Isoperimetric Properties of the Circle

by

Kimberly Holman

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Approved by

Andras Bezdek, Chair, C. Harry Knowles Professor of Mathematics
Dean Hoffman, Professor of Mathematics
Peter Johnson, Professor of Mathematics

Abstract

Jakob Steiner and Karl Weierstrass provided formal proofs of the isoperimetric property of the circle in the late 1830s. New mathematical tools, proof language, and concepts were needed to prove this rather obvious fact, that of all regions with a given perimeter the circle, and only the circle, has maximum area. We examine the historical context of the isoperimetric property, the proofs of Steiner and Weierstrass, and subsequent proofs using alternate methods. A comprehensive history and applications of the isoperimetric problem are presented. Several exercises are posed and solved using the isoperimetric property, relying heavily on methods used by Richard Demar in 1975.

Acknowledgments

I couldn't have asked for a better mentor and friend than Dr. András Bezdek. I encountered more than my share of hurdles, stumbling blocks, and other problems along this road and he has always been there to remind me that I am smart, capable, and worthy of this degree.

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I owe much thanks to my cat, Pascal. Yes, my cat. He knows just the right moment to curl up in my lap, and I spent many hours writing this paper with him softly purring, keeping me company.

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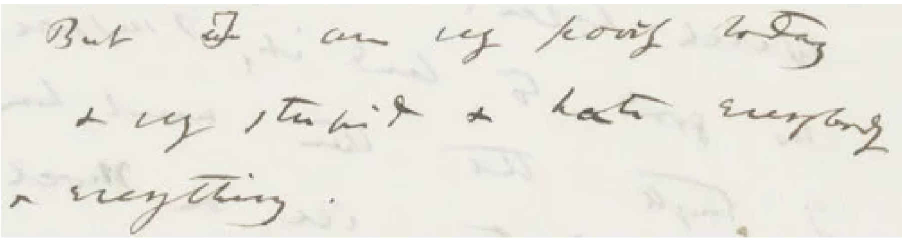
Dr. Anneliese Spaeth, Dr. William Young, Dr. Sydney Stubbs, Mr. Rick Stubbs, Mrs. Linda Alvear, and so many other math teachers I have encountered through the years: thank you.

Memes. I especially like this one, a quote from Charles Darwin in a letter dated October 1, 1861. We've all had days like this, even Darwin, two years after publishing *On the Origin of Species*.


But I am very poorly today
& very stupid & hate everybody
& everything.

"But I am very poorly today and very stupid and hate everybody and everything."

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Chapter 1

A summary of seven proofs concerning variants of the isoperimetric property of the circle

> The proof, the whole proof, and nothing but the proof. When it comes to mathematics that's the only thing that matters.
>
> *Włodzimierz Kuperberg*

Theorem 1. Among all regions of given perimeter, the circle has maximum area.

1.1 Jakob Steiner

First we call in the proposition that the isosceles triangle has the smallest leg-sum compared to triangles of the same baseline and either height or area, and vice-versa. Given an unequal triangle, transforms it to an isosceles triangle of the same baseline and height. The result is a triangle with a smaller leg-sum, and therefore a smaller perimeter, containing the same area.

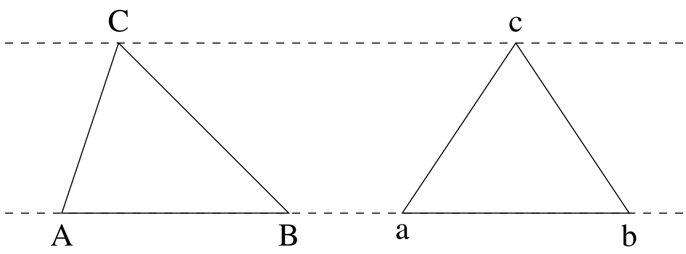

Figure 1.1.0.1: Maximum area triangles must be isosceles

We extend the principle above to show that every parallel trapezoid which does not have equal angles on one or the other baselines can be transformed into another of the same area and baselines; moreover, the new side lengths will have a smaller sum and the new parallel trapezoid has symmetry along the axis formed by the midpoints of the baselines.

In the figure below, the parallel trapezoid $ADBE$ has the property that there are no equal angles on the baseline AB and DE. The parallel trapezoid $adeb$ is the transformation of $ADEB$ such that $|AB| = |ab|$ and $|DE| = |de|$ as well as symmetry across the axis X formed by the midpoint of de, h, and the midpoint of ab, m.

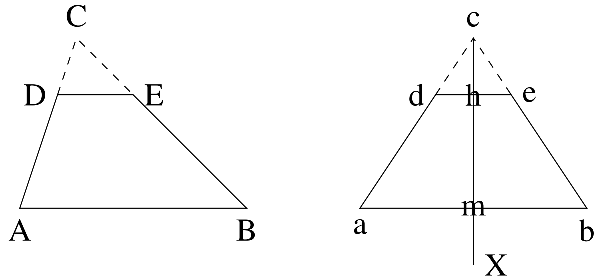

Figure 1.1.0.2: Maximum area parallel trapezoids must be symmetric

This follows directly from the previous assertion using properties of similar triangles, with the parallel trapezoids being contained within their respective triangles. The result is a new parallel trapezoid with a smaller perimeter and the same area than that of the original.

Thus, any arbitrary convex polygon V can be transformed into another polygon V_1 with smaller perimeter and the same area with respect to any one axis of symmetry. Examples follow.

Example 1.1.1. Given a triangle ABC, transform it into triangle abc and reflect along ac such that $|BD| = |bd|$ as shown below. The resulting tetragon $abcd$ has a smaller perimeter and the same area as $\triangle ABC$.

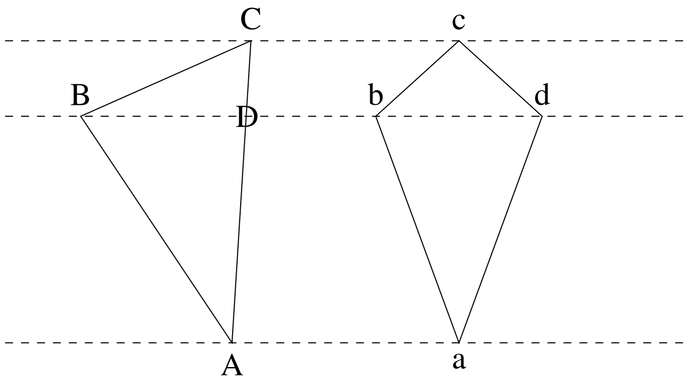

Figure 1.1.1.1: A triangle and a tetragon of the same area where the tetragon has smaller perimeter

Example 1.1.2. Using Example 1, label the line ac as axis Y. Create a new axis, X, perpendicular to axis Y and label the intersection of the axes as μ. The tetragon $abcd$ is transformed into a square $\alpha\beta\gamma\delta$, having the same area but smaller perimeter. Moreover, $\alpha\beta\gamma\delta$ is symmetric about axes X and Y thus a rhombus with center μ. Therefore, any arbitrary triangle ABC is transformed into a rhombus $\alpha\beta\gamma\delta$ of the same area but smaller perimeter.

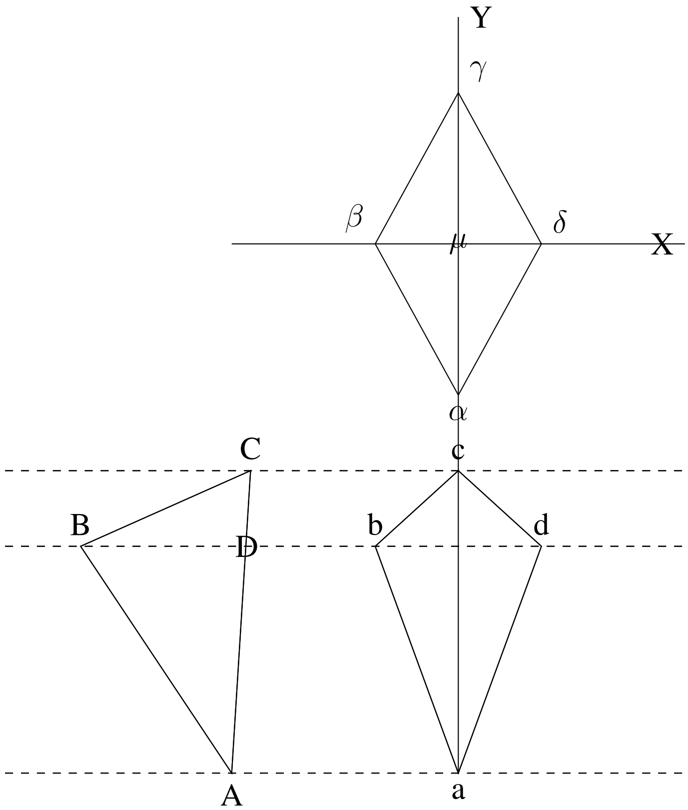

Figure 1.1.2.1: Transform the tetragon into a rhombus with the same area and a smaller perimeter

Example 1.1.3. Transform a hexagon into a symmetrical decagon according to the principles above. The resulting decagon is symmetric across one axis, has a smaller perimeter, and the same area as the hexagon.

If one were to apply the symmetry across a second, perpendicular axis, the resulting hexadecagon has the properties of the same area, smaller perimeter, and 2 lines of symmetry; moreover, the intersection of the perpendicular lines of symmetry (axes) is the center of the hexadecagon.

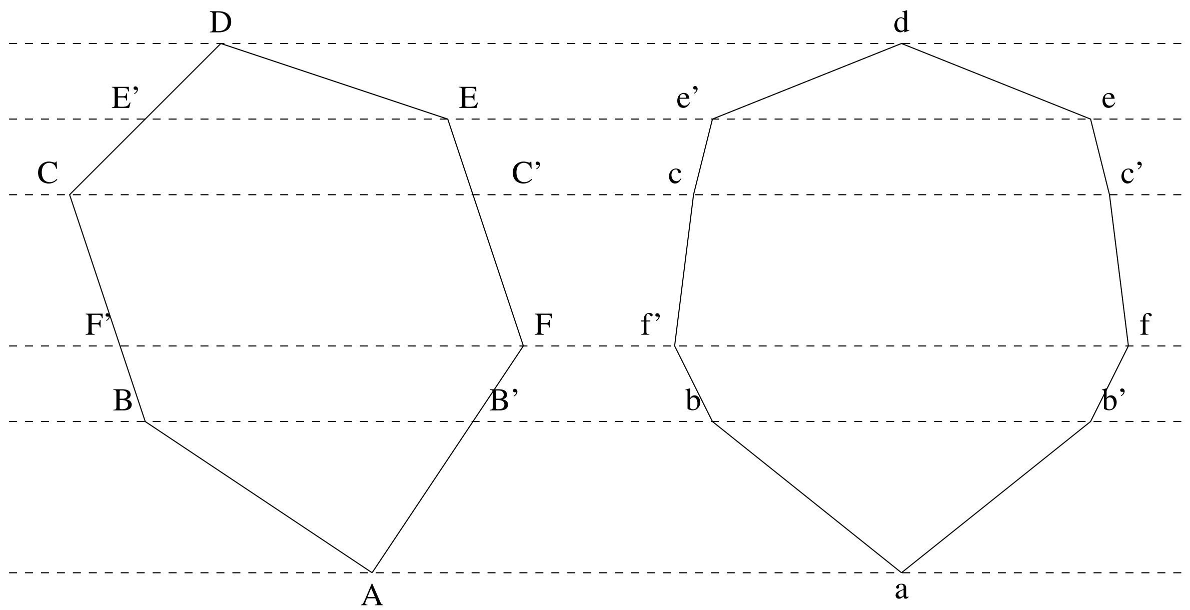

Figure 1.1.3.1: Transform a hexagon into a decagon with the same area and smaller perimeter

Example 1.1.4. Using the principles above, any given polygon V with any number n edges can be transformed using an axis X_1 into a symmetric polygon of the same area and smaller perimeter with at most $2n-2$ edges. Repeating the process to obtain the kth arbitrary axis X_k results in a polygon V_k, with a maximum number of edges $e_k = 2^k(n-2)+2$, the same area, and a smaller perimeter than any of the previous V_i.

If one picks X_2 to be perpendicular to X_1 then V_2 has its center at their intersection point M with two axes of symmetry, but at most $2(2n-4)$ edges. Moreover, every subsequent polygon $V_3, V_4, \ldots, V_k$ maintains its center point at M regardless whether any subsequent axes are mutually perpendicular.

We see that by following the iterative process described above we can transform any given polygon V into another polygon V_x with the same area and smaller perimeter with no limit to the number of edges. If the number of edges is large, perhaps infinitely large, and the perimeter shrinks, each edge becomes infinitely small. Therefore, the perimeter of the polygon V_x becomes, instead, a curve.

Conversely, we can imagine a given closed curve V as a polygon with infinitely many and infinitely small edges. Then this curve can be transformed via an arbitrary axis to another curve V_1 of the same area and smaller perimeter, and transformed again via a second axis perpendicular to the first yielding a curve V_2 with smaller perimeter than V_1, the same area, and mutually perpendicular axes of symmetry with center point M.

Repeating this process results in subsequent curves of the same area, each with a smaller perimeter than the one before. The diameters of the curve approach equality, quickly so if each arbitrary axis is selected to enlarge the smallest diameter and shrink the largest.

Therefore, every closed convex figure V - with straight or curved edges - can be transformed to yield a smaller perimeter containing the same area, until the figure has symmetrical axes in all directions. When this is the case, further transformations result in congruent shapes. However, the shape has the center point M where all axes intersect and all diameters are equal: namely, a circle. [14][15]

1.2 Karl Weierstrass

Relying on variational calculus, this proof defines area and length in terms of integrals. The result is two Euler-Lagrange equations which, when solved, provide an equation in the form of a circle and the length of the radius which optimizes the area given the length of the perimeter. [5]

1.3 Erhard Schmidt

Beginning with parameterised curves, a circle is defined using carefully selected intervals and bounding. Using advanced calculus, isoperimetric equality is obtained if all calculated isoperimetric inequality statements are equalities. [8]

1.4 Vladimir Boltjanski

Boltjanski[3] provides a summary of Steiner's isoperimetric proof. He begins with proving that the maximum area shape must be convex.

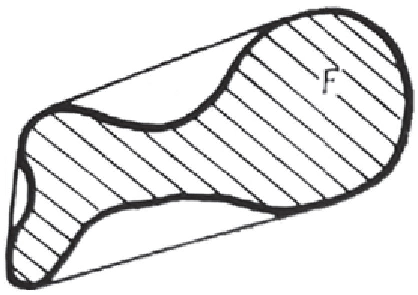

Figure 1.4.1: Solution must be convex

He moves on to a general shape. He makes a "cross-cut" that divides the boundary into two equal lengths. If every cross-cut has equal area then the shape is extremal. Otherwise, omit the side with smaller area and reflect the side with larger area across the cross-cut. We may continue in this way until every cross-cut has equal area.

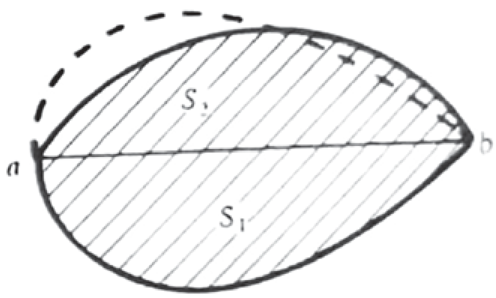

Figure 1.4.2: A chord which bisects the perimeter

Now he takes a fixed cross-cut, a, b, and any boundary point distinct from a and b, c, and demonstrates that the extremal must have $\angle acb$ equal to $90°$. If the angle is anything else, install a hinge and rotate about c until the angle measure of acb is $90°$. This does not change the area between $\triangle abc$ and the boundary of the shape but it does increase the area of $\triangle abc$.

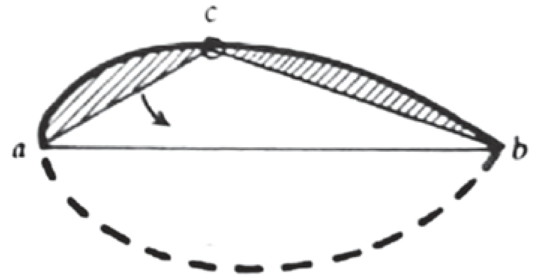

Figure 1.4.3: Install a hinge

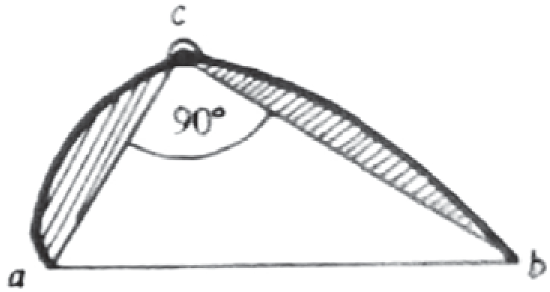

Figure 1.4.4: Rotate about c

Then, reflect over the cross-cut to achieve a shape with greater area and same perimeter.

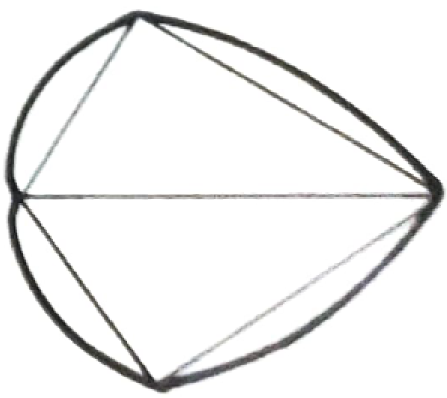
Figure 1.4.5: Reflect the cross-cut

1.5 Richard Demar

Demar begins with an indirect proof to show that the extremal is convex, then proceeds with a trivial example that the square is not extremal. While the square example is trivial and can easily be proved computationally, the trick Demar uses here will be used in subsequent proofs and is illustrated in Figures 1.5.1, 1.5.2, and 1.5.3.

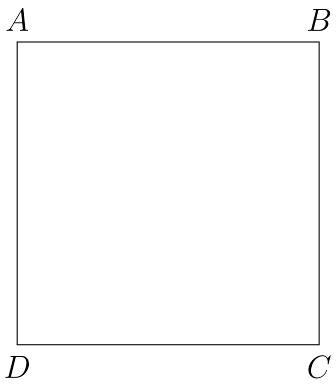

Figure 1.5.1: Square

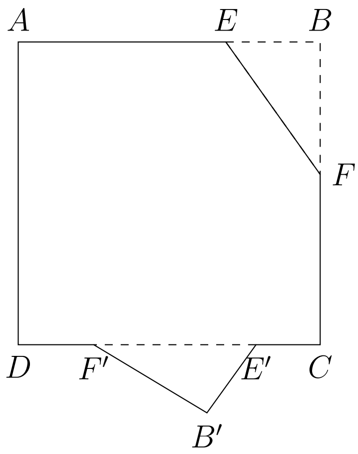

Figure 1.5.2: Replace

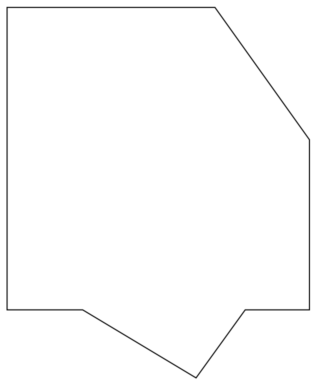

Figure 1.5.3: Nonconvex

Theorem 2. Among all n-sided polygons of perimeter p, the regular n-gon has maximum area.

Demar begins by showing that the solution is equilateral using an indirect method. Using reflections and replacements, which preserve length and area, he achieves a shape which is not convex. Therefore, the extremal must be equilateral.

The indirect approach is also used to show equiangularity. Using reflections and replacements, the shape is not convex, so the extremal must be equiangular.

As the n-gon must be both equilateral and equiangular, it is regular.

Theorem 3. Among all plane regions of perimeter p, a circular region has maximal area.

Let R be a region of maximum area among all regions of perimeter length p. Let γ be the boundary of R. Since R must be convex, there is a dense set of points on γ at each of which γ has a tangent. Let A and B be any two points of this dense set such that the tangents at A and B are not parallel. Let C denote their point of intersection.

The measures of $\angle CAB = \angle C'BA$ is shown by contradiction, using the property that reflection preserves area and length.

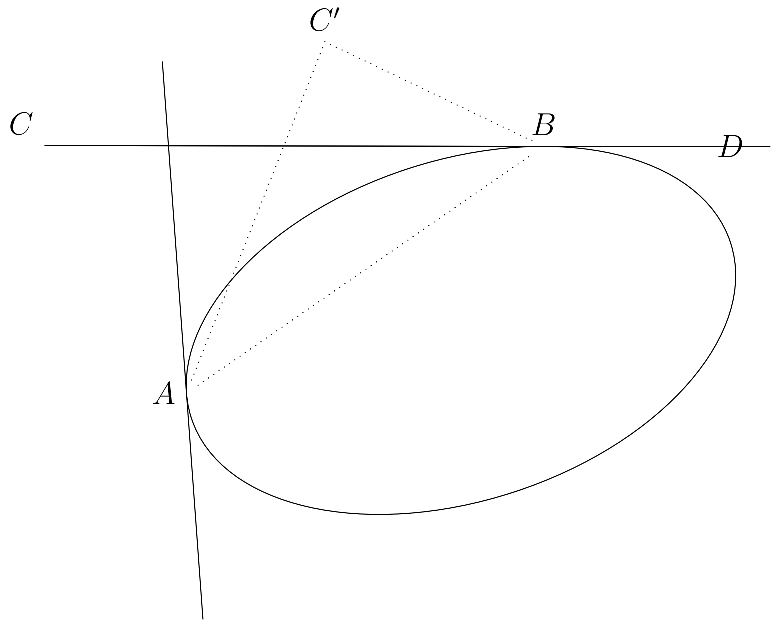

Figure 1.5.4: Reflect and replace to get a nonconvex region

It is shown that region is a circle by constructing perpendiculars to tangents and resulting with $\triangle FGH$ being degenerate.

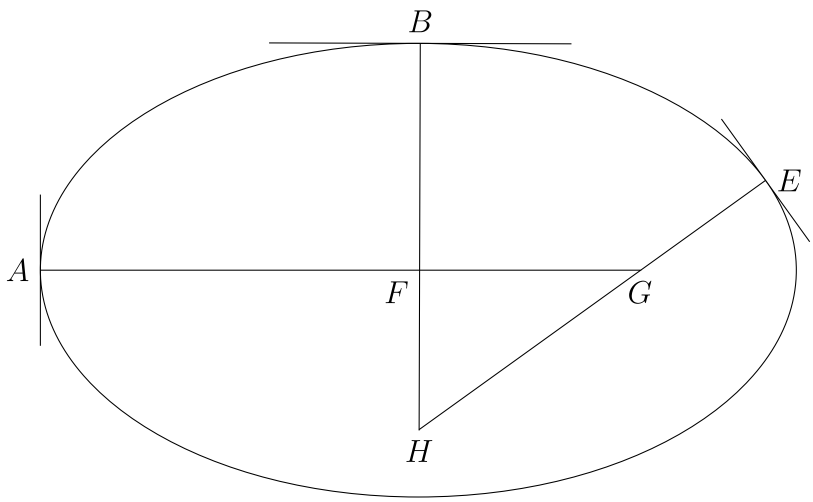

Figure 1.5.5: Proof of the fact that points F, G, H coincide

This proof relies on reflection and replacement principles, a trick which was demonstrated on the square (see Figs. 1.5.1, 1.5.2, 1.5.3). Each of the triangles formed by two points together with the intersection of their tangents lines produces an isosceles triangle, resulting in the triangle FGH being degenerate. Thus, $F = G = H$, and points A, B, E are equidistant from this point he calls G.

Finally, it is noted that every point on γ which has a tangent is also this same distance from G. As γ is continuous, all points on γ are equidistant from G, except finitely many with no tangent. He uses integral calculus to show that there are zero points without a tangent, and γ is a circle.

Theorem 4. Let T be a given triangular region with perimeter P and with a circumference of the inscribed circle equal to c. Let p be a number such that $c \leq p \leq P$. Then among all regions contained in T and having perimeter p, the region R of maximum area has boundary γ consisting of three circular arcs all of the same radius, each tangent to two adjacent sides of the boundary of T, together with the three segments of sides of T between endpoints of these arcs.

First, Demar cites the previous theorem to indicate that γ contains at least one circular arc. Using an indirect proof, he then shows that none of the vertices of T can coincide with γ, meaning, γ contains three circular arcs. This is achieved using replacement of a small corner piece at C and a small sector of the arc near B (see Fig. 1.5.6), which preserves length and area but creates a nonconvex region.

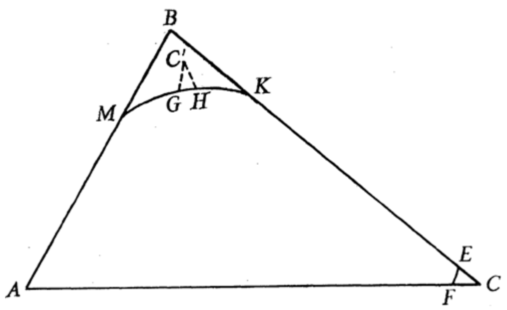

Figure 1.5.6: Replace a small corner with a small sector, and vice-versa

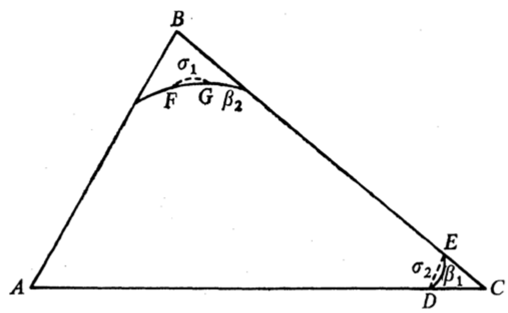

Figure 1.5.7: Replace two small sectors with each other

To show that each of the three arcs have equal radii, Demar again uses an indirect proof method. When the extremal is assumed to have different radii, using replacement of small sectors (see Fig. 1.5.7), a non-convex shape is achieved, implying that the radii are equal.

Lastly, to show that each of the circular arcs of γ is tangent to T, and indirect approach is used. Reflections and replacements yield a γ' which is not convex, hence, the arcs are tangent to T (see Fig. 1.5.8). [4]

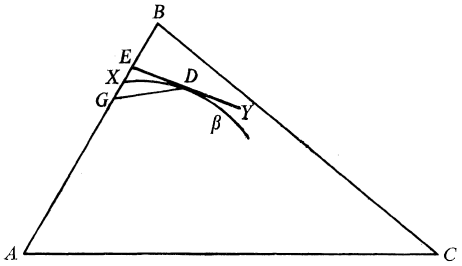

Figure 1.5.8: Drawing a tangent to the arc, segments on AB, and triangles from AB to the point of tangency resulting in a contradiction such that the point of tangency touches BC

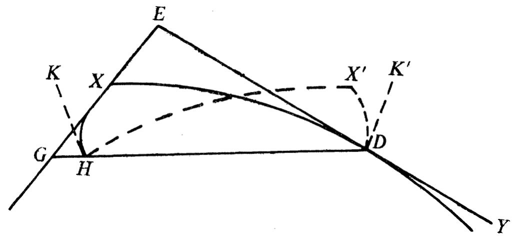

Figure 1.5.9: Zoomed in view where the arc is reflected and replaced resulting in a nonconvex region

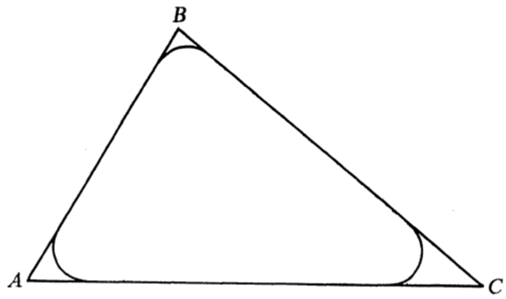

Figure 1.5.10: The maximum area solution

This work by Demar will be referenced heavily in exercises later on.

1.6 Alfred D. Garvin

Expanding on the work of Demar, physical experiments are performed to prove the statements. First, liquid mercury is slowly poured into a confined triangular reservoir and is expected to take on an isoperimetrically optimal configuration, which it does. However, the meniscus effect of the mercury slightly skews the endpoints of the arcs.

Next, the reservoir is partially filled with water. A viscous oil is poured slowly onto the water and is expected to take on an isoperimetrically optimal shape, which it does. Again, the meniscus effect of the oil slightly skews the endpoits of the arcs.

The third experiment uses a flexible, asymmetrical triangular wood fence and a sturdy-paper closed fence with perimeter less than that of the triangle. BB pellets are poured inside the paper fence, which is expected to take on an isoperimetrically optimal shape, which it does. This internal-pressure experiment does not yield any skewed results.

The three experiments yield the same description of the isoperimetrically optimal shape: the arcs have equal radii, the arcs have equal chords, and the arcs have equal length. [6]

1.7 David Singmaster and D. J. Souppouris

> A conjecture both deep and profound
> Is whether a circle is round.
> In a paper of Erdős
> Written in Kurdish
> A counterexample is found.
>
> ---
> Leo Moser[1]

The ratio of area to perimeter, A/P, is examined to find the set which produces the greatest number. Using advanced calculus and analysis, the maximal ratio of A/P is a measure of the circularity of the set. [13]

1.8 Gary Lawlor

A new proof of the planar isoperimetric property is presented. The method is a slicing and covering argument, and calls for strategically dividing the circle as well as another, arbitrary region of equal perimeter, into tiny pieces. Each pieces from non-circle region contains the same length of the region's boundary and has less area than the piece of the circle containing the same length of the circle's boundary. An advantage of this proof is, as it is comparative, it does not require a proof of existence of an optimum solution. [10]

Chapter 2

Some historical facts and remarks on geometry prior to proving the isoperimetric property of the circle

> The story so far: In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.
>
> Douglas Adams, *The Hitchhiker's Guide to the Galaxy*

The isoperimetric property of the circle is a fact that has been known for millennia but was only proven around 150 years ago, and shortly thereafter an addendum was made to satisfy the proof as being rigorous. This proof by Steiner is purely geometrical in nature, not relying on analysis; however, Weierstass's addendum required analysis to satisfy the proof as being rigorous. Subsequent proofs have been published that rely on other methods.

Many of the texts on the isoperimetric property of the circle begin with anecdotes about the ancient Greeks; they knew that the circle was the shape with maximum area for a fixed perimeter but were unable to prove it mathematically. However, it was an obvious enough property that they accepted it as fact without proof.

2.1 Thales

Thales of Miletus was a Greek mathematician, among other scholarly activities, living in Miletus in Ionia, Asia Minor. He is recognized as the first philosopher and the first actual mathematician, framing the logical structure of geometry. Also attributed to Thales is that, in 585 BCE, he was the first person to have predicted a solar eclipse. He was well-situated to travel to

Egypt, where he learned mathematics, and to Babylon, where he learned astronomy. There are five major theorems in geometry Thales is said to have proved:

1. A circle is bisected by a diameter.

2. The base angles of an isosceles triangle are equal.

3. The pairs of vertical angles formed by two intersecting lines are equal.

4. If two triangles are such that two angles and a side of one are equal, respectively, to two angles and a side of the other, then the triangles are congruent.

5. An angle inscribed across a circle's diameter is always a right angle.

No documents survive to verify that Thales himself accomplished this, but the unwavering mythology of his achievements survive.

Claims that Thales was the first mathematician are attributed to Proclus, who lived from 410-485 CE, at the beginning of his work *Commentary on the First Book of Euclid's Elements*. Whether Thales proved these theorems or codified the demonstration of theorems is uncertain, but he is the first person historically credited with these mathematical discoveries. [11] The five theorems listed above paved the way for the discovery of the isoperimetric property of the circle.

2.2 Pythagoras

Pythagoras came roughly 50 years after Thales. Their mathematical interests were similar, likely because Pythagoras also traveled to Babylon and Egypt - as well as India. He became not only a mathematician but also a religious figure, finding himself among contemporaries such as Confucius, Buddha, and Lao-tzu. Upon return from his travels he established a secret society, the Pythagoreans, which studied mathematics and philosophy, subjects which were inextricably linked. Due to the communal nature of this society, ideas attributed to Pythagoras arose from the Pythagoreans as a collective with the exact author unknown. They viewed mathematics as a love of wisdom, and their motto was "All is number." [11]

...

2.3 Zeno

Zenodorus (not to be confused with Zeno of Elea, famed for his paradoxes) lived from around 200 BCE to 120 BCE. His work *On isometric figures* was lost but is referenced so often that we know he proved the following:

- The regular polygon with most angles has the greatest area.
- The circle has greater area than any regular polygon of equal perimeter.
- The equilateral and equiangular (i.e., regular) polygon has the greatest area of any polygon with the same perimeter and number of sides.

He theorized that the circle is the solution to the isoperimetric problem in two dimensions and that the sphere is the solution in three dimensions. [5]

2.4 Thucydides

Thucydides lived around 100 BCE. He used the time needed to circumnavigate a city to determine its area, and determined the size of Sicily by the amount of time it takes to sail around it. He and other geometers of his time were mocked for measuring area according to perimeter. Additionally, it was uncommon that a person of that time knew that shapes of the same perimeter could have different areas and unfortunately, some people cheated others out of land because of this misbelief. [5]

2.5 Queen Dido

Queen Dido is a mythical figure who is most commonly remembered, in mathematics, by an anecdote in Virgil's *Æneid*. She had been a Phœnecian queen in the city of Tyre (in present day Lebanon) and fled to present day Tunisia. The Berber king Iarbas allowed her to stay and would give her as much land as she could bound with an ox's hide. She cut the hide into very thin strips, sewed them together, and outlined a perimeter shaped like a semicircle, using the flat and straight coastline as the remaining boundary. This provided her the greatest area of land, based on the isoperimetric property of the circle and the bound provided by the sea. This area that she bounded became the city of Carthage. [16]

2.5.1 Dido's Problem

There are many statements of "Dido's Problem," some of which are listed below:

- Find the figure bounded by a line which has the maximum area for a given perimeter. The solution is a semicircle.

  a region of the maximal area bounded by a straight line and a curvilinear arc whose endpoints belong to that line.

  A family of segments

- Let E be a collection of finite line segments located in $\mathbb{R}^2$. A formal definition of the area enclosed by E is as follows. Let U be the union of the bounded components of $\mathbb{R}^2 \backslash E$. Let W be the closure of the point set defined by U. Define the area of to be the area enclosed by W be the closure of the point set defined by E.

2.6 Pappus

> Bees ... by virtue of a certain geometrical forethought ... know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material.
>
> ---
>
> Pappus of Alexandria

Pappus of Alexandria published his seminal work, *Collection* (or *Synagogue*), around 320 CE, containing eight books; all of book one and half of book two are lost. Of note is book five, an extensive treatise on bees. He noted that bees have some degree of mathematical understanding as their cells are hexagonal prisms. Later in the book he concerns himself with problems of isoperimetry, going so far as to demonstrate the the circle has the largest area for a given perimeter, and it is from this book that we know of Zenodorus's *On isometric figures*. [11]

2.7 Duke Hengist

In the 5th century, German Duke Hengist offered himself to military service under King Vortigern in exchange for, much like Queen Dido, the land he could enclose with a bull's hide. He established the castle of Kaercorrei with this land, again parallel to the mythology of Dido. [12]

2.8 Dirichlet's principle

Dirichlet was a German mathematics professor, teaching the likes of Riemann, Dedekind, Kronecker, and Eisenstein, among others. His academic advisors were Poisson, Fourier, and Gauss. His contributions were in the fields of number theory, the theory of Fourier series, and analysis. He is credited as being one of the first mathematicians to give a formal definition of a function. [11]

While Dirichlet's principle states that, loosely speaking, if a function is finite then it has a minimum or maximum. Riemann named and popularized the Dirichlet Principle in the field of complex analysis. Steiner famously did not prove the existence of a maximum area figure in his proof of the isoperimetric property of the circle, as noted by Weierstrass, and assumed that such a figure does exist and that figure is the circle.

2.9 Steiner

> Unfortunately, [Lhulier's] work seems to have been quoted more often than the prevailing method has been correctly understood or duly appreciated and followed; for all his successors have deviated more or less from his simple, natural way of looking at things.
>
> Jacob Steiner[14]

Jacob Steiner was the first mathematician known to present a proof of the isoperimetric property of the circle. His paper was first published in *Crelle's Journal* in 1836. The quote from Steiner, above, foreshadows the way that Steiner's work has been discussed and similarly misunderstood.

Steiner was a Swiss mathematician, although he did not learn to write until he was 14 years of age. He began school in Heidelberg in 1818 where he was noted for his mathematical abilities, and he began tutoring mathematics in 1821; shortly thereafter he was appointed as a teacher in the Gewerbeakademi (trade academy). Along with Abel, he founded *Crelle's Journal*, where the two were leading contributors. His particular field of study became known as synthetic geometry, or axiomatic geometry, or pure geometry; the study of geometry without the use of coordinates or formulae, relying instead on the axiomatic method and tools directly related to them (the compass and straight-edge). He did not rely on analysis or calculus for any of his works or proofs. [11]

2.10 Weierstrass

Karl Weierstrass was the first to state openly that Steiner's proof of the isoperimetric property of the circle was flawed. Steiner relied on the existence of a maximal area region, and given that this region exists then the circle is the shape with maximal area. Weierstrass added the missing piece to Steiner's proof, showing that a region of maximal area in fact exists for a fixed perimeter; this part of the proof relies on modern mathematics and the notions of limits and compactness, and not in an elementary way.

The Steiner proof relied only on geometrical methods to prove the isoperimetric property of the circle, assuming a solution exists; Weierstrass used advanced calculus to rigorously finish Steiner's proof showing the existence of a solution. Subsequent mathematicians have published alternate proofs of Steiner's isoperimetric property using other analytical means.

2.11 Demar

Richard Demar and Alfred Garvin exchanged ideas regarding a new proof of the isoperimetric problem. Garvin performed several experiments and made notes that supported Demar's hypothesis by observation; namely, that when a region is bound inside a triangle, it achieves maximal area when it touches the sides of the triangle and has caps, each of equal radius, which are tangent to the edges of the triangle.

2.12 Lawlor

Gary Lawlor presented a thoroughly different proof of the isoperimetric property, using slices of a circle and an arbitrary region, having the same perimeter. By slicing the circle and the region into a covering of triangles, Lawlor showed that there is not a region more optimal than the circle.

Chapter 3

Applications of the isoperimetric property

> The circle is the most simple, and the most perfect figure.
>
> Dante

3.1 More from Steiner

An application of Steiner's treatise, *Geometrical constructions with a ruler, given a fixed circle with its center*, is the following.

Theorem 5. There is exactly one circle which connects a fixed chord AB and a fixed arc of length p.

Proof. With the given arc AB we must complete the circle; that is, we must find its center and radius. Pick a unique point on arc AB, call it C, and draw $\triangle ABC$. Construct the perpendicular bisectors of two sides of $\triangle ABC$ to find the center of the circle. This circle is unique, as the perpendicular bisectors of the sides of the triangle are unique. [2] $\square$

Steiner also introduced a new synthetic geometry term: the power of a point. This concept is used often, including the Law of Cosines and a solution to Apollonius's problem.

3.2 Apollonius's problem

Apollonius of Perga posed a problem: construct circles that are tangent to three given circles in a plane. He proved the construction but his original work was lost. What has survived are the report of his results by Pappus of Alexandria.

Statement (Generalized). Construct one or more circles that are tangent to three given objects in a plane, where an object may be a line, a point, or a circle of any size. These objects may be arranged in any way and may cross one another; however, they are usually taken to be distinct, meaning that they do not coincide. [5]

Solutions to Appolonius's problems are called Apollonius circles, but sometimes this label is assigned to other circles associated with Apollonius.

3.3 Lord Kelvin

Sir William Thompson (Baron Kelvin) applied the isoperimetric property to economics and land values. He noted that not all land is valued the same, so a greater acreage was not necessarily more valuable. He used the isoperimetric property to maximize the area of valuable land when delineating property lines and to more properly assess the value of land owned for tax collection. [9]

3.4 Bees

> Bees are perfect at economizing labor and wax.
>
> ---
>
> Charles Darwin

Known since ±39 BCE and proved by Thomas Hales in 1999: hexagon is the most efficient way to fill space using a minimum amount of material while being able to enclose a maximum amount of honey. [7]

3.5 Soap Bubbles

Soap bubble: maximize volume inside itself with a finite amount of soap; i.e., if you have a given surface area, what's the biggest volume you can make?

3.6 Archimedes and Schwartz

Archimedes knew in 200 BCE that the sphere (or ball) has the greatest surface area for a fixed volume. In 1884 Herman Schwartz published Proof of the theorem that the ball has less surface area than any other body of the same volume.

Chapter 4

Using the isoperimetric property

> Thinking like a mathematician or scientist is not an innate skill.

4.1 Definitions

Note that these definitions refer specifically to planar bodies; that is to say, objects in 2-dimensional Euclidean space.

A dense set of points with the properties that each is equidistant from a given point (O, which is called the center) and each has a tangent is called a <u>circle</u>.

A set of points which extend in opposite directions indefinitely is called a <u>line</u>.

A set of points with one end that extends indefinitely in one direction is called a <u>half line</u>.

A <u>curve</u> is a continuous dense set of points, similar to a line but not necessarily straight; a line is a straight curve. Alternately, the trace left by a moving point, or the image of an interval to a plane by a continuous function.

A curve is <u>smooth</u> when it does not have any corners; alternately, when a curve is continuously differentiable.

A curve that does not cross itself is called a <u>Jordan curve</u>. If it gets back to itself it is a closed Jordan curve.

A line passing through two points of a curve is called a <u>secant line</u>. On a circle, the secant line is known as a <u>chord</u>.

A line which intersects a smooth curve at a single point is called a tangent line. Alternately, the limiting secant line as the two points come together.

The point where a tangent line intersects a curve is called the point of tangency.

A line of support or support line contains at least one point of the disk while the rest of the set of points of the disk exists on one side of it.

Any smooth curve joining two points is an arc. On a circle, an arc is a portion of the circumference (perimeter) of the circle.

The part of a circle bounded by a chord and an arc is called a segment of a circle.

A sector of a circle is bounded by two radii and an arc.

A circular segment or cap (denoted $\frown$) is a region comprised of an arc $\widehat{AB}$ of a circle together with the line segment AB as its boundary.

The angle formed in the interior of a circle when two secant lines intersect on the circle is an inscribed angle. An inscribed angle can also be described as the angle subtended at a point on a circle by two given points on the circle; equivalently, it is defined by two chords of a circle sharing an endpoint.

4.2 Exercises

Exercise 1. Assume we have a line e and a number $p > 0$. Find two points A and B along e and a Jordan curve length p connecting A and B such that the Jordan curve together with segment AB encloses the largest area.

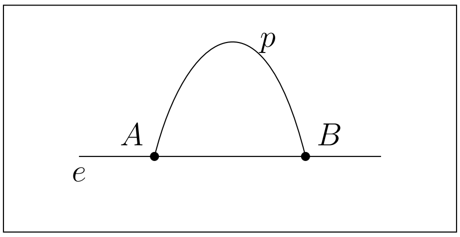

Figure 1.1: Illustration for Exercise 1

Solution of Exercise 1. Here we achieve maximal area with endpoints A and B on line e forming a semicircle.

We achieve this by reflecting the Jordan curve of length p to the other side of e to obtain a closed Jordan Curve with perimeter length $2p$. Consider the circle with radius $r = \frac{p}{\pi}$, denoted as C, with the line e coinciding with a diameter of C. The circle C has area $\mathbb{A}_C = \frac{p^2}{\pi}$, and by the isoperimetric theorem, $\mathbb{A}_C$ is greater than or equal to the area of the closed Jordan curve we reflected, with equality only when the closed Jordan curve is exactly C.

As C is the optimal shape with perimeter length $2p$, the semicircle with arc length p together with the segment AB along e is the solution.

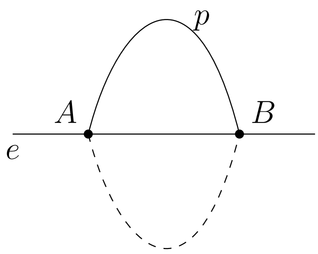

Figure 1.2: Reflect across e

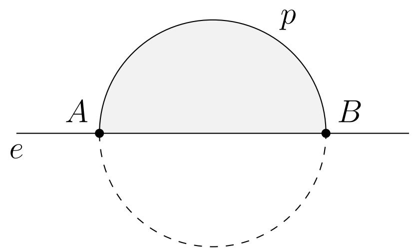

Figure 1.3: Expand into a semicircle with endpoints A and B

■

Exercise 2. Assume we have a segment AB and a number $p > 0$. Find the Jordan curve connecting A and B of length p which together with the segment AB encloses the largest area.

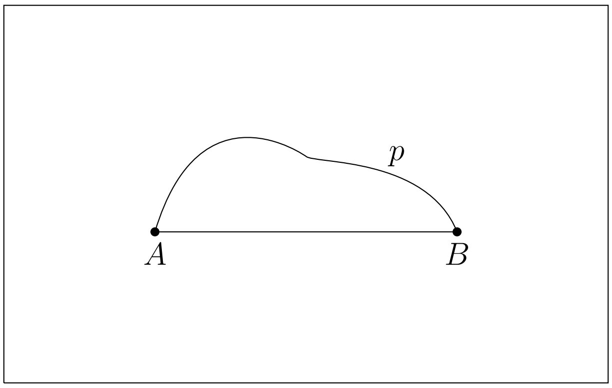

Figure 2.1: Illustration for Exercise 2

Solution of Exercise 2. Here we are going to show that a circular segment of length p with fixed endpoints A and B has maximal area.

We begin with a general Jordan curve of length p with fixed endpoints A and B (see Fig. 2.2). Applying the isoperimetric theorem, we get a unique circle which connects the fixed points A and B with arc length p (see Fig. 2.3).

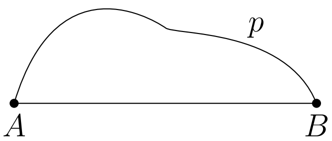

Figure 2.2: A general Jordan curve for Exercise 2

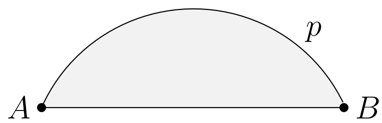

Figure 2.3: Arc length p forming a circular segment with fixed endpoints A and B

Now we will complete the circle corresponding to the arc of length p (see Fig. 2.4), copy it, and paste it onto the general Jordan curve of length p (see Fig. 2.5).

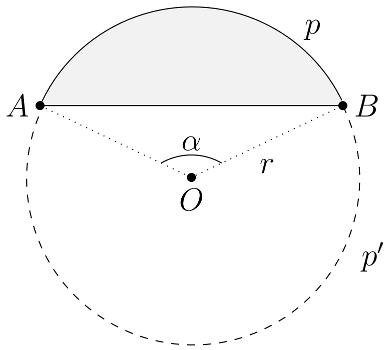

Figure 2.4: Complete the circle with p'

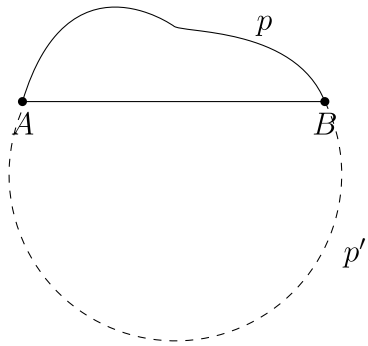

Figure 2.5: Arc length p forming a circular segment with fixed endpoints A and B

■

Exercise 3. Assume we have two half lines e and f starting at the same point O and perpendicular, and a number $p > 0$. Find a point A on half line e, a point B on half line f, and a Jordan curve of length p connecting A, B such that the Jordan curve together with the segments OA and OB enclose the largest area.

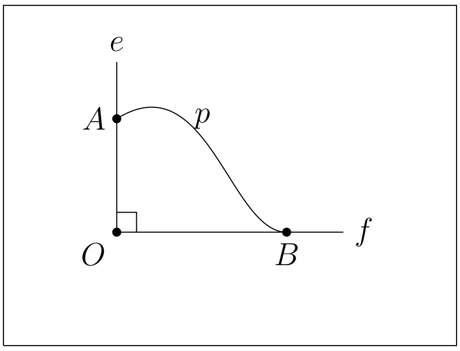

Figure 3.1: Illustration for Exercise 3

Solution of Exercise 3. Here we are going to show that the sector of a circle centered at O with interior angle $90°$ and subtended by an arc of length p connecting A and B has maximal area.

We know that there is exactly one circle which connects the fixed points A and B with arc length p, shown below with origin O. Reflect A and B such that we have four quadrants and a closed shape of perimeter $4p$, then expand the curve of length $4p$ into a circle centered about the origin.

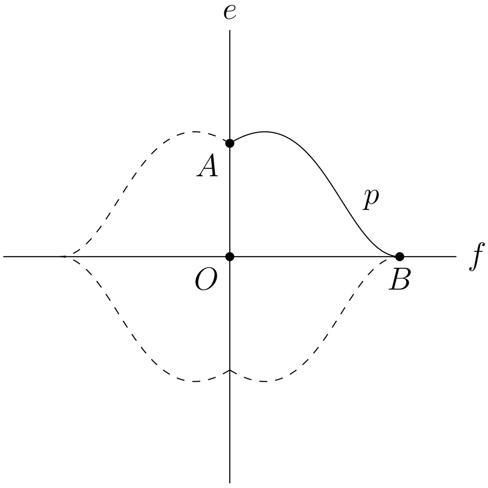

Figure 3.2: Reflect to quadrants

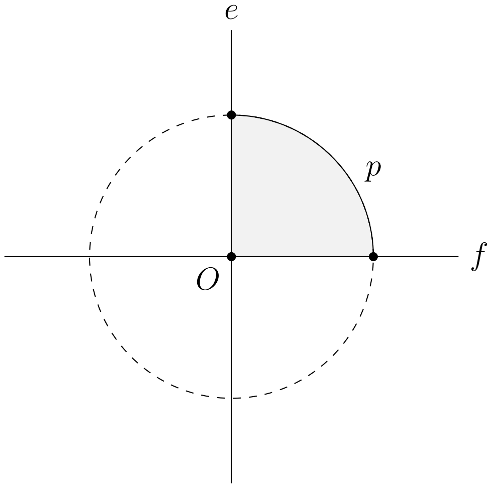

Figure 3.3: Expand into a circle

Now we have that the sector of a circle centered at O with interior angle $\frac{\pi}{2}$ and subtended by an arc of length p connecting A and B is equal to $\frac{1}{4}$ the area of the circle with perimeter $4p$.

When we arrange the points A, B on e, f, respectively, such that $\widehat{AB}$ of length p forms a one-quarter sector we get maximal area. $\blacksquare$

Exercise 4. Assume we have two half lines e and f starting at the same point O and creating a $60°$ angle, and a number $p > 0$. Find a point A on half line e, a point B on half line f, and a Jordan curve of length p connecting A, B such that the Jordan curve together with the segments OA and OB enclose the largest area.

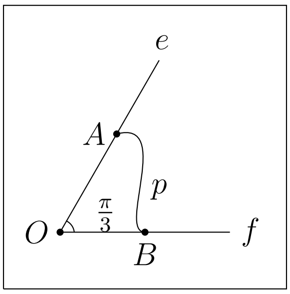

Figure 4.1: Illustration for Exercise 4

Solution of Exercise 4. Here we are going to show that the sector of a circle centered at O with interior angle $\frac{\pi}{3}$ and subtended by an arc of length p connecting A and B has maximal area.

First reflect 5 times to achieve a closed region of perimeter $6p$ (see Fig. 4.2). Then apply the isoperimetric theorem which results in a circle (see Fig. 4.3).

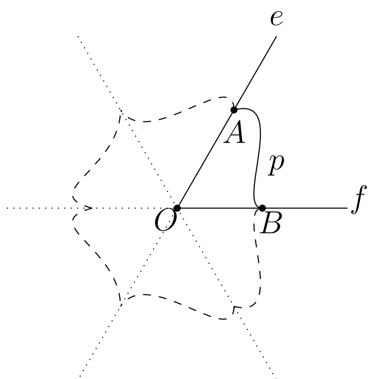

Figure 4.2: Reflect to make six parts

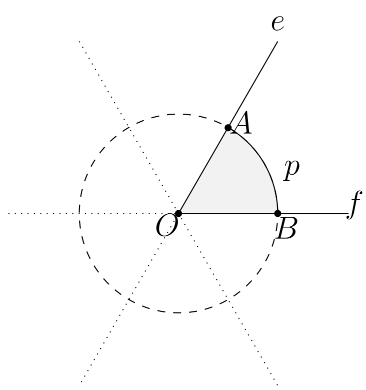

Figure 4.3: Expand the closed Jordan curve into a circle

When we arrange the points A, B on e, f, respectively, such that $\widehat{AB}$ of length p forms a one-sixth sector we get maximal area. ■

Exercise 5. Assume we have two half lines e and f starting at the same point O and creating a fixed angle α, and a number $p > 0$. Find a point A on half line e, a point B on half line f, and a Jordan curve of length p connecting A, B such that the Jordan curve together with the segments OA and OB enclose the largest area.

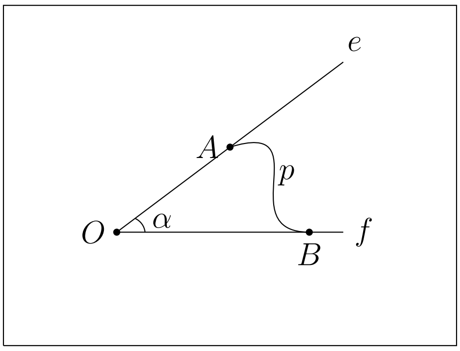

Figure 5.1: Illustration for Exercise 5

Solution of Exercise 5. Here we are going to show that the sector of a circle with center O with interior angle α and subtended by an arc of length p connecting A and B has maximal area.

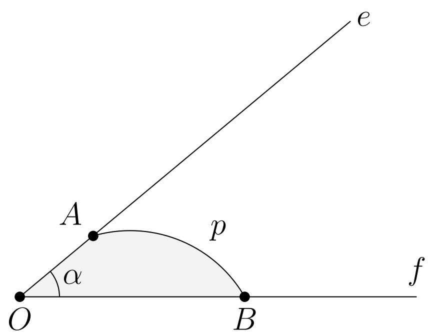

Figure 5.2: AB is connected with a circular arc of length p

In view of Exercise 2, we can assume that the optimal region is bounded by a circular arc AB (see Fig. 5.2). Fix points A and B in these positions to create $\triangle ABO$ and a circular sector. We reorient $\triangle ABO$ and maintain the fixed angle $AOB = \alpha$ (see Fig. 5.3).

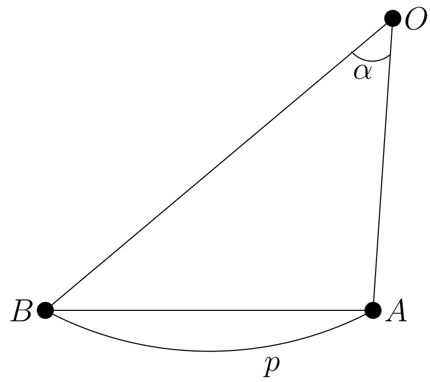

Figure 5.3: Triangle ABO and the circular sector AB with arc length p

The locus of points O with angle $AOB = \alpha$ is a circular arc. Clearly the midpoint of this arc lies the largest distance from AB and $\triangle ABO$ is isosceles (see Fig. 5.4).

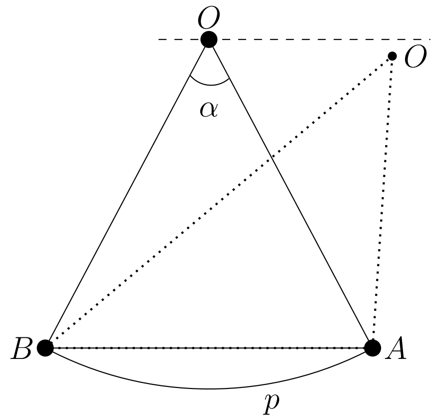

Figure 5.4: Modified triangle ABO and the circular sector AB with length p

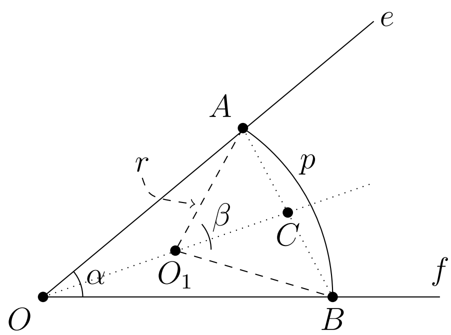

Figure 5.5: Circular arc of length p

We now have that the extremal must be isosceles. Using calculus, we get the largest area when $\alpha = \beta$.

Referencing Figure 5.5, denote the central angle corresponding to $\widehat{AB}$ by β. Let O_1 be the center of this circle. If $OA = OB$, then the center O_1 lies on the angle bisector of the angle AOB. As the distance $OA = OB$ increases, the length of the segment AB increases and β decreases. It is clear that β is largest when O_1A is perpendicular to OA. In this case $\beta = \pi + \alpha$, thus $0 < \beta \leq \pi + \alpha$.

Express the area of the enclosed region in terms of β. We will differentiate this expression in β and show that α is the only place where the derivative is equal to zero, meaning, $\beta = \alpha$ is the only critical number of the function and hence the maximum.

Let r be the radius O_1A and let C be the midpoint of AB. Notice that $r = \dfrac{p}{\beta}$, $AC = r \sin \dfrac{\beta}{2}$, $O_1C = r \cos \dfrac{\beta}{2}$, therefore the area between the chord AB and the circular arc AB is equal to

$$\frac{\beta}{2\pi}\left(\frac{p}{\beta}\right)^2 \pi - r^2 \sin\frac{\beta}{2}\cos\frac{\beta}{2},$$

and the area of $\triangle OAB$ is equal to $\left(\dfrac{p}{\beta}\right)^2 \sin^2\dfrac{\beta}{2}\cot\dfrac{\alpha}{2}$.

We need to maximize the area of the region which is enclosed by the segments OA, OB and the arc AB, which is a function $f(\beta)$.

$$f(\beta) = \frac{p^2}{2\beta} - \frac{p^2}{\beta^2}\frac{1}{2}\sin\beta + \frac{p^2}{\beta^2}\sin^2\frac{\beta}{2}\cot\frac{\alpha}{2}$$

By simple differentiation we get the following.

$$\begin{aligned} f'(\beta) &= \tfrac{p^2}{2}\left(-\tfrac{1}{\beta^2} - \tfrac{\cos\beta\cdot\beta^2 - \sin\beta\cdot 2\beta}{\beta^4} + \tfrac{2\cot\frac{\alpha}{2}\cdot\left(\sin\frac{\beta}{2}\cdot\cos\frac{\beta}{2}\cdot\beta - \sin^2\frac{\beta}{2}\cdot 2\beta\right)}{\beta^4}\right) \\ &= \tfrac{p^2}{2}\tfrac{-\beta-\beta\cos\beta+2\sin\beta+2\cot\frac{\alpha}{2}\cdot\left(\frac{1}{2}\sin\beta\cdot\beta-2\sin^2\frac{\beta}{2}\right)}{\beta^3} \end{aligned}$$

Now we check that $\beta = \alpha$ is a critical number.

$$f'(\alpha) = \frac{p^2}{2}\frac{-\alpha - \alpha\cos\alpha + 2\sin\alpha + \cot\frac{\alpha}{2}\cdot\sin\alpha\cdot\alpha - 4\cot\frac{\alpha}{2}\cdot\sin^2\frac{\alpha}{2}}{\alpha^3}$$

We make note that the third term in the numerator simplifies to $2\cos^2\frac{\alpha}{2}\cdot\alpha$ and the fourth term simplifies to $-2\sin\alpha$, thus $f'(\alpha) = 0$.

$$f'(\alpha) = \frac{p^2}{2}\frac{\alpha(-1 - \cos\alpha + 2\cos^2\frac{\alpha}{2})}{\alpha^3} = 0$$

Lastly, we need to check that $f(\beta)$ increases for $\beta < \alpha$ and decreases for $\beta > \alpha$. First we show that the derivative of $f'(\beta)$ is positive on the interval $0 < \beta < \alpha$. Since $\frac{p^2}{2\beta^3}$ is positive, it suffices to show that

$$-\beta - \beta\cos\beta + 2\sin\beta + 2\cot\frac{\alpha}{2}\cdot\left(\frac{1}{2}\sin\beta\cdot\beta - 2\sin^2\frac{\beta}{2}\right) > 0.$$

Closer inspection of this expression reveals that the coefficient of $\cot\frac{\alpha}{2}$ is negative.

$$\frac{1}{2}\sin\beta\cdot\beta - 2\sin^2\frac{\beta}{2} = \sin\frac{\beta}{2}\cdot\left(\cos\frac{\beta}{2}\cdot\beta - 2\sin\frac{\beta}{2}\right) = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\left(\frac{\beta}{2} - \tan\frac{\beta}{2}\right)$$

In this product the first two terms are positive, but the third term is negative, which shows that the coefficient of $\cot\frac{\alpha}{2}$ is negative. Now we rearrange the terms in $f'(\beta)$ and state that checking whether $f'(\beta)$ is positive on the interval $0 < \beta < \alpha$ is equivalent to checking, on the same interval, whether

$$\cot\frac{\alpha}{2} < \frac{1}{2}\cdot\frac{\beta + \beta\cos\beta - 2\sin\beta}{\frac{1}{2}\sin\beta\cdot\beta - 2\sin^2\frac{\beta}{2}}.$$

It was already proved that this inequality becomes an equation if α is replaced with β. Since $\cot(x)$ is a decreasing function, the above inequality is true, which we intended to show. The exact same argument also shows that $f'(\beta)$ decreases if $\alpha > \beta$, which completes the solution of Exercise 5.

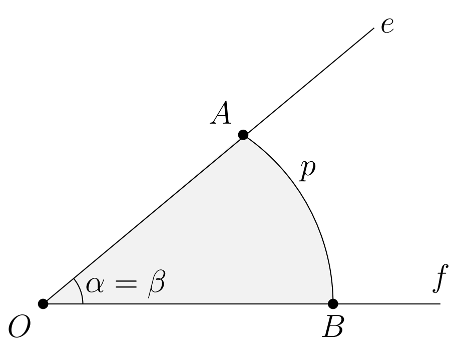

Figure 5.6: Maximal area solution

■

Exercise 6. Assume we have a unit equilateral triangle $\triangle ABC$ and a number $1 < p < 2$. Find the Jordan curve of length p connecting A, B such that the Jordan curve together with the edge AB encloses the largest area and where the Jordan curve stays within the bounds of $\triangle ABC$.

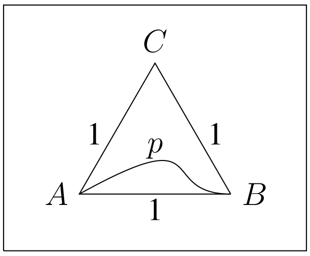

Figure 6.1: Illustration for Exercise 6

Solution of Exercise 6. Here we will show that the maximal area formed by the Jordan curve of length p connecting A and B together with edge AB is either a circular arc or a line segment-arc-line segment, where the line segments are of equal length and coincide with two edges of the triangle. The case where $\widehat{AB}$ together with edge AB meet the stated conditions is trivial as it is the case of Exercise 2.

Otherwise, we use the result of Demar (Theorem 4) to obtain a line segment-arc-line segment Jordan curve of length p which together with edge AB has maximal area. Applying the isoperimetric theorem and the result of Demar, we obtain line segments AA' and BB' on edges AC and BC, respectively, having the property $AA' = BB'$. We also have $\widehat{A'B'}$, a circular arc.

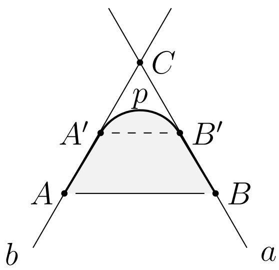

Figure 6.2: The Jordan curve is capped inside $\triangle ABC$

When we arrange A' on b and B' on a such that we achieve a segment-arc-segment for the Jordan curve we get maximal area. ■

Exercise 7. Assume we have a unit equilateral triangle $\triangle ABC$, its incirlce of perimeter $\frac{\pi}{\sqrt{3}}$, and a number $\frac{\pi}{\sqrt{3}} < p < 3$. Find the closed Jordan curve of length p which encloses the largest area and stays within the bounds of $\triangle ABC$.

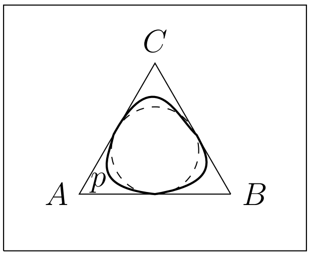

Figure 7.1: Illustration for Exercise 8

Solution of Exercise 7. Here we will show that the maximal area is obtained from the closed Jordan curve of length p with the properties: there is a line segment coinciding with each of the edges of $\triangle ABC$, each of these segments has its midpoint at the same point as the midpoint of its coinciding edge, each line segment is connected to the other two by a circular arc, and each of these circular arcs have the same radius.

We use the result of Demar (Theorem 4) together with the isoperimetric theorem to obtain line segments A_1A_2, B_1B_2, and C_1C_2 along edges A, B, and C, respectively. From Demar we get $AB_2 = AC_1 = BC_2 = BA_1 = CA_2 = CB_1$, isosceles caps near each corner, which are circular arcs of the same radius and arc length, and the closed Jordan curve does not contain any corner of $\triangle ABC$.

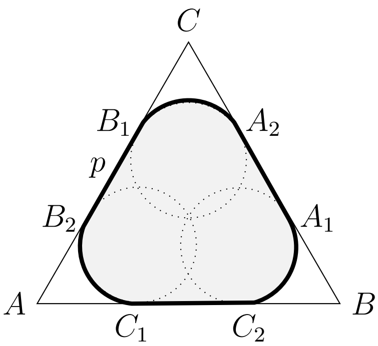

Figure 7.2: Extremal solution

■

Exercise 8. Assume we have a triangle $\triangle ABC$ with incircle perimeter length p_i and a number $p_i < p_j < AB + BC + CA$. Find the Jordan curve of length p_j which encloses the largest area and stays within the bounds of $\triangle ABC$.

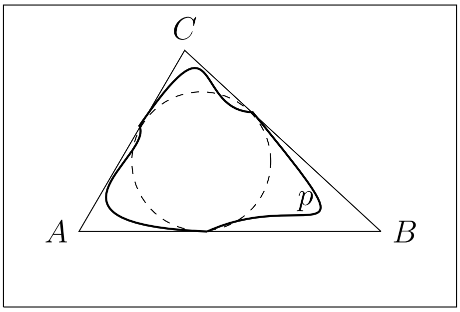

Figure 8.1: Illustration for Exercise 9

Solution of Exercise 8.

Theorem 6. Given a triangle $\triangle ABC$ with incircle perimeter length p_i and a number $p_i < p_j < AB + BC + CA$. The Jordan curve, γ, of length p_j which encloses the largest area and stays within the bounds of $\triangle ABC$ has the following properties:

Lemma 1.1. No vertices of $\triangle ABC$ are included in γ.

Lemma 1.2. The parts of γ that are not coincident to $\triangle ABC$ are circular arcs.

Lemma 1.3. The circular arcs of γ have the same radius.

Lemma 1.4. The edges of $\triangle ABC$ are tangent to the circular arcs of γ.

Proof. Let γ be the boundary of a solution and p_j the length of γ.

Proof of Lemma 1. First note that we cannot have all three corners of $\triangle ABC$ in γ as γ has length p_j which is strictly less than $AB + BC + CA$. Then there is at least one corner not included. Without loss of generality, suppose corner B is not in γ and corner C is in γ.

We take a small part of γ at C and move it to a part of γ near B. This cut is sufficiently small and its placement is such that γ remains inside the bounds of $\triangle ABC$. The area and perimeter length are preserved but we have a shape that is nonconvex. Hence, a shape which includes a corner can not be the solution, so no corners of $\triangle ABC$ are in the extremal.

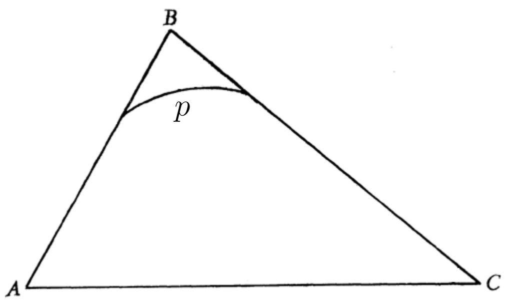

Figure 8.2: One corner not on γ

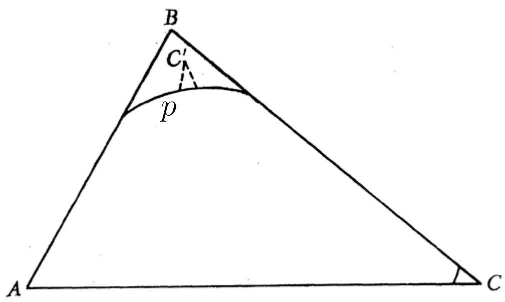

Figure 8.3: Cut and replace

Image credits: [4] ▲

Proof of Lemma 2. We cite Theorem 3 (Demar) that the extremal is a circular region. Thus, the parts of γ not coinciding with $\triangle ABC$ are circular arcs. ▲

Proof of Lemma 3. Suppose the circular arcs at B and C have different radii and let the radius of the arc at B be greater than the radius of the arc at C. We cut equal lengths of these arcs and replace them with the other. Area and perimeter length are preserved. The replacements result in a nonconvex. Therefore, each of the circular arcs must have the same radius

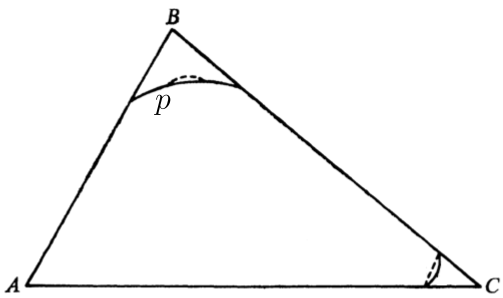

Figure 8.4: Cut and replace

Image credits: [4] ▲

Proof of Lemma 4. Suppose that γ is not tangent to $\triangle ABC$ near B. We cite Theorem 4 (Demar) to show that this orientation of γ is not extremal. Therefore, the circular arcs must be tangent to $\triangle ABC$ where each arc meets $\triangle ABC$. ▲

Thus, it must be the case that each of these four properties are properties of the extremal.

■

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Appendices

Appendix A

Notes on the history of the isoperimetric problem and Jakob Steiner

> Mathematical intelligence is neither necessary nor sufficient to deal with the trials and tribulations of the heart.
>
> Richard Tapia

The motivation for chapters 1 and 2 came from my research on the isoperimetric property, trying to find a primary source for Steiner's proof. I found numerous books, articles, and other materials that were structured in the same way but (a) did not contain a concise, structured begin and end of proof and (b) did not cite either Steiner or Weierstrass. It appeared as if the proof by Steiner was some kind of mythology or folklore.

I had the mathematics subject librarian, Patrician Hartman, in on this hunt as well. She and I both had difficulty finding the material, and when we did find it we had difficulty finding an English translation (or any language besides the original German). It was baffling to me, as a liberal-arts-trained researcher, that no one ever cited a primary source yet everyone had mostly the same things to say regarding this proof.

I was determined to find my primary source, and to find it in English. When Hartman and I were certain that it was not available in English anywhere, I spent untold hours with Google Translate creating my own. The translation of the full paper appears in Appendix C. The history of the isoperimetric problem was not intended to be part of this thesis when we began. It wasn't what we originally set out to do, but research evolves in its own way.

There is much irony in this quote from the opening paragraph of Steiner's paper:

> Unfortunately, [Lhulier's] work seems to have been quoted more often than the prevailing method has been correctly understood or duly appreciated and followed; for all his successors have deviated more or less from his simple, natural way of looking at things – apart from the fact that they also limited themselves to a much smaller number of tasks and propositions, – which also means the beautiful simplicity of the proofs, the intimate connection, to the same extent the sentences together with their inner reasoning disappeared.