<\epsilon<2\sqrt{a}$, let $\bar a$ and l be as in* (3.49) *and* (3.50)*. Let $0<\tilde a<a-\bar a$ be fixed. There is $\epsilon_0>0$ such that for any $0<\eta\le\epsilon_0$, any $(u_0,v_0)\in X_1^+\times X_2^+$, any $\xi\in S^{N-1}$, any $-2\sqrt{a}+\epsilon\le c\le 2\sqrt{a}-\epsilon$, any t_1,t_2 satisfying $T_0(u_0,v_0)\le t_1<t_2\le\infty$, and any ball $B_{2L(\eta)}$ with radius $2L(\eta)$ in $\mathbb{R}^N$, if*

$$\sup_{x\in B_{2L(\eta)}}\tilde u(x,t;\xi,c,u_0,v_0)\le\eta\quad\forall\, t_1\le t<t_2,$$

then

$$\sup_{x\in B_{L(\eta)}}\max\{\tilde v(x,t;\xi,c,u_0,v_0),|\partial_{x_i}\tilde v(x,t;\xi,c,u_0,v_0)|\}\le\tilde M\eta\quad\forall\, t_1+T(\eta)\le t<t_2 \quad (3.61)$$

and

$$\chi \sup_{x\in B_{L(\eta)}} \sum_{i,j=1}^{N} |\partial_{x_i x_j}\tilde{v}(x,t;\xi,c,u_0,v_0)| \le \tilde{a} \quad \forall\, t_1+T(\eta)+1\le t<t_2, \tag{3.62}$$

where

$$\tilde{M}=\max\big\{1+\frac{\mu M}{\lambda\pi^{\frac{N}{2}}}+\frac{\mu}{\lambda},\ 1+\frac{\mu}{\pi^{\frac{N}{2}}}\lambda^{-\frac{1}{2}}\Gamma(\frac{1}{2})M+\frac{\mu}{\pi^{\frac{N}{2}}}\lambda^{-\frac{1}{2}}\Gamma(\frac{1}{2})\big\}.$$

Proof. It suffices to prove the lemma for the ball centered at the origin with radius $2L(\eta)$. If not, we can make appropriate translation of $(\tilde{u}(x,t;\xi,c,u_0,v_0),\tilde{v}(x,t;\xi,c,u_0,v_0))$ for the space variable x to achieve this. We first prove that (3.61) holds for any $\eta>0$. Fix $t_1\ge T_0(u_0,v_0)$. Note that

$$\begin{aligned}
&\tilde{v}(x,t;\xi,c,u_0,v_0)\\
&=\int_{\mathbb{R}^N}\frac{e^{-\lambda(t-t_1)}}{(4\pi(t-t_1))^{\frac{N}{2}}}e^{-\frac{|x+c(t-t_1)\xi-y|^2}{4(t-t_1)}}\tilde{v}(y,t_1;\xi,c,u_0,v_0)dy\\
&\quad+\mu\int_{t_1}^{t}\int_{\mathbb{R}^N}\frac{e^{-\lambda(t-s)}}{(4\pi(t-s))^{\frac{N}{2}}}e^{-\frac{|x+c(t-s)\xi-y|^2}{4(t-s)}}\tilde{u}(y,s;\xi,c,u_0,v_0)dyds\\
&=\frac{1}{\pi^{\frac{N}{2}}}\int_{\mathbb{R}^N}e^{-\lambda(t-t_1)}e^{-|z|^2}\tilde{v}(x+c(t-t_1)\xi+2\sqrt{t-t_1}y,t_1;\xi,c,u_0,v_0)dz\\
&\quad+\frac{\mu}{\pi^{\frac{N}{2}}}\int_{t_1}^{t}\int_{\mathbb{R}^N}e^{-\lambda(t-s)}e^{-|z|^2}\tilde{u}(x+c(t-s)\xi+2\sqrt{t-s}z,s;\xi,c,u_0,v_0)dzds,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{x_i}\tilde{v}(x,t;\xi,c,u_0,v_0)\\
&=\int_{\mathbb{R}^N}\frac{(y_i-x_i-c(t-t_1)\xi)e^{-\lambda(t-t_1)}}{2(t-t_1)(4\pi(t-t_1))^{\frac{N}{2}}}e^{-\frac{|x+c(t-t_1)\xi-y|^2}{4(t-t_1)}}\tilde{v}(y,t_1;\xi,c,u_0,v_0)dy\\
&\quad+\mu\int_{t_1}^{t}\int_{\mathbb{R}^N}\frac{(y_i-x_i-c(t-s)\xi)e^{-\lambda(t-s)}}{2(t-s)(4\pi(t-s))^{\frac{N}{2}}}e^{-\frac{|x+c(t-s)\xi-y|^2}{4(t-s)}}\tilde{u}(y,s;\xi,c,u_0,v_0)dyds\\
&=\frac{1}{\pi^{\frac{N}{2}}}(t-t_1)^{-\frac{1}{2}}e^{-\lambda(t-t_1)}\int_{\mathbb{R}^N}ze^{-z^2}\tilde{v}(x+c(t-t_1)\xi+2\sqrt{t-t_1}z,t_1;\xi,c,u_0,v_0)dz\\
&\quad+\frac{\mu}{\pi^{\frac{N}{2}}}\int_{t_1}^{t}\int_{\mathbb{R}^N}(t-s)^{-\frac{1}{2}}e^{-\lambda(t-s)}ze^{-z^2}\tilde{u}(x+c(t-s)\xi+2\sqrt{t-s}z,s;\xi,c,u_0,v_0)dzds.
\end{aligned}$$

Hence, for $x \in B_L(0)$ and $t_1 + T \le t \le \min\{t_1 + 2T, t_2\}$, we have

$$\begin{aligned}\tilde{v}(x,t;\xi,c,u_0,v_0) \le e^{-\lambda T}M + \frac{\mu}{\pi^{\frac{N}{2}}}\left[\int_{t_1}^{t}\int_{\mathbb{R}^N\setminus B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)} e^{-\lambda(t-s)}e^{-|z|^2}dzds\right]M \\ + \frac{\mu}{\pi^{\frac{N}{2}}}\left[\int_{t_1}^{t}\int_{B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)} e^{-\lambda(t-s)}e^{-|z|^2}dzds\right]\sup_{t_1\le t<t_2,|z|\le 2L}\tilde{u}(z,t;\xi,c,u_0,v_0).\end{aligned}$$

By (3.59) and (3.60), if $\sup_{x\in B_{2L}(0)}\tilde{u}(x,t;\xi,c,u_0,v_0) \le \eta$ for any $t_1 \le t < t_2$, then

$$\tilde{v}(x,t;\xi,c,u_0,v_0) \le (1 + \frac{\mu M}{\lambda\pi^{\frac{N}{2}}} + \frac{\mu}{\lambda})\eta \quad \forall\ t_1 + T \le t \le \min\{t_1 + 2T, t_2\},\ \ |x| \le L. \tag{3.63}$$

For $t_1 + T \le t \le \min\{t_1 + 2T, t_2\}$, and $x \in B_L(0)$, we have

$$\begin{aligned}&|\partial_{x_i}\tilde{v}(x,t;\xi,c,u_0,v_0)| \\ &\le \frac{1}{\pi^{\frac{N}{2}}}T^{-\frac{1}{2}}e^{-\lambda T}M + \frac{\mu}{\pi^{\frac{N}{2}}}\left[\int_{t_1}^{t}\int_{\mathbb{R}^N\setminus B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)}(t-s)^{-\frac{1}{2}}e^{-\lambda(t-s)}|z|e^{-|z|^2}dzds\right]M \\ &+ \frac{\mu}{\pi^{\frac{N}{2}}}\left[\int_{t_1}^{t}\int_{B_{\frac{L-4T\sqrt{a}}{2\sqrt{2T}}}(0)}(t-s)^{-\frac{1}{2}}e^{-\lambda(t-s)}|z|e^{-|z|^2}dzds\right]\sup_{t_1\le t<t_2,|z|\le 2L}\tilde{u}(z,t;\xi,c,u_0,v_0).\end{aligned}$$

By (3.59) and (3.60), if $\sup_{x\in B_{2L}(0)}\tilde{u}(x,t;\xi,c,u_0,v_0) \le \eta$ for any $t_1 \le t < t_2$, then

$$|\partial_{x_i}\tilde{v}(x,t;\xi,c,u_0,v_0)| \le (1 + \frac{\mu}{\pi^{\frac{N}{2}}}\lambda^{-\frac{1}{2}}\Gamma(\frac{1}{2})M + \frac{\mu}{\pi^{\frac{N}{2}}}\lambda^{-\frac{1}{2}}\Gamma(\frac{1}{2}))\eta \tag{3.64}$$

for $t_1 + T \le t \le \min\{t_1 + 2T, t_2\}$ and $x \in B_L(0)$.

In the above arguments, replace t_1 by $t_1 + T$. We have (3.63) and (3.64) for $t_1 + 2T \le t \le \min\{t_1 + 3T, t_2\}$. Repeating this process, we have (3.63) and (3.64) for $t_1 + T \le t < t_2$. It then follows that (3.61) holds for any $\eta > 0$.

Next, we prove that there is $\epsilon_0 > 0$ such that (3.62) holds for $0 < \eta \le \epsilon_0$. Assume this is not true. Then there are $\eta_n \to 0$ as $n \to \infty$, $(u_n, v_n) \in X_1^+ \times X_2^+$, $\xi_n \in S^{N-1}$,

$-2\sqrt{a}+\epsilon \le c_n \le 2\sqrt{a}-\epsilon$, $T_0(u_n,v_n) \le t_{1n} < t_{1n}+T(\eta_n)+1 \le t_n < t_{2n}$ such that

$$\sup_{|x|\le 2L(\eta_n)} \tilde{u}(x,t;\xi_n,c_n,u_n,v_n) \le \eta_n, \quad \forall\, t_{1n} \le t < t_{2n}$$

and

$$\chi \sup_{|x|\le L(\eta_n)} \sum_{i,j=1}^{N} |\partial_{x_i x_j} \tilde{v}(x,t_n;\xi_n,c_n,u_n,v_n)| > \tilde{a}.$$

Let

$$(\tilde{u}_n(x,t),\tilde{v}_n(x,t)) = (\tilde{u}(x,t+t_n;\xi_n,c_n,u_n,v_n),\tilde{v}(x,t+t_n;\xi_n,c_n,u_n,v_n)).$$

Without loss of generality, we may assume that

$$(\tilde{u}_n(x,t),\tilde{v}_n(x,t)) \to (u^*(x,t),v^*(x,t))$$

as $n\to\infty$ locally uniformly on $(x,t)\in\mathbb{R}^N\times[-1,\infty)$, $\xi_n\to\xi^*$, $c_n\to c^*$ as $n\to\infty$ for some $\xi^*\in S^{N-1}$, $-2\sqrt{a}+\epsilon\le c^*\le 2\sqrt{a}-\epsilon$. Note that $v^*(x,t)$ satisfies

$$v_t^* = \Delta v^* + c^*\xi^*\cdot\nabla v^* - \lambda v^* + \mu u^*, \quad \forall\ x\in\mathbb{R}^N,\ t\ge -1$$

and

$$\chi \sup_{x\in\mathbb{R}^N} \sum_{i,j=1}^{N} |\partial_{x_i x_j} v^*(x,0)| \ge \tilde{a}.$$

By (3.61), we have

$$u^*(x,t)=0,\ v^*(x,t)=0 \quad \forall\ x\in\mathbb{R}^N,\ -1\le t\le 0.$$

Then by the comparison principle for parabolic equations,

$$v^*(x,t)=0 \quad \forall\, x\in\mathbb{R}^N,\ t\ge -1,$$

which is a contradiction. Hence (3.62) holds. □

Lemma 3.9. *For any given $0 < \epsilon < 2\sqrt{a}$, let $\bar{a}$ and l be as in* (3.49) *and* (3.50). *Let $\zeta_0 = \min_{-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon} \zeta(c, \bar{a}) > 0$, where $\zeta(c, \bar{a})$ is as in Lemma 3.6. Let $\tilde{T}_0 \ge 1$ be such that $e^{\zeta_0 \tilde{T}_0} \ge 4$. Let ϵ_0 be as in Lemma 3.8. For any $0 < \eta \le \epsilon_0$, there is $0 < \delta_\eta \le \epsilon_0$ such that for any $(u_0, v_0) \in X_1^+ \times X_2^+$, any $\xi \in S^{N-1}$, any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, any $t_0 \ge T_0(u_0, v_0)+2$, and any ball $B_{2L} \subset \mathbb{R}^N$ with radius $2L$, if*

$$\sup_{x \in B_{2L}} \tilde{u}(x, t_0; \xi, c, u_0, v_0) \ge \eta,$$

then

$$\inf_{x \in B_{2L}} \tilde{u}(x, t; \xi, c, u_0, v_0) \ge \delta_\eta \quad \forall\, t_0 \le t \le t_0 + T + \tilde{T}_0,$$

where $L = L(\eta)$ and $T = T(\eta)$.

Proof. Suppose on the contrary that the conclusion fails. Then there exist $0 < \eta_0 \le \epsilon_0$, $(u_{0n}, v_{0n}) \in X_1^+ \times X_2^+$, $\xi_n \in S^{N-1}$, $-2\sqrt{a} + \epsilon \le c_n \le 2\sqrt{a} - \epsilon$, $t_{0n} \ge T_0(u_{0n}, v_{0n}) + 2$, a sequence of ball $B^n_{2L(\eta_0)} \subset \mathbb{R}^N$ with radius $2L(\eta_0)$, x_n, $x_n^* \in \mathbb{R}$ with $x_n \in B^n_{2L(\eta_0)}$, $x_n^* \in B^n_{2L(\eta_0)}$, $t_n \in \mathbb{R}$ with $t_{0n} \le t_n \le t_{0n} + T(\eta_0) + \tilde{T}_0$ such that

$$\lim_{n\to\infty} \tilde{u}(x_n, t_{0n}; \xi_n, c_n, u_{0n}, v_{0n}) \ge \eta_0 \tag{3.65}$$

and

$$\lim_{n\to\infty} \tilde{u}(x_n^*, t_n; \xi_n, c_n, u_{0n}, v_{0n}) = 0. \tag{3.66}$$

Let $\tilde{u}_n(x,t) = \tilde{u}(x+x_n, t+t_{0n}-1; \xi_n, c_n, u_{0n}, v_{0n})$, $\tilde{v}_n(x,t) = \tilde{v}(x+x_n, t+t_{0n}-1; \xi_n, c_n, u_{0n}, v_{0n})$, and $T = T(\eta_0) + \tilde{T}_0$, $L = L(\eta_0)$. Without loss of generality, we may assume that

$$x_n^* - x_n \to x^*, \quad t_n - t_{0n} + 1 \to t^* \ge 1 \quad \text{as} \quad n \to \infty \tag{3.67}$$

and

$$(\tilde{u}_n(x,t), \tilde{v}_n(x,t)) \to (u^*(x,t), v^*(x,t)) \tag{3.68}$$

as $n \to \infty$ locally uniformly in $(x,t) \in \mathbb{R}^N \times [0,\infty)$, $\xi_n \to \xi^*$ and $c_n \to c^*$ as $n \to \infty$ for some $\xi^* \in S^{N-1}$, $-2\sqrt{a}+\epsilon \le c^* \le 2\sqrt{a}-\epsilon$. Then (u^*, v^*) is a solution of (3.48) with ξ being replaced by ξ^* and c being replaced by c^* for $t \ge 0$.

By (3.65), $u^*(0,1) \ge \eta_0$, it follows from comparison principle for parabolic equations that $u^*(x,t) > 0$ for $x \in \mathbb{R}^N$, $t > 0$. But by (3.66), $u^*(x^*,t^*) = 0$. This is a contraction. $\square$

Lemma 3.10. *For any given $0 < \epsilon < 2\sqrt{a}$, let $\bar{a}$ and l be as in* (3.49) *and* (3.50). *There is $0 < \tilde{\epsilon}_0 \le \epsilon_0$ such that for any $0 < \eta \le \tilde{\epsilon}_0$, there is $\tilde{\delta}_\eta > 0$ such that for any $(u_0, v_0) \in X_1^+ \times X_2^+$, any $\xi \in S^{N-1}$, any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, any t_1, t_2 satisfying that $T_0(u_0,v_0)+2 \le t_1 < t_2 \le \infty$, and any ball $B_{2L} \subset \mathbb{R}^N$ with radius $2L$, if*

$$\sup_{x \in B_{2L}} \tilde{u}(x,t_1;\xi,c,u_0,v_0) = \eta, \quad \sup_{x \in B_{2L}} \tilde{u}(x,t;\xi,c,u_0,v_0) \le \eta, \ \forall \ t_1 < t < t_2,$$

then

$$\inf_{x \in B_{2L}} \tilde{u}(x,t;\xi,c,u_0,v_0) \ge \tilde{\delta}_\eta \quad \forall \ t_1 \le t < t_2,$$

where $L = L(\eta)$.

Proof. It suffices to prove the lemma for the ball $B_{2L}(0)$ centered at the origin with the radius $2L$. If the ball with the radius $2L$ is not centered at the origin, we can make an appropriate translation of $(\tilde{u}(x,t;\xi,c,u_0,v_0), \tilde{v}(x,t;\xi,c,u_0,v_0))$ for the space variable x to shift the ball into the ball centered at the origin.

First, consider

$$\begin{cases} u_t = \Delta u + c\xi \cdot \nabla u + q(x,t) \cdot \nabla u + \bar{a}u, \quad x \in D_l, \ t > 0 \\ u(x,t) = 0, \quad x \in \partial D_l, \ t > 0, \\ u(x,0) = \bar{\phi}(x;\xi,c,\bar{a}), \quad x \in D_l, \end{cases} \tag{3.69}$$

where $\bar{\phi}(x;\xi,c,\bar{a}) = \frac{\phi(x;\xi,c,\bar{a})}{\|\phi\|_\infty}$ and $\phi(x;\xi,c,\bar{a})$ is as in Lemma 3.6. Let $\bar{u}(x,t;\xi,c,q)$ be the solution of (3.69). Let $\tilde{T}_0 \ge 1$ be as in lemma 3.9. We claim that there is $\tilde{\epsilon}_0 > 0$ such that for any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, any $\xi \in S^{N-1}$, any function $q(x,t)$ which is C^1 in x and Hölder continuous in t with exponent $0 < \theta < \frac{1}{2}$,

$$\sup_{t \geq 0} \|q(\cdot, t)\|_{C(\bar{D}_l)} \leq \chi \sqrt{N} \tilde{M} \tilde{\epsilon}_0 \tag{3.70}$$

($\tilde{M}$ is as in Lemma 3.8), and

$$\sup_{t,s \geq 0, t \neq s} \frac{\|q(\cdot, t) - q(\cdot, s)\|_{C(\bar{D}_l)}}{|t - s|^\theta} \leq \chi M M_1 \tag{3.71}$$

(M and M_1 are as in Lemma 3.7), there holds

$$\bar{u}(x, \tilde{T}_0; \xi, c, q) \geq 2\bar{\phi}(x; \xi, c, \bar{a}) \quad \forall\ x \in D_l. \tag{3.72}$$

In fact, assume this is not true. Then there are $\epsilon_n \to 0$ as $n \to \infty$, $x_n \in D_l$, $\xi_n \in S^{N-1}$, $-2\sqrt{a} + \epsilon \leq c_n \leq 2\sqrt{a} - \epsilon$, and $q_n(x, t)$ satisfying (3.71) and

$$\sup_{t \geq 0} \|q_n(\cdot, t)\|_{C(\bar{D}_l)} \leq \chi \sqrt{N} \tilde{M} \epsilon_n$$

such that

$$\bar{u}(x_n, \tilde{T}_0; \xi_n, c_n, q_n) < 2\bar{\phi}(x_n; \xi_n, c_n, \bar{a}) \quad \forall\ n \geq 1. \tag{3.73}$$

Let $u_n(x, t) = \bar{u}(x, t; \xi_n, c_n, q_n)$. Without loss of generality, we may assume that

$$u_n(x, t) \to u^*(x, t), \quad \partial_{x_j} u_n(x, t) \to \partial_{x_j} u^*(x, t) \quad \text{as} \quad n \to \infty$$

locally uniformly in $(x, t) \in \bar{D}_l \times [0, \infty)$, $\xi_n \to \xi^*$ and $c_n \to c^*$ as $n \to \infty$ for some $\xi^* \in S^{N-1}$, $-2\sqrt{a} + \epsilon \leq c^* \leq 2\sqrt{a} - \epsilon$. Note that $u^*(x, t) = \bar{u}(x, t; \xi^*, c^*, 0) = e^{\zeta(c^*, \bar{a})t} \bar{\phi}(x; \xi^*, c^*, \bar{a})$. Hence

$$u^*(x, \tilde{T}_0) \geq e^{\zeta_0 \tilde{T}_0} \bar{\phi}(x; \xi^*, c^*, \bar{a}) \geq 4\bar{\phi}(x; \xi^*, c^*, \bar{a}), \quad \forall\ x \in D_l.$$

This together with the Hopf's Lemma implies that

$$u_n(x, \tilde{T}_0) \geq 2\bar{\phi}(x; \xi_n, c_n, \bar{a}) \quad \forall\ x \in D_l,\ \ n \gg 1,$$

which is a contradiction. Hence the claim holds true.

Next, without loss of generality, we may assume that

$$a - \tilde{a} - b\tilde{\epsilon}_0 \geq \bar{a}.$$

Let $T = T(\eta)$. By Lemma 3.8, for any given $0 < \eta \leq \tilde{\epsilon}_0$, $\xi \in S^{N-1}$, $-2\sqrt{a}+\epsilon \leq c \leq 2\sqrt{a}-\epsilon$, $t_1 + T + 1 \leq t < t_2 \leq \infty$, and $x \in B_L(0)$,

$$\begin{aligned} \tilde{u}_t &= \Delta\tilde{u} + c\xi \cdot \nabla\tilde{u} - \chi\nabla\tilde{v} \cdot \nabla\tilde{u} + \tilde{u}(a - \chi\Delta\tilde{v} - b\tilde{u}) \\ &\geq \Delta\tilde{u} + c\xi \cdot \nabla\tilde{u} + q(x,t) \cdot \nabla\tilde{u} + \bar{a}\tilde{u}, \end{aligned} \tag{3.74}$$

where $q(x,t) = -\chi\nabla\tilde{v}(x,t;\xi,c,u_0,v_0)$. By Lemma 3.7 and Lemma 3.8, $q(\cdot,\cdot{+}t_1+T+1)$ satisfies (3.70) and (3.71). Let $n_0 \geq 0$ be such that

$$t_1 + T + 1 + n_0\tilde{T}_0 < t_2 \quad \text{and} \quad t_1 + T + 1 + (n_0+1)\tilde{T}_0 \geq t_2.$$

By Lemma 3.9,

$$\inf_{x\in B_{2L}(0)} \tilde{u}(x,t;\xi,c,u_0,v_0) \geq \delta_\eta \quad \forall\, t_1 \leq t \leq t_1 + T + 1.$$

This together with the comparison principle for parabolic equations and (3.72) implies that for any $-2\sqrt{a}+\epsilon \leq c \leq 2\sqrt{a}-\epsilon$, any $\xi \in S^{N-1}$,

$$\begin{aligned} \tilde{u}(x,t_1+T+1+k\tilde{T}_0;\xi,c,u_0,v_0) &\geq 2^{k-1}\delta_\eta\bar{u}(x,\tilde{T}_0;\xi,c,q(\cdot,\cdot+t_1+T+1+(k-1)\tilde{T}_0)) \\ &\geq 2^k\delta_\eta\bar{\phi}(x;\xi,c,\bar{a}) \quad \forall\ x \in D_l \end{aligned}$$

for $k = 1, 2, \cdots, n_0$, where δ_η is as in Lemma 3.9. By Lemma 3.9 again, we then have for any $-2\sqrt{a}+\epsilon \leq c \leq 2\sqrt{a}-\epsilon$, any $\xi \in S^{N-1}$,

$$\inf_{x\in B_{2L}(0)} \tilde{u}(x,t;\xi,c,u_0,v_0) \geq \tilde{\delta}_\eta := \min\{\delta_\eta, \delta_{\delta_\eta}\} \quad \forall\, t_1 \leq t < t_2.$$

□

We now prove Theorem 3.4 (1).

Proof of Theorem 3.4 (1). let $(u_0, v_0) \in C_{cp}^{+} \times C_{cp}^{+,1}$ be fixed. We first prove that for any $0 < \epsilon < 2\sqrt{a}$,

$$\liminf_{t\to\infty} \inf_{|x|\le(2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0) > 0. \tag{3.75}$$

For any $0 < \epsilon < 2\sqrt{a}$, let $\bar{a}$ and l be as in (3.49) and (3.50). Let $T_0 = T_0(u_0, v_0)$ and $\tilde{\epsilon}_0$ be as in Lemma 3.7 and Lemma 3.10, respectively. Let $T(\tilde{\epsilon}_0)$ be such that (3.59) holds. For any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, any $\xi \in S^{N-1}$, let

$$\tilde{\delta} := \tilde{\delta}(\xi, c) = \inf_{x\in\bar{D}_l} \tilde{u}(x, T_0 + T(\tilde{\epsilon}_0) + 3; \xi, c, u_0, v_0).$$

By the assumption $u_0(x) \ge 0$ and $u_0(x) \not\equiv 0$, $\tilde{\delta} > 0$. Let

$$k_0 = \inf\{k \in \mathbb{Z}^+ \mid 2^k\tilde{\delta} \ge \tilde{\epsilon}_0\} \quad \text{and} \quad T_{00} = T_0 + T(\tilde{\epsilon}_0) + 3 + k_0\tilde{T}_0.$$

where $\tilde{T}_0 \ge 1$ is as in lemma 3.9. We claim that for any $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, any $\xi \in S^{N-1}$,

$$\inf_{|x|\le 2L(\tilde{\epsilon}_0)} \tilde{u}(x,t;\xi,c,u_0,v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall\, t \ge T_{00}. \tag{3.76}$$

To prove the claim, for any given $-2\sqrt{a}+\epsilon \le c \le 2\sqrt{a}-\epsilon$, $\xi \in S^{N-1}$, let

$$I = \{t > T_0 + 2 \mid \sup_{|x|\le 2L(\tilde{\epsilon}_0)} \tilde{u}(x,t;\xi,c,u_0,v_0) < \tilde{\epsilon}_0\}.$$

Note that I is an open set. By Lemma 3.9,

$$\inf_{|x|\le 2L(\tilde{\epsilon}_0)} \tilde{u}(x,t;\xi,c,u_0,v_0) \ge \delta_{\tilde{\epsilon}_0} \quad \forall\, t \notin I \ \text{ for } t > T_0 + 2. \tag{3.77}$$

Hence, if $I = \emptyset$, then

$$\inf_{|x|\le 2L(\tilde{\epsilon}_0)} \tilde{u}(x,t;\xi,c,u_0,v_0) \ge \delta_{\tilde{\epsilon}_0} \quad \forall\ t \ge T_0 + 2. \tag{3.78}$$

If $I \neq \emptyset$, then $I = \cup(a_i, b_i)$. If $a_i \neq T_0 + 2$, then

$$\sup_{|x| \leq 2L(\tilde{\epsilon}_0)} \tilde{u}(x, a_i; \xi, c, u_0, v_0) = \tilde{\epsilon}_0 \quad \text{and} \quad \sup_{|x| \leq 2L(\tilde{\epsilon}_0)} \tilde{u}(x, t; \xi, c, u_0, v_0) < \tilde{\epsilon}_0 \quad \forall t \in (a_i, b_i).$$

By the statement in Lemma 3.10,

$$\inf_{|x| \leq 2L(\tilde{\epsilon}_0)} \tilde{u}(x, t; \xi, c, u_0, v_0) \geq \tilde{\delta}_{\tilde{\epsilon}_0} \quad \forall t \in (a_i, b_i) \text{ for } a_i \neq T_0 + 2. \tag{3.79}$$

If $a_i = T_0 + 2$, by the arguments in Lemma 3.10, there holds

$$\tilde{u}(x, T_0 + T(\tilde{\epsilon}_0) + 3 + k\tilde{T}_0; \xi, c, u_0, v_0) \geq 2^k \tilde{\delta} \phi(x; \xi, c, \bar{a}) \quad \forall \ x \in D_l$$

for $k = 0, 1, 2, \cdots, k_0$. This implies that $b_i \leq T_{00}$. This together with (3.77), (3.78), and (3.79) implies (3.76).

By (3.76) and $\tilde{u}(x, t; \xi, c, u_0, v_0) = u(x + ct\xi, t; u_0, v_0)$, we have for any $-2\sqrt{a} + \epsilon \leq c \leq 2\sqrt{a} - \epsilon$, any $\xi \in S^{N-1}$,

$$\inf_{|x - ct\xi| \leq 2L(\tilde{\epsilon}_0)} u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall t \geq T_{00}. \tag{3.80}$$

Thus for any $t \geq T_{00}$, any $|x| \leq (2\sqrt{a} - \epsilon)t$, there exist $c = \frac{|x|}{t}$ and $\xi = \frac{x}{|x|}$ such that $|x - ct\xi| \leq 2L(\tilde{\epsilon}_0)$, it then holds that

$$u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\},$$

which implies that

$$\inf_{|x| \leq (2\sqrt{a} - \epsilon)t} u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall \ t \geq T_{00}.$$

Hence,

$$\liminf_{t \to \infty} \inf_{|x| \leq (2\sqrt{a} - \epsilon)t} u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\}.$$

(3.75) is thus proved.

Finally, we prove that for any $0 < \epsilon < 2\sqrt{a}$,

$$\liminf_{t\to\infty} \inf_{|x|\le(2\sqrt{a}-\epsilon)t} v(x,t;u_0,v_0) > 0. \tag{3.81}$$

Suppose by contraction that the result does not hold. Then there are constant $0 < \epsilon < 2\sqrt{a}$ and a sequence $\{(x_n,t_n)\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$, $|x_n| \le (2\sqrt{a}-\epsilon)t_n$, and

$$v(x_n,t_n;u_0,v_0) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.82}$$

For every $n \ge 1$, let us define

$$u_n(x,t) = u(x+x_n,t+t_n;u_0,v_0), \quad \text{and} \quad v_n(x,t) = v(x+x_n,t+t_n;u_0,v_0)$$

for every $x \in \mathbb{R}^N$, $t \ge -t_n$. By a prior estimates for parabolic equations, without loss of generality, we may assume that $(u_n(x,t),v_n(x,t)) \to (u^*(x,t),v^*(x,t))$ locally uniformly in $C^{2,1}(\mathbb{R}^N\times\mathbb{R})$. Furthermore, $(u^*(t,x),v^*(t,x))$ is an entire solution of

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + u(a-bu), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R} \\ v_t = \Delta v - \lambda v + \mu u, \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}, \end{cases}$$

Choose $0 < \tilde{\epsilon} < \epsilon$. For every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} |x+x_n| &\le |x| + |x_n| \le |x| + (2\sqrt{a}-\epsilon)t_n \\ &= (2\sqrt{a}-\tilde{\epsilon})(t_n+t) - (\epsilon-\tilde{\epsilon})(t_n - \frac{|x|-(2\sqrt{a}-\tilde{\epsilon})t}{\epsilon-\tilde{\epsilon}}) \\ &\le (2\sqrt{a}-\tilde{\epsilon})(t_n+t) \end{aligned}$$

whenever $t_n \ge \frac{|x|+(2\sqrt{a}-\tilde{\epsilon})t}{\epsilon-\tilde{\epsilon}}$. By (3.75),

$$u^*(x,t) = \lim_{n\to\infty} u(x+x_n,t+t_n;u_0,v_0) \ge \liminf_{s\to\infty} \inf_{|y|\le(2\sqrt{a}-\tilde{\epsilon})s} u(y,s;u_0,v_0) > 0$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. It follows from comparison principle for parabolic equations that $v^*(x,t) > 0$ for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. In particular, $v^*(0,0) > 0$, which contradicts to (3.82). $\square$

We then prove Theorem 3.5 (1).

Proof of Theorem 3.5 (1). Let $\xi \in S^{N-1}$ and $(u_0, v_0) \in C^+_{fl}(\xi) \times C^{+,1}_{fl}(\xi)$ be fixed. We first prove that for any $0 < \epsilon < 2\sqrt{a}$,

$$\liminf_{t\to\infty} \inf_{x\cdot\xi \le (2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0) > 0. \tag{3.83}$$

Let $\tilde{u}(x,t) = u(x+(2\sqrt{a}-\epsilon)t\xi, t)$ and $\tilde{v}(x,t) = v(x+(2\sqrt{a}-\epsilon)t\xi, t)$. Then $(\tilde{u}(x,t), \tilde{v}(x,t))$ solves (3.48) with c being replaced by $2\sqrt{a}-\epsilon$. $(\tilde{u}(x,t;\xi,u_0,v_0), \tilde{v}(x,t;\xi,u_0,v_0))$ denotes the solution of (3.48) with c being replaced by $2\sqrt{a}-\epsilon$ and $(\tilde{u}(x,0;\xi,u_0,v_0), \tilde{v}(x,0;\xi,u_0,v_0)) = (u_0,v_0)$. Let $T_0 = T_0(u_0,v_0)$ and $\tilde{\epsilon}_0$ be as in Lemma 3.7 and Lemma 3.10, respectively. Let $T(\tilde{\epsilon}_0)$ be such that (3.59) holds. Let

$$\tilde{\delta} = \inf_{x\cdot\xi \le 2L(\tilde{\epsilon}_0)} \tilde{u}(x, T_0 + T(\tilde{\epsilon}_0) + 3; \xi, u_0, v_0).$$

Since $\liminf_{x\cdot\xi\to-\infty} u_0(x) > 0$, $\tilde{\delta} > 0$. Let

$$k_0 = \inf\{k \in \mathbb{Z}^+ \,|\, 2^k\tilde{\delta} \ge \tilde{\epsilon}_0\} \quad \text{and} \quad T_{00} = T_0 + T(\tilde{\epsilon}_0) + 3 + k_0\tilde{T}_0.$$

where $\tilde{T}_0 \ge 1$ is as in Lemma 3.9. By the similar arguments used in the proof of (3.76), we can prove for any ball $B_{2L(\tilde{\epsilon}_0)} \subset \{x \mid x\cdot\xi < 2L(\tilde{\epsilon}_0)\}$ with radius $2L(\tilde{\epsilon}_0)$, it holds that

$$\inf_{x\in B_{2L(\tilde{\epsilon}_0)}} \tilde{u}(x,t;\xi,u_0,v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall\, t \ge T_{00}.$$

For any $x \in \{x \mid x \cdot \xi < 2L(\tilde{\epsilon}_0)\}$, there exists a ball $B_{2L(\tilde{\epsilon}_0)} \subset \{x \mid x \cdot \xi < 2L(\tilde{\epsilon}_0)\}$ such that $x \in B_{2L(\tilde{\epsilon}_0)}$, we then obtain that

$$\tilde{u}(x,t;\xi,u_0,v_0) \geq \inf_{x \in B_{2L(\tilde{\epsilon}_0)}} \tilde{u}(x,t;\xi,u_0,v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall\, t \geq T_{00},$$

which implies that

$$\inf_{x \cdot \xi < 2L(\tilde{\epsilon}_0)} \tilde{u}(x,t;\xi,u_0,v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} > 0 \quad \forall\, t \geq T_{00}. \tag{3.84}$$

By (3.84) and $\tilde{u}(x,t;\xi,u_0,v_0) = u(x + (2\sqrt{a} - \epsilon)t\xi, t; u_0, v_0)$, we have

$$\inf_{x \cdot \xi < (2\sqrt{a}-\epsilon)t + 2L(\tilde{\epsilon}_0)} u(x,t;u_0,v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall\, t \geq T_{00}. \tag{3.85}$$

Hence,

$$\liminf_{t \to \infty} \inf_{x \cdot \xi \leq (2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\}.$$

(3.83) is thus proved.

Finally, it can be proved by the similar arguments used in proving (3.81) that for any $0 < \epsilon < 2\sqrt{a}$,

$$\liminf_{t \to \infty} \inf_{x \cdot \xi \leq (2\sqrt{a}-\epsilon)t} v(x,t;u_0,v_0) > 0.$$

□

We now prove Theorem 3.6 (1).

Proof of Theorem 3.6 (1). Let $\xi \in S^{N-1}$ and $(u_0, v_0) \in C^+(\xi) \times C^{+,1}(\xi)$ be fixed. We first prove that for any $0 < \epsilon < 2\sqrt{a}$,

$$\liminf_{t \to \infty} \inf_{|x \cdot \xi| \leq (2\sqrt{a}-\epsilon)t} u(x,t;u_0,v_0) > 0. \tag{3.86}$$

Let $T_0 = T_0(u_0, v_0)$ and $\tilde{\epsilon}_0$ be as in Lemma 3.7 and Lemma 3.10, respectively. Let $T(\tilde{\epsilon}_0)$ be such that (3.59) holds. For any $-2\sqrt{a} + \epsilon \le c \le 2\sqrt{a} - \epsilon$, let

$$\tilde{\delta} := \tilde{\delta}(\xi, c) = \inf_{|x\cdot\xi| \le 2L(\tilde{\epsilon}_0)} \tilde{u}(x, T_0 + T(\tilde{\epsilon}_0) + 3; \xi, c, u_0, v_0).$$

Since there exists $r > 0$ such that $\inf_{|x\cdot\xi|<r} u_0(x) > 0$, $\tilde{\delta} > 0$. Let

$$k_0 = \inf\{k \in \mathbb{Z}^+ \mid 2^k \tilde{\delta} \ge \tilde{\epsilon}_0\} \quad \text{and} \quad T_{00} = T_0 + T(\tilde{\epsilon}_0) + 3 + k_0 \tilde{T}_0.$$

where $\tilde{T}_0 \ge 1$ is as in Lemma 3.9. By the similar arguments used in the proof of (3.76), we can prove that for any $-2\sqrt{a} + \epsilon \le c \le 2\sqrt{a} - \epsilon$, any ball $B_{2L(\tilde{\epsilon}_0)} \subset \{x \mid |x \cdot \xi| < 2L(\tilde{\epsilon}_0)\}$ with radius $2L(\tilde{\epsilon}_0)$, it holds that

$$\inf_{x \in B_{2L(\tilde{\epsilon}_0)}} \tilde{u}(x, t; \xi, c, u_0, v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall t \ge T_{00}.$$

For any $x \in \{x \mid |x \cdot \xi| < 2L(\tilde{\epsilon}_0)\}$, there exists a ball $B_{2L(\tilde{\epsilon}_0)} \subset \{x \mid |x \cdot \xi| < 2L(\tilde{\epsilon}_0)\}$ such that $x \in B_{2L(\tilde{\epsilon}_0)}$, we then obtain that for any $-2\sqrt{a} + \epsilon \le c \le 2\sqrt{a} - \epsilon$,

$$\tilde{u}(x, t; \xi, c, u_0, v_0) \ge \inf_{x \in B_{2L(\tilde{\epsilon}_0)}} \tilde{u}(x, t; \xi, c, u_0, v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall t \ge T_{00},$$

which implies that for any $-2\sqrt{a} + \epsilon \le c \le 2\sqrt{a} - \epsilon$,

$$\inf_{|x\cdot\xi| < 2L(\tilde{\epsilon}_0)} \tilde{u}(x, t; \xi, c, u_0, v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} > 0 \quad \forall t \ge T_{00}. \tag{3.87}$$

By (3.87) and $\tilde{u}(x, t; \xi, c, u_0, v_0) = u(x + ct\xi, t; u_0, v_0)$, we have for any $-2\sqrt{a} + \epsilon \le c \le 2\sqrt{a} - \epsilon$,

$$\inf_{|x\cdot\xi - ct| < 2L(\tilde{\epsilon}_0)} u(x, t; u_0, v_0) \ge \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall t \ge T_{00}.$$

For any $t \geq T_{00}$, any $x \in \{x \,|\, |x \cdot \xi| \leq (2\sqrt{a} - \epsilon)t\}$, there exists $c = \frac{x \cdot \xi}{t}$ such that $x \in \{x \,|\, |x \cdot \xi - ct| < 2L(\tilde{\epsilon}_0)\}$, it then holds that

$$u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\},$$

which implies that

$$\inf_{|x \cdot \xi| \leq (2\sqrt{a} - \epsilon)t} u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\} \quad \forall\, t \geq T_{00}.$$

Hence,

$$\liminf_{t \to \infty} \inf_{|x \cdot \xi| \leq (2\sqrt{a} - \epsilon)t} u(x, t; u_0, v_0) \geq \min\{\delta_{\tilde{\epsilon}_0}, \tilde{\delta}_{\tilde{\epsilon}_0}\}.$$

(3.86) is thus proved.

Finally, it can be proved by the similar arguments used in proving (3.81) that

$$\liminf_{t \to \infty} \inf_{|x \cdot \xi| \leq (2\sqrt{a} - \epsilon)t} v(x, t; u_0, v_0) > 0.$$

□

Finally, we prove Theorem 3.7.

Proof of Theorem 3.7. It follows from the arguments similar to those in the proof of (3.75) with $c = 0$ and the ball $B_{2L(\tilde{\epsilon}_0)}(x_0)$ for any $x_0 \in \mathbb{R}$. □

3.7 Upper bounds of spreading speeds

This section is devoted to the study of upper bounds of spreading speeds of global classical solutions of (1.4) with different initial functions and prove Theorems 3.4 (2), 3.5 (2) and 3.6 (2). Throughout this section, we assume that $b > \frac{N\mu\chi}{4}$.

First, we present a lemma.

Lemma 3.11. *Let $w = u + \frac{\chi}{2\mu}|\nabla v|^2$. Then*

$$w_t \leq \Delta w + aw.$$

Proof. By the proof of Theorem 3.2, we have

$$\frac{d}{dt}\big[u+\frac{\chi}{2\mu}|\nabla v|^2\big] \le \Delta\big[u+\frac{\chi}{2\mu}|\nabla v|^2\big] - \frac{\chi\lambda}{\mu}|\nabla v|^2 - \Big(b-\frac{N\mu\chi}{4}\Big)u^2 + au.$$

Since $b > \frac{N\mu\chi}{4}$, then

$$\frac{d}{dt}\big[u+\frac{\chi}{2\mu}|\nabla v|^2\big] \le \Delta\big[u+\frac{\chi}{2\mu}|\nabla v|^2\big] + a\big[u+\frac{\chi}{2\mu}|\nabla v|^2\big].$$

The lemma then follows. □

We now prove Theorem 3.4 (2).

Proof of Theorem 3.4 (2). First of all, for any given $(u_0, v_0) \in C_{cp}^+ \times C_{cp}^{+,1}$ and $0 < k < \sqrt{a}$, let $M > 0$ be such that

$$u_0(x) + \frac{\chi}{2\mu}|\nabla v_0(x)|^2 \le \min\{Me^{-kx\cdot\xi},\ \xi \in S^{N-1}\} \quad \forall\, x \in \mathbb{R}^N.$$

Let

$$c = \frac{k^2+a}{k},$$

and

$$U(x,t,\xi) = Me^{-k(x\cdot\xi - ct)}.$$

Write $u = u(x,t;u_0,v_0)$, $v = v(x,t;u_0,v_0)$. Let $w = u + \frac{\chi}{2\mu}|\nabla v|^2$. By Lemma 3.11,

$$w_t \le \Delta w + aw.$$

It follows from the comparison principle for parabolic equations that

$$u(x,t;u_0,v_0) \le U(x,t,\xi) \quad \forall\, x \in \mathbb{R}^N,\ t>0,\ \xi \in S^{N-1}. \tag{3.88}$$

Let $\xi = \frac{x}{|x|}$, then

$$u(x,t;u_0,v_0) \le Me^{-k(|x|-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0.$$

For any $\varepsilon > 0$, there exists $0 < k < \sqrt{a}$ such that $2\sqrt{a} + \varepsilon > c$, it then holds that

$$\lim_{t\to\infty} \sup_{|x|\geq(2\sqrt{a}+\varepsilon)t} u(x,t;u_0,v_0) = 0. \tag{3.89}$$

Next, we prove that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \sup_{|x|\geq(2\sqrt{a}+\varepsilon)t} v(x,t;u_0,v_0) = 0. \tag{3.90}$$

Let $d \geq \frac{\mu M}{a+\lambda}$ be such that

$$v_0(x) \leq \min\{de^{-kx\cdot\xi},\ \xi \in S^{N-1}\} \quad \forall\, x \in \mathbb{R}^N.$$

By the second equation of (1.4) and (3.88),

$$v_t = \Delta v - \lambda v + \mu u \leq \Delta v - \lambda v + \mu M e^{-k(x\cdot\xi - ct)}.$$

Direct computation yields that $de^{-k(x\cdot\xi-ct)}$ satisfies

$$\frac{\partial}{\partial t}(de^{-k(x\cdot\xi-ct)}) \geq \Delta(de^{-k(x\cdot\xi-ct)}) - \lambda(de^{-k(x\cdot\xi-ct)}) + \mu M e^{-k(x\cdot\xi-ct)}.$$

It follows from the comparison principle for parabolic equations again that

$$v(x,t;u_0,v_0) \leq de^{-k(x\cdot\xi-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t > 0,\ \xi \in S^{N-1}.$$

Similar arguments as in deriving (3.89) yield that (3.90) holds. □

Next, we prove Theorem 3.5 (2).

Proof of Theorem 3.5 (2). For any given $\xi \in S^{N-1}$, $(u_0, v_0) \in C^+_{fl}(\xi) \times C^{+,1}_{fl}(\xi)$ and $0 < k < \sqrt{a}$, let

$$c = \frac{k^2 + a}{k},$$

and $M > 0$ be such that

$$u_0(x) + \frac{\chi}{2\mu}|\nabla v_0(x)|^2 \le Me^{-kx\cdot\xi} \quad \forall\ x \in \mathbb{R}^N.$$

Let $d \ge \frac{\mu M}{a+\lambda}$ be such that

$$v_0(x) \le de^{-kx\cdot\xi} \quad \forall\ x \in \mathbb{R}^N.$$

By similar arguments as those in Theorem 3.4 (2), we can prove that

$$u(x,t;u_0,v_0) \le Me^{-k(x\cdot\xi-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t > 0$$

and

$$v(x,t;u_0,v_0) \le de^{-k(x\cdot\xi-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t > 0.$$

For any $\varepsilon > 0$, there exists $0 < k < \sqrt{a}$ such that $2\sqrt{a} + \varepsilon > c$, Theorem 3.5 (2) thus follows. $\square$

Finally, we prove Theorem 3.6 (2).

Proof of Theorem 3.6 (2). For any given $\xi \in S^{N-1}$, $(u_0, v_0) \in C^+(\xi) \times C^{+,1}(\xi)$ and $0 < k < \sqrt{a}$, let

$$c = \frac{k^2 + a}{k},$$

and $M > 0$ be such that

$$u_0(x) + \frac{\chi}{2\mu}|\nabla v_0(x)|^2 \le \min\{Me^{-kx\cdot\xi},\, Me^{kx\cdot\xi}\} \quad \forall\ x \in \mathbb{R}^N.$$

Let $d \ge \frac{\mu M}{a+\lambda}$ be such that

$$v_0(x) \le \min\{de^{-kx\cdot\xi},\, de^{kx\cdot\xi}\} \quad \forall\ x \in \mathbb{R}^N.$$

By the similar arguments as those in Theorem 3.4 (2), we can prove that

$$u(x,t;u_0,v_0) \le Me^{-k(x\cdot\xi-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0,$$

$$u(x,t;u_0,v_0) \le Me^{k(x\cdot\xi+ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0,$$

$$v(x,t;u_0,v_0) \le de^{-k(x\cdot\xi-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0,$$

and

$$v(x,t;u_0,v_0) \le de^{k(x\cdot\xi+ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0.$$

It then follows that

$$u(x,t;u_0,v_0) \le Me^{-k(|x\cdot\xi|-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0,$$

and

$$v(x,t;u_0,v_0) \le de^{-k(|x\cdot\xi|-ct)} \quad \forall\, x \in \mathbb{R}^N,\ t>0.$$

For any $\varepsilon > 0$, there exists $0 < k < \sqrt{a}$ such that $2\sqrt{a}+\varepsilon > c$, Theorem 3.6 (2) thus follows. □

Chapter 4

Concluding remarks and future works

In this chapter, we make some remarks about our main results obtained in this dissertation and present some possible future works.

In chapter 2, we incorporated the climate change into the parabolic-elliptic chemotaxis model (1.1) and studied persistence, spreading speeds and the existence of forced waves in one-dimensional setting. We obtained some conditions under which the species can survive or become extinct. But there are some critical cases which are still open. For example, in case 1, can species keep up with the shifting environment if $c = 2\sqrt{r^*}$ and the species initially lives in a bounded region? In case 2, can species keep up with the shifting environment if $\zeta_\infty(r(\cdot), c) = 0$? We leave these problems as future works.

We also studied forced waves of (1.1). We proved forced waves exist in certain parameter ranges. Some numerical simulations are carried out to illustrate our theoretical results. In addition, numerical simulations indicate that forced waves can exist in a larger parameter ranges which are not covered in theoretical results. Numerical simulations also indicate that forced waves are unique and stable. But there is no theoretical results for the uniqueness and stability of forced waves in (1.1). It is interesting to study these findings indicated by numerical simulations theoretically.

As mentioned before, we studied (1.1) in one-dimensional setting. The natural question arise: consider chemotaxis model (1.1) in high-dimensional shifting environments, that is,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(r(x \cdot \xi - ct) - bu), \quad x \in \mathbb{R}^N \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $\xi \in \mathbb{R}^N$ is a unit vector, can the results obtained in chapter 2 be extended to (4.1)? We believe that the results obtained in chapter 2 can be extended to (4.1). For example, for given $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ with $u_0 \geq 0$ and $\{x : |x \cdot \xi| < r\} \subset \text{supp}(u_0) \subset \{x : |x \cdot \xi| < R\}$ for some $0 < r < R$, we believe that the statements in Theorem 2.1 hold with $x \leq \tilde{c}t$ (resp. $x \geq \tilde{c}t$) being replaced by $x \cdot \xi \leq \tilde{c}t$ (resp. $x \cdot \xi \geq \tilde{c}t$), where $\tilde{c} = c - \epsilon$, $c^* - \epsilon$, or $-c^* - \epsilon$ (resp. $c + \epsilon$, $c^* + \epsilon$, or $-c^* + \epsilon$). But due to the lack of comparison principle for chemotaxis models, more new techniques may need to be developed to prove such results. We leave the study of the extension of the results obtained in chapter 2 to high-dimensional space case for further investigation.

we studied parabolic-elliptic chemotaxis model (1.1) in two different shifting environments in chapter 2. The shifting speed of the environments is a constant in both cases. This motivates us to think if the shifting speed of the environments is not a constant, say a periodic or almost periodic function, what will happen? More precisely, it is interesting to study (1.1) in shifting environments with time-dependent shifting speed. That is the following model.

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + u(r(x - \int_0^t c(s)ds) - bu), & x \in \mathbb{R}, \\ 0 = v_{xx} - \lambda v + \mu u, & x \in \mathbb{R}. \end{cases} \tag{4.2}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a periodic or almost-periodic function. In a recent joint work [55] with Drs. W. Shen, Z. Shen and D. Zhou, we studied (4.2) for the case $\chi = 0$ (In fact, we studied more general reaction term). We established the persistence criterion in terms of the sign of the approximate top Lyapunov exponent and, in the case of persistence, proved the existence of a unique forced wave solution that dominates the population profile of species in the long run. We also studied the effects of fluctuation and yielded that fluctuations in the shifting speed or location have negative impacts on the persistence of species. As one of future works, we can study the persistence of species, the existence of forced waves and the effects of fluctuation in (4.2) with the presence of chemotaxis.

Consider the following general parabolic-parabolic chemotaxis model on the whole space

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \mathbb{R}^N \\ \tau v_t = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N, \end{cases} \tag{4.3}$$

where $\tau > 0$ is a positive constant related to the diffusion rate of the chemical substance. In chapter 3, we studied the dynamical aspects of (4.3) with $\tau = 1$. The natural questions arise: can the results obtained in chapter 3 be extended to (4.3) with $\tau > 0$ (τ is not necessarily 1)? Most existing works concerning (4.3) are on bounded domains. For example, Winkler [67] studied the system (4.3) in a smooth bounded convex domain $\Omega \subset \mathbb{R}^N$ with Neumann boundary condition $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ for $x \in \partial\Omega$ and established the global existence and boundedness of non-negative classical solutions of system (4.3) provided that b is large enough. In [73], Zheng, Li, Bao and Zou extended Winkler's global existence result to bounded domains (not necessarily convex) of $\mathbb{R}^N$ for $\chi > 0$ and proved that if the logistic dampening $b > \frac{(N-2)_+}{N}\chi[C_{\frac{N}{2}+1}]^{\frac{1}{\frac{N}{2}+1}}$, where $C_{\frac{N}{2}+1}$ is a positive constant which is corresponding to the maximal Sobolev regularity, then (4.3) admits a unique, smooth,and bounded global non-negative solution. Recently, Issa and Shen [23] extended the global existence results obtained in both [67] and [73] to the general full chemotaxis model (4.3) with $u(a - bu)$ being replaced by a local as well as nonlocal time and space dependent logistic source. We point out that the methods used in [23, 67, 73] can not be adapted to study the global existence of classical solutions of (4.3) on the whole space because those methods are based on the finite measure of the domain. It is also very difficult to adapt the method used in our result Theorem 3.2 to study the global existence of classical solutions of (4.3) due to the different diffusion rates of the biological species and the chemical substance. We leave these interesting and challenging problems as our future works. We also leave the study of (4.3) in shifting environments with time-independent or time-dependent shifting speed as possible future works.

In addition to the aforementioned problems associated with chemotaxis models (1.1) and (4.3), in the future, I also plan to study dynamics of other chemotaxis models or other mathematical models arising from biology such as epidemic models. For example, I want to study the dynamics of two species chemotaxis model with Lotka-Volterra type competition term on the whole space

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + u(a_0 - a_1 u - a_2 v), \quad x \in \mathbb{R}^N, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + v(b_0 - b_1 u - b_2 v), \quad x \in \mathbb{R}^N, \\ w_t = \Delta w - \lambda w + d_1 u + d_2 v, \quad x \in \mathbb{R}^N. \end{cases} \tag{4.4}$$

In model (4.4), the first two parabolic equations describe the evolution of two biological species "u" and "v". The third parabolic equation models the evolution of a chemical substance "w" which is produced over time by these two biological species. For model (4.4), I plan to study the global existence and boundedness of classical solutions with given nonnegative initial functions in some spaces, the persistence, coexistence and extinction dynamics, the traveling wave solutions, etc.

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