Mathematical Studies of Population Models in Stochastic Environments

by

Jiaqi Cheng

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
May 7, 2022

Keywords: stochastic lottery competition Model, random age-structured model

Approved by

Xiaoying Han, Chair, Professor of Department of Mathematics and Statistics
Ming Liao, Professor of Department of Mathematics and Statistics
Yanzhao Cao, Professor of Department of Mathematics and Statistics
Georg Hetzer, Professor of Department of Mathematics and Statistics

Abstract

This dissertation is devoted to the study of population models in stochastic environments. We will investigate a two-species lottery model in non-stationary stochastic environment, an $N-$species lottery model in stationary stochastic environment and an age-structured model in random environment.

First, a two-species lottery competition model with non-stationary environmental parameters is studied. We start with viewing the classical discrete lottery model as a Markov process. Then a diffusion process that represents the fraction of sites occupied by adults of the species, as the limit of the Markov process, is derived. A non-autonomous stochastic differential equation that describes the diffusion process, as well as a Fokker-Planck equation on its transitional probability are developed. Existence, uniqueness, and dynamics of solutions for the resulting stochastic differential equation and Fokker-Planck equation are investigated, from which sufficient conditions for coexistence are established. Numerical simulations are presented to illustrate the theoretical results.

Furthermore, a lottery competition model with $N \geq 2$ species in stationary stochastic environments is studied under the assumption that the environmental parameters are i.i.d.. We establish a system of nonlinear SDEs as the diffusion approximation for the discrete lottery model. Then the existence and uniqueness of the well-posed global solutions, along with asymptotic behaviors for the SDE system are investigated. Especially, sufficient conditions under which extinction and persistence occur are constructed, respectively.

Finally, a random age-structured model with random nonlinear birth rate is formulated. Its mathematical theories including well-posedness, co-cycle property and long time dynamics of the solution are developed. The emphasis is given to the asymptotic smoothness and the bounded dissipativeness of the solution to the model, which implies the existence of the random pullback attractor.

Acknowledgments

First, I would like to express my sincerest thanks to my advisor, Prof. Xiaoying Han for her professional guidance, endless patience throughout my Ph.D. study. This dissertation is almost impossible without her support and encouragement. In my five years at Auburn University, Prof. Han taught me how to develop my own eyesight in research and how to overcome diffculties in mathematics and life, which builds me a bright future. Moreover, I also want to thank Prof. Ming Liao's selfless help on my research. His impressive solid foundation in mathematics just make me want to be a better person like him.

I would also like to sincerely thank Prof. Yanzhao Cao, Prof. Georg Hezter, and Prof. Ming Liao for serving on my Ph.D. committee and providing constant help and guidance along the way. I would also like to sincerely thank Prof. Zhaofei Fan for being the university reader.

Finally, I want to thank my wife Na Xu who is thousands of miles away from me. You are the lighthouse that break all the darkness in my life. My biggest dream is having a happy and quiet life with you.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Lottery competition model

It is well known that environmental fluctuations play an important role in the formation of ecological communities (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein). To investigate the impact of uncertain environmental fluctuations on the structure of an ecological community, mathematical models have been widely used to describe interactions among competing species (see, e.g., [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and references therein). One seminal work on ecological competition in temporally fluctuating environments is the lottery model developed by Chesson and Warner in 1981 [26]. The term "lottery" was adopted due to the similarity between Sale's lottery system and ecological competition in patchy environments [26]. Since then the lottery model has been widely used to describe competition among species, as well as to understand interesting ecological phenomena such as organism competition, storage effect, and neutral theory (see, e.g., [27, 28, 29, 30, 26, 31]).

In the representative works [28, 26], Chesson showed that environmental variability had significant impacts on promoting coexistence provided the species had overlapping generations, by analyzing a two-species lottery model. The idea therein was modeling environmental fluctuations by temporally varying reproduction rates and adult mortality rates. Species may take the advantage of fluctuating rates during their favorable periods which are disadvantageous to others, and thus tend to persist. Coexistence, in turn, can result from the asymmetry effects caused by the environmental variability. Such type of asymmetry was referred to as the "storage effect" [29, 30]. Lottery models with more species and various types of competitions were further studied in [29] and [32], whereby a larger variety of environmental fluctuations such as pure spatial variation, pure temporal variation and the space-time interaction were considered. More general models and methods were later developed in [3] and [27], from which conclusions consistent with those for two-species models were drawn, that the storage effect is an essential mechanism for coexistence.

Another influential work on lottery models was [33] due to Hatfield, in which a sufficient condition on environmental variability ensuring the coexistence of two competing species was developed, and the probability density function of the stationary distribution was also established. The key idea employed in the analysis there was treating the space occupations of species as continuous evolving diffusion processes, whose drift and diffusion coefficients were derived from the discrete-time lottery model. Theory of diffusion processes was then utilized to establish sufficient conditions for coexistence, as well as to construct the probability density function of the stationary distribution. In fact, in the limit of accelerating and increasing numbers of breeding seasons, the discrete lottery model converges to the continuous lottery model. The coexistence of the species can then be implied by the existence of stochastic boundaries of the stationary distribution [34]. Note the conditions for coexistence in [33] matches precisely with the results derived by Chesson [27], though derived from a different approach. More works concerning the lottery model can be found in [35, 36, 37, 38, 39, 40, 41] and references therein.

The goal of Chapter 2 is to further study the lottery model under non-stationary environments. More precisely, the reproduction rates and adult mortality rates are assumed to be non-stationary stochastic processes with time-dependent moments. To that end, Chapter 2 is organised as follows. First, the classical discrete lottery model is formulated in Section 2.1. Then, an rigorous mathematical derivation for the transition from the discrete lottery model to the continuous lottery model, i.e. the process of the diffusion approximation, is established in Section 2.2. The existence and uniqueness of the global solution to the continuous lottery model is proved in Section 2.3. Long time dynamics of the solution and sufficient conditions for the coexistence of two competing species are also investigated in this section.

Moreover, lottery models with $N > 2$ were studied in [33] and [27], where conclusions that were consistent with the two-species lottery model were concluded. However, the results were drawn under the assumption that all the species should have same death rate, which might not be general enough to describe the practical problems. Thus, the goal of Chapter 3 is to provide more general sufficient conditions for the coexistence among more than two competing species. To that end, we will investigate dynamical behaviors of an N-species lottery model in stochastic environments, by applying theory and techniques of stochastic differential equations.

Chapter 3 is organized as follows. First in Section 3.1, a system of stochastic differential equations (SDEs) is derived from diffusion approximation to the discrete lottery system (3.1). Then the existence and uniqueness of a positive global solution to the resulting SDE system is established. The asymptotic behaviors of the SDE system are investigated in Section 3.2. Consequently, sufficient conditions for at least one species to extinct and for at least two species to coexist, are constructed respectively. In order to maintain a smooth presentation, some technical analysis is presented in the Appendix.

1.2 Age-structured population model

In 1798, the famous continuous Malthusian model was proposed assuming that the growth rate of population is proportional to the population size by an environmental factor λ:

$$P'(t) = \lambda P(t),\ t \geq 0. \tag{1.1}$$

However, (1.1) implied the exponential growth of the population, which did not consider the limited resources. Later on, in the logistic model

$$P'(t) = \lambda \left[1 - \frac{P(t)}{K}\right] P(t),\ t \geq 0, \tag{1.2}$$

the environmental parameter was assumed to depend on the maximum population K. This model provided a more practical way to describe the dynamics of the population when there was a carrying capacity of the environment. Indeed, the solution of (1.2) could approach the nontrivial equilibrium K when time goes to infinity.

Notice that the Malthus and logistic models did not provide any result about the age distribution of the population. In fact, it is more realistic to consider age dependent birth and death rates when we investigate the population like human. In 1911, the first continuous population model in which the birth and death rates were linear functions of the population densities was formulated by Lotka [42]. Actually, Lotka's model is similar to the Malthus model in the sense that the effects of competition for the limited resources are neglected.

Thus, in 1974, Gurtin [43] investigated a nonlinear age-structured population model in which the birth and death processes were nonlinear functions of population densities. As a consequence, Gurtin's model, corresponding to the logistic age-structured model, provided nontrivial age-dependent equilibrium state.

The third part of this work is devoted to the study of a continuous age-structured model where birth process is modeled by a random process and Ricker's function. It is organised as follows. In Section 4.1, a random age-structured model is formulated. The definition of the solution to the random age-structured model developed in Section 4.1 is given in Section 4.2. The existence and uniqueness of solutions are also presented in this section. Moreover, we verify the co-cycle property in Section 4.3. Finally, the existence of a random pullback attractor is established in Section 4.4, which is the highlight of this work.

Chapter 2

2 - D Lottery Model

2.1 The model

Consider two ecological species in a single habitat competing for a limited number of sites. For $t \geq 0$ and $i = 1,2$, let $X_i(t)$ be the fraction of the sites occupied by adults of the ith species at time t. Given any $h > 0$, denote by $\nu_i^h(t)$ the proportion of adults of the ith species dying during the time period $(t, t+h]$. Then the term $(1 - \nu_i^h(t))X_i(t)$ represents the fraction of sites occupied by surviving adults of the ith species during $(t, t+h]$, and the proportion of new sites available for settling by juveniles is $\sum_{i=1}^{2} \nu_i^h(t)X_i(t)$. Let $\beta_i^h(t) \in [0,1]$ be the per capita net reproduction by adults of the ith species during time $(t, t+h]$, i.e., for each adult of species i at time t, $\beta_i^h(t)$ larvae reach the settling stage during $(t, t+h]$. According to an intuitive ecological interpretation that the next generation is given by the sum of surviving adults and new recruitments [28, 26], the discrete-time lottery model for n competing species can be formulated as

$$X_i(t+h) = (1 - \nu_i^h(t))X_i(t) + \frac{\beta_i^h(t)X_i(t)}{\sum_{i=1}^{2} \beta_i^h(t)X_i(t)} \sum_{i=1}^{2} \nu_i^h(t)X_i(t), \quad i = 1,2,\ h > 0. \tag{2.1}$$

Let the initial time of the system (2.1) be t_0, and assume that initial fractions of species satisfy

$$X_i(t_0) := x_{i,0} > 0, \quad x_{1,0} + x_{2,0} = 1, \quad i = 1,2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$X_1(t) + X_2(t) = 1, \; \forall\, t \geq t_0.$$

For simplicity, throughout this chapter, we denote $X_1(t)$ as $X(t)$. Then, $X_2(t) = 1 - X(t)$.

Since the environment fluctuates stochastically about time, it is natural to assume the time dependent parameters $\nu_i^h(t)$ and $\beta_i^h(t)$ to be stochastic processes. It is worth to mention that most of existing works ([44, 45, 46]) on the system (2.1)–(2.2) considered stationary environment, in which both $\nu_i^h(t)$ and $\beta_i^h(t)$ are stationary processes with constant moments. Nonetheless, the stochastic process with constant moments are at most approximations to temporal environmental fluctuations. The lottery model with the non-stationary environmental parameters was first studied by Chesson in [44] in which the concept of AEDT was developed.

2.2 Diffusion approximation

2.2.1 Mathematical foundation

Since the lottery model (2.1) implies that the the proportion of ith species at $t+h$ only depends on the proportion at time t, it is intuitive to treat (2.1) as a Markov process. Let $(E, \mathscr{F}, \mathbb{P})$ be a probability space. For each $t \in \{jh : j \in \mathbb{N}\}$, let $(\nu^h(t), \beta^h(t))$ be nonnegative real-value random variables on $(E, \mathscr{F}, \mathbb{P})$. To be more specific, it is assumed that at any different discrete time instants t and s, $(\nu^h(s), \beta^h(s))$ and $(\nu^h(t), \beta^h(t))$ are independent. Due to such an independence structure, by (2.1), the process $X(t)$ is a discrete time Markov chain as the conditional distribution of $X(t+h)$, given $X(t)$, can be determined by $X(t)$ and the distribution of $(\nu^h(t), \beta^h(t))$.

Then, we construct a continuous counterpart of the discrete process $\{X(t)\}_{t \in \{jh: j \in \mathbb{N}\}}$. Define the space $\Omega = C([0,\infty), \mathbb{R})$ to be the collection of all continuous paths from $[0,\infty)$ into $\mathbb{R}$. For any real number $s \geq 0$ and $\omega \in \Omega$, set $x(s,\omega) = \omega(s)$ and define the metric on Ω by

$$\rho(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0 \leq s \leq n} |x(s,\omega) - x(s,\omega')|}{1 + \sup_{0 \leq s \leq n} |x(s,\omega) - x(s,\omega')|}, \quad \forall\, \omega,\ \omega' \in \Omega.$$

Consequently, (Ω,ρ) is a complete separable metric space [47]. Next, let $\mathscr{M}$ be the Borel σ-field on (Ω_s, D), i.e., $\mathscr{M} = \sigma[x(s) : s \geq 0]$ [47]. For any s_1, $s_2 > 0$, define

$$\mathscr{M}_{s_1,s_2} = \sigma[x(s) : s_1 \leq s \leq s_2] \text{ and } \mathscr{M}_{s_1} = \sigma[x(s) : s \geq s_1].$$

Let $x_{t_0,x}$ be the convex combination of $X(t)$ with $X(t_0) = x$, i.e., for any $q \in E$ let

$$x_{t_0,x}(s,q) = \begin{cases} X(t_0,q), & \text{for } s = t_0, \\ \frac{(j+1)h-s}{h} X(t_0+jh,q) + \frac{s-jh}{h} X(t_0+(j+1)h,q), & \text{for } jh \leq s < (j+1)h,\ j \in \mathbb{N}. \end{cases}$$

Then $x_{t_0,x}(s,q)$ is a continuous time process on $(E,\mathscr{F},P)$ and defines a measurable mapping from $(E,\mathscr{F})$ into $(\Omega,\mathscr{M}_{t_0})$ given by $q \mapsto x_{t_0,x}(\cdot,q)$. Therefore it induces a probability measure $P^h_{t_0,x}$ on $(\Omega,\mathscr{M}_{t_0})$ that satisfies

$$\begin{cases} P^h_{t_0,x}[x_{t_0,x}(t_0) = x] = 1, \\ P^h_{t_0,x}\left[x_{t_0,x}(s) = \frac{(j+1)h-s}{h} X(t_0+jh) + \frac{s-jh}{h} X(t_0+(j+1)h),\ jh \leq s < (j+1)h\right] = 1, \quad j \in \mathbb{N}, \\ P^h_{t_0,x}\left[x_{t_0,x}(t_0+(j+1)h) \in \Gamma \mid \mathscr{M}_{t_0+jh}\right] = \Pi_h(t_0,x,t_0+jh,\Gamma), \quad j \in \mathbb{N} \text{ and } \Gamma \in \mathscr{B}_{\mathbb{R}}, \end{cases}$$

where $\mathscr{B}_{\mathbb{R}}$ is the Borel σ-algebra in $\mathbb{R}$, and $\Pi_h(t_0,x,t_0+t,\cdot)$ is the probability transition function of $X(t)$ on $\mathbb{R}$ for any $t \in \{jh : j \in \mathbb{N}\}$. Finally, we cite Theorem 11.2.3 in [47].

Theorem 1. *Assume that in addition to being continuous, the drift and diffusion coefficients have the property that for each $x \in \mathbb{R}^N$ the martingale problem with the coefficients has exactly one solution P_x starting from x. Then, $P^h_{t_0,x} \to P_{t_0,x}$ as $h \to 0$ uniformly on compact subset of $\mathbb{R}^N$.*

Remark 1. *Theorem 11.2.3 in [47] states the weak convergence of $P^h_{t_0,x}$ to $P_{t_0,x}$ in the time-homogeneous case. But the proof can be extended to the time-inhomogeneous case as in this work without loss of generality.*

Remark 2. *The drift coefficient* (2.11) *and diffusion coefficient* (2.12) *will be derived in the following section. They can guarantee the existence and the uniqueness of the solution to* (2.14)*, which means the martingale problem with* (2.11) *and* (2.12) *has exactly one solution.*

2.2.2 Derivation of the diffusion approximation

Essentially, the diffusion approximation of $\{X(t)\}_{t\in\{jh:j\in\mathbb{N}\}}$ is a diffusion process characterized by its infinitesimal mean and variance, defined respectively as

$$f(t,x) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[X(t+h)-X(t)|X(t)=x], \tag{2.3}$$

$$g^2(t,x) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[(X(t+h)-X(t))^2|X(t)=x]. \tag{2.4}$$

It follows directly from (2.1) that

$$\begin{aligned} X(t+h)-X(t) &= -\nu_1^h(t)X(t)+\frac{(\nu_1^h(t)X(t)+\nu_2^h(t)(1-X(t)))\beta_1^h(t)X(t)}{\beta_1^h(t)X(t)+\beta_2^h(t)(1-X(t))} \\ &= \frac{X(t)(1-X(t))(\nu_2^h(t)\beta_1^h(t)-\nu_1^h(t)\beta_2^h(t))}{\beta_1^h(t)X(t)+\beta_2^h(t)(1-X(t))} \\ &= \frac{X(t)(1-X(t))\left(\frac{\nu_2^h(t)\beta_1^h(t)}{\nu_1^h(t)\beta_2^h(t)}-1\right)}{\frac{\beta_1^h(t)}{\nu_1^h(t)\beta_2^h(t)}X(t)+\frac{\beta_2^h(t)}{\nu_1^h(t)\beta_2^h(t)}(1-X(t))} \\ &= \frac{X(t)(1-X(t))\left(\frac{\nu_2^h(t)\beta_1^h(t)}{\nu_1^h(t)\beta_2^h(t)}-1\right)}{\frac{\beta_1^h(t)\nu_2^h(t)}{\nu_1^h(t)\beta_2^h(t)}\frac{1}{\nu_2^h(t)}X(t)+\frac{1}{\nu_1^h(t)}(1-X(t))} \end{aligned} \tag{2.5}$$

Here, we use the same notation in [34] and define the stochastic processes $\xi^h(t)$ and $\gamma_i^h(t)$ by

$$\xi^h(t)=\ln\frac{\beta_1^h(t)\nu_2^h(t)}{\beta_2^h(t)\nu_1^h(t)}, \quad \gamma_i^h(t)=\ln\nu_i^h(t)-\ln d_i^h(t), \quad i=1,2, \tag{2.6}$$

where $d_i^h(t)$ is the geometric mean of $\nu_i^h(t)$. Then $\mathbb{E}[\gamma_i^h(t)]=0$ for all $t\geq 0$ and $h>0$. $\xi^h(t)$ a crucial environmental parameter that describes interactions between species caused by the environmental fluctuations on a log time scale [26].

Then, plugging (2.6) into (2.5) gives

$$X(t+h)-X(t)=\frac{d_1^h(t)d_2^h(t)X(t)(1-X(t))\left(e^{\xi^h(t)}-1\right)}{d_1^h(t)X(t)e^{\xi^h(t)-\gamma_2^h(t)}+d_2^h(t)(1-X(t))e^{-\gamma_1^h(t)}},\qquad t\in\mathscr{R},\ h>0. \tag{2.7}$$

Consequently, (2.3) and (2.4) become

$$f(t,x)=\lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\frac{R_h(t)}{S_h(t)}\right],\qquad g^2(t,x)=\lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\frac{R_h^2(t)}{S_h^2(t)}\right], \tag{2.8}$$

where

$$\begin{aligned} R_h(t) &= d_1^h(t)d_2^h(t)x(1-x)\left(e^{\xi^h(t)}-1\right),\\ S_h(t) &= d_1^h(t)xe^{\xi^h(t)-\gamma_2^h(t)}+d_2^h(t)(1-x)e^{-\gamma_1^h(t)}. \end{aligned}$$

Here, we assume that $\xi^h(t)$ and $\gamma_i^h(t)$ are non-stationary stochastic processes which is different from [34] and other related works in the literature. In particular, assume

$$\mathbb{E}[\xi^h(t)]=h\mu(t),\quad \mathrm{Var}[\xi^h(t)]=h\sigma^2(t),\quad \mathrm{Var}[\gamma_i^h(t)]=h\sigma_i^2(t),\quad t\in\mathscr{R},\ h>0. \tag{2.9}$$

Moreover, suppose that

$$\mathrm{Cov}\left[\gamma_1^h(t),\xi^h(t)\right]=h\theta_1(t),\qquad \mathrm{Cov}\left[\gamma_2^h(t),-\xi^h(t)\right]=h\theta_2(t). \tag{2.10}$$

Furthermore, denote by $\nu_i(t)$ the instantaneous death rate at time t, i.e., $\nu_i(t)=\lim_{h\to 0}\nu_i^h(t)$ and let $d_i(t)$ be the geometric mean of the instantaneous death rate $\nu_i(t)$. Notice that for any $t\geq 0$, $\nu_i^h(t)$ is non-decreasing with respect to h. Thus by the dominated convergence theorem, $d_i(t)=\lim_{h\to 0}d_i^h(t)$ for all $t\geq 0$.

Now, whereby the assumptions (2.9)–(2.10), Taylor expansion of exponential functions, the following approximation [48, 49],

$$\mathbb{E}\left[\frac{R(t)}{S(t)}\right]\approx\frac{\mathbb{E}[R(t)]}{\mathbb{E}[S(t)]}-\frac{\mathrm{Cov}[R(t),S(t)]}{\mathbb{E}[S^2(t)]}+\frac{\mathrm{Var}[S(t)]\mathbb{E}[R(t)]}{\mathbb{E}[S^3(t)]},$$

and the technical calculations similar to those presented in [34], we obtain that

$$
\begin{aligned}
f(t,x) &\approx \lim_{h\to 0}\frac{1}{h}\left(\frac{\mathbb{E}[R_h(t)]}{\mathbb{E}[S_h(t)]}-\frac{\mathrm{Cov}[R_h(t),S_h(t)]}{\mathbb{E}[S_h^2(t)]}+\frac{\mathrm{Var}[S_h(t)]\mathbb{E}[R_h(t)]}{\mathbb{E}[S_h^3(t)]}\right)\\
&= \frac{d_1(t)d_2(t)x(1-x)}{(d_1(t)x+d_2(t)(1-x))^2}\big(d_1(t)xC_1(t)+d_2(t)(1-x)C_2(t)\big), \qquad (2.11)\\
g^2(t,x) &\approx \lim_{h\to 0}\frac{1}{h}\left(\frac{\mathbb{E}[R_h^2(t)]}{\mathbb{E}[S_h^2(t)]}-\frac{\mathrm{Cov}[R_h^2(t),S_h^2(t)]}{\mathbb{E}[S_h^4(t)]}+\frac{\mathrm{Var}[S_h^2(t)]\mathbb{E}[R_h^2(t)]}{\mathbb{E}[S_h^6(t)]}\right)\\
&= \left(\frac{d_1(t)d_2(t)x(1-x)}{d_1(t)x+d_2(t)(1-x)}\right)^2\sigma^2(t), \qquad (2.12)
\end{aligned}
$$

where

$$
C_1(t) := \mu(t)-\theta_2(t)-\frac{\sigma^2(t)}{2}, \qquad C_2(t) := \mu(t)+\theta_1(t)+\frac{\sigma^2(t)}{2}. \qquad (2.13)
$$

Remark 3. *From* (2.11)–(2.12)*, we observe that the drift and the diffusion coefficients do not depend on $\sigma_i(t)$, which, however, does not imply that the variances of death rates do not play a role in the diffusion process. Indeed, the death rate parameters $\nu_i^h(t)$ ($i=1,2$) do have influence on the parameter $\xi^h(t)$, whose variance $\sigma^2(t)$ appears in the coefficients of the diffusion process.*

Remark 4. *The drift and diffusion terms f and g^2 are the same as those obtained in [34], except that the moment functions μ, σ^2, d_i, θ_i $(i=1,2)$ here are all time-dependent.*

2.3 The lottery stochastic differential equation

Because of Theorem 11.2.3 in [47], the solution to the discrete lottery model (2.1) converges weakly to the diffusion process characterized by the drift and diffusion coefficients given by (2.11) and (2.12), respectively. Denote by $Y(t)$ the limiting diffusion process for the solution to the discrete lottery model. It also represents the fraction of the sites occupied by adults of the first species at time t. Then $Y(t)$ satisfies the following stochastic differential equation (SDE)

$$
\mathrm{d}Y(t) = f(t,Y(t))\mathrm{d}t + g(t,Y(t))\mathrm{d}W(t), \quad t\ge t_0, \quad Y(t_0)=y_0 := x_{1,0}\in(0,1), \qquad (2.14)
$$

where $W(t)$ is a one dimensional Brownian motion, and $f(t,\cdot)$ and $g(t,\cdot) > 0$ are defined by (2.11) and (2.12), respectively. In this section we first prove the existence of a unique non-negative solution to the SDE (2.14), and then investigate its asymptotic behaviors. Throughout this section we assume

(**A1**) the functions $\mu(t)$, $\sigma(t)$, $\theta_i(t)$, $d_i(t)$ $(i = 1,2)$ are bounded and continuously differentiable for all $t \geq t_0$ with

$$\begin{aligned} &\mu^m \leq \mu(t) \leq \mu^M, && 0 < \sigma_m^2 \leq \sigma^2(t) \leq \sigma_M^2, \\ &\theta_i^m \leq \theta_i(t) \leq \theta_i^M, && 0 < d_i^m \leq d_i(t) \leq d_i^M, \quad i = 1,2. \end{aligned}$$

Remark 5. *The non-stationary environmental parameters $\xi^h(t)$, $\gamma_i^h(t)$ may not be bounded, even though the moment functions $\mu(t)$, $\sigma(t)$, $d_i(t)$ and $\theta_i(t)$ $(i = 1,2)$ are assumed to be bounded.*

The following theorem proves the existence, uniqueness and positiveness of solutions to the SDE (2.14). We apply the same approach as in, e.g., [50, 25], with different technical calculations.

Theorem 2. *Let Assumption (**A1**) hold. Then for any $t_0 \geq 0$ and $y_0 \in (0,1)$, the SDE (2.14) has a pathwise unique solution $Y(t) = Y(t;t_0,y_0,\omega)$ on (t_0,∞). Moreover, the solution $Y(t) \in (0,1)$ for all $t \geq t_0$ almost surely.*

Proof. First, (**A1**) guarantees that the drift and diffusion coefficients $f(t,Y)$ and $g(t,Y)$ are continuously differentiable in t and locally Lipschitz in Y. Then the classical theory for SDEs (see, e.g., [20, 51]) shows that the initial value problem (2.14) possesses a unique local solution $Y(t)$ on $[0,\tau_e)$ with $Y(t) \in (0,1)$ a.s., where τ_e is the explosion time. The existence of a global solution will be proved by showing $\tau_e = \infty$ a.s. below.

Let $k_0 > 0$ be a positive integer satisfying $y_0 \in (1/k_0, 1-1/k_0)$. For any $k \geq k_0$, define the sequence of "stopping times", $\{\tau_k\}$, by

$$\tau_k = \inf\left\{t \in [0,\tau_e) : Y(t) \notin \left(\frac{1}{k}, 1-\frac{1}{k}\right)\right\},\ k = 1,\ 2,\ \cdots$$

Clearly $\{\tau_k\}$ is an increasing sequence. Denote by $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \le \tau_e$ a.s. Next, we show that $\tau_\infty = \infty$ a.s. by contradiction. Define

$$V(Y) = \frac{1}{Y} + \frac{1}{1-Y}.$$

So, $V(Y) > 0$ for any $Y \in (0,1)$. According to (2.14) and Itô's Lemma, we have

$$\mathrm{d}V(Y(t)) = \Big(f(t,Y(t))V'(Y(t)) + \frac{1}{2} g^2(t,Y(t))V''(Y(t)) \Big)\mathrm{d}t + g(t,Y(t))V'(Y(t))\mathrm{d}W(t). \tag{2.15}$$

It follows immediately from (**A1**) that the functions C_1 and C_2 defined in (2.13) satisfy

$$C_1^m := \mu^m - \theta_2^M - \frac{\sigma_M^2}{2} \le \quad C_1(t) \quad \le \mu^M - \theta_2^m - \frac{\sigma_m^2}{2} := C_1^M, \tag{2.16}$$

$$C_2^m := \mu^m + \theta_1^m + \frac{\sigma_m^2}{2} \le \quad C_2(t) \quad \le \mu^M + \theta_1^M + \frac{\sigma_M^2}{2} := C_2^M. \tag{2.17}$$

Therefore, for all $Y(t) \in (0,1)$ the drift terms on the right hand side of (2.15) satisfy, respectively,

$$\begin{aligned} f(t,Y(t))V'(Y(t)) &= \frac{d_1(t)d_2(t)(d_1(t)Y(t)C_1(t) + d_2(t)(1-Y(t))C_2(t))}{(d_1(t)Y(t) + d_2(t)(1-Y(t)))^2}(2Y(t)-1)V(Y(t)) \\ &\le \frac{d_1^M d_2^M \left(d_1^M + d_2^M\right) \max_{i=1,2}\{|C_i^m|, |C_i^M|\}}{\left(\min\left\{d_1^m, d_2^m\right\}\right)^2} V(Y(t)) := K_1 V(Y(t)), \end{aligned} \tag{2.18}$$

$$\begin{aligned} g^2(t,Y(t))V''(Y(t)) &= \frac{d_1^2(t)d_2^2(t)\sigma^2(t)}{(d_1Y(t) + d_2(1-Y(t)))^2} \cdot \left(\frac{(1-Y(t))^2}{Y(t)} + \frac{Y^2(t)}{1-Y(t)} \right) \\ &\le \frac{(d_1^M d_2^M)^2 \sigma_M^2}{\left(\min\left\{d_1^m, d_2^m\right\}\right)^2} V(Y(t)) := K_2 V(Y(t)). \end{aligned} \tag{2.19}$$

Combining (2.18), (2.19) and (2.15) results in

$$\mathrm{d}V(Y(t)) = KV(Y(t))\mathrm{d}t + g(t,Y(t))V'(Y(t))\mathrm{d}W(t), \quad K = \max\{K_1, K_2\}. \tag{2.20}$$

In order to show $\tau_\infty = \infty$ a.s., assume (for contradiction) that there exist $T > 0$ and $\varepsilon > 0$ such that $\mathbb{P}(\tau_\infty \leq T) > \varepsilon$. Since $\{\tau_k\}$ is an increasing sequence, there exists $N \geq k_0$ such that

$$\mathbb{P}(\tau_k \leq T) > \frac{\varepsilon}{2}, \text{ for all } k \geq N. \tag{2.21}$$

Then it follows from (2.20) that

$$V(Y(\tau_k \wedge T)) \leq V(Y(t_0)) + \int_{t_0}^{\tau_k \wedge T} KV(Y(t))\mathrm{d}t + \int_{t_0}^{\tau_k \wedge T} g(t, Y(t))V'(Y(t)\mathrm{d}W(t).$$

Taking expectations of the inequality above gives

$$\begin{aligned} \mathbb{E}[V(Y(\tau_k \wedge T))] &\leq V(Y(t_0)) + \mathbb{E}\Big[\int_{t_0}^{\tau_k \wedge T} KV(Y(t))\mathrm{d}t\Big] \\ &\leq V(Y(t_0)) + \int_\Omega \int_{t_0}^{T} \chi_{[t_0, \tau_k(\omega)\wedge T]}(t) KV(Y(t))\mathrm{d}t\mathrm{d}\mathbb{P}, \end{aligned} \tag{2.22}$$

where χ is the indicator function, Ω is the sample space and $\mathbb{P}$ the probability measure. Then, we divide Ω into

$$\Omega_1(t) = \{\omega \in \Omega : \tau_k(\omega) < t\}, \quad \Omega_2(t) = \{\omega \in \Omega : t \leq \tau_k(\omega) \leq T\}, \quad \Omega_3 = \{\omega \in \Omega : \tau_k(\omega) > T\},$$

and apply Fubini's Theorem to obtain

$$\begin{aligned} \mathbb{E}[V(Y(\tau_k \wedge T))] &\leq V(Y(t_0)) + \int_{t_0}^{T} \int_{\Omega_1(t)\cup\Omega_2(t)\cup\Omega_3} \chi_{[t_0, \tau_k(\omega)\wedge T]}(t) KV(Y(t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &\leq V(Y(t_0)) + \int_{t_0}^{T} \int_{\Omega_1(t)} KV(Y(\tau_k \wedge t))\mathrm{d}\mathbb{P}\mathrm{d}t + \int_{t_0}^{T} \int_{\Omega_3} \chi_{[t_0,T]}(t) KV(Y(t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &\quad + \int_{t_0}^{T} \int_{\Omega_2(t)} \chi_{[t_0, \tau_k(\omega)]}(t) KV(Y(t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &= V(Y(t_0)) + \int_{t_0}^{T} \int_{\Omega_1(t)\cup\Omega_2(t)\cup\Omega_3} KV(Y(\tau_k \wedge t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &= V(Y(t_0)) + K\int_{t_0}^{T} \mathbb{E}[V(Y(\tau_k \wedge t))]\mathrm{d}t. \end{aligned}$$

Applying Gronwall's Lemma to the above inequality gives

$$\mathbb{E}[V(Y(\tau_k \wedge T))] \leq V(t_0) \cdot e^{K(T-t_0)}. \tag{2.23}$$

Moreover, due to (2.21) we have

$$\mathbb{E}[V(Y(\tau_k \wedge T))] \geq \int_{\Omega_1(t)\cup\Omega_2(t)} V(Y(\tau_k \wedge T))\mathrm{d}\mathbb{P} = \int_{\Omega_1(t)\cup\Omega_2(t)} V(Y(\tau_k(\omega)))\mathrm{d}\mathbb{P} \geq \big(k + \frac{1}{1-\frac{1}{k}}\big)\frac{\varepsilon}{2}. \tag{2.24}$$

Combining (2.23) and (2.24) results in

$$V(t_0) \cdot e^{K(T-t_0)} \geq \left(k + \frac{1}{1-\frac{1}{k}}\right)\frac{\varepsilon}{2}, \quad k \geq N.$$

Because K is independent of k, the above inequality fails as $k \to \infty$. Thus, $\tau_\infty = \infty$ a.s. and as a result $\tau_e = \infty$. The proof is complete. □

Theorem 2 not only shows the existence and the uniqueness of the global pathwise solution to (2.3), but also implies that the species cannot extinct at any finite time. However, this result does not exclude the scenarios that the $Y(t)$ converge to 0 or 1 in the long run. When the environmental parameters μ, σ, and θ are constants, the probability density of the stationary distribution was obtained in [33]. Nonetheless, the non-autonomous SDE (2.14) with non-stationary environmental parameters $\mu(t)$, $\sigma(t)$, and $\theta(t)$ may not converge to a stationary process as time goes to infinity. Thus, it is more interesting but challenging to achieve the asymptotic behaviors of the non-autonomous SDE. In the following theorem, we prove the solutions to (2.1) with different initial values will converge to each other, which means the attractor consists of a single trajectory when the sample point ω is fixed. First, we construct sufficient conditions under which all solutions of (2.14) converge in $L^1(\Omega)$, i.e., given any $t_0 \geq 0$, and $y_1, y_2 \in (0,1)$ the solutions of (2.14) with initial conditions $Y(t_0) = y_1$ and $Y(t_0) = y_2$ satisfy

$$\lim_{t\to\infty} \int_\Omega |Y(t;t_0,y_1,\omega) - Y(t;t_0,y_2,\omega)|\,\mathrm{d}\omega = 0. \tag{2.25}$$

The following assumptions will be needed.

(A2) $d_1^M \sigma_M^2 - \sigma_m^2 < 2\mu^m + 2\theta_1^m$,

(A3) $\sigma_M^2 - d_2^m \sigma_m^2 \leq 2\mu^m - 2\theta_2^M$,

(A4) $\left(\frac{d_2^M}{2d_1^m} + d_1^M\right)\sigma_M^2 - \frac{3}{2}\sigma_m^2 < -\left(\frac{d_2^M}{d_1^m} + 1\right)\mu^M + 2\mu^m - \frac{d_2^M}{d_1^m}\theta_1^M + 2\theta_1^m + \theta_2^m$,

(A5) $d_2^M \sigma_M^2 - d_1^m \sigma_m^2 \leq -2(d_1^m + d_2^M)\mu^M + 2d_1^m\theta_2^m - 2d_2^M\theta_1^M$,

(A6) $d_1^M \leq 2d_2^m$.

Theorem 3. *Assume (**A1**)–(**A2**) hold. Then all solutions of equation* (2.14) *with different initial values converge in $L^1(\Omega)$ as $t \to \infty$, if either (**A3**)–(**A4**), or (**A5**)–(**A6**) hold.*

Proof. The following transformation [44] could help us reduce the complicity of analyzing dynamics of $Y(t)$. Let

$$Z(t) = Z(t;t_0,z_0) = \ln \frac{Y(t;t_0,y_0)}{1-Y(t;t_0,y_0)}, \quad z_0 = \ln \frac{y_0}{1-y_0}.$$

Notice that this is a monotone transformation that transform the domain of state from $Y \in (0,1)$ to $Z \in (-\infty,\infty)$. Then according to Itô's formula and (2.14),

$$\begin{aligned}
\mathrm{d}Z(t) &= \Bigg(\frac{1}{Y(t)(1-Y(t))}\frac{d_1(t)d_2(t)Y(t)(1-Y(t))}{(d_1(t)Y(t)+d_2(t)(1-Y(t)))^2}\big(d_1(t)Y(t)C_1(t)+d_2(t)(1-Y(t))C_2(t)\big) \\
&\quad + \frac{1}{2}\Big(-\frac{1}{Z^2(t)} + \frac{1}{(1-Y(t))^2}\Big)\frac{\big(d_1(t)d_2(t)Y(t)(1-Y(t))\big)^2}{\big(d_1(t)Y(t)+d_2(t)(1-Y(t))\big)^2}\sigma^2(t)\Bigg)\mathrm{d}t \\
&\quad + \Bigg(\frac{1}{Y(t)(1-Y(t))}\frac{d_1(t)d_2(t)Y(t)(1-Y(t))}{d_1(t)Y(t)+d_2(t)(1-Y(t))}\sigma(t)\Bigg)\mathrm{d}W(t) \\
&= \frac{F(t)Y(t)-G(t)}{\big(d_1(t)Y(t)+d_2(t)(1-Y(t))\big)^2}\mathrm{d}t + \frac{d_1(t)d_2(t)\sigma(t)}{d_1(t)Y(t)+d_2(t)(1-Y(t))}\mathrm{d}W(t),
\end{aligned} \tag{2.26}$$

where $C_1(t)$ and $C_2(t)$ are defined in (2.13), $Y(t) = \frac{e^{Z(t)}}{1+e^{Z(t)}}$, and

$$F(t) = d_1^2(t)d_2(t)C_1(t) + d_1^2(t)d_2^2(t)\sigma^2(t) - d_1(t)d_2^2(t)C_2(t), \tag{2.27}$$

$$G(t) = -d_1(t)d_2^2(t)C_2(t) + \frac{1}{2}d_1^2(t)d_2^2(t)\sigma^2(t). \tag{2.28}$$

Assume $z_0^1,\ z_0^2 \in \mathscr{R}$ with $z_0^1 > z_0^2$ and $\omega \in \Omega$, let $Z_1(t) = Z(t;t_0,z_0^1,\omega)$ and $Z_2(t) = Z(t;t_0,z_0^2,\omega)$ be two solutions of the SDE (2.26) with initial values $Z(t_0) = z_0^1$ and $Z(t_0) = z_0^2$, respectively. Denote by $\Delta_Z(t) = Z_1(t) - Z_2(t)$, then Δ_Z satisfies

$$
\begin{aligned}
\mathrm{d}\Delta_Z(t) &= \frac{(F(t)Y_1 - G(t))\big((d_1(t)-d_2(t))Y_2 + d_2(t)\big)^2}{\big(d_1(t)Y_1 + d_2(t)(1-Y_1)\big)^2\big(d_1(t)Y_2 + d_2(t)(1-Y_2)\big)^2}\mathrm{d}t \\
&\quad - \frac{(F(t)Y_2 - G(t))\big((d_1(t)-d_2(t))Y_1^2 + d_2(t)\big)^2}{\big(d_1(t)Y_1 + d_2(t)(1-Y_1)\big)^2\big(d_1(t)Y_2 + d_2(t)(1-Y_2)\big)^2}\mathrm{d}t \\
&\quad + \frac{d_1(t)d_2(t)\sigma(t)(d_1(t)-d_2(t))(Y_2 - Y_1)}{\big(d_1(t)Y_1 + d_2(t)(1-Y_1)\big)\big(d_1(t)Y_2 + d_2(t)(1-Y_2)\big)}\mathrm{d}W(t), \\
&= \frac{F(t)H_1(t) + G(t)H_2(t)}{H_3^2(t)}\Delta_Y(t)\mathrm{d}t + H_4(t)\Delta_Y(t)\mathrm{d}W(t),
\end{aligned} \tag{2.29}
$$

where $Y_i = Y_i(t) = \frac{e^{Z_i(t)}}{1+e^{Z_i(t)}}$ for $i = 1,2$, $\Delta_Y(t) = Y_1(t) - Y_2(t)$ and

$$
\begin{aligned}
H_1(t) &= -\big(d_1(t) - d_2(t)\big)^2 Y_1 Y_2 + d_2^2(t), \\
H_2(t) &= \big(d_1(t) - d_2(t)\big)^2 (Y_1 + Y_2) + 2(d_1(t) - d_2(t))d_2(t), \\
H_3(t) &= \big(d_1(t)Y_1 + d_2(t)(1-Y_1)\big)\big(d_1(t)Y_2 + d_2(t)(1-Y_2)\big), \\
H_4(t) &= \frac{d_1(t)d_2(t)\sigma(t)(d_2(t) - d_1(t))}{H_3(t)}.
\end{aligned}
$$

Next, integrating the SDE (2.29) gives

$$
\Delta_Z(t) = z_0^1 - z_0^2 + \int_{t_0}^{t} \frac{F(s)H_1(s) + G(s)H_2(s)}{H_3^2(s)}\Delta_Y(s)\mathrm{d}s + \int_{t_0}^{t} H_4(s)\Delta_Y(s)\mathrm{d}W(s). \tag{2.30}
$$

$F(s)H_1(s)$ and $G(s)H_2(s)$ can be simplified to be

$$
\begin{aligned}
F(s)H_1(s) &= d_1(s)d_2(s)\big(d_1(s)C_1(s) + d_1(s)d_2(s)\sigma^2(s) - d_2(s)C_2(s)\big) \\
&\qquad \cdot \Big(d_2^2(s) - \big(d_1(s) - d_2(s)\big)^2 Y_1 Y_2\Big), \\
G(s)H_2(s) &= \frac{1}{2}d_1(s)d_2^2(s)\left(-2C_2(s) + d_1(s)\sigma^2(s)\right) \\
&\qquad \cdot \left((d_1(s) - d_2(s))^2(Y_1 + Y_2) + 2(d_1(s) - d_2(s))d_2(s)\right),
\end{aligned}
$$

which, after being rearranged, give

$$\begin{aligned}F(s)H_1(s)+G(s)H_2(s) &= d_1(s)d_2^2(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right)(d_1(s)-d_2(s))^2\big(\frac{Y_1+Y_2}{2}-Y_1Y_2\big)\\ &\quad +d_1(s)d_2^3(s)\,(d_1C_1+d_2C_2)+d_1^2(s)d_2^3(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right)\\ &\quad -d_1(s)d_2(s)\,(d_1C_1+d_2C_2)\left(d_1(s)-d_2(s)\right)^2 Y_1Y_2. \qquad (2.31)\end{aligned}$$

Plugging (2.31) into (2.30) then taking expectation of the resulting equation gives

$$\mathbb{E}[\Delta_Z(t)] = z_0^1 - z_0^2 + \mathbb{E}\left[\int_{t_0}^t \frac{C_3(s)+C_4(s)}{H_3^2(s)}\Delta_Y(s)\mathrm{d}s\right], \qquad (2.32)$$

where

$$\begin{aligned}C_3(s) &= d_1(s)d_2^2(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right)(d_1(s)-d_2(s))^2\left(\frac{Y_1+Y_2}{2}-Y_1Y_2\right),\\ C_4(s) &= d_1(s)d_2^3(s)\,(d_1(s)C_1(s)+d_2(s)C_2(s))+d_1^2(s)d_2^3(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right)\\ &\quad -d_1(s)d_2(s)(d_1(s)-d_2(s))^2(d_1(s)C_1(s)+d_2(s)C_2(s))Y_1Y_2.\end{aligned}$$

Note that under Assumptions (**A1**) and (**A2**)

$$-2C_2(s)+d_1(s)\sigma^2(s) \le -2\mu^m - 2\theta_1^m - \sigma_m^2 + d_1^M\sigma_M^2 := K_3 < 0. \qquad (2.33)$$

Moreover, since $d_i(t) > 0$ for $i = 1,2$ and $Y_1, Y_2 \le 1$, then $\frac{Y_1+Y_2}{2} - Y_1Y_2 \ge 0$ and thus

$$C_3(s) \le 0. \qquad (2.34)$$

Next, we estimate $C_4(s)$ under two cases, where $d_1(s)C_1(s)+d_2(s)C_2(s)$ is positive or negative, respectively. Indeed, $d_1(s)C_1(s)+d_2(s)C_2(s) > 0$ under Assumption (**A3**), and $d_1(s)C_1(s)+d_2(s)C_2(s) < 0$ under Assumption (**A5**).

(i) Notice the inequality (2.33) shows that $C_2(s) > \frac{d_1(s)\sigma^2(s)}{2}$. Thus by Assumptions (**A1**) and (**A3**), and (2.16)–(2.17)

$$\begin{aligned} d_1(s)C_1(s)+d_2(s)C_2(s) &> d_1(s)\left(C_1(s)+d_2(s)\frac{\sigma^2(s)}{2}\right) \\ &\geq \frac{d_1(s)}{2}\left(2\mu^m-2\theta_2^M-\sigma_M^2+d_2^m\sigma_m^2\right)\geq 0, \end{aligned}$$

and because of $Y_1, Y_2 \geq 0$, (2.16)–(2.17) and Assumptions (**A1**) and (**A4**), we get

$$\begin{aligned} C_4(s) &\leq d_1^2(s)d_2^3(s)\left(C_1(s)+\frac{d_2(s)}{d_1(s)}C_2(s)-2C_2(s)+d_1(s)\sigma^2(s)\right) \\ &\leq (d_1^m)^2(d_2^m)^3\left(C_1^M+\frac{d_2^M}{d_1^m}C_2^M-2C_2^m+d_1^M\sigma_M^2\right) := K_4 < 0. \qquad (2.35) \end{aligned}$$

(ii) Since $C_2(s) \geq 0$, then by Assumption (**A5**) and (2.16)–(2.17) we have

$$\begin{aligned} d_1(s)C_1(s)+d_2(s)C_2(s) &\leq d_1(s)\left(C_1^M+\frac{d_2^M}{d_1^m}C_2^M\right) \\ &= d_1(s)\left(\mu^M-\theta_2^m-\frac{1}{2}\sigma_m^2+\frac{d_2^M}{d_1^m}\left(\mu^M+\theta_1^M+\frac{1}{2}\sigma_M^2\right)\right) \\ &\leq 0, \qquad (2.36) \end{aligned}$$

and therefore, from $Y_1, Y_2 \leq 1$ it follows that

$$\begin{aligned} C_4(s) \leq\ & d_1^2(s)d_2(s)\left(d_1(s)C_1(s)+d_2(s)C_2(s)\right)\left(-d_1(s)+2d_2(s)\right) \\ & +d_1^2(s)d_2^3(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right). \qquad (2.37) \end{aligned}$$

Assumptions (**A1**) and (**A6**) guarantee that

$$-d_1(s)+2d_2(s) \geq -d_1^M+2d_2^m \geq 0,$$

and thus, (2.37) and (2.33) imply that

$$C_4(s) \leq d_1^2(s)d_2^3(s)\left(-2C_2(s)+d_1(s)\sigma^2(s)\right) \leq (d_1^m)^2(d_2^m)^3K_3 < 0.$$

Since $z_0^1 - z_0^2 > 0$, then $\Delta_Z(t) = \Delta_Z(t;t_0,z_0^1 - z_0^2,\omega) \geq 0$ for all $t \geq t_0$ and any $\omega \in \Omega$ due to the uniqueness of solutions. The monotonicity of the transformation $Z(t)$ implies that $\Delta_Y = \Delta_Y(t;t_0,y_0^1 - y_0^2,\omega) \geq 0$ for all $t \geq t_0$ and any $\omega \in \Omega$. Note that $H_3(t) \leq (d_1^M + d_2^M)^2$, and inserting (2.34) and (2.35) into (2.32) gives

$$\mathbb{E}[\Delta_Z(t)] \leq z_0^1 - z_0^2 + \frac{K_4}{(d_1^M + d_2^M)^4}\mathbb{E}\left[\int_{t_0}^{t} \Delta_Y(s)\mathrm{d}s\right]. \tag{2.38}$$

Similarly, inserting (2.34) and (2.37) into (2.32) shows

$$\mathbb{E}[\Delta_Z(t)] \leq z_0^1 - z_0^2 + \frac{(d_1^m)^2(d_2^m)^3K_3}{(d_1^M + d_2^M)^4}\mathbb{E}\left[\int_{t_0}^{t} \Delta_Y(s)\mathrm{d}s\right]. \tag{2.39}$$

Moreover, applying the mean value Theorem results in the existence of $y \in (0,1)$ such that

$$\mathbb{E}[\Delta_Z(t)] = \mathbb{E}\left[\left(\frac{1}{y} + \frac{1}{1-y}\right)\Delta_Y(t)\right] \geq 4\mathbb{E}[\Delta_Y(t)].$$

The above inequality, along with (2.38)–(2.39) and the Fubini Theorem give

$$\mathbb{E}[\Delta_Y(t)] \leq \frac{z_0^1 - z_0^2}{4} + \frac{\alpha}{4}\int_{t_0}^{t} \mathbb{E}[\Delta_Y(s)]\mathrm{d}s \text{ with } \alpha := \frac{\max\left\{K_4,(d_1^m)^2(d_2^m)^3K_3\right\}}{(d_1^M + d_2^M)^4} < 0.$$

It follows immediately from Gronwall's Lemma that

$$\mathbb{E}[\Delta_Y(t)] \leq \frac{z_0^1 - z_0^2}{4}e^{\frac{\alpha}{4}(t-t_0)} \to 0, \text{ as } t \to \infty,$$

which confirms that all solutions with different initial values converge in $L^1(\Omega)$. The proof is complete. □

The assumptions in the theorem above only consist of the upper and lower bounds of the moments of the non-stationary parameters. The following corollary provides a set of time-dependent conditions, which are weaker than assumptions in Theorem 3.

(B2) there exists $\varepsilon > 0$ such that $\big(1 - d_1(t)\big)\sigma^2(t) + 2\mu(t) + 2\theta_1(t) \geq \varepsilon$ for $t \in [t_0,T]$,

(B3) $\big(1 - d_2(t)\big)\sigma^2(t) \leq 2\mu(t) - 2\theta_2(t)$ for $t \in [t_0,T]$,

(B4) there exists $\varepsilon > 0$ such that for $t \in [t_0, T]$,

$\big(d_1(t) - d_2(t)\big)\mu(t) + \big(2d_1(t) - d_2(t)\big)\theta_1(t) + d_1(t)\theta_2(t) + \left(\frac{3d_1(t)}{2} - d_1^2(t) - \frac{d_2(t)}{2}\right)\sigma^2(t)$

is greater than ε,

(B5) $(d_2(t) - d_1(t))\sigma^2(t) \le -2(d_1(t) + d_2(t))\mu(t) + 2(d_1(t)\theta_2(t) - d_2(t)\theta_1(t))$ for $t \in [t_0, T]$,

(B6) $d_1(t) \le 2d_2(t)$ for $t \in [t_0, T]$.

Corollary 1. *Assume (**A1**) and (**B2**) hold. Then all solutions of equation* (2.14) *with different initial values converge in $L^1(\Omega)$ as $t \to \infty$, if either (**B3**)–(**B4**), or (**B5**)–(**B6**) hold.*

Theorem 3 and Corollary 1 show that for each fixed $\omega \in \Omega$, there is a limiting trajectory that attracts all the solutions start from different initial value. Consequently, the existence of a time-dependent limiting process for the non-autonomous SDE (2.14) as $t \to \infty$ is ensured. On the other side, the coexistence of two species means the proportion of the first species $Y(t)$ is stochastic persistent, i.e., $\Pr(\lim_{t\to\infty} Y(t) = 1) = \Pr(\lim_{t\to\infty} Y(t) = 0) = 0$ [34]. Therefore, Theorem 3 and Corollary 1 cannot exclude the case in which $Y(t)$ approaches 0 or 1 as $t \to \infty$ and hence do not confirm the coexistence of the two species. In the following theorem, we further establish sufficiently conditions for coexistence in the sense of stochastic persistence of $Y(t)$.

Theorem 4. *Let Assumptions (**A1**), (**A2**), and (**A5**) hold. In addition, assume that*

(A7) $d_2(t) \le d_1(t) \le 2d_2(t)$ for all $t \ge t_0$.

Then there exists a deterministic function $Y^*(t) \in [Y_m^*, Y_M^*]$ for all $t \ge t_0$ with $Y_m^* > 0$ and $Y_M^* < 1$ such that

$$\limsup_{t\to\infty}\big(Y(t) - Y^*(t)\big) \ge 0, \quad \liminf_{t\to\infty}\big(Y(t) - Y^*(t)\big) \le 0, \quad a.s.$$

Proof. To reduce the complexity, we still apply the same transformation in Theorem 2.10. Let $Z(Y(t)) = \ln \frac{Y(t)}{1-Y(t)}$. Then by Itô's Lemma, $Z(t)$ satisfies the SDE

$$\mathrm{d}Z(t) = U(Y(t),t)\mathrm{d}t + \frac{d_1(t)d_2(t)\sigma(t)}{d_1(t)Y(t)+d_2(t)(1-Y(t))}\mathrm{d}W(t), \quad \text{with} \tag{2.40}$$

$$U(Y(t),t) := \frac{F(t)Y(t)-G(t)}{\big(d_1(t)Y(t)+d_2(t)(1-Y(t))\big)^2}, \tag{2.41}$$

where $F(t)$ and $G(t)$ are defined by (2.27)–(2.28).

Whereby Assumptions (**A1**) and (**A2**), the inequality (2.33) still holds and thus

$$G(t) \le \frac{1}{2}d_1^m(d_2^m)^2\left(-2\mu^m - 2\theta_1^m - \sigma_m^2 + d_1^M\sigma_M^2\right) := G^M < 0, \quad \forall t \ge t_0. \tag{2.42}$$

Moreover, by Assumption (**A1**) and (2.42) we get

$$0 > G(t) \ge \frac{1}{2}d_1^M(d_2^M)^2\left(-2\mu^M - 2\theta_1^M - \sigma_M^2 + d_1^m\sigma_m^2\right) := G^m, \quad \forall t \ge t_0. \tag{2.43}$$

The inequalities (2.42) and (2.33) along with Assumption (**A5**) imply that

$$C_3(t) := d_1(t)C_1(t) + \frac{1}{2}d_1(t)d_2(t)\sigma^2(t) < d_1(t)C_1(t) + d_2(t)C_2(t) \le 0, \quad \forall t \ge t_0.$$

Combining (2.43) and (2.36) gives

$$d_1^M d_2^M C_3^m + G^m < d_1(t)d_2(t)C_3(t) + G(t) = F(t) < 0, \quad \forall t \ge t_0, \tag{2.44}$$

where $C_3^m = d_1^M\left(\mu^m - \theta_2^M - \frac{\sigma_M^2}{2}\right) + \frac{1}{2}d_1^m d_2^m \sigma_m^2$.

Consequently, $U(Y(t),t) = 0$ always has a positive root $Y^*(t) = G(t)/F(t)$. Furthermore, due to (2.42) and (2.44)

$$Y^*(t) \ge \frac{G^M}{d_1^M d_2^M C_3^m + G_m} := Y_m^* > 0, \quad \forall t \ge t_0. \tag{2.45}$$

Besides, using that $C_3(t) \leq d_1^m \left(\mu^M - \theta_2^m - \frac{\sigma_m^2}{2}\right) + \frac{1}{2} d_1^M d_2^M \sigma_M^2 := C_3^M < 0$, and noticing that $Y^*(t)$ is a decreasing function with respect to G result in

$$Y^*(t) \leq \frac{G^m}{d_1^m d_2^m C_3^M + G^m} := Y_M^* < 1, \quad \forall t \geq t_0. \tag{2.46}$$

Next, we show that $\limsup_{t\to\infty} \big(Y(t) - Y^*(t)\big) \geq 0$ a.s. by contradiction. Assume it is not true. Then there exists $\varepsilon \in (0,1)$ sufficiently small, such that

$$\mathbb{P}[\Omega_4(t)] > \frac{1}{2}\varepsilon \quad \text{for} \quad \Omega_4(t) = \left\{\omega : \limsup_{t\to\infty} \big(Y(t,\omega) - Y^*(t)\big) \leq -\varepsilon\right\}.$$

Hence, for every $\omega \in \Omega_4(t)$, there exists $T_1(\omega,\varepsilon)$ such that

$$Y(t,\omega) - Y^*(t) \leq -\frac{1}{2}\varepsilon, \quad \forall\, t \geq T_1(\omega,\varepsilon). \tag{2.47}$$

Due to (2.42) and (2.44), the function U defined in (2.41) satisfies $U(Y=0,t) > 0$ and $U(Y=1,t) < 0$ for all $t \geq t_0$. Moreover, under Assumption (**A7**)

$$\begin{aligned}
\frac{\partial U}{\partial Y} &= \frac{-F(t)d_1(t)Y(t) + F(t)d_2(t)Y(t) + 2G(t)d_1(t) + F(t)d_2(t) - 2G(t)d_2(t)}{\big(d_1(t)Y(t) + d_2(t)(1-Y(t))\big)^3} \\
&= \frac{F(t)Y(t)(2d_2(t) - d_1(t)) + F(t)d_2(t)(1-Y(t)) + 2G(t)(d_1(t) - d_2(t))}{\big(d_1(t)Y(t) + d_2(t)(1-Y(t))\big)^3} < 0,
\end{aligned}$$

which means that $U(Y,t)$ is a decreasing function with respect to Y for any $Y \in (0,1)$ and $t \geq t_0$. This together with (2.47) give

$$U(Y(t,\omega),t) \geq U\left(Y^*(t) - \frac{1}{2}\varepsilon, t\right) > U(Y^*(t)) = 0, \quad \forall\, t \geq T_1(\omega,\varepsilon). \tag{2.48}$$

Plugging (2.48) into (2.40) gives

$$\begin{aligned}
Z(Y(t,\omega)) \;\geq\; & Z(y_0) + \int_{t_0}^{T_1(\omega,\varepsilon)} U(Y(s),s)\mathrm{d}s + U\left(Y^*(t) - \frac{1}{2}\varepsilon, t\right)(t - T_1(\omega,\varepsilon)) \\
& + \int_{t_0}^{t} \frac{d_1(s)d_2(s)\sigma(s)}{d_1(s)Y(s) + d_2(s)(1-Y(s))} \mathrm{d}W(s,\omega), \quad \forall\, t \geq T_1(\omega,\varepsilon).
\end{aligned}$$

Then the large number theorem of martingales implies there exists $\Omega_5 \in \Omega$ with $\mathbb{P}[\Omega_5] = 1$ such that

$$\liminf_{t\to\infty} \frac{1}{t} Z(Y(t,\omega)) \geq U\left(Y^*(t) - \frac{1}{2}\varepsilon, t\right) > 0, \quad \omega \in \Omega_4(t) \cap \Omega_5,\ t \geq T_1(\omega,\varepsilon).$$

Consequently, $\lim_{t\to\infty} Z(Y(t,\omega)) = \infty$, i.e., $\lim_{t\to\infty} Y(t,\omega) = 1$, which contradicts (2.46) and (2.47). Therefore $\limsup_{t\to\infty} \left(Y(t) - Y^*(t)\right) \geq 0$ a.s.

In order to prove $\liminf_{t\to\infty} \left(Y(t) - Y^*(t)\right) \leq 0$ a.s. We again assume for contradiction that it does not hold. Then there exists $\varepsilon \in (0,1)$ sufficiently small, such that

$$\mathbb{P}[\Omega_6(t)] > \frac{1}{2}\varepsilon \quad \text{for} \quad \Omega_6(t) = \left\{\omega : \liminf_{t\to\infty} \left(Y(t,\omega) - Y^*(t)\right) \geq \varepsilon\right\}.$$

Hence, for every $\omega \in \Omega_6(t)$, there exists $T_2(\omega,\varepsilon)$ such that

$$Y(t,\omega) - Y^*(t) \geq \frac{1}{2}\varepsilon, \quad \forall\, t \geq T_2(\omega,\varepsilon). \tag{2.49}$$

Going through similar process as above, we obtain that $\lim_{t\to\infty} Z(Y(t,\omega)) = -\infty$, i.e., $\lim_{t\to\infty} Y(t,\omega) = 0$, which contradiccts (2.45) and (2.49). Therefore

$$\liminf_{t\to\infty} \left(Y(t) - Y^*(t)\right) \leq 0 \quad \text{a.s.}$$

The proof is complete. □

Theorem 4 shows that that $Y(t)$ will oscillate infinitely often around the time-dependent trajectory $Y^*(t)$ that is away from 0 and 1, which implies the coexistence of 2 species. When $Y^*(t)$ is a constant, this means $Y(t)$ is stochastic bounded [28]. Here, because $0 < Y^*_m \leq Y^*(t) \leq Y^*_M < 1$, Theorem 4 generalizes this concept of coexistence to the case with $Y^*(t)$ time-dependent. Hence, the two species also coexist in the sense of generalized stochastic boundedness.

Clearly, the Assumption (**A7**) in Theorem 4 is stronger that the Assumption (**A6**) in Theorem 3. Therefore, under the assumptions of Theorem 4, we can also obtain the limiting process that attracts all solutions with differential values. Thus, the additional condition of $d_1(t) \geq d_2(t)$ in Assumption (**A7**) essentially puts a balance between the death rates of two species, which promotes their coexistence [52]. It is worth noting that for the special non-stationary case where the SDE (2.14) has periodic parameters, i.e., $\sigma(t)$, $d_i(t)$, $\mu(t)$, and $\theta_i(t)$ are periodic functions with the same period, conditions weaker than those in Theorem 4 for the coexistence may be obtained by utilizing the method of Lyapunov exponents [53].

2.4 Fokker-Planck equation for the lottery model

In the previous section, we establish the unique global solution to SDE (2.1) and analyze the long-term dynamics of the solution. Indeed, the fraction of sites occupied by each species, i.e., path-wise dynamics of the states are studied. This section is devoted to the probabilistic behavior of states. In particular, we first derive the Fokker-Planck equation on the transition probability density of the diffusion process, and then prove the existence and uniqueness of the solution.

Note that $Y(t)$is the diffusion process determined by the drift and diffusion coefficients f and g defined in (2.11) and (2.12). For any $(s,x) \in \mathbb{R}^+ \times (0,1)$ and $(t,y) \in (s,\infty) \times (0,1)$, let $P(t,y|s,x) = \mathbb{P}[Y(t) \leq y | Y(s) = x]$ be the transition probability and denote $p(t,y|s,x)$ as the corresponding transition probability density. The following lemma shows $p(t,y|s,x)$ satisfies Fokker-Planck equation.

Lemma 1. *For any $(s,x) \in (0,\infty) \times (0,1)$ and $(t,y) \in (s,\infty) \times (0,1)$, if the continuous partial derivatives $\frac{\partial}{\partial t} p(t,y|s,x)$, $\frac{\partial}{\partial y}(f(t,y)p(t,y|s,x))$, $\frac{\partial^2}{\partial^2 y}\left(g^2(t,y)p(t,y|s,x)\right)$ exist, the transition probability density $p(t,y|s,x)$ satisfies the Fokker-Planck equation*

$$\frac{\partial}{\partial t} p(t,y|s,x) = -\frac{\partial}{\partial y}(f(t,y)p(t,y|s,x)) + \frac{1}{2}\frac{\partial^2}{\partial^2 y}\left(g^2(t,y)p(t,y|s,x)\right),$$

with the boundary condition $p(s,y|s,x) = \delta(y-x)$, where δ is the Dirac function.

Proof. Since $p(t,y|s,x)$ is the transition probability density function, $p(s,y|s,x) = \delta(y-x)$ holds automatically. Next, for any $\phi \in C_c^2(0,1)$ we estimate

$$\begin{aligned}
\mathscr{I} &:= \int_0^1 \frac{\partial}{\partial t}\phi(y)p(t,y|s,x)\mathrm{d}y = \frac{\partial}{\partial t}\int_0^1 \phi(y)p(t,y|s,x)\mathrm{d}y \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_0^1 \phi(y)p(t+h,y|s,x)\mathrm{d}y - \int_0^1 \phi(z)p(t,z|s,x)\mathrm{d}z\right).
\end{aligned}$$

To that end, first use the Kolmogorov-Chapman equation and Fubini's Theorem to obtain

$$\begin{aligned}
\mathscr{I} &= \lim_{h\to 0}\frac{1}{h}\left(\int_0^1 \phi(y)\Big(\int_0^1 p(t+h,y|t,z)p(t,z|s,x)dz\Big)\mathrm{d}y - \int_0^1 \phi(z)p(t,z|s,x)\mathrm{d}z\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_0^1 \Big(\int_0^1 \phi(y)p(t+h,y|t,z)\mathrm{d}y\Big)p(t,z|s,x)\mathrm{d}z - \int_0^1 \phi(z)p(t,z|s,x)\mathrm{d}z\right) \\
&= \lim_{h\to 0}\frac{1}{h}\int_0^1 \Big(\int_0^1 \phi(y)p(t+h,y|t,z)\mathrm{d}y - \phi(z)\Big)p(t,z|s,x)\mathrm{d}z \\
&= \int_0^1 \Big(\lim_{h\to 0}\frac{1}{h}\int_0^1 \big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y\Big)p(t,z|s,x)\mathrm{d}z. \qquad (2.50)
\end{aligned}$$

To further estimate $\mathscr{I}$, we split the inner integral in (2.50) into two parts. More precisely, let ε be an arbitrary positive number and write

$$\begin{aligned}
\int_0^1 \big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y &= \int_{|z-y|>\varepsilon} \big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y \\
&\quad + \int_{|z-y|\le\varepsilon} \big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y. \qquad (2.51)
\end{aligned}$$

Then by using the property of the probability density of a diffusion process [54], there exists $K = K(\phi)$ such that

$$\int_{|z-y|>\varepsilon} \big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y \le K\int_{|z-y|>\varepsilon} p(t+h,y|t,z)\mathrm{d}y \le \mathbf{o}(h). \qquad (2.52)$$

On the other side, by Taylor's expansion of ϕ at z and (2.3) – (2.4) again,

$$\begin{aligned}
& \frac{1}{h}\int_{|z-y|\le\varepsilon}\big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y \\
= \quad & \frac{1}{h}\int_{|z-y|\le\varepsilon}\big(\phi'(z)(y-z)+\frac{1}{2}\phi''(z)(y-z)^2\big)p(t+h,y|t,z)\mathrm{d}y+\mathbf{o}(\varepsilon^2) \\
= \quad & \phi'(z)f(t,z)+\frac{1}{2}\phi''(z)g^2(t,z)+\mathbf{o}(\varepsilon^2). \qquad (2.53)
\end{aligned}$$

Applying estimates (2.52) and (2.53) to (2.51) and letting $\varepsilon \to 0$ gives

$$\lim_{h\to 0}\frac{1}{h}\int_0^1\big(\phi(y)-\phi(z)\big)p(t+h,y|t,z)\mathrm{d}y=\phi'(z)f(t,z)+\frac{1}{2}\phi''(z)g^2(t,z). \qquad (2.54)$$

Now inserting (2.54) into (2.50) and integrating by parts results in

$$\begin{aligned}
\int_0^1\frac{\partial}{\partial t}\phi(y)p(t,y|s,x)\mathrm{d}y &= \int_0^1\left(f(t,z)\phi'(z)+\frac{1}{2}g^2(t,z)\phi''(z)\right)p(t,z|s,x)\mathrm{d}z \\
&= \int_0^1\left(-\frac{\partial}{\partial z}(a(t,z)p(t,z|s,x))+\frac{\partial^2}{\partial^2 z}(\frac{1}{2}b^2(t,z)p(t,z|s,x))\right)f(z)dz \\
&= \int_0^1\left(-\frac{\partial}{\partial y}(f(t,y)p(t,y|s,x))+\frac{\partial^2}{\partial^2 y}(\frac{1}{2}g^2(t,y)p(t,y|s,x))\right)\phi(y)\mathrm{d}y.
\end{aligned}$$

Since ϕ is an arbitrary function in $C_c^2(0,1)$ and $p(t,y|s,x)$ is continuous, the above equation implies that

$$\frac{\partial}{\partial t}p(t,y|s,x)=-\frac{\partial}{\partial y}(f(t,y)p(t,y|s,x))+\frac{1}{2}\frac{\partial^2}{\partial^2 y}\big(g^2(t,y)p(t,y|s,x)\big) \qquad (2.55)$$

for all $(s,x)\in(0,\infty)\times(0,1)$ and $(t,y)\in(s,\infty)\times(0,1)$. The proof is complete. □

Recall that $Y(t)$ is the diffusion approximation of the discrete lottery model. Hence, 0 and 1 are invariant for Y, i.e., if $Y(\tau)=0$ or $Y(\tau)=1$ for some $\tau>t_0$, then $Y(t)$ cannot get back to $(0,1)$ for $t>\tau$, which implies that we have an absorbing barrier problem [55]. From (2.55) we can formulate the Fokker-Planck equation corresponding to the SDE (2.14) as

$$\frac{\partial}{\partial t}p(t,y|t_0,y_0)=-\frac{\partial}{\partial y}\big(f(t,y)p(t,y|t_0,y_0)\big)+\frac{1}{2}\frac{\partial^2}{\partial^2 y}\big(g^2(t,y)p(t,y|t_0,y_0)\big)\ \text{in } (0,\infty)\times(0,1) \qquad (2.56)$$

with boundary and initial conditions

$$p(t,0|t_0,y_0) = p(t,1|t_0,y_0) = 0 \text{ for } t > t_0, \qquad p(t_0,y|t_0,y_0) = \delta(y-y_0) \text{ for } y \in (0,1). \quad (2.57)$$

Next, we prove the existence and uniqueness of solutions for the above partial differential equation system (2.56)–(2.57). The diffusion coefficient satisfies $g(t,0) = g(t,1)$ which yields that (2.56) is a degenerate parabolic equation. We cannot apply the classical existence theorems directly. However, the transformation

$$Z(t) = \ln \frac{Y(t)}{1-Y(t)}, \quad t \geq t_0,$$

will help us overcome the technical difficulties resulted from the degeneracy of (2.56).

Denote by $Q(t,z|s,x) = \mathbb{P}[Z(t) \leq z | Z(s) = x]$ the transition probability of the process $Z(t)$, and by $q(t,z|s,x)$ the corresponding transition probability density of $Z(t)$. Since Z is strictly increasing with respect to Y, and given any $y_0 \in (0,1)$,

$$\begin{aligned} P(t,y|t_0,y_0) &= \mathbb{P}[Y(t) \leq y | Y(t_0) = y_0] \\ &= \mathbb{P}\left[Z(t) \leq \ln \frac{y}{1-y} \Big| Z(t_0) = z_0\right] = Q\left(t, \ln \frac{y}{1-y} \Big| t_0, \ln \frac{y_0}{1-y_0}\right). \end{aligned}$$

Differentiating both sides of the above equations results in the following relation between the transition probability densities of $Z(t)$ and $Y(t)$

$$p(t,y|t_0,y_0) = q\left(t, \ln \frac{y}{1-y} \Big| t_0, \ln \frac{y_0}{1-y_0}\right) \frac{1}{y(1-y)}. \quad (2.58)$$

Since, $q(t,z|t_0,z_0)$ is a transition probability density,

$$\begin{aligned} \int_0^1 p(t,y|t_0,u_0)\mathrm{d}y &= \int_0^1 q\Big(t, \ln \frac{y}{1-y} | t_0, \ln \frac{y_0}{1-y_0}\Big) \frac{1}{y(1-y)} \mathrm{d}y \\ &= \int_0^1 q\Big(t, \ln \frac{y}{1-y} | t_0, \ln \frac{y_0}{1-y_0}\Big) \, \mathrm{d}\big(\ln \frac{y}{1-y}\big) \\ &= \int_0^1 q\Big(t, z | t_0, \ln \frac{y_0}{1-y_0}\Big) \mathrm{d}z = 1. \end{aligned}$$

Thus, it suffices to prove the existence of solutions for the equation satisfied by $q(t,z|t_0,z_0)$. Note that $Z(t)$ satisfies the SDE (2.26) which is equivalent to

$$\mathrm{d}Z(t) = \tilde{f}(t,Z)\mathrm{d}t + \tilde{g}(t,Z)\mathrm{d}W(t)$$

with

$$\begin{aligned}
\tilde{f}(t,z) &= \frac{d_1(t)d_2(t)\,(1+e^z)}{(d_1(t)e^z+d_2(t))^2}\Big(e^z d_1(t)C_1(t)+d_2(t)C_2(t)+\frac{1}{2}d_1(t)d_2(t)\sigma^2(t)\,(e^z-1)\Big),\\
\tilde{g}(t,z) &= \frac{d_1(t)d_2(t)(1+e^z)\sigma(t)}{d_1(t)e^z+d_2(t)}.
\end{aligned}$$

It follows from (2.55) that $q(t,z|t_0,z_0)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}q(t,z|t_0,z_0) = -\frac{\partial}{\partial z}(\tilde{f}(t,z)q(t,z|t_0,z_0)) + \frac{1}{2}\frac{\partial^2}{\partial^2 z}(\tilde{g}^2(t,z)q(t,z|t_0,z_0)) \text{ in } (0,\infty)\times\mathbb{R}, \quad (2.59)$$

with the initial condition

$$q(t_0,z|t_0,z_0) = \delta(z-z_0) \quad \text{for } z\in\mathscr{R}. \qquad (2.60)$$

Notice that there is no boundary condition in the new PDE (2.59) that possesses the whole real line as its boundary condition. In addition, assumption [(**A1**)] includes that $\sigma(t)\geq\sigma_m^2>0$ for all $t\geq t_0$. Then the problem (2.59)–(2.60) is uniformly parabolic on $[0,\infty)\times\mathbb{R}$. The existence of classical solutions for (2.59)–(2.60) is ensured by the results in [56] (Theorem 11, Chapter 1) and [57] (Theorem 3), and the uniqueness of a classical solution is ensured by [58] (Theorem 9.4.3, Chapter 9). Moreover, the proof in [58] (Theorem 6.6.2, Chapter 6) shows that the unique classical solution of (2.59)–(2.60) is a probability density. We conclude the above result in the following theorem.

Theorem 5. *Let Assumption (***A1***) hold. Then, the problem* (2.56)–(2.57) *has a unique classical solution* $p(t,y|t_0,y_0)$. *Moreover,* $p(t,y|t_0,y_0)$ *is a probability density.*

In fact, the solution to (2.59)–(2.60) may exist uniquely in more general sense (see, e.g., [58], Theorem 6.6.2, Chapter 6 and [58], Theorem 9.4.3, Chapter 9). Hence, Theorem 5 can then be generalized to the following version. But the solution may not be a probability density.

Theorem 6. *Let $\mu(t), \sigma(t), \theta_i(t), d_i(t)$ ($i = 1, 2$) be bounded Lebesgue measurable functions on $[t_0, \infty)$, and assume that $\inf_{t \geq t_0}\{\sigma(t)\} > 0$. Then the problem* (2.56)–(2.57) *has a unique solution $p(t, y|t_0, y_0)$. However, $p(t, y|t_0, y_0)$ is a sub-probability density with $\int_0^1 p(t, y|t_0, y_0)\mathrm{d}y \leq 1$.*

Remark 6. *In Theorem 6, the solution of the problem* (2.56)–(2.57) *exists in the sense that for every function $\varphi \in C_c^\infty(0, 1)$ there exists a set of full measure subset J_φ (see [58]) of $(0, \infty)$ such that for all $t \in J_\varphi$ we have*

$$\int_0^1 \varphi \cdot p(t, y|t_0, y_0)\mathrm{d}y - \int_0^1 \varphi \cdot \delta(y - y_0)\mathrm{d}y = \lim_{\tau \to 0+} \int_\tau^t \int_0^1 L_{f,g}(\varphi) \cdot p(r, y|t_0, y_0)\mathrm{d}y\mathrm{d}r,$$

where $L_{f,g}(\varphi) = \frac{1}{2}g^2(t, y)\frac{\partial^2 \varphi}{\partial y^2} + f(t, y)\frac{\partial \varphi}{\partial y}$.

Notice that in Theorem 6, weaker assumptions on the environment parameters result in a sub-probability density. To be specific, the environment parameters in Theorem 6 do not need to be continuous, which implies one of the species may extinct in an extreme scenario with fierce environmental fluctuations.

2.5 Numerical simulations

In this section, numerical simulations are presented to illustrate the theoretical results obtained in Section 2.3. Throughout this section, the initial time is assumed to be $t_0 = 0$. Moreover, we choose the following set of time-dependent moment functions that satisfy assumptions (**A1**), (**B2**), (**B5**), (**B6**), and (**A7**), for the non-stationary environmental processes.

$$\begin{cases} \mu(t) = \frac{1+2t}{1+t} + \frac{0.5\cos e^t}{2+\cos(\sqrt{2}e^t)}, & \sigma(t) = \frac{15+20t}{1+t} + \frac{\cos e^t}{2+\cos(\sqrt{2}e^t)}, \\ d_1(t) = \frac{1+3t}{5+10t} + \frac{0.1\cos e^t}{2+\cos(\sqrt{2}e^t)}, & d_2(t) = \frac{1+t}{7+5t} + \frac{0.05\cos e^t}{2+\cos(\sqrt{2}e^t)}, \\ \theta_1(t) = \frac{0.5\cos e^t}{2+\cos(\sqrt{2}e^t)}, & \theta_2(t) = -\frac{0.25\cos e^t}{2+\cos(\sqrt{2}e^t)}. \end{cases} \tag{2.61}$$

First, in Figure 2.1 below, for a fixed sample point $\omega \in \Omega$, simulated solutions of the SDE (2.14) with 10 different initial values $\{y_j\}_{j=1,\cdots,10}$ are presented. We can easily observe that 10 solutions converge to a non-stationary process, as time goes on, even though they have different starting points. This phenomenon can be addressed by AEDT in [44], that describes asymptotic dynamics of $Y(t)$ only determined by the environment rather than the initial states in the context of lottery models.

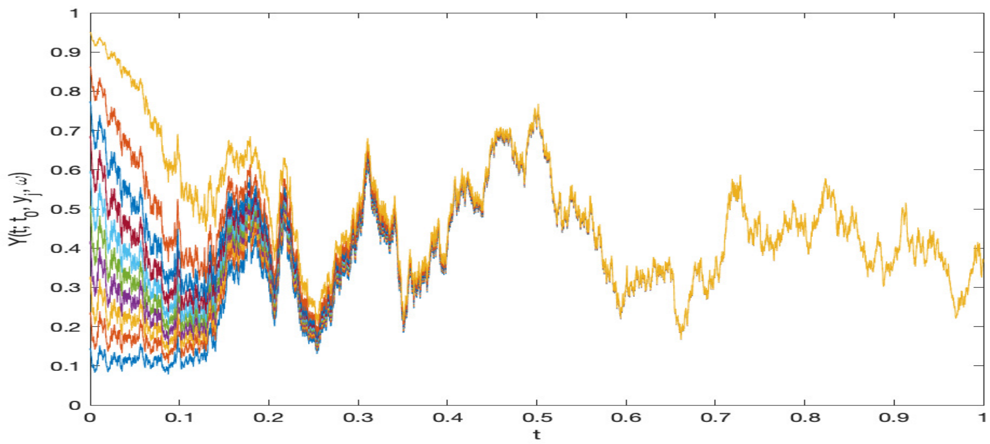


Figure 2.1: Solutions to (2.14) using 10 different initial values with the same sample $\omega \in \Omega$.

Furthermore, it is more convincing to describe the observation from Figure 2.1 with statistics. More precisely, in Figure 2.2 below, with the sample point $\omega \in \Omega$ fixed, the empirical average and variance of 100 solutions with 100 different initial values are plotted. Given initial values $y_j \in (0,1)$ for $j = 1, \cdots, 100$, the average of solution $\bar{Y}(t) = \frac{1}{100}\sum_{j=1}^{100} Y(t;t_0,y_j,\omega)$ is plotted on the left, and the variance $V(Y(t)) = \frac{1}{100}\sum_{j=1}^{100}\left(Y(t;t_0,y_j,\omega) - \bar{Y}(t)\right)^2$ is plotted on the right. As the variance approaches 0, all solutions converge pathwise.

Notice that the convergence in Figure 2.1 is in fact almost sure convergence with respect to the initial values, which is different as the convergence described in Theorem 3. In order to show the convergence in mean, we should let $\omega \in \Omega$ be free. So, we choose two arbitrary initial conditions $y_1,\ y_2 \in (0,1)$ and compute the mean of their difference $|Y(t;t_0,y_1,\omega_i) - Y(t;t_0,y_2,\omega_i)|$ with 1000 difference samples $\omega_i \in \Omega$, $i = 1, \cdots, 1000$. In Figure 2.3 below, the 1000 difference $|Y(t;t_0,y_1,\omega_i) - Y(t;t_0,y_2,\omega_i)|$ are presented on the left, and the average of the difference $\frac{1}{1000}\sum_{i=1}^{1000} |Y(t;t_0,y_1,\omega_i) - Y(t;t_0,y_2,\omega_i)|$ as an simulation of the mean difference

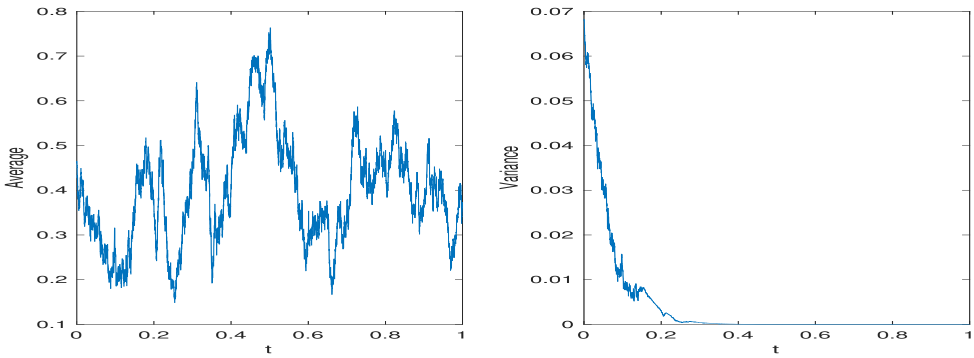


Figure 2.2: Average and variance of solutions to (2.14) using 100 different initial values with the same sample $\omega \in \Omega$.

$\mathbb{E}[|Y(t;t_0,y_1) - Y(t;t_0,y_2)|]$ is plotted on the right. Thus, the right part of the above figure

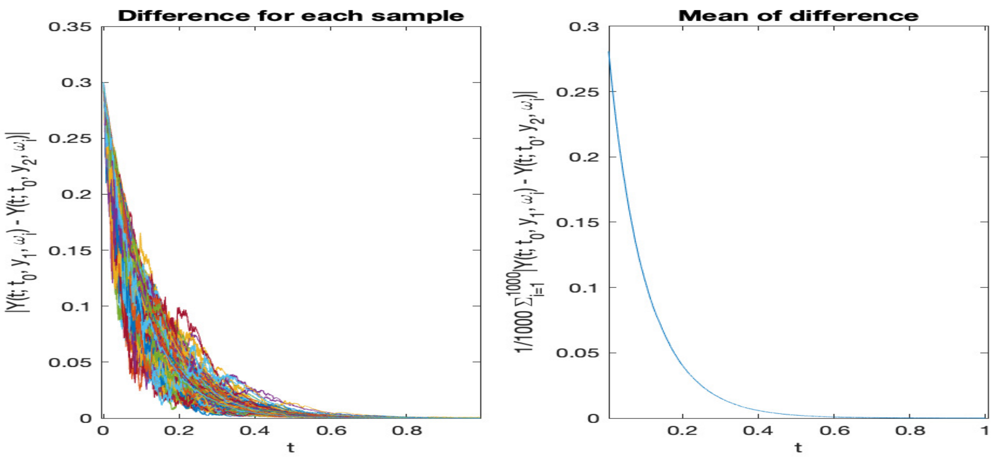


Figure 2.3: Convergence in mean of solutions of (2.14).

confirms the convergence in mean for two solutions to (2.14) with two initial values.

In addition, to investigate the influence of environmental fluctuations on coexistence of species, we compare the continuous deterministic lottery model with (2.14). To that end, we just need to let the variances of the environmental parameters be zero. Therefore, $v_i^h(t)$ and $\xi^h(t)$ are deterministic functions and

$$\xi^h = \mathbb{E}[\xi^h(t)] = h\mu(t), \quad \text{and} \quad v_i^h(t) = e^{\mathbb{E}[\ln v_i^h(t)]} = d_i^h(t), \quad \gamma_i(t) = \mathbb{E}[\gamma_i(t)] = 0, \quad i = 1,2.$$

Inserting the above equations into the difference equation (2.7) we obtain its deterministic counterpart

$$X(t+h)-X(t)=\frac{d_1^h(t)d_2^h(t)X(t)(1-X(t)\left(e^{h\mu(t)}-1\right)}{d_1^h(t)X(t)e^{h\mu(t)}+d_2^h(t)(1-X(t))}.$$

Then, we divide both sides of the above equation by h and let $h\to 0$. Applying

$$\lim_{h\to 0} d_i^h(t)=d_i(t),\ \text{for } i=1,\ 2\ \text{ and } \lim_{h\to 0}\frac{e^{h\mu(t)}-1}{h}=\mu(t)$$

results in the continuous deterministic lottery model

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t}=\frac{d_1(t)d_2(t)X(t)(1-X(t)\mu(t)}{d_1(t)X(t)+d_2(t)(1-X(t))}. \tag{2.62}$$

With the same set of functions $\mu(t)$, $d_1(t)$ and $d_2(t)$ as in (2.61), we simulate solutions of the ODE (2.62) using three different initial values. Then, we compare them with solutions to solutions of the SDE (2.14) with the same initial values. Figure 2.4 shows that even if $X(t)$ starts from different the initial values, eventually, it approaches 1 in the deterministic case. Recall that $X(t)$ represents that fraction of the sites that occupied by the first species. Thus, the second species eventually becomes extinct. This phenomena is also consistent with Chesson's coexistence theory that environmental fluctuations promote coexistence of species [28].

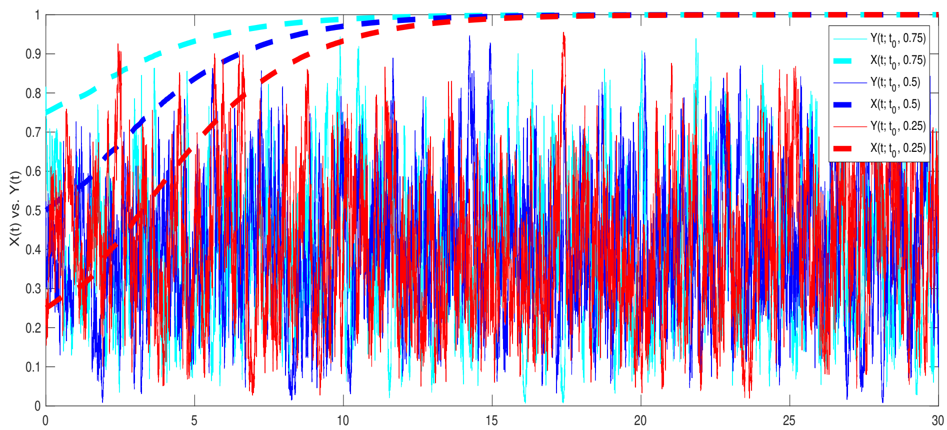
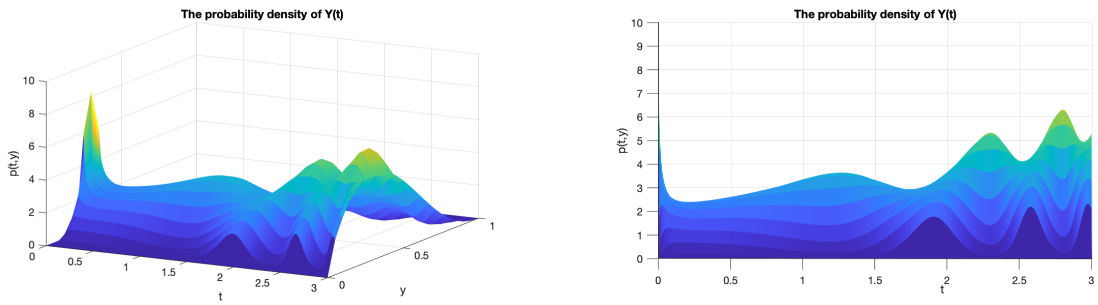


Figure 2.4: Solutions to the ODE (2.62) with initial values 0.25, 0.5, 0.75 (plotted by dashed lines) compared with solutions to the SDE (2.14) with the same initial values (plotted by solid lines).



(a) A 3-dimensional view (b) A 2-dimensional view on *p*-*t* plane.

Figure 2.5: Numerical solutions of PDE system (2.56)–(2.57) with initial condition $p_0(y)$.

Finally, we simulate the Fokker-Planck equation (2.56) with coefficients determined by functions in (2.61). We choose 0.3 is the initial value of (2.14). In order to approximate the initial condition $p_0(y) = \delta(y-0.3)$, we apply the piecewise continuous function that peaks at $y = 0.3$ and satisfies $\int_0^1 p_0(y)\mathrm{d}y = 1$:

$$p_0(y) = \begin{cases} 0, & 0 \le y \le 0.2, \\ 22.523e^{\frac{1}{100(y-0.3)^2-1}}, & 0.2 < y < 0.4, \\ 0, & 0.4 \le x \le 1. \end{cases}$$

The numerical solution of PDE system (2.56)–(2.57) with the initial condition $p_0(y) = \delta(y-0.3)$ is shown in Figure 2.5a. A two dimensional view from the time axis of the 3 dimensional graph in Figure 2.5a on the *p*-*t* plane is shown in Figure 2.5b to better illustrate the non-stationarity property of the asymptotic probability density function. Therefore, the existence of the stationary distribution is not possible with coefficients in (2.61).

Chapter 3

N-D Lottery Model

The N-D ($N > 2$) lottery model is formulated in the same way as 2-D case in the previous chapter. For $t \geq 0$, let $X_i(t)$ be the fraction of the sites occupied by adults of the ith species at time t. For any $h > 0$, let $\nu_i^h(t)$ be the proportion of adults of the ith species dying during the time period $(t, t+h]$. Denote by $\beta_i^h(t) \in [0,1]$ the per capita net reproduction by adults of the ith species during time $(t, t+h]$. Following the logic that the next generation is given by the sum of surviving adults and new recruitments [28, 26], the original discrete-time lottery model for N competing species is established as

$$X_i(t+h) = (1-\nu_i^h(t))X_i(t) + \frac{\beta_i^h(t)X_i(t)}{\sum_{i=1}^N \beta_i^h(t)X_i(t)} \sum_{i=1}^N \nu_i^h(t)X_i(t),\ i = 1,\cdots,N,\ h > 0. \tag{3.1}$$

Assume the initial time of the system (3.1) to be t_0, and also let the initial fractions of species satisfy

$$X_i(t_0) := x_{i,0} > 0 \text{ for } i = 1,\cdots,N \text{ and } \sum_{i=1}^N x_{i,0} = 1. \tag{3.2}$$

Clearly, under the above assumption (3.2), if any solution of the system (3.1) exists, it will satisfy

$$\sum_{i=1}^N X_i(t) = 1, \quad \text{for all } t \geq t_0. \tag{3.3}$$

3.1 Diffusion Approximation

Similarly to Chapter 2, for each $i = 1,\cdots,N$ and any different discrete time instant t and s, $\nu_i^h(t)$ and $\nu_i^h(s)$ are assumed to be independent. Similarly, $\beta_i^h(t)$ and $\beta_i^h(t)$ are also independent. Due to Remark 2, we can repeat the same process in Section 2.1 to obtain the existence of the diffusion approximation of the N-D lottery model. Note that the diffusion approximation of a stochastic process $\mathbf{X}(t) = (X_1(t), \ldots, X_N(t))$ is characterized by its drift and diffusion coefficients, defined respectively as

$$f_i(t,\mathbf{x}) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[X_i(t+h) - X_i(t)|\mathbf{X}(t) = \mathbf{x}], \tag{3.4}$$

$$\mathfrak{a}_{ij}(t,\mathbf{x}) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[(X_i(t+h) - X_i(t))(X_j(t+h) - X_j(t))|\mathbf{X}(t) = \mathbf{x}], \tag{3.5}$$

for $i,\ j = 1,\cdots,N$.

Next, we calculate the (3.4) and (3.5) whereby (3.2). To that end, taking into account (3.3), it is more convenient to rewrite the system (2.1) in an equivalent differential-algebraic form

$$\begin{aligned} X_i(t+h) - X_i(t) &= X_i(t)\nu_i^h(t)\frac{\sum_{n=1}^{N-1} A_{in}(t)X_n + B_{in}(t)(1-\sum_{n=1}^{N-1}X_n)}{\sum_{n=1}^{N-1}\left(\frac{\beta_n^h(t)\nu_N^h(t)}{\beta_N^h(t)\nu_n^h(t)}\cdot\frac{\nu_n^h(t)}{\nu_N^h(t)}\right)X_n + 1 - \sum_{n=1}^{N-1}X_n}, \\ &\quad \text{for} \quad i = 1,\cdots,N-1, \\ X_N(t) &= 1 - \sum_{n=1}^{N-1}X_n(t), \end{aligned} \tag{3.6}$$

where

$$A_{in}(t) = \frac{\nu_n^h(t)}{\nu_N^h(t)}\left(\frac{\beta_i^h(t)\nu_N^h(t)}{\nu_i^h(t)\beta_N^h(t)} - \frac{\beta_n^h(t)\nu_N^h(t)}{\nu_n^h(t)\beta_N^h(t)}\right), \quad B_{in}(t) = \frac{\beta_i^h(t)\nu_N^h(t)}{\nu_i^h(t)\beta_N^h(t)} - 1.$$

Notice that in reality, the environmental fluctuations could probability be non-stationary [52, 44, 59]. Thus, it is natural to assume the stochastic processes $\beta_i^h(t)$ and $\nu_i^h(t)$ to be non-stationary as Chapter 2. However, the simplified stationary assumption of $\beta_i^h(t)$ and $\nu_i^h(t)$ still provide a great deal of useful information, because we can investigate more details of dynamics for a higher dimensional system. Throughout this section, it is assumed that the environment has stationary temporal fluctuations, for which the stochastic processes $\beta_i^h(t)$ and $\nu_i^h(t)$ are stationary.

Here, we follow the ideas of Refs. [52, 34] but apply the different set-up. The stochastic processes $\rho_i^h(t)$ and $\gamma_i^h(t)$ are defined as

$$\rho_i^h(t) = \ln \frac{\beta_i^h(t)\nu_N^h(t)}{\nu_i^h(t)\beta_N^h(t)}, \quad \nu_i^h(t) = d_i e^{\gamma_i^h(t)}, \quad i = 1, \cdots, N-1$$

where d_i is the geometric mean of $\nu_i^h(t)$. The equations in (3.6) then becomes

$$X_i(t+h) - X_i(t) = \frac{R_i^h(t)}{S^h(t)}, \quad \text{for} \quad i = 1, \cdots, N-1 \tag{3.7}$$

where

$$R_i^h(t) = X_i d_i e^{\gamma_i} \left(\sum_{n=1}^{N-1} d_n e^{\gamma_n - \gamma_N} \left(e^{\rho_i} - e^{\rho_n}\right) X_n + d_N X_N \left(e^{\rho_i} - 1\right) \right), \tag{3.8}$$

$$S^h(t) = \sum_{n=1}^{N-1} d_n e^{\rho_n} e^{\gamma_n - \gamma_N} X_n + d_N X_N, \tag{3.9}$$

and $X_N = 1 - \sum_{n=1}^{N-1} X_n$.

Since $\gamma_i^h(t) = \ln \nu_i^h(t) - \ln d_i$ and d_i is the geometric mean of $\nu_i^h(t)$, we have $\mathbb{E}[\gamma_i^h(t)] = 0$. Throughout this section it is assumed that

(**A8**) the stochastic processes $\rho_i^h(t)$ and $\nu_i^h(t)$ are stationary with

$$\mathbb{E}[\rho_i^h(t)] = h\mu_i, \qquad \text{Var}[\rho_i^h(t)] = h\sigma_i^2, \qquad \text{Cov}[\rho_i^h(t), \rho_j^h(t)] = h\sigma_{ij},$$

$$\text{Var}[\gamma_i^h(t)] = h\alpha_i^2, \qquad \text{Cov}[\gamma_i^h(t), \gamma_j^h(t)] = 0, \quad \text{Cov}[\gamma_i^h(t), \rho_j^h(t)] = h\theta_{ij}.$$

Applying the approximation (3.10) below and (**A8**) to each of the ratios $\frac{R_i^h(t)}{S^h(t)}$, $\frac{(R_i^h(t))^2}{(S^h(t))^2}$, and $\frac{R_i^h(t)R_j^h(t)}{(S^h(t))^2}$, respectively,

$$\mathbb{E}\left[\frac{R(t)}{S(t)}\right] \approx \frac{\mathbb{E}[R(t)]}{\mathbb{E}[S(t)]} - \frac{\text{Cov}[R(t), S(t)]}{\mathbb{E}[S^2(t)]} + \frac{\text{Var}[S(t)]\mathbb{E}[R(t)]}{\mathbb{E}[S^3(t)]}, \tag{3.10}$$

we obtain that (see Appendix A for detailed calculations)

$$
\begin{aligned}
f_i(\mathbf{x}) &\approx x_i d_i\Big(\mu_i + \frac{1}{2}\sigma_i^2 - \frac{\sum_{n=1}^{N-1} d_n x_n(\mu_n + \frac{1}{2}\sigma_n^2)}{\sum_{n=1}^{N} d_n x_n} \\
&\qquad + \frac{\sum_{n,m=1}^{N-1} d_n d_m x_n x_m C_{inm} + \sum_{n=1}^{N-1} d_n x_n D_{in} + d_N^2 x_N^2 \theta_{ii}}{(\sum_{n=1}^{N} d_n x_n)^2}\Big), && (3.11)\\
\mathfrak{a}_{ij}(\mathbf{x}) &\approx d_i d_j x_i x_j\Big(\sigma_{ij} - \frac{\sum_{n=1}^{N-1} d_n x_n(\sigma_{in} + \sigma_{jn})}{\sum_{n=1}^{N} d_n x_n} \\
&\qquad + \frac{\sum_{n,m=1}^{N-1} d_n d_m x_n x_m \sigma_{mn}}{(\sum_{n=1}^{N} d_n x_n)^2}\Big), && (3.12)
\end{aligned}
$$

where $x_N = 1 - \sum_{n=1}^{N-1} x_n$ and

$$
\begin{aligned}
C_{inm} &= \theta_{ii} - \theta_{in} + \theta_{ni} - \theta_{nn} - \theta_{mi} + \theta_{mn} - \sigma_{mi} + \sigma_{mn}, && (3.13)\\
D_{in} &= 2\theta_{ii} - \theta_{in} - \theta_{nn} + \theta_{Nn} - \sigma_{ni}. && (3.14)
\end{aligned}
$$

Especially, for $i = j$, the relation (3.12) still holds with the convention $\sigma_{ii} = \sigma_i^2$.

Because of Theorem 11.2.3 in [47], the solutions to the discrete lottery model (3.6), $\mathbf{X}^h(t) = (X_i^h(t))_{i=1}^{N-1}$, converge weakly to the diffusion process $\mathbf{Y}(t) = (Y_i(t))_{i=1}^{N-1}$ with the drift and diffusion coefficients given by (3.4) – (3.5), as $h \to 0$. Then, $Y_i(t)$ represents the fraction of the sites occupied by adults of the ith species at time t. And $\mathbf{Y}(t) = (Y_1, \ldots, Y_{N-1})$ satisfies the following system of nonlinear autonomous stochastic differential equations

$$
\begin{aligned}
\mathrm{d}Y_i(t) &= f_i(\mathbf{Y}(t))\mathrm{d}t + \sum_{j=1}^{N-1} g_{ij}(\mathbf{Y}(t))\mathrm{d}W_j(t),\ t \geq 0,\ i = 1, \cdots, N-1, && (3.15)\\
\mathbf{Y}(0) &= \mathbf{y}_0 \in (0,1)^{N-1},
\end{aligned}
$$

where f_i is defined as in (2.12), $\mathfrak{G} = (g_{ij})_{(N-1)\times(N-1)}$ is a "square root" of the matrix $\mathfrak{A} = (\mathfrak{a}_{ij})_{(N-1)\times(N-1)}$ with components $\mathfrak{a}_{ij}$ defined by (3.12), and $\mathbf{W}(t) = (W_1, \ldots, W_{N-1})$ is an $(N-1)$-dimensional standard Brownian motion.

Remark 7. *The "square root" $\mathfrak{G}$ of $\mathfrak{A}$ is view in the sense that $\mathfrak{G}\mathfrak{G}^T = \mathfrak{A}$. The existence of such a square root is ensured by Proposition 6.2 in Chapter 4 of Ref. [60]. But, such a decomposition of $\mathfrak{A}$ may not be unique. Nonetheless, thanks to Theorem 6.1 in Chapter 4 of*

Ref. [60], any two solutions of the system (3.15) *with two different square roots of* $\mathfrak{A}$ *have the same probability distribution.*

According to Proposition 6.2 in [60], the positive semi-definite property of the matrix $\mathfrak{A}$ is a necessary condition for the existence of the square root of $\mathfrak{A}$. To ensure $\mathfrak{A}$ is positive semi-definite, as well as to facilitate analysis in the sequel, throughout this section it is assumed that

(**A9**). The moments in Assumption (**A8**) satisfy

$$\begin{aligned}\sigma_{ii}^2 &= \sigma^2, \quad \sigma_{ij} = \lambda\sigma^2 \text{ for } i \neq j \text{ with } 0 < \lambda < 1, \\ \theta_{ii} &= \theta, \quad \theta_{ij} = \varepsilon\theta \quad \text{for } i \neq j \text{ with } 0 < \varepsilon < 1.\end{aligned}$$

With the Assumption (**A9**), the coefficients C_{inm} and D_{in} in (3.13)–(3.14) can be simplified to

$$C_{inm} = \begin{cases} -(1-\varepsilon)\theta - (1-\lambda)\sigma^2, & n \neq i, m = i, \\ (1-\varepsilon)\theta + (1-\lambda)\sigma^2, & n \neq i, m \neq i, m = n \\ 0, & \text{otherwise} \end{cases}, \quad D_{in} = \begin{cases} \varepsilon\theta - \sigma^2, & n = i \\ \theta - \lambda\sigma^2, & n \neq i \end{cases}.$$

For simplicity of exposition for $\mathbf{Y} = (Y_i)_{i=1}^{N-1}$, let

$$S_{N-1}(\mathbf{Y}) := \sum_{n=1}^{N-1} d_n Y_n, \quad S_N(\mathbf{Y}) := S_{N-1}(\mathbf{Y}) + d_N \left(1 - \sum_{n=1}^{N-1} Y_n\right).$$

It follows that drift and diffusion coefficients in (3.11) – (3.12) become, respectively,

$$\begin{aligned} f_i(\mathbf{Y}) &= d_iY_i\Bigg(\mu_i+\frac{1}{2}\sigma^2\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big)-\frac{\sum_{n=1}^{N-1}d_n\mu_nY_n}{S_N(\mathbf{Y})} \\ &\quad+\frac{\sum_{n=1}^{N-1}d_nY_n\Big(\zeta(d_nY_n-d_iY_i)+\theta-\lambda\sigma^2\Big)-\zeta d_iY_i+d_N^2Y_N^2\theta}{S_N^2(\mathbf{Y})}\Bigg), \qquad (3.16) \\ &\quad \text{with}\quad \zeta=(1-\varepsilon)\theta+(1-\lambda)\sigma^2, \\ \mathfrak{a}_{ii}(\mathbf{Y}) &= d_i^2Y_i^2\sigma^2\left(\lambda\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big)^2\right. \\ &\quad\left.+(1-\lambda)\Big(1-\frac{2Y_id_i}{S_N(\mathbf{Y})}+\frac{\sum_{n=1}^{N-1}d_n^2Y_n^2}{S_N^2(\mathbf{Y})}\Big)\right), \qquad (3.17) \\ \mathfrak{a}_{ij}(\mathbf{Y}) &= d_id_jY_iY_j\sigma^2\left(\lambda\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big)^2\right. \\ &\quad\left.+(1-\lambda)\Big(-\frac{Y_id_i+Y_jd_j}{S_N(\mathbf{Y})}+\frac{\sum_{n=1}^{N-1}d_n^2Y_n^2}{S_N^2(\mathbf{Y})}\Big)\right),\quad i\neq j. \qquad (3.18) \end{aligned}$$

Lemma 2. *For each fixed vector $\boldsymbol{Y}\in[0,1]^{N-1}$, the matrix $\mathfrak{A}(\boldsymbol{Y})=(\mathfrak{a}_{ij}(\boldsymbol{Y}))_{(N-1)\times(N-1)}$ is non-negative definite.*

Proof. First, we denote $u_i=\frac{d_iY_i}{S_N(\mathbf{Y})}$, $z_i=d_iY_i$ and set $\delta_{ij}=\begin{cases}1, \text{ if } i=j\\ 0, \text{ if } i\neq j\end{cases}$.

Then, the matrix $\mathfrak{A}$ can be separated as

$$\mathfrak{A}=\lambda\sigma^2P+(1-\lambda)\sigma^2Q,$$

where each element of P and Q are defined, respectively, as

$$p_{ij}=z_iz_j\left(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\right)^2,\quad q_{ij}=z_iz_j\left(\delta_{ij}-u_i-u_j+\sum_{n=1}^{N-1}u_n^2\right).$$

Clearly, P is already non-negative definite. It is sufficient to prove Q is non-negative definite. In fact, for an arbitrary $\xi = (\xi_i)_{i=1,\cdots,N-1} \in [0,1]^{N-1}$, we have

$$\begin{aligned}\sum_{i,j=1}^{N-1} q_{ij}\xi_i\xi_j &= \sum_{i=1}^{N-1} \xi_i^2 z_i^2 - 2\sum_{j=1}^{N-1} \xi_j z_j \sum_{i=1}^{N-1} u_i\xi_i z_i + \sum_{n=1}^{N-1} u_n^2 \left(\sum_{i=1}^{N-1} \xi_i z_i\right)^2 \\ &= \sum_{i=1}^{N-1} \left(\xi_i z_i - (\sum_{j=1}^{N-1} \xi_j z_j) u_i\right)^2 \geq 0,\end{aligned}$$

Therefore, the matrix Q is non-negative definite. The proof is complete. □

Remark 8. *The non-negativeness of* $\mathfrak{A}$ *can also be shown from* (3.5).

Lemma 2 shows that $\mathfrak{G} = (g_{ij})_{(N-1)\times(N-1)}$ in (3.15) exists. The rest of this section is devoted to study properties of solutions to the SDE system (3.15). The global existence and uniqueness of a positive bounded solution will be proved in Theorem 7 below. Then the asymptotic behaviors of the solution will be studied in Section 3.2. According to Itô's formula, for a non-negative C^2 mapping $\mathscr{V} : \mathbb{R}^{N-1} \to \mathbb{R}$, the process $\mathscr{V}(\mathbf{Y}(t))$ satisfies

$$\mathrm{d}\mathscr{V}(\mathbf{Y}(t)) = \sum_{i=1}^{N-1} \frac{\partial \mathscr{V}}{\partial Y_i}\mathrm{d}Y_i + \frac{1}{2}\sum_{i,j=1}^{N-1} \frac{\partial^2 \mathscr{V}}{\partial Y_i \partial Y_j}\mathrm{d}Y_i\mathrm{d}Y_j.$$

Inserting (3.16) – (3.18) into the above equation gives

$$\mathrm{d}\mathscr{V}(\mathbf{Y}(t)) = \mathscr{L}\mathscr{V}(\mathbf{Y}(t))\mathrm{d}t + \sum_{i=1}^{N-1} \left(\frac{\partial \mathscr{V}(\mathbf{Y}(t))}{\partial Y_i} \sum_{j=1}^{N-1} g_{ij}(\mathbf{Y}(t))\mathrm{d}W_j\right), \tag{3.19}$$

where

$$\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) = \sum_{i=1}^{N-1} f_i(\mathbf{Y}(t))\frac{\partial \mathscr{V}}{\partial Y_i}(\mathbf{Y}(t)) + \frac{1}{2}\sum_{i,j}^{N-1} \mathfrak{a}_{ij}(\mathbf{Y}(t))\frac{\partial^2 \mathscr{V}}{\partial Y_i \partial Y_j}(\mathbf{Y}(t)). \tag{3.20}$$

Theorem 7. *For any* $y_0 \in (0,1)^{N-1}$*, the system* (3.15) *has a unique solution* $\mathbf{Y}(t) = \mathbf{Y}(t;\, \mathbf{y}_0,\ \omega)$. *Moreover, the solution* $\mathbf{Y}(t) \in (0,\ 1)^{N-1}$ *almost surely for all* $t \geq 0$.

Proof. Lemma 2 along with Proposition 6.2 in Chapter 4 of [60] ensure that the system (3.15) has a unique local solution $\mathbf{Y}(t)$ on $[0, \tau_e)$ with $\mathbf{Y}(t) \in (0,1)^{N-1}$ a.s., where τ_e is the explosion time. Then the global existence of the solution will be shown by proving $\tau_e = \infty$ a.s. Let $k_0 > 0$ be a positive integer such that $\mathbf{y}_0 \in (1/k_0,\ k_0)^{N-1}$. For any $k \geq k_0$, define the sequence of "stopping times", $\{\tau_k\}$, by

$$\tau_k = \inf\left\{ t \in [0,\tau_e) : \mathbf{Y}(t) \notin \left(\frac{1}{k}, 1-\frac{1}{k}\right)^{N-1} \right\}.$$

Clearly $\{\tau_k\}$ is an increasing sequence. Let $\tau_\infty = \lim_{k\to\infty}\tau_k$, then $\tau_\infty \le \tau_e$. We prove $\tau_\infty = \infty$ a.s. by contradiction. To that end, define

$$\mathscr{V}(\mathbf{x}) = \frac{1}{\prod_{n=1}^{N-1} x_n}, \text{ for any } \mathbf{x} = (x_1,\ldots,\ x_{N-1}) \in (0,\ 1)^{N-1}.$$

It follows immediately from (3.16) that for any $\mathbf{Y}(t) \in (0,1)^{N-1}$ it holds

$$\begin{aligned}
&\sum_{i=1}^{N-1} f_i(t,\mathbf{Y}(t))\frac{\partial \mathscr{V}}{\partial Y_i}(\mathbf{Y}(t)) \\
= \;& -\sum_{i=1}^{N-1}\frac{d_i}{\prod_{n=1}^{N-1}Y_n}\cdot\Bigg(\mu_i + \frac{1}{2}\sigma^2\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big) - \frac{\sum_{n=1}^{N-1} d_n\mu_n Y_n}{S_N(\mathbf{Y})} \\
&\quad + \frac{\sum_{n=1}^{N-1} d_n Y_n\Big(\zeta(d_nY_n - d_iY_i) + \theta - \lambda\sigma^2\Big) - \zeta d_i Y_i + d_N^2 Y_N^2\theta}{S_N^2(\mathbf{Y})}\Bigg) \\
\le \;& K_1\mathscr{V}(\mathbf{Y}(t)),
\end{aligned} \tag{3.21}$$

where K_1 is a constant depending on $\{\mu_i\}_{i=1,\cdots,N-1}$, $\{d_i\}_{i=1,\cdots,N}$, θ, σ^2, ε, and λ. Next, by using (3.17)–(3.18), we have that for all $\mathbf{Y}(t) \in (0,1)^{N-1}$

$$\begin{aligned}
&\sum_{i,j=1}^{N-1} \mathfrak{a}_{ij}(t,\mathbf{Y}(t))\frac{\partial^2\mathscr{V}}{\partial Y_i\partial Y_j}(\mathbf{Y}(t)) \\
= \;& \sum_{\substack{i=1\\ i\neq j}}^{N-1}\frac{d_id_j\sigma^2}{\prod_{n=1}^{N-1}Y_n}\left(\lambda\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big)^2 + (1-\lambda)\Big(-\frac{Y_id_i+Y_jd_j}{S_N(\mathbf{Y})} + \frac{\sum_{n=1}^{N-1}d_n^2Y_n^2}{S_N^2(\mathbf{Y})}\Big)\right) \\
&+\sum_{i=1}^{N-1}\frac{\sigma^2 d_i^2}{2\prod_{n=1}^{N-1}Y_n}\left(\lambda\Big(1-\frac{S_{N-1}(\mathbf{Y})}{S_N(\mathbf{Y})}\Big)^2 + (1-\lambda)\Big(1-\frac{2Y_id_i}{S_N(\mathbf{Y})} + \frac{\sum_{n=1}^{N-1}d_n^2Y_n^2}{S_N^2(\mathbf{Y})}\Big)\right) \\
\le \;& K_2\mathscr{V}(\mathbf{Y}(t)),
\end{aligned} \tag{3.22}$$

where K_2 is a constant depending on $\{d_i\}_{i=1,\cdots,N-1}$, σ^2, and λ.

Inserting (3.21) and (3.22) into (3.20) results in

$$\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) \leq K\mathscr{V}(\mathbf{Y}(t)), \quad K = \max\{K_1, \frac{K_2}{2}\}. \tag{3.23}$$

With a focus on showing $\tau_\infty = \infty$ a.s., assume that there exist $T > 0$ and $\varepsilon > 0$ such that $\mathbb{P}(\tau_\infty \leq T) > \varepsilon$. Since $\{\tau_k\}$ is an increasing sequence, there exists $N > k_0$ such that

$$\mathbb{P}(\tau_k \leq T) > \frac{\varepsilon}{2} \text{ for any } k > N. \tag{3.24}$$

Combining (3.23) and (3.19) shows

$$\begin{aligned} \mathscr{V}(\mathbf{Y}(\tau_k \wedge T)) \;\; \leq \;\; & \mathscr{V}(\mathbf{y}_0) + K\int_0^{\tau_k\wedge T} \mathscr{V}(\mathbf{Y}(t))\mathrm{d}t \\ & + \sum_{i=1}^{N-1} \int_0^{\tau_k\wedge T} \frac{\partial \mathscr{V}}{\partial Y_i}(\mathbf{Y}(t)) \sum_{j=1}^{N-1} g_{ij}(t, \mathbf{Y}(t))\mathrm{d}W_j(t). \end{aligned}$$

Taking the expectation of the inequality above gives

$$\begin{aligned} \mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))] \;\; \leq \;\; & \mathscr{V}(\mathbf{y}_0) + K\int_0^T \int_\Omega \chi_{[0,\tau_k(\omega)\wedge T]}(t)\mathscr{V}(\mathbf{Y}(t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ = \;\; & \mathscr{V}(\mathbf{y}_0) + K\int_0^T \int_{\Omega_1\cup\Omega_2} \chi_{[0,\tau_k(\omega)\wedge T]}(t)\mathscr{V}(\mathbf{Y}(t))\mathrm{d}\mathbb{P}\mathrm{d}t, \end{aligned} \tag{3.25}$$

where χ is the indicator function with $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ for $x \notin I$, and

$$\Omega_1 = \{\omega \in \Omega : \tau_k(\omega) \leq T\}, \quad \Omega_2 = \{\omega \in \Omega : \tau_k(\omega) > T\}.$$

So, (3.25) can be written as

$$\begin{aligned} \mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))] \;\; \leq \;\; & \mathscr{V}(\mathbf{y}_0) + K\int_0^T \int_{\Omega_1} \chi_{[0,\tau_k(\omega)]}(t)\mathscr{V}(\mathbf{Y}(t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ & + K\int_0^T \int_{\Omega_2} \chi_{[0,\, T]}(t)\mathscr{V}(\mathbf{Y}(t))\mathrm{d}\mathbb{P}\mathrm{d}t \end{aligned}$$

It then follows directly that

$$\begin{aligned}\mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))] &\leq \mathscr{V}(\mathbf{y}_0) + K\int_0^T \int_{\tau_k < T} \mathscr{V}(\mathbf{Y}(\tau_k \wedge t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &\quad + K\int_{t_0}^T \int_{\tau_k > T} \mathscr{V}(\mathbf{Y}(\tau_k \wedge t))\mathrm{d}\mathbb{P}\mathrm{d}t \\ &= \mathscr{V}(\mathbf{y}_0) + K\int_0^T \mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))]\mathrm{d}t.\end{aligned}$$

Applying Gronwall's Lemma to the above inequality yields

$$\mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))] \leq \mathscr{V}(\mathbf{y}_0)\, e^{KT}. \tag{3.26}$$

In addition, (3.24) along with fact that $\sum_{i=1}^N Y_i(t) = 1$ imply

$$\mathbb{E}[\mathscr{V}(\mathbf{Y}(\tau_k \wedge T))] \geq \int_{\tau_k \leq T} \mathscr{V}(\mathbf{Y}(\tau_k \wedge T))\mathrm{d}\mathbb{P} = \int_{\tau_k \leq T} \mathscr{V}(\mathbf{Y}(\tau_k(\omega)))\mathrm{d}\mathbb{P} \geq \frac{k \cdot \varepsilon}{2}. \tag{3.27}$$

Combining (3.26) and (3.27) gives

$$\mathscr{V}(\mathbf{y}_0) \cdot e^{KT} \geq \frac{k \cdot \varepsilon}{2}, \text{ for any } k \geq N,$$

which provides the contradiction as $k \to \infty$. Thus, $\tau_\infty = \infty$ a.s. The proof is complete. □

3.2 Dynamical behavior

This section is devoted to study the asymptotic behaviors of the system (3.15). Lemma 2 shows that $\mathfrak{A}$ is non-negative, which could only ensure the existence of the square root of $\mathfrak{A}$. However, we cannot obtain the explicit formulation of the square root, $\mathfrak{G} = (g_{ij})_{(N-1)\times(N-1)}$. Consequently, we will not apply the same method used in the 2-D lottery model. In turn, methods of Lyapunov functions will be applied to investigate the long time dynamical behaviors of (3.15). More specifically, we will provide sufficient conditions for the extinction of at least one species, and conditions for coexistence among at least two species.

3.2.1 Extinction

In order to prove the extinction of the ith species, the following assumptions are needed

$$(\mathbf{A10}) \quad \mu_i + \frac{\sigma^2}{2} \le 0 \text{ and } \mu_i = \min_{n=1,\cdots,N-1}\{\mu_n\}.$$

$$(\mathbf{A11}) \quad \frac{\lambda-1}{1-\varepsilon}\sigma^2 \le \; \theta \le \min\left\{\frac{d_i\lambda}{2}, \frac{1-(1-\lambda)d_i}{\varepsilon}, \min_{n=1,\cdots,N-1}\left\{\frac{\lambda-(1-\lambda)d_n}{1+(1-\varepsilon)d_n}\right\}\right\}\sigma^2.$$

Theorem 8. *(Extinction) Let Assumptions (**A8**) – (**A11**) hold. Then for any initial value* $\mathbf{y}_0 \in (0,1)^{N-1}$*, the ith component of the solution* $\mathbf{Y}(t)$ *satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\ln Y_i(t) < 0 \quad a.\ s.,$$

i.e., the ith species dies out exponentially with probability one.

Proof. Let $\mathscr{V}(\mathbf{Y}(t)) = \ln Y_i(t)$. We recall (3.19) to obtain

$$\ln Y_i(t) = \ln y_{0,i} + \int_0^t \mathscr{L}\mathscr{V}(\mathbf{Y}(s))\mathrm{d}s + \int_0^t \frac{1}{Y_i}\sum_{j=1}^{N-1} g_{ij}(\mathbf{Y}(s))\mathrm{d}W_j. \tag{3.28}$$

First, it follow from (3.20) that (details presented in B)

$$\begin{aligned}\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) &= \frac{1}{Y_i}f_i - \frac{1}{2Y_i^2}\mathfrak{a}_{ii} \\ &\le d_i\left(\mathscr{J}_1+\mathscr{J}_2+\mathscr{J}_3+\mathscr{J}_4+\mathscr{J}_5+\mathscr{J}_6-\frac{d_i(1-\lambda)\sigma^2}{2}\right),\end{aligned} \tag{3.29}$$

where for $i=1,\cdots,N-1$,

$$\begin{aligned}
\mathscr{J}_1 &= \mu_i + \frac{\sigma^2}{2} - \min_{\substack{n=1,\cdots,N-1\\ \mu_n<0}} \Big(\mu_n + \frac{\sigma^2}{2}\Big) \frac{\sum_{\substack{n=1\\ \mu_n<0}}^{N-1} d_n Y_n}{\sum_{n=1}^{N} d_n Y_n},\\
\mathscr{J}_2 &= -\frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n d_i Y_i\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2},\\
\mathscr{J}_3 &= \frac{d_i Y_i(\varepsilon\theta - \sigma^2 + d_i(1-\lambda)\sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2},\\
\mathscr{J}_4 &= \frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n\big(\theta - \lambda\sigma^2 + d_n(1-\lambda)\sigma^2 + d_n(1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2},\\
\mathscr{J}_5 &= -\frac{\sum_{n=1}^{N-1} d_n^2 Y_n^2 d_i(1-\lambda)\sigma^2}{2(\sum_{n=1}^{N} d_n Y_n)^2},\\
\mathscr{J}_6 &= \frac{d_N^2 Y_N^2(2\theta - d_i\lambda\sigma^2)}{2(\sum_{n=1}^{N} d_n Y_n)^2}.
\end{aligned}$$

Clearly, $\mathscr{J}_5 \le 0$ for all $\mathbf{Y} \in (0,1)^{N-1}$. Then, Assumption (**A10**) guarantees that

$$\mathscr{J}_1 \le \Big(\mu_i + \frac{\sigma^2}{2}\Big)\left(1 - \frac{\sum_{\substack{n=1\\ \mu_n<0}}^{N-1} d_n Y_n}{\sum_{n=1}^{N} d_n Y_n}\right) \le 0.$$

The condition $\theta \ge \frac{\lambda-1}{1-\varepsilon}\sigma^2$ in Assumption (**A11**) provides that $(1-\lambda)\sigma^2 + (1-\varepsilon)\theta \ge 0$, and hence

$$\mathscr{J}_2 \le 0.$$

The conditions $\theta \le \frac{1-(1-\lambda)d_i}{\varepsilon}\sigma^2$, $\theta \le \min_{n=1,\cdots,N-1}\left\{\frac{\lambda-(1-\lambda)d_n}{1+(1-\varepsilon)d_n}\right\}\sigma^2$, and $\theta \le \frac{d_i\lambda}{2}\sigma^2$ in Assumption (**A4**) confirm, respectively, that

$$\mathscr{J}_3 \le 0, \quad \mathscr{J}_4 \le 0, \quad \mathscr{J}_6 \le 0.$$

Collecting the above gives

$$\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) \le -\frac{d_i^2(1-\lambda)\sigma^2}{2}, \quad t \ge 0. \tag{3.30}$$

Applying the Cauchy inequality shows that the quadratic variation of the stochastic integral in (3.28) satisfies

$$\begin{aligned} \frac{1}{t}\int_0^t \frac{1}{Y_i^2}\Big(\sum_{j=1}^{N-1} g_{ij}\Big)^2 \mathrm{d}s &\leq \frac{N-1}{t}\int_0^t \frac{1}{Y_i^2}\sum_{j=1}^{N-1}(g_{ij})^2 \mathrm{d}s \\ &= \frac{N-1}{t}\int_0^t \frac{\mathfrak{a}_{ii}(\mathbf{Y}(s))}{Y_i^2}\mathrm{d}s < \infty. \end{aligned}$$

According to the law of large numbers for local martingales [61] (Theorem 3.4 on Page 12), we have

$$\frac{1}{t}\int_0^t \frac{1}{Y_i}\sum_{j=1}^{N-1} g_{ij}\mathrm{d}W_i(s) \to 0 \text{ a.s. as } t \to \infty. \tag{3.31}$$

Inserting (3.30) and (3.31) into (3.28) yields

$$\limsup_{t\to\infty} \frac{1}{t}\ln Y_i(t) \leq -\frac{d_i^2(1-\lambda)\sigma^2}{2} < 0,$$

which means Y_i tends to zero exponentially. The proof is complete. □

3.2.2 Persistence

In this subsection we provide the sufficient conditions for the coexistence of N ($N \geq 3$) by proving that the solution to (3.15) is positive recurrent by Lyapunov functions [62]. Moreover, the positive recurrence of the solution with respect to an appropriate set implies that (3.15) has a unique stationary distribution [63]. Recall that a stochastic process $\mathbf{Y}(t)$ is said to be recurrent with respect to a nonempty bounded open set $E \subset \mathbb{R}^{N-1}$ if $\mathbb{P}\{\tau_E < \infty\} = 1$, where $\tau_E := \inf\{t > 0 : \mathbf{Y}(t) \in E\}$. A stochastic process $\mathbf{Y}(t)$ is said to be positive recurrent with respect to E if $\mathbb{E}\{\tau_E\} < \infty$.

The persistence will be proved in two steps. First, it will be shown that the solution to (3.15) is positive recurrent about the nonempty bounded set E_1 defined by

$$E_1 := \left\{ (y_n)_{n=1}^{N-1} \in (0,1)^{N-1} : 1 - \sum_{n=1}^{N-1} Y_n > \ell_y \right\}, \tag{3.32}$$

where $d_{min} = \min_{n=1,\cdots,N-1}\{d_n\}$, and

$$\ell_y = \frac{d_{min}^2(1-\lambda)\sigma^2}{d_{min}^2(1-\lambda)\sigma^2 + d_N\big((6-d_{min}(1-\lambda))\sigma^2 + 12|\theta|\big)}. \tag{3.33}$$

Second, we will show that the solution to (3.15) is also positive recurrent with respect to the nonempty bounded set E_2 defined by

$$E_2 := \left\{ (y_n)_{n=1}^{N-1} \in (0,1)^{N-1} : 1 - \sum_{n=1}^{N-1} Y_n < \chi_y \right\}, \tag{3.34}$$

where

$$\chi_y = \frac{4(1-\lambda)(d_{max}+2)+6}{4(1-\lambda)(d_{max}+2)+7-d_{max}}. \tag{3.35}$$

Notice that the fact $d_n \leq 1$ for all $n \in 1,\cdots,N$, implies $\ell_y < 1$ and $\chi_y < 1$.

Lemma 3. *Let Assumptions (**A8**) – (**A9**) hold. In addition, assume that*

$$\begin{aligned}
&(\boldsymbol{A12}) \quad \sigma^2 \geq \frac{24|\mu|_{max}}{d_{min}^2(1-\lambda)}, \quad \textit{with} \quad |\mu|_{max} = \max_{n=1,\cdots,N-1}\{|\mu_n|\} \\
&(\boldsymbol{A13}) \quad \frac{\lambda-1}{1-\varepsilon}\sigma^2 \leq \theta \leq \min_{1\leq n\leq N-1}\left\{\frac{1-(2-\lambda)d_n}{2\varepsilon}, \frac{\lambda-d_n(1-\lambda)}{1+d_n(1-\varepsilon)}\right\}\sigma^2.
\end{aligned}$$

Then the solution to the system (3.15) *is positive recurrent with respect to E_1 defined in* (3.32).

Proof. Let $\mathscr{V}(\mathbf{Y}(t)) = \sum_{i=1}^{N-1}\sqrt{Y_i(t)}$. Recalling (3.20) gives (details presented in Appendix C)

$$\begin{aligned}
\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) &= \sum_{i=1}^{N-1}\left(\frac{f_i}{2\sqrt{Y_i}} - \frac{1}{8\sqrt{Y_i^3}}\mathfrak{a}_{ii}\right) \\
&\leq \sum_{i=1}^{N-1}\frac{d_i\sqrt{Y_i}}{2}\left(\mathscr{R}_1+\mathscr{R}_2+\mathscr{R}_3+\mathscr{R}_4+\mathscr{R}_5-\frac{d_i(1-\lambda)\sigma^2}{12}\right),
\end{aligned} \tag{3.36}$$

where for $i = 1, \cdots, N-1$,

$$\begin{aligned}
\mathscr{R}_1 &= \mu_i - \frac{\sum_{n=1}^{N-1} d_n Y_n \mu_n}{\sum_{n=1}^{N} d_n Y_n} - \frac{d_i(1-\lambda)\sigma^2}{12}, \\
\mathscr{R}_2 &= \frac{d_N Y_N \sigma^2}{2\sum_{n=1}^{N} d_n Y_n} + \frac{d_N^2 Y_N^2 \theta}{(\sum_{n=1}^{N} d_n Y_n)^2} - \frac{d_i(1-\lambda)\sigma^2}{12}, \\
\mathscr{R}_3 &= \frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n \big(d_n(1-\lambda)\sigma^2 + d_n(1-\varepsilon)\theta + (\theta - \lambda\sigma^2)\big)}{(\sum_{n=1}^{N} d_n Y_n)^2}, \\
\mathscr{R}_4 &= -\frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_i Y_i d_n Y_n \big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2}, \\
\mathscr{R}_5 &= \frac{d_i Y_i(\varepsilon\theta - \sigma^2) + \frac{d_i^2}{2} Y_i (1-\lambda)\sigma^2}{(\sum_{n=1}^{N} d_n Y_n)^2}.
\end{aligned}$$

First, using Assumption (**A4**) shows

$$\mu_i - \frac{\sum_{n=1}^{N-1} d_n Y_n \mu_n}{\sum_{n=1}^{N} d_n Y_n} \le 2|\mu|_{max} \le \frac{d_i(1-\lambda)\sigma^2}{12},$$

and hence

$$\mathscr{R}_1 \le 0.$$

Then the conditions $\theta \ge \frac{\lambda-1}{1-\varepsilon}\sigma^2$, $\theta \le \min_{n=1,\cdots,N-1}\left\{\frac{\lambda - d_n(1-\lambda)}{1+d_n(1-\varepsilon)}\right\}\sigma^2$ and $\theta \le \min_{n=1,\cdots,N-1}\{\frac{1-(2-\lambda)d_n}{2\varepsilon}\}\sigma^2$ in Assumption (**A13**), guarantee, respectively, that

$$\mathscr{R}_4 \le 0, \quad \mathscr{R}_3 \le 0 \quad \text{and} \quad \mathscr{R}_5 \le 0.$$

If for given any $\mathbf{Y} \notin E_1$, where E_1 is defined as in (3.32), we have $y_N \le \ell_y$, and hence

$$\frac{d_N Y_N}{\sum_{n=1}^{N} d_n Y_n} \le \frac{d_N Y_N}{d_{min}(1-Y_N) + d_N Y_N} \le \frac{d_N \ell_y}{d_{min}(1-\ell_y) + d_N \ell_y} \le 1.$$

Consequently

$$\begin{aligned}\mathscr{R}_2 &\leq \frac{\sigma^2}{2}\frac{d_N\ell_y}{d_{min}(1-\ell_y)+d_N\ell_y}+|\theta|\left(\frac{d_N\ell_y}{d_{min}(1-\ell_y)+d_N\ell_y}\right)^2-\frac{d_{min}(1-\lambda)\sigma^2}{12}\\ &\leq \left(\frac{\sigma^2}{2}+|\theta|\right)\frac{d_N\ell_y}{d_{min}(1-\ell_y)+d_N\ell_y}-\frac{d_{min}(1-\lambda)\sigma^2}{12}.\end{aligned} \tag{3.37}$$

It follows from the definition of ℓ_y in (3.33) that

$$\frac{d_N\ell_y}{d_{min}(1-\ell_y)+d_N\ell_y}=\frac{d_{min}(1-\lambda)\sigma^2}{6(\sigma^2+2|\theta|)},$$

which, in turn, ensures that $\mathscr{R}_2 \leq 0$ by (3.37). Therefore, given any $\mathbf{Y} \notin E_1$ there is at least one $n \in \{1,\cdots,N-1\}$ such that $y_n \geq \frac{1-\ell_y}{N-1}$, adding the above estimations on $\mathscr{R}_j$ for $j = 1,\cdots,5$, gives

$$\mathscr{L}\mathscr{V}(\mathbf{Y}(t)) \leq -\frac{d_{min}^2\sqrt{1-\ell_y}}{24\sqrt{N-1}}(1-\lambda)\sigma^2 := -K.$$

According to Itô's Lemma,

$$\mathbb{E}[\mathscr{V}(\mathbf{Y}(t\wedge\tau_{E_1}))]-\mathscr{V}(\mathbf{y}_0)=\mathbb{E}\Big[\int_0^{t\wedge\tau_{E_1}}\mathscr{L}\mathscr{V}(\mathbf{Y}(s))\mathrm{d}s\Big] \leq -K\mathbb{E}[t\wedge\tau_{E_1}].$$

Since the function $\mathscr{V}$ is non-negative, we reorganize the above terms to obtain

$$K\mathbb{E}[t\wedge\tau_E] \leq K\mathbb{E}[t\wedge\tau_{E_1}]+\mathbb{E}[\mathscr{V}(\mathbf{Y}(t\wedge\tau_{E_1}))] \leq \mathscr{V}(\mathbf{y}_0).$$

Letting $t\to\infty$ shows $\mathbb{E}[\tau_{E_1}] < \infty$, which completes the proof. □

Lemma 3 proves that if there is a time t such that $Y_N(t) < \ell_y$, after finite amount of time, Y_N will go above ℓ_y with probability 1, which implies the persistence of the Nth species. Nonetheless, this does not ensure the persistence of other species or coexistence among multiple species. In the following lemma, we will show the proportion Y_N cannot approach 1, which implies the coexistence of at least two species.

Lemma 4. *Let Assumptions (A8) – (A9) hold. In addition, assume that*

$$
\begin{aligned}
&(\boldsymbol{A14}) \quad \sigma^2 \geq \frac{18\big(\mu_{max}-\mu_{min}(d_N\chi_y+1)\big)}{d_N\chi_y(1-d_{max})}\\
&(\boldsymbol{A15}) \quad |\theta| \leq \sigma^2\cdot\min\left\{\frac{1-\lambda}{1-\varepsilon}, \frac{d_N\chi_y(1-d_{max})}{18(2(1-\varepsilon)d_{max}+1)}\right\},
\end{aligned}
$$

where $d_{max}=\max_{n=1,\cdots,N-1}\{d_n\}$*,* $\mu_{min}=\min_{n=1,\cdots,N-1}\{\mu_n\}$*,* $\mu_{max}=\max_{n=1,\cdots,N-1}\{\mu_n\}$ *and* χ_y *is defined in* (3.35)*. Then the solution to the system* (3.15) *is positive recurrent with respect to* E_2 *defined in* (3.34)*.*

Proof. Let $\mathscr{V}(\mathbf{Y}(t))=\sum_{i=1}^{N-1}\frac{1}{\iota}Y_i^{-\iota}$, where ι is a positive constant to be determined later. Whereby (3.20), we get

$$
\mathscr{L}\mathscr{V}(\mathbf{Y}(t))=\sum_{i=1}^{N-1}\frac{-1}{Y_i^{\iota+1}}f_i+\frac{\iota+1}{2Y_i^{\iota+2}}\mathfrak{a}_{ii}, \tag{3.38}
$$

It follows from detailed analysis presented in Appendix D that

$$
\mathscr{L}\mathscr{V}(\mathbf{Y}(t))\leq\sum_{i=1}^{N-1}\frac{-d_i}{Y_i^{\iota}}\left(\frac{\sigma^2}{12}\mathscr{K}_0+\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3\right), \tag{3.39}
$$

where

$$
\begin{aligned}
\mathscr{K}_0 &= \frac{d_NY_N\big(1-(\iota+1)d_i\big)}{\sum_{n=1}^{N}d_nY_n},\\
\mathscr{K}_1 &= \sigma^2\left(\frac{1}{4}\mathscr{K}_0-\frac{\sum_{n=1}^{N-1}d_nY_n(1-\lambda)(1+d_i(\iota+1))}{\sum_{n=1}^{N}d_nY_n}-\frac{\sum_{n=1}^{N-1}d_nY_n}{\sum_{n=1}^{N}d_nY_n}\right),\\
\mathscr{K}_2 &= \mu_i-\frac{\sum_{n=1}^{N-1}d_nY_n\mu_n}{\sum_{n=1}^{N}d_nY_n}+\frac{\sigma^2}{12}\mathscr{K}_0,\\
\mathscr{K}_3 &= -\frac{\sum_{n=1}^{N-1}d_iY_id_nY_n(1-\varepsilon)\theta}{(\sum_{n=1}^{N}d_nY_n)^2}+\frac{d_iY_i\varepsilon\theta+\sum_{n\neq i}^{N-1}d_nY_n\theta+d_N^2Y_N^2\theta}{\sum_{n=1}^{N}d_nY_n}+\frac{\sigma^2}{12}\mathscr{K}_0.
\end{aligned}
$$

Chose $\iota=\min\{\frac{1}{2},\frac{1-d_{max}}{3d_{max}}\}$. Clearly $\iota>0$ for $d_{max}<1$, and hence

$$
\mathscr{K}_0\geq\frac{2}{3}\frac{d_NY_N}{\sum_{n=1}^{N}d_nY_n}(1-d_{max})>0. \tag{3.40}
$$

Applying (3.40) and the fact that $1+d_i(\iota+1)\leq\frac{2}{3}(d_{max}+2)$ in $\mathscr{K}_1$ gives

$$
\begin{aligned}
\mathscr{K}_1 \;&\geq\; \frac{\sigma^2}{\sum_{n=1}^N d_nY_n}\left(\frac{1}{6}d_NY_N(1-d_{max})-\big(\frac{2}{3}(d_{max}+2)(1-\lambda)+1\big)\sum_{n=1}^{N-1}d_nY_n\right)\\
&\geq\; \frac{\sigma^2}{\sum_{n=1}^N d_nY_n}\Bigg(d_N\chi_y\Big(\frac{1-d_{max}}{6}+\frac{2}{3}(d_{max}+2)(1-\lambda)+1\Big)\\
&\qquad\qquad -\big(\frac{2}{3}(d_{max}+2)(1-\lambda)+1\big)\sum_{n=1}^{N}d_nY_n\Bigg).
\end{aligned}
$$

Combining (3.35) and the fact that $\sum_{n=1}^N d_nY_n\leq 1$ shows $\mathscr{K}_1\geq 0$. Next, applying (3.40) in $\mathscr{K}_2$ gives

$$
\begin{aligned}
\mathscr{K}_2 \;&\geq\; \frac{1}{\sum_{n=1}^N d_nY_n}\left(\sum_{n=1}^{N-1}(\mu_i-\mu_n)d_nY_n+\mu_id_NY_N+\frac{\sigma^2}{18}d_NY_N(1-d_{max})\right)\\
&\geq\; \frac{1}{\sum_{n=1}^N d_nY_n}\left(\mu_{min}-\mu_{max}+\mu_{min}d_N\chi_y+\frac{\sigma^2}{18}d_N\chi_y(1-d_{max})\right).
\end{aligned}
$$

$\mathscr{K}_2\geq 0$ is ensured by the Assumption (**A14**).

We still need to verify that $\mathscr{K}_3\geq 0$ under the Assumption (**A8**). Indeed, whereby (3.40) and $\sum_{n=1}^N d_nY_n\leq 1$ we have

$$
\begin{aligned}
\mathscr{K}_3 \;&\geq\; \frac{\theta}{\sum_{n=1}^N d_nY_n}\left(-(1-\varepsilon)d_iY_i\Big(\sum_{n=1}^{N-1}d_nY_n+1\Big)+\sum_{n=1}^{N-1}d_nY_n+d_N^2Y_N^2\right)\\
&\quad+\frac{\sigma^2}{18\sum_{n=1}^N d_nY_n}d_NY_N(1-d_{max})\\
&\geq\; \frac{1}{\sum_{n=1}^N d_nY_n}\left(-|\theta|\big(2(1-\varepsilon)d_{max}+1\big)+\frac{\sigma^2}{18}d_N\chi_y(1-d_{max})\right).
\end{aligned}
$$

The second condition in Assumption (**A15**) then guarantees $\mathscr{K}_3\geq 0$. Adding the above estimations on $\mathscr{K}_1$, $\mathscr{K}_2$ and $\mathscr{K}_3$ gives

$$
\mathscr{L}\mathscr{V}(\mathbf{Y}(t))\leq\sum_{i=1}^{N-1}\frac{-d_i}{Y_i^{\iota}}\frac{\sigma^2}{12}\mathscr{K}_0\leq-\frac{d_N\chi_y(1-(\iota+1)d_{max})\sigma^2}{12\max_{1\leq i\leq N}\{d_i\}}\sum_{i=1}^{N-1}d_i=:-K.
$$

In conclusion, we have

$$\mathbb{E}[\mathscr{V}(\mathbf{Y}(t\wedge\tau_{E_2}))]-\mathscr{V}(\mathbf{y}_0)=\mathbb{E}\left[\int_0^{t\wedge\tau_{E_2}}\mathscr{L}\mathscr{V}(\mathbf{Y}(s))\mathrm{d}s\right]\leq -K\mathbb{E}\left[t\wedge\tau_{E_2}\right].$$

Since the function $\mathscr{V}$ is non-negative, we reorganize the above terms to obtain

$$\mathbb{E}[t\wedge\tau_{E_2}]\leq\frac{1}{K}\mathscr{V}(\mathbf{y}_0).$$

Letting $t\to\infty$ gives $\mathbb{E}[\tau_{E_2}]<\infty$. □

Finally, combining 3 and 4 gives

Theorem 9. *Let Assumptions (**A8**) – (**A9**) and (**A12**) – (**A15**) hold. Moreover, assume that $\ell_y\leq\chi_y$. Then the solution to the system* (3.15) *is positive recurrent with respect to the non-empty bounded set $E_1\bigcap E_2$.*

Chapter 4

Random Age-structured Model

This section is devoted to studying an age-structured model with a random birth rate of mature individuals. More specifically, we will investigate the well-posedness, co-cycle property, and the asymptotic behaviors of the solution to the age-structure model when the function of birth rate is driven by a random flow $\theta = (\theta_t)_{t\in\mathbb{R}}$ on a probability space.

4.1 The random model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space where $\mathscr{F}$ is the σ-algebra of measurable subset of Ω and $\mathbb{P}$ is the probability measure on $\mathscr{F}$. To connect the state ω in the probability space Ω at time 0 with its state after a time of t elapses, we define a driving dynamical system [17] below.

Definition 1. *A flow $\theta = (\theta_t)_{t\in\mathbb{R}}$ on Ω is called driving dynamical system, if for each θ_t being a mapping: $\Omega \to \Omega$, satisfies*

1. $\theta_0 = Id_\Omega$,
2. $\theta_s \circ \theta_t = \theta_{t+s}$ *for all* $s,t \in \mathbb{R}$,
3. *the mapping* $(t,\omega) \to \theta_t\omega$ *is measurable,*
4. *the probability measure* $\mathbb{P}$ *is preserved by* θ_t*, i.e.* $\theta_t\mathbb{P} = \mathbb{P}$.

This set-up establishes a time-dependent family θ that tracks the noise, and $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ is called a metric dynamical system [64].

Now we model the parameter of the birth rate of the adults, β, as a stochastic process $\beta(\theta_t\omega)$, where θ_t is defined as above. Here is the assumption we will need later.

(**A16**) $\beta \in L^1(\Omega; C([0,\infty);\mathbb{R}_+))$ and for each fixed $\omega \in \Omega$, $\beta(\theta_t\omega) \in [\beta_m, \beta_M]$ with $\beta_m > 0$.

For each fixed $\omega \in \Omega$, denote by $u(a,t,\omega)$ the age density function of a population with age a at time $t > 0$. Once the sample point ω is fixed, $u(a,t,\omega)$ can be viewed as a classical deterministic age-structure model. According to [65], $u(a,t,\omega)$ satisfies

$$\begin{cases} \dfrac{\partial u(a,t,\omega)}{\partial t} + \dfrac{\partial u(a,t,\omega)}{\partial a} = -\mu(a)u(a,t,\omega), \text{ for } t,a > 0, \quad \omega \in \Omega, \\ u(0,t,\omega) = \beta(\theta_t\omega)\, h\left(\int_0^\infty \delta(a)u(t,a,\omega)\mathrm{d}a\right), \quad t \geq 0, \ \omega \in \Omega, \\ u(a,0,\omega) = \phi(a,\omega) \quad a \geq 0, \ \omega \in \Omega. \end{cases} \tag{4.1}$$

Here, we also assume that the initial value is in the space of all the positive integrable functions, i.e.

$$(\mathbf{A17})\phi(\cdot,\omega) \in L^1_+(0,\infty).$$

Where $L^1_+(0,\infty)$ collects all of the almost everywhere non-negative integrable functions on $(0,\infty)$. Moreover, the function

$$(\mathbf{A18})\ \mu \in L^1_{+,loc}(0,\infty)$$

is the death rate of the population. And the function

$$(\mathbf{A19})\ \delta \in L^\infty_+(0,\infty) \cap L^1(0,\infty)$$

is the probability for an individual of a certain age to be mature. Finally, $h(x)$ is the Ricker's function of the form

$$h(x) = xe^{-x},$$

which describe the non-linear cannibalism [66].

4.2 Solution to the random model

This section is devoted to investigate the existence and uniqueness of the solution to (4.1). First, the definition of solution will be given by the method of characteristic curve. Then, existence and uniqueness of the solution will be proved by the fixed point theorem.

For each fixed $\omega \in \Omega$, let

$$v(a,t,\omega) = e^{\int_0^a \mu(\sigma)\mathrm{d}\sigma} u(a,t,\omega). \tag{4.2}$$

It follows directly that

$$\begin{aligned} \frac{\partial v}{\partial a} &= \mu(a) e^{\int_0^a \mu(\sigma)\mathrm{d}\sigma} u(a,t,\omega) + e^{\int_0^a \mu(\sigma)\mathrm{d}\sigma} \frac{\partial u}{\partial a}, & (4.3) \\ \frac{\partial v}{\partial t} &= e^{\int_0^a \mu(\sigma)\mathrm{d}\sigma} \frac{\partial u}{\partial t}. & (4.4) \end{aligned}$$

Adding (4.3) and (4.4) results in

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = 0, \text{ for } t, a > 0, \quad \omega \in \Omega, \\ v(0,t,\omega) = u(0,t,\omega), \quad t \geq 0, \ \omega \in \Omega, \\ v(a,0,\omega) = e^{\int_0^a \mu(\sigma)\mathrm{d}\sigma} \phi(a,\omega), \quad a \geq 0, \ \omega \in \Omega. \end{cases} \tag{4.5}$$

Clearly, (4.5) can be solved as

$$v(a,t,\omega) = \begin{cases} e^{\int_0^{a-t} \mu(\sigma)\mathrm{d}\sigma} \phi(a-t,\omega), \quad \text{if } a \geq t, \\ u(0,t-a,\omega), \quad \text{if } 0 \leq a < t. \end{cases} \tag{4.6}$$

For simplicity, denote $\Pi(a) = e^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma}$, $\forall a \in [0,\infty)$. Thus,

$$u(a,t,\omega) = \begin{cases} \frac{\Pi(a)}{\Pi(a-t)} \phi(a-t,\omega), \quad \text{if } a \geq t, \\ \Pi(a) \beta\left(\theta_{t-a}\omega\right) h\left(\int_0^\infty \delta(s) u(t-a,s,\omega)\mathrm{d}s\right), \quad \text{if } 0 \leq a < t. \end{cases} \tag{4.7}$$

Here, we treat the above equation as the solution of the age-structured model.

Definition 2. *A funciton $u \in L^1_+(0,\infty)$ is said to be the solution to* (4.1) *if it satisfies* (4.7).

Remark 9. *A function that satisfies the above definition is also a mild solution defined in [65].*

Actually, solving for $u(a,t,\omega)$ is equivalent to know what $b_\phi(t,\omega) = \int_0^\infty \delta(s)u(s,t,\omega)\mathrm{d}s$ is. Moreover, $b_\phi(t,\omega)$ satisfies the Volterra integral equation

$$\begin{aligned} b_\phi(t,\omega) &= \int_0^t \delta(s)u(s,t,\omega)\mathrm{d}s + \int_t^\infty \delta(s)u(s,t,\omega)\mathrm{d}s \\ &= \int_0^t \delta(s)\Pi(s)\beta(\theta_{t-s}\omega)h\left(\int_0^\infty \delta(\sigma)u(t-s,\sigma,\omega)\mathrm{d}\sigma\right)\mathrm{d}s \\ &\quad + \int_t^\infty \delta(s)\frac{\Pi(a)}{\Pi(a-t)}u_0(a-t,\omega)\mathrm{d}s \\ &= \int_0^t K(s)\beta(\theta_{t-s}\omega)h\left(b_\phi(t-s,\omega)\right)\mathrm{d}s + g(t,\omega), \end{aligned} \tag{4.8}$$

where $g(t,\omega) := \int_t^\infty \delta(s)\frac{\Pi(a)}{\Pi(a-t)}u_0(a-t,\omega)\mathrm{d}s$ and $K(a) = \delta(a)\Pi(a)$. The following theorem prove the existence and uniqueness of $b_\phi(t,\omega)$ in the space

$$A_\eta := \{f \in C[0,\infty] | \sup_{t\geq 0} e^{-\eta t}|f(t)| < \infty\},\ \forall \eta > 0.$$

Theorem 10. *Under Assumption **(A16)** – **(A19)**, for any fixed $\omega \in (\Omega,\mathscr{F},\mathbb{P})$, there is a unique solution to the equation* (4.8) *in A_η, where $\eta > 0$ satisfies $\beta_M\|h\|_{Lip}\int_0^\infty e^{-\eta s}K(s)\mathrm{d}s < 1$.*

Proof. For any $f \in A_\eta$, we define $T : A_\eta \to A_\eta$ as

$$\begin{aligned} T(f)(t,w) &= g(t,\omega) + \int_0^t \beta(\theta_{t-a}\omega)\delta(a)\Pi(a)h(f(t-a,\omega))\mathrm{d}a \\ &= g(t,\omega) + \int_0^t \delta(t-a)\Pi(t-a)\beta(\theta_a\omega)h(f(a,\omega))\mathrm{d}a \\ &= g(t,\omega) + \int_0^t K(t-a)\beta(\theta_a\omega)h(f(a,\omega))\mathrm{d}a \\ &= g(t,\omega) + K*(\beta h(f))(t,w). \end{aligned}$$

Then, we verify that T is a contraction mapping on A_η. For any f_1, f_2 in A_η and any $t \geq 0$,

$$\begin{aligned}
|e^{-\eta t}T(f_1)(t) - e^{-\eta t}T(f_2)(t)| &= e^{-\eta t}|K*(\beta h(f_1))(t,\omega) - K*(\beta h(f_2))(t,w)| \\
&\leq \int_0^t e^{-\eta(t-a)}K(t-a)e^{-\eta a}|\beta(\theta_a\omega)||h(f_1(a)) - h(f_2(a))|\mathrm{d}a \\
&\leq \beta_M\|h\|_{Lip}\|f_1 - f_2\|_{A_\eta}\int_0^t e^{-\eta(t-a)}K(t-a)\mathrm{d}a \\
&\leq \|f_1 - f_2\|_{A_\eta}\beta_M\|h\|_{Lip}\int_0^\infty e^{-\eta s}K(s)\mathrm{d}a.
\end{aligned}$$

Then, Banach Contraction Mapping Theorem implies the equation (4.13) has a unique solution in A_η. □

Thus, the existence and uniqueness of the solution to (4.1) is ensured by the above theorem.

4.3 Co-cycle property of the solution

In this section, we will show that the solution to (4.1), $u(a,t,\omega) =: U(t,\omega)\phi$, forms a random dynamical system [17] on $L^1_+(0,\infty)$, which is defined below.

Definition 3. *The pair (θ, U) is called a random dynamical system on $L^1_+(0,\infty)$ if it consists of a driving dynamical system $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ defined in Definition 1 and co-cycle mapping : $\mathbb{R}^+ \times \Omega \times L^1_+(0,\infty) \to L^1_+(0,\infty)$ satisfying*

1. *initial condition: $U(0,\omega)\phi = \phi$ for $\omega \in \Omega$ and $\phi \in L^1_+(0,\infty)$,*
2. *co-cycle property: $U(s+t,\omega)\phi = U(t,\theta_s\omega)\circ U(s,\omega)\phi$, for all $t,\ s \geq 0,\ \omega \in \Omega$ and $\phi \in L^1_+(0,\infty)$,*
3. *measurability: $(t,\omega,\phi) \to U(t,\omega)\phi$ is measurable,*
4. *continuity: $\phi \to U(t,\omega)\phi$ is continuous for all $(t,\omega) \in \mathbb{R}^+ \times \Omega$.*

The following Lemma 5 and Theorem 11 will prove $U(t,\omega)$ is continuous on $L^1_+(0,\infty)$.

Lemma 5. *Let $T > 0$ and $\omega \in \Omega$. For any $r > 0$, $t \in [0,T]$, if $\phi \in B(0,r) \subset L^1_+(0,\infty)$ and $\phi_n \xrightarrow{L^1(0,\infty)} \phi$, then $b_{\phi_n}(t,\omega) \xrightarrow{\mathbb{R}^1} b_\phi(t,\omega)$ uniformly on $[0,T]$.*

Proof. For any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$, such that $\forall m > N$, $\|\phi_m - \phi\|_{L^1(0,\infty)} < \frac{\varepsilon}{\|\delta\|_{L^\infty(0,\infty)}}$. It follows from (4.13) that

$$
\begin{aligned}
|b_{\phi_m}(t,\omega) - b_\phi(t,\omega)| &\leq |g_{\phi_m}(t,\omega) - g_\phi(t,\omega)| + \int_0^t K(t-a)\beta_M \|h\|_{Lip} |b_{\phi_m}(a,\omega) - b_\phi(a,\omega)| \mathrm{d}a, \\
|g_{\phi_m}(t,\omega) - g_\phi(t,\omega)| &\leq \int_0^\infty \frac{K(a+t)}{\Pi(a)} |\phi_m(a) - \phi(a)| \mathrm{d}a \leq \|\delta\|_{L^\infty(0,\infty)} \int_0^\infty |\phi_m(a) - \phi(a)| \mathrm{d}a < \varepsilon.
\end{aligned}
$$

For simplicity, we denote $B = |b_{\phi_m}(t,\omega) - b_\phi(t,\omega)|$, $f = |g_{\phi_m}(t,\omega) - g_\phi(t,\omega)|$ and $k(a-t) = \beta_M \|h\|_{Lip} K(a-t)$. Then we have the equivalent inequality

$$
B(t,\omega) \leq f(t,\omega) + \int_0^t k(t-a) B(a,\omega) \mathrm{d}a. \tag{4.9}
$$

Applying the iterated property of (4.9) gives

$$
\begin{aligned}
B(t,\omega) &\leq f(t,\omega) + \int_0^t k(t-a) f(a,\omega) \mathrm{d}a + \int_0^t k(t-a) \int_0^a k(a-v) B(v,\omega) \mathrm{d}v \mathrm{d}a \\
&= f(t,\omega) + \int_0^t k(t-a) f(a,\omega) \mathrm{d}a + \int_0^t \int_0^a k(t-a) k(a-v) B(v,\omega) \mathrm{d}v \mathrm{d}a \\
&= f(t,\omega) + \int_0^t k(t-a) f(a,\omega) \mathrm{d}a + \int_0^t \int_0^t \chi_{[0,a]}(v) k(t-a) k(a-v) B(v,\omega) \mathrm{d}v \mathrm{d}a \\
&= f(t,\omega) + \int_0^t k(t-a) f(a,\omega) \mathrm{d}a + \int_0^t \int_v^t k(t-a) k(a-v) B(v,\omega) \mathrm{d}a \mathrm{d}v.
\end{aligned}
$$

For any $0 \leq s \leq v \leq t \leq T$, by induction,

$$
k_1(t-s) = k(t-s),\; k_n(t-s) = \int_s^t k_{n-1}(t-v) k(v-s) \mathrm{d}v = \int_0^{t-s} k_{n-1}(t-s-v) k(v) \mathrm{d}v,
$$

$n \geq 2,$ and $n \in \mathbb{N}$.

Therefore, for each $n \geq 2$ and $n \in \mathbb{N}$, (4.9) also has the form

$$
B(t,\omega) \leq f(t,\omega) + \int_0^t \sum_{j=1}^n k_j(t-a) f(a,\omega) \mathrm{d}a + \int_0^t k_{n+1}(t-a) B(a,\omega) \mathrm{d}a. \tag{4.10}
$$

From the definition of the function k, we know that $|k_1| \leq \beta_M \|h\|_{Lip} \|\delta\|_{L^\infty(0,\infty)} =: \tilde{k}$. If we assume for any $n \geq 2$ and $n \in \mathbb{N}$,

$$|k_n(t-s)| \leq \tilde{k}^n \frac{(t-s)^{n-1}}{(n-1)!}.$$

is true, then

$$|k_{n+1}(t-s)| \leq \int_s^t |k_{n-1}(t-v)|k(v-s)\mathrm{d}v \leq \frac{\tilde{k}^{n+1}}{(n-1)!} \int_s^t (t-v)^{n-1}\mathrm{d}v = \tilde{k}^{n+1} \frac{(t-s)^n}{n!}. \quad (4.11)$$

Inserting (4.11) into (4.10) and letting $n \to \infty$ give us

$$\begin{aligned} B(t,\omega) &\leq f(t,\omega) + \int_0^t \sum_{j=1}^{\infty} k_j(t-a) f(a,\omega)\mathrm{d}a \\ &= f(t,\omega) + \int_0^t r(t-a) f(a,\omega)\mathrm{d}a \\ &\leq f(t,\omega) + \int_0^t \tilde{k} e^{\tilde{k}T} f(a,\omega)\mathrm{d}a \\ &\leq \varepsilon + \varepsilon C(T,\tilde{k}), \end{aligned}$$

where $r(t)$ is the resolvent kernel [67]. Thus, $b_{\phi_n}(t) \to b_\phi(t)$ uniformly on $[0,T]$. The proof is complete. □

We will apply the uniform continuity in the following theorem.

Theorem 11. *Under assumptions (**A16**) – (**A19**), for* $\forall \omega \in \Omega$, $U(t,\omega)\phi$ *is continuous from* $[0,\infty) \times L^1_+(0,\infty)$ *to* $L^1_+(0,\infty)$.

Proof. For any $t \geq 0$ and $\phi \in L^1_+(0,\infty)$, assume $t_n \to t$ (W.L.O.G., $t_n \leq t$) and $\phi_n \xrightarrow{L^1(0,\infty)} \phi$.

$$
\begin{aligned}
\int_0^\infty |u_{\phi_n}(t_n,a,\omega) - u_\phi(t,a,\omega)| \mathrm{d}a \; = \; & \underbrace{\int_0^{t_n} \Pi(a) |G(b_{\phi_n}(t_n - a)) - G(b_\phi(t-a))| \mathrm{d}a}_{I_1} \\
& + \underbrace{\int_{t_n}^{t} \left| \frac{\Pi(a)\phi_n(a-t_n)}{\Pi(a-t_n)} - \Pi(a) G(b_\phi(t-a)) \right| \mathrm{d}a}_{I_2} \\
& + \underbrace{\int_t^\infty \left| \frac{\Pi(a)\phi_n(a-t_n)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a}_{I_3}.
\end{aligned}
$$

First, because of the boundedness of $\|\phi_n\|_{L^1(0,\infty)}$ and the continuity of $G(b_\phi)$, $I_2 \to 0$ as $n \to \infty$. Second,

$$
\begin{aligned}
I_1 \; \leq \; & \int_0^{t_n} \beta_M \|h\|_{Lip} |b_{\phi_n}(t_n - a) - b_\phi(t-a)| \mathrm{d}a \\
= \; & \beta_M \|h\|_{Lip} \left(\underbrace{\int_0^{t_n} |b_{\phi_n}(t_n - a) - b_\phi(t_n - a)| \mathrm{d}a}_{①} + \underbrace{\int_0^{t_n} |b_\phi(t_n - a) - b_\phi(t-a)| \mathrm{d}a}_{②} \right).
\end{aligned}
$$

Invoking Lemma 5 gives ① $\to 0$ as $n \to 0$. Applying the Dominate Convergence Theorem shows ② $\to 0$ as $n \to 0$.

Moreover,

$$
\begin{aligned}
I_3 &= \int_t^\infty \left| \frac{\Pi(a)\phi_n(a-t_n)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a \\
&\leq \int_t^\infty \left| \frac{\Pi(a)\phi_n(a-t_n)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t_n)} \right| \mathrm{d}a + \int_t^\infty \left| \frac{\Pi(a)\phi(a-t)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a \\
&\leq \int_t^\infty |\phi_n(a-t_n) - \phi(a-t)| \,\mathrm{d}a + \int_t^\infty \left| \frac{\Pi(a)\phi(a-t)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a \\
&\leq \int_{t_n}^\infty |\phi_n(a-t_n) - \phi(a-t_n)| \,\mathrm{d}a + \int_t^\infty |\phi(a-t_n) - \phi(a-t)| \,\mathrm{d}a \\
&\quad + \int_t^\infty \left| \frac{\Pi(a)\phi(a-t)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a \\
&= \int_0^\infty |\phi_n(a) - \phi(a)| \,\mathrm{d}a + \int_t^\infty |\phi(a-t_n) - \phi(a-t)| \,\mathrm{d}a \\
&\quad + \int_t^\infty \left| \frac{\Pi(a)\phi(a-t)}{\Pi(a-t_n)} - \frac{\Pi(a)\phi(a-t)}{\Pi(a-t)} \right| \mathrm{d}a
\end{aligned}
$$

Whereby $\phi_n \xrightarrow{L^1(0,\infty)} \phi$, the L^1-norm continuity and Dominate Convergence Theorem, $I_3 \to 0$ as $n \to \infty$. □

In the following Lemma 6 and Theorem 4.15, we will verify that the solution to (4.13) satisfies the co-cycle property defined in Definition 3.

Lemma 6. *For every $\omega \in \Omega$, $\phi \in L^1_+(0,\infty)$, and $0 \leq s,\ t < \infty$,*

$$b_\phi(t+s,\omega) = b_{U(s,\omega)\phi}(t,\theta_s\omega).$$

Proof. **Method 1**. For simplicity, in the following calculation, we denote $b_{U(s,\omega)\phi}(\cdot,\theta_s\omega) = b(\cdot,\theta_s\omega)$.

$$
\begin{aligned}
b_{U(s,\omega)\phi}(t,\theta_s\omega) &= g_{U(s,\omega)\phi}(t,\theta_s\omega)+\int_0^t K(a)\beta(\theta_{t-a}\circ\theta_s\omega)h(b(t-a,\theta_s\omega))\mathrm{d}a \\
&= \int_t^\infty \frac{K(a)}{\Pi(a-t)}U(s,\omega)\phi(a-t)\mathrm{d}a \\
&\quad +\int_0^t K(a)\beta(\theta_{t-a+s}\omega)h(b(t-a,\theta_s\omega))\mathrm{d}a \\
&= \int_0^\infty \frac{K(a+t)}{\Pi(a)}U(s,\omega)\phi(a)\mathrm{d}a+\int_0^t K(t-a)\beta(\theta_{a+s}\omega)h(b(a,\theta_s\omega))\mathrm{d}a \\
&= \int_0^s \frac{K(a+t)}{\Pi(a)}\Pi(a)\beta(\theta_{s-a}\omega)h(b_\phi(s-a,\omega))\mathrm{d}a \\
&\quad +\int_s^\infty \frac{K(a+t)}{\Pi(a)}\frac{\Pi(a)}{\Pi(a-s)}\phi(a-s)\mathrm{d}a \\
&\quad +\int_0^t K(t-a)\beta(\theta_{a+s}\omega)h(b(a,\theta_s\omega))\mathrm{d}a \\
&= \int_0^s K(a+t)\beta(\theta_{s-a}\omega)h(b_\phi(s-a,\omega))\mathrm{d}a+\int_s^\infty \frac{K(a+t)}{\Pi(a-s)}\phi(a-s)\mathrm{d}a \\
&\quad +\int_0^t K(t-a)\beta(\theta_{a+s}\omega)h(b(a,\theta_s\omega))\mathrm{d}a \\
&= \int_0^s K(s+t-a)\beta(\theta_a\omega)h(b_\phi(a,\omega))\mathrm{d}a+\int_0^\infty \frac{K(a+s+t)}{\Pi(a)}\phi(a)\mathrm{d}a \\
&\quad +\int_s^{s+t} K(s+t-a)\beta(\theta_a\omega)h(b(a-s,\theta_s\omega))\mathrm{d}a \\
&= g_\phi(s+t,\omega)+\int_0^{s+t} K(s+t-a)\beta(\theta_a\omega)h(\tilde{b}_\phi(a,\omega))\mathrm{d}a, \qquad (4.12)
\end{aligned}
$$

where

$$
\tilde{b}_\phi(x,w)=\begin{cases} b_\phi(x,\omega),\ 0\le x\le s, \\ b_{U(s,\omega)\phi}(x-s,\theta_s\omega),\ s\le x\le s+t. \end{cases}
$$

Then, we verify that $\tilde{b}_\phi$ is a solution to

$$
\tilde{b}_\phi(x,\omega)=g_\phi(x,\omega)+\int_0^x K(x-a)\beta(\theta_a\omega)h(\tilde{b}_\phi(a,\omega))\mathrm{d}a,\ \forall x\in[0,s+t]. \qquad (4.13)
$$

Clearly, (4.13) is true when $x\in[0,s]$. If $x\in[s,s+t]$, replacing t with $x-s$ in (4.12) gives

$$
\begin{aligned}
\tilde{b}_\phi(x,w) &= b_{U(s,\omega)\phi}(x-s,\theta_s\omega) \\
&= g_\phi(x,\omega)+\int_0^x K(x-a)\beta(\theta_a\omega)h(\tilde{b}_\phi(a,\omega))\mathrm{d}a.
\end{aligned}
$$

Since the solution to (4.13) is unique, we have for any $x \in [s, s+t]$,

$$b_\phi(x,w) = \tilde{b}_\phi(x,w) = b_{U(s,\omega)}(x-s, \theta_s\omega)$$

Method 2.

$$\begin{aligned}
b_\phi(s+t,\omega) &= g_\phi(s+t,\omega) + \int_0^{s+t} K(s+t-a)\beta(\theta_a\omega)h(b_\phi(a,\omega))\mathrm{d}a \\
&= \int_{s+t}^{\infty} \frac{K(a)}{\Pi(a-s-t)}\phi(a-s-t)\mathrm{d}a + \int_0^{s} K(s+t-a)\beta(\theta_a\omega)h(b_\phi(a,\omega))\mathrm{d}a \\
&\quad + \int_s^{s+t} K(s+t-a)\beta(\theta_a\omega)h(b_\phi(a,\omega))\mathrm{d}a \\
&= \int_s^{\infty} \frac{K(a+t)}{\Pi(a)}\frac{\Pi(a)}{\Pi(a-s)}\phi(a-s)\mathrm{d}a \\
&\quad + \int_0^{s} \frac{K(a+t)}{\Pi(a)}\Pi(a)\beta(\theta_{s-a}\omega)h(b_\phi(s-a,\omega))\mathrm{d}a \\
&\quad + \int_s^{s+t} K(s+t-a)\beta(\theta_a\omega)h(b_\phi(a,\omega))\mathrm{d}a \\
&= \int_0^{\infty} \frac{K(a+t)}{\Pi(a)}U(s,\omega)\phi(a)\mathrm{d}a + \int_0^{t} K(t-a)\beta(\theta_{a+s}\omega)h(b_\phi(a+s,\omega))\mathrm{d}a \\
&= g_{U(s,\omega)\phi}(t,\theta_s\omega) + \int_0^{t} K(t-a)\beta(\theta_a\circ\theta_s\omega)h(b_\phi(a+s,\omega))\mathrm{d}a \\
&= g_{\bar{\phi}}(t,\bar{\omega}) + \int_0^{t} K(t-a)\beta(\theta_a\bar{\omega})h(B_{\bar{\phi}}(a,\bar{\omega}))\mathrm{d}a,
\end{aligned}$$

where $\bar{\omega} := \theta_s\omega$, $\bar{\phi} := U(s,\omega)\phi$, and $B_{\bar{\phi}}(a,\bar{\omega}) := b_\phi(a+s,\omega)$. Thus, the calculations above implies

$$B_{\bar{\phi}}(t,\bar{\omega}) = g_{\bar{\phi}}(t,\bar{\omega}) + \int_0^{t} K(t-a)\beta(\theta_a\bar{\omega})h(B_{\bar{\phi}}(a,\bar{\omega}))\mathrm{d}a. \tag{4.14}$$

The uniqueness of the solution to (4.14) implies

$$b_\phi(s+t,\omega) = B_{\bar{\phi}}(t,\bar{\omega}) = b_{\bar{\phi}}(t,\bar{\omega}) = b_{U(s,\omega)\phi}(t,\theta_s\omega).$$

The proof is complete. □

Lemma 6 shows that b_ϕ satisfies the co-cycle property, which also implies the co-cycle property of $U(t,\omega)$ in the following theorem.

Theorem 12. *For every $\omega \in \Omega$ and $0 \leq s,\ t < \infty$,*

$$U(s+t,\omega)\phi = U(t,\theta_s\omega) \circ U(s,\omega)\phi,\ \forall \phi \in L^1_+(0,\infty). \tag{4.15}$$

Proof. First,

$$U(s+t,\omega) = \begin{cases} \Pi(a)\beta(\theta_{s+t-a}\omega)h(b_\phi(t+s-a,\omega)),\ 0 \leq a \leq s+t, \\ \frac{\Pi(a)}{\Pi(a-s-t)}\phi(a-s-t),\ a \geq s+t. \end{cases}$$

On the other hand,

$$\begin{aligned} U(t,\theta_s\omega) \circ U(s,\omega)\phi &= \begin{cases} \Pi(a)\beta(\theta_{t-a} \circ \theta_s\omega)h(b_{U(s,\omega)\phi}(t-a,\theta_s\omega)),\ 0 \leq a \leq t, \\ \frac{\Pi(a)}{\Pi(a-t)}U(s,\omega)\phi(a-t),\ a \geq t. \end{cases} \\ &= \begin{cases} \Pi(a)\beta(\theta_{s+t-a}\omega)h(b_{U(s,\omega)\phi}(t-a,\theta_s\omega)),\ 0 \leq a \leq t, \\ \frac{\Pi(a)}{\Pi(a-t)}\Pi(a-t)\beta(\theta_{s+t-a}\omega)h(b_\phi(s+t-a,\omega)),\ t \leq a \leq s+t, \\ \frac{\Pi(a)}{\Pi(a-t)}\frac{\Pi(a-t)}{\Pi(a-s-t)}\phi(a-s-t),\ a \geq s+t. \end{cases} \end{aligned}$$

Clearly, (4.15) is true when $a \geq s+t$. On the other hand, Lemma 6 says (4.15) is also true if $0 \leq a \leq s+t$. □

To conclude the section, we summarize the lemmas and theorems above as

Theorem 13. *Under assumptions **(A16)** – **(A19)**, the solution to* (4.1) *defines a continuous random dynamical system [64] on $L^1_+(0,\infty)$.*

4.4 Asymptotic behaviors of $U(t,\omega)\phi$

This section is devoted to investigate the asymptotic behaviors of the solution to the age-structured model. First, we introduce concepts that describe the long time dynamics of the solution. Then a natural decomposition of the solution to (4.1) will be applied to show a trajectory is compact in $L^1(0,\infty)$. Moreover, a series of lemmas will show the omega-limit set of the invariant part of the absorbing set is the attracor.

4.4.1 Preliminaries

we begin with a fundamental concept.

Definition 4. *A family $D = \{D(\omega)\}_{\omega\in\Omega}$ of non-empty subsets of $L^1_+(0,\infty)$ is called uniformly bounded, if $\cup_{\omega\in\Omega}D(\omega)$ is a bounded set in $L^1_+(0,\infty)$.*

Throughout this section, we denote $\mathscr{B}$ the universe that collects all of uniformly bounded families defined above. Then, we can define the positive invariant family below.

Definition 5. *A family $D = \{D(\omega)\}_{\omega\in\Omega}$ of non-empty subsets of $L^1_+(0,\infty)$ is called positive invariant if*

$$U(t,\omega)D(\omega) \subset D(\theta_t\omega), \text{ for all } t \geq 0, \text{ and } \omega \in \Omega.$$

Now, we can define the random pullback attractor for the universe $\mathscr{B}$.

Definition 6. *Let (θ,U) be an random dynamical system on $L^1_+(0,\infty)$ and $\mathscr{B}$ be the universe of all uniformly bounded families. A family of compact sets $\mathscr{A} \in \mathscr{B}$ is said to be a random pullback attractor for $\mathscr{B}$ if*

1. *it is invariant: $U(t,\omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega)$ for all $t \geq 0$, $\mathbb{P}-$a.s.,*

2. *it attracts all the families of the universe $\mathscr{B}$ in the pullback sense. i.e.,*

$$\lim_{t\to\infty} dist(U(t,\theta_{-t}\omega)B(\theta_{-t}\omega),\mathscr{A}(\omega)) = 0, \ \mathbb{P}-a.s., \text{ for every } B \in \mathscr{B},$$

where $dist(C,D) := \sup_{c\in C}\left\{\inf_{d\in D}\{\|c-d\|_{L^1}\}\right\}$, for any C, $D \subset L^1_+(0,\infty)$.

The existence of a random attractor usually depends on the existence of a random pullback absorbing set which is defined below.

Definition 7. *A family $K \in \mathscr{B}$ is said to be pullback absorbing for the universe $\mathscr{B}$ if for every $B \in \mathscr{B}$ and any $\omega \in \Omega$, there exists $t(B,\omega) > 0$ such that*

$$U(t,\theta_{-t})B(\theta_{-t}\omega) \subset K(\omega), \text{ for all } t \geq t(B,\omega).$$

Actually, in this work, the following construction of a omega-limit set of the absorbing set will be the random attractor. We will also apply the Kuratowski measurable of non-cpmpactness [68].

Definition 8. $\alpha(A) = \inf\{d > 0 \mid A \textit{ has a finite cover of diameter} < d\}$, $\forall A \subset L^1_+(0,\infty)$.

We can verify that the the Kuratowski measurable of non-cpmpactness satisfies, for A, $B \subset L^1_+(0,\infty)$,

1. $\alpha(A) = 0$ for $A \subset L^1_+(0,\infty)$ iff A is pre-compact,
2. $\alpha(A+B) = \alpha(A) + \alpha(B)$,
3. $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$,
4. $\alpha(\overline{A}) = \alpha(A)$,
5. If $A_1 \supset A_2 \supset \cdots$ are non-empty closed set of $L^1_+(0,\infty)$ such that $\alpha(A_n) \to 0$ as $n \to \infty$, then $\cap_{n\geq 1} A_n$ is non-empty and compact.

Definition 9. *Let $B = \{B(\omega)\}_{\omega\in\Omega}$ be a family of subsets of $L^1_+(0,\infty)$, the omega-limit set of B is*

$$\Omega_B(\omega) = \cap_{T\geq 0}\overline{\cup_{t\geq T} U(t,\theta_{-t}\omega)B(\theta_{-t}\omega)}$$

4.4.2 Existence of the random pullback attractor

Enlightened by Proposition 3.13 in [69] (page 100), since $U(t,\omega)\phi$ is defined in terms of a step function, it is intuitive to write

$$U(t,\omega)\phi = S(t,\omega)\phi + W(t,\omega)\phi,$$

where

$$[S(t,\omega)\phi](a) = \begin{cases} 0, \text{ a.e. } a \in [0,t] \\ [U(t,\omega)\phi](a), \text{ a.e. } a \in (t,\infty), \end{cases} = \begin{cases} 0, \text{ a.e. } a \in [0,t] \\ \frac{\Pi(a)}{\Pi(a-t)}\phi(a-t), \text{ a.e. } a \in (t,\infty), \end{cases}$$

$$[W(t,\omega)\phi](a) = \begin{cases} [U(t,\omega)\phi](a), \text{ a.e. } a \in [0,t] \\ 0, \text{ a.e. } a \in (t,\infty) \end{cases} = \begin{cases} \Pi(a)\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega)), \text{ a.e. } a \in [0,t] \\ 0, \text{ a.e. } a \in (t,\infty). \end{cases}$$

The following lemma verifies $S(t,\omega)$ satisfies the property (3.60) of Proposition 3.13 in [69]. We also need additional assumptions.

(**A20**) $\delta \in C[0,\infty)$;

(**A21**) $\inf_{\sigma\geq 0}\mu(\sigma) = \tilde{\mu} > 0$;

(**A22**) There exists $T_\mu \geq 0$ such that the function μ is strictly increasing after time T_μ, and $\lim_{t\to\infty}\mu(t) = \infty$.

Remark 10. *(A22) is mentioned in [70] in the case where the death rate of the population increases drastically after a certain age.*

Lemma 7. *Under (**A17**), (**A18**) and (**A21**), there exists a function $f : [0,\infty)\times[0,\infty)\to[0,\infty)$ such that for any fixed $r \geq 0$, $\lim_{t\to\infty} f(r,\ t) = 0$ and $\|S(t,\omega)\phi\|_{L^1(0,\infty)} \leq f(r,\ t)$ for any $\|\phi\|_{L^1(0,\infty)} \leq r$, and $t \geq 0$.*

Proof. Let $f(r,t) = re^{-\tilde{\mu}t}$, $\forall r,\ t \geq 0$. For any $\|\phi\|_{L^1(0,\infty)} \leq r$,

$$\begin{aligned} \|S(t,\omega)\phi\|_{L^1(0,\infty)} &= \int_t^\infty \frac{\Pi(a)}{\Pi(a-t)}\phi(a-t)\mathrm{d}a \\ &= \int_0^\infty \frac{\Pi(a+t)}{\Pi(a)}\phi(a)\mathrm{d}a \\ &= \int_0^\infty e^{-\int_a^{a+t}\mu(\sigma)\mathrm{d}\sigma}\phi(a)\mathrm{d}a \\ &\leq e^{-\tilde{\mu}t}\int_0^\infty \phi(a)\mathrm{d}a \\ &\leq e^{-\tilde{\mu}t}\|\phi\|_{L^1(0,\infty)} \leq f(r,\ t). \end{aligned}$$

□

In order to obtain more information of the solution to (4.1), we still need to investigate $W(t,\omega)$.

Lemma 8. *Under assumptions (**A16**) – (**A22**), for any fixed $\omega \in \Omega$ and $t > 0$, $W(t,\omega)$ maps bounded sets of $L^1_+(0,\infty)$ into pre-compact sets in $L^1_+(0,\infty)$.*

Proof. Assume $A \subset L^1_+(0,\infty)$ is bounded. For any $\phi \in A$, we extend $W(t,\omega)\phi$ as

$$[\tilde{W}(t,\omega)\phi](a) = \begin{cases} \Pi(a)\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega)), \text{ a.e. } a \in [0,t] \\ 0, \text{ a.e. } a \in (-\infty,0)\cup(t,\infty). \end{cases}$$

Clearly, $\tilde{W}(t,\omega)A$ is pre-compact in $L^1(\mathbb{R}^1)$ is equivalent to $W(t,\omega)A$ is pre-compact in $L^1_+(0,\infty)$.

Step 1. We prove the $\tilde{W}(t,\omega)A$ is bounded in $L^1_+(0,\infty)$. Throughout this proof, we denote $C := \beta_M \|h\|_{L^\infty(0,\infty)}$.

For any $\phi \in A$,

$$\begin{aligned} \int_{\mathbb{R}^1} |[\tilde{W}(t,\omega)\phi](a)|\mathrm{d}a &= \int_0^t e^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))\mathrm{d}a \\ &\leq C\int_0^t e^{-a\tilde{\mu}}\mathrm{d}a = \frac{C}{\tilde{\mu}} < \infty, \end{aligned}$$

Step 2. Here, we prove for any $\frac{C}{\tilde{\mu}} > \varepsilon > 0$ and $t > 0$, there exists $T_1 > 0$ such that

$$\|\tilde{W}(t,\omega)\phi\|_{L^1(\mathbb{R}^1/[0,T_1])} < \varepsilon.$$

Certainly, we can find $T_1 > 0$, such that for $\forall \phi \in A,\ t \geq T_1$,

$$\begin{aligned} \int_{T_1}^t |[\tilde{W}(t,\omega)\phi](a)|\mathrm{d}a &= \int_{T_1}^t e^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))\mathrm{d}a \\ &\leq C\int_{T_1}^\infty e^{-a\tilde{\mu}}\mathrm{d}a = \frac{C}{\tilde{\mu}}e^{-T_1\tilde{\mu}} < \varepsilon. \end{aligned}$$

In addition, for $0 < t < T_1$, $\|\tilde{W}(t,\omega)\phi\|_{L^1(\mathbb{R}^1/[0,T])} = 0 < \varepsilon$.

Step 3. Finally, we prove for any fixed $t > 0$ and $0 < \tau \leq \min\{t,1\}$

$$\int_{\mathbb{R}^1} |[\tilde{W}(t,\omega)\phi](a+\tau) - [\tilde{W}(t,\omega)\phi](a)|\mathrm{d}a \to 0$$

uniformly for $\phi \in A$.

$$\begin{aligned}
& \int_{\mathbb{R}^1} |[\tilde{W}(t,\omega)\phi](a+\tau) - [\tilde{W}(t,\omega)\phi](a)|\mathrm{d}a \\
= & \int_{-\tau}^{0} |[W(t,\omega)\phi](a+\tau)|\mathrm{d}a + \int_{0}^{t-\tau} |[W(t,\omega)\phi](a+\tau) - [W(t,\omega)\phi](a)|\mathrm{d}a \\
& + \int_{t-\tau}^{t} |[W(t,\omega)\phi](a)|\mathrm{d}a.
\end{aligned}$$

Then, we deal with those three integrals one by one.

$$\begin{aligned}
\int_{-\tau}^{0} |[W(t,\omega)\phi](a+\tau)|\mathrm{d}a &= \int_{-\tau}^{0} e^{-\int_0^{a+\tau}\mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega))\mathrm{d}a \\
&\le C\int_{-\tau}^{0} e^{-(a+\tau)\tilde{\mu}}\mathrm{d}a \\
&= C\int_{0}^{\tau} e^{-a\tilde{\mu}}\mathrm{d}a \to 0
\end{aligned}$$

uniformly for $\phi \in A$ as $\tau \to 0$. Similarly,

$$\begin{aligned}
\int_{t-\tau}^{t} |[W(t,\omega)\phi](a)|\mathrm{d}a &= \int_{t-\tau}^{t} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))\mathrm{d}a \\
&\le C\int_{t-\tau}^{t} e^{-a\tilde{\mu}}\mathrm{d}a \to 0
\end{aligned}$$

uniformly for $\phi \in A$ as $\tau \to 0$.

$$
\begin{aligned}
&\int_0^{t-\tau} |[W(t,\omega)\phi](a+\tau) - [W(t,\omega)\phi](a)|\mathrm{d}a \\
= &\int_0^{t-\tau} |e^{-\int_0^{a+\tau}\mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega)) \\
&-e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))|\mathrm{d}a \\
\le &\int_0^{t-\tau} |e^{-\int_0^{a+\tau}\mu(\sigma)\mathrm{d}\sigma} - e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega))\mathrm{d}a \\
&+\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega)) \\
&-\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))|\mathrm{d}a \\
\le &C\int_0^{\infty} |e^{-\int_0^{a+\tau}\mu(\sigma)\mathrm{d}\sigma} - e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\mathrm{d}a \\
&+\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega)) \\
&-\beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))|\mathrm{d}a \qquad (4.16)
\end{aligned}
$$

It follows from (4.16) that

$$
\begin{aligned}
&\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega)) - \beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))|\mathrm{d}a \\
\le &\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega)h(b_\phi(t-a-\tau,\omega)) - \beta(\theta_{t-a}\omega)h(b_\phi(t-a-\tau,\omega))|\mathrm{d}a \\
&+\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a}\omega)h(b_\phi(t-a-\tau,\omega)) - \beta(\theta_{t-a}\omega)h(b_\phi(t-a,\omega))|\mathrm{d}a \\
\le &\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}|\beta(\theta_{t-a-\tau}\omega) - \beta(\theta_{t-a}\omega)|h(b_\phi(t-a-\tau,\omega))\mathrm{d}a \qquad (4.17) \\
&+\int_0^{t-\tau} e^{-\int_0^{a}\mu(\sigma)\mathrm{d}\sigma}\beta(\theta_{t-a}\omega)|h(b_\phi(t-a-\tau,\omega)) - h(b_\phi(t-a,\omega))|\mathrm{d}a \qquad (4.18)
\end{aligned}
$$

The continuity of $\beta(\theta_t\omega)$ implies (4.17) converges to 0 uniformly for $\phi \in A$ as $\tau \to 0$. Besides, according to the dominate convergence theorem we verify that for each fixed $0 < a < t-\tau$,

$|h(b_\phi(t-a-\tau,\omega))-h(b_\phi(t-a,\omega))| \to 0$ uniformly as $\tau \to 0$ in the following estimation.

$$\begin{aligned}
& |h(b_\phi(t-a-\tau,\omega))-h(b_\phi(t-a,\omega))| \\
\leq\ & \|h\|_{Lip}\left|\int_0^\infty \delta(s+t-a-\tau)e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s \right.\\
& \left. -\int_0^\infty \delta(s+t-a)e^{-\int_s^{s+t-a}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s\right| \qquad (4.19)\\
& +\|h\|_{Lip}\left|\int_0^{t-a-\tau} K(t-a-\tau-s)\beta(\theta_s\omega)h(b_\phi(s,\omega))\mathrm{d}s \right.\\
& \left. -\int_0^{t-a} K(t-a-s)\beta(\theta_s\omega)h(b_\phi(s,\omega))\mathrm{d}s\right| \qquad (4.20)
\end{aligned}$$

On one hand,

$$\begin{aligned}
(4.19)\ \leq\ & \|h\|_{Lip}\int_0^\infty |\delta(s+t-a-\tau)e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}-\delta(s+t-a)e^{-\int_s^{s+t-a}\mu(\sigma)\mathrm{d}\sigma}|\phi(s)\mathrm{d}s\\
\leq\ & \|h\|_{Lip}\int_0^\infty |\delta(s+t-a-\tau)-\delta(s+t-a)|e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s \qquad (4.21)\\
& +\|h\|_{Lip}\int_0^\infty \delta(s+t-a)|e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}-e^{-\int_s^{s+t-a}\mu(\sigma)\mathrm{d}\sigma}|\phi(s)\mathrm{d}s. \qquad (4.22)
\end{aligned}$$

(**A22**) implies that for arbitrary $\varepsilon > 0$, there exists T_2 such that μ is strictly increasing after T_2 and $e^{-(t-a)\mu(T_2)} < \varepsilon$. Therefore,

$$\begin{aligned}
(4.21)\ \leq\ & \|h\|_{Lip}\int_0^{T_2} |\delta(s+t-a-\tau)-\delta(s+t-a)|e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s \qquad (4.23)\\
& +\|h\|_{Lip}\int_{T_2}^\infty |\delta(s+t-a-\tau)-\delta(s+t-a)|e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s \qquad (4.24)
\end{aligned}$$

The uniform continuity of δ, (**A20**), on the finite intervals implies that (4.23) converges to 0 as $\tau \to 0$. Moreover,

$$\begin{aligned}
(4.24)\ \leq\ & 2\|h\|_{Lip}\|\delta\|_{L^\infty}\int_{T_2}^\infty e^{-\int_s^{s+t-a-\tau}\mu(\sigma)\mathrm{d}\sigma}\phi(s)\mathrm{d}s\\
\leq\ & 2\|h\|_{Lip}\|\delta\|_{L^\infty}e^{\tau}e^{-(t-a)\mu(T_2)}\int_{T_2}^\infty \phi(s)\mathrm{d}s.
\end{aligned}$$

The Arbitrariness of ε gives (4.24) $\to 0$ as $\tau \to 0$. With similar process, we can also prove (4.22) $\to 0$ as $\tau \to 0$. Invoking the dominate convergence theorem shows (4.22) $\to 0$ as $\tau \to 0$.

On the other hand,

$$
\begin{aligned}
(4.20) \quad \leq\ & \|h\|_{Lip}\int_0^{t-a-\tau}|K(t-a-\tau-s)-K(t-a-s)|\beta(\theta_s\omega)h(b_\phi(s,\omega))\mathrm{d}s\\
&+\|h\|_{Lip}\int_{t-a-\tau}^{t-a}K(t-a-s)\beta(\theta_s\omega)h(b_\phi(s,\omega))\mathrm{d}s\\
\leq\ & +\|h\|_{Lip}C\int_0^{t-a-\tau}|K(t-a-\tau-s)-K(t-a-s)|\mathrm{d}s\\
&+\|h\|_{Lip}C\int_{t-a-\tau}^{t-a}K(t-a-s)\mathrm{d}s\\
\leq\ & +\|h\|_{Lip}C\int_0^{t-a}\chi_{[\tau,t-a](s)}|K(s-\tau)-K(s)|\mathrm{d}s\\
&+\|h\|_{Lip}C\int_0^{\tau}K(s)\mathrm{d}s.
\end{aligned}
$$

Those two terms converge to 0 uniformly for $\phi\in A$ as τ goes to 0. In conclusion, for any $t>0$, $\tilde{W}(t,\omega)A$ is pre-compact in $L^1(\mathbb{R}^1)$, which is equivalent to $W(t,\omega)A$ is pre-compact in $L^1_+(0,\infty)$. □

In addition, the proof to Lemma 7 and the first step of Lemma 8 provide us that the existence of the absorbing set.

Lemma 9. *Under (**A16**) – (**A19**) and (**A21**) for any uniformly bound set $D\subset L^1_+(0,\infty)$, there exists $T(D)\geq 0$ such that if $t>T(D)$,*

$$U(t,\theta_{-t}\omega)D\subset B:=\{\phi\in L^1_+(0,\infty)|\|\phi\|_{L^1(0,\infty)}\leq 2\frac{\beta_M\|h\|_{L^\infty}}{\tilde{\mu}}\}.$$

Unfortunately, due to the closed and bounded set in $L^1_+(0,\infty)$ may not be compact, we cannot obtain the attractor directly by constructing the omega-limit set of the absorbing set. Nonetheless, the omega-limit set of invariant set defined below will be the attractor.

Let $L=\{L(\omega)\}_{\omega\in\Omega}$, where

$$L(\omega)=\{\phi\in L^1_+(0,\infty)|U(s,\omega)\phi\in B,\forall s\geq 0\},\forall\omega\in\Omega,$$

and denote the $\varepsilon-$neighborhood of a subset A of $L^1_+(0,\infty)$

$$N(\varepsilon,A) := \{\phi \in L^1_+(0,\infty) | \inf_{f\in A} \|\phi - f\|_{L^1} < \varepsilon\}.$$

Lemma 10. *Under assumptions (**A16**) – (**A22**), L*

(1) is uniformly bounded,

(2) is positive invariant,

(3) attracts points of $L^1_+(0,\infty)$.

Proof. (1) is a direct result of Lemma 7 and Lemma 8. In fact, $L(\theta_t\omega) \subset \frac{3}{2}B$, $\forall\, t > 0$. Then, choose an arbitrary $\phi \in L(\omega)$. we have

$$U(s,\theta_t\omega)U(t,\omega)\phi = U(s+t,\omega)\phi \in B, \forall s \geq 0.$$

Thus, $U(t,\omega)L(\omega) \subset L(\theta_t\omega)$. (2) is verified. Finally, we prove (3) by contradiction. Because B is an absorbing set, W.L.O.G., assume there exists a point $\phi_0 \in B$ and $\varepsilon_0 > 0$ such that $U(t_n,\theta_{-t_n}\omega)\phi_0 \notin N(\varepsilon_0,L(w))$, $t_n \to \infty$ as $n \to \infty$, where $N(\varepsilon_0,L(w))$. Let $A_l = \{n \geq l | U(t_n,\theta_{-t_n}\omega)\}$, $l \in \mathbb{N}^+$. We estimate the Kuratowski measure of non-compactness $\overline{A_l}$ as

$$\alpha(\overline{A_l}) = \alpha(\overline{U(l,\theta_{-l}\;\omega)\{U(t_{t_n-l},\theta_{-t_n}\omega)\phi_0 | t_n \geq l\}}) \leq \alpha(U(l,\theta_{-l}\;\omega)\frac{3}{2}B) = \alpha(S(l)\frac{3}{2}B).$$

It follows from Lemma 7 that $\alpha(\overline{A_l})$ converges to 0, as $l \to \infty$. So, the omega-limit set of $\{U(t_n,\theta_{-t_n}\omega)\phi_0\}$ is non-empty and pre-compact. Suppose $U(t_n,\theta_{-t_n}\omega)\phi_0 \to \phi \in B$. We also know that $\phi \notin L(\omega)$. Then, there exist $t_0 > 0$ so that $U(t_0,\omega)\phi \notin B$. Therefore, we can find a integer $N > l$ so that

$$U(t_n+t_0,\theta_{-t_n}\omega)\phi_0 = U(t_0,\omega)U(t_n,\theta_{-t_n}\omega)\phi_0 \notin B,\ \forall n > N,$$

which is contradict to the fact that B is an absorbing set. □

We will follow two steps to prove the omega-limit set of $L(\omega)$ is the attractor. First, the following lemma will show it attracts $L(\omega)$.

Lemma 11. *Under assumptions (**A16**) – (**A22**),* $\Omega_L(\omega)$

(1) is compact,

(2) is invariant,

(3) attracts $L(\omega)$.

Proof. First, we prove that $\Omega_L(\omega)$ is compact. Since $L(\omega)$ is positive invariant, we have for any $T > 0$,

$$\begin{aligned}
\cup_{t\geq T} U(t,\theta_{-t}\omega)L(\theta_{-t}\omega) &= \cup_{t\geq T} U(T,\theta_{-T}\omega)U(t-T,\theta_{-t}\omega)L(\theta_{-t}\omega) \\
&= U(T,\theta_{-T}\omega)\cup_{t\geq T} U(t-T,\theta_{-t}\omega)L(\theta_{-t}\omega) \\
&\subset U(T,\theta_{-T}\omega)L(\theta_{-T}\omega).
\end{aligned}$$

Applying Lemma 7 and Lemma 8 gives, for any $t > 0$,

$$\alpha(\cup_{t\geq T} U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)) \leq \alpha(U(T,\theta_{-T}\omega)L(\theta_{-T}\omega)) = \alpha(S(T)L(\theta_{-T}\omega)) \to 0,$$

as $T \to \infty$. Therefore,

$$\alpha(\cap_{T\geq 0}\overline{\cup_{t\geq T} U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)}) = 0,$$

$\Omega_L(\omega)$ is compact.

Then, we prove (3) by contradiction. If there exists $\varepsilon_0 > 0$ and sequence $\{\phi_n\}$, each $\phi_n \in L(\theta_{-t_n}\omega)$, such that $U(t_n,\theta_{-t_n}\omega)\phi_n \notin N(\varepsilon_0,\, \Omega_L(\omega))$, $0 < t_1 < t_2 < \cdots < t_n \to \infty$ as $n \to \infty$. Denote $A_l = \{U(t_n,\theta_{-t_n}\omega)\phi_n | t_n \geq l\}$. Whereby,

$$\begin{aligned}
\alpha(\overline{A_l}) = \alpha(\overline{U(l,\theta_{-l}\,\omega)\cup_{t_n\geq l} U(t_{t_n-l},\theta_{-t_n}\omega)\phi_n}) \leq \alpha(\overline{U(l,\theta_{-l}\,\omega)L(\theta_{-l}\omega)}) \\
\leq \alpha(U(l,\theta_{-l}\,\omega)B) \leq \alpha(S(l)B) \to 0, \text{ as } l \to \infty.
\end{aligned}$$

So, the omega-limit set of $\{U(t_n,\theta_{-t_n}\omega)\phi_n\}$ is non-empty and pre-compact. Then, W.L.O.G., there exists $y \in L^1_+(0,\infty)$ such that $U(t_n,\theta_{-t_n}\omega)\phi_n \to y$ as $n \to \infty$, which is contradict to the choice of $\{U(t_n,\theta_{-t_n}\omega)\phi_n\}$.

Finally, we verify $U(s,\omega)\Omega_L(\omega) = \Omega_L(\theta_s\omega)$, $\forall s \geq 0$. The positive invariance of $L(\omega)$ implies that

$$
\begin{aligned}
U(s,\omega)\cap_{T\geq 0}\overline{\cup_{t\geq T}U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)} &\subset \cap_{T\geq 0}U(s,\omega)\overline{\cup_{t\geq T}U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)}\\
&\subset \cap_{T\geq 0}\overline{U(s,\omega)\cup_{t\geq T}U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)}\\
&\subset \cap_{T\geq 0}\overline{\cup_{t\geq T}U(s,\omega)U(t,\theta_{-t}\omega)L(\theta_{-t}\omega)}\\
&\subset \cap_{T\geq 0}\overline{\cup_{t\geq T}U(s+t,\theta_{-t}\omega)L(\theta_{-t}\omega)}\\
&= \cap_{T\geq 0}\overline{\cup_{t\geq T}U(s+t,\theta_{-t-s}\theta_s\omega)L(\theta_{-t-s}\theta_s\omega)}\\
&= \cap_{T\geq s}\overline{\cup_{t\geq T}U(t,\theta_{-t}\theta_s\omega)L(\theta_{-t}\theta_s\omega)}\\
&= \Omega_L(\theta_s\omega).
\end{aligned}
$$

In addition, for any $\phi \in \Omega_L(\theta_s\omega)$,

$$
\begin{aligned}
\phi &= \lim_{n\to\infty} U(t_n,\theta_{-t_n}\theta_s\omega)\phi_n\\
&= U(s,\omega)\lim_{n\to\infty} U(t_n - s,\theta_{s-t_n}\omega)\phi_n
\end{aligned}
$$

where each $t_n \geq s$ and $\phi_n \in L(\theta_{s-t_n}\omega)$. Because $\Omega_L(\omega)$ is compact and attracts $L(\omega)$, W.L.O.G., suppose $U(t_n - s,\theta_{s-t_n}\omega)\phi_n \to z \in \Omega_L(\omega)$. Thus, $\phi = U(s,\omega)z$ which implies $\Omega_L(\theta_s\omega) \subset U(s,\omega)\Omega_L(\omega)$. □

Second, combining the fact that $L(\omega)$ attracts all of the bounded sets in $L^1_+(0,\infty)$ results in

Lemma 12. *Under assumptions (A16) – (A22),* $\Omega_L(\omega)$ *attracts any uniformly bounded set of* $L^1_+(0,\infty)$.

Proof. Assume $D = \{D(\omega)\}_{\omega\in\Omega}$ is an uniformly bounded set of $L^1_+(0,\infty)$. Due to Lemma 7 and Lemma 8, there exists $T(D) > 0$ such that

$$U(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B, \ \forall\, t \geq T(D).$$

Furthermore, we have

$$U(s,\omega)U(t,\theta_{-t}\omega)D(\theta_{-t}\omega) = U(s+t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B, \ \forall\, s \geq 0.$$

Thus,

$$U(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset L(\omega), \ \forall\, t \geq T(D).$$

Applying Lemma 11 shows that $\Omega_L(\omega)$ attracts D. □

Finally, we conclude this section with the existence of the random pullback attractor.

Theorem 14. *If the same assumptions (**A16**) – (**A22**) are satisfied, $\Omega_L(\omega)$ is a random pullback attractor for the random dynamical system (θ,U) generated by the solution to* (4.1).

References

[1] H. G. Andrewartha and L. C. Birch. *The distribution and abundance of animals.* University of Chicago Press, Chicago, Ill., USA, 1954.

[2] Jordi Bascompte, Hugh Possingham, and Joan Roughgarden. Patchy populations in stochastic environments: Critical number of patches for persistence. *The American Naturalist*, 159(2):128–137, 2002.

[3] Peter L Chesson. Interactions between environment and competition: how fluctuations mediate coexistence and competitive exclusion. In A. Hastings, editor, *Community ecology*, volume 77, pages 51–71. Springer, Berlin, Heidelberg, 1988.

[4] Miguel A Fortuna, Carola Gómez-Rodríguez, and Jordi Bascompte. Spatial network structure and amphibian persistence in stochastic environments. *Proceedings of the Royal Society B: Biological Sciences*, 273(1592):1429–1434, 2006.

[5] Peter F. Sale. Maintenance of high diversity in coral reef fish communities. *The American Naturalist*, 111(978):337–359, 1977.

[6] Peter F. Sale and William A. Douglas. Temporal variability in the community structure of fish on coral patch reefs and the relation of community structure to reef structure. *Ecology*, 65(2):409–422, 1984.

[7] Shripad Tuljapurkar, Jean-Michel Gaillard, and Tim Coulson. From stochastic environments to life histories and back. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 364(1523):1499–1509, 2009.

[8] Michael Turelli. Does environmental variability limit niche overlap? *Proceedings of the National Academy of Sciences*, 75(10):5085–5089, 1978.

[9] M. Turelli. Stochastic community theory: a partially guided tour. In T. G. Hallam and S. A. Levin, editors, *Mathematical Ecology. Biomathematics*. Springer, Berlin, Heidelberg, 1986.

[10] AJ UNDERWOOD. Paradigms, explanations and generalizations in models for the structure of intertidal communities on rocky shore. *Ecological communities : conceptual issues and the evidence*, pages 151–180, 1984.

[11] Benjamin C. Victor. Larval settlement and juvenile mortality in a recruitment-limited coral reef fish population. *Ecological Monographs*, 56(2):145–160, 1986.

[12] John A. Wiens. On competition and variable environments: Populations may experience ”ecological crunches” in variable climates, nullifying the assumptions of competition theory and limiting the usefulness of short-term studies of population patterns. *American Scientist*, 65(5):590–597, 1977.

[13] Jin Yoshimura and Vincent A. A. Jansen. Evolution and population dynamics in stochastic environments. *Population Ecology*, 38(2):165–182, 1996.

[14] Trang Bui and George Yin. Hybrid competitive Lotka-Volterra ecosystems: countable switching states and two-time-scale models. *Stoch. Anal. Appl.*, 37(2):219–242, 2019.

[15] Tomás Caraballo, Renato Colucci, and Xiaoying Han. Predation with indirect effects in fluctuating environments. *Nonlinear Dynam.*, 84(1):115–126, 2016.

[16] Tomás Caraballo, Renato Colucci, Javier López-de-la Cruz, and Alain Rapaport. A way to model stochastic perturbations in population dynamics models with bounded realizations. *Commun. Nonlinear Sci. Numer. Simul.*, 77:239–257, 2019.

[17] Tomás Caraballo and Xiaoying Han. *Applied nonautonomous and random dynamical systems*. SpringerBriefs in Mathematics. Springer, Cham, 2016. Applied dynamical systems.

[18] Tomás Caraballo, Xiaoying Han, and Peter E. Kloeden. Chemostats with random inputs and wall growth. *Math. Methods Appl. Sci.*, 38(16):3538–3550, 2015.

[19] Alexandru Hening, Ky Quan Tran, Tien Trong Phan, and George Yin. Harvesting of interacting stochastic populations. *J. Math. Biol.*, 79(2):533–570, 2019.

[20] Xuerong Mao. *Stochastic differential equations and applications*. Horwood Publishing Limited, Chichester, second edition, 2008.

[21] Peter E. Kloeden and Martin Rasmussen. *Nonautonomous dynamical systems*, volume 176 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[22] Robert M May. Stability in randomly fluctuating versus deterministic environments. *The American Naturalist*, 107(957):621–650, 1973.

[23] Dang H. Nguyen and George Yin. Coexistence and exclusion of stochastic competitive Lotka-Volterra models. *J. Differential Equations*, 262(3):1192–1225, 2017.

[24] George Yin and Zhexin Wen. Stochastic Kolmogorov systems driven by wideband noises. *Phys. A*, 531:121746, 11, 2019.

[25] C. Zhu and G. Yin. On hybrid competitive lotka-volterra ecosystems. *Nonlinear Analysis: Theory, Methods & Applications*, 71:1370–1379, 2009.

[26] Peter L Chesson and Robert R Warner. Environmental variability promotes coexistence in lottery competitive systems. *American Naturalist*, 117(6):923–943, 1981.

[27] Peter L Chesson. Multispecies competition in variable environments. *Theoretical population biology*, 45(3):227–276, 1994.

[28] Peter L Chesson. The stabilizing effect of a random environment. *Journal of Mathematical Biology*, 15(1):1–36, 1982.

[29] Peter L Chesson. Coexistence of competitors in a stochastic environment: the storage effect, 1983.

[30] Peter L Chesson. The storage effect in stochastic population models, 1984.

[31] Stephen P. Hubbell. *The unified neutral theory of biodiversity and biogeography*. Princeton University Press, 2001.

[32] Peter L Chesson. Coexistence of competitors in spatially and temporally varying environments: a look at the combined effects of different sorts of variability. *Theoretical Population Biology*, 28(3):263–287, 1985.

[33] Jeffrey Scott Hatfield. Diffusion analysis and stationary distribution of the lottery competition model, 1986.

[34] Jeff S Hatfield and Peter L Chesson. Diffusion analysis and stationary distribution of the two-species lottery competition model. *Theoretical Population Biology*, 36(3):251–266, 1989.

[35] J Bertram and J Masel. A lottery model of density-dependent selection in evolutionary genetics. *bioRxiv*, 2017.

[36] Peter L Chesson. Scale transition theory with special reference to species coexistence in a variable environment. *J. Biol. Dyn.*, 3(2-3):149–163, 2009.

[37] Peter L Chesson. Contributions to nonstationary community theory. *J. Biol. Dyn.*, 13(suppl. 1):123–150, 2019.

[38] Sonya Dewi and Peter L Chesson. The age-structured lottery model. *Theoretical Population Biology*, 64(3):331–343, 2003.

[39] A. Bradley Duthie, Karen C. Abbott, John D. Nason, Associate Editor: William F. Morris, and Editor: Judith L. Bronstein. Trade-offs and coexistence: A lottery model applied to fig wasp communities. *The American Naturalist*, 183(6):826–841, 2014.

[40] Shigehide Iwata, Yasuhiro Takeuchi, and Ryusuke Kon. Analysis of a lottery competition model with limited nutrient availability. *Journal of Biological Dynamics*, 1(1):133–156, 2007.

[41] Yun Kang and Hal Smith. Global dynamics of a discrete two-species lottery-ricker competition model. *Journal of Biological Dynamics*, 6(2):358–376, 2012.

[42] F. R. Sharpe and Lotka A. J. A problem in age-distribution. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 21(124):435–438, 1911.

[43] M.E. Gurtin and R.C. Maccamy. Non-linear age-dependent population dynamics. *Arch. Rational Mech. Anal*, 54:281–300, 1974.

[44] Peter L Chesson. Aedt: A new concept for ecological dynamics in the ever-changing world. *PLoS biology*, 15(5):e2002634, 2017.

[45] Peter L Chesson and Patricia J. Yang. Populations as fluid on a landscape under global environmental change. *Front. Ecol. Evol.*, 27, 2019.

[46] Michael A. Litzow, Lorenzo Ciannelli, Patricia Puerta, Justin J. Wettstein, Ryan R. Rykaczewski, and Michael Opiekun. Nonstationary environmental and community relationships in the north pacific ocean. *Ecology*, 100(8):e02760, 2019.

[47] D. W. Strook and S. R. S. Varadhan. *Multidimensional Diffusion Processes*. Springer-Verlag, Berlin Heidelberg, 2006.

[48] Regina C Elandt-Johnson and Norman L Johnson. *Survival models and data analysis*, volume 110. John Wiley & Sons, 1980.

[49] Alan Stuart, Maurice G Kendall, et al. *The advanced theory of statistics*. Edward Arnold, 6 edition, 1998.

[50] Xiaoyue Li and Xuerong Mao. Population dynamical behavior of non-autonomous lotka-volterra competitive system with random perturbation. *Discrete and Continuous Dynamical Systems-Series A*, 24(2):523–593, 2009.

[51] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 6 edition, 2003.

[52] J. Cheng, P. Chesson, and X. Han. The lottery model for ecological competition in non-stationary environments. to appear.

[53] Michel Benaim. Stochastic persistence, 2018.

[54] Iosif I. Gikhman and Anatoly V. Skorokhod. *Introduction to the theory of random processes*. Dover, 1996.

[55] Crispin Gardiner. *Stochastic methods*, volume 4. Springer, Berlin, 2009.

[56] Avner Friedman. *Partial differential equations of parabolic type*. Courier Dover Publications, 2008.

[57] Francis G Dressel. The fundamental solution of the parabolic equation, ii,. *Duke Mathematical Journal*, 13:61–70, 1946.

[58] Vladimir I Bogachev, Nicolai V Krylov, Michael Röckner, and Stanislav V Shaposhnikov. *Fokker-Planck-Kolmogorov Equations*, volume 207. American Mathematical Soc., 2015.

[59] P. Chesson. Contributions to nonstationary community theory. *Journal of Biological Dynamics*, 13(sup1):123–150, 2019.

[60] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Process*. North-Holland Publishing Company, 2 edition, 1989.

[61] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Applications*. Woodhead Publishing, 2 edition, 2007.

[62] W. M. Wonham. Liapunov criteria for weak stochastic stability. *Journal of Differential Equations*, 2:195–207, 1966.

[63] R. Z. Khasminskii. Ergodic properties of recurrent diffusion processes and stabilization of the solution to the cauchy problem for parabolic equations. *Theory of Probability and Its Applications*, 5:179–196, 1960.

[64] Ludwig Arnold. *Random Dynamical Systems*. Springer-Verlag Berlin Heidelberg, 1998.

[65] Pierre Magal and Shigui Ruan. *Theory and Applications of Abstract Semilinear Cauchy Problems*. Springer, Cham, 2018.

[66] W. E. Ricker. Computation and interpretation of biological statistics of fish populations. In *Bulletin of the Fisheries Research Board of Canda*, volume 191. Environment Canda, Ottawa, 1975.

[67] Brunner Hermann. *Volterra integral Equations: an introduction to theory and applications*. Cambridge University Press, 2017.

[68] Jack K. Hale. *Asymptotic Behavior of Dissipative Systems*. American Mathematical Society, 2009.

[69] Glenn F. Webb. *Theory of Nonlinear Age-Dependent Population Dynamics*. Marcel Dekker, Inc, 1985.

[70] Mimmo Iannelli. Mathematical theory of age-structured population dynamics. *Giardini editori e stampatori in Pisa*, 1995.

Appendix A

Derivations of (3.11) and (3.12)

Throughout the derivations, terms of the following form are considered as higher order terms which can be neglected as $h \to 0$

$$\mathbb{E}[(\rho_i^h(t))^{k_1}(\gamma_j^h(t))^{k_2}] = \mathbf{o}(h) \quad \text{for } k_1+k_2 \geq 3, \quad i,j=1,\cdots,N.$$

Recall the definition of the drift and diffusion coefficients (3.4) and (3.5). We should begin with the difference

$$\begin{aligned}
X_i(t+h)-X_i(t) &= -\nu_i X_i(t) + \frac{\beta_i X_i(t)}{\sum_{n=1}^{N}\beta_n X_n}\sum_{n=1}^{N}\nu_n X_n(t) \\
&= \frac{1}{\sum_{n=1}^{N}\beta_n X_n}[\sum_{n=1}^{N}\nu_n X_n \beta_i X_i - \beta_n X_n \nu_i X_i] \\
&= \frac{(\sum_{n=1}^{N-1}\nu_n X_n \beta_i X_i - \beta_n X_n \nu_i X_i) + (\nu_N \beta_i X_i - \beta_N \nu_i X_i)(1-\sum_{n=1}^{N-1}X_n)}{(\sum_{n=1}^{N-1}\beta_n X_n) + \beta_N(1-\sum_{n=1}^{N-1}X_n)} \\
&= \frac{\left(\sum_{n=1}^{N-1}(\nu_n\beta_i - \beta_n\nu_i - \nu_N\beta_i + \beta_N\nu_i)X_iX_n\right) + (\nu_N\beta_i X_i - \beta_N\nu_i X_i)}{\left(\sum_{n=1}^{N-1}(\beta_n-\beta_N)X_n\right) + \beta_N} \\
&= \frac{\left(\sum_{n=1}^{N-1}(\frac{\nu_n\beta_i}{\nu_i\beta_N} - \frac{\beta_n\nu_i}{\nu_i\beta_N} - \frac{\nu_N\beta_i}{\nu_i\beta_N} + \frac{\beta_N\nu_i}{\nu_i\beta_N})X_iX_n\right) + (\frac{\nu_N\beta_i}{\nu_i\beta_N}X_i - \frac{\beta_N\nu_i}{\nu_i\beta_N}X_i)}{\left[\left(\sum_{n=1}^{N-1}(\beta_n-\beta_N)X_n\right)+\beta_N\right]/\nu_i\beta_N} \\
&= \frac{X_i\nu_i\left[\left(\sum_{n=1}^{N-1}(\frac{\nu_N\beta_i}{\nu_i\beta_N}\frac{\nu_n}{\nu_N} - \frac{\beta_n\nu_N}{\nu_n\beta_N}\frac{\nu_n}{\nu_N} - \frac{\nu_N\beta_i}{\nu_i\beta_N} + \frac{\beta_N\nu_i}{\nu_i\beta_N})X_n\right) + (\frac{\nu_N\beta_i}{\nu_i\beta_N}-1)\right]}{\left(\sum_{n=1}^{N-1}(\frac{\beta_n\nu_N}{\beta_N\nu_n}\frac{\nu_n}{\nu_N}-1)X_n\right)+1} \\
&= \frac{R_i^h(t)}{S^h(t)},
\end{aligned}$$

where,

$$
\begin{aligned}
R_i^h(t) &= X_i\nu_i\left[\left(\sum_{n=1}^{N-1}(\frac{\nu_N\beta_i}{\nu_i\beta_N}\frac{\nu_n}{\nu_N}-\frac{\beta_n\nu_N}{\nu_n\beta_N}\frac{\nu_n}{\nu_N}-\frac{\nu_N\beta_i}{\nu_i\beta_N}+\frac{\beta_N\nu_i}{\nu_i\beta_N})X_n\right)+(\frac{\nu_N\beta_i}{\nu_i\beta_N}-1)\right],\\
&= X_i\nu_i\left[\left(\sum_{n=1}^{N-1}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1)X_n\right)+(e^{\rho_i^h}-1)\right]\\
S^h(t) &= \left(\sum_{n=1}^{N-1}(\frac{\beta_n\nu_N}{\beta_N\nu_n}\frac{\nu_n}{\nu_N}-1)X_n\right)+1\\
&= \left(\sum_{n=1}^{N-1}(e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-1)X_n\right)+1.
\end{aligned}
$$

According to formula (3.10), we compute $\mathbb{E}[R_i^h(t)]$, $\mathbb{E}[S^h(t)]$, $\mathbb{E}[R_i^h(t)S^h(t)]$, $\mathbb{E}[(R_i^h(t))^2]$ and $\mathbb{E}[(S^h(t))^2]$ as follows. Applying Taylor expansions for the exponential functions yields

$$
\begin{aligned}
\mathbb{E}[R_i^h(t)] &= \mathbb{E}\left(X_i\nu_i\left[\left(\sum_{n=1}^{N-1}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1)X_n\right)+(e^{\rho_i^h}-1)\right]\right)\\
&= \mathbb{E}\left(d_iX_ie^{\gamma_i^h}\left[\left(\sum_{n=1}^{N-1}\left((e^{\rho_i^h}-e^{\rho_n^h})\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1\right)X_n\right)+(e^{\rho_i^h}-1)\right]\right)\\
&= \mathbb{E}\left(d_iX_i(1+\gamma_i^h+\frac{(\gamma_i^h)^2}{2})\left[\left(\sum_{n=1}^{N-1}\left(\frac{d_n}{d_N}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2})\right.\right.\right.\right.\\
&\quad \left.\left.\left.\left.\cdot(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2})-\ \rho_i^h-\frac{(\rho_i^h)^2}{2}\right)X_n\right)+(\rho_i^h+\frac{(\rho_i^h)^2}{2})\right]\right)+\mathbf{o}(h)\\
&= \mathbb{E}\left(X_id_i\left[\left(\sum_{n=1}^{N-1}\left(\frac{d_n}{d_N}\left(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2}+(\gamma_n^h-\gamma_N^h)(\rho_i^h-\rho_n^h)\right)\right.\right.\right.\right.\\
&\quad \left.\left.\left.\left.-\ \rho_i^h-\frac{(\rho_i^h)^2}{2}\right)X_n\right)+(\rho_i^h+\frac{(\rho_i^h)^2}{2})\right]\right)\\
&\quad +\mathbb{E}\left(X_id_i\left[\left(\sum_{n=1}^{N-1}\left(\frac{d_n}{d_N}\gamma_i^h\left(\rho_i^h-\rho_n^h\right)-\gamma_i^h\rho_i^h\right)X_n\right)+\gamma_i^h\rho_i^h\right]\right)+\mathbf{o}(h)\\
&= X_id_i\left[\left(\sum_{n=1}^{N-1}\left(\frac{d_n}{d_N}\left(h\mu_i-h\mu_n+\frac{h}{2}\sigma_i^2-\frac{h}{2}\sigma_n^2+h\theta_{ni}-h\theta_{nn}-h\theta_{Ni}\right.\right.\right.\right.\\
&\quad \left.\left.\left.\left.+h\theta_{Nn}+h\theta_{ii}-h\theta_{in}\right)-h\mu_i-\frac{h}{2}\sigma_i^2-h\theta_{ii}\right)X_n\right)+\left(h\mu_i+\frac{h}{2}\sigma_i^2+h\theta_{ii}\right)\right]+\mathbf{o}(h)\\
&= hX_id_i\left[\left(\sum_{n=1}^{N-1}\left(\frac{d_n}{d_N}\left(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in}\right)\right.\right.\right.\\
&\quad \left.\left.\left.-\ \mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\right)X_n\right)+\left(\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\right)\right]+\mathbf{o}(h).
\end{aligned}
$$

Inserting $x_N = 1 - \sum_{n=1}^{N-1} x_n$ into above equation shows

$$
\begin{aligned}
\mathbb{E}[R_i^h(t)] &= x_i d_i \left\{ \sum_{n=1}^{N-1} x_n d_n \mathbb{E}\left[(1+\gamma_i+\gamma_n-\gamma_N)\big(\rho_i-\rho_n+\frac{(\rho_i-\rho_n)^2}{2}\big) \right] \right. \\
&\quad \left. + d_N x_N \, \mathbb{E}\left[(1+\gamma_i+\frac{1}{2}\gamma_i^2)(\rho_i+\frac{1}{2}\rho_i^2) \right] \right\} + \mathbf{o}(h) \\
&= h x_i d_i \left\{ \sum_{n=1}^{N-1} x_n d_n \Big(\mu_i - \mu_n + \frac{1}{2}\sigma_i^2 - \frac{1}{2}\sigma_n^2 + \Theta_1 \Big) \right. \\
&\quad \left. + d_N x_N \left(\mu_i + \frac{1}{2}\sigma_i^2 + \theta_{ii} \right) \right\} + \mathbf{o}(h),
\end{aligned}
\tag{A.1}
$$

with $\Theta_1 = \theta_{ii} - \theta_{in} + \theta_{ni} - \theta_{nn} - \theta_{Ni} + \theta_{Nn}$. Then, we compute the expectation of $S^h(t)$.

$$
\begin{aligned}
\mathrm{E}[S^h(t)] &= \mathbb{E}\left(\sum_{n=1}^{N-1} (e^{\rho_n^h} \frac{d_n}{d_N} e^{\gamma_n^h - \gamma_N^h} - 1) X_n \right) + 1 \\
&= \mathbb{E}\left(\sum_{n=1}^{N-1} \left(\frac{d_n}{d_N}(1+\rho_n^h+\frac{(\rho_n^h)^2}{2})(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2}) - 1 \right) X_n \right) + 1 + \mathbf{o}(h) \\
&= \mathbb{E}\left(\sum_{n=1}^{N-1} \left(\frac{d_n}{d_N}(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2}+\rho_n^h+\rho_n^h\gamma_n^h-\rho_n^h\gamma_N^h+\frac{(\rho_n^h)^2}{2}) - 1 \right) X_n \right) \\
&\quad + 1 + \mathbf{o}(h) \\
&= 1 + \sum_{n=1}^{N-1} \left(\frac{d_n}{d_N}(1+\frac{h}{2}\alpha_n^2+\frac{h}{2}\alpha_N^2+h\mu_n+h\theta_{nn}-h\theta_{Nn}+\frac{h}{2}\sigma_n^2) - 1 \right) X_n + \mathbf{o}(h).
\end{aligned}
$$

Moreover, following similar technical calculations as above it can be derived that

$$
\lim_{h\to 0} \mathbb{E}[(S^h(t))^k] = \left(\sum_{n=1}^{N} x_n d_n \right)^k + \mathbf{o}(h), \; k = 2,3,4,\cdots. \tag{A.2}
$$

Inserting $x_N = 1 - \sum_{n=1}^{N-1} x_n$ into above equation gives

$$
\begin{aligned}
\mathbb{E}[S^h(t)] &= d_N x_N + \sum_{n=1}^{N-1} x_n d_n \mathbb{E}\left[1+\rho_n+\gamma_n-\gamma_N+\frac{1}{2}(\rho_n+\gamma_n-\gamma_N)^2 \right] + \mathbf{o}(h) \\
&= \sum_{n=1}^{N-1} x_n d_n \left(1 + h\big(\mu_n+\frac{1}{2}\sigma_n^2+\frac{1}{2}\alpha_n^2+\frac{1}{2}\alpha_N^2+\theta_{nn}-\theta_{nN}\big) \right) + d_N x_N + \mathbf{o}(h).
\end{aligned}
\tag{A.3}
$$

Next, we calculate $\mathbb{E}[R_i^h(t)S^h(t)]$.

$$\begin{aligned}
R_i^h(t)S^h(t) &= X_i d_i \left[\left(\sum_{n=1}^{N-1} e^{\gamma_i^h}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_i^h}+1)X_n\right) + e^{\gamma_i^h}(e^{\rho_i^h}-1)\right] \\
&\quad \cdot \left[\left(\sum_{m=1}^{N-1}(e^{\rho_n^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)X_m\right)+1\right] \\
&= X_i d_i \left[\left(\sum_{n=1}^{N-1} e^{\gamma_i^h}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_i^h}+1)X_n\right)\right. \\
&\quad \cdot \left(\sum_{m=1}^{N-1}(e^{\rho_n^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)X_m\right) \text{①} \\
&\quad + \left(\sum_{n=1}^{N-1} e^{\gamma_i^h}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_i^h}+1)X_n\right) \text{②} \\
&\quad \left. + e^{\gamma_i^h}(e^{\rho_i^h}-1)\left(\sum_{m=1}^{N-1}(e^{\rho_m^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)X_m\right) + e^{\gamma_i^h}(e^{\rho_i^h}-1) \text{③}\right],
\end{aligned}$$

where

$$\begin{aligned}
\text{①} &= X_i d_i \left(\sum_{n=1}^{N-1} e^{\gamma_i^h}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_i^h}+1)X_n\right)\left(\sum_{m=1}^{N-1}(e^{\rho_m^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)X_m\right) \\
&= X_i d_i \left[\sum_{n,m=1}^{N-1} X_n X_m e^{\gamma_i^h}(\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}(e^{\rho_i^h}-e^{\rho_n^h}) - e^{\rho_i^h}+1)(e^{\rho_m^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)\right] \\
&= X_i d_i \left[\sum_{n,m=1}^{N-1} X_n X_m (1+\gamma_i^h+\frac{(\gamma_i^h)^2}{2})\left(\frac{d_n}{d_N}(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2})(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}\right.\right. \\
&\quad \left.\left. -\frac{(\rho_n^h)^2}{2}) - \rho_i^h - \frac{(\rho_i^h)^2}{2}\right)\left(\frac{d_m}{d_N}(1+\rho_m^h+\frac{(\rho_m^h)^2}{2})(1+\gamma_m^h-\gamma_N^h+\frac{(\gamma_m^h-\gamma_N^h)^2}{2})-1\right)\right] + \mathbf{o}(h) \\
&= X_i d_i \sum_{n,m=1}^{N-1} X_n X_m \left[\left(\frac{d_n}{d_N}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2}\right.\right. \\
&\quad \left. +\gamma_n^h\rho_i^h-\gamma_n^h\rho_n^h-\gamma_N^h\rho_i^h+\gamma_N^h\rho_n^h+\gamma_i^h\rho_i^h-\gamma_i^h\rho_n^h) - \rho_i^h - \frac{(\rho_i^h)^2}{2} - \gamma_i^h\rho_i^h\right)(\frac{d_m}{d_N}-1) \\
&\quad + \left(\frac{d_n}{d_N}\frac{d_m}{d_N}\rho_m^h(\rho_i^h-\rho_n^h) + \frac{d_n}{d_N}\frac{d_m}{d_N}(\gamma_m^h-\gamma_N^h)(\rho_i^h-\rho_n^h)\right) \\
&\quad \left. - \left(\frac{d_m}{d_N}\rho_m^h\rho_i^h + \frac{d_m}{d_N}(\gamma_m^h-\gamma_N^h)\rho_i^h\right)\right] + \mathbf{o}(h),
\end{aligned}$$

$$
\begin{aligned}
\textcircled{2} &= \sum_{n=1}^{N-1} e^{\gamma_i^h}(e^{\rho_i^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_n^h}\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h} - e^{\rho_i^h}+1)X_n \\
&= \sum_{n=1}^{N-1} e^{\gamma_i^h}(\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}(e^{\rho_i^h}-e^{\rho_n^h}) - e^{\rho_i^h}+1)X_n \\
&= \sum_{n=1}^{N-1}(1+\gamma_i^h+\frac{(\gamma_i^h)^2}{2})\left(\frac{d_n}{d_N}(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2})(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2})\right. \\
&\quad \left. -\rho_i^h-\frac{(\rho_i^h)^2}{2}\right)X_n+\mathbf{o}(h) \\
&= \sum_{n=1}^{N-1}X_n\left(\frac{d_n}{d_N}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2}+\gamma_n^h\rho_i^h-\gamma_n^h\rho_n^h-\gamma_N^h\rho_i^h+\gamma_N^h\rho_n^h+\gamma_i^h\rho_i^h-\gamma_i^h\rho_n^h)\right. \\
&\quad \left. -\rho_i^h-\frac{(\rho_i^h)^2}{2}-\gamma_i^h\rho_i^h\right)+\mathbf{o}(h),
\end{aligned}
$$

and

$$
\begin{aligned}
\textcircled{3} &= e^{\gamma_i^h}(e^{\rho_i^h}-1)\left(\sum_{m=1}^{N-1}(e^{\rho_m^h}\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-1)X_m\right)+e^{\gamma_i^h}(e^{\rho_i^h}-1) \\
&= \rho_i^h+\frac{(\rho_i^h)^2}{2}+\gamma_i^h\rho_i^h+(1+\gamma_i^h+\frac{(\gamma_i^h)^2}{2})(\rho_i^h+\frac{(\rho_i^h)^2}{2})\sum_{m=1}^{N-1}X_m\left(\frac{d_m}{d_N}(1+\rho_m^h+\frac{(\rho_m^h)^2}{2})\right. \\
&\quad \left. \cdot(1+\gamma_m^h-\gamma_N^h+\frac{(\gamma_m^h-\gamma_N^h)^2}{2})-1\right)+\mathbf{o}(h) \\
&= \rho_i^h+\frac{(\rho_i^h)^2}{2}+\gamma_i^h\rho_i^h+\sum_{m=1}^{N-1}X_m\left(\frac{d_m}{d_N}(\rho_i^h+\rho_i^h\rho_m^h+\gamma_m^h\rho_i^h-\gamma_N^h\rho_i^h+\frac{(\rho_i^h)^2}{2}+\gamma_i^h\rho_i^h)\right. \\
&\quad \left. -\rho_i^h-\frac{(\rho_i^h)^2}{2}-\gamma_i^h\rho_i^h\right)+\mathbf{o}(h).
\end{aligned}
$$

Then, we calculate the expectations of $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$ respectively.

$$
\begin{aligned}
\mathbb{E}\left[\textcircled{1}\right] &= X_i d_i \sum_{n,m=1}^{N-1} X_n X_m \left[(\frac{d_m}{d_N}-1)\left(\frac{d_n}{d_N}(h\mu_i-h\mu_n+\frac{h}{2}\sigma_i^2-\frac{h}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}\right.\right. \\
&\quad \left. +\theta_{ii}-\theta_{in})-h\mu_i-\frac{h}{2}\sigma_i^2-\theta_{ii}\right) \\
&\quad \left. +\frac{d_n d_m}{d_N^2}(\sigma_{mi}-\sigma_{mn}+\theta_{mi}-\theta_{mn}-\theta_{Ni}+\theta_{Nn})-\frac{d_m}{d_N}(\sigma_{mi}+\theta_{mi}-\theta_{Ni})\right]+\mathbf{o}(h).
\end{aligned}
$$

$$\begin{aligned}\mathbb{E}\left[\textcircled{2}\right] &= \sum_{n=1}^{N-1} X_n\left(\frac{d_n}{d_N}(h\mu_i - h\mu_n + h\frac{h}{2}\sigma_i^2 - \frac{h}{2}\sigma_n^2 + \theta_{ni} - \theta_{nn} - \theta_{Ni} + \theta_{Nn} + \theta_{ii} - \theta_{in})\right.\\ &\quad \left. -h\mu_i - \frac{h}{2}\sigma_i^2 - \theta_{ii}\right) + \mathbf{o}(h).\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\textcircled{3}\right] &= h\mu_i + \frac{h}{2}\sigma_i^2 + \theta_{ii} + \sum_{m=1}^{N-1} X_m\left(\frac{d_m}{d_N}(h\mu_i + \frac{h}{2}\sigma_i^2 + h\sigma_{im} + h\theta_{mi} - h\theta_{Ni} + h\theta_{ii})\right.\\ &\quad \left. -h\mu_i - \frac{h}{2}\sigma_i^2 - h\theta_{ii}\right)\\ &= h\mu_i + \frac{h}{2}\sigma_i^2 + \theta_{ii} + \sum_{n=1}^{N-1} X_n\left(\frac{d_n}{d_N}(h\mu_i + \frac{h}{2}\sigma_i^2 + h\sigma_{in} + h\theta_{ni} - h\theta_{Ni} + h\theta_{ii})\right.\\ &\quad \left. -h\mu_i - \frac{h}{2}\sigma_i^2 - h\theta_{ii}\right) + \mathbf{o}(h).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[R_i^h(t)S^h(t)] &= \mathbb{E}\left[\textcircled{1}\right] + \mathbb{E}\left[\textcircled{2}\right] + \mathbb{E}\left[\textcircled{3}\right]\\ &= hX_i d_i\left\{\sum_{n,m=1}^{N-1} X_n X_m\left[(\frac{d_m}{d_N} - 1)\left(\frac{d_n}{d_N}(\mu_i - \mu_n + \frac{1}{2}\sigma_i^2 - \frac{1}{2}\sigma_n^2 + \theta_{ni} - \theta_{nn} - \theta_{Ni}\right.\right.\right.\\ &\quad \left. +\theta_{Nn} + \theta_{ii} - \theta_{in}) - h\mu_i - \frac{h}{2}\sigma_i^2 - \theta_{ii}\right)\\ &\quad \left. + \frac{d_n d_m}{d_N^2}(\sigma_{mi} - \sigma_{mn} + \theta_{mi} - \theta_{mn} - \theta_{Ni} + \theta_{Nn}) - \frac{d_m}{d_N}(\sigma_{mi} + \theta_{mi} - \theta_{Ni})\right]\\ &\quad + \sum_{n=1}^{N-1} X_n(\frac{d_n}{d_N}(2\mu_i - \mu_n + \sigma_i^2 - \frac{1}{2}\sigma_n^2 + \sigma_{in} + 2\theta_{ni} - \theta_{nn} - 2\theta_{Ni} + \theta_{Nn} + 2\theta_{ii}\\ &\quad -\theta_{in}) - 2\mu_i - \sigma_i^2 - 2\theta_{ii})\\ &\quad \left. + \mu_i + \frac{1}{2}\sigma_i^2 + \theta_{ii}\right\} + \mathbf{o}(h).\end{aligned}$$

In addition,

$$
\begin{aligned}
\lim_{h\to 0}\frac{\mathbb{E}[R_i^h(t)]\mathbb{E}[S^h(t)]}{h} &= X_i d_i \Bigg[\Bigg(\sum_{n=1}^{N-1}\Bigg(\frac{d_n}{d_N}\Bigg(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn} \\
&\quad -\ \mu_i+\theta_{ii}-\theta_{in})-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg)X_n\Bigg)+\Bigg(\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\Bigg)\Bigg] \\
&\quad \cdot\Bigg(1+\sum_{m=1}^{N-1}(\frac{d_m}{d_N}-1)X_m\Bigg) \\
&= X_i d_i\Bigg\{\sum_{n,m=1}^{N-1}X_nX_m(\frac{d_m}{d_N}-1)\Bigg(\frac{d_n}{d_N}(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2 \\
&\quad +\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in})-\mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg) \\
&\quad +\sum_{n=1}^{N-1}X_n(\frac{d_n}{d_N}-1)(\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}) \\
&\quad +\sum_{n=1}^{N-1}X_n\Bigg(\frac{d_n}{d_N}(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in}) \\
&\quad -\ \mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg) \\
&\quad +\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\Bigg\} \\
&= X_i d_i\Bigg\{\sum_{n,m=1}^{N-1}X_nX_m(\frac{d_m}{d_N}-1)\Bigg(\frac{d_n}{d_N}(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2 \\
&\quad +\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in})-\mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg) \\
&\quad +\sum_{n=1}^{N-1}X_n\Bigg(\frac{d_n}{d_N}(2\mu_i-\mu_n+\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+2\theta_{ii}-\theta_{in}) \\
&\quad -\ 2\mu_i-\sigma_i^2-2\theta_{ii}) \\
&\quad +\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\Bigg\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_i &= \lim_{h\to 0}\frac{2\mathbb{E}[R_i^h(t)]\mathbb{E}[S^h(t)]-\mathbb{E}[R_i^h(t)S^h(t)]}{h}\\
&= X_id_i\Bigg\{\sum_{n,m=1}^{N-1}X_nX_m(\frac{d_m}{d_N}-1)\Bigg(\frac{d_n}{d_N}(2\mu_i-2\mu_n+\sigma_i^2-\sigma_n^2\\
&\quad +2\theta_{ni}-2\theta_{nn}-2\theta_{Ni}+2\theta_{Nn}+2\theta_{ii}-2\theta_{in})-2\mu_i-\sigma_i^2-2\theta_{ii}\Big)\\
&\quad +\sum_{n=1}^{N-1}X_n\Bigg(\frac{d_n}{d_N}(4\mu_i-2\mu_n+2\sigma_i^2-\sigma_n^2+2\theta_{ni}-2\theta_{nn}-2\theta_{Ni}+2\theta_{Nn}+4\theta_{ii}-2\theta_{in})\\
&\quad -\ 4\mu_i-2\sigma_i^2-4\theta_{ii}\Big)\\
&\quad +2\mu_i+\sigma_i^2+2\theta_{ii}\Big\}\\
&\quad -X_id_i\Bigg\{\sum_{n,m=1}^{N-1}X_nX_m\Bigg[(\frac{d_m}{d_N}-1)\Bigg(\frac{d_n}{d_N}(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in})\\
&\quad -\mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg)+\frac{d_nd_m}{d_N^2}(\sigma_{mi}-\sigma_{mn}+\theta_{mi}-\theta_{mn}-\theta_{Ni}+\theta_{Nn})-\frac{d_m}{d_N}(\sigma_{mi}+\theta_{mi}-\theta_{Ni})\Bigg]\\
&\quad +\sum_{n=1}^{N-1}X_n(\frac{d_n}{d_N}(2\mu_i-\mu_n+\sigma_i^2-\frac{1}{2}\sigma_n^2+\sigma_{in}+2\theta_{ni}-\theta_{nn}-2\theta_{Ni}+\theta_{Nn}+2\theta_{ii}-\theta_{in})\\
&\quad -2\mu_i-\sigma_i^2-2\theta_{ii})\\
&\quad +\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\Bigg\}\\
&= X_id_i\Bigg\{\sum_{n,m=1}^{N-1}X_nX_m\Bigg[(\frac{d_m}{d_N}-1)\Bigg(\frac{d_n}{d_N}(\mu_i-\mu_n+\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_n^2+\theta_{ni}-\theta_{nn}-\theta_{Ni}+\theta_{Nn}+\theta_{ii}-\theta_{in})\\
&\quad -\mu_i-\frac{1}{2}\sigma_i^2-\theta_{ii}\Bigg)-\frac{d_nd_m}{d_N^2}(\sigma_{mi}-\sigma_{mn}+\theta_{mi}-\theta_{mn}-\theta_{Ni}+\theta_{Nn})+\frac{d_m}{d_N}(\sigma_{mi}+\theta_{mi}-\theta_{Ni})\Bigg]\\
&\quad +\sum_{n=1}^{N-1}X_n(\frac{d_n}{d_N}(2\mu_i-\mu_n+\sigma_i^2-\frac{1}{2}\sigma_n^2-\sigma_{in}-\theta_{nn}+\theta_{Nn}+2\theta_{ii}-\theta_{in})-2\mu_i-\sigma_i^2-2\theta_{ii})\\
&\quad +\mu_i+\frac{1}{2}\sigma_i^2+\theta_{ii}\Bigg\}\\
&= x_id_i\Big(\mu_i+\frac{1}{2}\sigma_i^2-\frac{\sum_{n=1}^{N-1}d_nx_n(\mu_n+\frac{1}{2}\sigma_n^2)}{\sum_{n=1}^{N}d_nx_n}\\
&\qquad +\frac{\sum_{n,m=1}^{N-1}d_nd_mx_nx_mC_{inm}+\sum_{n=1}^{N-1}d_nx_nD_{in}+d_N^2x_N^2\theta_{ii}}{(\sum_{n=1}^{N}d_nx_n)^2}\Big),
\end{aligned}
$$

where $x_N = 1 - \sum_{n=1}^{N-1} x_n$ and

$$
\begin{aligned}
C_{inm} &= \theta_{ii} - \theta_{in} + \theta_{ni} - \theta_{nn} - \theta_{mi} + \theta_{mn} - \sigma_{mi} + \sigma_{mn}, \\
D_{in} &= 2\theta_{ii} - \theta_{in} - \theta_{nn} + \theta_{Nn} - \sigma_{ni}.
\end{aligned}
$$

Our next job is to calculate the diffusion coefficients $\mathfrak{a}_{ij}$ with the help of (3.10) and (A.2). We just need to compute $\mathbb{E}[(R_i^h(t))^2]$ and $\mathbb{E}[R_i^h(t)R_j^h(t)]$.

$$
\begin{aligned}
(R_i^h(t))^2 &= d_i^2 X_i^2 e^{2\gamma_i^h} \left[\left(\sum_{n=1}^{N-1} \left((e^{\rho_i^h} - e^{\rho_n^h}) \frac{d_n}{d_N} e^{\gamma_n^h - \gamma_N^h} - e^{\rho_i^h} + 1 \right) X_n \right) + (e^{\rho_i^h} - 1) \right] \\
&\quad \cdot \left[\left(\sum_{m=1}^{N-1} \left((e^{\rho_i^h} - e^{\rho_m^h}) \frac{d_m}{d_N} e^{\gamma_m^h - \gamma_N^h} - e^{\rho_i^h} + 1 \right) X_n \right) + (e^{\rho_i^h} - 1) \right] \\
&= d_i^2 X_i^2 e^{2\gamma_i^h} \left[\sum_{n,m=1}^{N-1} X_m X_n \left((e^{\rho_i^h} - e^{\rho_n^h}) \frac{d_n}{d_N} e^{\gamma_n^h - \gamma_N^h} - e^{\rho_i^h} + 1 \right) \right. \\
&\quad \cdot \left((e^{\rho_i^h} - e^{\rho_m^h}) \frac{d_m}{d_N} e^{\gamma_m^h - \gamma_N^h} - e^{\rho_i^h} + 1 \right) \\
&\quad + 2 \sum_{n=1}^{N-1} X_n \left((e^{\rho_i^h} - e^{\rho_n^h}) \frac{d_n}{d_N} e^{\gamma_n^h - \gamma_N^h} - e^{\rho_i^h} + 1 \right) (e^{\rho_i^h} - 1) \\
&\quad \left. + (e^{\rho_i^h} - 1)^2 \right] + \mathbf{o}(h) \\
&= d_i^2 X_i^2 \left\{ \sum_{n,m=1}^{N-1} X_m X_n (1 + 2\gamma_i^h + 2(\gamma_i^h)^2) \left[\frac{d_n d_m}{d_N^2} (\rho_i^h - \rho_n^h + \frac{(\rho_i^h)^2}{2} - \frac{(\rho_n^h)^2}{2}) \right. \right. \\
&\quad \cdot (\rho_i^h - \rho_m^h + \frac{(\rho_i^h)^2}{2} - \frac{(\rho_m^h)^2}{2})(1 + \gamma_n^h - \gamma_N^h + \gamma_m^h - \gamma_N^h + \frac{(\gamma_n^h - \gamma_N^h + \gamma_m^h - \gamma_N^h)^2}{2}) \\
&\quad - 2 \frac{d_n}{d_N} (\rho_i^h - \rho_n^h + \frac{(\rho_i^h)^2}{2} - \frac{(\rho_n^h)^2}{2})(1 + \gamma_n^h - \gamma_N^h + \frac{(\gamma_n^h - \gamma_N^h)^2}{2}) \\
&\quad \left. (\rho_i^h + \frac{(\rho_i^h)^2}{2}) + (\rho_i^h + \frac{(\rho_i^h)^2}{2})^2 \right] \\
&\quad + 2 \sum_{n=1}^{N-1} X_n \left(\frac{d_n}{d_N} (\rho_i^h - \rho_n^h + \frac{(\rho_i^h)^2}{2} - \frac{(\rho_n^h)^2}{2})(1 + \gamma_n^h - \gamma_N^h + \frac{(\gamma_n^h - \gamma_N^h)^2}{2}) \right. \\
&\quad \left. \left. - \rho_i^h - \frac{(\rho_i^h)^2}{2} \right) \cdot (\rho_i^h + \frac{(\rho_i^h)^2}{2}) + (\rho_i^h + \frac{(\rho_i^h)^2}{2})^2 \right\} + \mathbf{o}(h).
\end{aligned}
$$

It follows that

$$\begin{aligned}\mathbb{E}\left[(R_i^h(t))^2\right] &= hd_i^2X_i^2\left\{\sum_{n,m=1}^{N-1}X_mX_n[\frac{d_nd_m}{d_N^2}(\sigma_i^2-\sigma_{im}-\sigma_{ni}+\sigma_{nm})-2\frac{d_n}{d_N}(\sigma_i^2-\sigma_{ni})+\sigma_i^2]\right.\\ &\quad +2\sum_{n=1}^{N-1}X_n[\frac{d_n}{d_N}(\sigma_i^2-\sigma_{ni})-\sigma_i^2]\\ &\quad \left.+\sigma_i^2\right\}+\mathbf{o}(h).\end{aligned}$$

$\mathbb{E}[R_i^h(t)R_j^h(t)]$ is calculated as follows.

$$
\begin{aligned}
R_i^h(t)R_j^h(t) &= d_iX_id_jX_je^{\gamma_i^h+\gamma_j^h}\left[\left(\sum_{n=1}^{N-1}\left((e^{\rho_i^h}-e^{\rho_n^h})\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1\right)X_n\right)+(e^{\rho_i^h}-1)\right] \\
&\cdot\left[\left(\sum_{m=1}^{N-1}\left((e^{\rho_j^h}-e^{\rho_m^h})\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-e^{\rho_j^h}+1\right)X_n\right)+(e^{\rho_j^h}-1)\right] \\
&= d_iX_id_jX_je^{\gamma_i^h+\gamma_j^h}\left[\sum_{n,m=1}^{N-1}X_mX_n\left((e^{\rho_i^h}-e^{\rho_n^h})\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1\right)\right. \\
&\cdot\left((e^{\rho_j^h}-e^{\rho_m^h})\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-e^{\rho_j^h}+1\right) \\
&+\sum_{n=1}^{N-1}X_n\left((e^{\rho_i^h}-e^{\rho_n^h})\frac{d_n}{d_N}e^{\gamma_n^h-\gamma_N^h}-e^{\rho_i^h}+1\right)(e^{\rho_j^h}-1) \\
&+\sum_{m=1}^{N-1}X_m\left((e^{\rho_j^h}-e^{\rho_m^h})\frac{d_m}{d_N}e^{\gamma_m^h-\gamma_N^h}-e^{\rho_j^h}+1\right)(e^{\rho_i^h}-1) \\
&\left.+(e^{\rho_i^h}-1)(e^{\rho_j^h}-1)\right]+\mathbf{o}(h) \\
&= d_iX_id_jX_j\left\{\sum_{n,m=1}^{N-1}X_mX_n(1+\gamma_i^h+\gamma_j^h+\frac{(\gamma_i^h+\gamma_j^h)^2}{2})\left[\frac{d_nd_m}{d_N^2}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2})\right.\right. \\
&\cdot(\rho_j^h-\rho_m^h+\frac{(\rho_j^h)^2}{2}-\frac{(\rho_m^h)^2}{2})(1+\gamma_n^h-\gamma_N^h+\gamma_m^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h+\gamma_m^h-\gamma_N^h)^2}{2}) \\
&-\frac{d_n}{d_N}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2})(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2})(\rho_j^h+\frac{(\rho_j^h)^2}{2}) \\
&-\frac{d_m}{d_N}(\rho_j^h-\rho_m^h+\frac{(\rho_j^h)^2}{2}-\frac{(\rho_m^h)^2}{2})(1+\gamma_m^h-\gamma_N^h+\frac{(\gamma_m^h-\gamma_N^h)^2}{2})(\rho_i^h+\frac{(\rho_i^h)^2}{2}) \\
&\left.(\rho_i^h+\frac{(\rho_i^h)^2}{2})(\rho_j^h+\frac{(\rho_j^h)^2}{2})\right] \\
&+\sum_{n=1}^{N-1}X_n\left(\frac{d_n}{d_N}(\rho_i^h-\rho_n^h+\frac{(\rho_i^h)^2}{2}-\frac{(\rho_n^h)^2}{2})(1+\gamma_n^h-\gamma_N^h+\frac{(\gamma_n^h-\gamma_N^h)^2}{2})\right. \\
&\left.-\rho_i^h-\frac{(\rho_i^h)^2}{2}\right)\cdot(\rho_j^h+\frac{(\rho_j^h)^2}{2}) \\
&+\sum_{m=1}^{N-1}X_m\left(\frac{d_m}{d_N}(\rho_j^h-\rho_m^h+\frac{(\rho_j^h)^2}{2}-\frac{(\rho_m^h)^2}{2})(1+\gamma_m^h-\gamma_N^h+\frac{(\gamma_m^h-\gamma_N^h)^2}{2})\right. \\
&\left.-\rho_j^h-\frac{(\rho_j^h)^2}{2}\right)\cdot(\rho_i^h+\frac{(\rho_i^h)^2}{2}) \\
&\left.+(\rho_j^h+\frac{(\rho_j^h)^2}{2})(\rho_j^h+\frac{(\rho_j^h)^2}{2})\right\}+\mathbf{o}(h).
\end{aligned}
$$

So,

$$\begin{aligned}
\mathbb{E}[R_i^h(t)R_j^h(t)] &= hd_iX_id_jX_j\Big\{\sum_{n,m=1}^{N-1}X_mX_n[\frac{d_nd_m}{d_N^2}(\sigma_{ij}-\sigma_{im}-\sigma_{nj}+\sigma_{nm}) \\
&\quad -\frac{d_n}{d_N}(\sigma_{ij}-\sigma_{nj})-\frac{d_m}{d_N}(\sigma_{ij}-\sigma_{mi})+\sigma_{ij}] \\
&\quad +\sum_{n=1}^{N-1}X_n[\frac{d_n}{d_N}(\sigma_{ij}-\sigma_{nj})-\sigma_{ij}]+\sum_{m=1}^{N-1}X_m[\frac{d_m}{d_N}(\sigma_{ij}-\sigma_{mi})-\sigma_{ij}] \\
&\quad +\sigma_{ij}\Big\}+\mathbf{o}(h).
\end{aligned}$$

Observing that

$$\lim_{h\to 0}\mathrm{Cov}[(R_i^h)^2(t),(S^h)^2(t)]=0,$$

$$\lim_{h\to 0}\mathrm{Cov}[R_i^h(t)R_j^h(t),S^h(t)S_j^h(t)]=0.$$

Finally, assembling the above calculations as (3.10) yields

$$\begin{aligned}
\mathfrak{a}_{ij} &= \frac{x_ix_jd_id_j}{[\sum_{n=1}^{N-1}x_n(\frac{d_n}{d_N}-1)+1]^2}\Big\{\sum_{n,m=1}^{N-1}X_mX_n[\frac{d_nd_m}{d_N^2}(\sigma_{ij}-\sigma_{im}-\sigma_{nj}+\sigma_{nm}) \\
&\quad -\frac{d_n}{d_N}(\sigma_{ij}-\sigma_{nj})-\frac{d_m}{d_N}(\sigma_{ij}-\sigma_{mi})+\sigma_{ij}] \\
&\quad +\sum_{n=1}^{N-1}X_n[\frac{d_n}{d_N}(\sigma_{ij}-\sigma_{nj})-\sigma_{ij}]+\sum_{m=1}^{N-1}X_m[\frac{d_m}{d_N}(\sigma_{ij}-\sigma_{mi})-\sigma_{ij}] \\
&\quad +\sigma_{ij}\Big\} \\
&= d_id_jx_ix_j\Big(\sigma_{ij}-\frac{\sum_{n=1}^{N-1}d_nx_n(\sigma_{in}+\sigma_{jn})}{\sum_{n=1}^{N}d_nx_n}+\frac{\sum_{n,m=1}^{N-1}d_nd_mx_nx_m\sigma_{mn}}{(\sum_{n=1}^{N}d_nx_n)^2}\Big).
\end{aligned}$$

Appendix B

Derivation of (3.29)

First, by (3.16) and (3.17),

$$
\begin{aligned}
\frac{1}{Y_i} f_i - \frac{1}{2Y_i^2}\mathfrak{a}_{ii} \quad = \quad & d_i \left\{ \mu_i + \frac{\sigma^2}{2} - \frac{\sum_{n=1}^{N-1} d_n Y_n (\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n} \right. \\
& + \frac{\sum_{n=1}^{N-1} d_n Y_n (d_n Y_n - d_i Y_i)\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2} \\
& \left. + \frac{d_N^2 Y_N^2 \theta + d_i Y_i (\varepsilon\theta - \sigma^2) + \sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n Y_n (\theta - \lambda\sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2} \right\} \\
& - \frac{d_i^2 \sigma^2}{2} \left\{ \frac{d_N^2 Y_N^2 \lambda}{(\sum_{n=1}^{N} d_n Y_n)^2} + (1-\lambda) \right. \\
& \left. + (1-\lambda)\left(-\frac{2 Y_i d_i}{\sum_{n=1}^{N} d_n Y_n} + \frac{\sum_{n=1}^{N-1} d_n^2 Y_n^2}{\sum_{n=1}^{N} (d_n Y_n)^2} \right) \right\}.
\end{aligned}
$$

Rearranging terms on the right hand side gives

$$
\begin{aligned}
& \frac{1}{d_i}\left(\frac{1}{Y_i} f_i - \frac{1}{2Y_i^2}\mathfrak{a}_{ii} \right) \\
= \quad & \underbrace{\mu_i + \frac{\sigma^2}{2} - \frac{\sum_{n=1}^{N-1} d_n Y_n (\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n}}_{(\mathbf{iii})} + \underbrace{\frac{d_i Y_i (\varepsilon\theta - \sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2} + \frac{d_i^2 \sigma^2 (1-\lambda) Y_i}{\sum_{n=1}^{N} d_n Y_n}}_{(\mathbf{iv})} \\
& + \mathscr{J}_2 + \underbrace{\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big) \frac{\sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n^2 Y_n^2}{(\sum_{n=1}^{N} d_n Y_n)^2} + (\theta - \lambda\sigma^2) \frac{\sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n Y_n}{(\sum_{n=1}^{N} d_n Y_n)^2}}_{(\mathbf{v})} \\
& + \mathscr{J}_5 + \underbrace{\frac{d_N^2 Y_N^2 \theta}{(\sum_{n=1}^{N} d_n Y_n)^2} - \frac{d_i \lambda \sigma^2 d_N^2 Y_N^2}{2(\sum_{n=1}^{N} d_n Y_n)^2}}_{(\mathbf{vi})},
\end{aligned}
\tag{B.1}
$$

where $\mathscr{J}_2$ and $\mathscr{J}_5$ are as defined in (3.30) and (3.30), respectively.

Notice that since $d_n \le 1$, then $\sum_{n=1}^{N} d_n Y_n \le \sum_{n=1}^{N} Y_n = 1$, and thus $(\sum_{n=1}^{N} d_n Y_n)^2 \le \sum_{n=1}^{N} d_n Y_n$. Also, recall that $Y_n \le 1$ for all $n = 1, \cdots, N$. The rest of the terms on the right hand side of (B.1) satisfy

$$\begin{aligned}
(\mathbf{iii}) &= \mu_i + \frac{\sigma^2}{2} - \frac{\sum_{\substack{n=1\\ \mu_n>0}}^{N-1} d_n Y_n(\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n} - \frac{\sum_{\substack{n=1\\ \mu_n<0}}^{N-1} d_n Y_n(\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n}, \\
&\le \mu_i + \frac{\sigma^2}{2} - \frac{\sum_{\substack{n=1\\ \mu_n<0}}^{N-1} d_n Y_n(\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n} \le \mathscr{J}_1, &\text{(B.2)}\\
(\mathbf{iv}) &\le \frac{d_i Y_i}{(\sum_{n=1}^{N} d_n Y_n)^2}\big(\varepsilon\theta - \sigma^2 + d_i\sigma^2(1-\lambda)\big) = \mathscr{J}_3, &\text{(B.3)}\\
(\mathbf{v}) &\le \frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n}{(\sum_{n=1}^{N} d_n Y_n)^2}\big(d_n((1-\lambda)\sigma^2 + (1-\varepsilon)\theta) + \theta - \lambda\sigma^2\big) = \mathscr{J}_4, &\text{(B.4)}\\
(\mathbf{vi}) &= \frac{d_N^2 Y_N^2}{2(\sum_{n=1}^{N} d_n Y_n)^2}\big(2\theta - d_i\lambda\sigma^2\big) = \mathscr{J}_6, &\text{(B.5)}
\end{aligned}$$

where $\mathscr{J}_1$, $\mathscr{J}_3$, $\mathscr{J}_4$, and $\mathscr{J}_6$ are as defined in (3.30), (3.30), (3.30), and (3.30), respectively. Collecting (B.2)–(B.5) and inserting into (B.1) results in (3.29).

Appendix C

Derivation of (3.36)

First, by using (3.16) and (3.17), we have

$$
\begin{aligned}
\frac{f_i}{2\sqrt{Y_i}} - \frac{\mathfrak{a}_{ii}}{8\sqrt{Y_i^3}} \quad = \quad & \frac{d_i}{2}\sqrt{Y_i}\Bigg\{\mu_i + \frac{\sigma^2}{2} - \frac{\sum_{n=1}^{N-1} d_n Y_n(\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n} \\
& + \frac{\sum_{n=1}^{N-1} d_n Y_n (d_n Y_n - d_i Y_i)\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2} \\
& + \frac{d_N^2 Y_N^2 \theta + d_i Y_i(\varepsilon\theta - \sigma^2) + \sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n Y_n (\theta - \lambda\sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2} \\
& - \frac{d_i\sigma^2}{4}(1-\lambda)\Big(1 - \frac{2Y_i d_i}{\sum_{n=1}^{N} d_n Y_n}\Big) \\
& - \frac{d_i\sigma^2}{4}\left(\frac{d_N^2 Y_N^2 \lambda}{(\sum_{n=1}^{N} d_n Y_n)^2} + (1-\lambda)\frac{\sum_{n=1}^{N-1} d_n^2 Y_n^2}{(\sum_{n=1}^{N} d_n Y_n)^2}\right)\Bigg\}
\end{aligned}
$$

Notice that the last term in the above equation is non-positive. Removing the last term and rearranging the remaining terms gives

$$
\begin{aligned}
&\frac{f_i}{2\sqrt{Y_i}} - \frac{\mathfrak{a}_{ii}}{8\sqrt{Y_i^3}} \\
\leq\ & \frac{d_i}{2}\sqrt{Y_i}\left\{ \underbrace{\mu_i - \frac{\sum_{n=1}^{N-1} d_n Y_n \mu_n}{\sum_{n=1}^{N} d_n Y_n}}_{(\mathbf{vii})} + \underbrace{\frac{\sigma^2}{2} - \sum_{n=1}^{N-1} \frac{d_n Y_n \sigma^2}{2\sum_{n=1}^{N} d_n Y_n} + \frac{d_N^2 Y_N^2 \theta}{(\sum_{n=1}^{N} d_n Y_n)^2}}_{(\mathbf{viii})} \right. \\
& \qquad + \underbrace{\frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n^2 Y_n^2 \big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2} + \frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n (\theta - \lambda\sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2}}_{(\mathbf{ix})} \\
& \qquad \left. + \underbrace{\frac{d_i Y_i (\varepsilon\theta - \sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2} + \frac{d_i^2 \sigma^2}{2} \frac{(1-\lambda) Y_i}{\sum_{n=1}^{N} d_n Y_n}}_{(\mathbf{x})} + \mathscr{R}_4 - \frac{d_i \sigma^2}{4}(1-\lambda) \right\},
\end{aligned}
\tag{C.1}
$$

where $\mathscr{R}_4$ is as defined in (3.37), and the term **(x)** is the same as $\mathscr{R}_5$ defined in (3.37).

Next, using $Y_n \leq 1$ for all n,

$$
(\mathbf{ix}) \leq \frac{\sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n^2 Y_n \big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta \sum_{\substack{n=1\\ n\neq i}}^{N-1} d_n Y_n (\theta - \lambda\sigma^2)\big)}{(\sum_{n=1}^{N} d_n Y_n)^2} \leq \mathscr{R}_3,
$$

where $\mathscr{R}_3$ is as defined in (3.37).

Now, splitting the term $-\frac{d_i\sigma^2}{4}(1-\lambda)$ into three of $-\frac{d_i\sigma^2}{12}(1-\lambda)$, then combining two of them with the terms **(vii)** and **(viii)**, respectively, we have

$$
(\mathbf{vii}) - \frac{d_i\sigma^2}{12}(1-\lambda) = \mathscr{R}_1, \qquad (\mathbf{viii}) - \frac{d_i\sigma^2}{12}(1-\lambda) = \mathscr{R}_2,
$$

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are defined as in (3.37) and (3.37), respectively.

Inserting the above estimates into (C.1) and summing over i from 1 to $N-1$ then results in (3.36).

Appendix D

Derivations of (3.39)

First, using (3.16) and (3.17) in (3.38) gives

$$\begin{aligned}
\mathscr{L}\mathscr{V}(\mathbf{Y}) \;=\; & \sum_{i=1}^{N-1} \frac{-d_i}{Y_i^{\iota}} \Bigg\{ \mu_i + \frac{\sigma^2}{2} - \frac{\sum_{n=1}^{N-1} d_n Y_n (\mu_n + \frac{\sigma^2}{2})}{\sum_{n=1}^{N} d_n Y_n} \\
& + \frac{\sum_{n=1}^{N-1} d_n Y_n (d_n Y_n - d_i Y_i)\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big)}{(\sum_{n=1}^{N} d_n Y_n)^2} \\
& + \frac{d_N^2 Y_N^2 \theta + d_i Y_i (\varepsilon\theta - \sigma^2) + \sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n Y_n (\theta - \lambda\sigma^2)}{(\sum_{n=1}^{N} d_n Y_n)^2} \Bigg\} \\
& + \frac{(\iota+1)\sigma^2}{2} \sum_{i=1}^{N-1} \frac{d_i^2}{Y_i^{\iota}} \Bigg\{ \frac{d_N^2 Y_N^2 \lambda}{(\sum_{n=1}^{N} d_n Y_n)^2} + (1-\lambda) \\
& + (1-\lambda)\left(-\frac{2Y_i d_i}{\sum_{n=1}^{N} d_n Y_n} + \frac{\sum_{n=1}^{N-1} d_n^2 Y_n^2}{\sum_{n=1}^{N} (d_n Y_n)^2} \right) \Bigg\}.
\end{aligned}$$

Using the facts that $d_n^2 Y_n^2\big((1-\lambda)\sigma^2 + (1-\varepsilon)\theta\big) > 0$ under the first condition of Assumption (**A8**), and that $-d_i Y_i \sigma^2 - \lambda\sigma^2 \sum_{\substack{n=1 \\ n\neq i}}^{N-1} d_n Y_n \geq \sigma^2 \sum_{n=1}^{N-1} d_n Y_n$ since $\lambda \leq 1$, and regrouping terms on the right hand side in the above equation we have

$$\begin{aligned}
\mathscr{L}\mathscr{V}(\mathbf{Y}) \;\leq\; & \sum_{i=1}^{N-1} \frac{-d_i}{Y_i^{\iota}} \Bigg\{ \mu_i - \frac{\sum_{n=1}^{N-1} d_n Y_n \mu_n}{\sum_{n=1}^{N} d_n Y_n} + \frac{d_N Y_N \frac{\sigma^2}{2}}{\sum_{n=1}^{N} d_n Y_n} - \frac{(\iota+1) d_i d_N^2 Y_N^2 \lambda\sigma^2}{2(\sum_{n=1}^{N} d_n Y_n)^2} \\
& - \frac{\sum_{n=1}^{N-1} d_i Y_i d_n Y_n (1-\lambda)\sigma^2}{(\sum_{n=1}^{N} d_n Y_n)^2} - \frac{\sum_{n=1}^{N-1} d_n Y_n \sigma^2}{\sum_{n=1}^{N} d_n Y_n} \\
& - \frac{\sum_{n=1}^{N-1} d_i Y_i d_n Y_n (1-\varepsilon)\theta}{(\sum_{n=1}^{N} d_n Y_n)^2} + \frac{d_i Y_i \varepsilon\theta + \sum_{n\neq i}^{N-1} d_n Y_n \theta + d_N^2 Y_N^2 \theta}{(\sum_{n=1}^{N} d_n Y_n)^2} \\
& - \frac{(1-\lambda)(\iota+1) d_i \sigma^2}{2} \left(\left(1 - \frac{Y_i d_i}{\sum_{n=1}^{N} d_n Y_n}\right)^2 + \frac{\sum_{n\neq i}^{N-1} d_n^2 Y_n^2}{(\sum_{n=1}^{N} d_n Y_n)^2} \right) \Bigg\}.
\end{aligned}$$

Noticing that

$$\frac{d_N^2 Y_N^2}{(\sum_{n=1}^N d_n Y_n)^2} \le \frac{d_N Y_N}{\sum_{n=1}^N d_n Y_n}, \quad \frac{\sum_{n\neq i}^{N-1} d_n^2 Y_n^2}{(\sum_{n=1}^N d_n Y_n)^2} \le \frac{(\sum_{n\neq i}^{N-1} d_n Y_n)^2}{(\sum_{n=1}^N d_n Y_n)^2} \le \frac{\sum_{n\neq i}^{N-1} d_n Y_n}{\sum_{n=1}^N d_n Y_n}$$

we get

$$\begin{aligned}\mathscr{L}\mathscr{V}(\mathbf{Y}) \quad \le \quad & \sum_{i=1}^{N-1} \frac{-d_i}{Y_i^\iota} \Bigg\{ \mu_i - \frac{\sum_{n=1}^{N-1} d_n Y_n \mu_n}{\sum_{n=1}^N d_n Y_n} + \underbrace{\frac{1}{2} \frac{d_N Y_N \sigma^2 \big(1-(\iota+1)d_i\big)}{\sum_{n=1}^N d_n Y_n}}_{(\mathbf{xi})} \\ & - \frac{\sum_{n=1}^{N-1} d_i Y_i d_n Y_n (1-\lambda)\sigma^2}{(\sum_{n=1}^N d_n Y_n)^2} - \frac{\sum_{n=1}^{N-1} d_n Y_n \sigma^2}{\sum_{n=1}^N d_n Y_n} \\ & - \frac{\sum_{n=1}^{N-1} d_i Y_i d_n Y_n (1-\varepsilon)\theta}{(\sum_{n=1}^N d_n Y_n)^2} + \frac{d_i Y_i \varepsilon\theta + \sum_{n\neq i}^{N-1} d_n Y_n \theta + d_N^2 Y_N^2 \theta}{(\sum_{n=1}^N d_n Y_n)^2} \\ & -(1-\lambda)(\iota+1) d_i \sigma^2 \frac{\sum_{n\neq i}^{N-1} Y_n d_n}{\sum_{n=1}^N d_n Y_n} \Bigg\}.\end{aligned}$$

Splitting the $\frac{1}{2}$ in term (**xi**) into $\frac{1}{12}$, $\frac{1}{12}$, $\frac{1}{12}$ and $\frac{1}{4}$ results in (3.39).