Maximal Sets of Hamilton Cycles in Complete Multipartite Graphs IV

by

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Abstract

Finding the values of μ for which there exists a maximal set of μ edge-disjoint Hamilton cycles in the complete multipartite graph K_n^p has been considered in papers for over 20 years. This paper finally settles the problem by finding such a set in the last remaining open case, namely where μ is as small as possible (so its existence was still in doubt) when $n = 3$ and the number of parts, p, is 3 (mod 4).

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Chapter 1

History

A cycle containing every vertex of a graph G is called a Hamilton cycle of G. Let S be a set of edge-disjoint Hamilton cycles of graph G, and let $E(S)$ be the set of edges which form the cycles in S. S is called maximal if $G - E(S)$ contains no Hamilton cycle. Let K_n^p denote the complete p-partite graph with n vertices in each part. Decades of effort have been put into determining the spectrum for maximal sets of Hamilton cycles in K_n^p, namely finding the integers μ for which there exists a maximal set of μ of edge-disjoint Hamilton cycles in K_n^p. The point of this paper is to solve the final case, thereby completely settling this problem. An underlying motivation for finding the sizes of maximal sets in this and other settings is the algorithmic issue of how badly one might fail when using a greedy approach to find a Hamilton decomposition of G.

This spectrum problem originated in a paper of Hoffman, Rodger, and Rosa [7] who showed that there exists a maximal set of μ edge-disjoint Hamilton cycles in $K_n = K_1^n$ if and only if $\lfloor (n+3)/4 \rfloor \leq \mu \leq \lfloor (n-1)/2 \rfloor$. Bryant, El-Zanati, and Rodger [4] showed such a set exists in the complete bipartite graph $K_{n,n} = K_n^2$ if and only if $n/4 < \mu \leq n/2$. This result was extended by Daven, MacDougall and Rodger [6], when they showed that a maximal set of μ edge-disjoint Hamilton cycles exists in K_n^p if and only if

- $\lceil n(p-1)/4 \rceil \leq \mu \leq \lfloor n(p-1)/2 \rfloor$, and
- $\mu > n(p-1)/4$ if either n is odd and $p \equiv 1 \pmod 4$ or $p = 2$ or $n = 1$,

except possibly for the undecided case:

- $n \geq 3$ is odd, p is odd and $\mu \leq ((n+1)(p-1)-2)/4$.

The open cases in [6] were greatly reduced when Jarrell and Rodger [9] solved the problem for $n \geq 5$ and for all but the smallest possible values when $n = 3$, showing that a maximal set of Hamilton cycles of size μ exists when $n = 3$ and $\lceil n(p-1)/4 \rceil + 1 \leq \mu \leq \lfloor ((n+1)(p-1)-2)/4 \rfloor$ if $p \cong 3 \pmod 4$, and $\lceil n(p-1)/4 \rceil + 1 < \mu \leq \lfloor ((n+1)(p-1)-2)/4 \rfloor$ when $p \cong 1 \pmod 4$. Following this the only cases that remained in doubt were when $\mu = \lceil n(p-1)/4 \rceil + 1$ with $n = 3$, $p \equiv 1 \pmod 4$, and when $\mu = \lceil n(p-1)/4 \rceil$ with $n = 3$ and $p \equiv 3 \pmod 4$. Noble and Rodger [12] solved the case when $n = 3$, $p \equiv 1 \pmod 4$, and $\mu = \lceil n(p-1)/4 \rceil + 1$. In her dissertation, Noble [11] also found solutions for the smallest values of μ when $n = 3$ and $p = 7$ or $p = 11$ or $p \equiv 7 \pmod 8$ for $p \geq 31$. Thus the only unsettled cases are when $\mu = \lceil n(p-1)/4 \rceil$, $n = 3$, and p is either $3 \pmod 8$ or is in $\{15, 23\}$.

This paper will complete the spectrum problem by providing a solution to the entire $p \equiv 3 \pmod 4$ case, specifically by showing that when $n = 3$ and $p = 4m - 1$ for any $m \geq 2$, a maximal set of Hamilton cycles of K_n^p exists of size $\mu = \lceil n(p-1)/4 \rceil = 3m - 1$. The construction of Noble provided in [11] when $p \equiv 7 \pmod 8$ relied on the existence of constructions of smaller cases of p, and the maximality of the set of Hamilton cycles was shown by proving that the removal of the Hamilton cycles resulted in a graph with cut vertex. In this paper a direct construction for $p \equiv 3 \pmod 4$ will be given, and maximality will be ensured by obtaining a disconnected graph upon the removal of the edges in the Hamilton cycles.

The culmination of these results combined with Theorem 2.0.2 results in the following theorem stating the complete spectrum of sizes for maximal sets of Hamilton cycles of K_n^p. (Note the case when $n = 1$ from [6], in the second bullet point above, is unneeded in Theorem 1.0.1 as it is implied by (i) in Theorem 1.0.1.)

Theorem 1.0.1. *There exists a maximal set of edge-disjoint Hamilton cycles of size μ of K_n^p if and only if $\lceil n(p-1)/4 \rceil \leq \mu \leq \lfloor n(p-1)/2 \rfloor$ and $n(p-1)/4 < \mu$ when*

(i) *n is odd and $p \equiv 1 \pmod 4$, or*

(ii) $p = 2$.

1.1 Preliminary Results and Definitions

An H-decomposition of a graph G is a partition of $E(G)$, each element of which induces a copy of H. G is said to be decomposed into H if there exists an H-decomposition of G. Of particular interest in this paper are Hamilton Cycle decompositions (i.e. H-decompositions where H is a Hamilton cycle).

The following pair of lemmas are variants of well known results due to Laskar [2]. However, since particular 1-factors are of interest later, proofs are presented here.

Throughout this paper, let $F_1 = \{\{(0,p),(1,p)\} \mid 0 \leq p \leq 2m-1\}$ and $F_2 = \{\{(0,p+1),(1,p)\} \mid 0 \leq p \leq 2m-1\}$, reducing the sum (mod $2m$). The following result will be of use in Section 3, where F_2 will play a pivotal role in ensuring that some color classes are connected. For any graph G let λG denote the multigraph formed from G by replacing each edge with λ edges.

Lemma 1.1.1. *$2K_{2m,2m} - 2F_1 - 2F_2$ on the vertex set $\mathbb{Z}_{2m} \times \mathbb{Z}_{2m}$ can be decomposed into $2m-2$ Hamilton cycles.*

Proof. Observe that the graph induced by $F_1 \cup F_2$ is a Hamilton cycle. Also, $K_{2m,2m}$ can easily be decomposed into m Hamilton cycles. So the result follows by naming the vertices so that the first Hamilton cycle has edge set $F_1 \cup F_2$, then taking two copies of each Hamilton cycle. □

Throughout this paper, let $F_3 = \{\{(0,p),(1,p)\} \mid 2m \leq p \leq 4m-2\}$. The next result will also be used in Section 3.

Lemma 1.1.2. *$2K_{2m-1,2m-1} - 2F_3$ with vertex set $(\mathbb{Z}_{4m-1} \setminus \mathbb{Z}_{2m}) \times (\mathbb{Z}_{4m-1} \setminus \mathbb{Z}_{2m})$ can be decomposed into $2m-2$ Hamilton cycles.*

Proof. $K_{2m-1,2m-1}$ can easily be decomposed into $m-1$ Hamilton cycles and a 1-factor, so $K_{2m-1,2m-1} - F_3$ can be decomposed into $m-1$ Hamilton cycles. Therefore the result follows by taking two copies of each Hamilton cycle. □

Given a path P, with $|E(P)| \geq 2$, the two edges incident with a vertex of degree one are called end edges, and all other edges of P are called interior edges. The vertices of a path P of degree one will be called endpoints of P.

Let $\overline{B} = K_{2m,2m}$ with bi-partition of the vertex set being $\{U, L\}$, where $U = \{(0, j) \mid 0 \leq j \leq 2m-1\}$ and $L = \{(1, k) \mid 0 \leq k \leq 2m-1\}$. Let $\overline{\mathcal{P}} = \{\overline{P_i} \mid 0 \leq i \leq m-1\}$ where $\overline{P_i}$ is the Hamilton cycle in $\overline{B}$ defined as follows: for $0 \leq i \leq m-1$, $E(\overline{P_i}) = \{\{(0, j), (1, k)\} \mid k \equiv j + 2i \pmod{2m}$, or $k \equiv j + 2i + 1 \pmod{2m}, 0 \leq j \leq 2m-1\}$. $\overline{\mathcal{P}}$ is a well known Hamilton cycle decomposition of $K_{2m,2m}$. Let $B = K_{2m-1,2m}$ be formed from $\overline{B}$ by deleting vertex $(0, 2m-1)$, let P_i be the path formed by $\overline{P_i} - (0, 2m-1)$, and let $\mathcal{P} = \{P_i \mid 0 \leq i \leq m-1\}$.

Lemma 1.1.3. *$\mathcal{P}$ is a Hamilton path decomposition of $K_{2m-1,2m}$, Furthermore:*

1. *Each vertex in L is an endpoint of exactly one path in $\mathcal{P}$, and*

2. *The end edges of P_i are $\{(0, 0), (1, 2i)\}$ and $\{(0, 2m-2), (1, 2i-1)\}$.*

Proof. Since $\mathcal{P}$ is formed by deleting $(0, 2m-1)$ from each cycle of the Hamilton cycle decomposition $\overline{\mathcal{P}}$, it is a Hamilton path decomposition of B. Clearly, each vertex, w, in the larger part of B, namely L, is in a Hamilton cycle $\overline{P_i}$ containing the edge $\{w, (0, 2m-2)\}$; so w is an endpoint of P_i.

By construction of $\overline{P_i}$, in $\overline{P_i}$ the neighbor set of $(0, 2m-1)$ is $\{(1, 2i), (1, 2i-1)\}$, so $(1, 2i)$ and $(1, 2i-1)$ are the endpoints of P_i. In $\overline{P_i}$ the neighbors of $(1, 2i)$ and $(1, 2i-1)$ are $(0, 0)$ and $(0, 2m-2)$ respectively. Therefore in P_i $\{(0, 0), (1, 2i)\}$ and $\{(0, 2m-2), (1, 2i-1)\}$ are end edges. □

The following is an array with a variety of properties, each of which will be used in the proof of Theorem 2.0.2. Assuming $m \geq 3$, form the $(2m-1) \times 2m$ array $A(\mathcal{P})$, with rows and columns indexed by $U \setminus (0, 2m-2)$ and L respectively, by defining cell $((0, j), (1, k))$ to contain symbol i if and only if $\{(0, j), (1, k)\} \in E(P_i)$. The following result provides useful properties of $A(\mathcal{P})$.

Lemma 1.1.4. *Let $m \geq 3$. There exists a set, D, of $2m$ cells in $A(\mathcal{P})$ with the following properties.*

1. *Each symbol i, $0 \leq i \leq m-1$ appears in exactly two cells of D.*

2. *Each column of $A(\mathcal{P})$ contains exactly one cell of D.*

3. *Regarding the rows of $A(\mathcal{P})$:*

   (a) *Exactly one row of $A(\mathcal{P})$ contains no cells of D, specifically row 0*

   (b) *Exactly two rows of $A(\mathcal{P})$ contain exactly two cells of D, the two cells containing different symbols in each such row, specifically rows 1 and $2m-2$ and*

   (c) *Each of the remaining $2m-4 \neq 0$ rows contains exactly one cell of D.*

4. *If $\{e_1, e_2\} \subset P_i \cap D$ then in P_i there are an odd number of edges between e_1 and e_2.*

5. *Each edge corresponding to an element of D is an internal edge in a path in $\mathcal{P}$.*

6. *For each symbol i, the rows containing a cell of D containing i are not consecutive.*

Proof. Let $D = D_1 \cup D_2$ with $D_1 = \{((0, 2m-2-k), (1, k)) \mid 0 \leq k \leq 2m-3\}$ and $D_2 = \{((0, 2m-2), (1, 2m-2)), ((0, 1), (1, 2m-1))\}$. So $|D| = 2m$ as required. Each property will be considered in turn.

By the construction of P_i, cell $((0, 2m-2-k), (1, k))$ of D_1 contains symbol i if and only if $k \equiv 2m-2-k+2i \pmod{2m}$, or $k \equiv 2m-2-k+2i+1 \pmod{2m}$. The latter case implies $2k+2 \equiv 2i+1 \pmod{2m}$ which cannot happen. In the former case, $k \equiv 2m-2-k+2i \pmod{2m}$ must be satisfied, implying that $k+1 \equiv i \pmod{m}$. So, since $0 \leq k \leq 2m-3$, every symbol i appears twice in D_1 except $i = 0$ and $i = m-1$ which each appear once. From D_2, cell $((0, 2m-2), (1, 2m-2))$ contains symbol 0, and cell $((0, 1), (1, 2m-1))$ contains symbol $m-1$. Note these two cells also satisfy $k \equiv j + 2i \pmod{2m}$; this fact will be useful for showing Property (4). Therefore each symbol i, $0 \leq i \leq m-1$, appears in exactly two cells of D. So the first property is satisfied.

For $0 \leq k \leq 2m-3$ D_1 contains a cell in column k, and D_2 contains a cell in each of column $2m-2$ and $2m-1$. So the second property is satisfied.

No element of D is contained in row 0. Two elements of D are contained in each of rows 1 and $2m-2$, once in D_1, and once in D_2. Note, cell $((0, 2m-2), (1, 2m-2))$ contains symbol 0 and cell $((0, 2m-2), (1, 0))$ contains symbol 1, while cell $((0, 1), (1, 2m-1))$ contains symbol $m-1$ and cell $((0, 1), (1, 2m-3))$ contains symbol $m-2$. So if a row of $A(\mathcal{P})$ contains two cells of D, then those cells contain different symbols. The remaining nonempty set of rows each contain exactly one element of D. So the third property is satisfied.

By (1), let $e_1, e_2 \in E(P_i) \cap D$. By the argument for the first property, all cells $(j,k) \in D$ satisfy $k \equiv j + 2i \pmod{2m}$. Since edges $\{(1,k),(0,j)\}$ of P_i alternate satisfying $k \equiv j + 2i \pmod{2m}$, and $k \equiv j + 2i + 1 \pmod{2m}$, e_1 and e_2 must have an odd number of edges between them. So the fourth property is satisfied.

By Lemma 1.1.3 (2), end edges of P_i correspond to cells $((0,0),(1,x))$ and $((0,2m-2),(1,y))$ where x is even, and y is odd. Since row $(0,0)$ of $A(\mathcal{P})$ contains no cells of D and since the two elements of D contained in row $(0,2m-2)$ are $((0,2m-2),(1,0))$ and $((0,2m-2),(1,2m-2))$, all cells of D represent internal edges of a path in $\mathcal{P}$. So the fifth property is satisfied.

Consider cells of D_1 which are in consecutive rows of $A(\mathcal{P})$, namely $((0,2m-2-k),(1,k))$ and $((0,2m-2-(k+1)),(1,k+1))$ for some k, $0 \leq k \leq 2m-3$. If symbol i_1 appears in $((0,2m-2-k),(1,k))$ then i_1 satisfies $k+1 \equiv i_1 \pmod{m}$. If symbol i_2 appears in $((0,2m-2-(k+1)),(1,k+1))$ then i_2 satisfies $k+2 \equiv i_2 \pmod{m}$. Since $m \geq 3$, $i_1 \neq i_2$. So for two cells of D_1 in consecutive rows of $A(\mathcal{P})$, the symbols contained in those cells are different. Now we consider cells in D_2, and the cell of D_1 in the adjacent row. Cell $((0,2m-2),(1,2m-2))$ contains symbol 0, and cell $((0,2m-3),(1,1))$ contains symbol 2. Cell $((0,1),(1,2m-1))$ contains the symbol $m-1$, and cell $((0,2),(1,2m-4))$ contains symbol $m-3$. Therefore, for $0 \leq i \leq m-1$, the two cells of D containing symbol i are not in consecutive rows of $A(\mathcal{P})$. So the sixth property is satisfied. □

The following notation will be used throughout the paper. If G is an edge-colored graph then let G_i denote the i^{th} color class of G (i.e. the subgraph of G induced by the edges colored i.) At times, $d_{G_i}(v)$ will be denoted by $d_i(v)$ if it is clear which graph is being considered. An amalgamation of a graph G with associated amalgamation function $\psi : V(G) \to V(A)$ is the graph A with vertex set $V(A)$ and edge multi-set $E(A) = \{\{\psi(a), \psi(b)\} \mid \{a,b\} \in E(G)\}$, where $\{\psi(a), \psi(b)\}$ is a loop if $\psi(a) = \psi(b)$; so $|E(A)| = |E(G)|$. For any amalgamation function there is an associated amalgamation number function η_ψ defined by $\eta_\psi(c) = |\psi^{-1}(c)|$, for $c \in V(A)$. When ψ is clear, η_ψ will be denoted η, and G may be referred to as an η-detachment of A. The terms detachment and disentangling of an amalgamated graph will be used interchangeably, typically in line with whichever term was prevalent in relevant literature. The number of components in a graph G will be denoted by $\omega(G)$. The number of loops adjacent to vertex $w \in V(G)$ will be denoted $\ell_G(w)$, each contributing 2 to the degree of w. For vertices $u, v \in V(G)$, the multiplicity of edge $\{u, v\}$ will be denoted by $m_G(u, v)$.

Lemma 1.1.5. *Given an amalgamated graph A with amalgamation function ψ and edge cut set $\mathcal{E}$, then the disentangled edge of $\mathcal{E}$ are a cut set of edges in any detachment of A.*

Proof. Towards a contradiction suppose there is a detachment G of A in which the disentangled edges of $\mathcal{E}$ do not form a cut set of edges in G. Let A^- be the disconnected amalgamated graph formed by removing all edges in $\mathcal{E}$ from $E(A)$. Let $u', v' \in V(A)$ such that u' and v' are in separate components of A^-. By assumption for $u \in \psi^{-1}(u')$ and $v \in \psi^{-1}(v')$ there exists a path P connecting u and v which contains no edge of the disentangled $\mathcal{E}$. Without loss of generality it may be assumed this path has length 1. If it is longer than 1 say it has length n where $P = (u = u_0, u_1, \ldots, u_{n-1}, u_n = v)$. Now if $\psi(u)$ and $\psi(u_1)$ are in separate components of A^- replace v with u_1 and proceed with this path of length 1, else replace u with u_1 and a shorter path in G is obtained continue this process until a path of length 1 is found. Let this path of length one be called e. Note $e \notin \mathcal{E}$, yet in A $e = \{u', v'\} \Rightarrow\Leftarrow$ to $\mathcal{E}$ being a cut set. Therefore the disentangled edges of $\mathcal{E}$ are a cut set of edges of any detachment of A. $\square$

The following result will be crucial in showing that the edge-colorings defined for an amalgamation of K_n^p can be disentangled to an edge-coloring of K_n^p in which the color classes represent a maximal set of Hamilton cycles.

Previous results on maximal sets of Hamilton cycles used early, less general results like Theorem 1.1.6 in their proofs. But in all cases, the main stumbling block has been to find suitable edge-colorings of the relevant amalgamated graph to which a version of Corollary 2.6 can be applied, and indeed that is the focus of the proof of the main result in this paper. But also an issue in the cases settled here was to carefully choose the relevant amalgamated graph together with then choosing the appropriate edge-cut; neither of these choices was immediately evident.

In the following $x \approx y$ means that $\lfloor y \rfloor \leq x \leq \lceil y \rceil$.

Theorem 1.1.6. *[3] Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $w \in V(H)$, $\eta(w) = 1$ implies $\ell_H(w) = 0$. Then there exists a loopless η-detachment G of H with amalgamation function $\psi : V(G) \to V(H)$, such that G satisfies the following conditions:*

1. *$d_{G(j)}(u) \approx d_{H(j)}(w)/\eta(w)$ for each $w \in V(H)$, each $u \in \psi^{-1}(w)$, and each $j \in \mathbb{Z}_k$;*

2. *$m_G(u, v) \approx m_H(w, z)/(\eta(w)\eta(z))$ for every pair of distinct vertices $w, z \in V(H)$, each $u \in \psi^{-1}(w)$, each $v \in \psi^{-1}(z)$; and*

3. *If for some $j \in \mathbb{Z}_k$, $d_{H(j)}(w)/\eta(w)$ is an even integer for each $w \in V(H)$, then $\omega(G_j) = \omega(H_j)$.*

In the following corollary, note that since the amalgamation A of Γ is loopless only vertices in the same part of Γ are amalgamated.

Corollary 1.1.7. *Let Γ be a complete multipartite graph and A a loopless amalgamation of Γ with amalgamation function ψ, and associated amalgamation number function η. Let $\mathcal{E}$ be a cut set of edges of A. Suppose an edge-coloring of a spanning subgraph H of A exists using colors in $\{1, 2, \ldots c\}$, where for $1 \leq i \leq c$:*

1. *$d_i(v) = 2\eta(v)$, for each $v \in A$,*

2. *Color class i is connected, and*

3. *Every edge of $\mathcal{E}$ is in $E(H)$.*

Then there exists a maximal set of c Hamilton cycles in Γ.

Proof. Since an edge-coloring of A is required in order to apply Theorem 1.1.6, color the edges of $A - E(H)$ color 0. Since A is loopless, apply Theorem 1.1.6 to A to produce an η-detachment, G. Let $u, v \in V(G)$ so that $u \in \psi^{-1}(w)$ and $v \in \psi^{-1}(z)$. Since A is a loopless amalgamation of a complete multipartite graph, if w and z each contain a vertex from the same part of Γ then $m_A(w,z) = 0$, so by Theorem 1.1.6 (2) $m_G(u,v) = 0$. Similarly, if w and z each contain a vertex from different parts of Γ, then $m_A(w,z) = \eta(w)\eta(z)$, so by Theorem 1.1.6 (2) $m_G(u,v) = m_A(w,z)/(\eta(w)\eta(z)) = 1$. Therefore $G \cong \Gamma$.

Now suppose $1 \leq i \leq c$. Since $d_i(v) = 2\eta(v)$ for each $v \in A$, by Theorem 1.1.6 (1) $d_{G(i)}(v) = 2$ for each $v \in G$. Since each color class i is connected in A and since $d_i(v)/\eta(v) = 2$ for each i and each $v \in V(A)$, by Theorem 1.1.6 (3) each color class is connected in G. Therefore each color class is a Hamilton cycle of G after the detachment. Since $\mathcal{E}$ is a cut set of edges of A, when disentangled these edges will also be a cut set of edges of G. Since every edge of $\mathcal{E}$ is colored, the removal of all colored edges from G results in a disconnected graph. Therefore the set of Hamilton cycles which have been colored is maximal. $\square$

Chapter 2

Main Result

Using Corollary 1.1.7, Theorem 2.0.2 provides a construction for a maximal set S of $3m-1$ of Hamilton cycles of K_3^{4m-1} when $m \geq 3$ in which the complement $K_3^{4m-1} - E(S)$ is disconnected. While Noble [11] provided a solution for the case when $m = 2$, the complement in her example contained a cut vertex. The general proof of Theorem 2.0.2 does not quite work in the case $p = 7$, so it is handled separately here. Although it could be proved by explicitly defining 5 Hamilton cycles, the proof here uses Corollary 1.1.7 as a way of introducing readers to the proof technique used in Theorem 2.0.2 Throughout the remainder of the paper the vertex set of K_n^p will be defined as $V(K_n^p) = \{(u,v) \mid 0 \leq u \leq n, 0 \leq v \leq p\}$.

Lemma 2.0.1. *There exists a maximal set S of five edge-disjoint Hamilton cycles in K_3^7 for which the complement $K_3^7 - E(S)$ is disconnected.*

Proof. Let A be the amalgamation of K_3^7 formed by the amalgamation function f defined as follows: for $0 \leq p \leq 3$, $f(0,p) = f(1,p) = (0,p)$ and $f(2,p) = (1,p)$; for $4 \leq p \leq 6$, $f(0,p) = (0,p)$ and $f(1,p) = f(2,p) = (1,p)$. So, for $0 \leq p \leq 3$, $\eta(0,p) = 2$ and $\eta(1,p) = 1$, and for $4 \leq p \leq 6$, $\eta(0,p) = 1$ and $\eta(1,p) = 2$. Each of the five color classes will be defined by a union of walks as follows:

1. $((0,0),(1,1),(0,3),(1,0),(0,2),(1,3),(0,1),(1,2),(0,0)) \cup$
   $((0,0),(1,4),(0,1),(1,5),(0,2),(1,6),(0,3),(0,0)) \cup$
   $((0,4),(1,5),(0,6),(1,4),(0,5),(1,6),(0,4))$

2. $((0,0),(1,1),(0,3),(1,0),(0,2),(1,3),(0,1),(1,2),(0,0))\cup$

   $((0,1),(1,6),(0,0),(1,5),(0,3),(1,4),(0,2),(0,1))\cup$

   $((0,4),(1,5),(0,6),(1,4),(0,5),(1,6),(0,4))$

3. $((0,0),(1,4),(0,1),(1,5),(0,2),(1,6),(0,3))\cup$

   $((0,1),(1,6),(0,0),(1,5),(0,3),(1,4),(0,2))\cup$

   $((0,0),(1,3),(0,6),(0,5),(1,2),(0,3))\cup$

   $((0,1),(1,0),(0,4),(1,1),(0,2))$

4. $((0,0),(1,4),(0,1),(1,5),(0,2),(1,6),(0,3))\cup$

   $((0,1),(1,6),(0,0),(1,5),(0,3),(1,4),(0,2))\cup$

   $((0,0),(1,3),(0,5),(0,4),(1,2),(0,3))\cup$

   $((0,1),(1,0),(0,6),(1,1),(0,2))$

5. $((0,0),(1,4),(0,1),(1,5),(0,2),(1,6),(0,3),(0,1),(1,6),(0,0),(1,5),(0,3),(1,4),(0,2))\cup$

   $((0,0),(0,4),(1,3),(1,0),(0,5),(1,1),(1,2),(0,6),(0,2))$

Each of the three properties of Corollary 1.1.7 is now considered in turn.

It is readily checked for $1 \le i \le 5$ and each $v \in V(A)$ that $d_i(v) = 2\eta(v)$.

In the first color class, since $(0,0)$ is in the first and second walks, and $(1,4)$ is in the second and third walks, the first color class is connected. Similarly, to see that each other color class is connected, consider the vertices: $(0,0)$ and $(1,4)$ in the 2^{nd} color class; $(0,0)$ and $(0,1)$ in the 3^{rd} and 4^{th} color classes; and $(0,0)$ in the 5^{th} color class.

The set of edges of A of the set $\mathcal{E} = \{(0,p),(1,q)\}$ $0 \le p,q \le 6$ are a cut set of edges in A. It is also readily checked that all edges of $\mathcal{E}$ are colored.

Therefore, by Corollary 1.1.7, there exists a maximal set S of five Hamilton cycles of K_3^7 for which the complement, $K_3^7 - E(S)$ is disconnected. □

The approach which has been used in previous papers to find a maximal set $\mathcal{H}$ of Hamilton cycles in Γ is to ensure maximality by forcing the complement of $\mathcal{H}$ to be disconnected or to contain a cut vertex. A similar approach will be used here. K_3^{4m-1} will be amalgamated to form A. A cut set of edges $\mathcal{E}$ in A will be identified; G will be used denote $A[\mathcal{E}]$. An edge in A

is said to be a pure edge if it is not contained in $\mathcal{E}$. A subgraph H of A will be defined, induced by edges of G together with some carefully chosen pure edges in $E(A) - \mathcal{E}$. Then $3m - 1$ colors will be used to color edges of H. More precisely, for $1 \leq i \leq 3m-1$, exactly $3(4m-1)$ edges in A will be colored i so that:

1. Color classes are edge-disjoint,
2. All of the edges of the cut set $\mathcal{E}$ are colored,
3. $d_{A_i}(v) = 2\eta(v)$ for all $v \in V(A)$, and
4. Each color class is connected.

Note that not every edge of A is colored. Using Corollary 1.1.7 it will be shown that A can be disentangled to form K_3^{4m-1} so that each color class is a Hamilton cycle of K_3^{4m-1}, and since a cut set of edges is contained within them, this will be a maximal set of Hamilton cycles in K_3^{4m-1}.

In the following proof, spanning cycles/paths of various graphs are discussed. To help avoid confusion, the term Hamilton cycle/path will be used to describe cycles/paths of K_3^{4m-1} and its amalgamation A, whereas they will be referred to as spanning cycles/paths in smaller subgraphs of A.

Theorem 2.0.2. *Let $m \geq 2$. There exists a maximal set S of $3m-1$ Hamilton Cycles of K_3^{4m-1} for which $K_3^{4m-1} - E(S)$ is disconnected.*

Proof. By Lemma 2.0.1 assume that $m \geq 3$. Let $\Gamma = K_3^{4m-1}$. Let A be the amalgamation of Γ formed by f defined as follows: for $0 \leq p \leq 2m-1$, $f(0,p) = f(1,p) = (0,p)$ and $f(2,p) = (1,p)$; for $2m \leq p \leq 4m-2$, $f(0,p) = (0,p)$ and $f(1,p) = f(2,p) = (1,p)$. The amalgamation conveniently partitions the vertices into four sets, defined as follows: $S_1 = \{(0,p) \mid 0 \leq p \leq 2m-1\}$, $S_2 = \{(1,p) \mid 0 \leq p \leq 2m-1\}$, $S_3 = \{(0,p) \mid 2m \leq p \leq 4m-2\}$, and $S_4 = \{(1,p) \mid 2m \leq p \leq 4m-2\}$. So $\eta(v) = 2$ if $v \in S_1 \cup S_4$ and $\eta(v) = 1$ if $v \in S_2 \cup S_3$.

Let $\mathcal{E} = \{\{(0,p),(1,q)\} \mid 0 \leq p,q \leq 4m-2, p \neq q\}$; $\mathcal{E}$ is the set of all edges for which one endpoint is in $S_1 \cup S_3$ and the other endpoint is in $S_2 \cup S_4$. Therefore $\mathcal{E}$ is a cut set of edges in A. Let $G = A[\mathcal{E}]$.

In what follows, a spanning subgraph H of A containing all the edges $\mathcal{E}$ will be colored to satisfy the conditions of Corollary 1.1.7, thus proving the theorem.

Since $3m-1$ Hamilton cycles are required in the desired maximal set of K_3^{4m-1}, H will be given a $(3m-1)$-edge-coloring. Then from Corollary 1.1.7, it is necessary that for all $v \in V(A)$, exactly $2(3m-1)\eta(v)$ edges incident with v are colored (namely $2\eta(v)$ for each color i for $1 \leq i \leq 3m-1$). However, in G, the degree of each vertex in $S_1 \cup S_3, S_2$, and S_4, is $2(3m-1)\eta(v)-2, 2(3m-1)\eta(v)-1$, and $2(3m-1)\eta(v)$ respectively. Therefore G will be supplemented with pure edges to form H so that for all v in A, v is incident with $2(3m-1)\eta(v)$ colored edges. To be precise, for all v in $S_1 \cup S_3, S_2$, and S_4 v must be incident with $2, 1$, and 0 colored pure edges respectively, to be able to apply Corollary 1.1.7 to the resulting spanning subgraph of A.

The edges of $\mathcal{E}$ will be partitioned into color classes and pure edges will be determined using three steps.

Step 1 In this step, $2m-2$ color classes are defined, each of which will contain:

1. A spanning cycle of $G[S_1 \cup S_2]$,
2. A spanning cycle of $G[S_3 \cup S_4]$,
3. A spanning path of $G[S_1 \cup S_4]$, and
4. A pure edge contained in $A[S_1]$.

Note that $G[S_1 \cup S_2] \cong 2K_{2m,2m} - 2F_1$. By Lemma 1.1.1, $G[S_1 \cup S_2]$ can be decomposed into $2m-2$ spanning cycles and $2F_2$. Let $T_1, T_2, \ldots, T_{2m-2}$ be these cycles and edge-color T_i with color i for $1 \leq i \leq 2m-2$. The $4m$ edges of $2F_2$ will be colored in Step 2.

Note that $G[S_3 \cup S_4] \cong 2K_{2m-1,2m-1} - 2F_3$. By Lemma 1.1.2, $G[S_3 \cup S_4]$ can be decomposed into $2m-2$ spanning cycles. Let $S_1, S_2, \ldots, S_{2m-2}$ be these cycles and edge-color S_i with color i for $1 \leq i \leq 2m-2$.

Note that $G[S_1 \cup S_4] \cong 4K_{2m,2m-1}$. By Lemma 1.1.3, $K_{2m,2m-1}$, can be decomposed into m spanning paths. So $2K_{2m,2m-1}$ on the vertex set with bipartition $\{S_1, S_4\}$ can be decomposed into the $2m$ spanning paths $P_1, P_2, \ldots, P_{2m}$ where $P_{2i-1} = P_{2i}$ for $1 \leq i \leq m$. Let $(0, p(i))$

and $(0, q(i))$ be the two distinct endpoints of P_i for $0 \leq i \leq 2m$. Edge-color P_i with color i for $1 \leq i \leq 2m-2$. P_{2m} and P_{2m-1} will be edge-colored in Step 3, and the remaining $2K_{2m,2m-1}$ will be edge-colored in Step 2.

For $1 \leq i \leq 2m-2$ color the pure edge $\{(0, p(i)), (0, q(i))\}$ in A with color i.

Consider color class H_i for $1 \leq i \leq 2m-2$. Since H_i contains a spanning path on each of $G[S_1 \cup S_2]$, $G[S_3 \cup S_4]$ and $G[S_1 \cup S_4]$, it is connected. For each $v \in S_3 \cup S_2$, v is in a cycle colored i; therefore $d_i(v) = 2 = 2\eta(v)$. For each $v \in S_1 \cup S_4$, v is in a cycle colored i, and is an internal vertex of a path colored i unless v is one of the two endpoints of P_i, namely $(0, p(i))$ and $(0, q(i))$. These endpoints are joined by a pure edge colored i. Therefore $d_i(v) = 4 = 2\eta(v)$ for each $v \in S_1 \cup S_4$. Therefore for $1 \leq i \leq 2m-2$, H_i is connected and for all $v \in A$ $d_i(v) = 2\eta(v)$.

Step 2 In this step, m color classes are defined, each of which contains:

1. Four edges of $2F_2$,

2. Two spanning paths of $G[S_1 \cup S_4]$, and

3. A modified spanning path of $G[S_2 \cup S_3]$, modified by replacing two carefully chosen edges with by a pure edge in $A[S_3]$.

Note that $G[S_2 \cup S_3] \cong K_{2m,2m-1}$. Apply Lemma 1.1.3 to $K_{2m,2m-1}$ with $L = S_2$ and $U = S_3$ to decompose it into the set $\mathcal{P}$ of m spanning paths, where $\mathcal{P} = \{\rho_{2m-1}, \rho_{2m}, \ldots, \rho_{3m-2}\}$. Let $(1, p(i))$ and $(1, q(i))$ be the endpoints of ρ_i for $2m-1 \leq i \leq 3m-2$ (necessarily these endpoints are in S_2). Create the array $A(\mathcal{P})$, with columns indexed by S_2 and rows indexed by S_3. Let D be the set of cells in $A(\mathcal{P})$ as described in Lemma 1.1.4. For $2m-1 \leq i \leq 3m-2$, except for the two edges in ρ_i (using Lemma 1.1.4 (1)) which correspond to elements of D, color all other edges of ρ_i with color i. By Lemma 1.1.4 (1), let $((0, j_1(i)), (1, k_1(i)))$ and $((0, j_2(i)), (1, k_2(i)))$ be the two cells of D containing entry i. By Lemma 1.1.4 (6) $j_1(i) \neq j_2(i)$ and by Lemma 1.1.4 (2) $k_1(i) \neq k_2(i)$. Color the pure edge $\{(0, j_1(i)), (0, j_2(i))\}$ with i; doing so ensures that $d_i(v) = 2 = 2\eta(v)$ for $2m-1 \leq i \leq 3m-2$ and all $v \in S_3$. As described before Step 1, vertices of S_3 need to be incident with exactly two pure edges in the final edge-coloring. Lemma 1.1.4 (3) guarantees that no vertex of S_3 has more than two incident pure edges colored in this step. Specifically $(0, 2m)$ has no incident pure edges colored, $(0, 2m+1)$ and $(0, 4m-2)$ have two incident pure edges colored, and the remaining elements of S_3 all have one incident pure edge colored in this step. If G is a graph with vertex set $\mathbb{Z}_a \times \mathbb{Z}_b$, then two vertices (x, y) and (x, z) are said to be consecutive if $|y - z| = 1$. It will be important later to note that by Lemma 1.1.4 (6):

$$\text{vertices } (0, j_1(i)) \text{ and, } (0, j_2(i)) \text{ are not consecutive.} \qquad (*)$$

Since ρ_i is a spanning path of $G[S_2 \cup S_3]$, and since by Lemma 1.1.4 (5), edges $e_1(i) = \{(0, j_1(i)), (1, k_1(i))\}$ and $e_2(i) = \{(0, j_2(i)), (1, k_2(i))\}$ are both internal edges of ρ_i, by Lemma 1.1.4 (6), $\phi(i) = \rho_i - e_1(i) - e_2(i)$ consists of three vertex-disjoint paths which together span $V(G[S_2 \cup S_3])$. By Lemma 1.1.4 (4), $(0, j_1(i))$ and $(0, j_2(i))$ are in separate components of ϕ_i, so the addition of the pure edge $e_3(i) = \{(0, j_1(i)), (0, j_2(i))\}$ to $\phi(i)$ will not create a cycle. So $\phi'(i) = \rho_i - e_1(i) - e_2(i) + e_3(i)$ consists of two vertex disjoint-paths, the union of their vertex sets being $S_2 \cup S_3$, whose four endpoints are the vertices in $\{(1, p(i)), (1, q(i)), (1, k_1(i)), (1, k_2(i))\}$.

By Lemma 1.1.4 (2,5), vertices $(1, p(i)), (1, q(i)), (1, k_1(i))$ and $(1, k_2(i))$ are independent of each other. Also since, each $v \in S_2$ is an endpoint of some ρ exactly once, and by Lemma 1.1.4 (2), each $v \in S_2$ is an endpoint of an edge represented by a cell of D exactly once. Therefore the edges of $2F_2$ are colored as follows: for $2m - 1 \leq i \leq 3m - 2$ use color i to color the edges $\{(1, p(i)), (0, p(i) + 1)\}$ $\{(1, q(i)), (0, q(i) + 1)\}$ $\{(1, k_1(i)), (0, k_1(i) + 1)\}$ $\{(1, k_2(i)), (0, k_2(i) + 1)\}$, where addition is done (mod $2m$). Note $F_2 = \{\{(1, p(i)), (0, p(i) + 1)\}, \{(1, q(i)), (0, q(i)+1)\} \mid 2m-1 \leq i \leq 3m-2\} = \{\{(1, k_1(i)), (0, k_1(i)+1)\}, \{(1, k_2(i)), (0, k_2(i)+1)\} \mid 2m - 1 \leq i \leq 3m - 2\}$.

As previously noted $G[S_1 \cup S_4] \cong 4K_{2m,2m-1}$. At this juncture, the uncolored portion of $G[S_1 \cup S_4]$ is isomorphic to $2K_{2m,2m-1}$. These edges are colored as follows. Let $\sigma_{2m-1}, \sigma_{2m}, \ldots, \sigma_{3m-2}$ be a decomposition of $K_{2m,2m-1}$ into spanning paths (renaming vertices as needed) so that the endpoints of σ_i are $(0, p(i)+1)$ and $(0, q(i)+1)$. Let $\tau_{2m-1}, \tau_{2m}, \ldots, \tau_{3m-2}$ be a decomposition of $K_{2m,2m-1}$ into spanning paths (again renaming vertices as needed) so that the endpoints of τ_i are $(0, k_1(i) + 1)$ and $(0, k_2(i) + 1)$, again with addition (mod $2m$). Edge-color σ_i, and τ_i with color i for $2m - 1 \leq i \leq 3m - 2$.

Consider color class H_i for $2m - 1 \leq i \leq 3m - 2$. H_i is connected since, contains a spanning path of $G[S_1 \cup S_4]$, and since $\phi'(i)$ spans the vertices of $S_2 \cup S_3$ and must be in the same component of H_i as S_1 due to the 4 edges of $2F_2$ colored i. For each $v \in S_1$, v is either an internal vertex of both σ_i and τ_i, or is an endpoint of one and an internal vertex of the other the endpoint being incident to an edge of $2F_2$ colored i; in both cases $d_i(v) = 4 = 2\eta(v)$. For, each $v \in S_2$, v is either an internal vertex of a path in $\phi'(i)$, or is an endpoint of a path in $\phi'(i)$, and adjacent to an edge of $2F_2$ colored i; again in both cases $d_i(v) = 2 = 2\eta(v)$. For each $v \in S_3$, v is an internal vertex of a path in $\phi'(i)$, so $d_i(v) = 2 = 2\eta(v)$. For each $v \in S_4$, v is an internal vertex of σ_i and of τ_i, so $d_i(v) = 4 = 2\eta(v)$. Therefore, for $2m - 1 \leq i \leq 3m - 2$, H_i is connected and $d_i(v) = 2\eta(v)$ for all $v \in A$.

Step 3 In this step, the final color class H_{3m-1} is defined and contains:

1. $E(P_{2m-1})$ and $E(P_{2m})$, each of which is a spanning path of $G[S_1 \cup S_4]$,

2. Edges corresponding to elements of D, and

3. $2m + 1$ carefully chosen pure edges.

Let Υ be the bipartite subgraph of $G[S_2 \cup S_3]$ with edge set corresponding to the elements of D. By Lemma 1.1.4 (3), the $2m - 1$ components of Υ are: two paths of length 2, $2m - 4$ paths of length 1, and the isolated vertex $(0, 2m)$. $2m - 2$ pure edges are required to connect these $2m - 1$ components. Since rows $2m + 2$ through $4m - 2$ of $A(\mathcal{P})$ only contain one element of D, $\{(0, i) \mid 2m + 2 \leq i \leq 4m - 2\}$ is the set of vertices which are the endpoints in S_3 of the paths of length one. Also, it means that these vertices are incident with a single colored pure edge from Step 2, and therefore each such vertex, v, must be incident with an additional colored pure edge in order that $d_H(v) = 2\eta(v)$. So, add the $m - 3$ pure edges in $\{\{(0, 2m + 3 + 2i), (0, 2m + 4 + 2i)\} \mid 0 \leq i \leq m - 4\}$ to Υ to form Υ_1 (this set of edges is empty when $m = 3$.) Since each of these $m - 3$ pure edges was chosen to connect two paths of length 1, they form paths of length 3. Add the pure edge $\{(0, 2m), (0, 2m + 2)\}$ to Υ_1, to form

Υ_2; this edge joins the previously isolated vertex to a path of length 1. It will be important to note the pure edges with both endpoints in S_3 colored in this step satisfy:

all edges join consecutive vertices, except for one edge which is incident to $(0, 2m)$. (†)

Each of the $m + 1$ components of Υ_2 is a path. All endpoints of these paths are in S_2, except for the two exceptional components each of which contains exactly one endpoint in S_2 (the other endpoint is in S_3.) Name the two exceptional components C_1 and C_{m+1}, and let their endpoints in S_2 be ω_1 and υ_{m+1} respectively. Name the rest of the components $C_2, C_3, \ldots C_m$ arbitrarily, with the endpoints of path C_i being, say, ω_i and υ_i for $2 \leq i \leq m$. Now add the m pure edges in $\{\{\omega_i, \upsilon_{i+1}\} \mid 1 \leq i \leq m\}$ to Υ_2, to form Υ_3. Since the endpoints of these m pure edges are in different components of Υ_2, Υ_3 is connected, so Υ_3 is a spanning path of $A[S_2 \cup S_3]$ whose endpoints are $(0, 2m)$ and $(0, 4m - 3)$.

To ensure Υ_3 and P_{2m-1} are in the same component, color the edges $\{(0, p(2m-1)), (0, 2m)\}$ and $\{(0, q(2m - 1)), (0, 4m - 3)\}$ with $3m - 1$. Finally, color with $3m - 1$ an edge joining the endpoints of P_{2m}, namely the edge $\{(0, p(2m)), (0, q(2m))\}$. This ensures that $d_{3m-1}(v) = 2\eta(v)$ for all $v \in V(A)$.

In order to apply Corollary 1.1.7, H must be a subgraph of A; that is, between each pair of vertices, the number of edges in H must be at most the number of edges in A. For $v, w \in V(A)$, $m_A(v, w) = \eta(v)\eta(w)$. Since all edges in $\mathcal{E}$ have been colored $m_H(v, w) = m_A(v, w)$ for all $v \in S_1 \cup S_3$ and all $w \in S_2 \cup S_4$. So, it remains to show that the pure edges defined to be part of H are also in $E(A)$. This is easy to establish for $v \in S_1 \cup S_2 \cup S_4$ since $d_{H-\mathcal{E}}(v) \leq \eta(v)$, so for all $v \in V(A)$, $m_H(v, w) \leq \eta(v) \leq \eta(v)\eta(w) = m_A(v, w)$. For $v, w \in S_3$, $\eta(v)\eta(w) = 1$ so it needs to be shown that no pair of vertices in S_3 is joined by more than one colored edge. By (*), all pure edges with both endpoints in S_3 colored in Step 2 are non-consecutive. By (†), each pure edge with both endpoints in S_3 colored in Step 3 either joins consecutive vertices, or is

$\{(0, 2m), (0, 2m+2)\}$. Therefore, $H[S_3]$ is a simple graph. So it is the case that $E(H) \subset E(A)$.

Since:

1. H is a subgraph of A,
2. H_i is connected for $0 \leq i \leq 3m - 2$, and
3. $d_i(v) = 2\eta(v)$ for $0 \leq i \leq 3m - 2$ and $v \in V(H)$,

by Corollary 1.1.7 there exists a maximal set of $3m - 1$ Hamilton cycles of K_3^{4n-1}. $\square$

Chapter 3

An Attempt Which Generated More Questions Than Answers

This chapter details an initial attempt to complete the spectrum of sizes for maximal sets of Hamilton cycles by following the steps Noble used in [11]. In order to gain familiarity with the problem and the technique of amalgamations, some of Noble's results were independently redone. Let G be the complete multipartite graph K_3^{8m-1}. Let $\bar{A}$ be the amalgamated graph of G formed by amalgamation function f defined as follows: for $i < m$, $f(0,i) = f(1,i) = f(2,i) = (0,i)$; for $m \leq i < 4m-1$, $f(0,i) = f(1,i) = (0,i)$ and $f(2,i) = (1,i)$; for $0 \leq j \leq 2, i = 4m-1$, $f(j, 4m-1) = (j, 4m-1)$; for $4m \leq i < 7m-1$, $f(0,i) = (0,i)$ and $f(1,i) = f(2,i) = (1,i)$; for $i \geq 7m-1$, $f(0,i) = f(1,i) = f(2,i) = (1,i)$. Note for $i < m$, $\eta(0,i) = 3$; for $m \leq i < 4m-1$, $\eta(0,i) = 2$ and $\eta(1,i) = 1$; for $i = 4m-1$ $\eta(j,i) = 1$; for $4m \leq i < 7m-1$, $\eta(0,i) = 1$ and $\eta(1,i) = 2$; and for $i \geq 7m-1$, $\eta(1,i) = 3$.

Let A be the subgraph of $\bar{A}$ induced by the union of three sets of edges namely, $A = \bar{A}[E]$ where $E = E_1 \cup E_2 \cup E_3$. Firstly, for $0 \leq i \leq 8m-2, 0 \leq j \leq 8m-2$, $E_1 = \{\{(0,i),(1,j)\} \mid i \neq j\}$, secondly $E_2 = \{\{(2,4m-1),(j,i)\} \mid \eta(j,i) = 2\} \cup \{\{(2,4m-1),(m,1)\},\{(2,4m-1),(7m-2,0)\}\}$, and lastly the pure edges $E_3 = \{\{(0,i),(0,i+1)\} \mid 4m-1 \leq i < 7m-1\} \cup \{\{(1,i),(1,i+1)\} \mid m \leq i \leq 4m-1\}$. Figure 3.1 shows the amalgamated graph A in the case $m = 1$, not including the edges of E_1 which are all the edges from a top vertex to bottom vertex as long as said vertices are in different parts.

Now an edge-coloring of A using $6m-1$ colors is required so that for each vertex v and each color i, $d_i(v) = 2\eta(v)(6m-1)$, and each color class is connected. Then applying Theorem 1.1.6 when the color classes are disentangled they form a maximal set of Hamilton

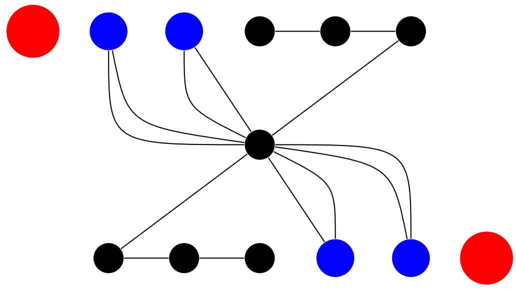

Figure 3.1: $A \setminus E_1$

cycles of G. Since $G \setminus E_1$ will contain $(2, 4m-1)$ as a cut vertex. The following theorem simplifies the task of finding such an edge coloring.

Theorem 3.0.1. *If there exist $6m-1$ edge-disjoint Hamilton cycles in A in which all edges of E_2 and E_3 are colored, then there exists a maximal set of $6m-1$ edge-disjoint Hamilton cycles in G.*

Proof. Let H be a set of $6m-1$ edge-disjoint Hamilton cycles in A, in which all edges of E_2 and E_3 are colored. Now certainly every color class is connected as it contains a Hamilton cycle. Note in A each vertex has degree $2\eta(v)(6m-1)$. Since H colors $2(6m-1)$ edges incident with each v, the only vertices which have uncolored edges are those vertices v such that $\eta(v) > 1$. The only uncolored edges are those in E_1. Since the number of uncolored edges at a vertex is $2(\eta(v)-1)(6m-1)$, any evenly equitable coloring of the uncolored edges will result $d_i(v) = 2\eta(v)(6m-1)$ for each v. Since $A[E_1]$ is a bipartite graph, the uncolored edges can be evenly equitably colored. Therefore by Theorem 1.1.6 there exists a maximal set of $6m-1$ Hamilton cycles of G. $\square$

Theorem 3.0.1 simplifies the process of finding the coloring of A since the only edges which explicitly need to be colored are those in the Hamilton cycles of A, and the remainder can be colored with any evenly equitable coloring. For K_3^7 and K_3^{15} there are $6m-1$ edge-disjoint Hamilton cycles which contain all edges of E_2 and E_3 for their corresponding amalgamation A, these cycles are contained in Appendix A. It is worth noting that prior to Theorem 2 this solved the open case regarding K_3^{15}.

At this point the strategy was to use the now known smaller cases to construct Hamilton cycles of the amalgamated larger cases. However, every attempt to do so resulted in requiring each color to appear at least twice on a specific set of edges, of the unamalgamated graph. When using Theorem 3.0.1 the color on many of the edges is not explicitly stated, combined with the fact that Theorem 1.1.6 simply states such a disentangling exists and makes no claim as to exactly what color an edge of the disentangled graph is, Theorem 3.0.1 could not be used to ensure specific edges received certain colors. After one look at the forty-five vertices and nine-hundred forty-five edges of K_3^{15}, practicality dictated this line of reasoning be abandoned in favor of the methods used in Chapter 2.

During this process many interesting questions were raised about how a Hamilton decomposition of a smaller graph can be used to create a Hamilton decomposition of a larger graph. Some of these questions are addressed in the following chapter. The most pertinent in regards to this method is what, if anything, can be said about the edges of each decomposition?

Chapter 4

Preserving Hamilton Cycles

Stemming from the idea in Chapter 3 of using known Hamilton decompositions to construct Hamilton decompositions of larger graphs, the question arises of when can a Hamilton decomposition be found in which many edges are partitioned in accordance with a smaller decomposition.

Let H be a proper subgraph of G. Let P be an edge-coloring of H with colors in $\{1, \ldots, p\}$ where P_j is the j^{th} color class, and Q be an edge-coloring of G with colors in $\{1, \ldots q\}$, $p < q$. P is said to be preserved in Q if for each color there exists $e \in E(H)$ so that $P(e) = Q(e)$. So, the color of edge e in the smaller graph is preserved in the larger graph.

P is said to be well-preserved in Q if

1. P is preserved in Q, and
2. for each edge coloring F of G,

$$\sum_{j=1}^{p} |P_j \cap F_j| \leq \sum_{j=1}^{p} |P_j \cap Q_j|.$$

For the purposes of this chapter, a Hamilton decomposition is thought of as an edge-coloring, P, in which the j^{th} color class, P_j, is a Hamilton cycle for each color j. It is clear that a Hamilton decomposition of K_n cannot be embedded in a Hamilton decomposition of K_{n+2i}, where $i \in \mathbb{N}, n$ is odd. So instead, the question to ask is: can an arbitrary Hamilton decomposition of K_n be well-preserved in a Hamilton decomposition of K_{n+2i}? The following result from Hilton [8] will be crucial in showing that it is possible to well-preserve Hamilton decompositions.

Theorem 4.0.1. *Let $1 \leq r < 2m+1$. An edge-coloring of K_r with m colors $c_1, \ldots, c_m$ can be extended to a Hamilton decomposition of K_{2m+1} in which each color class of the edge coloring of K_r is incorporated into a Hamilton cycle of K_{2m+1} if and only if each color class of the edge-coloring of K_r consists of at most $2m+1-r$ disjoint paths, considering a vertex of K_r with no incident edges colored c_j as a path of length zero.*

Since Hamilton cycles are of interest here the following corollary to Theorem 4.0.1 will be of use.

Corollary 4.0.2. *An edge-coloring of K_{2m+1} using $m+i$ colors can be embedded in a Hamilton cycle decomposition of $K_{2m+1+2i}$ if each color class is a linear forest containing at least $2m+1-2i$ edges.*

Proof. By Theorem 4.0.1 each color class must consist of at most $(2m+1+2i)-(2m+1) = 2i$ disjoint paths. This is equivalent to saying each color class is a linear forest with at most $2i$ components. Since there are $2m+1$ vertices, and every edge of the linear forest reduces the number of components by one, $2m+1-2i$ edges are needed for each color. □

Theorem 4.0.3. *It is possible to preserve a Hamilton decomposition of K_{2m+1} in a Hamilton decomposition of $K_{2m+1+2i}$, for $i \geq 1$.*

Proof. Let P be a Hamilton decomposition of K_{2m+1}, using colors in $\{1, \ldots, m\}$. Consider an edge-colored subgraph H induced by any set of $\min(m+i, 2m)$ vertices. Since $\min(m+i, 2m) > (2m+1)/2$ each of the original m colors appear on at least one edge of H. To complete the proof Theorem 4.0.1 is used to show this coloring can be embedded in $K_{2m+1+2i}$. Since H only contains $m+i$ vertices, and each original color appears on at least one edge of H, each color class consists of at most $m+i$ disjoint paths. Therefore by Theorem 4.0.1 this coloring can be incorporated in a Hamilton decomposition Q of $K_{2m+1+2i}$. Since each color in $\{1, \ldots, m\}$ appeared on H, at least one edge of each color is preserved from K_{2m+1} in a Hamilton cycle of $K_{2m+1+2i}$. □

Corollary 4.0.4. *It is possible to well-preserve a Hamilton decomposition of K_{2m+1} in a Hamilton decomposition of $K_{2m+1+2i}$, for $i \geq 1$.*

Proof. Since only finite graphs are considered here a maximum for $\sum_{j=1}^{m} |P_j \cap Q_j|$ must exist. $\square$

When dealing with preserving Hamilton decompositions it is clear that some edges of the smaller cycles must change color in order to preserve them in larger cycles, this suggests the following definitions. Let H be a Hamilton cycle decomposition of K_{2m+1}, then $wp(m, i, H)$ will denote the minimum number of edges of K_{2m+1} which must be recolored in order to well-preserve H in $K_{2m+1+2i}$. Let $wp(m, i) = \min\{wp(m, i, H) \mid H$ is a Hamilton decomposition of $K_{2m+1}\}$. By the previous Corollary $wp(m, i)$ exists. Now the question looms, what is $wm(p, i)$ numerically?

Lemma 4.0.5. $wp(m, i) \geq \max\{m, i(2m + 1 - 2i)\}$.

Proof. Theorem 4.0.1 states that to extend an edge-coloring each component of each color class must be a path. Since the m original color classes on K_{2m+1} are all cycles, each original color class must have at least one edge change color. Therefore $wp(m, i) \geq m$. By Corollary 4.0.2 each of the i new colors (those which will appear in the Hamilton decomposition of $K_{2m+1+2i}$ and did not appear in the Hamilton decomposition of K_m) must appear on at least $2m + 1 - 2i$ edges of K_{2m+1}. Therefore $wp(m, i) \geq i(2m + 1 - 2i)$. $\square$

Theorem 4.0.6. *Let $n = 2m + 1$. For each $i \geq n/4$, $wp(m, i) = \max\{m, i(2m + 1 - 2i)\}$.*

Proof. Let $H = \{h_k \mid 1 \leq k < m\}$ be a Hamilton decomposition of K_n, of size m. The aim is to recolor edges of K_{2m+1} in order to apply Theorem 4.0.1. Since each color class requires at least $n - 2i$ edges, the original color classes can never have more than $2i$ edges change color. Thus for $i \geq n/4$ we have $2i \geq n - 2i$. For $n/4 \leq i \leq m$, let $j = m - i$. Edges of K_n will be colored with the i new colors in three steps:

1. For $1 \leq k \leq m - 2j$, color $n - 2i$ edges of h_k with color c_{m+k}.

2. For $m - 2j + 1 \leq k \leq m - j$, color $n - 2i - 1$ edges of h_k with color c_{m+k}.

3. $m-j+1 \leq k \leq m$, color a single edge of h_k with color c_{m+k-j}.

Note that Steps 2 and 3 use the same new colors. Since $i \geq n/4$,

$$n - 2i - 1 \leq n - \frac{n}{2} - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

thus in Step 2 the $n-2i-1$ edges may be obtained by choosing every other edge of h_k, and thus are independent edges. Since the edges in Step 2 were chosen independently, the union of them with the correspondingly colored edge from Step 3 will result in a union of paths. Therefore each color class has $n-2i$ edges and since $i \geq n/4$ none of the original color classes lost more than $2i$ edges.

Since i new colors were added by Lemma 4.0.5 at least $i(n-2i)$ edges from the original coloring must change color. Since the total number of edges changed was

$$(m-2j)(n-2i) + j(n-2i-1) + j = mn - 2mi - jn + 2ji.$$

Substituting $j = m - i$,

$$mn - 2mi - mn + in + 2mi - 2i^2 = i(n-2i) = i(2m+1-2i).$$

So it is true that the number of edges which received a new color is equal to the minimum number of edges required to change color.

For $i \geq m$, since $\max\{m, i(2m+1-2i)\} = m$, recolor a single edge of each of the m elements of H with a new color, never repeating a new color. Then each of the original m color classes have lost an edge, and each of the new i color classes have gained at least $n-2i$ edges. (Note: for $i > m$, $2m+1-2i < 0$ and for $i = m$, $2m+1-2i = 1$.)

Therefore, for $i \geq n/4$, $wp(m,i) = \max\{m, i(2m+1-2i)\}$. □

Since Theorem 4.0.6 is true for every Hamilton decomposition of K_{2m+1}, the following conjecture is reasonable to make.

Conjecture 4.0.7. *For every Hamilton decomposition H of K_{2m+1}, $wp(m,i) = wp(m,i,H)$.*

However, in the cases when $i < n/4$, Conjecture 4.0.7 is not proved as easily. So, it will be shown that $wp(m,i) = \max\{m, i(2m+1-2i)\}$. Consider the lower extreme of well-preserving a Hamilton decomposition of K_n in K_{n+2}; that is find $wp(m,1)$. Then there is only one new color class. Since all of the original color classes originally contained $2m+1$ edges, to apply Corollary 4.0.2 each of these original color classes must contain $2m-1$ edges after recoloring some edges with color $m+1$, so no more than 2 edges from each color class maybe recolored. The new color class will need

$$n - 2 = 2m - 1$$

edges. Since there are m original color classes, the new color class will contain two edges from each of the original color classes, with the exception of a single color class which would have only a single edge changed to the new color. The necessity of changing (nearly) the same number of edges from each color class motivates the following useful definition. A path in an edge-colored graph is said to be a double-rainbow path if every color appears on exactly two edges of the path.

Lemma 4.0.8. *If a Hamilton decomposition of K_{2m+1} contains a double-rainbow path P, then* $wp(m,1) = \max\{m, 1(2m+1-2(1))\} = 2m-1$.

Proof. Since there are m colors, P is of length $2m$ and is therefore a Hamilton Path. By recoloring all but one of the edges of P with the new color, the new color will have attained the required $2m-1$ edges, and since these edges were chosen from a path the new color class will be the union of paths. Since every original color appeared twice in P and all but one edge was recolored we have that each of the original m Hamilton cycles has had at least one edge recolored. So each of the original colors is a linear forest of size at least $2m-1$. So the result follows by Corollary 4.0.2 and Lemma 4.0.5. □

Conjecture 4.0.9. *Every Hamilton decomposition of K_n, for n odd contains a double rainbow path.*

Let $[v_1, v_2, ...v_n]$ denote a path of length $n-1$, and let $(v_1, v_2, ...v_n)$ denote the Hamilton cycle of K_n, $(v_1, v_2, ...v_n)$, in which v_i is adjacent to v_{i-1} and v_{i+1}, and v_1 and v_n are adjacent. For evidence supporting the preceding conjecture, consider the following. The Hamilton decomposition of K_5 $\{(0, 1, 2, 3, 4), (0, 2, 4, 1, 3,)\}$ contains the double-rainbow path $P = [0, 1, 3, 4, 2]$ where edges in $\{\{0, 1\}, \{3, 4\}\}$ and $\{\{1, 3\}, \{4, 2\}\}$ are colored 1 and 2 respectively as they are in the first and second Hamilton cycles respectively. More succinctly, P is denoted by $01, 34/24, 13$. The two non-isomorphic Hamilton decompositions of K_7 each contain a double-rainbow path: $(0, 1, 2, 3, 4, 5, 6)$, $(0, 2, 4, 6, 1, 3, 5)$, $(0, 3, 6, 2, 5, 1, 4)$ contains $01, 23/46, 50/25, 36$

$(0, 1, 2, 3, 4, 5, 6)$, $(0, 2, 4, 1, 6, 3, 5)$, $(0, 3, 1, 5, 2, 6, 4)$ contains $01, 23/41, 50/25, 46$.

The attached copies in Appendix B of all 122 non-isomorphic Hamilton decompositions of K_9 originally stated by Colbourn in [5] each contain double-rainbow path. (For brevity, in Appendix B the Hamilton cycles are indicated without the parenthesis or commas.)

The following Hamilton decomposition of K_{2m+1}, due to Walecki and restated by Alspach in [1], is of interest and so will be stated explicitly here. Let $G = K_{2m+1}$, and $V(G) = \{\infty\} \cup \{i \mid 0 \leq i \leq 2m-1\}$. Let $W = \cup W_i$ where for $0 \leq i \leq m-1$, $W_i = (\infty, i, i+1, i-1, i+2, i-2, \ldots, i+(m+1), i-(m-1), m+i)$ where all arithmetic is done (mod $2m$). The Walecki decomposition is W.

Given edge $e = \{i, j\}$ of K_n, if $|d| \leq n/2$ such that $i - j + d \cong 0 (\text{mod } n)$, then e is said to have edge difference d.

Lemma 4.0.10. *For each $m \geq 1$ there exists at least one Hamilton decomposition of K_{2m+1} containing a double-rainbow path.*

Proof. Consider the Walecki Hamilton decomposition W. So, $V(K_{2m+1}) = \{u_i \mid 0 \leq i \leq 2m-1, \} \cup \{v_\infty\}$. Let D be the set of edges with edge difference 1. D is a cycle of length $2m$ not including v_∞. Note that each color class contains exactly two edges in D, and note the edges $\{v_\infty, v_0\}$ and $\{v_0, v_1\}$ are both contained in W_1. Therefore $D \setminus \{v_0, v_1\} \cup \{v_\infty, v_0\}$ is a double-rainbow path.

□

After Theorem 4.0.6, the exact size $wp(m, i)$ only remains in doubt when $1 \leq i \leq m/2$. The following result is true for $1 \leq i \leq m$. However, at this stage consideration towards settling uncompleted cases is prioritized.

Lemma 4.0.11. *If a Hamilton decomposition of K_{2m+1} contains $1 \leq i \leq m/2$ edge-disjoint double-rainbow paths $p_1, p_2, \dots, p_i$ then for $1 \leq i \leq m/2$, $wp(m, i) = \max\{m, i(2m + 1 - 2i)\}$.*

Proof. Let D be a Hamilton decomposition of K_{2m+1}, with i edge-disjoint double rainbow paths $p_1, p_2, \dots, p_i$. For $1 \leq j \leq i$, recolor $2m+1-2i$ edges of the edges of p_j in K_{2m+1} with color $m + j$, ensuring that in p_j at least one edge of each color is recolored with the new color. This is possible since p_j has length $2m$, and at least $m + 1$ edges of p_j are being recolored. Since the edges now colored $m + j$ were contained in a path, after recoloring each of the new colors is a linear forest of size $2m + 1 - 2i$. The original m color classes were Hamilton cycles containing $2m + 1$ edges. Each original color has had at least one edge recolored with a new color so the original color classes are linear forests after the recoloring. For each double rainbow path at most 2 edges of a given color have been recolored, so after recoloring each original color contains at least $2m + 1 - 2i$ edges. After recoloring K_{2m+1} with $m + i$ colors, every color class is a linear forest containing at least $2m+1-2i$ edges. By Corollary 4.0.2 this recoloring of D can be embedded in $K_{2m+1+2i}$. Since $i(2m+1-2i)$ edges were recolored and since $i \leq m/2$ it is the case that $wp(m, i) = \max\{m, i(2m + 1 - 2i)\} = i(2m + 1 - 2i)$. □

The following was brought about after Parik Chalise noticed that for prime $p = 2m+1$, the well known Hamilton decomposition of K_p with cycles R_i given by $R_i = \{e_i \mid e_i \text{ has edge difference } i\}$ for $1 \leq i \leq m$, will always contain a double-rainbow path. In fact it will be shown this decomposition contains m edge-disjoint double-rainbow paths.

Lemma 4.0.12. *For $p = 2m + 1$ a prime, the Hamilton decomposition of K_p given by $R = \{R_i \mid 1 \leq i \leq m\}$ contains m edge-disjoint double-rainbow paths.*

Proof. Let R be the given Hamilton decomposition of K_p. Let $V(K_p) = \{v_i \mid 0 \leq i \leq 2m\}$. Note all edges of difference i are colored i, so if 2 edges of each difference are included in the path then it will be a double-rainbow path. For $0 \leq i \leq m - 1$ let $DR_i =$

$\{\{v_{i-j}, v_{i+j+1}\}, \{v_{i+j+1}, v_{i-j-1}\} \mid 0 \leq j \leq m-1\}$. Note DR_i consists of pairs of incident edges, that is paths of length 2. Since v_{i-j-1} is an end of such a path for j and $j+1$, the union of these paths form a path, therefore DR_i is a path. For a given j edge $\{v_{i-j+1}, v_{i+j+1}\}$ and edge $\{v_{i-j}, v_{i+j+1}\}$ both have difference j, so DR_i contains 2 edges of each difference. Therefore DR_i is a double-rainbow path. Since each DR_i consists of a pairs of edges of fixed differences, and each DR_i begins at a unique vertex v_i the double-rainbow paths are edge-disjoint. Therefore the decomposition given by R contains m edge-disjoint double-rainbow paths.

□

Corollary 4.0.13. *For p prime and any $i > 0$, $wp(m, i) = \max\{m, i(2m+1-2i)\}$.*

Proof. The result follows from Theorem 4.0.6, Lemma 4.0.11, and Lemma 4.0.12. □

While the double-rainbow paths are effective as shown by Lemma 4.0.11, and give a certain aesthetically pleasing visual image, they are a bit of overkill as the following lemma shows.

Lemma 4.0.14. *Let H be a Hamilton cycle decomposition of K_{2m+1} using m colors. Let $1 \leq i \leq m$. If H contains i edge-disjoint linear forests of size $2m+1-2i$, in which each color appears in at least one forest, and at most twice in each forest. Then $wp(m, i) = \max\{m, i(2m+1-2i)\}$.*

Proof. Let $F_1, F_2, \ldots, F_i$ be the postulated edge-disjoint linear forests. Recolor the edges of F_i with color $m+i$. Since each forest is of size $2m+1-2i$, each of the new colors appears on a linear forest of size $2m+1-2i$. Since each of the original colors was a Hamilton cycle which has had at least one edge recolored, each of the original colors are now linear forests. Since each original color appeared at most twice in a given forest no original color has lost more than $2i$ edges, so the linear forests of the original colors are now of size at least $2m+1-2i$. Note $i(2m+1-2i)$ edges have been recolored. Since $1 \leq i \leq m$, $\max\{m, i(2m+1-2i)\} = i(2m+1-2i)$. Therefore, $wp(m, i) = \max\{m, i(2m+1-2i)\}$. □

Again, the following result is true if $m/2$ is replaced by m. However, at this stage considering just the unfinished cases it will be shown for $m/2$.

Lemma 4.0.15. *For $i \leq m/2$, the edge-coloring of K_{2m+1} produced by the Walecki decomposition contains i edge-disjoint linear forests each of size $2m+1-2i$ where each color appears in at least one forest, and at most twice in each forest.*

Proof. Edge-color K_{2m+1} according to the Walecki decomposition. For $j < m$ let U_j be the subgraph induced by the set of edges of difference j; note each color appears exactly twice in U_j. Each U_j is a vertex cover of $K_{2m+1} \setminus \{\infty\}$, so $|E(U_j)| = 2m$. By Corollary 10.3.4 in [10], the components of U_j are cycles, and there are no more than j components in U_j. The U_j's will be used to construct the linear forests with the properties required in Lemma 4.0.14. For $i \leq m/2$, $m+1 \leq 2m+1-2i \leq 2m-1$. Consider $\{U_j \mid j \leq i\}$. Let F_1 be any $2m+1-2i$ edges of U_1, F_1 will contain every color at least once since $2m+1-2i \geq m+1$, and $E(U_1)$ was of size $2m$ containing every color exactly twice. Since $2m+1-2i < 2m$ not all edges of U_1 are in F_1; therefore F_1 is a linear forest. For $j > 1$ let F_j be $2m+1-2i$ edges of U_j so that F_j does not contain all edges of any component of U_j. This is possible since $|E(U_j)| - (2m+1-2i) = 2i-1 \geq i$, and since i is less than or equal to the number of components of U_j, at least one edge from each component of U_j can be omitted from F_j. Thus, F_j is a linear forest. Therefore $\{F_j \mid 1 \leq j \leq i\}$ is a set of i linear forests of size $2m+1-2i$ so that no color appears more than twice on any forest, and every color appears in the union.

□

Theorem 4.0.16. *For $m, i > 0$, $wp(m,i) = \max\{m, i(2m+1-2i)\}$.*

Proof. The result follows from Theorem 4.0.6, Lemma 4.0.14, Lemma 4.0.15. □

Chapter 5

Questions Needing Answers

Though the spectrum for the size of maximal sets of Hamilton cycles of multipartite graphs is now settled, there are yet interesting questions along this vein. Similarly to how Noble proceeded in the case when $p \equiv 1 (\text{mod } 4)$, is it possible to find a recursive construction when $p \equiv 3 (\text{mod } 8)$, where the removal of the edges of the maximal set of Hamilton cycles results in a graph with a cut vertex? Specifically, can Theorem 3.0.1 be used to aid in finding Maximal sets of Hamilton cycles? Can Theorem 2.0.2 be modified to work in the case when $p \equiv 1 (\text{mod } 4)$? A required modification would be the number of pure edges necessary, vertices of S_1, S_2, S_3, S_4 would need to be incident with $4, 2, 3, 2$ pure edges respectively.

There are classes of graphs besides complete graphs which are known to contain Hamilton cycles, for a given class is it possible to determine the spectrum of μ? For instance $\mu = 1$ if the graph is a cycle, but there are certainly more interesting graphs than cycles to consider. What if multi-graphs are allowed? Particularly multi-edges since loops are insignificant when searching for Hamilton cycles. If λK_n^p is the complete multipartite graph where each edge appears λ times, are the bounds for μ simply multiplied by λ?

Cycles are not as intuitive in hypergraphs however there are defined (in multiple ways). Can maximal sets of Hamilton Berge cycles or Hamilton γ-cycles be found in a d-regular k-uniform hypergraph?

Conjecture 4.0.9 is an easily understood conjecture, but one which is seemingly difficult to prove or disprove. Does any decomposition of K_{2m+1} contain $(2m + 1)/4$ edge-disjoint double-rainbow paths? If so, Conjecture 4.0.7 would be true for such a decomposition.

What more can be said about Conjecture 4.0.7, and can the lower bound of where it is true be moved beneath $n/4$?

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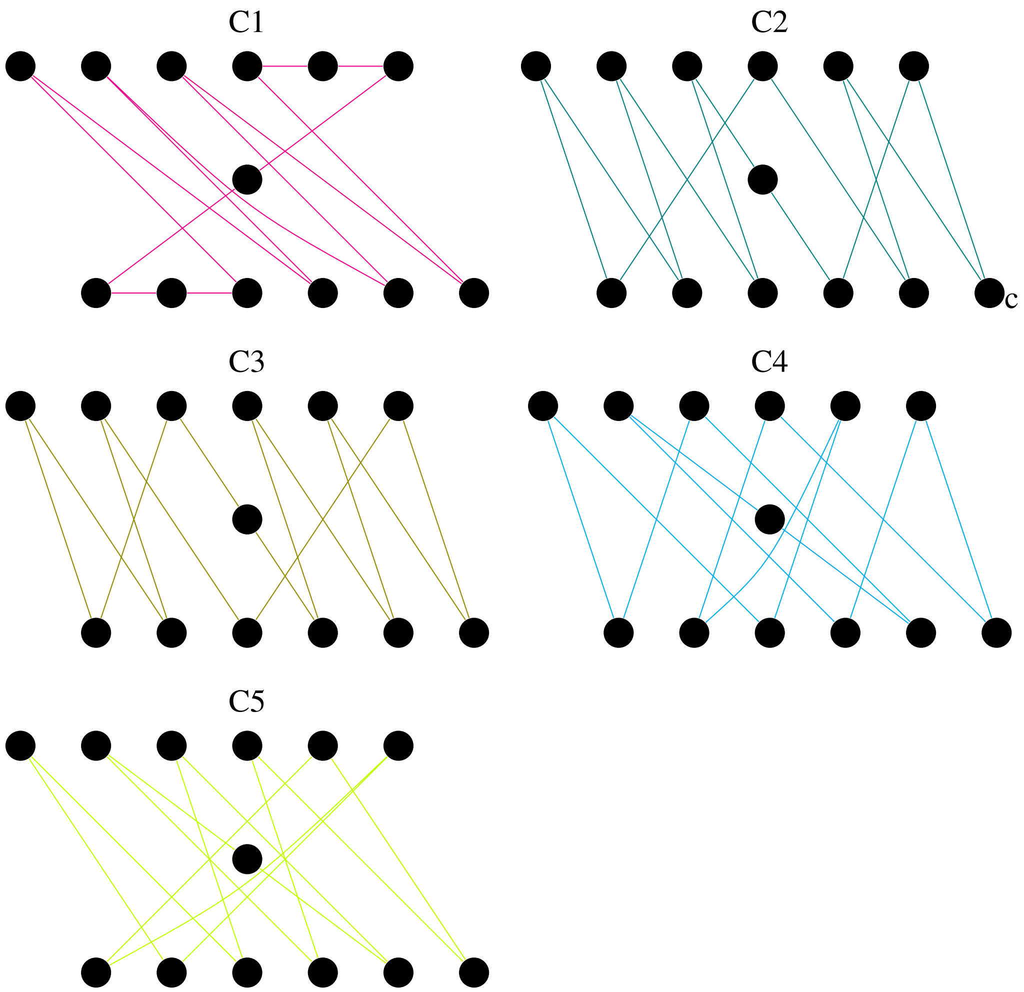
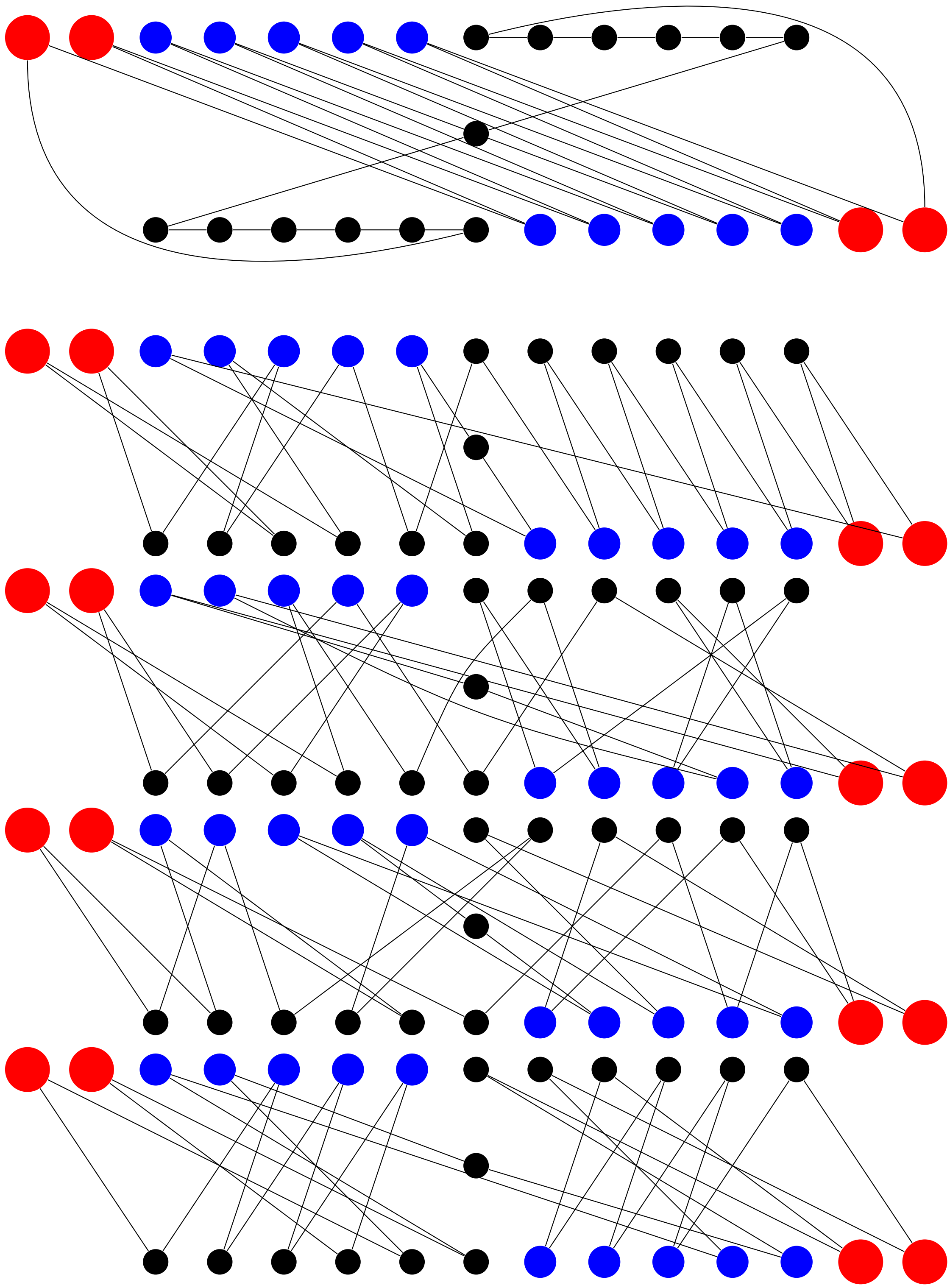
Appendices

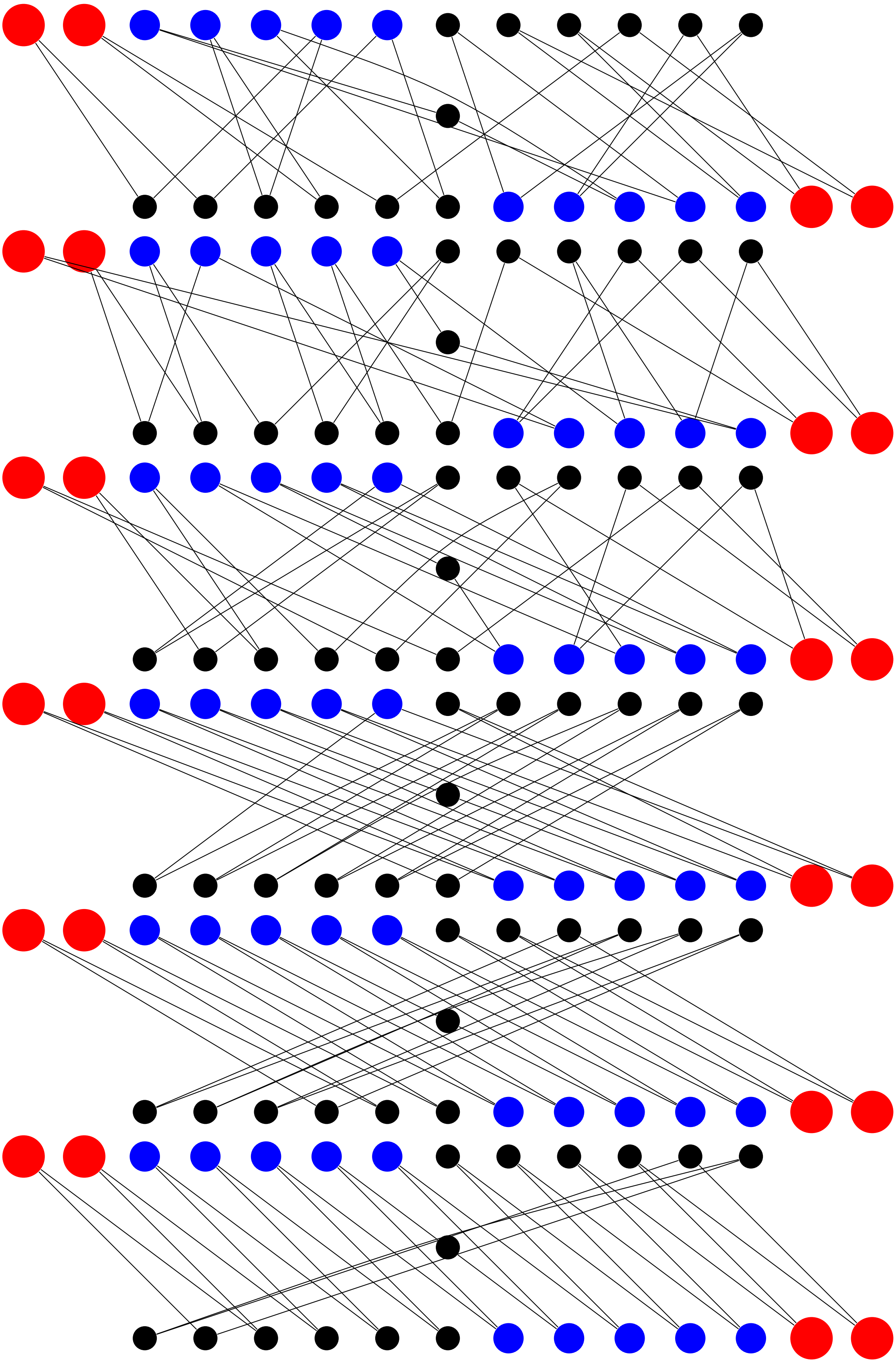
Appendix A

Pictures of amalgamated grpahs.

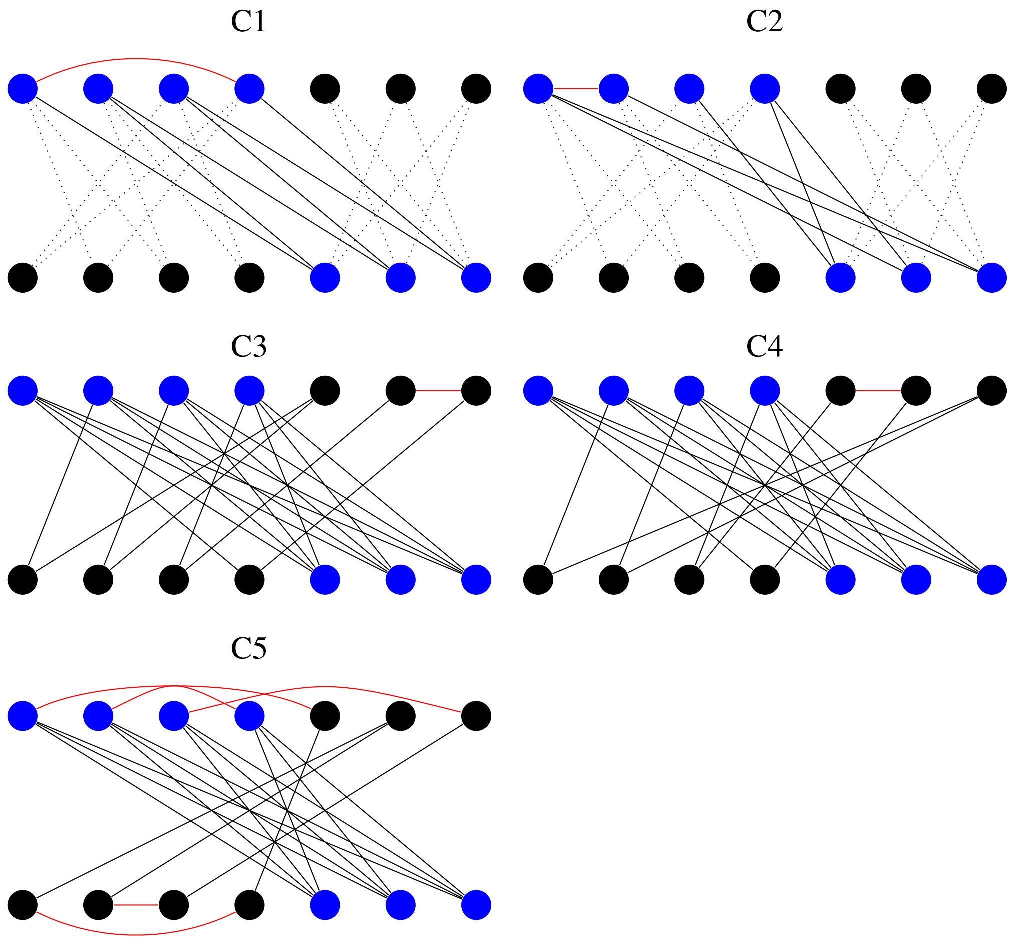


Five Hamilton cycles of amalgamated K_3^7.





Eleven Hamilton cycles of amalgamated K_3^{15}.



Entangled color classes K_3^7, using the Chapter 2 construction.

Appendix B

Double rainbow path in all 122 non-isomorphic copies of K_9

,

012345678, 024136857, 035172846, 047381625	01,23/ 68,57/ 46,82/ 73,50
012345678, 024136857, 035174826, 046183725	01,23/ 68,57/ 48,17/ 04,25
012345678, 024136857, 035182746, 048371625	01,23/ 68,57/ 46,82/ 04,37
012345678, 024136857, 035184726, 046173825	01,23/ 68,57/ 51,74/ 46,82
012345678, 024136857, 035274816, 046283715	01,23/ 68,57/ 48,61/ 50,73
012345678, 024136857, 035281746, 048372615	01,23/ 68,57/ 47,52/ 48,61
012345678, 024136857, 035284716, 046273815	01,23/ 68,57/ 47,52/ 40,18
012345678, 024136857, 037184625, 047283516	01,23/ 68,57/ 48,52/ 47,61
012345678, 024136857, 037284615, 047183526	01,23/ 68,57/ 48,61/ 47,52
012345678, 024136857, 037462815, 048352716	01,23/ 68,57/ 46,82/ 04,71
012345678, 024136857, 037481526, 046172835	01,23/ 68,57/ 48,26/ 04,17
012345678, 024136857, 037482516, 046271835	01,23/ 68,57/ 48,25/ 04,71
012345678, 024136857, 038174625, 048273516	01,23/ 68,57/ 47,25/ 04,16
012345678, 024136857, 038472516, 046281735	01,23/ 68,57/ 47,25/ 04,81
012345678, 024137586, 035162847, 046381725	01,23/ 68,57/ 47,26/ 04,81
012345678, 024137586, 035164827, 047183625	01,23/ 68,57/ 46,28/ 04,71
012345678, 024137586, 035182647, 048361725	01,23/ 68,57/ 46,28/ 04,71
012345678, 024137586, 035184627, 047163825	01,23/ 68,57/ 46,27/ 50,83
012345678, 024137586, 035261847, 046382715	01,23/ 68,57/ 48,26/ 04,15
012345678, 024137586, 035264817, 047283615	01,23/ 68,57/ 48,26/ 04,15

012345678, 024137586, 035281647, 048362715	01,23/ 68,57/ 46,28/ 04,71
012345678, 024137586, 035284617, 047263815	01,23/ 68,57/ 46,28/ 04,15
012345678, 024137586, 036284715, 046183527	01,23/ 68,57/ 47,26/ 04,81
012345678, 024137586, 036471825, 048351627	01,23/ 68,57/ 47,25/ 04,16
012345678, 024137586, 038164725, 048263517	01,23/ 68,57/ 47,16/ 28,04
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012345678, 024157386, 031746285, 048163527	01,23/ 68,57/ 47,26/ 04,81
012345678, 024157386, 031852647, 048271635	01,23/ 68,57/ 46,25/ 04,71
012345678, 024157386, 035261847, 046317285	01,23/ 68,57/ 48,25/ 04,71
012345678, 024157386, 035284617, 047263185	01,23/ 68,57/ 46,25/ 74,81
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