Solvability and exact moment asymptotics for the interpolated stochastic heat and wave equation and the existence of an invariant measure for a special case.

by

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Abstract

This thesis will consist of two main projects, Chapters 2 and 4, and a smaller project in Chapter 3. We will be studying a general space-time fractional stochastic partial differential equation in Chapters 2 and 3 and the stochastic heat equation in Chapter 4, which is a special case of the just mentioned space-time fractional equation. The aim of this thesis is to handle the following: solvability of the equations, deriving exact moment asymptotics and proving the existence of an invariant measure.

In Chapter 2, we study a class of space-time fractional stochastic partial differential equations subject to some time-independent multiplicative Gaussian noise. We derive sharp conditions, under which a unique global $L^p(\Omega)$-solution exists for all $p \geqslant 2$. In this case, we derive exact moment asymptotics following the same strategy as that in a recent work by Balan *et al* [BCC22]. In the case when there exists only a local solution, we determine the precise deterministic time, T_2, before which a unique $L^2(\Omega)$-solution exists, but after which the series corresponding to the $L^2(\Omega)$ moment of the solution blows up. By properly choosing the parameters, results in this chapter interpolate the known results for both stochastic heat and wave equations.

In Chapter 3, we will again be studying the space-time fractional equation but driven by a space-time white noise. The goal of this project is to show the global existence of the solution when the diffusion term has super-linear growth. The work follows closely a recent work by Millet and Sanz-Solé [MS21].

Chapter 4 deals with the long term behavior of the solution to the nonlinear stochastic heat equation with no drift term that is driven by a Gaussian noise that is white in time and colored in space. Using the theory of the stochastic integral laid out by John Walsh, we provide conditions which will guarantee the existence of an invariant measure for a broad range of initial conditions, which includes bounded L^2_ρ functions as well as the Dirac delta distribution δ_0.

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Chapter 1

Introduction

The field of *stochastic partial differential equations* (SPDEs) is a relatively new field that has proven to be extremely useful. In the 1960's a Japanese mathematician named Kyosi Ito pioneered what would later be referred to as *Ito calculus*. This new and innovative material would then inspire other mathematicians, such as J. Walsh and R. Dalang, to set the framework for stochastic partial differential equations. Today, SPDEs is as active as ever and the field has even seen a Field's medal winner, Martin Hairer [Hai13], and several Nobel Memorial Prize in Economics winners, Myron Scholes and Robert Merton [1] [BS73].

Let's consider one application that will lead us to the first project presented in Chapter 2. Consider the situation where an infinitesimally thin piece of wire is initially heated. Also suppose that there is no external source of heat. Can we model the temperature of the wire as a function of position and time? In other words, does there exist a function, $u(t,x)$, such that $u(t,x)$ gives the temperature of the wire at a position x and at a time t? The answer is yes, and this scenario can be modeled by the well known heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0,x) = \zeta(x) & x \in \mathbb{R}, \end{cases}$$

where $\zeta(x)$ describes the initial temperature at a position x.

However, the above system is completely deterministic in the sense that there is no randomness involved. In other words, every single simulation of modeling the evolution of heat starting from $\zeta(x)$ will be the same. In practice this is unrealistic and for sure there will be

[1] Fischer Black would have also been Awarded the Nobel Memorial Prize in Economics but he passed away prior to the presentation of the award.

some randomness involved that will affect the evolution of the temperature. The need to model this randomness is the motivation behind the field of stochastic partial differential equations.

We will use a centerd, or 0-mean, Gaussian noise, which we denote as $\dot{W}$, to introduce randomness into our system. The noise will always be uniquely defined by associating it with an appropriate positive-definite covariance functional of the following form:

$$\mathbb{E}\left[\dot{W}(\psi)\,\dot{W}(\phi)\right] = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\psi(s,\cdot) * \widetilde{\phi}(s,\cdot))(x), \tag{1.0.1}$$

where ψ and ϕ are Schwartz functions, $\widetilde{\phi}(s,\cdot))(x) = \phi(s,-x)$ and $'*'$ denotes the spatial convolution. We will always assume that $\Gamma(\mathrm{d}x) = f(x)\mathrm{d}x$ is a non-negative and non-negative definite tempered measure and we will refer to f as the *correlation function* and its Fourier transform, $\widehat{f}(\xi) = \int_{\mathbb{R}^d} \exp(-ix\cdot\xi)f(x)\mathrm{d}x$, will be denoted as the *spectral density*.

Definition 1.0.1. The noise $\dot{W}$ has the following names under the following scenarios.

1. *Space-time white* : $f(x) = \delta_0(x)$ where δ_0 is the *Dirac delta distribution*,

2. *White in time and colored in space* : When $f(x)$ is a non-negative and nonnegative definite function. For example, f may take the form of any of the following well known kernels: *Riesz kernel, Poisson kernel, Ornstein-Uhlenbeck kernel, or Bessel kernel.*

3. *Time-independent* : When the integral in (1.0.1) is indepnedent of s, in other words,

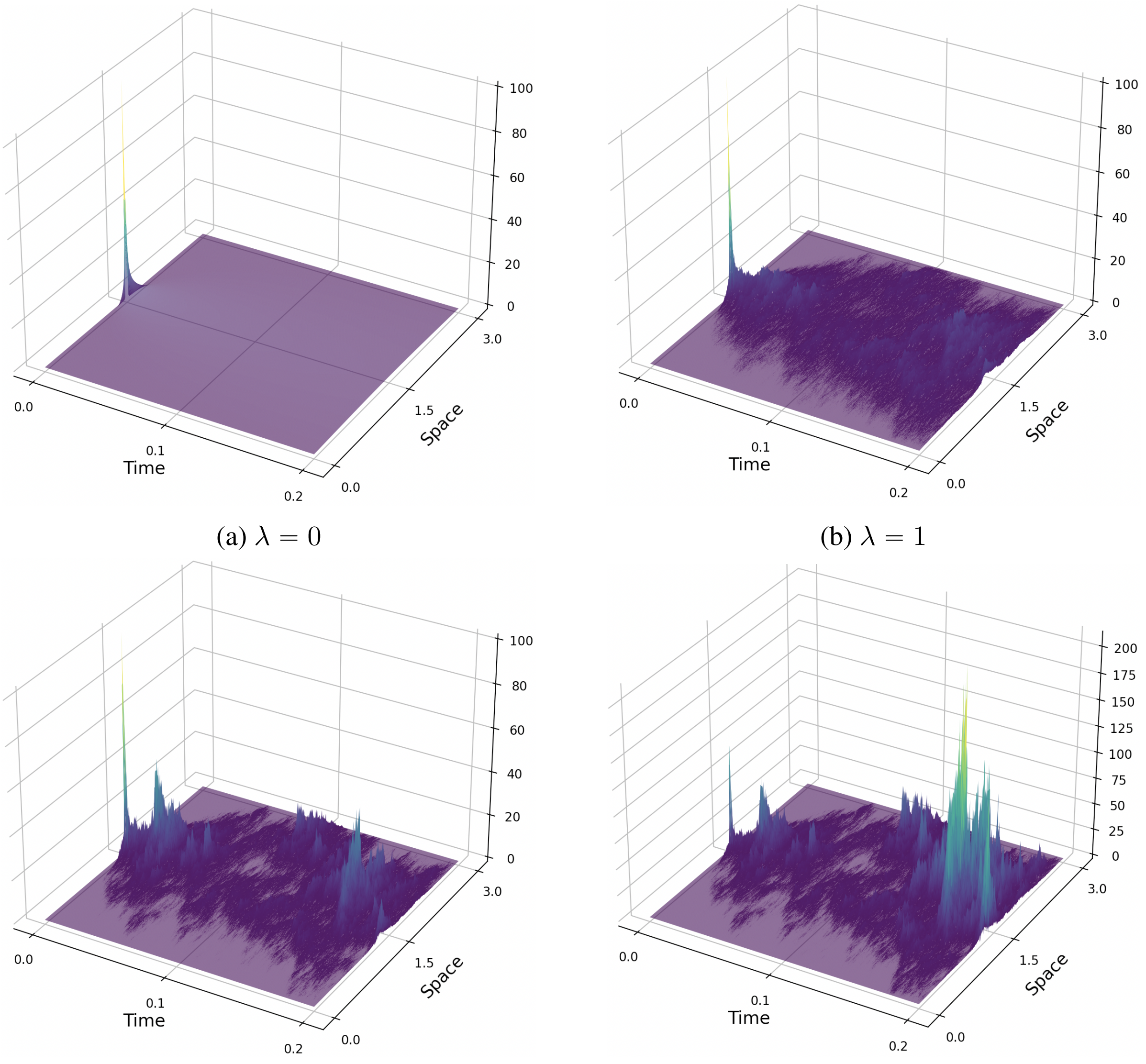
$$\mathbb{E}\left[\dot{W}(\psi)\,\dot{W}(\phi)\right] = \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\psi(\cdot) * \widetilde{\phi}(\cdot))(x),$$

   and $\Gamma(\mathrm{d}x)$ can be as in either of the two cases above.

Following Walsh [Wal86], we incorporate the noise into our system in the following way:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x = b(u(t,x))\dot{W}(t,x) & t \geqslant 0,\ x \in \mathbb{R}^d \\ u(0,x) = \zeta(x) & x \in \mathbb{R}^d, \end{cases} \tag{1.0.2}$$

where we assume the diffusion term, b, to be Lipschitz continuous with Lipschitz constant L_b. The above equation is purely notational and can be legitimately interpreted as the following



(a) $\lambda = 0$

(b) $\lambda = 1$

(c) $\lambda = 1.8$

(d) $\lambda = 2$

Figure 1.1: Simulations of Equation (1.0.2) with $d = 1$, $\zeta(\cdot) = \delta_0(\cdot)$ and $b(u) = \lambda u$.

stochastic integral equation:

$$u(t,x) = \int_{\mathbb{R}^d} G(t, x-y)\zeta(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) b(u(s,y)) W(\mathrm{d}s, \mathrm{d}y), \qquad (1.0.3)$$

where the stochastic integral above is the *Walsh integral* and $G(t,x)$ is the Gaussian heat kernel

$$G(t,x) = \frac{\exp\left(-\frac{|x|^2}{2t}\right)}{\sqrt{2\pi t}}, \quad t > 0, \ x \in \mathbb{R}^d.$$

Equation 1.0.2 is referred to as the *stochastic heat equation* (SHE) and is a widely studied stochastic partial differential equation, see for example [Kho14; Wal86; CD13; CD14; CM94; HHN16; BC18] and also Chapter 4 below, where we prove the existence of an invariant measure for the stochastic heat equation.

The noise term, $\dot{W}$, brings fundamental changes to the solution $u(t,x)$ which can be seen from the simulations in Figure 1.1. In particular, one can notice that as we increase λ, or the level of noise that we allow in our system, then the formation of taller and taller peaks begin to form. This phenomena is referred to as *intermittency*, see [CK19; KK15; CM95]. This concept will be one of the main focuses of Chapter 2, which I will now introduce.

1.0.1 A brief overview of Chapter 2

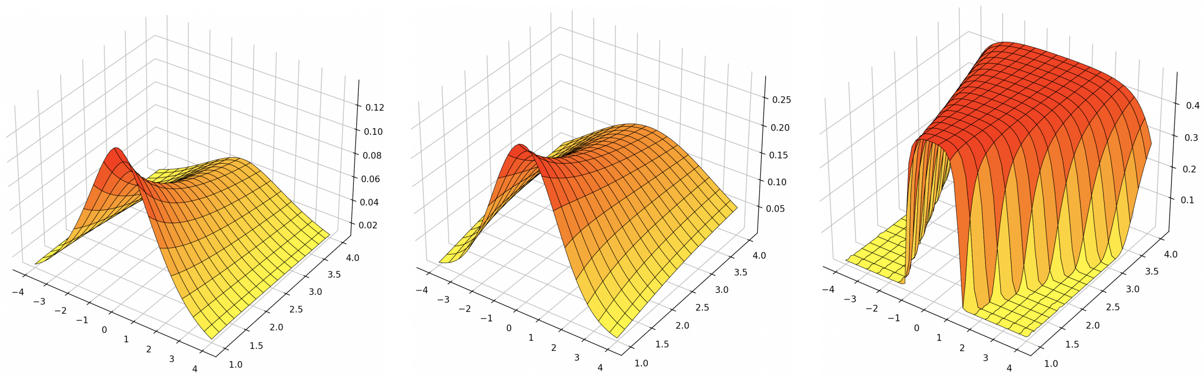
In the following chapter we will study the a space-time fractional stochastic partial differential equation driven by a *time-independent noise*, $\dot{W}(x)$. The equation interpolates both the stochastic heat and wave equations and has the following form:

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r\left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\, t>0, \\ u(0,\cdot) = 1 & b \in (0,1], \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2), \end{cases} \tag{1.0.4}$$

where ∂_t^b is the *Caputo* derivative, $(-\Delta)^{a/2}$ is the *fractional Laplacian* and I_t^r is the *Riemann-Liouville fractional integral*. Similar to the stochastic heat equation above, (1.0.4) is understood through the following stochastic integral equation:

$$u(t,x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta y) \right) \mathrm{d}s, \tag{1.0.5}$$

where, because of the choice of noise, the stochastic integral above is the *Skorohod* integral [NN18] and here $G = G_{a,b,r,\nu,d}$ is the fundamental solution which is given through the Fox H-function (see Figures 1.2 and 1.3 and [CHN19, Theorem 4.1]). One can see that when $a = 2$,



(a) b = .5 (b) b = 1 (c) b = 1.9

Figure 1.2: Some plots of $G_{2,b,0,2,1}(t,x)$ with $-4 \leqslant x \leqslant 4$ and $1 \leqslant t \leqslant 4$.

$b = 1$ and $r = 0$ then (1.0.4) reduces to the stochastic heat equation and when $a = b = 2$ and $r = 0$ then it reduces to the stochastic wave equation.

An important key feature of the solution, which is due to the choice of noise and simplified initial conditions, is the following Wiener Chaos expansion of the solution:

$$u(t,x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot, x, t)), \tag{1.0.6}$$

where I_k denotes the k^{th} order Skorohod integral and the kernels f_k are calculated through a standard Picard iteration procedure (see the discussion after Definition 2.3.1 below). Because of an orthogonality result that is exhibited by the integrals I_k (e.g. [NN18, Equation 4.1]), one may deduce the following useful expression for the second moment of the solution (see Theorem 2.3.3 below):

$$\mathbb{E}\left(|u(t,x)|^2\right) = \sum_{n \geqslant 0} \theta^n n! \left\| \widetilde{f}_n(\cdot, x, t) \right\|^2_{\mathcal{H}^{\otimes n}} \quad \text{for all } (t,x) \in (0,T) \times \mathbb{R}^d.$$

The main goals will be to prove existence and uniqueness of the solution (Theorem 2.1.6) and to calculate the following limits:

$$\lim_{t \to \infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) \quad \text{and} \quad \lim_{p \to \infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right),$$

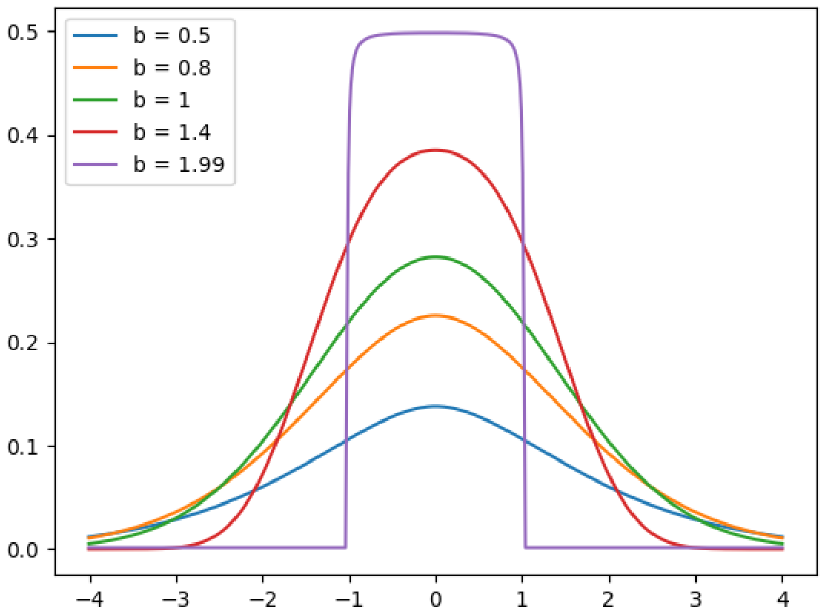


Figure 1.3: Cross sections of $G_{2,b,0,2,1}(t,x)$ with $-4 \leqslant x \leqslant 4$ and $t = 1$.

where β is a constant that is to be determined (see Theorem 2.1.7 and Corollary 2.1.8). One should note that both the positivity and the finiteness of the above limits imply that the following functions exhibit exponential growth:

$$t \mapsto \mathbb{E}\left(|u(t,x)|^p\right) \quad \text{and} \quad p \mapsto \mathbb{E}\left(|u(t,x)|^p\right).$$

The calculation of these limits is a very long and lengthly process so we hold any further discussion until Chapter 2.

However, as for the solvability of (1.0.4), we prove that the solution may uniquely exist either *globally* or *locally*. We say a global solution exists if for all $p \geqslant 2$, $x \in \mathbb{R}^d$ and $t > 0$, the p^{th} moment, $\|u(t,x)\|_p$, exists. On the other hand, we say that a local solution exists when there exists two times $T_{1,p} < T_{2,p}$ such that for all $p \geqslant 2$ and $x \in \mathbb{R}^d$, the p^{th} moment exists for $t < T_{1,p}$ and blows up for $t > T_{2,p}$. We prove that $T_{1,2} = T_{2,2}$ and in addition, for all $p \geqslant 2$, we calculate $T_{1,p}$ and give precise conditions on when a solution will be global or local. It is worth mentioning that the calculation of the cutoff time, $T_{1,p}$, is only possible due to our ability to express the solution in terms of its Wiener Chaos expansion as in (1.0.6).

1.0.2 A brief overview of Chapter 3

In this chapter we will study the same space-time fractional SPDE as in the previous chapter, however, in this case we will assume the noise is space-time white and that the diffusion term exhibits the following super-linear growth property as $|x|,\ |z| \to \infty$:

$$|\sigma(x) - \sigma(z)| \leqslant \sigma_2 |x - z| [\ln_+(|x - z|)]^\delta,$$

where $\sigma_2, \delta > 0$ and $\ln_+(z) := \ln(z \vee e)$ for $z > 0$. Moreover, we consider more general initial data, u_0 and v_0, than from the equation studied in Chapter 2. More precisely, we allow for any Borel-measurable initial conditions which satisfy for all $T > 0$ the following:

$$\sup_{(x,y)\in[0,T]\times\mathbb{R}^d} |J_0(t, x)| < \infty,$$

where J_0 is the solution to the homogeneous equation (see (1.0.8) below). With these slight changes, the equation takes the following form:

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t, x) = I_t^r \left[\sigma(u(t, x))\, \dot{W}(t, x)\right] & x \in \mathbb{R}^d, t > 0, \\ u(0, \cdot) = u_0(\cdot) & b \in (0, 1], \\ u(0, \cdot) = u_0(\cdot), \quad \partial_t u(0, \cdot) = v_0(\cdot) & b \in (1, 2). \end{cases} \tag{1.0.7}$$

As before, this is purely notational, and Equation (1.0.7) is legitimately viewed as the following stochastic integral equation:

$$u(t, x) = J_0(t, x) + I(t, x), \tag{1.0.8}$$

where

$$J_0(t, x) = \begin{cases} [Z(t, \cdot) * u_0](x) & \beta \in (0, 1] \\ [Z(t, \cdot) * v_0](x) + [Z^*(t, \cdot) * u_0](x) & \beta \in (1, 2) \end{cases}$$

and

$$I(t,x) = \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y).$$

The goal of this chapter is to prove the existence and uniqueness of a global solution to (1.0.7) (see Theorem 3.1.2 below). The work is highly motivated by the recent work by Millet and Sanz-Solé [MS21]. It is a well studied phenomena that either a super-linear drift or diffusion term may cause blow-up of the solution. As for the stochastic heat equation, we direct the reader to [FN21; MS93; DKZ19; BG09]. On the other hand, the only other work that we are aware of that is dedicated to proving some non-existence results of (1.0.7) is [AMN20]. So, to the best of our knowledge, this is the first work on proving the global existence of a solution to (1.0.7) with a super-linear diffusion term. However, as we will see, the calculations given below are essentially identical to those performed in [MS21], which is the cause for our motivation.

We will end this section with some comments on trying to prove Theorem 3.1.2 below but for a noise that is white in time and colored in space. This problem was answered in [MS21, Theorem 4.13] for the stochastic wave equation where global existence and uniqueness is proven. The challenge with applying their techniques to the space-time fractional equation, as in (1.0.7), comes from the increased complexity of the fundamental solutions. In particular, proving corresponding upper bounds for $R_1, \cdots, R_4$ from the proof of Theorem 4.10 *ibid* becomes less clear and will be saved for a future project.

1.0.3 A brief overview of Chapter 4

In the final chapter, we will turn our focus towards proving the existence of an invariant measure for the stochastic heat equation (1.0.2) on the whole space $\mathbb{R}^d$. Equations such as the SHE can be used by researchers to describe dynamical systems. Moreover, there often is a strong need to know the ergodic behavior of the dynamical system and no ergodic behavior can exist without an invariant measure. Hence proving the existence of such a measure is a crucial starting point.

Since we work on the whole space, one needs to introduce a positive, bounded and continuous weight $\rho \in L^1(\mathbb{R}^d)$ and the corresponding ρ-weighted space of square integrable functions

$L^2_\rho(\mathbb{R}^d)$ (see Section 4.2.1). We will see below in Theorem 4.1.2, that the solution starting from the initial condition μ, which we denote as $u(t,\cdot;\mu)$, is almost surely in this weighted space for a broad range of initial conditions, which includes all bounded functions, some unbounded functions such as $|x|^{-\alpha}$ with $0 < \alpha < d/2$ and even some measures such as the Dirac delta measure.

A crucial property of the system that must be exhibited for an invariant measure to exist is the *Markov property*. Essentially this means that one must be able to stop and restart their system whenever they choose and that the system then must act the same as it would if it were allowed to run uninterrupted. It makes sense that the SHE would satisfy this property since heat always moves from hot to cold. Thus no matter when we restart our system, the transfer of heat will just continue as if nothing happened.

However, this restart property is not always satisfied. Consider the situation where we pull back a string that is tightly secured at both end points. When we release the string, it will vibrate back and fourth until it loses its momentum and ends back at rest. The problem that presents itself here is that two vibrating strings could have the same position but different velocities (see Figure 1.4).

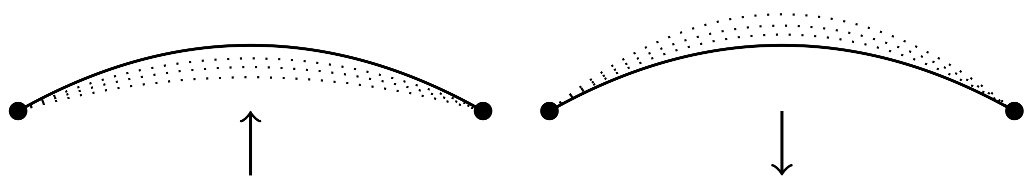


Figure 1.4: A still shot of two vibrating strings at the same position but moving in opposite directions.

For example, suppose we froze both strings illustrated above. When we initiate the restart, both stings will not continue as they were prior to the restart. What will happen is that they both will start moving in the direction of least resistance (see Figure 1.5). Thus the system will appear differently as it would if it were never restarted at all. This heuristically shows us that this system can not satisfy the Markov property and therefore an invariant measure will not exist in this setting.

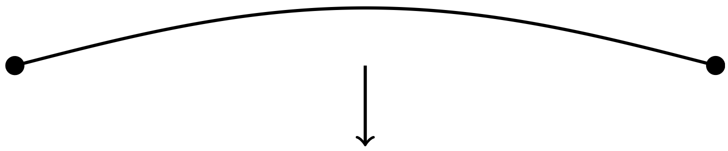


Figure 1.5: Direction of movement when re-releasing the string, regardless of the prior velocity.

We define the probability laws of the solution as

$$\mathscr{L}(u(t,\cdot;\zeta))(A) = \mathbb{P}\left[\omega \in \Omega : u(t,\cdot;\zeta)(\omega) \in A\right], \quad A \in \mathscr{B}(L^2_\rho(\mathbb{R}^d)),$$

where $\mathscr{B}(L^2_\rho(\mathbb{R}^d))$ denotes the Borel subsets of $L^2_\rho(\mathbb{R}^d)$ and ζ is the initial condition. Because of the Markov property of the solution, the laws form a family of *Markovian transition functions*, $P_t(\zeta, A) := \mathscr{L}(u(t,\cdot;\zeta))(A)$, for $\zeta \in L^2_\rho(\mathbb{R}^d)$. Moreover, the transition functions form a *transition semi-group* defined as follows:

$$P_t\varphi(x) = \int_{L^2_\rho(\mathbb{R}^d)} P_t(x, \mathrm{d}y)\varphi(y), \quad \varphi \in \mathcal{B}(L^2_\rho(\mathbb{R}^d)),$$

where $\mathcal{B}(L^2_\rho(\mathbb{R}^d))$ denotes the bounded Borel measurable functions on $L^2_\rho(\mathbb{R}^d)$.

With this said, we say that a probability measure, η, on $\mathscr{B}(L^2_\rho(\mathbb{R}^d))$ is *invariant* for (1.0.2) when the following holds:

$$\int_{L^2_\rho(\mathbb{R}^d)} P_t\varphi(x)\eta(\mathrm{d}x) = \int_{L^2_\rho(\mathbb{R}^d)} \varphi(x)\eta(\mathrm{d}x), \quad \text{for all } t \geqslant 0 \text{ and } \varphi \in \mathcal{B}(L^2_\rho)(\mathbb{R}^d)\,. \tag{1.0.9}$$

Recall that by definition $P_0\varphi(x) = \varphi(x)$ and so (1.0.9) is essentially saying that the transition semi-group is *time-invariant* with respect to the invariant measure η. We mention that this is a key feature of ergodicity.

We should also mention now that there is an equivalent way that one can define the invariant measure, and in fact it is the definition that we will choose to use for the remainder of this thesis. Any probability measure, η, on the Borel σ-field $\mathscr{B}(L^2_\rho(\mathbb{R}^d))$ is said to also be invariant for (1.0.2) if

$$\eta(A) = \int_{L^2_\rho(\mathbb{R}^d)} \mathscr{L}(u(t,\cdot;\zeta))(A)\,\eta(\mathrm{d}\zeta), \quad \text{for all } t \geqslant 0 \text{ and } A \in \mathscr{B}(L^2_\rho)(\mathbb{R}^d). \tag{1.0.10}$$

For more on these equivilent definitions, we direct the reader to [DZ14, Section 11.1].

Tessitore and Zabczyk laid forth a schematic to prove the existence of an invariant measure in their paper [TZ98], which revolves around showing the following:

1. the probability laws of the solution form a family of *Markovian transition functions*,
2. the solution, $u(t,x;\mu)$, satisfies a boundedness in probability condition.

Moreover, when one verifies the above steps, then by applying the Krylov–Bogoliubov existence theorem ([DZ14, Theorem 11.7]), it can be shown that the invariant measure, η, takes the following form for any $t_0 > 0$:

$$\eta(A) = \lim_{n\to\infty} \frac{1}{T_n} \int_{t_0}^{T_n+t_0} \mathscr{L}(u(t,\cdot;\mu))(A)\mathrm{d}t, \quad A \subset \mathscr{B}(L^2_\rho(\mathbb{R}^d)), \tag{1.0.11}$$

where $\{T_n\}_{n\geqslant 1}$ is an appropriately chosen sequence with $T_n \uparrow \infty$ (see Theorem 4.1.3 below).

Note that we already discussed that item 1 above is satisfied in our setting. On the other hand, the second item will require some effort to show. A sufficient condition that is more easily verified which implies this boundedness condition is the following:

$$\sup_{t>0} \mathbb{E}\left(\|u(t,\cdot)\|_\rho^2\right) < \infty. \tag{1.0.12}$$

We remind the reader that for the SHE, moments usually grow exponentially in time (e.g. see [CK19, Theorem 1.3]), and so for the above finiteness to occur, we will need additional assumptions (e.g. see (4.1.10) below).

In fact, in order for Tessitore and Zabczyk to provide a specific scenario where an invariant measure exists (e.g. [TZ98, Theorem 3.3]), they had to significantly strengthen their assumptions and require the following:

$$d \geqslant 3 \quad \text{and} \quad L_b^{-2} > \frac{\Gamma(d/2-1)2^{d/2-2}}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \left(\left| \mathcal{F}\left(\sqrt{\widehat{f}}\right) \right| * \left| \mathcal{F}\left(\sqrt{\widehat{f}}\right) \right| \right) (\zeta) |\zeta|^{2-d} \mathrm{d}\zeta, \tag{1.0.13}$$

where L_b is the Lipschitz constant of the diffusion term and $\hat{f}$ is the spectral density of the noise (see Section 4.2.4 below). Moreover, under this strengthened assumption, they proved that there exists an invariant measure for the SHE starting from the constant 1 initial condition (e.g. (1.0.2) with $\zeta(x) = 1$). However, due to its complexity, (1.0.13) was not calculated for any specific $\widehat{f}$ and without being able to do so, one may not apply this result for a specific noise. This is clearly a huge set back as picking the noise of your system is the main reason one would want to study a SPDE.

Our results vastly improve on this as we provide much simpler and more easily verifiable conditions in our Theorem 4.1.3 below that will guarantee the existence of an invariant measure. Moreover in Section 4.2.3, we are able to give specific examples of spectral densities, $\widehat{f}$, that satisfy the conditions of our Theorem 4.1.3. In addition, in Section 4.2.2 we enlarge the space of allowable initial conditions to include all bounded functions, some unbounded functions and even some measures such as the Dirac delta measure.

Chapter 2

Exact Moment Asymptotics for the Interpolated Stochastic Heat and Wave Equation

2.1 Introduction and main results

In this chapter we study the following *stochastic partial differential equation* (SPDE) with fractional differential operators:

$$
\begin{cases}
\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r\left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\, t>0,\\
u(0,\cdot) = 1 & b \in (0,1],\\
u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2),
\end{cases} \tag{2.1.1}
$$

where $a \in (0,2]$, $b \in (0,2)$, $r \geqslant 0$, $\nu > 0$ and $\theta > 0$. Here the noise $W = \left\{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\right\}$ is a centered and time-independent Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with mean zero and covariance

$$
\mathbb{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi) =: \langle \phi, \psi \rangle_{\mathcal{H}},
$$

where μ refers to the *spectral measure*, which is assumed to be a nonnegative and nonnegative definite tempered measure on $\mathbb{R}^d$. Let γ be the Fourier transform of μ (see Section 2.3.1), which is also a nonnegative and nonnegative definite measure on $\mathbb{R}^d$ thanks to Bochner's theorem. Throughout this chapter, we will use $\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^d} \exp(-ix\xi)\phi(x)\mathrm{d}x$ to denote the Fourier transform of a test function ϕ.

In (2.1.1), $(-\Delta)^{a/2}$ refers to the *fractional Laplacian* of order a, ∂_t^b denotes the *Caputo fractional differential operator*

$$\partial_t^b f(t) := \begin{cases} \dfrac{1}{\Gamma(m-b)} \int_0^t \mathrm{d}\tau \dfrac{f^{(m)}(\tau)}{(t-\tau)^{b+1-m}} & \text{if } m-1 < b < m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } b = m, \end{cases}$$

where m is an integer, and I_t^r refers to the *Riemann-Liouville fractional integral* of order $r > 0$

$$I_t^r f(t) := \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)\mathrm{d}s, \quad \text{for } t > 0,$$

with the convention that when $r = 0$, $I_t^0 = \mathrm{Id}$ reduces to the identity operator. The fundamental solution to (2.1.1) is expressed explicitly in terms of the *Fox H-function*, $H_{p,q}^{m,n}(z)$, which is much more complicated than the Green's function for either the heat or wave equation. We denote the fundamental solution as

$$G(t,x) := G_{a,b,r,\nu,d}(t,x), \tag{2.1.2}$$

where

$$G_{a,b,r,\nu,d}(t,x) = \pi^{-d/2}|x|^{-d} t^{b+r-1} H_{2,3}^{2,1}\left(\frac{|x|^a}{2^{a-1}\nu t^b} \,\middle|\, \begin{matrix} (1,1),\ (b+r,b) \\ (d/2,a/2),\ (1,1),\ (1,a/2) \end{matrix}\right).$$

We direct the reader to Theorem 4.1[1] of [CHN19] for more details. Since we are interested in the constant one initial condition (and zero initial velocity when $b > 1$), Theorem 4.1 (*ibid.*) implies that the corresponding solution to the homogeneous equation (i.e. the solution when there is no driving source) is equal to the constant one. Hence through superposition, (2.1.1) can be written as the following stochastic integral equation:

$$u(t,x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta y) \right) \mathrm{d}s, \tag{2.1.3}$$

[1] $G(t,x)$ corresponds to $Y_{a,b,r,\nu,d}(t,x)$ from [CHN19].

where the stochastic integral is in the *Skorohod* sense; see Definition 2.3.1 below. In the following, the fundamental solution will exclusively refer to $G(t,x)$, which is indeed a smooth function for $x \neq 0$. Our results rely on the following assumption for the nonnegativity of $G(t,x)$:

Assumption 2.1.1 (Nonnegativity)**.** Assume that the fundamental solution $G(t,x)$ is nonnegative for all $t > 0$ and $x \in \mathbb{R}^d$.

Remark 2.1.2. Thanks to Theorem 4.6 of [CHN19] (see also Theorem 3.1 of [Che+17] for the case when $r = 0$), we have the following four groups of sufficient conditions [2], under either group of which $G(t,\cdot)$ is nonnegative (see Figure 2.1 for an illustration) :

1. $d \geqslant 1$, $b \in (0,1]$, $a \in (0,2]$, $r \geqslant 0$;
2. $1 \leqslant d \leqslant 3$, $1 < b < a \leqslant 2$, $r > 0$;
3. $1 \leqslant d \leqslant 3$, $1 < b = a < 2$, $r > \dfrac{d+3}{2} - b$.

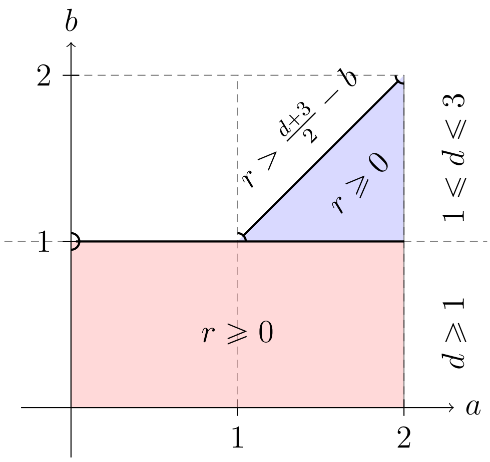


Figure 2.1: Illustration of the sufficient conditions (Remark 2.1.2) for $G(t,\cdot)$ to be nonnegative.

Regarding the noise, we formulate the following assumption in order to cover the Riesz kernel case, the fractional noise and a mixture of them:

[2]Note that when $d \geqslant 1$, $b = 1$ and $a \in (0,2]$, part (1) of [CHN19, Theorem 4.6] says that the fundamental solution Y, which is the fundamental solution G in this chapter, is nonnegative provided $r = 0$ or $r > 1$. Indeed, because in this case Z is always nonnegative, for $r > 0$, Y as a fractional integral of Z (see (4.5), *ibid.*), Y, or our G, should also be nonnegative. We thank Guannan Hu who pointed out to us this observation.

Assumption 2.1.3 (Noise)**.** Let $k \in \{1, \cdots, d\}$ and partition the d-coordinates of $x = (x_1, \cdots, x_d)$ into k distinct groups of size d_i so that $d_1 + \cdots + d_k = d$. Denote $x_{(i)} = (x_{i_1}, \cdots, x_{i_{d_i}})$ to be the coordinates in the i^{th} partition. Assume that the correlation function of the Gaussian noise is given by

$$\gamma(x) = \prod_{i=1}^{k} \left|x_{(i)}\right|^{-\alpha_i} \qquad \text{with } \alpha_i \in (0, d_i). \tag{2.1.4}$$

Define $\alpha := \sum_{i=1}^{k} \alpha_i$.

Remark 2.1.4 (Spectral density and decomposition)**.** Recall that the *spectral density* of γ from (2.1.4), which by definition is $\mathcal{F}\gamma$, takes the following form:

$$\mu(\mathrm{d}\xi) = \varphi(\xi)\mathrm{d}\xi \quad \text{with} \quad \varphi(\xi) = \prod_{i=1}^{k} C_{\alpha_i,d_i} |\xi_{(i)}|^{-(d_i-\alpha_i)}. \tag{2.1.5}$$

Moreover, in the derivations below, we need to find a nonnegative and nonnegative definite K such that $\gamma = K * K$ where '$*$' denotes the spatial convolution. Indeed, one can choose

$$K(x) = \prod_{i=1}^{k} \beta_{\alpha_i,d_i} \left|x_{(i)}\right|^{-(d_i+\alpha_i)/2}. \tag{2.1.6}$$

The two constants in both (2.1.5) and (2.1.6) are defined as

$$C_{\alpha,d} = \pi^{-d/2} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \quad \text{and} \quad \beta_{\alpha,d} = \pi^{-d/4} \frac{\Gamma((d+\alpha)/4)}{\Gamma((d-\alpha)/4)} \sqrt{\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}}. \tag{2.1.7}$$

Example 2.1.5 (Noises)**.** We have the following special cases: (1) Setting $k = 1$ in (2.1.4) and (2.1.5) recovers the Riesz kernel case. In this case,

$$\gamma(x) = |x|^{-\alpha}, \quad \varphi(x) = C_{\alpha,d}|x|^{-(d-\alpha)} \quad \text{and} \quad K(x) = \beta_{\alpha,d}|x|^{-(d+\alpha)/2}. \tag{2.1.8}$$

(2) Setting $k = d$ in (2.1.4) and (2.1.5) recovers the time-independent fractional noise. The corresponding SHE with such noise was earlier studied by Hu come back to this citation [Hu01].

For this noise, we have that

$$\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\alpha_i}, \quad \varphi(\xi) = \prod_{i=1}^{d} C_{\alpha_i,1}|\xi_i|^{-(1-\alpha_i)} \quad \text{and} \quad K(x) = \prod_{i=1}^{d} \beta_{\alpha_i,1}|x_i|^{-(1+\alpha_i)/2}. \tag{2.1.9}$$

In a recent work by Balan *et al* [BCC22], the same equation as (2.1.1), but exclusively for the *stochastic wave equation* (SWE), namely, the case when $a = b = \nu = 2$ and $r = 0$, has been studied, where both the well-posedness and the exact moment asymptotics have been obtained. The corresponding *stochastic heat equation* (SHE), namely, the case when $a = 2$, $b = \nu = 1$ and $r = 0$, has been earlier studied by Hu [Hu01], but only for the well-posedness and exclusively for the fractional noise (2.1.9). The corresponding moment asymptotics have been obtained by X. Chen [Che17b] as a special case by setting $\alpha_0 = 0$. One may check Remark 1.9 of Balan *et al* [BCC22] for the explicit expressions in terms of notation of the current thesis. In this chapter, by working on a more general class of SPDEs, we are able to interpolate the asymptotics for both SWE and SHE; see Section 2.2.2 below for more details. Moreover, we give the sharp conditions under which there exists only a local $L^2(\Omega)$ solution.

The moment asymptotics obtained by X. Chen, such as those in [Che17b; Che19], rely crucially on the Feynman-Kac representation of the moments of the solution. However, whenever $b \neq 1$, especially for the case when $b \in (1,2)$, we are not aware of any such Feynman-Kac formula for the moments. Instead, in the recent work by Balan *et al* [BCC22], this difficulty has been overcome by studying the Wiener chaos expansion of the solution. In this chapter, we follow the same strategy laid out by Balan *et al* (*ibid.*). The challenge comes from the much more involved parametric form of the fundamental solution.

Now let us state the main results of this chapter. The first main result deals with the well-posedness of the SPDE (2.1.1) (or (2.1.3)) as stated in the following theorem. For this, we need to introduce the following variational constant (see Section 2.3.2 for more details):

$$\mathcal{M}_{a,d}(f) := \sup_{g\in\mathcal{F}_a} \left\{ \left\langle g^2 * g^2, f \right\rangle_{L^2(\mathbb{R}^d)}^{1/2} - \frac{1}{2}\,\mathcal{E}_a(g,g) \right\}. \tag{2.1.10}$$

We use the convention that $\mathcal{M}_a(f) := \mathcal{M}_{a,d}(f)$ when the dimension is clear from the context, and $\mathcal{M}_a := \mathcal{M}_a(\gamma)$, where γ is defined in (2.1.4). It is important to note that by Theorem 2.3.5, stated and proven below, that $\mathcal{M}_a < \infty$.

Theorem 2.1.6 (Solvability)**.** *Assume that both Assumptions 2.1.1 and 2.1.3 hold.*

(1) (2.1.1) *has a unique (global) solution $u(t,x)$ in $L^p(\Omega)$ for all $p \geqslant 2$, $t > 0$, and $x \in \mathbb{R}^d$ provided that*

$$0 < \alpha < \min\left(\frac{a}{b}\left[2(b+r)-1\right], 2a, d\right). \tag{2.1.11}$$

(2) Otherwise, if

$$r \in [0, 1/2) \qquad \text{and} \qquad 0 < \alpha = \frac{a}{b}[2(b+r)-1] \leqslant d, \tag{2.1.12}$$

then (2.1.1) *has a local solution in the sense that*

(2-i) For any $p \geqslant 2$, (2.1.1) *has a unique solution $u(t,x)$ in $L^p(\Omega)$ for all $p \geqslant 2$ and $x \in \mathbb{R}^d$, but only for $t \in (0, T_p)$ where*

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}. \tag{2.1.13}$$

(2-ii) For any $t > T_2$, the series (2.3.9) *below diverges, that is, the $L^2(\Omega)$-solution $u(t,x)$ to* (2.1.1) *does not exist whenever $t > T_2$.*

The second main result of this chapter is about the moment asymptotics. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$ moments.

Theorem 2.1.7. *Under Assumptions 2.1.1 and 2.1.3, if condition* (2.1.11) *holds, then we have that*

$$\begin{aligned}\lim_{t_p\to\infty} t_p^{-\beta}\log\|u(t,x)\|_p = &\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\ &\times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right),\end{aligned} \tag{2.1.14}$$

where

$$\beta := \frac{2(b+r)-\dfrac{b\alpha}{a}}{2(b+r)-\dfrac{b\alpha}{a}-1} \qquad \textit{and} \qquad t_p := (p-1)^{1-1/\beta}\, t. \tag{2.1.15}$$

Proof. We prove the matching upper bound (2.5.1) and the lower bound (2.6.1) of (2.1.14) at the end of Sections 2.5 and 2.6 below, respectively, which together prove (2.1.14). □

As a direct consequence of (2.1.14), one can send either t or p to infinity as follows:

Corollary 2.1.8. *Under both Assumptions 2.1.1 and 2.1.3, if condition (2.1.11) holds, then*

(1) For all $p \geqslant 2$ fixed, it holds that

$$\begin{aligned}\lim_{t\to\infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) =& p(p-1)^{\frac{1}{2(b+r)-\frac{b\alpha}{a}-1}} \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\ &\times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right);\end{aligned} \tag{2.1.16}$$

(2) For all $t > 0$ fixed, it holds that

$$\begin{aligned}\lim_{p\to\infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) =& t^{\beta} \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\ &\times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).\end{aligned} \tag{2.1.17}$$

This chapter is organized as follows. In Section 2.2, we first give some concrete examples, where one can find many explicit formulas for either moment asymptotics in the case of global solutions or the expressions for the critical time T_p in the case of local solutions. Then in Section 2.3, we present some preliminaries, including the *Skorohod* integral, definition of the mild solution, and some asymptotics with corresponding variational constants. We prove part (1) and part (2) of Theorem 2.1.6 in Sections 2.4 and 2.5, respectively. The upper bound and lower bounds for (2.1.14) are established in Sections 2.5 and 2.6, respectively. Finally, in the appendix — Section 2.7, we list a few proofs of results that will be used.

2.2 Examples on solvability and asymptotics

In this section, we will give various examples to illustrate our main results. The cases with $b = 1$ and $r = 0$ are mostly known, which will be pointed out in the example below and will be used as test examples for our results. To the best of our knowledge, all results in this chapter for either $b \neq 1, 2$ or $r > 0$ should be new.

2.2.1 Examples on solvability

In this part, we list some concrete examples regarding the solvability — Theorem 2.1.6.

Example 2.2.1 (SHE)**.** By setting $a = 2$, $b = 1$ and $r = 0$ in (2.1.12), we obtain the following condition for the SHE under which there only exists a local solution:

$$\alpha = 2 \leqslant d. \tag{2.2.1}$$

Clearly, the fundamental solutions in this case are nonnegative for all $d \geqslant 1$. Hence, the picture is slightly more complicated since we need to check all possible dimensions $d \geqslant 1$. We illustrate possible cases in Figure 2.2. In particular, let us explain a few cases:

(a) When $d = 2$, condition (2.2.1) says that the 2-dimensional SHE driven by white noise has only a local $L^p(\Omega)$ solution. By applying (2.1.13) to this case, the critical time T_p becomes

$$T_p = \frac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}, \qquad p \geqslant 2. \tag{2.2.2}$$

Note that in part 2) of Theorem 4.1 of Hu [Hu02], some lower and upper bounds for T_2 were obtained. More precisely, by setting additionally that $\theta = 1$ and $\nu = 1$, Hu (*ibid.*) proved that when $t < 2$, an $L^2(\Omega)$ solution exists but when $t > 2\pi$, the second moment of the solution blows up. It is an interesting exercise to show that

$$2 \leqslant T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \leqslant 2\pi, \qquad \text{where } d = 2.$$

This case is covered as a special time-independent case (i.e., $H_0 = 1$) by Chen *et al* [Che+21, Theorem 3.4 and Remark 3.13].

(b) Recall that the white noise driven SHE corresponds to when $\alpha = d$. Therefore by examining (2.1.11) and (2.1.12), we see that when $d \geqslant 3$, the SHE driven by white noise no longer has any $L^2(\Omega)$-solution. In addition, local solutions exist only when $\alpha = 2$ and the noise is not white. This is illustrated in Figure 2.2 below. In addition, the critical time T_p takes the same expression as (2.2.2) but one needs to replace δ_0 by γ.

Remark 2.2.2. Note that we use the Skorohod integral to interpret the multiplication of the solution with the noise in (2.1.1). Multiplication interpreted in this way is traditionally called the *Wick product* which is consistent with the *Itô* or *Walsh* integral (see, e.g., [Dal+09]) when the noise is white in time. One can also interpret this product as the usual product. In order to handle the singularities caused by this multiplication, one needs to carry out certain renormalization processes. In fact, for the standard SHE with white noise in $\mathbb{R}^d$ (i.e., $a = 2$, $b = 1$, $\alpha = d$ and $r = 0$), Hairer and Labbé constructed pathwise solutions using the regularity structure for both cases $d = 2, 3$ in [HL15] and [HL18], respectively. The relation between these two types of solution is left for future work.

Example 2.2.3 (SWE)**.** By setting $a = 2$ and formally setting $b = 2$ in (2.1.12), we obtain the following condition for the stochastic wave equation under which there only exists a local solution:

$$\alpha = 3 + 2r \leqslant d \quad \text{and} \quad r \in [0, 1/2]. \tag{2.2.3}$$

We recall that results in Balan *et al* [BCC22] require $d \leqslant 3$, and likewise, Assumption 2.1.1 and all known sufficient conditions for the nonnegativity of the fundamental solution (see Remark

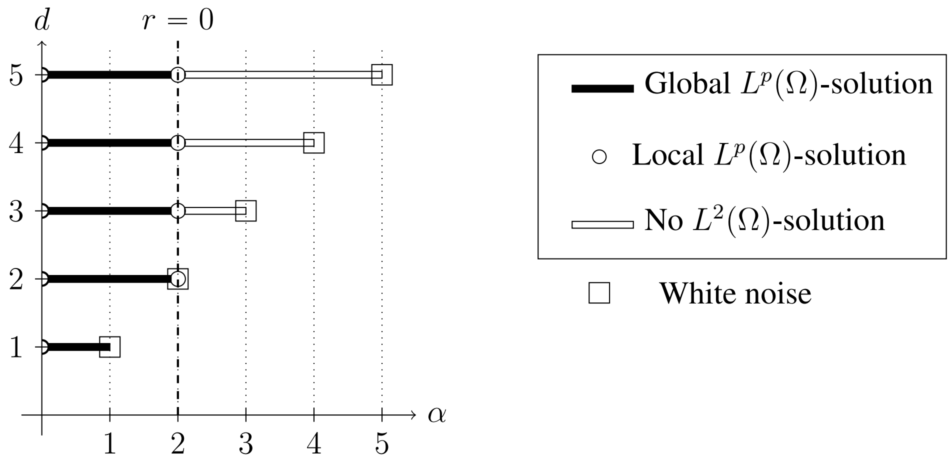


Figure 2.2: Solvability for the stochastic heat equation (i.e., $a = 2$, $b = 1$ and $r = 0$) with $p \geqslant 2$.

2.1.2) also require $d \leqslant 3$ in case of $b \in (1, 2)$. With this restriction, conditions (2.2.3) reduce to

$$\alpha = 3 = d \quad \text{and} \quad r = 0,$$

which says that at dimension $d = 3$, when $\dot{W}$ is a white noise, there exists only a local $L^p(\Omega)$ solution for all $p \geqslant 2$. See Figure 2.3 for an illustration. Moreover, one can check easily that the expression for the critical time T_p in (2.1.13) in this case reduces to

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \qquad p \geqslant 2, \tag{2.2.4}$$

which is identical to (1.12) (*ibid.*) when setting $\nu = 2$.

Example 2.2.4 (Fractional SPDEs with $r = \lceil b \rceil - b$ and $a = 2$)**.** For the fractional SDPEs with $b \neq 1$, many known works focus on the case when $r = \lceil b \rceil - b$, where $\lceil b \rceil$ is the ceiling function; see, e,g., [Che17a; MN15]. To facilitate the discussions here, we will only focus on the case when $a = 2$. In particular, by setting $r = \lceil b \rceil - b$ and $a = 2$, conditions in (2.1.12) become

$$\begin{cases} \alpha = \dfrac{2}{b} \leqslant d \quad \text{and} \quad b \in [1/2, 1], \\[2ex] \alpha = \dfrac{6}{b} \leqslant d \quad \text{and} \quad b \in [3/2, 2). \end{cases} \tag{2.2.5}$$

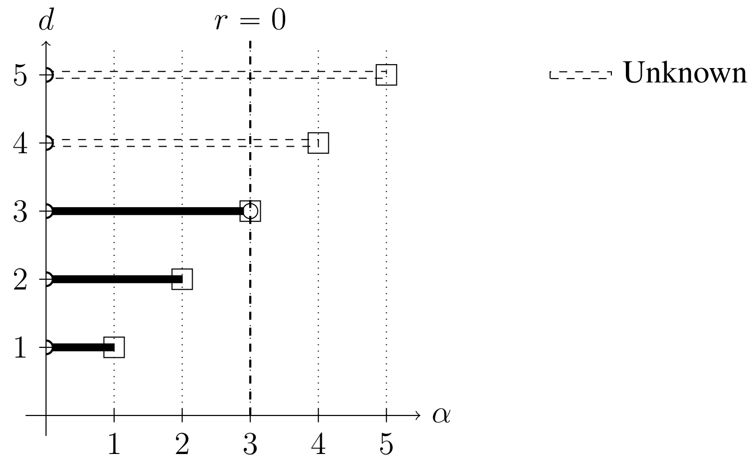


Figure 2.3: Solvability for the stochastic wave equation (i.e., $a = b = 2$ and $r = 0$). See Figure 2.2 for an additional legend.

When $b = 1$, we have $r = 0$ and the fundamental solution is the standard heat kernel. Hence, Assumption 2.1.1 is satisfied for all $d \geqslant 1$. When $b < 1$, sufficient conditions in Remark 2.1.2 guarantees Assumption 2.1.1 for all $d \geqslant 1$. However, when $b > 1$ and $a = 2$, from Remark 2.1.2 we see that the fundamental solution is nonnegative only for $d \leqslant 3$. The solvability for this case is illustrated in Figure 2.4 and the critical time T_p in case of local solution (hence, only for the case when $b \in [1/2, 1]$) is equal to

$$T_p = \frac{\nu^{\alpha/2}}{2\theta(p-1)\mathcal{M}_{2,d}^{2-\alpha/2}}. \tag{2.2.6}$$

Example 2.2.5 (Fractional SPDEs with $r = 0$ and $a = 2$)**.** In this example, we study the special case of the fractional SPDEs when $r = 0$. The choice of $r = 0$ has been used in, e.g., [Che+17]. We will only consider the case $a = 2$ for simplicity. Now by setting $r = 0$ and $a = 2$ and restricting $b \leqslant 1$, conditions in (2.1.12) become

$$\alpha = 4 - \frac{2}{b} \leqslant d \quad \text{and} \quad b \in (0, 1]. \tag{2.2.7}$$

As discussed in Example 2.2.4, Assumption 2.1.1 is satisfied for all $d \geqslant 1$ when $b \leqslant 1$ but only for $d \leqslant 3$ when $b > 1$. The solvability for this case is illustrated in Figure 2.5 with T_p given in (2.2.6). In particular, for the example in the second figure in Figure 2.5, namely, when $b = 2/3$

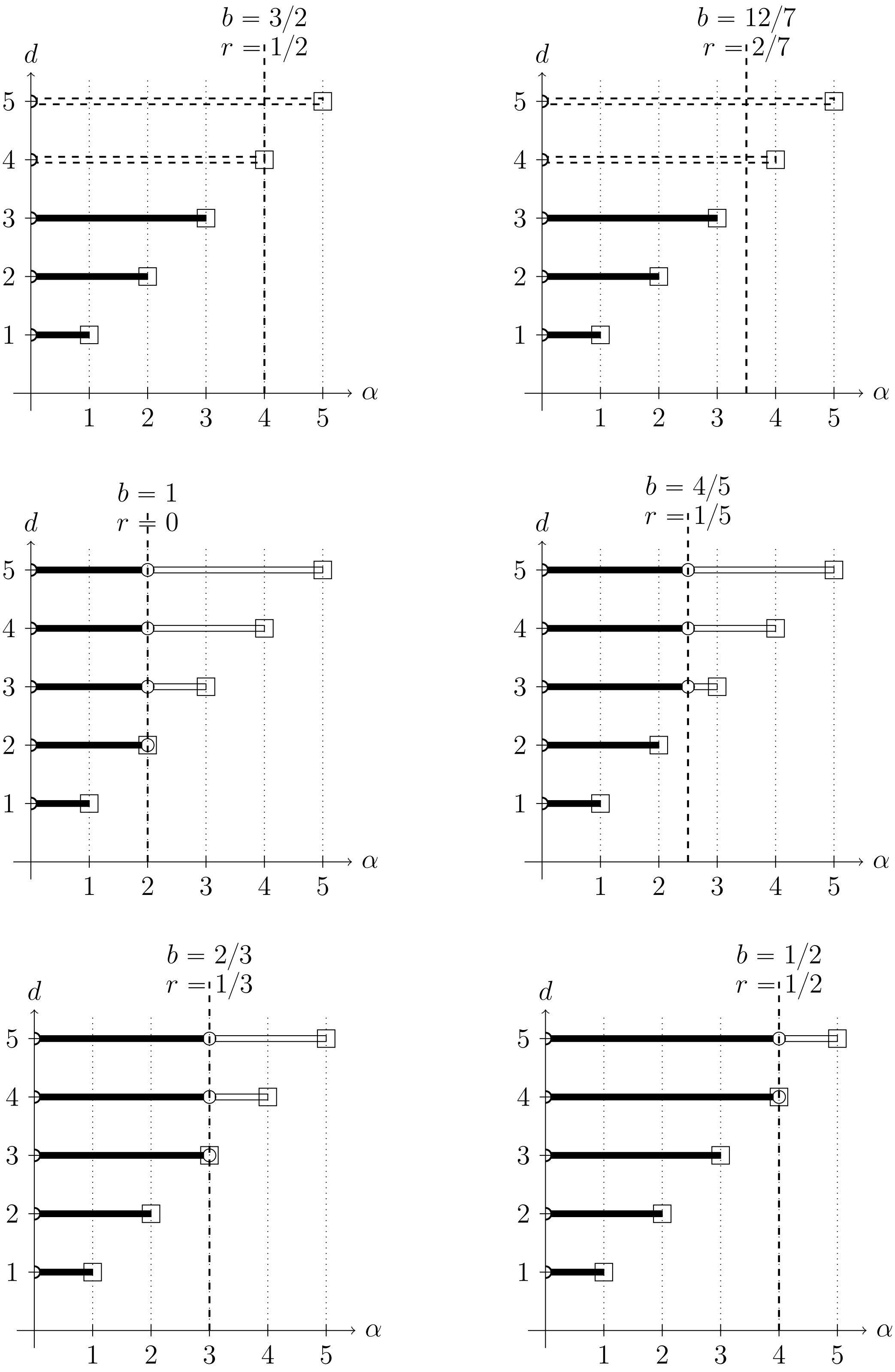


Figure 2.4: Solvability for the fractional SPDEs in case of $a = 2$ and $r = [b] - b$. See Figures 2.2 and 2.3 for the legend.

and $\alpha = d = 1$, the white noise driven SHE has a local solution with

$$T_p = \frac{2^{5/2}\sqrt{\nu}}{3(p-1)\theta}, \qquad \text{for all } p \geqslant 2, \tag{2.2.8}$$

where we have applied (2.2.6) and the relation (2.3.22).

More examples regarding the solvability can be studied in a similar way, which are left to the interested readers.

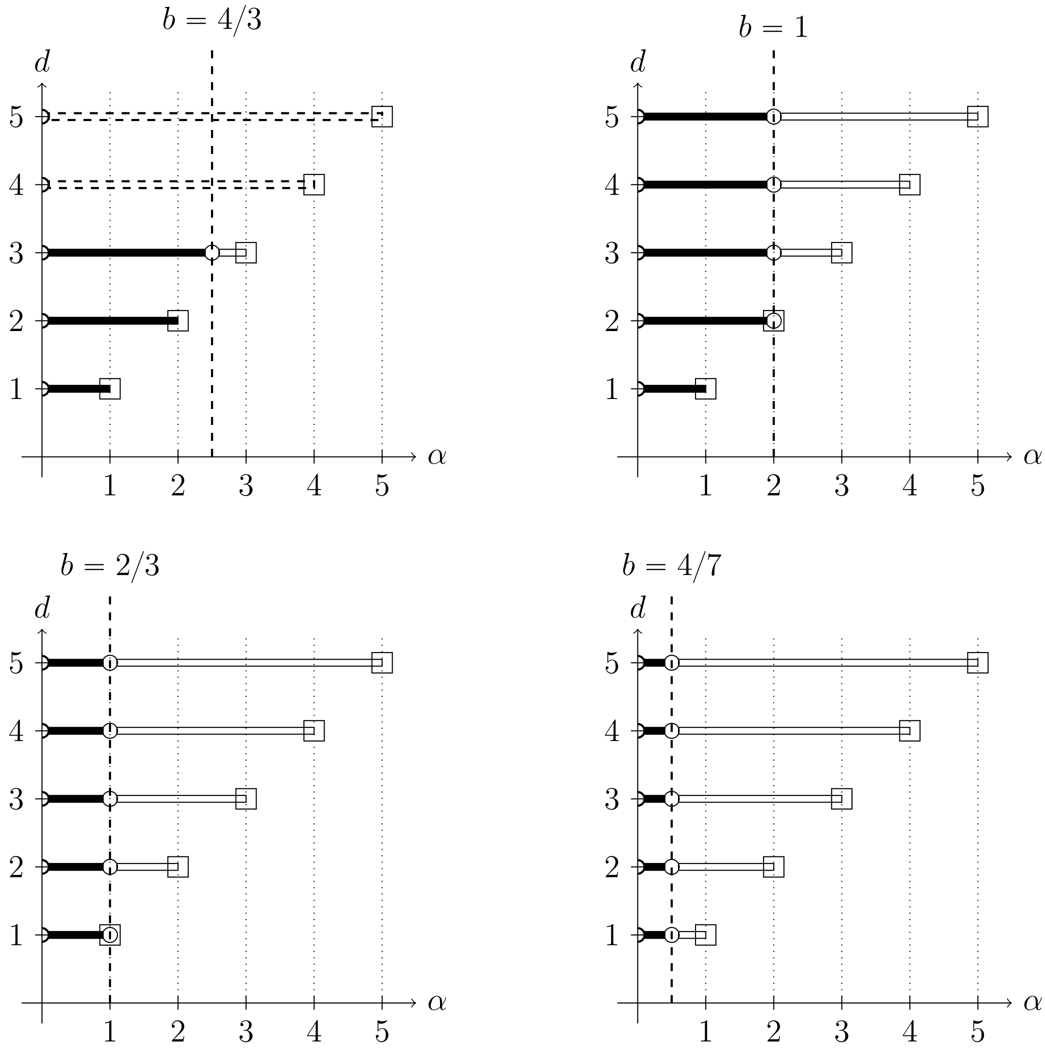


Figure 2.5: Solvability for the fractional SPDEs in case of $a = 2$ and $r = 0$. See Figures 2.2 and 2.3 for the legend.

Example 2.2.6 (SHE with fractional Laplacian)**.** The stochastic heat equation with fractional Laplacian (i.e., the case when $b = 1$, $r = 0$ and $a \in (0, 2]$) has been widely studied in the literature, but possibly with different noises. In this case, the fundamental solutions are transition densities for the alpha-stable jump processes, which are necessarily to be nonnegative. This is also consistent with the sufficient conditions for nonnegativity in Remark 2.1.2. By setting $b = 1$ and $r = 0$ in (2.1.12), we have the following condition:

$$\alpha = a \leqslant d$$

The solvability for this case is illustrated in Figure 2.6.

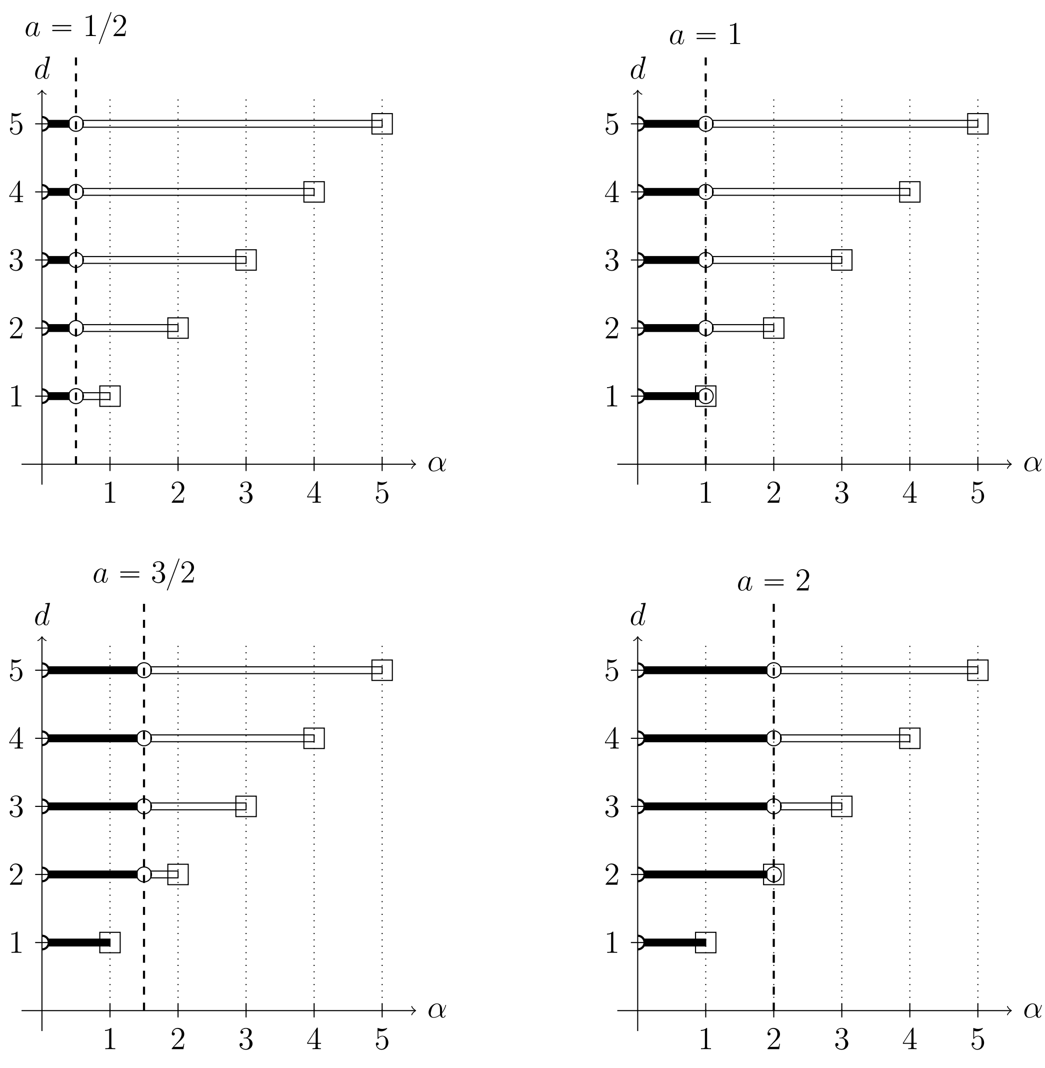


Figure 2.6: Solvability for the stochastic heat equation with fractional Laplacian, i.e, the case when $b = 1$ and $r = 0$. See Figures 2.2 and 2.3 for the legend.

2.2.2 Examples on asymptotics

In this part, we list several examples for the asymptotics when global solutions exist. In particular, we will show that the asymptotics in (2.1.14) interpolates the corresponding results for both stochastic wave and heat equations.

Example 2.2.7 (Asymptotics for SWE)**.** Even though our results requires b to be strictly less than 2, but by formally setting

$$a = b = \nu = 2 \quad \text{and} \quad r = 0,$$

we have that

$$\beta = \frac{4-\alpha}{3-\alpha} \quad \text{and} \quad t_p = (p-1)^{1/(4-\alpha)} t,$$

and results in (2.1.14), (2.1.16), and (2.1.17) recover the corresponding results for the stochastic wave equation, namely, (1.9), (1.10), and (1.11) of [BCC22], respectively. Due to the importance of white noise and for the future references, here we list two special cases regarding white noise:

(1) The SWE with white noise in $\mathbb{R}$: By further setting $d = \alpha = 1$, we see that

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t^{3/2}} = \frac{p(p-1)^{1/2}\sqrt{\theta}}{3(2\nu)^{1/4}} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{p^{3/2}} = \frac{t^{3/2}\sqrt{\theta}}{3(2\nu)^{1/4}}, \tag{2.2.9}$$

where we have applied (2.3.22).

(2) The SWE with white noise in $\mathbb{R}^2$: Similarly, by setting $d = \alpha = 2$, we see that

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t^2} = \frac{p(p-1)\theta \mathcal{M}_{2,2}(\delta_0)}{2\nu} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{p^2} = \frac{t^2\theta \mathcal{M}_{2,2}(\delta_0)}{2\nu}. \tag{2.2.10}$$

Example 2.2.8 (Asymptotics for SHE)**.** As for the stochastic heat equation case, by setting

$$a = 2, \quad b = \nu = 1 \quad \text{and} \quad r = 0,$$

we have that

$$\beta = \frac{4-\alpha}{2-\alpha} \quad \text{and} \quad t_p = (p-1)^{2/(4-\alpha)} t,$$

and results in (2.1.14) and (2.1.16) recover the corresponding conjectured results for SHE, namely, (1.16) and (1.17) of Balan *et al* [BCC22], respectively, which are equivalent to Theorem 1.1 and 1.2 of X. Chen [Che17b] when setting $\alpha_0 = 0$ and using [Che+15, Lemma A.2] to rewrite the constant $\mathcal{E}$ in [Che17b] in terms of $\mathcal{M}_a$. Due to the importance of the white noise case, we list the corresponding asymptotics here. When $\alpha = d = 1$,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^3} = \frac{p(p-1)^2\theta^2}{24\nu} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{p^3} = \frac{t^3\theta^2}{24\nu}, \tag{2.2.11}$$

where we have applied (2.3.22). Note that some upper and lower bounds for the first limit in (2.2.11) in case of $p = 2$ were earlier obtained by Hu [Hu02, part 1) of Theorem 4.1].

Example 2.2.9 (Asymptotics for SHE with fractional Laplacian)**.** In this example we restrict ourselves to the case when $b = 1$, $a \in (0,2]$, $\alpha < d$, and $r = 0$, which is the 1-dimensional SHE with fractional Laplace. With this set up we have

$$\beta = \frac{2a-\alpha}{a-\alpha} \qquad \text{and} \qquad t_p = (p-1)^{\frac{a}{2a-\alpha}} t,$$

and by Corollary 2.1.8,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^{\frac{2a-\alpha}{a-\alpha}}} = p(p-1)^{\frac{a}{a-\alpha}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}} \left[\theta \nu^{-\alpha/a} \mathcal{M}_{a,d}^{\frac{2a-\alpha}{a}}\right]^{\frac{a}{a-\alpha}} \left(\frac{a-\alpha}{a}\right) \tag{2.2.12}$$

and

$$\lim_{p\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{p^{\frac{2a-\alpha}{a-\alpha}}} = t^{\frac{2a-\alpha}{a-\alpha}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}} \left[\theta \nu^{-\alpha/a} \mathcal{M}_{a,d}^{\frac{2a-\alpha}{a}}\right]^{\frac{a}{a-\alpha}} \left(\frac{a-\alpha}{a}\right). \quad (2.2.13)$$

this setup has been studied in [Che+18] for the case of a time-dependent noise where the covariance function is given by

$$\mathbb{E}[\dot{W}(r,x)\dot{W}(s,y)] = |r-s|^{-\alpha_0}\gamma(x-y)$$

and $\gamma(x)$ is defined to be either of the following:

$$\gamma(x) := \begin{cases} |x|^{-\alpha} & \text{where } \alpha \in (0,d) \ \text{ or} \\ \prod_{j=1}^{d} |x_j|^{\alpha_j} & \text{where } \alpha_j \in (0,1). \end{cases} \quad (2.2.14)$$

They proved that for $\alpha < \min\{a,d\}$ and let $p \geqslant 2$,

$$\lim_{t\to\infty} t^{-\frac{2a-\alpha-\alpha\alpha_0}{a-\alpha}} \log \mathbb{E}[|u(t,x)|^p] = p(p-1)^{\frac{a}{a-\alpha}} \mathbf{M}(a,\alpha_0,d,\gamma), \quad (2.2.15)$$

where the variational constant is given by

$$\mathbf{M}(a,\alpha_0,d,\gamma) = \sup_{g\in\mathcal{A}_{a,d}} \left\{ \frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^{2d}} \frac{\gamma(x-y)}{|r-s|^{\alpha_0}} g^2(s,x)g^2(r,y)\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s \right. \\ \left. - (2\pi)^{-d}\int_0^1\int_{\mathbb{R}^d} |x|^a |\mathcal{F}g(s,\xi)|^2 \mathrm{d}\xi\mathrm{d}s \right\}$$

with[3]

$$\mathcal{A}_{a,d} := \left\{ g(s,x) : \int_{\mathbb{R}^d} g^2(s,x)\mathrm{d}x = 1, \forall s \in [0,1] \text{ and } (2\pi)^{-d}\int_0^1\int_{\mathbb{R}^d} |x|^a|\mathcal{F}g(s,\xi)|^2\mathrm{d}\xi\mathrm{d}s < \infty \right\}.$$

By setting $\alpha_0 = 0$ and letting $g(s,x) = g(x) \in \mathcal{F}_a$ be independent in s, which is the time-independent setup, then Equation (2.2.12) and Lemma 2.3.9 together recover (2.2.15). Indeed,

[3] Note that the Fourier transform is defined differently in [Che+18].

by observing (2.3.29) and (2.3.33), we see that

$$
\begin{aligned}
\mathbf{M}(a,\alpha_0,d,\gamma) &= \mathbf{E}_{a,d}\left(\frac{1}{2}\gamma,2\right)\\
&= 2^{\frac{-a}{a-\alpha}}\left(\frac{a-\alpha}{a}\right)\left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}}\mathcal{M}_{a,d}\left(\gamma,1\right)^{\frac{2a-\alpha}{a-\alpha}}
\end{aligned}
\tag{2.2.16}
$$

and by rewriting (2.2.15) with (2.2.16) yields (2.2.12). Finally, we note that condition (2.2.14) can be relaxed to allow white noise in one dimensional case, namely, $\alpha = d = 1$. In this case, one can simply replace α and d in both (2.2.12) and (2.2.20) by 1 and in addition replace $\mathcal{M}_{a,d}$ by $\mathcal{M}_{a,1}(\delta_0)$.

Example 2.2.10 (Asymptotics for SPDEs with $r = \lceil b\rceil - b$ and white noise)**.** In this example, we consider the case when $a = 2$, $d = \alpha = 1$ (white noise), and $r = \lceil b\rceil - b$. As seen in Example 2.2.4, there exists a global solution. In this case,

$$
\beta = \frac{4\lceil b\rceil - b}{4\lceil b\rceil - b - 2} \qquad \text{and} \qquad t_p = (p-1)^{\frac{2}{4\lceil b\rceil - b}}\, t,
$$

and by (2.3.22) and Corollary 2.1.8,

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^{\frac{4\lceil b\rceil - b}{4\lceil b\rceil - b-2}}} =& p(p-1)^{\frac{2}{4\lceil b\rceil - b-2}}\\
&\times\left(\frac{9\theta^2}{8\nu}\right)^{\frac{1}{4\lceil b\rceil - b-2}}\left(4\lceil b\rceil - b - 2\right)\left(4\lceil b\rceil - b\right)^{-\frac{4\lceil b\rceil - b}{4\lceil b\rceil - b-2}},
\end{aligned}
\tag{2.2.17}
$$

and

$$
\lim_{p\to\infty}\frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{p^{\frac{4\lceil b\rceil - b}{4\lceil b\rceil - b-2}}} = t^{\frac{4\lceil b\rceil - b}{4\lceil b\rceil - b-2}}\left(\frac{9\theta^2}{8\nu}\right)^{\frac{1}{4\lceil b\rceil - b-2}}\left(4\lceil b\rceil - b - 2\right)\left(4\lceil b\rceil - b\right)^{-\frac{4\lceil b\rceil - b}{4\lceil b\rceil - b-2}}. \tag{2.2.18}
$$

Example 2.2.11 (Asymptotics for SPDEs with $r = 0$ and white noise)**.** From Example 2.2.5, we see that when $a = 2$, $r = 0$, $d = \alpha = 1$ (white noise), the global solution exists when $b \in (2/3, 2)$. In this case, we have that

$$
\beta = \frac{3b}{3b-2} \qquad \text{and} \qquad t_p = (p-1)^{2/(3b)}\, t,
$$

and by (2.3.22) and Corollary 2.1.8,

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\left(|u(t,x)|^p\right)}{t^{3b/(3b-2)}}=p(p-1)^{\frac{2}{3b}}\left(b-\frac{2}{3}\right)b^{-\frac{3b}{3b-2}}\left(\frac{\theta^2}{8\nu}\right)^{\frac{1}{3b-2}}\quad\text{and}\tag{2.2.19}$$

$$\lim_{p\to\infty}\frac{\log\mathbb{E}\left(|u(t,x)|^p\right)}{p^{3b/(3b-2)}}=t^{\frac{3b}{3b-2}}\left(b-\frac{2}{3}\right)b^{-\frac{3b}{3b-2}}\left(\frac{\theta^2}{8\nu}\right)^{\frac{1}{3b-2}}.\tag{2.2.20}$$

2.3 Some preliminaries

2.3.1 Skorohod integral and mild solution

We start with a nonnegative and nonnegative definite tempered measure Γ with density γ in the sense that $\Gamma(\mathrm{d}x)=\gamma(x)\mathrm{d}x$ and

$$\int_{\mathbb{R}^d}\Gamma(\mathrm{d}x)(\phi*\widetilde{\phi})(x)\geqslant 0\quad\text{for all }\phi\in\mathscr{S}(\mathbb{R}^d)$$

where $\widetilde{\phi}(x):=\phi(-x)$. According to the Bochner theorem, there exists a nonnegative and nonnegative definite measure μ, often referred as the *spectral measure* on $\mathbb{R}^d$ whose Fourier transform (in the weak sense) is Γ, namely, that for any $\phi\in\mathcal{D}(\mathbb{R}^d)$ (the space of test functions),

$$\int_{\mathbb{R}^d}\Gamma(\mathrm{d}x)\phi(x)=\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)\mathcal{F}\phi(\xi).$$

Since μ is nonnegative definite, the following functional

$$C(\phi,\psi)=\int_{\mathbb{R}}\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi),\qquad\text{for all }\phi,\psi\in\mathcal{D}(\mathbb{R}^d)\tag{2.3.1}$$

is nonnegative-definite and thus one can associate it with a zero-mean Gaussian processes, $W:=\left\{W(\phi):\phi\in\mathcal{D}(\mathbb{R}^d)\right\}$, with the covariance functional of W given by (2.3.1). In other words,

$$\mathbb{E}\left(W(\phi)W(\psi)\right)=\int_{\mathbb{R}}\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi)=:\langle\phi,\psi\rangle_{\mathcal{H}}.$$

Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and thus we see $\phi \mapsto W(\phi)$ is an isometry from $\mathcal{D}(\mathbb{R}^d)$ to $L^2(\Omega)$, that is, $\mathbb{E}\left(W(\phi)^2\right) = \|\phi\|_{\mathcal{H}}^2$ for $\phi \in \mathcal{D}(\mathbb{R}^d)$. One can extend this isometry from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{H}$. We refer the interested readers to [Dal+09] and references therein.

We denote δ the *Skorohod integral* with respect to W and denote its domain by $\mathrm{Dom}(\delta)$. u is called *Skorohod integrable* if $u \in \mathrm{Dom}(\delta)$, in which case we write $\delta(u) = \int_{\mathbb{R}^d} u(x) W(\delta x)$ and by isometry, $\mathbb{E}\left(\delta(u)^2\right) = \mathbb{E}(\|u\|_{\mathcal{H}}^2)$. For a complete treatment of the Skorohod integral, see Nualart *et al* [NN18].

Definition 2.3.1 (Mild, local and global solutions)**.** (1) For $T \in (0, \infty]$, a random field $u = \left\{u(t,x) : t \in (0,T),\ x \in \mathbb{R}^d\right\}$ is called *a mild solution* to the equation (2.1.1) if for all $x \in \mathbb{R}^d$ and s,t fixed with $0 < s \leqslant t < T$, $y \to G(t-s, x-y)u(s,y)$ is Skorohod integrable and the following stochastic integral equation holds almost surely

$$u(t,x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta y) \right) \mathrm{d}s. \tag{2.3.2}$$

(2) Let $u(t,x)$ be a mild solution to (2.1.1) (or (2.3.2)) and fix $p \geqslant 1$. We call $u(t,x)$ a *global* $L^p(\Omega)$*-solution*, or simply an $L^p(\Omega)$*-solution* if

$$\|u(t,x)\|_p < \infty \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{2.3.3}$$

(3) If there exist $0 < T_1 \leqslant T_2 < \infty$ such that $\|u(t,x)\|_p$ is finite for all $t \in (0, T_1)$ and $x \in \mathbb{R}^d$, but $\|u(t,x)\|_p$ diverges to infinity whenever $t > T_2$, the mild solution $u(t,x)$ in this case is called a *local* $L^p(\Omega)$*-solution*.

Note that through construction of the Skorohod integral δ, a mild solution is necessarily to be an $L^2(\Omega)$-solution. For more details, one may check, e.g., Nualart [NN18, Chapter 3].

Through the standard *Picard iteration* scheme, the solution can be expressed by the following *Wiener chaos expansion* [4]:

$$u(t,x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot,x,t)), \tag{2.3.4}$$

where $I_k : \mathcal{H}^{\otimes k}\left((\mathbb{R}^d)^k\right) \to \mathcal{H}_k$ is the k^{th} order Skorohod integral and $\mathcal{H}_k$ is the k^{th} Wiener chaos space and the kernels $f_k(\cdot,x,t)$, obtained through the iteration, are equal to

$$\begin{aligned} f_n(x_1,...,x_k;x,t) &= \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} G(t-t_n, x-x_n)\cdots G(t_2-t_1, x_2-x_1)\mathrm{d}t_1\cdots \mathrm{d}t_n \\ &= \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} G(t_1, x-x_n)\cdots G(t_n-t_{n-1}, x_2-x_1)\mathrm{d}t_1\cdots \mathrm{d}t_n. \end{aligned}$$

For ease of notation, throughout this article, we may write the above integrals as

$$\begin{aligned} f_n(x_1,...,x_n;x,t) &= \int_{[0,t]^n_<} G(t-t_n, x-x_n)\cdots G(t_2-t_1, x_2-x_1)\mathrm{d}\vec{t} \\ &= \int_{[0,t]^n_<} G(t_1, x-x_n)\cdots G(t_n-t_{n-1}, x_2-x_1)\mathrm{d}\vec{t}, \end{aligned}$$

where $[0,t]^n_< := \{(t_1,\cdots,t_n) \in [0,t]^n : \ t_1 < \cdots < t_n\}$. As usual, we use $\widetilde{f}_n(\cdot,x,t)$ to denote the symmetrization of $f_n(\cdot,x,t)$:

$$\begin{aligned} \widetilde{f}_n(\cdot;x,t) &= \frac{1}{n!}\sum_{\rho\in\Sigma_n} f_n\left(x_{\rho(1)},\cdots,x_{\rho(n)}\right) \\ &= \frac{1}{n!}\sum_{\rho\in\Sigma_n} \int_{[0,t]^n_<} G\left(t-t_n, x-x_{\rho(n)}\right)\cdots G\left(t_2-t_1, x_{\rho(2)}-x_{\rho(1)}\right)\mathrm{d}\vec{t}, \end{aligned}$$

where Σ_n is the set of all permutations of $\{1,\cdots,n\}$. By setting $t_{n+1} = t$, the Fourier transform of the kernels, f_n, is given by

$$\mathcal{F}f_n(\cdot;x,t)(\xi_1,...,\xi_n) = e^{-i\left(\sum_{j=1}^n \xi_j\right)\cdot x} \int_{[0,t]^n_<} \prod_{k=1}^{n} \overline{\mathcal{F}G(t_{k+1}-t_k,\cdot)\left(\sum_{j=1}^{k}\xi_j\right)}\mathrm{d}\vec{t}. \tag{2.3.5}$$

[4] Wiener chaos expansion has been widely to solve the linear stochastic partial differential equations. We direct interested readers to [BS17, Section 5] for a presentation of this procedure.

Recall the notation above in (2.1.2) that $G(t,x) = G_{a,b,r,d}(t,x)$. The following scaling properties for both $\mathcal{F}G(t,\cdot)(\xi)$ and $\left\|\widetilde{f}_n(\cdot,x,t)\right\|_{\mathcal{H}^{\otimes n}}$ play an important role in this work.

Lemma 2.3.2. *For any $c, t > 0$, $n \geqslant 1$, $\xi, \xi_1, \cdots, \xi_n \in \mathbb{R}^d$, the following scaling properties hold:*

$$\mathcal{F}G(t,\cdot)(c\xi) = c^{-\frac{a}{b}(b+r-1)}\mathcal{F}G\left(c^{\frac{a}{b}}t,\cdot\right)(\xi) \quad \text{and} \quad \mathcal{F}G(ct,\cdot)(\xi) = c^{b+r-1}\mathcal{F}G(t,\cdot)(c^{b/a}\xi), \tag{2.3.6}$$

$$\mathcal{F}\widetilde{f}_n(\cdot,0,ct)(\xi_1,\cdots,\xi_n) = c^{n(b+r)}\mathcal{F}\widetilde{f}_n(\cdot,0,t)(c^{b/a}\xi_1,\cdots,c^{b/a}\xi_n), \tag{2.3.7}$$

$$\left\|\widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}}^2 = t^{[2(b+r)-b\alpha/a]n}\left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2. \tag{2.3.8}$$

Proof. The scaling properties in (2.3.6) are direct consequences of the explicit expression of $\mathcal{F}G(t,\cdot)(\xi)$ as in [CHN19, (4.8)]. Property (2.3.7) is an easy exercise of change of variables on (2.3.5). Property (2.3.8) is a direct consequence of (2.3.6), (2.3.7), and the scaling property of the spectral measure μ. We leave the details for the interested readers. □

Finally, let us recall the following standard result about the existence and uniqueness of the solution to (2.1.1) (or (2.3.2)) when it can be written as the Wiener chaos expansion (2.3.4).

Theorem 2.3.3. *Fix any $T \in (0,\infty]$. Suppose that $f_n(\cdot,x,t) \in \mathcal{H}^{\otimes n}$ for any $t \in (0,T)$, $x \in \mathbb{R}^d$ and $n \geqslant 1$. Then (2.1.1) (or (2.3.2)) has a unique $L^2(\Omega)$-solution on $(0,T)\times\mathbb{R}^d$ if and only if the series (2.3.4) converges in $L^2(\Omega)$ for any $(t,x) \in (0,T)\times\mathbb{R}^d$, which is equivalent to the convergence of the series (2.3.9). In this case, the solution is given by (2.3.4) with the second moment given by*

$$\mathbb{E}\left(u(t,x)^2\right) = \sum_{n\geqslant 0}\theta^n n!\left\|\widetilde{f}_n(\cdot,x,t)\right\|_{\mathcal{H}^{\otimes n}}^2 \quad \text{for all } (t,x) \in (0,T)\times\mathbb{R}^d. \tag{2.3.9}$$

2.3.2 Some asymptotics and variational constants

Recall that the correlation function γ satisfies Assumption 2.1.3 and that the corresponding spectral measure is μ; see Remark 2.1.4. Define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x) \tag{2.3.10}$$

and

$$\begin{aligned} \mathcal{M}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a} \left\{ \left(\iint_{\mathbb{R}^{2d}} g^2(x)g^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y \right)^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g) \right\} \\ &= \sup_{g\in\mathcal{F}_a} \left\{ \left\langle g^2 * g^2, \gamma \right\rangle_{L^2(\mathbb{R}^d)}^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g) \right\}, \end{aligned} \tag{2.3.11}$$

where

$$\mathcal{E}_a(g,g) := (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^a |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi \quad \text{and} \tag{2.3.12}$$

$$\mathcal{F}_a := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{L^2(\mathbb{R}^d)} = 1,\ \mathcal{E}_a(f,f) < \infty \right\}. \tag{2.3.13}$$

We often omit the dimension d in $\mathcal{M}_{a,d}$ when it is clear from context. We use the convention that $\mathcal{M}_a(f) := \mathcal{M}_a(f,1)$ to be consistent with notation (2.1.10). By a similar argument as the proof of [BCC22, Lemma 2.3], one can show that

$$\mathcal{M}_a(\Theta\gamma,\theta) = \Theta^{\frac{a}{2a-\alpha}}\theta^{-\frac{\alpha}{2a-\alpha}}\mathcal{M}_a(\gamma,1), \qquad \text{for all } \theta \text{ and } \Theta > 0. \tag{2.3.14}$$

For the Riesz kernel case (see Example 2.1.5), Bass, Chen and Rosen [BCR09] established that when $a \in (0,2]$, $\nu = 2$ and $\alpha < \min\{2a, d\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+|\sum_{j=k}^n \xi_{\sigma(j)}|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}) \right] = \log\left(\rho_{2,a}\left(|\cdot|^{-\alpha}\right)\right), \tag{2.3.15}$$

and [5]

$$\rho_{2,a}\left(|\cdot|^{-\alpha}\right) = \mathcal{M}_a^{2-(\alpha/a)}\left(|\cdot|^{-\alpha},2\right), \tag{2.3.16}$$

where

$$\mu(\mathrm{d}\vec{\xi}) = \prod_{j=1}^n \mu(\mathrm{d}\xi_j) = \prod_{j=1}^n \varphi(\xi_j)\mathrm{d}\xi_j. \tag{2.3.17}$$

We first apply some scaling arguments to accommodate the parameter ν in both (2.3.15) and (2.3.16), the proof of which can be found in Appendix:

Lemma 2.3.4 (The Riesz kernel case). *If $\gamma(x) = |x|^{-\alpha}$ for some $\alpha \in (0,d)$, then for any $\nu > 0$ and $a \in (0,2]$,*

$$\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) = \left(\frac{\nu}{2}\right)^{-\alpha/a} \mathcal{M}_a^{2-(\alpha/a)}\left(|\cdot|^{-\alpha},2\right) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(|\cdot|^{-\alpha}), \tag{2.3.18}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^n\xi_{\sigma(j)}|^a}\right)^2\mu(\mathrm{d}\vec{\xi})\right] = \log\left(\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right)\right). \tag{2.3.19}$$

More generally we have the following theorem:

Theorem 2.3.5. *Suppose that the correlation function γ satisfies Assumption 2.1.3 and is such that $\alpha < \min\{2a,d\}$. Then both (2.3.18) and (2.3.19) hold with $|\cdot|^{-\alpha}$ and μ replaced by γ and μ as in (2.1.5), respectively. More precisely, it holds that*

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty, \tag{2.3.20}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^n\xi_{\sigma(j)}|^a}\right)^2\mu(\mathrm{d}\vec{\xi})\right] = \log\left(\rho_{\nu,a}\left(\gamma\right)\right). \tag{2.3.21}$$

[5] In Theorem 1.5 or eq. (1.20) of Bass *et al* [BCR09], the factor $(2\pi)^{-d}$ should not be present.

Remark 2.3.6. It is often very difficult to obtain the exact value for the variational constant $\mathcal{M}_{a,d}(\gamma)$. To the best of our knowledge, only in case of $a = 2$ and $\alpha = d = 1$ (white noise), one can compute explicitly that

$$\mathcal{M}_{2,1}(\delta_0) = (3/4)(1/6)^{1/3}, \tag{2.3.22}$$

which is a consequence of Chen and Li [CL04, Lemma .2] with $p = 2$. When $d \geqslant 2$, the value of $\mathcal{M}_{2,d}(\delta_0)$ can be expressed using the best constant for the classical *Gagliardo-Nirenberg* inequality; see Remark 3.13 of Chen *et al* [Che+21] for more details.

Sketch of the proof of Theorem 2.3.5. The proof of this theorem follows essentially the identical proof as Bass *et al* [BCR09], which is exclusively for the Riesz kernel. One simplification is that we only need to handle the case $p = 2$ thanks to the hypercontractivity property. For our slight extension to the noise given in Assumption 2.1.3, there is no need to repeat their paper. Instead we will only point out the differences and necessary changes. For your convenience, the correspondence of parameters between Bass *et al* [BCR09] and the current chapter is listed in the following Table 2.1.

Table 2.1: Notation correspondence.

	Laplace		Noise			Moment	Variational Const.
Bass *et al* [BCR09]	β	2	σ	$\lvert\cdot\rvert^{-\sigma}$	$\varphi_{d-\sigma}$	p	Λ_σ
Current chapter	a	ν	α	$\gamma(\cdot)$	φ	2	$\mathcal{M}_a\left(\lvert\cdot\rvert^{-\alpha}, 2\right)$

Theorem 2.3.5 is proven by showing the following claims: for γ given in Assumption 2.1.3,

(i) $\rho_{\nu,a}(\gamma) < \infty$;

(ii) $\liminf_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}\lvert\sum_{j=k}^n \xi_{\sigma(j)}\rvert^a} \right)^2 \mu(\mathrm{d}\vec{\xi}\,) \right] \geqslant \log\left(\rho_{\nu,a}(\gamma)\right)$;

(iii) $\limsup_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}\lvert\sum_{j=k}^n \xi_{\sigma(j)}\rvert^a} \right)^2 \mu(\mathrm{d}\vec{\xi}\,) \right] \leqslant \log\left(\rho_{\nu,a}(\gamma)\right)$;

(iv) $\mathcal{M}_a(\gamma) < \infty$;

(v) $\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma)$.

Part (i) which corresponds to Lemma 1.6 (*ibid.*) is established by Lemma 2.3.7 below.

Following exactly the same arguments as those in Section 3 (*ibid.*) with $\varphi_{d-\sigma}$ (*ibid.*) replaced by our φ as in (2.1.5), one can prove part (ii) for $\nu = 2$. Then an application of the scaling property as the proof of Lemma 2.3.4 shows the general case $\nu > 0$.

The proof of the upper bound, namely part (iii), is more challenging. This part corresponds to Sections 5 and 6 (*ibid.*). By examining these two sections carefully, we need to make some changes in Section 5 (*ibid.*), where as the arguments in Section 6 (*ibid.*) follow unchanged. For Section 5 (*ibid.*), we need to use the following decomposition of φ as opposed to (5.4) (*ibid.*):

$$\varphi(\xi) = \prod_{i=1}^{k} C_{\alpha_i,d_i} |\xi_{(i)}|^{-(d_i-\alpha_i)} = \prod_{i=1}^{k} C_{\alpha_i,d_i} (\mathcal{P}_i * \mathcal{P}_i) \left(\xi_{(i)}\right),$$

with

$$\mathcal{P}_i\left(\xi_{(i)}\right) = \beta_{d_i-\alpha_i,d_i} \left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)};$$

see (2.1.7) for the constants. Or equivalently,

$$\varphi(\xi) = (\mathcal{P} * \mathcal{P})(\xi) \quad \text{with} \quad \mathcal{P}(\xi) := \prod_{i=1}^{k} \sqrt{C_{\alpha_i,d_i}}\, \beta_{d_i-\alpha_i,d_i} \left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)}.$$

Now (5.5) (*ibid.*) should be written as

$$\mathcal{P}_{\beta,\epsilon}(\xi) = \hat{h}(\epsilon\xi) \prod_{i=1}^{k} \frac{\sqrt{C_{\alpha_i,d_i}}\, \beta_{d_i-\alpha_i,d_i}}{\beta + \left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)}}, \qquad \text{for all } \beta, \epsilon \geqslant 0,$$

where $h(\cdot)$ is defined in (5.2) (*ibid.*). So $\mathcal{P}(\xi) = \mathcal{P}_{0,0}(\xi)$ and

$$(\mathcal{P}_{\beta,\epsilon} * \mathcal{P}_{\beta,\epsilon})(\xi) \leqslant (\mathcal{P}_{\beta,0} * \mathcal{P}_{\beta,0})(\xi) \leqslant (\mathcal{P}_{0,0} * \mathcal{P}_{0,0})(\xi) = \varphi(\xi);$$

see (5.6) (*ibid.*). With these changes, one can update accordingly the proof of Lemma 5.1 (*ibid.*) without any difficulty. Then the rest of Section 5 (*ibid.*) follows unchanged. In this way, we establish part (iii) for $\nu = 2$. Finally, a scaling argument as in part (ii) proves part (iii) for all $\nu > 0$.

Parts (iv) and (v) correspond to Section 7 (*ibid.*). In particular, part (iv) corresponds to Lemma 7.1 (*ibid.*). Note that we only need to study the case $p = 2$. By (2.3.24) with $\varphi(x-y)$ replaced by $\gamma(x-y)$, we see that

$$\begin{aligned}\int_{(\mathbb{R}^d)^2} g^2(x)g^2(-y)\gamma(x-y)\mathrm{d}x\mathrm{d}y &\leqslant C\left\|\tilde{g}^2\right\|_{L^{2d/(2d-\alpha)}(\mathbb{R}^d)}\left\|g^2\right\|_{L^{2d/(2d-\alpha)}(\mathbb{R}^d)} \\ &= C\left\|g\right\|_{L^{4d/(2d-\alpha)}(\mathbb{R}^d)}^4,\end{aligned} \tag{2.3.23}$$

where $\tilde{g}^2(x) = g(-x)$. Note that we must have $\alpha < 2a$ to ensure that the right hand side of the above is finite. This is seen by applying $h = \gamma$ in Lemma 2.3.7. Thus equation (7.1) (*ibid.*) can be applied in our setting . The rest of the proof of Lemma 7.1 (*ibid.*) remains unchanged.

It remains to update the proof of Theorem 1.5 in Section 7 (*ibid.*). For this, one needs only to update the four appearances of $1/(|\cdot|^\sigma)$ in (7.15), (7.22) and (7.23) (*ibid.*) to $\gamma(\cdot)$. Note that the factor $(2\pi)^{-d(p+1)}$ in the first equation of (7.22) (*ibid.*) should be $(2\pi)^{-dp}$. With this, we complete the sketch proof of Theorem 2.3.5. □

Note that the proof of [BCR09, Lemma 1.6] relies on inequality (1.27) on p. 630 (*ibid.*), which was a consequence of *Sobolev's inequality*. For the more general noises studied in this chapter, we can no longer apply this inequality. Instead, we prove the following lemma using the *weak Young's inequality* (see, e.g., [LL97, p.107]) as a generalization of Lemma 1.6 (*ibid.*). Even though we only need the case $p = 2$, the following lemma is proven for all $p \geqslant 2$.

Lemma 2.3.7. *For any f, g, h with $h \geqslant 0$, for φ given as in* (2.1.5) *(see also Assumption 2.1.3), and for all $p \geqslant 2$, it holds that*

$$\left(\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{|f(x+y)g(y)|}{\sqrt{h(x+y)}\sqrt{h(y)}}\mathrm{d}y\right]^p\varphi(x)\mathrm{d}x\right)^{1/p} \leqslant C\left\|f\right\|_{L^2(\mathbb{R}^d)}\left\|g\right\|_{L^2(\mathbb{R}^d)}\left\|h^{-1}\right\|_{L^{pd/\alpha}(\mathbb{R}^d)}.$$

Proof. By observing the proof of Lemma 1.6 of [BCR09], we only need to prove that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F(y)G(x)\varphi\left(x-y\right)\mathrm{d}y\mathrm{d}x \leqslant C\left\|F\right\|_{L^{2d/(d+\alpha)}(\mathbb{R}^d)}\left\|G\right\|_{L^{2d/(d+\alpha)}(\mathbb{R}^d)}, \tag{2.3.24}$$

where

$$F(x) = \frac{|f(x)|}{(h(x))^{p/2}} \qquad \text{and} \qquad G(x) = \frac{|g(x)|}{(h(x))^{p/2}}.$$

Note that when $\varphi(x) = C|x|^{-(d-\alpha)}$, (2.3.24) is nothing but (1.27) (*ibid.*). Here we need to handle more general φ as given in (2.1.5). To prove (2.3.24), we need to apply the *weak version of Young's inequality* (see, e.g., [LL97, eq. (7) on p. 107]), which says that for all $p, q, r > 1$ with $1/p + 1/q + 1/r = 2$, it holds that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x)b(x-y)c(y)\mathrm{d}x\mathrm{d}y \right| \leqslant K_{p,q,r,d} \|a\|_{L^p(\mathbb{R}^d)} \|b\|_{q,w} \|c\|_{L^r(\mathbb{R}^d)}, \tag{2.3.25}$$

where

$$\|b\|_{q,w} := \sup_A |A|^{-1/q'} \int_A |b(x)|\mathrm{d}x, \qquad \text{with } 1/q + 1/(q') = 1,$$

and A is an arbitrary Borel set of finite measure $|A| < \infty$. Now we apply (2.3.25) with

$$a = F, \quad c = G, \quad b = \varphi, \quad p = r = 2d/(d+\alpha), \quad \text{and} \quad q = \frac{d}{d - \sum_{j=1}^d \alpha_j} = d/(d-\alpha).$$

By (2.3.25) above, it suffices to prove that $\|\varphi\|_{q,w}$ is finite with $q = d/(d-\alpha)$ and $1/q' = 1 - 1/q = \alpha/d$.

Recall that according to Assumption 2.1.3, the d coordinates are partitioned into k groups. Define $A_R := A_1 \times \cdots \times A_k$ where $A_i = B_{R,d_i}(0)$ is the ball in $\mathbb{R}^{d_i}$ centered at the origin with radius R. With this we have that

$$\int_{A_i} |x_{(i)}|^{-(d_i-\alpha_i)}\mathrm{d}x_{(i)} = |\mathbb{S}^{d_i-1}| \frac{R^{\alpha_i}}{\alpha_i}, \tag{2.3.26}$$

where we have used polar coordinated to calculate the integral and $\left|\mathbb{S}^{d_i-1}\right| = 2\pi^{d_i/2}/\Gamma(d_i/2)$ is the surface area of the unit sphere in $\mathbb{R}^{d_i}$ (clearly, when $d_i = 1$, $|S^0| = 2$). Moreover, by the formula for the volume of balls in $\mathbb{R}^{d_i}$, we see that

$$|A_i| = \frac{\pi^{d_i/2}}{\Gamma\left(1+\frac{d_i}{2}\right)} R^{d_i} = |\mathbb{S}^{d_i-1}| \frac{R^{d_i}}{d_i}. \tag{2.3.27}$$

Recall that $1/q' = \alpha/d$. Then a combination of (2.3.26), (2.3.27) and (2.1.5) shows that

$$\begin{aligned}
|A_R|^{-1/q'} \int_{A_R} \varphi(x)\mathrm{d}x &= \prod_{i=1}^k C_{\alpha_i,d_i} |A_i|^{-1/q'} \int_{A_i} |x_{(i)}|^{-(d_i-\alpha_i)} \mathrm{d}x_{(i)} \\
&= \prod_{i=1}^k C_{\alpha_i,d_i} \alpha_i^{-1} |S^{d_i-1}|^{1-\alpha/d} R^{\alpha_i - \frac{d_i}{d}\alpha} d_i^{\alpha/d} \\
&= \prod_{i=1}^k C_{\alpha_i,d_i} \alpha_i^{-1} |S^{d_i-1}|^{1-\alpha/d} d_i^{\alpha/d} =: K,
\end{aligned}$$

where the constants C_{α_i,d_i} are defined in (2.1.7) and the final constant K does not depend on R. Finally, by symmetry of φ, we have that

$$\|\varphi\|_{q,w} = \sup_{R>0} |A_R|^{-1/q'} \int_{A_R} \varphi(x)\mathrm{d}x = K < \infty. \tag{2.3.28}$$

Hence, $\varphi \in L_{q,w}(\mathbb{R}^d)$ with $q = d/(d-\alpha)$. This completes the proof of Lemma 2.3.7. □

Remark 2.3.8. Note that when there is only one partition (i.e., $k = 1$), or equivalently when γ itself is the Riesz kernel, by [LL97, (6) on p. 107], we see that

$$\left\| |\cdot|^{-(d-\alpha)} \right\|_{\frac{d}{d-\alpha},w} = \alpha^{-1} |S^{d-1}|^{1-\alpha/d} d^{\alpha/d},$$

which is consistent with the norm we find in (2.3.28) up to a constant $C_{\alpha,d}$.

In order to compare our results with known results (see, e.g., Example 2.2.9), let us introduce another commonly used variational constant

$$\begin{aligned}
\mathbf{E}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a} \left\{ \iint_{\mathbb{R}^{2d}} g^2(x)g^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y - \frac{\theta}{2}\,\mathcal{E}_a(g,g) \right\} \\
&= \sup_{g\in\mathcal{F}_a} \left\{ \left\langle g^2 * g^2, \gamma \right\rangle_{L^2(\mathbb{R}^d)} - \frac{\theta}{2}\,\mathcal{E}_a(g,g) \right\}.
\end{aligned} \tag{2.3.29}$$

By using the same techniques used to derive (2.3.14), one can show that for any $\Theta > 0$ and $\theta > 0$ that

$$\mathbf{E}_{a,d}(\Theta\gamma,\theta) = \Theta^{\frac{a}{a-\alpha}}\theta^{-\frac{\alpha}{a-\alpha}}\mathbf{E}_{a,d}(\gamma,1). \tag{2.3.30}$$

The relation between $\mathbf{E}_{a,d}(\gamma,\theta)$ and $\mathcal{M}_{a,d}(\gamma,\theta)$ can be established in a similar way as [Che+15, Lemma A.2], which is stated in the following lemma:

Lemma 2.3.9. *Under Assumption 2.1.3 and assuming $\alpha < \min\{a,d\}$, the following three expressions hold:*

$$\mathbf{E}_{a,d}(\Theta\gamma,\theta) = \Theta^{\frac{a}{a-\alpha}}\theta^{-\frac{\alpha}{a-\alpha}}\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}\frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)}, \tag{2.3.31}$$

$$\mathcal{M}_{a,d}(\Theta\gamma,\theta) = \Theta^{\frac{a}{2a-\alpha}}\theta^{-\frac{\alpha}{2a-\alpha}}\left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)}\frac{2a-\alpha}{2a}\sigma(a,d,\alpha)^{a/(2a-\alpha)}, \tag{2.3.32}$$

$$\mathbf{E}_{a,d}(\Theta\gamma,\theta) = \left(\frac{a-\alpha}{a}\right)2^{\alpha/(a-\alpha)}\left(\frac{2a-\alpha}{2a}\right)^{-(2a-\alpha)/(a-\alpha)}\mathcal{M}_{a,d}(\Theta\gamma,\theta)^{\frac{2a-\alpha}{a-\alpha}} \tag{2.3.33}$$

where $\sigma(a,d,\alpha)$ is defined in the following Lemma.

We need to prove two lemmas in order to prove Lemma 2.3.9.

Lemma 2.3.10. *Under Assumption 2.1.3, for any $f \in L^2(\mathbb{R}^d)$ with $\mathcal{E}_a(f,f) < \infty$, it holds that*

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \leqslant C\,\|f\|_2^{2-(2\alpha/a)}\,\mathcal{E}_a(f,f)^{\alpha/a}, \tag{2.3.34}$$

where the constant C only depends on a, d and α with $\alpha < \min\{a,d\}$. Denote the best constant in (2.3.34) by $\sigma(a,d,\alpha)$.

Proof. The proof of this result follows the scheme laid out in the proof of [Che12, Lemma A.3]. By the same techniques presented above in Lemma 2.3.5, one can show the following quantity is finite:

$$\begin{aligned}\Lambda &:= \sup_{h\in\mathcal{F}_a}\left\{\int_{\mathbb{R}^d} h^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}|x|^a|\mathcal{F}h(x)|^2\mathrm{d}x\right\}\\ &= \sup_{h\in\mathcal{F}_a}\left\{\int_{\mathbb{R}^d} h^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}\mathcal{E}_a(h,h)\right\} < \infty.\end{aligned}$$

Fix an arbitrary $f \in \mathcal{F}_a$. Clearly, $\|f\|_2 = 1$ and $\mathcal{E}_a(f,f) < \infty$. Let C_f be the constant such that

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x = C_f \mathcal{E}_a(f,f)^{\alpha/a}.$$

Now for $g(x) := t^{d/2} f(tx)$, it is easy to see that $\|g\|_2 = 1$ and

$$\mathcal{E}_a(g,g) = t^a \mathcal{E}_a(f,f) \quad \text{and} \quad \int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x = t^\alpha \int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x.$$

From this we can deduce that

$$\int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x = C_f \mathcal{E}_a(g,g)^{\alpha/a}.$$

Next we note that

$$\begin{aligned}
\Lambda &\geqslant \int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}\mathcal{E}_a(g,g) \\
&= t^\alpha \int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2} t^a \mathcal{E}_a(f,f) \\
&= C_f t^\alpha \mathcal{E}_a(f,f)^{\alpha/a} - \frac{1}{2} t^a \mathcal{E}_a(f,f) \\
&= C_f \left(t\mathcal{E}_a(f,f)^{1/a}\right)^\alpha - \frac{1}{2}\left(t\mathcal{E}_a(f,f)^{1/a}\right)^a.
\end{aligned}$$

Since $t > 0$, then $t\mathcal{E}_a(f,f)^{1/a}$ runs through all of $\mathbb{R}_+$ and thus we have that

$$\Lambda \geqslant \sup_{x>0}\left\{C_f x^\alpha - \frac{1}{2}x^a\right\} = \frac{a-\alpha}{a} C_f^{a/(a-\alpha)} \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}.$$

Note that this reduces to the equation present in the proof of Lemma A.3 [Che12] when $a = 2$. By taking the sup over all $f \in \mathcal{F}_a$ we see that

$$\infty > \Lambda \geqslant \frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)} \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}.$$

where

$$\sup_{f \in \mathcal{F}_a} C_f = \sigma(a, d, \alpha)$$

and finally we conclude that for any $f \in \mathcal{F}_a$

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \leqslant \sigma(a, d, \alpha)\mathcal{E}_a(f, f)^{\alpha/a} < \infty. \tag{2.3.35}$$

For arbitrary $f \in L^2(\mathbb{R}^2)$ with $\mathcal{E}_a(f, f) < \infty$ we apply (2.3.35) to $f / \|f\|_2$ and see that

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \leqslant \sigma(a, d, \alpha) \|f\|_2^{2-(2\alpha/a)} \mathcal{E}_a(f, f)^{\alpha/a}$$

which again reduces to the equation A.4 [Che12] when $a = 2$. □

Lemma 2.3.11. *For any $f \in \mathcal{F}_a$ and for $\alpha < \min\{a, d\}$ we have*

$$\int_{\mathbb{R}^{2d}} \gamma(x + y)f^2(x)f^2(y)\mathrm{d}x\mathrm{d}y \leqslant \sigma(a, d, \alpha)\mathcal{E}_a(f, f)^{\alpha/a} \tag{2.3.36}$$

and $\sigma(a, d, \alpha)$ is the sharpest such constant.

Proof. Suppose that $f \in L^2(\mathbb{R}^d)$ and suppose that $\mathcal{E}_a(f, f) < \infty$ and let $y \in \mathbb{R}^d$ be arbitrary. Recall the translation property of the Fourier transform

$$|\mathcal{F}f(\cdot)(\xi)| = |\mathcal{F}f(\cdot - y)(\xi)|.$$

Then by applying a change of variables and recalling (2.3.34) we see that

$$\begin{aligned}\int_{\mathbb{R}^d} f^2(x)\gamma(x + y)\mathrm{d}x &= \int_{\mathbb{R}^d} f^2(x - y)\gamma(x)\mathrm{d}x \\ &\leqslant \sigma(a, d, \alpha) \|f(\cdot - y)\|_2^{2-(2\alpha/a)} \mathcal{E}_a\big(f(\cdot - y), f(\cdot - y)\big)^{\alpha/a} \\ &= \sigma(a, d, \alpha) \|f\|_2^{2-(2\alpha/a)} \mathcal{E}_a\big(f, f\big)^{\alpha/a},\end{aligned}$$

and in return,

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} f^2(x)\gamma(x+y) \leqslant \sigma(a,d,\alpha)\,\|f\|_2^{2-(2\alpha/a)}\,\mathcal{E}_a(f,f)^{\alpha/a}.$$

Next, notice that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f^2(x)f^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}\mathrm{d}x f^2(x)\int_{\mathbb{R}^d}\mathrm{d}y f^2(y)\gamma(x+y)\\ &\leqslant \sigma(a,d,\alpha)\,\|f\|_2^{4-(2\alpha/a)}\,\mathcal{E}_a(f,f)^{\alpha/a}\end{aligned}$$

and when $f\in\mathcal{F}_a$, and thus $\|f\|_2 = 1$, we see that this reduces to

$$\int_{\mathbb{R}^{2d}} \gamma(x+y)f^2(x)f^2(y)\mathrm{d}x\mathrm{d}y \leqslant \sigma(a,d,\alpha)\mathcal{E}_a(f,f)^{\alpha/a}.$$

We note that the sharpness of $\sigma(a,d,\alpha)$ follows immediately from Lemma 2.3.10. In addition, this reduces to equation (A.1) [Che+15] for the time independent case when $a = 2$. □

Proof of Lemma 2.3.9. We only prove the case of $\Theta = \theta = 1$, the general case can be proven by applying the scaling properties (2.3.14) and (2.3.30).

We have that

$$\begin{aligned}E_{a,d}(\gamma,1) &\leqslant \sup_{g\in\mathcal{F}_a}\left\{\sigma(a,d,\alpha)\mathcal{E}_a(g,g)^{\alpha/a} - \frac{1}{2}\left(\mathcal{E}_a(g,g)^{1/a}\right)^a\right\}\\ &\leqslant \sup_{x>0}\left\{\sigma(a,d,\alpha)x^\alpha - \frac{1}{2}x^a\right\}\\ &= \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}\frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)}\end{aligned} \tag{2.3.37}$$

and similarly

$$\begin{aligned}\mathcal{M}_{a,d}(\gamma,1) &\leqslant \sup_{x>0}\left\{\sigma(a,d,\alpha)^{1/2}x^{\alpha/2} - \frac{1}{2}x^a\right\}\\ &= \left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)}\frac{2a-\alpha}{2a}\sigma(a,d,\alpha)^{a/(2a-\alpha)}.\end{aligned} \tag{2.3.38}$$

Recalling Lemma 2.3.11 above, one can choose $0 < \epsilon < \sigma(a, d, \alpha)$ and $f \in \mathcal{F}_a$ such that

$$\int_{\mathbb{R}^{2d}} \gamma(x+y) f^2(x) f^2(y) \mathrm{d}x\mathrm{d}y \geqslant (\sigma(a,d,\alpha) - \epsilon)\mathcal{E}_a(f,f)^{\alpha/a}.$$

Now define

$$g(x) = t^{d/2} f(tx).$$

Notice that

$$\begin{aligned}
E_{a,d}(\gamma, 1) &\geqslant \int_{\mathbb{R}^{2d}} g^2(x) g^2(y) \gamma(x-y) \mathrm{d}x\mathrm{d}y - \frac{1}{2}\,\mathcal{E}_a(g,g) \\
&= t^\alpha \int_{\mathbb{R}^{2d}} \gamma(x-y) f^2(x) f^2(y) - \frac{1}{2} t^a \mathcal{E}_a(f,f) \\
&\geqslant (\sigma(a,d,\alpha) - \epsilon) t^\alpha \mathcal{E}_a(f,f)^{\alpha/a} - \frac{1}{2} t^a \mathcal{E}_a(f,f)
\end{aligned}$$

and this is true for all $t > 0$ so we can say that

$$\begin{aligned}
E_{a,d}(\gamma, 1) &\geqslant \sup_{x>0} \left\{ (\sigma(a,d,\alpha) - \epsilon) x^\alpha - \frac{1}{2} x^a \right\} \\
&= \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)} \frac{a-\alpha}{a} (\sigma(a,d,\alpha) - \epsilon)^{a/(a-\alpha)}
\end{aligned}$$

and be letting $\epsilon \to 0$ gives us

$$E_{a,d}(\gamma, 1) \geqslant \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)} \frac{a-\alpha}{a} \sigma(a,d,\alpha)^{a/(a-\alpha)} \tag{2.3.39}$$

and if we combine this with (2.3.37) then we see that

$$E_{a,d}(\gamma, 1) = \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)} \frac{a-\alpha}{a} \sigma(a,d,\alpha)^{a/(a-\alpha)}. \tag{2.3.40}$$

Similarly we show that

$$\mathcal{M}_{a,d}(\gamma, 1) = \left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)} \frac{2a-\alpha}{2a} \sigma(a,d,\alpha)^{a/(2a-\alpha)}. \tag{2.3.41}$$

Lastly, by combining (2.3.40) and (2.3.41), we see that

$$E_{a,d}(\gamma,1) = \left(\frac{a-\alpha}{a}\right)(2)^{\alpha/(a-\alpha)}\left(\frac{2a-\alpha}{2a}\right)^{-(2a-\alpha)/(a-\alpha)} \mathcal{M}_{a,d}(\gamma,1)^{(2a-\alpha)/(a-\alpha)}. \quad (2.3.42)$$

Equations (2.3.41), (2.3.40) and (2.3.42) recover equations (A.2), (A.3) and (A.4) of [Che+15] respectively when $a = 2$. □

2.4 Existence and uniqueness of the solution

In this section, we will prove part (1) of Theorem 2.1.6. The proof will need the following lemma:

Lemma 2.4.1 (Lemma 3.5 of [BCC22]). *If $H : [0,\infty) \to [0,\infty)$ is a non-decreasing function, then*

$$2\int_0^\infty e^{-2t} H^2(t)\mathrm{d}t \leqslant \left(\int_0^\infty e^{-t}H(t)\mathrm{d}t\right)^2. \quad (2.4.1)$$

The proof of Theorem 2.1.6 follows the same strategy as [BCC22, Section 3] with minor changes such as

$$\frac{1}{1+|\xi|^2} \quad \text{replaced by} \quad \frac{1}{1+\frac{\nu}{2}|\xi|^a}. \quad (2.4.2)$$

Nevertheless, for completeness, here we streamline and reorganize this proof as follows.

Proof of Theorem 2.1.6. We first introduce some notation. Let $L(x)$ be the Laplace transform of $G(\cdot,x)$ evaluated at one and calculate its Fourier transform $\mathcal{F}L(\xi)$ as follows:

$$L(x) = \int_0^\infty e^{-t}G(t,x)\mathrm{d}t \quad \text{and} \quad \mathcal{F}L(\xi) = \int_0^\infty e^{-t}\mathcal{F}G(t,\cdot)(\xi)\mathrm{d}t = \frac{1}{1+\frac{\nu}{2}|\xi|^a}; \quad (2.4.3)$$

see the proof of Theorem 4.1 of [CHN19] for the last equality. Similarly, let $L_n(\vec{y})$ to be the Laplace transform of $\widetilde{f}_n(\vec{y},0,\cdot)$ evaluated at one, namely,

$$L_n(\vec{y}) := n!\int_0^\infty e^{-t}\widetilde{f}_n(\vec{y};0,t)\mathrm{d}t$$

$$= \sum_{\sigma\in\Sigma_n} \int_0^\infty e^{-t} \int_{[0,t]^n_<} \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{s}\mathrm{d}t$$

with the convention that $s_0 = 0$ and $y_{\sigma(0)} = 0$. By the relation of convolution and the Laplace transform (or through a change of variables), we see that

$$L_n(\vec{y}) = \sum_{\sigma\in\Sigma_n} L\left(y_{\sigma(1)}\right) L\left(y_{\sigma(2)} - y_{\sigma(1)}\right) \cdots L\left(y_{\sigma(n)} - y_{\sigma(n-1)}\right). \tag{2.4.4}$$

Hence, from (2.4.3),

$$\mathcal{F}L_n(\vec{\xi}) = \sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1 + \frac{\nu}{2}\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}. \tag{2.4.5}$$

Moreover, define

$$\begin{aligned} H_n(t, \vec{x}) &= n! \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \widetilde{f}_n(\vec{y}; 0, t) \mathrm{d}\vec{y} \\ &= \sum_{\sigma\in\Sigma_n} \int_{[0,t]^n_<} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{y}\mathrm{d}\vec{s}, \end{aligned} \tag{2.4.6}$$

where recall that K is defined in (2.1.6). Under the nonnegativity assumption — Assumption 2.1.1, we see that for any $\vec{x} \in \mathbb{R}^{nd}$ fixed, the function $t \to H_n(t, \vec{x})$ is a non-decreasing function for $t \geqslant 0$. For this function, we are about to apply Lemma 2.4.1.

Step 1. We first compute the corresponding part to the right-hand side of (2.4.1). By Fubini's theorem,

$$\begin{aligned} &\int_0^\infty e^{-t} H_n(t, \vec{x}) \mathrm{d}t \\ &= \sum_{\sigma\in\Sigma_n} \int_0^\infty e^{-t} \int_{[0,t]^n_<} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{y}\mathrm{d}\vec{s}\mathrm{d}t \\ &= \int_{\mathbb{R}^{dn}} \prod_{k=1}^n K(x_k - y_k) L_n(\vec{y})\, \mathrm{d}\vec{y}. \end{aligned}$$

Then an application of the Plancherel's theorem and the fact that $K * K = \gamma$ shows that

$$\int_{(\mathbb{R}^d)^n} \left[\int_0^\infty e^{-t} H_n(t, \vec{x}) \mathrm{d}t\right]^2 \mathrm{d}\vec{x} = \int_{(\mathbb{R}^d)^n} |\mathcal{F}L_n(\vec{\xi})|^2 \mu(\mathrm{d}\vec{\xi}). \tag{2.4.7}$$

One may check the proof of Lemma 3.3 of [BCC22] for more details.

Step 2. Now we compute the corresponding part to the left-hand side of (2.4.1). First, using the fact that $K * K = \gamma$, we see that

$$\left\|\widetilde{f}_n(\cdot, 0; t)\right\|_{\mathcal{H}^{\otimes n}}^2 = \frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n} H_n^2(t, \vec{x})\mathrm{d}\vec{x}; \tag{2.4.8}$$

one may check the proof of Lemma 3.4 of [BCC22] for more details. By the scaling property for $\mathcal{F}\widetilde{f}_n$ in (2.3.7), one can show that

$$\int_0^\infty e^{-t}\left\|\widetilde{f}_n(\cdot, 0, t)\right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t = \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2}\int_0^\infty 2e^{-2t}\int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2\mathrm{d}\vec{x}\mathrm{d}t; \tag{2.4.9}$$

see Appendix for the proof.

Step 3. Now we can apply Fubini's theorem and Lemma 2.4.1 to the function $t \to H_n(t, \vec{x})$ to see that

$$\int_0^\infty 2e^{-2t}\int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2\mathrm{d}\vec{x}\mathrm{d}t \leqslant \int_{\mathbb{R}^{nd}}\left[\int_0^\infty e^{-t}H_n(t, \vec{x})\mathrm{d}t\right]^2 \mathrm{d}\vec{x}. \tag{2.4.10}$$

Therefore, combining (2.4.7), (2.4.8), and (2.4.10) shows that

$$\begin{aligned}\int_0^\infty e^{-t}\left\|\widetilde{f}_n(\cdot, 0, t)\right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t &\leqslant \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2}\int_{(\mathbb{R}^d)^n} |\mathcal{F}L_n(\vec{\xi})|^2\mu(\mathrm{d}\vec{\xi}) \\ &=: \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2}T_n(\nu, a),\end{aligned} \tag{2.4.11}$$

where

$$T_n(\nu, a) = \int_{(\mathbb{R}^d)^n}\left[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}\right]^2 \mu(\mathrm{d}\vec{\xi}). \tag{2.4.12}$$

By the same arguments as those of Lemma 3.6 of [BCC22] with the replacement (2.4.2), we see that

$$T_n(\nu, a) \leqslant (n!)^2 C_\mu^n(\nu, a) \qquad \text{with} \quad C_\mu(\nu, a) := \int_{(\mathbb{R}^d)}\left(\frac{1}{1+\frac{\nu}{2}|\xi|^a}\right)^2 \mu(\mathrm{d}\xi). \tag{2.4.13}$$

Notice that

$$\int_{(\mathbb{R}^d)} \left(\frac{1}{1+|\xi|^a}\right)^2 \mu(\mathrm{d}\xi) = C\int_0^\infty \frac{\rho^{\alpha-1}}{(1+\rho^a)^2}\mathrm{d}\rho < \infty \quad \Longleftrightarrow \quad 2a-\alpha+1>1. \tag{2.4.14}$$

Therefore, conditions in (2.1.11) imply that $C_\mu(\nu, a) < \infty$. Combining (2.4.11) and (2.4.13) gives that

$$\int_0^\infty e^{-t}\left\|\widetilde{f}_n(\cdot,0;t)\right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t \leqslant 2^{n\left[2(b+r)-\frac{b\alpha}{a}\right]}C_\mu^n(\nu,a) < +\infty. \tag{2.4.15}$$

Step 4. From the scaling property (2.3.8) we see that

$$\begin{aligned}\int_0^\infty e^{-t}\left\|\widetilde{f}_n\left(\cdot,0,t\right)\right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t &= \left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 \int_0^\infty e^{-t}t^{\left[2(b+r)-\frac{b\alpha}{a}\right]n}\mathrm{d}t\\ &= \left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 \Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right),\end{aligned} \tag{2.4.16}$$

which entails another part of the conditions in (2.1.11):

$$2(b+r)-\frac{b\alpha}{a} > 0. \tag{2.4.17}$$

From (2.4.15) and (2.4.16), we deduce that

$$\begin{aligned}\left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 &= \frac{1}{\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)}\int_0^\infty e^{-t}\left\|\widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t\\ &\leqslant \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}C_\mu(\nu,a)\right)^n}{\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)} \leqslant C^n\frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}C_\mu(\nu,a)\right)^n}{(n!)^{2(b+r)-b\alpha/a}},\end{aligned}$$

where the constant C depends only on the value of $2(b+r)-b\alpha/a$ and the last inequality is due to Stirling's formula (see (2.5.2) below).

Because of the constant one initial condition, $\|u(t,x)\|_2 = \|u(t,0)\|_2$ for all $x\in\mathbb{R}^d$ and $t>0$. Therefore, by (2.3.9), (2.3.8), and the above inequality,

$$\|u(t,x)\|_2^2 = \sum_{n\geqslant 0}\theta^n n!\, t^{[2(b+r)-b\alpha/a]n}\left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2$$

$$\leqslant \sum_{n\geqslant 0} \theta^n C^n t^{[2(b+r)-b\alpha/a]n} \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^n}{(n!)^{2(b+r)-1-b\alpha/a}}, \tag{2.4.18}$$

which is finite provided that (see (2.1.11))

$$2(b+r)-1-b\alpha/a > 0. \tag{2.4.19}$$

Finally, an application of the Minkowski inequality and the hypercontractivity (see [BCC22, Theorem B.1] or [Lê16] for the case of the SHE) shows that for all $p \geqslant 2$,

$$\begin{aligned} \|u(t,x)\|_p &\leqslant \sum_{n\geqslant 0} \theta^{n/2}(p-1)^{n/2}\sqrt{n!}\left\|\widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}} \\ &\leqslant \sum_{n\geqslant 0} \theta^{n/2} C^{n/2}(p-1)^{n/2} t^{[2(b+r)-b\alpha/a]n/2} \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^{n/2}}{(n!)^{\frac{1}{2}[2(b+r)-1-b\alpha/a]}} < \infty. \end{aligned} \tag{2.4.20}$$

Therefore, under condition (2.1.11), (2.1.1) has an unique $L^p(\Omega)$-solution $u(t,x)$ for all $p \geqslant 2$, $t > 0$ and $x \in \mathbb{R}^d$. This proves part (1) of Theorem 2.1.6.

Step 5. The proof of part (2) of Theorem 2.1.6 will be postponed to part (ii) of Lemma 2.5.1 below. □

2.5 Upper bound of the asymptotics

In this section, we will give the proof of part 2 of Theorem 2.1.6 and establish the upper bound of (2.1.14) (under Assumptions 2.1.1 and 2.1.3, and condition (2.1.11)), namely,

$$\begin{aligned} \limsup_{t_p\to\infty} t_p^{-\beta} \log \|u(t,x)\|_p \leqslant \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^\beta \\ \times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r)-\frac{b\alpha}{a}-1\right). \end{aligned} \tag{2.5.1}$$

As for the upper bound, we will first establish the corresponding result for $p = 2$ in Lemma 2.5.2 and then apply the hypercontractivity property given by Theorem B.1 in [BCC22] to obtain the general case for $p \geqslant 2$. To prove the next lemma, we will need the following equality

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{\Gamma(an+1)}{(n!)^a}\right)=a\log(a),\quad \text{for all } a>0, \tag{2.5.2}$$

which is a direct consequence of Stirling's formula.

Lemma 2.5.1. *Assume Assumptions 2.1.1 and 2.1.3 hold and in addition that $\alpha<\min\{2a,d\}$. Let ρ be the constant defined in (2.3.10). Then the following identities hold true:*

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\left\|\widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\right)=\log\left(2^{2(b+r)-\frac{b\alpha}{a}}\rho\right)\quad \textit{and} \tag{2.5.3}$$

$$\lim_{n\to\infty}\frac{1}{n}\log\left((n!)^{2(b+r)-\frac{b\alpha}{a}}\left\|\widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}\right)=\log\left(\left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}}\right)+\log\rho. \tag{2.5.4}$$

Proof. The proof follows the same arguments as those of [BCC22, Lemma 4.3]. Nevertheless, we sketch the proof here for completeness. Recall the definition of $T_n(\nu,a)$ defined in (2.4.12). From (2.3.21), we see that

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{T_n(\nu,a)}{(n!)^2}\right]=\log\left(\rho_{\nu,a}\right). \tag{2.5.5}$$

As a consequence of (2.4.11) and (2.4.16) in the proof of Theorem 2.1.6, we see that

$$\left\|\widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\leqslant\frac{2^{n\left[2(b+r)-\frac{b\alpha}{a}\right]}}{(n!)^2}T_n(\nu,a).$$

Combining this and (2.3.21) we see that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\left[\left\|\widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\right]\\ \leqslant\log\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}\right)+\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{T_n(\nu,a)}{(n!)^2}\right)\end{aligned}$$

$$= \log\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}\right) + \log(\rho_{\nu,a}),$$

which proves the upper bound for (2.5.3).

Now we prove the lower bound for (2.5.3). Let τ and $\widetilde{\tau}$ be independent exponential random variables with mean one. In the following, we will compute $\mathbb{E}[J_n(\tau,\widetilde{\tau})]$ in two ways, where

$$J_n(t,t') := \int_{\mathbb{R}^{nd}} H_n(t,x)H_n(t',x)\mathrm{d}x, \qquad t,t'>0;$$

see (4.2.6) for the definition of the function H_n. Notice that using the above notation, (2.4.8) can be rewritten as

$$\left\|\widetilde{f}_n(\cdot,0;t)\right\|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n} H_n(t,\vec{x})^2\mathrm{d}\vec{x} = \frac{1}{(n!)^2}J_n(t,t).$$

On the one hand, the Cauchy-Schwartz inequality implies that

$$J_n(t,t') \leqslant J_n(t,t)^{1/2}J_n(t',t')^{1/2} = t^{[2(b+r)-b\alpha/a]n/2}(t')^{[2(b+r)-b\alpha/a]n/2}J_n(1,1).$$

where we have used the scaling property of $J_n(t,t)$ inherited from that of $\left\|\widetilde{f}_n(\cdot,0;t)\right\|_{\mathcal{H}^{\otimes n}}$ as in (2.3.8). Hence,

$$\begin{aligned}\mathbb{E}[J_n(\tau,\widetilde{\tau})] &\leqslant \mathbb{E}\left[\tau^{[2(b+r)-b\alpha/a]n/2}\right]\mathbb{E}\left[\widetilde{\tau}^{[2(b+r)-b\alpha/a]n/2}\right]J_n(1,1)\\ &= \Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2}+1\right)^2 (n!)^2\left\|\widetilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}.\end{aligned}$$

On the other hand, by (2.4.7) and (2.4.12), we see that

$$\mathbb{E}[J_n(\tau,\widetilde{\tau})] = \int_0^\infty\int_0^\infty e^{-t}e^{-\widetilde{t}}J_n(t,\widetilde{t})\mathrm{d}t\mathrm{d}\widetilde{t} = \int_{\mathbb{R}^{dn}}\left[\int_0^\infty e^{-t}H_n(t,x)\mathrm{d}t\right]^2\mathrm{d}x = T_n(\nu,a).$$

Therefore,

$$\frac{T_n(\nu,a)}{(n!)^2} \leqslant \Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2}+1\right)^2\left\|\widetilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}. \tag{2.5.6}$$

Now, an application of Stirling's formula as in (2.5.2) to see that, as $n \to \infty$,

$$\Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2}+1\right)^2 \sim \Gamma\left([2(b+r)-b\alpha/a]n+1\right) 2^{-[2(b+r)-b\alpha/a]n} C_n, \qquad (2.5.7)$$

where $C_n = 2^{-1}[2(b+r)-b\alpha/a]^{1/2}(2\pi n)^{1/2}$. Then an application of (2.5.5), (2.5.6) and (2.5.7) proves the lower bound of (2.5.3). Lastly, (2.5.4) follows from (2.5.3) and the limit (2.5.2). This proves Lemma 2.5.1. ☐

Now we are ready to prove part (2) of Theorem 2.1.6.

Proof of part (2) of Theorem 2.1.6. The critical case happens when the exponent of $n!$ in (2.4.20) vanishes, namely,

$$\alpha = \frac{a}{b}\left[2(b+r)-1\right].$$

Among the three inequalities in (2.1.11), we also need to make sure that the minimum is achieved by $\frac{a}{b}\left[2(b+r)-1\right]$, for which, we need to additionally require $\frac{a}{b}\left[2(b+r)-1\right] \leqslant d$ and

$$\frac{a}{b}\left[2(b+r)-1\right] < 2a \quad \Longleftrightarrow \quad r \in [0, 1/2).$$

The reason for having a strict inequality above is our need to apply (2.5.4) and Theorem 2.3.5 later on in this proof. Putting these conditions together gives the conditions stated in (2.1.12).

We start by proving part *(2-i)*. Let u_λ for $\lambda > 0$ be the solution of the SPDE (2.1.1) with θ replaced with λ and $u = u_\theta$. By the hypercontractivity property (see [BCC22, Lemma B.1]), we have that

$$\left\|u(t,x)\right\|_p \leqslant \left\|u_{(p-1)\theta}(t,x)\right\|_2, \quad \text{for all } p \geqslant 2,\ t>0 \text{ and } x \in \mathbb{R}^d. \qquad (2.5.8)$$

Now by recalling Theorem 2.3.3 and by applying $\alpha = \frac{a}{b}[2(b+r)-1]$ and the scaling property (2.3.8), we see that

$$\begin{aligned}\left\|u_{(p-1)\theta}(t,x)\right\|_2^2 &= \sum_{n\geqslant 0}[\theta(p-1)]^n\, n! \left\|\tilde{f}_n(\cdot,0;t)\right\|_{\mathcal{H}^{\otimes n}}^2 \\ &= \sum_{n\geqslant 0}[t\theta(p-1)]^n\, n! \left\|\tilde{f}_n(\cdot,0;1)\right\|_{\mathcal{H}^{\otimes n}}^2 \\ &=: \sum_{n\geqslant 0}[t\theta(p-1)]^n R_n,\end{aligned}$$

with $R_n = n! \left\|\tilde{f}_n(\cdot,0;1)\right\|_{\mathcal{H}^{\otimes n}}^2$. By the Cauchy-Hadamard theorem, this series converges for $\theta t(p-1) < \limsup_{n\to\infty}|R_n|^{-1/n}$. However, by (2.5.4) and Theorem 2.3.5, we see that

$$\lim_{n\to\infty}\frac{1}{n}\log(|R_n|) = \log(2\rho) = \log\left(2\nu^{-\alpha/a}\mathcal{M}_a^{(2a-\alpha)/a}\right).$$

Therefore, $\limsup_{n\to\infty}|R_n|^{-1/n} = (2\nu^{-\alpha/a}\mathcal{M}_a^{(2a-\alpha)/a})^{-1}$ and $\|u(t,x)\|_p$ converges for

$$t < \frac{1}{2\theta(p-1)\nu^{-\alpha/a}\mathcal{M}_a^{(2a-\alpha)/a}} =: T_p\,; \qquad \text{see (2.1.13).}$$

To show part *(2-ii)*, we use the Cauchy-Hadamard theorem and the same techniques above to see that the radius of convergence of the series

$$\|u(t,x)\|_2^2 = \sum_{n\geqslant 0}\theta^n n! \left\|\tilde{f}_n(\cdot,0;t)\right\|_{\mathcal{H}^{\otimes n}}^2,$$

is precisely T_2. This completes the proof of part (2) of Theorem 2.1.6. □

Lemma 2.5.2. *Assume Assumptions 2.1.1 and 2.1.3 hold. Let ρ be the constant defined in* (2.3.10)*. Under condition* (2.1.11)*, we have that*

$$\lim_{t\to\infty}\frac{1}{t^\beta}\log\mathbb{E}(|u(t,x)|^2) = \left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}(\theta\rho)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).$$

Proof. By part (1) of Theorem 2.1.6, there is an $L^2(\Omega)$ solution $u(t,x)$. By the scaling property (2.3.8),

$$\mathbb{E}(|u(t,x)|^2) = \sum_{n\geqslant 0} \theta^n (n!) \left\| \widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}}^2 = \sum_{n\geqslant 0} \theta^n (n!) t^{(2(b+r)-b\alpha/a)n} \left\| \widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2$$
$$= \sum_{n\geqslant 0} z_n R_n t^{(2(b+r)-b\alpha/a)n}$$

where

$$R_n = (n!)^{2(b+r)-b\alpha/a} \left\| \widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 \quad \text{and} \quad z_n = \frac{\theta^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}.$$

Notice that (2.5.4) above says that

$$\frac{1}{n}\log(R_n) \to \log\left(\left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}} \rho\right) \quad \text{as } n\to\infty.$$

Now define R to be

$$R = \left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}} \rho.$$

We want to find a β and A so that

$$\lim_{t\to\infty} \frac{1}{t^\beta} \log \sum_{n\geqslant 0} z_n R^n \left(t^{(2(b+r)-b\alpha/a)}\right)^n = A.$$

Indeed, by the following limit (see [BCC22, Lemma A.3]),

$$\lim_{t\to\infty} t^{-1/\gamma} \log \sum_{n\geqslant 0} (n!)^{-\gamma} t^n = \gamma, \qquad \text{for all } \gamma > 0,$$

we see that

$$\lim_{t\to\infty} \left[\frac{1}{(\theta R) t^{2(b+r)-b\alpha/a}}\right]^{\frac{1}{2(b+r)-(b\alpha/a)-1}} \log \sum_{n\geqslant 0} \frac{\left[(\theta R) t^{2(b+r)-b\alpha/a}\right]^n}{(n!)^{2(b+r)-(b\alpha/a)-1}} = 2(b+r)-(b\alpha/a)-1,$$

which, by an easy algebraic manipulation, is equivalent to

$$\lim_{t\to\infty}\left[\frac{1}{t^{2(b+r)-b\alpha/a}}\right]^{\frac{1}{2(b+r)-(b\alpha/a)-1}}\log\sum_{n\geqslant 0}\frac{\left[(\theta R)t^{2(b+r)-b\alpha/a}\right]^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}$$
$$=[2(b+r)-(b\alpha/a)-1](\theta R)^{\frac{1}{2(b+r)-(b\alpha/a)-1}}.$$

Hence,

$$\beta=\frac{2a(b+r)-b\alpha}{2a(b+r)-(b\alpha)-a}\quad\text{and}\quad A=[2(b+r)-(b\alpha/a)-1](\theta R)^{\frac{a}{2a(b+r)-b\alpha-a}}.$$

Finally, an application of [BCC22, Lemma A.2] shows that

$$\lim_{t\to\infty}\frac{1}{t^{\beta}}\log\mathbb{E}(|u(t,x)|^2)$$
$$=\theta^{\frac{a}{2a(b+r)-b\alpha-a}}\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}\rho^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right),$$

which proves Lemma 2.5.2. □

Now we are ready to prove (2.5.1).

Proof of (2.5.1). By the hypercontractivity (2.5.8) and the scaling property (2.3.8), we see that for all $p\geqslant 2$,

$$\begin{aligned}\|u(t,0)\|_p^2\leqslant\left\|u_{(p-1)\theta}(t,0)\right\|_2^2&=\sum_{n\geqslant 0}(n!)^n\theta^n(p-1)^n\left\|\widetilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}}\\&=\sum_{n\geqslant 0}(n!)^n\theta^n\left\|\widetilde{f}_n\left(\cdot,0,t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}}\right)\right\|^2_{\mathcal{H}^{\otimes n}}\\&=\left\|u\left(t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}},0\right)\right\|_2^2.\end{aligned}$$

Hence,

$$\|u(t,0)\|_p\leqslant\left\|u\left(t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}},0\right)\right\|_2=:\|u(t_p,0)\|_2, \tag{2.5.9}$$

where t_p is defined in (2.1.15). Finally, an application of Lemma 2.5.2 proves (2.5.1). □

2.6 Lower bound of the asymptotics

In this section, we will prove the lower bound of (2.1.14), namely,

$$\begin{aligned}\liminf_{t_p\to\infty} t_p^{-\beta}\log\|u(t,x)\|_p \geqslant \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\ \times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).\end{aligned} \tag{2.6.1}$$

Through out this section, we assume that Assumptions 2.1.1 and 2.1.3, and condition (2.1.11) hold.

We start by defining the function $W_n(t,\phi)$ on $(0,\infty)\times L^2_{\mathbb{C}}(\mu)$ by

$$W_n(t,\phi) := \int_{[0,t]^n_<}\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n\phi(\xi_k)\prod_{k=1}\mathcal{F}G(s_k-s_{k-1},\cdot)(\xi_k+\cdots+\xi_n)\mu(\mathrm{d}\xi_1)\cdots\mu(\mathrm{d}\xi_n)\mathrm{d}s. \tag{2.6.2}$$

with $s_0=0$. We now give conditions under which W_n is well defined.

Lemma 2.6.1. *If the measure μ satisfies the relation in (2.4.13), then $W_n(t,\phi)$ is well defined and for any $d\geqslant 1$, $t>0$ and $\phi\in L^2_{\mathbb{C}}(\mu)$. Moreover,*

$$\int_0^\infty e^{-t}W_n(t,\phi)\mathrm{d}t = \int_{(\mathbb{R}^d)^n}\prod_{k=1}^n\phi(\xi_k)\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\xi_k+\cdots+\xi_n|^a}\mu(\mathrm{d}\xi)\cdots\mu(\mathrm{d}\xi_n). \tag{2.6.3}$$

Proof. The proof follows the exact arguments as those in the proof of Lemma 6.2 of [BCC22] except that one needs to use the following Laplace transform:

$$\int_0^\infty e^{-t}\mathcal{F}G(t,\cdot)(\xi)\,\mathrm{d}t = \frac{1}{1+\frac{\nu}{2}|\xi|^a};$$

see the second equation in (2.4.3). □

The next proposition is a restatement of Proposition 6.3 of [BCC22]. The proof follows the same proof as that of Proposition 6.3 of [BCC22]. We will not repeat it here.

Proposition 2.6.2. *For $f \in \mathcal{H}$, $t > 0$, and $p, q > 1$ with $p^{-1} + q^{-1} = 1$, it holds that*

$$\|u(t,0)\|_p \geqslant \exp\left\{-\frac{1}{2}(q-1)\|f\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geqslant 0}\theta^{n/2}W_n(t,\mathcal{F}f)\right| \tag{2.6.4}$$

and as a consequence, the series $\left|\sum_{n\geqslant 0}\theta^{n/2}W_n(t,\mathcal{F}f)\right|$ converges provided that $\|u(t,0)\|_p < \infty$, which is the case under Theorem 2.1.6.

Now we are going to apply a scaling argument to (2.6.4) in order to put t and p together, from which we can determine the constants t_p and β defined in (2.1.15).

Proposition 2.6.3. *For $p, q > 1$, $p^{-1} + q^{-1} = 1$ and for any $f \in \mathcal{H}$,*

$$\|u(t,0)\|_p \geqslant \exp\left\{-\frac{1}{2}t_p^{\beta}\|f\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geqslant 0}\theta^{n/2}W_n\left(t_p^{\beta},\mathcal{F}f\right)\right|, \tag{2.6.5}$$

where the constants β and t_p are defined in (2.1.15).

Proof. From Proposition 2.6.2, we see that for any $f \in \mathcal{H}$, the inequality (2.6.4) holds. For some constants $V, W > 0$, which will be determined in this proof, let $f_\tau(x) := \tau^V f(\tau^W x)$ be a scaled version of f. It is clear that $f_\tau \in \mathcal{H}$. By some elementary scaling arguments (see the proof of the Lemma 6.4 of [BCC22]), one can show that

$$\|f_\tau\|_{\mathcal{H}}^2 = \tau^{2(V-dW)+W\alpha}\|f\|_{\mathcal{H}}^2 \quad \text{and} \tag{2.6.6}$$

$$W_n(t,\mathcal{F}f_\tau) = \tau^{n\left[V-W\left((d-\alpha)+\frac{a}{b}(b+r)\right)\right]}W_n\left(t\tau^{\frac{a}{b}W},\mathcal{F}f\right). \tag{2.6.7}$$

Hence, an application of Proposition 2.6.2 to f_τ shows that

$$\|u(t,0)\|_p \geqslant \exp\left\{-\frac{1}{2}(q-1)\|f_\tau\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geqslant 0}\theta^{n/2}W_n(t,\mathcal{F}f_\tau)\right|$$
$$= \exp\left\{-\frac{1}{2}(q-1)\tau^{2(V-dW)+W\alpha}\|f\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geqslant 0}\theta^{n/2}\tau^{n\left[V-W\left((d-\alpha)+\frac{a}{b}(b+r)\right)\right]}W_n\left(t\tau^{\frac{a}{b}W},\mathcal{F}f\right)\right|.$$

Comparing the above lower bound with that in (2.6.5), we obtain the following two relations with three unknowns W, V and τ:

$$V - W\left((d-\alpha) + \frac{a}{b}(b+r)\right) = 0, \tag{2.6.8}$$

$$(p-1)^{-1}\tau^{2(V-dW)+W\alpha} = t\tau^{\frac{a}{b}W}. \tag{2.6.9}$$

Since (2.6.9) should hold for all $t > 0$ and $p \geqslant 2$, one can choose $\tau = (p-1)t$ to reduce the relation (2.6.9) to

$$\tau^{2(V-dW)+W\alpha} = \tau^{\frac{a}{b}W+1},$$

which then gives the following equation

$$2(V-dW) + W\alpha = 1 + \frac{a}{b}W. \tag{2.6.10}$$

Now solve the linear equations (2.6.8) and (2.6.10) for W and V to see that

$$\begin{cases} W = \dfrac{b}{a}\left(\beta - 1\right), \\ V = \left(\dfrac{a}{b}(b+r) - \alpha + d\right)\dfrac{b}{a}(\beta-1), \end{cases} \qquad \text{with } \beta := \frac{2(b+r) - \dfrac{b\alpha}{a}}{2(b+r) - \dfrac{b\alpha}{a} - 1}.$$

Therefore, the scaling f_τ and t_p^β should be

$$f_\tau(x) = \tau^{\left(\frac{a}{b}(b+r)-\alpha+d\right)\frac{b}{a}(\beta-1)} f\left(\tau^{\frac{b}{a}(\beta-1)}x\right) \quad \text{and} \quad t_p^\beta := t\tau^{\frac{a}{b}W} = (p-1)^{\beta-1}t^\beta,$$

respectively. This completes the proof of Proposition 2.6.3. $\square$

To remove the absolute value sign in Proposition 2.6.3, we want to identify all the $f \in \mathcal{H}$ for which $W_n(t, \mathcal{F}f)$ is nonnegative. In fact, if we consider the space

$$\mathcal{H}_+ = \{f \in \mathcal{H} : f \text{ is nonnegative and nonnegative definite}\},$$

then by Plancherel's theorem, for $f \in \mathcal{H}_+$,

$$W_n(t, \mathcal{F}f) = \int_{[0,t]^n_<} \int_{\mathbb{R}^{nd}} \prod_{k=1}^{n} (f * \gamma)(x_k) \prod_{k=1} G(s_k - s_{k-1}, x_k - x_{k-1}) \mathrm{d}\vec{x}\mathrm{d}\vec{s} =: U_n(t, f) \geqslant 0$$

with the convention that $s_0 = 0$ and $x_0 = 0$, where we have used the fact that the fundamental solution $G(t, x)$ is nonnegative (under Assumption 2.1.1). Now define

$$W_n(t) := \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}}=1} W_n(t, \mathcal{F}(f)) \quad \text{and} \quad U_n(t) := \sup_{f \in \mathcal{H}_+, \|f\|_{\mathcal{H}}=1} U_n(t, f).$$

It is clear that $W_n(t) \geqslant U_n(t) \geqslant 0$.

Proposition 2.6.4. *If τ is an exponential random variable with mean one, then*

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}(U_n(\tau)) \geqslant \log(\rho^{1/2}),$$

where ρ is the constant defined in (2.3.10).

Proof. We start by letting τ be an exponential random variable with mean one. With this,

$$\mathbb{E}\left(U_n(\tau)\right) = \int_0^\infty e^{-t} U_n(t) \mathrm{d}t \geqslant \sup_{f \in \mathcal{H}_+,\, \|f\|_{\mathcal{H}}=1} \int_0^\infty e^{-t} U_n(t, f) \mathrm{d}t.$$

For any $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = 1$, by Bochner's theorem, $\mathcal{F}f$ is nonnegative and nonnegative definite, which further implies that $\mathcal{F}f$ is even. In addition, Lemma 2.6.1 gives us that

$$\begin{aligned} \int_0^\infty e^{-t} U_n(t, f) \mathrm{d}t &= \int_0^\infty e^{-t} W_n(t, \mathcal{F}f) \mathrm{d}t \\ &= \int_{\mathbb{R}^{dn}} \prod_{k=1}^{n} \mathcal{F}f(\xi_k) \prod_{k=1}^{n} \frac{1}{1 + \frac{\nu}{2}|\xi_k + \cdots + \xi_n|^a} \mu(\mathrm{d}\xi_1) \cdots \mu(\mathrm{d}\xi_n). \end{aligned}$$

Notice that the right hand side of the above equation takes the form as [BCR09, Equation (3.3)]. By the same arguments that follow in the proof of Theorem 2.3 *ibid* with the replacement (2.4.2), we have that

$$\liminf_{n \to \infty} \frac{1}{n} \log \int_{\mathbb{R}^{dn}} \prod_{k=1}^{n} (\mathcal{F}f)(\xi_k) \prod_{k=1}^{n} \frac{1}{1 + \frac{\nu}{2}|\xi_k + \cdots + \xi_n|^a} \mu(\mathrm{d}\xi_1) \cdots \mu(\mathrm{d}\xi_n) \geqslant \log \rho(\mathcal{F}f),$$

where $\rho(\cdot)$ (check (2.3.10) for a comparison) is defined as,

$$\rho(g) := \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} g(\xi) \left[\int_{\mathbb{R}^d} \frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}} \mathrm{d}\eta \right] \mu(\mathrm{d}\xi), \tag{2.6.11}$$

for all nonnegative and nonnegative-definite functions $g \in L^2(\mu(\mathrm{d}\xi))$. Before we proceed, we first make a few comments:

(1) $g \in L^2(\mu(\mathrm{d}\xi))$ if and only if $\mathcal{F}^{-1}g \in \mathcal{H}$. Moreover, $\|g\|_{L^2(\mu)} = \|\mathcal{F}^{-1}g\|_{\mathcal{H}}$.

(2) For $g \in L^2(\mu(\mathrm{d}\xi))$, g is nonnegative definite if and only if $\mathcal{F}^{-1}g \in \mathcal{H}_+$.

(3) For $g \in L^2(\mu(\mathrm{d}\xi))$, g is nonnegative if and only if $\mathcal{F}^{-1}g$ is nonnegative definite.

(4) Finding the best nonnegative and nonnegative-definite function g with $\|g\|_{L^2(\mu)} = 1$ to maximize $\rho(g)$ is equivalent to finding the best nonnegative and nonnegative-definite $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = 1$ to maximize $\rho(\mathcal{F}f)$.

(5) $\rho(g)$ is well defined (i.e., finite) because for any $h \in L^2(\mathbb{R}^d)$ with $\|h\|_{L^2(\mathbb{R}^d)} = 1$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} g(\xi) \left[\int_{\mathbb{R}^d} \frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}} \mathrm{d}\eta \right] \mu(\mathrm{d}\xi) \right| \\
&\leqslant \left(\int_{\mathbb{R}^d} g(\xi)^2 \mu(\mathrm{d}\xi) \right)^{1/2} \left(\int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}} \mathrm{d}\eta \right]^2 \mu(\mathrm{d}\xi) \right)^{1/2} \\
&\leqslant \|\mathcal{F}g\|_{\mathcal{H}} \sqrt{\rho_{\nu,a}} < \infty,
\end{aligned}$$

where the upper bound does not depends on h, and $\rho_{\nu,a}$ is the constant defined in (2.3.10) which is finite due to Theorem 2.3.5 (see (2.3.20)).

Note that since both μ and g are nonnegative in (2.6.11), the supremum in (2.6.11) has to be achieved by some nonnegative function h. Hence, we may assume h is also nonnegative for the remainder of this proof. With this being said, we see that for any $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = 1$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}(U_n(\tau)) \geqslant \log \rho(\mathcal{F}f).$$

We now need to calculate

$$\sup_{f\in\mathcal{H}_+,\,\|f\|_{\mathcal{H}}=1} \rho\left(\mathcal{F}f\right).$$

Consider a nonnegative function $h \in L^2(\mathbb{R}^d)$. The function $h(\cdot)/\sqrt{1+\frac{\nu}{2}|\cdot|^a} \in L^2(\mathbb{R}^d)$ so that $g_h(x) := (2\pi)^{d/2}\mathcal{F}^{-1}\left(\frac{h(\cdot)}{\sqrt{1+\frac{\nu}{2}|\cdot|^a}}\right)(x)$ is well defined. Under these conditions, $g_h \in W^{1,a}(\mathbb{R}^d)$ with

$$\|g_h\|_{W^{1,a}(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+|\xi|^a)|\mathcal{F}g_h(\xi)|^2\mathrm{d}\xi = \int_{\mathbb{R}^d}\frac{1+|\xi|^a}{1+\frac{\nu}{2}|\xi|^a}|h(\xi)|^2\mathrm{d}\xi \leqslant C_\nu\,\|h\|^2_{L^2(\mathbb{R}^d)} < \infty.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F}g_h(\xi-\eta)\mathcal{F}g_h(-\eta)\mathrm{d}\eta\\ &= (2\pi)^{-d}(\mathcal{F}g_h * \widetilde{\mathcal{F}g_h})(\xi),\end{aligned}$$

where we have used the notation that $\widetilde{h}(x) = h(-x)$. Since $h(\cdot)$ is real valued, we see that $\widetilde{\mathcal{F}g_h} = \overline{\mathcal{F}g_h} = \mathcal{F}\bar{g}_h$. Using this and the fact that $\mathcal{F}(fg) = (2\pi)^{-d}\mathcal{F}(f) * \mathcal{F}(g)$, we see that $(2\pi)^{-d}(\mathcal{F}g_h * \widetilde{\mathcal{F}g_h})(\xi) = \mathcal{F}\left[|g_h|^2\right](\xi)$. Hence, from (2.3.20), we see that $\mathcal{F}g_h * \widetilde{\mathcal{F}g_h} \in L^2\left(\mu(d\xi)\right)$ or equivalently $|g_h|^2 \in \mathcal{H}$. Then,

$$\begin{aligned}\rho(\mathcal{F}f) &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}(2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F}f(\xi)(\mathcal{F}g_h * \widetilde{\mathcal{F}g_h})(\xi)\mu(\mathrm{d}\xi)\\ &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}\mathcal{F}f(\xi)\mathcal{F}[|g_h|^2](\xi)\mu(\mathrm{d}\xi)\\ &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\langle f, |g_h|^2\rangle_{\mathcal{H}}.\end{aligned}$$

With this we see that

$$\sup_{f\in\mathcal{H},\|f\|_{\mathcal{H}=1}}\rho(\mathcal{F}f) = \sup_{f\in\mathcal{H},\|f\|_{\mathcal{H}=1}}\ \sup_{\|h\|_{L^2}=1}\langle f, |g_h|^2\rangle_{\mathcal{H}} = \sup_{\|h\|_{L^2}=1}\langle |g_h|^2, |g_h|^2\rangle^{1/2}_{\mathcal{H}} \tag{2.6.12}$$

where the optimal f is chosen to be $|g_h|^2 / \| |g_h|^2 \|_{\mathcal{H}}$. Now we claim that

$$|g_h|^2 \in \mathcal{H}_+, \tag{2.6.13}$$

which implies that the supremum in (2.6.12) can be restricted to $\mathcal{H}_+$ and

$$\sup_{f\in\mathcal{H},\, \|f\|_{\mathcal{H}}=1} \rho(\mathcal{F}f) = \sup_{f\in\mathcal{H}_+,\, \|f\|_{\mathcal{H}}=1} \rho(\mathcal{F}f) = \sup_{\|h\|_{L^2}=1} \left\langle |g_h|^2, |g_h|^2 \right\rangle_{\mathcal{H}}^{1/2}.$$

Then notice that

$$\mathcal{F}(|g_h|^2)(\xi) = (2\pi)^{-d}(\mathcal{F}g_h * \widetilde{\mathcal{F}g_h})(\xi) = \int_{\mathbb{R}^d} \frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}} \mathrm{d}\eta. \tag{2.6.14}$$

Hence, from (2.3.10) and (2.6.14), we see that

$$\sup_{\|h\|_{L^2(\mathbb{R}^d)}=1} \left\langle |g_h|^2, |g_h|^2 \right\rangle_{\mathcal{H}} = \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left|\mathcal{F}(|g_h|^2)(\xi)\right|^2 \mu(\mathrm{d}\xi) = \rho_{\nu,a}$$

which then leads us to the lower bound:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}(U_n(\tau)) \geqslant \log\left(\rho_{\nu,a}^{1/2}\right).$$

Therefore, it remains to proving (2.6.13). First notice that from the above we see that

$$\int_{\mathbb{R}^d} \left|\mathcal{F}\left(|g_h|^2\right)(\xi)\right|^2 \mu(\mathrm{d}\xi) \leqslant \rho_{\nu,a} < \infty \quad \Longrightarrow \quad |g_h(\cdot)|^2 \in \mathcal{H}.$$

Moreover, since h is nonnegative, from (2.6.14), we see that $\mathcal{F}\left(|g_h|^2\right)(\cdot)$ is also nonnegative. The Bochner-Schwarz theorem then implies that $|g_h(\cdot)|^2$ is nonnegative definite. It is clear that $|g_h(\cdot)|^2$ is nonnegative. This shows that $|g_h(\cdot)|^2 \in \mathcal{H}_+$. This completes the whole proof of Proposition 2.6.4. □

Lemma 2.6.5. *It holds that*

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[(n!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}U_n(1)\right]\geqslant\log\left(\frac{\rho_{\nu,a}^{1/2}}{\left(\frac{b}{a}[a-\frac{\alpha}{2}+\frac{a}{b}r]\right)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}\right).\tag{2.6.15}$$

Proof. Let τ be an exponential random variable with mean one. Notice that by some elementary scaling arguments, we have that for all $t>0$,

$$W_n(t)=t^{n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}W_n(1)\quad\text{and}\quad U_n(t)=t^{n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}U_n(1),\tag{2.6.16}$$

which then imply that

$$\mathbb{E}\left[U_n(\tau)\right]=\mathbb{E}\left(\tau^{n\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}\right)U_n(1)=\Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right)U_n(1).$$

Then by Proposition 2.6.4,

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[\Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right)U_n(1)\right]\geqslant\log\left(\rho_{\nu,a}^{1/2}\right).\tag{2.6.17}$$

Therefore, (2.6.15) is proven by noticing that, thanks to (2.5.2),

$$\liminf_{n\to\infty}\frac{1}{n}\log\left(\frac{\Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right)}{(n!)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}}\right)=\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log\left(\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\right),$$

where the condition $\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)>0$ is guaranteed by (2.4.19) (or (2.1.11)). □

We need one last lemma before the proof of the lower bound:

Lemma 2.6.6. *For any $k,\theta>0$, there exists a constant $c_1=c_1(\alpha,\mathcal{M},k,\theta)>0$ such that, by setting $n_t=[c_1t]$, it holds that*

$$\liminf_{t\to\infty}\frac{1}{t}\log(k^{n_t}\theta^{n_t/2}U_{n_t}(t))\geqslant\left(k\sqrt{\theta}\right)^{\frac{2a}{2ab-\alpha b+2ar}}\left(\rho_{\nu,a}^{1/2}\right)^{\frac{2a}{2ab-\alpha b+2ar}}.\tag{2.6.18}$$

Proof. Fix an arbitrary $\epsilon > 0$. Lemma 2.6.5 guarantees the existence of an $N_\epsilon > 0$ so for all $n > N_\epsilon$

$$(n!)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}U_n(1) \geqslant \exp\left(n(\log(R)-\epsilon)\right) = R^n e^{-n\epsilon} \tag{2.6.19}$$

where

$$R = \rho_{\nu,a}^{1/2}\left(\frac{b}{a}\left[a-\frac{\alpha}{2}+\frac{a}{b}r\right]\right)^{-\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}.$$

Now fix a $c > 0$ and let $n_t := [ct]$. Notice that $n_t \geqslant N_\epsilon$ for any $t > t_\epsilon := (N_\epsilon + 1)/c$. For $t > t_\epsilon$, from (2.6.19), we have

$$k^{n_t}\theta^{n_t/2}U_{n_t}(t) = k^{n_t}\theta^{n_t/2}t^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)n_t}U_{n_t}(1) \geqslant \frac{k^{n_t}\theta^{n_t/2}t^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)n_t}}{(n_t!)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}}R^{n_t}e^{-n_t\epsilon}. \tag{2.6.20}$$

Notice that $[ct]/t \to c$ as $t \to \infty$ which means that $n_t/t \to c$ as $t \to \infty$. With this we can say

$$\liminf_{t\to\infty}\frac{1}{t}\log\left(k^{n_t}\theta^{n_t/2}U_{n_t}(t)\right) = c\liminf_{t\to\infty}\frac{1}{n_t}\log\left(k^{n_t}\theta^{n_t/2}U_{n_t}(t)\right) =: I(n_t).$$

Now, by (2.6.20), we have that

$$\begin{aligned}
I(n_t) &\geqslant c\liminf_{t\to\infty}\frac{1}{n_t}\log\left((kR\sqrt{\theta})^{n_t}\frac{t^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)n_t}}{(n_t!)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}}\right) - c\epsilon \\
&= c\log(k\sqrt{\theta}R) + c\liminf_{t\to\infty}\frac{1}{n_t}\log\left[\left(\frac{t}{n_t}\right)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)n_t}\frac{n_t^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)n_t}}{(n_t!)^{\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}}\right] - c\epsilon \\
&= c\log(k\sqrt{\theta}R) - c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\liminf_{t\to\infty}\frac{1}{n_t}\log\left(\frac{n_t^{n_t}}{(n_t)!}\right) - c\epsilon \\
&= c\log(k\sqrt{\theta}R) - c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right) - c\epsilon
\end{aligned}$$

and letting ϵ tend to 0 we see that

$$I(n_t) \geqslant c\left[\log(k\sqrt{\theta}R) - \frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + \frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\right] =: h(c).$$

In order to maximize $h(c)$, notice that

$$h'(c) = 0 \quad \Longleftrightarrow \quad c^* = (k\sqrt{\theta}R)^{\frac{a}{b\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}}.$$

After plugging c^* and replacing R we arrive at the following inequality

$$I(n_t) \geqslant (k\sqrt{\theta})^{\frac{a}{b\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}} (\rho_{\nu,a}^{1/2})^{\frac{a}{b\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)}},$$

which proves (2.6.18) after some simplification. □

We are now ready to prove (2.6.1).

Proof of (2.6.1). By proposition 2.6.3, for and $p, q > 0$ with $p^{-1} + q^{-1} = 1$ we have that

$$\|u(t,0)\|_p \geqslant \exp\left\{-\frac{1}{2}t_p^\beta \|f\|_{\mathcal{H}}^2\right\} \left|\sum_{n\geqslant 0} \theta^{n/2} W_n(t_p^\beta, \mathcal{F}f)\right|.$$

We now take the supremum over all $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = k > 0$. Recall that $W_n(t,\phi) = k^n W_n(t, \phi/k)$ for $\phi \in L^2(\mu)$ and the non-negativity of $W_n(t,\cdot)$ on $\mathcal{H}_+$. Let $c > 0$.

$$\begin{aligned}
\|u(t,0)\|_p &\geqslant \exp\left\{-\frac{1}{2}t_p^\beta k^2\right\} \sup_{f\in\mathcal{H}_+\ \|f\|_{\mathcal{H}}=k} \sum_{n\geqslant 0} \theta^{n/2} W_n(t_p^\beta, \mathcal{F}f) \\
&= \exp\left\{-\frac{1}{2}t_p^\beta k^2\right\} \sup_{f\in\mathcal{H}_+\ \|f\|_{\mathcal{H}}=1} \sum_{n\geqslant 0} k^n \theta^{n/2} W_n(t_p^\beta, \mathcal{F}f) \\
&\geqslant \exp\left\{-\frac{1}{2}t_p^\beta k^2\right\} \sup_{f\in\mathcal{H}_+\ \|f\|_{\mathcal{H}}=1} k^{n_t} \theta^{n_t/2} U_{n_t}(t_p^\beta, f)
\end{aligned}$$

where $n_t = [ct_p^\beta]$. Now by choosing c as in Lemma 2.6.6 we get that

$$\begin{aligned}
\liminf_{t_p\to\infty} t_p^{-\beta} \log \|u(t,x)\|_p &\geqslant \liminf_{t_p\to\infty} \left(\frac{-\frac{1}{2}t_p^\beta k^2}{t_p^\beta} + \frac{1}{t_p^\beta} \log\left[\sup_{f\in\mathcal{H}_+\ \|f\|_{\mathcal{H}}=1} k^{n_t}\theta^{n_t/2} U_{n_t}(t_p^\beta, f)\right]\right) \\
&= -\frac{1}{2}k^2 + k^{\frac{2a}{2ab-\alpha b+2ar}} (\rho\theta)^{\frac{a}{2ab-\alpha b+2ar}} =: h(k).
\end{aligned}$$

By maximizing h for $k > 0$, we see that h is maximized at the point

$$k^* = \left(\frac{2ab - \alpha b + 2ar}{2aB}\right)^{\frac{2ab-\alpha b+2ar}{2a-2[2ab-\alpha b+2ar]}} \quad \text{with } B = (\theta\rho)^{\frac{a}{2ab-\alpha b+2ar}}.$$

Inserting k^* into h gives us that

$$h(k^*) = B^{\beta}\left(\frac{2a}{2ab - \alpha b + 2ar}\right)^{\beta}\left(\frac{2ab - \alpha b + 2ar - a}{2a}\right)$$

and plugging in the value for B proves (2.6.1). □

2.7 Appendix

Proof of Lemma 2.3.4. In this proof, $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x = C_{\alpha,d}|x|^{-(d-\alpha)}\mathrm{d}x$. By the change of variables $x' = (\nu/2)^{1/a}\, x$ and $y' = (\nu/2)^{1/a}\, y$, we see that

$$\begin{aligned}
\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) &= \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}}\mathrm{d}y\right]^2\mu(\mathrm{d}x)\\
&= \left(\frac{\nu}{2}\right)^{-\alpha/a}\sup_{\|f\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{f\left(\left(\frac{\nu}{2}\right)^{-1/a}(x+y)\right)f\left(\left(\frac{\nu}{2}\right)^{-1/a}y\right)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}}\left(\frac{\nu}{2}\right)^{-d/a}\mathrm{d}y\right]^2\varphi(x)\mathrm{d}x.
\end{aligned}$$

By setting $f^*(x) = (\nu/2)^{-d/(2a)}\, f\left((\nu/2)^{-1/a}\, x\right)$, we see that

$$\begin{aligned}
\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) &= \left(\frac{\nu}{2}\right)^{-\alpha/a}\sup_{\|f\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{f^*(x+y)\,f^*(y)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}}\mathrm{d}y\right]^2\varphi(x)\mathrm{d}x\\
&= \left(\frac{\nu}{2}\right)^{-\alpha/a}\sup_{\|f^*\|_2=1}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{f^*(x+y)\,f^*(y)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}}\mathrm{d}y\right]^2\varphi(x)\mathrm{d}x\\
&= \left(\frac{\nu}{2}\right)^{-\alpha/a}\rho_{2,a}\left(|\cdot|^{-\alpha}\right),
\end{aligned}$$

where the second equality is due to the fact that $\int_{\mathbb{R}^d} f(x)^2\mathrm{d}x = \int_{\mathbb{R}^d} f^*(x)^2\mathrm{d}x$. Then an application of (2.3.16) proves (2.3.18).

Similarly, for (2.3.21), by change of variables $\xi'_{\sigma(j)} = (\nu/2)^{1/a}\,\xi_{\sigma(j)}$, we see that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n}\xi_{\sigma(j)}|^a}\right)^2\mu(\mathrm{d}\vec{\xi}\,)\right]\\
&=\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{(n!)^2}\left(\frac{\nu}{2}\right)^{-n\alpha/a}\int_{(\mathbb{R}^d)^n}\left[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\left|\sum_{j=k}^{n}\xi_{\sigma(j)}\right|^a}\right]^2\mu(\mathrm{d}\vec{\xi})\right]\\
&=\log\left[\left(\frac{\nu}{2}\right)^{-\alpha/a}\right]+\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\left|\sum_{j=k}^{n}\xi_{\sigma(j)}\right|^a}\right]^2\mu(\mathrm{d}\vec{\xi})\right]\\
&=\log\left(\left(\frac{\nu}{2}\right)^{-\alpha/a}\right)+\log\left(\rho_{2,a}\left(|\cdot|^{-\alpha}\right)\right)=\log\left(\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right)\right),
\end{aligned}$$

where we have applied (2.3.15) and (2.3.18). This completes the proof of Lemma 2.3.4. □

Proof of (2.4.9). Starting from (2.3.5), by the change of variables $t'_i = t_i/c$ and the scaling property in (2.3.7), we have that

$$\begin{aligned}
\mathcal{F}f_n(\cdot,0,ct)(\xi_1,\cdots,\xi_n) &= \int_{[0,ct]^n_<}\overline{\prod_{k=1}^{n}\mathcal{F}G(t_{k+1}-t_k,\cdot)\left(\sum_{j=1}^{k}\xi_j\right)}\mathrm{d}\vec{t}\\
&=\int_{[0,t]^n_<}\overline{\prod_{k=1}^{n}\mathcal{F}G\left(c(t_{k+1}-t_k),\cdot\right)\left(\sum_{j=1}^{k}\xi_j\right)}c^n\mathrm{d}\vec{t}\\
&=\int_{[0,t]^n_<}\overline{\prod_{k=1}^{n}c^{b+r-1}\mathcal{F}G\left(t_{k+1}-t_k,\cdot\right)\left(c^{b/a}\sum_{j=1}^{k}\xi_j\right)}c^n\mathrm{d}\vec{t}
\end{aligned}$$

where in the last line we applied (2.3.6). Now,

$$\begin{aligned}
\mathcal{F}f_n(\cdot,0,ct)(\xi_1,\cdots,\xi_n) &= \int_{[0,t]^n_<}\overline{\prod_{k=1}^{n}c^{b+r-1}\mathcal{F}G\left(t_{k+1}-t_k,\cdot\right)\left(c^{b/a}\sum_{j=1}^{k}\xi_j\right)}c^n\mathrm{d}\vec{t}\\
&=c^{n(b+r)}\mathcal{F}f_n\left(\cdot,0,t\right)\left(c^{b/a}\xi_1,\cdots,c^{b/a}\xi_n\right),
\end{aligned}$$

from which we see that

$$\int_0^\infty e^{-t}\left\|\widetilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}}\mathrm{d}t = \int_0^\infty e^{-t}\int_{\mathbb{R}^{nd}}\left|\mathcal{F}\widetilde{f}_n(\cdot,0;t)(\xi_1,\cdots,\xi_n)\right|^2\mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$

$$= \int_0^\infty e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0; 2t)(\xi_1, \cdots, \xi_n) \right|^2 \mu(\mathrm{d}\vec{\xi}\,)\, 2\, \mathrm{d}t$$
$$= 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0, t)(2^{b/a}\xi_1, \cdots, 2^{b/a}\xi_n) \right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$

where in the last line we have applied (2.3.7). Now,

$$\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot, 0, t) \right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t = 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0, t)(2^{b/a}\xi_1, \cdots, 2^{b/a}\xi_n) \right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$
$$= 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n) \right|^2 2^{\frac{-nbd}{a}} 2^{\frac{nb(d-\alpha)}{a}} \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$

where the last line follows from a change of variables and recalling that $\mu(\mathrm{d}\vec{\xi}\,)$ in (2.3.17). Lastly,

$$\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot, 0, t) \right\|_{\mathcal{H}^{\otimes n}}^2 \mathrm{d}t = 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n) \right|^2 2^{\frac{-nbd}{a}} 2^{\frac{nb(d-\alpha)}{a}} \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$
$$= 2^{n(2(b+r)-b\alpha/a)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left| \mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n) \right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$
$$= \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2 \mathrm{d}\vec{x}\mathrm{d}t,$$

which proves (2.4.9). □

Chapter 3

Global Solutions of the Interpolated Stochastic Heat and Wave Equation with a Super-linear Diffusion Term

3.1 Introduction and main result

In this chapter, we study the interpolated stochastic heat and wave equation (ISHWE) on the whole space $\mathbb{R}^d$,

$$\begin{cases} \left(\partial_t^{\beta} + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^{\gamma}\left[\sigma(u(t,x))\,\dot{W}(t,x)\right] & x \in \mathbb{R}^d,\, t>0 \\ u(0,\cdot) = u_0(x) & \beta \in (0,1] \\ u(0,\cdot) = u_0(x), \quad \partial_t u(0,\cdot) = v_0(x) & \beta \in (1,2), \end{cases} \tag{3.1.1}$$

where the fractional differential operators ∂_t^{β}, $(-\Delta)^{\alpha/2}$ and I_t^{γ} respectively denote the *Caputo derivative*, the *fractional Laplacian* and the *Riemann-Liouville fractional integral operator*. The noise, $\dot{W}$, is a centered Gaussian noise that is taken to be white in time and colored in space. In other words, W is a 0-mean Gaussian processes on the space of Schwartz functions, $\mathscr{S}(\mathbb{R}^{d+1})$, with the following covariance functional:

$$\mathbb{E}\left(W(\phi)W(\psi)\right) = \int_0^{\infty} \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}y\, \phi(x)\psi(y)\delta_0(x-y), \quad \phi,\psi \in \mathscr{S}(\mathbb{R}^{d+1}),$$

where δ_0 denotes the *Dirac delta distribution*.

The initial data, u_0 and v_0, are assumed to be Borel measurable functions which are Bounded on $\mathbb{R}^d$. The nonlinearity, σ, is *locally Lipschitz* in the sense that the following asymptotic relation holds as $|x|, |z| \to \infty$:

$$|\sigma(x) - \sigma(z)| \leqslant \sigma_2 |x - z| [\ln_+(|x - z|)]^\delta, \tag{3.1.2}$$

where $\sigma_2, \delta > 0$ and $\ln_+(z) := \ln(z \vee e)$ for $z > 0$. Further, this implies that as $|x| \to \infty$,

$$|\sigma(x)| \leqslant \sigma_1 + \sigma_2 |x| [\ln_+(|x|)]^\delta, \tag{3.1.3}$$

where $\sigma_1 = \sigma(0)$.

For any $T > 0$, with $t \in [0, T]$ and $x \in \mathbb{R}^d$, the solution to (3.1.1) is understood in the *mild* sense as the solution to the following stochastic integral equation:

$$u(t, x) = J_0(t, x) + I(t, x),$$

where

$$J_0(t, x) = \begin{cases} [Z(t, \cdot) * u_0](x) & \beta \in (0, 1] \\ [Z(t, \cdot) * v_0](x) + [Z^*(t, \cdot) * u_0](x) & \beta \in (1, 2) \end{cases}$$

and

$$I(t, x) = \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) \sigma(u(s, y)) W(\mathrm{d}s, \mathrm{d}y).$$

The stochastic integral above is a *Walsh integral* (e.g. see [Wal86]) and the fundamental solution consists of a triple, $\{Z, Z^*, Y\}$, and each member of the triple is defined through *Fox-H functions*, however, one can more compactly define them through their Fourier transforms [CHN19, Theorem 4.1]:

$$\mathcal{F}Z(t, \cdot)(\xi) = t^{\lceil \beta \rceil - 1} E_{\beta, \lceil \beta \rceil} \left(-2^{-1} \nu t^\beta |\xi|^\alpha \right),$$
$$\mathcal{F}Z^*(t, \cdot)(\xi) = E_{\beta, 1} \left(-2^{-1} \nu t^\beta |\xi|^\alpha \right),$$
$$\mathcal{F}Y(t, \cdot)(\xi) = t^{\beta + \gamma - 1} E_{\beta. \beta + \gamma} \left(-2^{-1} \nu t^\beta |\xi|^\alpha \right),$$

where we use $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} \exp(-ix\cdot\xi)f(x)\mathrm{d}x$ to denote the Fourier transform, $E_{a,b}(z)$ denotes the *two-parameter Mittag-Leffler function* [Pod99]:

$$E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad a > 0,\ b > 0,\ z \in \mathbb{C},$$

and $\Gamma(\cdot)$ is the standard *Gamma function*.

Remark 3.1.1. We mention that under the special case where $\alpha \in (1,2)$, $\beta \in (0,1]$ and $\gamma = 1-\beta$, that $Z = Z^* = Y$, which is easily seen from their Fourier transforms given above. Because of these equalities, we simply let Y denote the fundamental solution under this situation. Moreover, under this case there is a probabilistic representation for Y which is used by the authors in [FLN19; FN17; MN15; FLO17; MN16]. Indeed, let X_t denote a symmetric α-stable process with density $p(t,x)$. Let $D = \{D_r,\ r \geqslant 0\}$ denote a β-stable subordinator and E_t its first passage time. Then it is known that the density of the time changed processes X_{E_t} is given by $Y(t,x)$ and we have that

$$Y(t,x) = \int_0^\infty p(s,x) f_{E_t}(s)\,\mathrm{d}s,$$

where

$$f_{E_t}(s) = t\beta^{-1}x^{-1-1/\beta} g_\beta\left(tx^{-1/\beta}\right),$$

where $g_\beta(\cdot)$ is the density function of D_1 and is smooth on the real line with $g_\beta(x) = 0$ for $x \leqslant 0$.

The goal of this paper is to prove the existence and uniqueness of a global solution to (3.1.1). The work is highly motivated by the recent work by Millet and Sanz-Solé [MS21]. It is a well studied phenomena that either a super-linear drift or diffusion term may cause blow-up of the solution. As for the stochastic heat equation, we direct the reader to [FN21; MS93; DKZ19; BG09]. On the other hand, the only other work that we are aware of that is dedicated to proving some non-existence results of (3.1.1) is [AMN20]. So, to the best of our knowledge, this is the first work on proving the global existence of a solution to (3.1.1) with a super-linear diffusion term (see Theorem 3.1.2 below). However, as we will see, the calculations given below are essentially identical to those performed in [MS21] which is the cause for our motivation.

Here is the main result of this chapter. The proof will be postponed until Section 3.3 below.

Theorem 3.1.2. *Suppose that σ satisfies (3.1.2) and the initial data, u_0 and v_0, are Borel measurable functions and satisfy for all $T > 0$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |J_0(t,x)| < \infty,$$

which is true if u_0 and v_0 are bounded. Then for any $R > 0$ and $(t,x) \in [0,T] \times \bar{B}_R(0)$, where $\bar{B}_R(0)$ denotes the closed d-dimensional ball centered around the origin, there exists a random field solution to (3.1.1) denoted as $(u(t,x) : (t,x) \in [0,T] \times \bar{B}_R(0))$. This solution is unique and satisfies

$$\sup_{t\in[0,T],\ |x|\leqslant R} |u(t,x)| < \infty, \quad \textit{almost surely.} \tag{3.1.4}$$

The proof of Theorem 3.1.2 uses a standard stopping time argument along with a series of truncations of the super-linear term, σ, in order to construct a solution of (3.1.1). Indeed, we define for $N > 0$,

$$\sigma_N(x) = \sigma(x)\mathbf{1}_{\{|x|\leqslant N\}} + \sigma(N)\mathbf{1}_{\{x>N\}} + \sigma(-N)\mathbf{1}_{\{x<-N\}}.$$

Then we associate for each $N > 0$, the corresponding global solution to (3.1.1) with σ replaced my σ_N as $u_N(t,x)$, whose existence follows due to Theorem 3.2.1 below. We then define the stopping time τ_N as follows:

$$\tau_N := \inf\left\{t > 0 : \sup_{|x|\leqslant R} |u_N(t,x)| \geqslant N\right\} \wedge T,$$

and prove that $\{t < \tau_N\} \uparrow \Omega$ and then show that the solution to (3.1.1) exists pathwise for $\omega \in \Omega$. We remind the reader that in the case of a globally Lipschitz diffusion term, one usually proves that the solution exists as a limit in $L^2(\Omega)$. However, the stopping-time argument applied here will no longer guarantee the $L^2(\Omega)$ existence of the solution. Moreover, because $L^2(\Omega)$ existence is no longer guaranteed, the solution may no longer satisfy the standard Itô isometry.

3.2 Preliminary results for a globally Lipschitz drift term

In this section, we assume that the coefficient σ is globally Lipschitz and therefore satisfies the following inequality:

$$|\sigma(x)| \leqslant c(\sigma) + L(\sigma)|x|, \tag{3.2.1}$$

where $c(\sigma) = |\sigma(0)|$ and $L(\sigma)$ is the Lipschitz constant of σ and we assume that $L(\sigma) > 0$. The equation (3.1.1) under the assumption that σ satisfies (3.2.1) has been studied in [CHN19] where they establish in Theorem 4.1 *ibid* the existence and uniqueness of global random field solutions. In order to do so, one needs to show *Dalang's condition*:

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y |Y(s,y)|^2 < \infty, \quad \text{for all } t > 0, \tag{3.2.2}$$

which is equivalent to (e.g. see Lemma 5.3 *ibid*)

$$d < 2\alpha + \frac{\alpha}{\beta}\min\{2\gamma - 1, 0\} =: \Theta, \tag{3.2.3}$$

which is further equivalent to

$$\rho(d) > 0 \quad \text{and} \quad d < 2\alpha, \tag{3.2.4}$$

where

$$\rho(x) := 2\beta + 2\gamma - 1 - \beta x/\alpha. \tag{3.2.5}$$

We now state the existence and uniqueness result proven in [CHN19].

Theorem 3.2.1. *[CHN19, Theorem 4.1] Under (3.2.2), the SPDE (3.1.1) has a unique (in the sense of versions) random field solution $\{u(t,x) : (t,x) \in (0,\infty) \times \mathbb{R}^d\}$ if the initial data are such that*

$$\widehat{C}_T := \sup_{t\in[0,T],\, x\in\mathbb{R}^d} |J_0(t,x)| < \infty. \tag{3.2.6}$$

We now state a lemma proven in [CHN19] that calculates the $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ norms of the fundamental solutions. The lemma will be needed below in some of our calculations.

Lemma 3.2.2. *[CHN19, Theorem 4.1 and Lemma 5.5] Assume that $d < 2\alpha$, $\beta \in (0,2)$, and $\gamma \geqslant 0$. Then*

$$\int_{\mathbb{R}^d} Y(t,x)\mathrm{d}x = t^{\beta+\gamma-1} \quad \text{and} \quad \int_{\mathbb{R}^d} Z(t,x)\mathrm{d}x = t^{\lceil\beta\rceil-1}$$

and

$$\int_{\mathbb{R}^d} Y^2(t,x)\mathrm{d}x = C_{\#} t^{2(\beta+\gamma-1)-d\beta/\alpha} = C_{\#} t^{\rho-1},$$

for all $t > 0$, where $\rho = \rho(d)$ and

$$C_{\#} := \frac{2}{\Gamma(d/2)(4\pi)^{d/2}} \int_0^\infty u^{d-1} E^2_{\beta,\beta+\gamma}(-u^\alpha)\mathrm{d}u.$$

Moreover, when $\beta \in (1,2)$,

$$\int_{\mathbb{R}^d} Z^*(t,x)\mathrm{d}x = 1.$$

Remark 3.2.3. When $\alpha = \beta = 2$, $\gamma = 0$ and $d = 1$ then Lemma 3.2.2 gives us that $\int_{\mathbb{R}} Y^2(t,x)\mathrm{d}x = t/2$ which coincides with the $L^2(\mathbb{R})$ norm of the wave kernel.

3.2.1 Some moment bounds

In this section, we will prove some moment bounds and a continuity result for the solution with a globally Lipschitz diffusion term. In the proof of the next proposition, we will use the following two facts:

$$\sup_{t\geqslant 0} t^k e^{-at} = k^k(ea)^{-k} \text{ for } k > 0 \quad \text{and} \quad \sup_{t\geqslant 0} \int_0^t s e^{-as}\mathrm{d}s = a^{-2} \text{ for } a > 0. \tag{3.2.7}$$

Proposition 3.2.4. [MS21, Proposition 3.2] *Let u_0 and v_0 be Borel functions satisfying $\|u_0\|_\infty + \|v_0\|_\infty < \infty$. Suppose that $d < 2\alpha$ and $\rho = \rho(d) > 0$, where $\rho(x)$ is defined in (3.2.5). Then there exists a universal constant $K := K(\alpha,\beta,\gamma,d) > 0$ such that for any*

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \text{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

we have that

$$N_{a,p}(u) \leqslant 2\mathcal{T}_0 + \frac{c(\sigma)}{L(\sigma)}, \tag{3.2.8}$$

where

$$\mathcal{T}_0 = \mathcal{T}_0(a, \beta, u_0, v_0) := \begin{cases} \|u_0\|_\infty & \beta \in (0,1] \\ (ea)^{-1} \|v_0\|_\infty + \|u_0\|_\infty & \beta \in (1,2). \end{cases}$$

Moreover, for all $T > 0$*,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(|u(t,x)|^p) \leqslant \exp(apt) \left[2\mathcal{T}_0 + \frac{c(\sigma)}{L(\sigma)}\right]^p. \tag{3.2.9}$$

Proof. First we fix an $a > 0$ and $p \in [2, \infty)$. Using Lemma 3.2.2 we see that,

$$\begin{aligned} |J_0(t,x)| &= \begin{cases} |(Z(t,\cdot) * u_0)(x)| & \beta \in (0,1] \\ |(Z(t,\cdot) * v_0)(x) + (Z^*(t,\cdot) * u_0)(x)| & \beta \in (1,2) \end{cases} \\ &\leqslant \begin{cases} \|u_0\|_\infty & \beta \in (0,1] \\ t\|v_0\|_\infty + \|u_0\|_\infty & \beta \in (1,2). \end{cases} \end{aligned}$$

Now using (3.2.7), we see that

$$\begin{aligned} N_{a,p}(J_0) &\leqslant \begin{cases} \sup_{t>0,\, x\in\mathbb{R}^d} \|u_0\|_\infty e^{-at} & \beta \in (0,1] \\ \sup_{t>0,\, x\in\mathbb{R}^d} (t\|v_0\|_\infty + \|u_0\|_\infty) e^{-at} & \beta \in (1,2) \end{cases} \\ &= \begin{cases} \|u_0\|_\infty & \beta \in (0,1] \\ (ea)^{-1}\|v_0\|_\infty + \|u_0\|_\infty & \beta \in (1,2) \end{cases} \\ &= \mathcal{T}_0. \end{aligned} \tag{3.2.10}$$

Now we find an upper bound for $N_{a,p}(I)$. By simultaneously applying the Burkholder-Davies-Gundy inequality and triangle inequality, we get that

$$\begin{aligned}\|I(t,x)\|_p^2 &\leqslant 4p \left\| \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; Y^2(t-s,x-y)\sigma^2(u(s,y)) \right\|_{p/2} \\ &\leqslant 4p \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; Y^2(t-s,x-y) \left\|\sigma^2(u(s,y))\right\|_{p/2}.\end{aligned}$$

Now by applying Lemma 3.2.2 and the Lipschitz property of σ we get that

$$\|I(t,x)\|_p^2 \leqslant 8pc^2(\sigma)C_{\#} \int_0^t (t-s)^{\rho-1}\mathrm{d}s + 8pL^2(\sigma)N_{a,p}^2(u) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; e^{2as}Y^2(t-s,x-y).$$

Now under Dalang's condition (3.2.4), the first integral above can be calculated as

$$\int_0^t (t-s)^{\rho-1}\mathrm{d}s = \frac{t^\rho}{\rho}.$$

By applying Lemma 3.2.2, we can calculate the second integral as

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; e^{2as}Y^2(t-s,x-y) = C_{\#} \int_0^t \mathrm{d}s \; e^{2as}(t-s)^{\rho-1}.$$

Putting this together gives us that

$$\|I(t,x)\|_p^2 \leqslant 8pc^2(\sigma)C_{\#}\frac{t^\rho}{\rho} + 8pL^2(\sigma)N_{a,p}^2(u)C_{\#} \int_0^t \mathrm{d}s \; e^{2as}(t-s)^{\rho-1}.$$

Now by multiplying both sides by e^{-2at}, we see that

$$\begin{aligned}\|I(t,x)\|_p^2 e^{-2at} \leqslant \frac{8pc^2(\sigma)C_{\#}}{\rho} \; t^\rho e^{-2at} \\ + 8pL^2(\sigma)N_{a,p}^2(u)C_{\#} \int_0^t \mathrm{d}s \; e^{-2a(t-s)}(t-s)^{\rho-1}.\end{aligned}$$

We now do a change of variables and again recall the definition of the Gamma function to see that

$$\|I(t,x)\|_p^2 e^{-2at} \leqslant \frac{8pc^2(\sigma)C_\#}{\rho} t^\rho e^{-2at} + 8pL^2(\sigma)N_{a,p}^2(u)C_\# \int_0^\infty \mathrm{d}u\, e^{-2au}u^{\rho-1}$$
$$= \frac{8pc^2(\sigma)C_\#}{\rho} t^\rho e^{-2at} + 8pL^2(\sigma)N_{a,p}^2(u)\frac{C_\#\Gamma(\rho)}{(2a)^\rho},$$

where the last line is true if Dalang's condition (3.2.4) holds. If we now apply (3.2.7) and also recall the subadditivity of the square root, namely $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for all $a, b \geqslant 0$, then we see that

$$N_{a,p}(I) \leqslant c(\sigma)\sqrt{\frac{8pC_\#}{\rho}}\left(\frac{\rho}{2ea}\right)^{\rho/2} + L(\sigma)N_{a,p}(u)\sqrt{\frac{8pC_\#\Gamma(\rho)}{(2a)^\rho}}. \tag{3.2.11}$$

To simplify things, we now let

$$K := \max\left\{\sqrt{\frac{8C_\#}{\rho}}\left(\frac{\rho}{2e}\right)^{\rho/2}, \sqrt{\frac{8C_\#\Gamma(\rho)}{2^\rho}}\right\}.$$

Then we may bound $N_{a,p}(I)$ by the following:

$$N_{a,p}(I) \leqslant \frac{K\sqrt{p}}{a^{\rho/2}}\left(c(\sigma) + L(\sigma)N_{a,p}(u)\right). \tag{3.2.12}$$

Now by combining (3.2.10) and (3.2.12), we see that

$$N_{a,p}(u) \leqslant \mathcal{T}_0 + K\frac{\sqrt{p}c(\sigma)}{a^{\rho/2}} + K\frac{\sqrt{p}L(\sigma)}{a^{\rho/2}}N_{a,p}(u). \tag{3.2.13}$$

Note that if we now restrict ourselves to the case of $\alpha = \beta = 2$, $d = 1$ and $\gamma = 0$, then (3.2.11) and (3.2.13) recover [MS21, (3.9) and (3.10)] respectively. Now choose $a > 0$ such that

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho}.$$

This implies that the following interval is nonempty:

$$\left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right].$$

In addition, for any p in this interval, we have that

$$\frac{\sqrt{p}L(\sigma)K}{a^{\rho/2}} \leqslant \frac{1}{2} \quad \text{and} \quad \frac{\sqrt{p}}{a^{\rho/2}} \leqslant \frac{1}{2KL(\sigma)}.$$

This along with (3.2.13) implies that

$$N_{a,p}(u) \leqslant 2\mathcal{T}_0 + \frac{c(\sigma)}{L(\sigma)}$$

and this is precisely (3.2.8). Note that (3.2.9) follows immediately from (3.2.8). □

We will need to recall the following result proved in [CHN19] in order to prove Proposition 3.2.6 below.

Proposition 3.2.5. [CH21, Propositions 2.1 and 2.2] *Suppose that $\alpha \in (0, 2]$, $\beta \in (0, 2)$, $\gamma > 0$, and (3.2.3) holds. Consider $\rho = \rho(d)$ which is defined above in (3.2.5). Then $Y(t, x)$ satisfies the following:*

1. *Suppose that $0 < \theta < (\Theta - d) \wedge 2$ where Θ is given in (3.2.3), $0 < t, r \leqslant T$ for some $T > 0$, $\beta \leqslant 1$ and $\gamma \leqslant \lceil \beta \rceil - \beta$. Then there exists some $C := C(\alpha, \beta, \gamma, \nu, d, \theta, T)$ such that*

$$\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y |Y(t-s, x-y) - Y(r-s, z-y)|^2 \leqslant C\left(|t-r|^\rho + |x-z|^\theta\right).$$

2. *If we only assume that $\alpha \in (0, 2]$, $\beta \in (0, 2)$, $\gamma > 0$, $0 < t, r \leqslant T$ for some $T > 0$ and that (3.2.3) holds, then for $0 < \theta < (\Theta - d) \wedge 2$, there exists some $C := C(\alpha, \beta, \gamma, \nu, d, \theta, T)$ such that*

$$\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y |Y(t-s, x-y) - Y(r-s, z-y)|^2 \leqslant C\left(|t-r|^q + |x-z|^\theta\right),$$

where

$$0 < q < \rho.$$

We now state and prove a proposition that will be used to prove the Hölder continuity of the stochastic integral, $I(t,x)$.

Proposition 3.2.6. *Suppose that* $0 < \theta < (\Theta - d) \wedge 2$ *and* $\rho = \rho(d) > 0$. *Then for all* $x, z \in \mathbb{R}^d$, $a > 0$, $p \geqslant 2$ *and for all* $T > 0$ *with* $t \in [0,T]$, *the stochastic integral,* $I(t,x)$, *satisfies the following:*

$$\frac{\|I(t,x) - I(r,z)\|_p}{\left(|t-r|^q + |x-z|^\theta\right)^{1/2}} \leqslant C(p,\theta,T)\left[\mathcal{M}_1 + \mathcal{M}_2 e^{aT} N_{a,p}(u)\right], \tag{3.2.14}$$

where

$$\mathcal{M}_1 = \sqrt{p}c(\sigma) \quad \textit{and} \quad \mathcal{M}_2 = \sqrt{p}L(\sigma),$$

and $q \in (0,\rho]$ *(resp.* $q \in (0,\rho)$*) under Case 1 (resp. Case 2) of Proposition 3.2.5. Moreover, if we consider the case when*

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \textit{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

then we have that

$$\frac{\|I(t,x) - I(r,z)\|_p}{\left(|t-r|^q + |x-z|^\theta\right)^{1/2}} \leqslant C(p,\theta,T)\left[\mathcal{M}_1 + \mathcal{M}_2 e^{aT}\left(2 + \frac{c(\sigma)}{L(\sigma)}\right)\right]. \tag{3.2.15}$$

Proof. We apply the Burkholder-Davies-Gundy inequality along with the triangle inequality, as was done in the proof of Proposition 3.2.4, to see that

$$\begin{aligned}
\|I(t,x) - I(r,z)\|_p^2 &\leqslant 4p\left\|\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, |Y(t-s,x-y) - Y(r-s,z-y)|^2\, \sigma^2(u(s,y))\right\|_{p/2} \\
&\leqslant 4p\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, |Y(t-s,x-y) - Y(r-s,z-y)|^2 \left\|\sigma^2(u(s,y))\right\|_{p/2}.
\end{aligned}$$

Now by applying the Lipchitz condition on σ gives us that

$$\|I(t,x) - I(r,z)\|_p^2 \leqslant 8p\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, |Y(t-s,x-y) - Y(r-s,z-y)|^2 \left(c^2(\sigma) + L^2(\sigma)\, \|u(s,y)\|_p^2\right)$$

$$\leqslant 8p\Bigg[c^2(\sigma)\int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\,|Y(t-s,x-y)-Y(r-s,z-y)|^2$$
$$+L^2(\sigma)\int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\,|Y(t-s,x-y)-Y(r-s,z-y)|^2\exp(2as)\,N_{a,p}^2(u)\Bigg]$$
$$\leqslant 8p\Bigg[c^2(\sigma)\int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\,|Y(t-s,x-y)-Y(r-s,z-y)|^2$$
$$+L^2(\sigma)\exp(2aT)\,N_{a,p}^2(u)\int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\,|Y(t-s,x-y)-Y(r-s,z-y)|^2\Bigg].$$

We now apply Proposition 3.2.5 to see that

$$\|I(t,x)-I(r,z)\|_p^2 \leqslant 8p\Big[c^2(\sigma)C(\theta,T)\left(|t-r|^q+|x-z|^\theta\right)$$
$$+L^2(\sigma)\exp(2aT)\,N_{a,p}^2(u)C(\theta,T)\left(|t-r|^q+|x-z|^\theta\right)\Big],$$

where $C(\theta,T)$ is as in Proposition 3.2.5. Now by taking the square root of both sides and applying the identity $\sqrt{x+y}\leqslant\sqrt{x}+\sqrt{y}$, we see that

$$\|I(t,x)-I(r,z)\|_p \leqslant 2\sqrt{2C(\theta,T)p}\Big[c(\sigma)+L(\sigma)\exp(aT)\,N_{a,p}(u)\Big]\left(|t-r|^q+|x-z|^\theta\right)^{1/2}.$$

This proves (3.2.14). It is clear that (3.2.15) follows directly from (3.2.14) and Proposition 3.2.4. □

Proposition 3.2.7. *The stochastic integral, I, has a version, still denoted by I that is η_1-Hölder continuous in time and η_2-Hölder continuous in space with $0<\eta_1<q/2$ and $0<\eta_2<\theta/2$ where $0<\theta<(\Theta-d)\wedge 2$ and $q\in(0,\rho]$ (resp. $q\in(0,\rho)$) under Case 1 (resp. Case 2) of Proposition 3.2.5. Moreover, if we consider the case when*

$$a>\left(8L^2(\sigma)K^2\right)^{1/\rho}\quad \textit{and}\quad p\in\left[2,\frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

where K is the constant from Proposition 3.2.4, then

$$\mathbb{E}\left[\sup_{t\in[0,T],\,|x|\leqslant R}|u(t,x)|^p\right]\leqslant 2^{p-1}\widehat{C}_T^p+C(p,\theta,T,R)\left[\mathcal{M}_1^p+\mathcal{M}_2^p e^{apT}\left(2+\frac{c(\sigma)}{L(\sigma)}\right)^p\right],$$

where $\widehat{C}_T$ is defined in (3.2.6).

Proof. Proposition 3.2.6 implies that

$$\|I(t,x)-I(r,z)\|_p \leqslant C(p,\theta,T)\left[\mathcal{M}_1+\mathcal{M}_2 e^{aT}N_{a,p}(u)\right]\left(|t-r|^q+|x-z|^\theta\right)^{1/2}. \quad (3.2.16)$$

By Kolmogorov's continuity theorem, there exists a version of I, which we denote by I, that is η_1-Hölder continuous in time and η_2-Hölder continuous in space with with $0<\eta_1<q/2$ and $0<\eta_2<\theta/2$.

Next, note that by triangle inequality,

$$\begin{aligned}|u(t,x)|^p &\leqslant 2^{p-1}\left(\|J_0(\cdot,\circ)\|_\infty^p+|u(t,x)-J_0(t,x)|^p\right)\\ &\leqslant 2^{p-1}\left(\widehat{C}_T^p+\sup_{t\in[0,T],\ |x|\leqslant R}|I(t,x)|^p\right)\\ &= 2^{p-1}\left(\widehat{C}_T^p+\sup_{t\in[0,T],\ |x|\leqslant R}|I(t,x)|^p\right),\end{aligned}$$

where we recall that $\widehat{C}_T$ defined in (3.2.6). Due to the continuity of I on the compact set $[0,T]\times\bar{B}_R(0)$, there exists a point $(t_0,x_0)\in[0,T]\times\bar{B}_R(0)$ such that

$$\sup_{t\in[0,T],\ |x|\leqslant R}|I(t,x)|^p=|I(t_0,x_0)|^p.$$

Thus, for any $(t,x)\in[0,T]\times\bar{B}_R(0)$,

$$|u(t,x)|^p\leqslant 2^{p-1}\left(\widehat{C}_T^p+|I(t_0,x_0)|^p\right),$$

which in return implies that

$$\sup_{t\in[0,T],\ |x|\leqslant R}|u(t,x)|^p\leqslant 2^{p-1}\left(\widehat{C}_T^p+|I(t_0,x_0)|^p\right).$$

Now by taking the expectation of both sides and applying Proposition 3.2.6 gives us

$$\mathbb{E}\left[\sup_{t\in[0,T],\ |x|\leqslant R}|u(t,x)|^p\right] \leqslant 2^{p-1}\widehat{C}_T^p + 2^{p-1}C(p,\theta,T)^p\left((2T)^{q/2}+(2R)^{\theta/2}\right)^p\left[\mathcal{M}_1+\mathcal{M}_2e^{aT}N_{a,p}(u)\right]^p$$
$$= 2^{p-1}\widehat{C}_T^p + C(p,\theta,T,R)\left[\mathcal{M}_1^p+\mathcal{M}_2^pe^{apT}N_{a,p}(u)^p\right].$$

Now if we consider

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \text{and} \quad p\in\left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

then by Proposition 3.2.4, we have that

$$\mathbb{E}\left[\sup_{t\in[0,T],\ |x|\leqslant R}|u(t,x)|^p\right] \leqslant 2^{p-1}\widehat{C}_T^p + C(p,\theta,T,R)\left[\mathcal{M}_1^p+\mathcal{M}_2^pe^{apT}\left(2+\frac{c(\sigma)}{L(\sigma)}\right)^p\right].$$

□

Remark 3.2.8. The Hölder continuity in time proven above in Proposition 3.2.7 recovers the result proven by Foondun and Nane in [FN17, Theorem 1.9] if we restrict ourself to the case where $\beta\in(0,1)$ and $\gamma = 1-\beta$.

3.3 Proof of Theorem 3.1.2

Proof. We first solve a *truncated* version of (3.1.1). By this, we mean (3.1.1) with σ replaced by σ_N where for any $N>0$,

$$\sigma_N(x) = \sigma(x)\mathbf{1}_{|x|\leqslant N} + \sigma(N)\mathbf{1}_{x>N} + \sigma(-N)\mathbf{1}_{x<-N}.$$

Note that since σ satisfies (3.1.2), then σ_N is Lipschitz continuous with Lipschitz constant $L(\sigma_N) := \sigma_2\ln(2N)^\delta$. In other words,

$$|\sigma_N(x)| \leqslant \sigma_1 + \sigma_2\ln(2N)^\delta|x|,$$

where σ_1 is chosen so that

$$\sup_{|x|\leqslant N} |\sigma(x)| \leqslant \sigma_1 < \infty.$$

Hence we can apply Theorem 3.2.1 and see that there exists an unique solution of the truncated version of (3.1.1), which we denote by $u_N := \{u_N(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\}$. In addition, Proposition 3.2.7 implies that the solution has a version, still denoted by u_N, which is η_1-Hölder continuous in time and η_2-Hölder continuous in space with $0 < \eta_1 < q/2$ and $0 < \eta_2 < \theta/2$.

We will now apply Proposition 3.2.7 to find an upper bound on the p-norm of u_N. For this, consider

$$a > \left(8L^2(\sigma_N)K^2\right)^{1/\rho} \quad \text{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma_N)K^2}\right].$$

Then Proposition 3.2.7 implies that

$$\mathbb{E}\left(\sup_{t\in[0,T],\ |x|\leqslant R} |u_N(t,x)|^p\right) \leqslant 2^{p-1} + C(p,\theta,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p(N)\exp(apT)\left(2 + \frac{\sigma_1}{L(\sigma_N)}\right)\right], \tag{3.3.1}$$

where we recall that

$$\mathcal{M}_1 = \sqrt{p}\sigma_1 \quad \text{and} \quad \mathcal{M}_2(N) = \sqrt{pL(\sigma_N)} = \sqrt{p\sigma_2 \ln(2N)^\delta}. \tag{3.3.2}$$

We now use the above to prove the existence and uniqueness of (3.1.1) with superlinear σ satisfying (3.1.2). For any $T > 0$ and $N \geqslant 2$ with $N \in \mathbb{N}$, we define the following stopping time:

$$\tau_N := \inf\left\{t > 0 : \sup_{|x|\leqslant R} |u_N(t,x)| \geqslant N\right\} \wedge T. \tag{3.3.3}$$

The uniqueness of the solution and the local property of the stochastic integral imply that on $\{t < \tau_N\}$, $u_N(t,x) = u_{N+k}(t,x)$ for any $k \in \mathbb{N}$. Hence, $(\tau_N)_{N\geqslant 2}$ is increasing and bounded above by T.

We now momentarily assume that $\sup_N \tau_N = T$, which would imply that $\{t < \tau_N\} \uparrow \Omega$. On each $\{t < \tau_N\}$, define $(u(t,x) : (t,x) \in [0,T) \times \mathbb{R}^d)$ by $u(t,x) = u_N(t,x)$ and hence

$u(t,x) = u_{N+k}(t,x)$ for any $k \in \mathbb{N}$. This implies that on $\{t < \tau_N\}$

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \, Y(t-s, x-y)\sigma_N(u(s,y))W(\mathrm{d}s, \mathrm{d}y).$$

However, on $\{t < \tau_N\}$, we have that $\sigma_N(u_N(t,x)) = \sigma(u(t,x))$ and so u satisfies

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \, Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y), \quad \{t < \tau_N\}.$$

Lastly, since $\{t < \tau_N\} \uparrow \Omega$, we conclude that

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \, Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y), \quad (t,x) \in [0,T) \times \mathbb{R}^d. \tag{3.3.4}$$

See the proof of [MS21, Theorem 3.5] for a comment pertaining to the issue that the solution no longer exists in $L^2(\Omega)$.

We now focus on proving that $\sup_N \tau_N = T$, or equivalently that $\mathbb{P}(\tau_N < T) \to 0$ as $N \to \infty$. To do this, note that by Chebychev's inequality that

$$\mathbb{P}(\tau_N < T) \leqslant \mathbb{P}\left(\sup_{t\in[0,T],\ |x|\leqslant R} |u_N(t,x)| \geqslant N\right) \leqslant N^{-p}\mathbb{E}\left(\sup_{t\in[0,T],\ |x|\leqslant R} |u_N(t,x)|^p\right).$$

Then an application of (3.3.1) gives

$$\begin{aligned} N^{-p}\mathbb{E}&\left(\sup_{t\in[0,T],\ |x|\leqslant R} |u_N(t,x)|^p\right) \\ &\leqslant N^{-p}\left(2^{p-1} + C(p,\theta,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2(N)^p \exp\{apT\}\left(2 + \frac{\sigma_1}{L(\sigma_N)}\right)^p\right]\right) \\ &= \frac{2^{p-1} + C(p,\theta,T,R)\mathcal{M}_1^p}{N^p} + \frac{C(p,T,R)\mathcal{M}_2(N)^p \exp\{apT\}\left(2 + \frac{\sigma_1}{L(\sigma_N)}\right)^p}{N^p}. \end{aligned}$$

The first term clearly converges to 0 as $N \to \infty$. As for the second term, we expand using (3.3.2) to see that

$$\frac{\mathcal{M}_2(N)^p \exp\{apT\}\left(2 + \frac{\sigma_1}{L(\sigma_N)}\right)^p}{N^p} = \frac{\left(\sqrt{p\sigma_2 \ln(2N)^\delta}\right)^p \exp\{apT\}\left(2 + \frac{\sigma_1}{\sqrt{p\sigma_2 \ln(2N)^\delta}}\right)^p}{N^p}$$

$$\sim \frac{\ln(2N)^{p\delta/2}}{N^p} \to 0,$$

where above we use the symbol $\sim$ to mean asymptotically equivalent as $N \to \infty$. This completes the proof of Theorem 3.1.2. □

Chapter 4

Invariant Measures for the Stochastic Heat Equation

4.1 Introduction and main results

In this chapter we consider the following *stochastic heat equation* (SHE):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{1}{2}\Delta u(t,x) = b(x,u(t,x))\dot{W}(t,x) & x \in \mathbb{R}^d, t > 0, \\ u(0,\cdot) = \mu(\cdot). \end{cases} \tag{4.1.1}$$

The noise, $\dot{W}(t,x)$, is a centered Gaussian noise that is white in time and homogeneously colored in space defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by the noise. Its covariance structure, J, is defined as follows:

$$J(\psi,\phi) := \mathbb{E}\left[\dot{W}(\psi)\,\dot{W}(\phi)\right] = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\psi(s,\cdot) * \widetilde{\phi}(s,\cdot))(x), \tag{4.1.2}$$

for all continuous and rapidly decreasing functions ψ and ϕ, where $\widetilde{\phi}(x) := \phi(-x)$, "$*$" refers to the convolution in spatial variable, and Γ is a nonnegative and nonnegative definite tempered measure on $\mathbb{R}^d$ that is commonly referred to as the *correlation measure*. The Fourier transform of Γ (in the generalized sense) is also a nonnegative and nonnegative definite tempered measure, which is usually called the *spectral measure* and is denoted by $\widehat{f}(\mathrm{d}\xi)$ [1]. Moreover, in the case where Γ has density f, namely, $\Gamma(\mathrm{d}x) = f(x)\mathrm{d}x$, then $\widehat{f}(\mathrm{d}\xi) = \widehat{f}(\xi)\mathrm{d}\xi$. The initial condition μ is a deterministic, locally finite, regular, signed Borel measure that satisfies the following

[1]See Remark 4.2.10 for the convention of Fourier transform that is used in this chapter.

integrability condition at the infinity (see (4.3.1) below).

$$\int_{\mathbb{R}^d} \exp\left(-a|x|^2\right) |\mu|(\mathrm{d}x) < \infty \quad \text{for all } a > 0,$$

where $|\mu| = \mu_+ + \mu_-$ and $\mu = \mu_+ - \mu_-$ refers to the *Hahn decomposition* of the measure μ. The function $b(x, u)$ is uniformly bounded in the first variable and Lipschitz continuous in the second variable, i.e., for some constants $L_b > 0$ and $L_0 \geqslant 0$,

$$|b(x,u) - b(x,v)| < L_b|u - v| \quad \text{and} \quad |b(x,0)| \leqslant L_0 \quad \text{for all } u, v \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{4.1.3}$$

In particular, our assumption allows the linear case $b(x, u) = \lambda u$, which is usually referred to as the *parabolic Anderson model* [CM94]. The SPDE (4.1.1) is understood in its *mild form*:

$$u(t,x) = J_0(t,x;\mu) + \int_0^t \int_{\mathbb{R}^d} b(y, u(s,y)) G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y), \tag{4.1.4}$$

where $G(t, x) = (2\pi t)^{-d/2} \exp\left(- (2t)^{-1} |x|^2\right)$ is the heat kernel,

$$J_0(t,x) = J_0(t,x;\mu) := (G(t,\cdot) * \mu)(x) = \int_{\mathbb{R}^d} G(t, x-y)\mu(\mathrm{d}y) \tag{4.1.5}$$

is the solution to the homogeneous equation, and the stochastic integral is the *Walsh integral*. We refer the interested readers to [Wal86; Dal99; Dal+09; CK19] for more details of the set up.

The aim of this chapter is to investigate the conditions required to guarantee the existence of an invariant measure for the solution to (4.1.1), which is a crucial step towards the study of the ergodicity of the system. The existence of invariant measure under the setting of the entire space $\mathbb{R}^d$ has been much less studied. In addition to Tessitore and Zabczyk [TZ98], with which we will follow closely in this chapter, a few other papers that consider the whole space include [AM03; MSY20; MSY16]. One thing to note is that the just mentioned papers use the theory of the stochastic integral developed by Daprato et al [DZ14] while we use the theory developed by Walsh [Wal86]. As far as we know, this is the first work which handles proving the existence of an invariant measure using Walsh's theory. Lastly, since the space $\mathbb{R}^d$ is not compact, one needs to work on a weighted space as in [TZ98]:

Definition 4.1.1. $\rho : \mathbb{R}^d \mapsto \mathbb{R}$ is called an *admissible weight* if ρ is a strictly positive, bounded and continuous function in $L^1(\mathbb{R}^d)$ such that for $T > 0$, there exists a constant $C_\rho(T)$ such that

$$\big(G(t,\cdot) * \rho(\cdot)\big)(x) \leqslant C_\rho(T)\rho(x) \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d. \tag{4.1.6}$$

Let ρ be an admissible weight function. Then we may introduce the Hilbert space $H := L^2_\rho(\mathbb{R}^d)$ with $\langle f, g\rangle_\rho := \int_{\mathbb{R}^d} f(x)g(x)\rho(x)\,\mathrm{d}x$ and $\|f\|_\rho^2 := \int_{\mathbb{R}^d} f(x)^2\rho(x)\,\mathrm{d}x$. A continued discussion about the weight function can be found in Section 4.2.1. Let $\mathcal{B}(H)$ be the space of all bounded Borel functions on H. Following [DZ14], a probability measure η on the Borel σ-field $\mathcal{B}(H)$ is said to be *invariant* for (4.1.1) if

$$\eta(A) = \int_H \mathscr{L}(u(t,\cdot;\zeta))(A)\,\eta(\mathrm{d}\zeta), \quad \text{for all } t \geqslant 0 \text{ and } A \in \mathcal{B}(H), \tag{4.1.7}$$

where we use the notation $u(t,x;\zeta)$ with a third argument to emphasize its dependence on the initial condition $u(0,\cdot) = \zeta$ and $\mathscr{L}(u(t,\cdot;\zeta))(A)$ denotes the law of the solution:

$$\mathscr{L}(u(t,\cdot;\zeta))(A) = \mathbb{P}\left[\omega \in \Omega : u(t,\cdot;\zeta)(\omega) \in A\right], \quad A \in \mathcal{B}(H).$$

Due the *Krylov-Bogoliubov theorem* (see, e.g., [DZ14, Theorem 11.7]), once the tightness of $\{\mathcal{L}(u(t,\cdot;\mu)\}_{t>t_0}$ for some $t_0 \geqslant 0$ is established, one can construct the invariant measure via

$$\eta(A) = \lim_{n\to\infty} \frac{1}{T_n} \int_{t_0}^{T_n+t_0} \mathscr{L}(u(t,\cdot;\mu))(A)\mathrm{d}t, \tag{4.1.8}$$

for some sequence $\{T_n\}_{n\geqslant 1}$ with $T_n \uparrow \infty$. A critical step in obtaining tightness is to show that the following moment is uniformly bounded in time:

$$\sup_{t>0} \mathbb{E}\left(\|u(t,\cdot)\|_\rho^2\right) < \infty. \tag{4.1.9}$$

Note that the uniqueness of the invariant measure is a much harder question; see, e.g., [HM06].

It is known that the solution to (4.1.1) is usually *intermittent*, namely, the probability moments to (4.1.1) have a certain exponential growth in t; see, e.g., [CM94; FK09]. In [AM03; MSY20; MSY16], a drift (or reaction-diffusion) term $f(x,u)$ is included in (4.1.1) in a crucial way to help cancel the otherwise exponential growing moments. The absence of such a drift term in this chapter makes the problem more challenging. In order to have moments bounded in time, as required in (4.1.9), one has to first identify the exact conditions, under which the moments to (4.1.1) do not possess exponential growth. This question has been answered in [CK19, Theorem 1.3 and Lemma 2.5], where necessary and sufficient conditions are given. More precisely, for bounded second moments, one has to have the spatial dimension $d \geqslant 3$, and in addition, the spectral measure, $\hat{f}$, and Lipschitz constant, L_b, (the Lyapunov exponent of b) in (4.1.3) have to satisfy the following two conditions:

$$\Upsilon(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^2} < \infty \tag{4.1.10a}$$

$$\text{and} \quad 64 L_b^2 < \frac{1}{2\Upsilon(0)}\,. \tag{4.1.10b}$$

Note that condition (4.1.10a) is a strengthened version of *Dalang's condition*:

$$\Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{\beta + |\xi|^2} < \infty, \quad \text{for some (and hence) all } \beta > 0. \tag{4.1.11}$$

Recall that in order to obtain the Hölder continuity of the solution, one needs to strengthen (4.1.11) in a different way. Indeed, what is required is that for some $\alpha \in (0,1]$,

$$\Upsilon_\alpha(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{\left(\beta + |\xi|^2\right)^{1-\alpha}} < \infty \quad \text{for some (hence all) } \beta > 0; \tag{4.1.12}$$

see Theorem 1.8 of [CH19] or [SS02]. Similarly, one can further strengthen condition (4.1.12) to

$$\Upsilon_\alpha(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-\alpha)}} < \infty \quad \text{for some } \alpha \in (0,1]. \tag{4.1.13}$$

We use the convention that when $\alpha = 0$, we simply drop it from the expression $\Upsilon_\alpha(\beta)$, i.e., $\Upsilon(\beta) = \Upsilon_0(\beta)$. The relations of these conditions are shown in Figure 4.1.

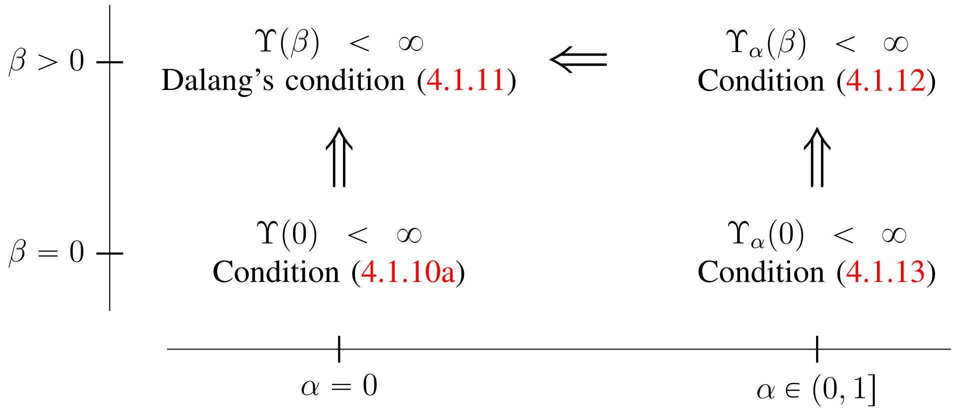


Figure 4.1: Relations among conditions (4.1.11), (4.1.12), (4.1.13) and (4.1.10a). Check also Lemma 4.4.5 for the relation between $\Upsilon(0) < \infty$ and $\Upsilon_\alpha(0) < \infty$.

Note that the two conditions in (4.1.10) guarantee the existence of the following non-empty open interval:

$$\left(2^7 L_b^2 \Upsilon(0), 1\right) \neq \varnothing. \tag{4.1.14}$$

Now we are ready to state our two main results of this chapter.

Theorem 4.1.2. *Let $u(t, x; \mu)$ be the solution to (4.1.1) starting from μ. Assume that*

(i) $\rho : \mathbb{R}^d \to \mathbb{R}_+$ is a nonnegative $L^1(\mathbb{R}^d)$ function;

(ii) the initial condition μ satisfies

$$\mathcal{G}_\rho(t; \mu) := \int_{\mathbb{R}^d} J_0^2\,(t, x; 1 + |\mu|)\, \rho(x)\, \mathrm{d}x < \infty \quad \text{for all } t > 0; \tag{4.1.15}$$

(iii) the spectral measure $\widehat{f}$ and the Lipschitz constant L_b satisfy the two conditions in (4.1.10).

Then there exists an unique $L^2(\Omega)$-continuous solution $u(t, x)$ such that for some constant $C > 0$, which does not depend on t, the following holds:

$$\mathbb{E}\left(\|u(t, \cdot; \mu)\|_\rho^2\right) \leqslant C \mathcal{G}_\rho(t; \mu) < \infty, \quad \text{for any } t > 0. \tag{4.1.16}$$

This theorem will be proved in Section 4.3.

Theorem 4.1.3. *Let $u(t,x)$ be the solution to (4.1.1) starting from μ and let ρ be an admissible weight function. Assume that*

(i) there exists another admissible weight $\tilde{\rho}$ such that

$$\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \mathrm{d}x < \infty; \tag{4.1.17}$$

(ii) the weight function $\tilde{\rho}$ and the initial condition satisfy the following condition:

$$\limsup_{t>0} \mathcal{G}_{\tilde{\rho}}(t;\mu) < \infty; \tag{4.1.18}$$

(iii) the spectral measure $\widehat{f}$ and the Lipschitz constant L_b satisfy the two conditions in (4.1.10);

(iv) for some $\alpha \in (2^7\Upsilon(0)L_b^2, 1)$ (see (4.1.14)), the spectral measure $\widehat{f}$ satisfies (4.1.13).

Then we have that

(a) for any $\tau > 0$, the sequence of laws of $\{\mathcal{L}u(t,\cdot;\mu)\}_{t\geqslant\tau}$ is tight, i.e., for any $\epsilon \in (0,1)$, there exists a compact set $\mathscr{K} \subset L^2_\rho(\mathbb{R}^d)$ such that

$$\mathcal{L}u(t,\cdot;\mu)(\mathcal{K}) := \mathbb{P}\left(u(t,\cdot;\mu) \in \mathscr{K}\right) \geqslant 1-\epsilon, \qquad \text{for all } t \geqslant \tau > 0; \tag{4.1.19}$$

(b) there exists an invariant measure for the laws $\{\mathcal{L}u(t,\cdot;\mu)\}_{t>0}$ in $L^2_\rho(\mathbb{R}^d)$.

This theorem will be proved in Section 4.5.

The work by Tessitore and Zabczyk [TZ98] has a strong influence in the current work. We formulate and solve the problem using the random field language with some state-of-the-art moment estimates. The improvements over [TZ98] consist of the following aspects:

1. Results in [TZ98] allow essentially all bounded functions as the initial conditions, though Theorem 3.3 (*ibid.*) was proved only for the constant one initial condition. Here we give

precise conditions on the initial condition, namely, (4.1.18), which allows a much wider class of initial conditions, including unbounded functions and measures such as the Dirac delta measure; see Examples 4.2.4 and 4.2.5. We emphasize that the Dirac delta initial measure plays a very prominent role in the study of the stochastic heat equation; see, e.g., [ACQ11].

2. We give a more easily verifiable condition in (4.1.10a) on the spectral density $\hat{f}$ and provide examples of suitable $\hat{f}$ in Section 4.2.3. Recall that due to the difficult nature of the condition (3.4) in [TZ98], no specific spectral densities were given. We further discuss this in Section 4.2.4.

This chapter is organized as follows: In Section 4.2, we will further discuss our main results and provide some examples. In particular, in Section 4.2.1 we make some comments on the weight function and in Section 4.2.2 we show that our results could include a wider class of initial conditions. Finally, the two main Theorems 4.1.2 and 4.1.3 will be proved in Sections 4.3 and 4.5, respectively.

We end this section by introducing some notation and formulas that we use throughout the paper. We will use $\|X\|_p$ to denote the $L^p(\Omega)$ norm, namely, $(\mathbb{E}(|X|^p))^{1/p}$. We will also use the following factorization property of the heat kernel

$$G(t,x)G(s,y) = G\left(\frac{ts}{t+s}, \frac{sx+ty}{t+s}\right) G\left(t+s, x-y\right), \tag{4.1.20}$$

which can be easily verified. Next, we remind the reader of the following spherical coordinate integration formula

$$\int_{\mathbb{R}^d} f(|x|)\mathrm{d}x = \sigma(\mathbb{S}^{d-1}) \int_0^\infty f(r) r^{d-1}\mathrm{d}r,$$

which we will use often and where $\sigma(\mathbb{S}^{d-1}) = 2(\pi)^{d/2}/\Gamma(d/2)$. The convention of Fourier transform is given in Remark 4.2.10 below. Lastly, $\Gamma(z)$ will be used to denote the Gamma function.

4.2 Discussions and Examples

4.2.1 Weight functions

Here are several examples of admissible weights (see Example in Section 2 of [TZ98])

$$\begin{aligned} \rho(x) &= \exp(-a|x|), \qquad a > 0 \quad \text{and} \\ \rho(x) &= (1 + |x|^a)^{-1}, \qquad a > d. \end{aligned} \tag{4.2.1}$$

Remark 4.2.1. The smaller the weight function $\rho(\cdot)$ (not necessarily admissible) is, the larger the space $L^2_\rho(\mathbb{R}^d)$ is. For example, one may choose ρ to be either a nonnegative function with compact support or the heat kernel itself $G(1,\cdot)$. In both cases, ρ is smaller than those in (4.2.1) (up to a constant). However, one can easily check that the admissible condition (4.1.6) excludes these two cases. Lastly, we should mention that numerical results will lead one to believe that the following functions are not admissible:

$$\rho(x) = \exp\left(-a|x|^b\right), \quad x \in \mathbb{R}^d, \quad \text{with } a > 0 \text{ and } b \in (1,2) \text{ fixed,}$$

but a proof of this has not yet been given.

The admissible condition 4.1.6 is needed in this chapter due to the following result:

Proposition 4.2.2 (Proposition 2.1 of [TZ98])**.** *For any admissible weight ρ, the operators on $L^2_\rho(\mathbb{R}^d)$ defined by $\varphi \mapsto \big(G(t,\cdot) * \varphi(\cdot)\big)(x)$ can be extended to a C_0 − semigroup on L^2_ρ. Moreover, if $\tilde{\rho}$ is another admissible weight such that*

$$\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \mathrm{d}x < \infty,$$

then for any $t > 0$, the operators defined above are compact from $L^2_{\tilde{\rho}}(\mathbb{R}^d)$ to $L^2_\rho(\mathbb{R}^d)$.

4.2.2 Various initial conditions

Example 4.2.3 ($L^\infty(\mathbb{R}^d)$ initial condition)**.** We emphasize that if the initial condition μ is deterministic and is such that $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ with $\varphi \in L^\infty(\mathbb{R}^d)$, then all conditions related to $\mathcal{G}_\rho(\cdot)$

in both Theorems 4.1.2 and 4.1.3 are trivially satisfied. To be more precise, both conditions (4.1.15) and (4.1.18) hold because

$$\mathcal{G}_\rho(t;\varphi) \leqslant \|\varphi\|_{L^\infty(\mathbb{R}^d)}^2 \|\rho\|_{L^1(\mathbb{R}^d)} < \infty \quad \text{uniformly for all } t \geqslant 0.$$

Example 4.2.4 (Delta initial condition)**.** In this example, we study the case when the initial condition μ is the Dirac delta measure at zero, namely, δ_0. Let ρ be a nonnegative $L^1(\mathbb{R}^d)$ function. Since

$$\mathcal{G}_\rho(t;\delta_0) = \int_{\mathbb{R}^d} G(t,x)^2\rho(x)\mathrm{d}x \leqslant G(t,0)^2 \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0,$$

we see that both conditions (4.1.15) and (4.1.18) are satisfied. In particular, $\limsup_{t>0}\mathcal{G}_\rho(t;\delta_0) = 0$.

Example 4.2.5 (More initial conditions not in $L^2_\rho(\mathbb{R}^d)$)**.** In this example, we study the case when $\mu(\mathrm{d}x) = |x|^{-\alpha}\mathrm{d}x$ for some $\alpha \in (0,d)$. It is clear that when $\alpha \in (d/2,d)$, $\mu \notin L^2_\rho(\mathbb{R}^d)$. However, in this case, we have

$$J_0(t,x) = \left(G(t,\cdot) * |\cdot|^{-\alpha}\right)(x) \leqslant \left(G(t,\cdot) * |\cdot|^{-\alpha}\right)(0).$$

On the other hand, Hence,

$$\left(G(t,\cdot) * |\cdot|^{-\alpha}\right)(0) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \times (2\pi t)^{-d/2} \int_0^\infty e^{-\frac{r^2}{2t}} r^{-\alpha+d-1}\mathrm{d}r = C_* t^{-\alpha/2},$$

with $C_* = 2^{-\alpha/2}\Gamma\left((d-\alpha)/2\right)/\Gamma(d/2)$, which implies that

$$\mathcal{G}_\rho\left(t;|\cdot|^{-\alpha}\right) \leqslant \int_{\mathbb{R}^d} J_0^2(t,0)\rho(x)\mathrm{d}x = C_*^2 t^{-\alpha} \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0.$$

Therefore, we see that both conditions (4.1.15) and (4.1.18) are satisfied.

The following proposition shows that for initial conditions with unbounded tails, condition (4.1.15) may hold while condition (4.1.18) may fail.

Proposition 4.2.6. *Suppose that $\rho = \exp(-|x|)$, which is an admissible weight function. Let the initial condition μ be given as*

$$\mu(\mathrm{d}x) = |x|^{\alpha}\mathrm{d}x \quad \text{with } \alpha > 0.$$

Then for some constants $C, C' > 0$ that depend on d and α, it holds that

$$C'(1+t^{\alpha}) \leqslant \mathcal{G}_{\rho}(t;\mu) \leqslant C(1+t^{\alpha}), \qquad \text{for all } t > 0. \tag{4.2.2}$$

In particular, this implies that condition (4.1.15) *is satisfied, but condition* (4.1.18) *fails.*

Proof. Notice that by scaling arguments,

$$\begin{aligned}
\mathcal{G}_{\rho}(t;\mu) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} G(t,x-y)|y|^{\alpha}\mathrm{d}y\right)^2 e^{-|x|}\mathrm{d}x \\
&= \int_{\mathbb{R}^d} t^{\alpha}\left(\int_{\mathbb{R}^d} G\left(1,\frac{x}{\sqrt{t}}-z\right)|z|^{\alpha}\mathrm{d}z\right)^2 e^{-|x|}\mathrm{d}x \\
&= \int_{\mathbb{R}^d} t^{\alpha+d/2}\left(\int_{\mathbb{R}^d} G\left(1,\xi-z\right)|z|^{\alpha}\mathrm{d}z\right)^2 e^{-\sqrt{t}|\xi|}\mathrm{d}\xi.
\end{aligned}$$

In the following, let C_d, C_α, C'_α, $C_{\alpha,d}$ and $C'_{\alpha,d}$ be generic constants that may depend on α and d and may change their value at each appearance.

Upper bound: Because

$$\int_{\mathbb{R}^d} G\left(1,z\right)|\xi-z|^{\alpha}\mathrm{d}z \leqslant C_\alpha \int_{\mathbb{R}^d} G\left(1,z\right)\left(|\xi|^{\alpha}+|z|^{\alpha}\right)\mathrm{d}z \leqslant C'_\alpha\left(1+|\xi|^{\alpha}\right),$$

we see that

$$\begin{aligned}
\mathcal{G}_{\rho}(t;\mu) &\leqslant C_\alpha \int_{\mathbb{R}^d} t^{\alpha+d/2}\left(1+|\xi|^{2\alpha}\right)e^{-\sqrt{t}|\xi|}\mathrm{d}\xi \\
&= C_{\alpha,d}\int_0^{\infty} t^{\alpha+d/2}\left(1+r^{2\alpha}\right)e^{-\sqrt{t}r}r^{d-1}\mathrm{d}r \\
&= C_{\alpha,d}\left(t^{\alpha}\Gamma(d)+\Gamma(d+2\alpha)\right) \\
&= C'_{\alpha,d}(1+t^{\alpha}) < \infty.
\end{aligned}$$

Hence, this proves the upper bound in (4.2.2).

Lower bound: Now we prove the lower bound in (4.2.2). Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} G(1,z)\,|\xi - z|^{\alpha}\mathrm{d}z &\geqslant \int_{\mathbb{R}^d} G(1,z)\,\big||\xi| - |z|\big|^{\alpha}\mathrm{d}z \\ &\geqslant C_d \int_0^{\infty} \big||\xi| - x\big|^{\alpha} e^{-\frac{x^2}{2}} x^{d-1}\mathrm{d}x \\ &\geqslant C_d \int_1^{2} \big||\xi| - x\big|^{\alpha}\mathrm{d}x \\ &= \frac{C_d}{1+\alpha}\psi\left(|\xi|\right),\end{aligned}$$

where, by considering three cases, we have

$$\begin{aligned}\psi(r) &= \begin{cases} (2-r)^{\alpha+1} - (1-r)^{\alpha+1} & \text{if } 0 < r < 1, \\ (2-r)^{\alpha+1} + (r-1)^{\alpha+1} & \text{if } 1 \leqslant r \leqslant 2, \\ (r-1)^{\alpha+1} - (r-2)^{\alpha+1} & \text{if } r > 2, \end{cases} \\ &= \operatorname{sgn}(2-r)|r-2|^{\alpha+1} + \operatorname{sgn}(r-1)|r-1|^{\alpha+1}.\end{aligned}$$

We claim that

$$\inf_{r\geqslant 0} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} > 0. \tag{4.2.3}$$

With (4.2.3), we have that

$$\int_{\mathbb{R}^d} G(1,z)\,|\xi - z|^{\alpha}\mathrm{d}z \geqslant C_{\alpha,d}\sqrt{1+|\xi|^{2\alpha}}.$$

Then, by the same arguments as above for the upper bound, we obtain the lower bound in (4.2.2). It remains to prove (4.2.3), which will be proved in three cases.

When $r > 2$, we see that

$$\frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} \geqslant C_{\alpha}\frac{(r-1)^{\alpha+1} - (r-2)^{\alpha+1}}{(1+r)^{\alpha}}$$

$$
\begin{aligned}
&\geqslant C_\alpha \frac{(r-1)^\alpha (r-1) - (r-1)^\alpha (r-2)}{(1+r)^\alpha} \\
&= C_\alpha \left(\frac{r-1}{1+r}\right)^\alpha = C_\alpha \left(1 - \frac{2}{1+r}\right)^\alpha \geqslant C_\alpha \left(1 - \frac{2}{3}\right)^\alpha .
\end{aligned}
$$

Note that in the first inequality above, we have considered two cases: $2\alpha \geqslant 1$ and $2\alpha < 1$. When $2\alpha < 1$, we have used the concavity of $x^{2\alpha}$, namely, $(1 + r^{2\alpha})^{2\alpha}/2 \leqslant ((1+r)/2)^{2\alpha}$; when $2\alpha \geqslant 1$, we have used the super-additiity of $x^{2\alpha}$: namely, that for $a, b > 0$ that $(a+b)^{2\alpha} \geqslant a^{2\alpha} + b^{2\alpha}$. This shows that $\inf_{r>2} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} > 0$.

When $r \in (1,2]$, elementary calculations show that the minimum of $\psi(r)$ is achieved at $r = 3/2$. Hence,

$$
\inf_{r\in(1,2]} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} \geqslant \frac{\psi(3/2)}{\sqrt{1+4^\alpha}} > 0.
$$

Similarly, when $r \in (0,1]$, by differentiation, one finds that the function $\psi(r)$ is nonincreasing. Hence, the minimum is achieved at $r = 1$:

$$
\inf_{r\in(0,1]} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} \geqslant \frac{\psi(1)}{\sqrt{2}} > 0.
$$

Combing the above three cases proves (4.2.3). This completes the proof of Proposition 4.2.6. □

4.2.3 Bessel kernel and Matérn class of correlation functions

Example 4.2.7 (Bessel kernel as a correlation function)**.** Let f_s denote the Bessel kernel with a strictly positive parameter $s > 0$. It is known that (see, e.g., Section 1.2.2 of [Gra14])

1. $f_s(x) > 0$ for all $x \in \mathbb{R}^d$ and $\|f_s\|_{L^1(\mathbb{R}^d)} = 1$;

2. there exists a constant $C(s,d) > 0$ such that

$$
f_s(x) \leqslant C(s,d)\exp(-|x|/2) \quad \text{for} \quad |x| \geqslant 2;
$$

3. there exists a constant $c(s,d)>0$ such that

$$\frac{1}{c(s,d)} \leqslant \frac{f_s(x)}{H_s(x)} \leqslant c(s,d) \quad \text{for} \quad |x|\leqslant 2, \quad \text{with}$$
$$H_s(x) = \begin{cases} |x|^{s-d} + 1 + O(|x|^{s-d+2}) & \text{for } 0<s<d, \\ \log\left(\frac{2}{|x|}\right) + 1 + O(|x|^2) & \text{for } s=d, \\ 1 + O(|x|^{s-d}) & \text{for } s>d; \end{cases}$$

4. the Fourier transform of f_s is strictly positive:

$$\mathcal{F}f_s(\xi) = \frac{1}{(1+|\xi|^2)^{s/2}}. \tag{4.2.4}$$

Note that one can use (4.2.4) as the definition of the Bessel kernel. Properties 1 and 4 ensure that f_s is a nonnegative and nonnegative-definite tempered measure for all $s>0$. From Property 4, we see that

$$\Upsilon(0)<\infty \quad \Longleftrightarrow \quad s>d-2.$$

In the following, we will assume that $s>d-2$ and $d\geqslant 3$.

Example 4.2.8 (Matérn class of correlation functions)**.** The *Matérn class of correlation functions* has been widely used in spatial statistics; one may check the recent work [LSW21] for references. Following Section 2.10 of [Ste99], this class of correlation functions is given by

$$K(x) = \phi\cdot(\alpha|x|)^{\nu}\mathcal{K}_{\nu}(\alpha|x|), \quad \text{for } x\in\mathbb{R}^d \text{ with } \phi>0, \alpha>0, \nu>0, \tag{4.2.5}$$

where $\mathcal{K}_{\nu}(\cdot)$ is the modified Bessel function of second type, and α and ν refer to the *scaling and smoothness parameters* respectively. From the inversion formula (see p. 46 *ibid.*), one sees that

$$\mathcal{F}K(\xi) = (2\pi)^d\mathcal{F}^{-1}K(\xi) = (2\pi)^d f(|\xi|) \quad \text{with} \quad f(\xi) = \frac{2^{\nu-1}\phi\Gamma(\nu+d/2)\,\alpha^{2\nu}}{\pi^{d/2}\left(\alpha^2+|\xi|^2\right)^{\nu+d/2}}, \quad \xi\in\mathbb{R}^d.$$

Comparing the above expression with (4.2.4), we see that the class of Bessel kernels f_s, with $s > d - 2$ and $d \geqslant 3$, includes the Matérn class (4.2.5) as a special case under the following choice of parameters:

$$\alpha = 1, \quad \nu = (s-d)/2, \quad \text{and} \quad \phi = 2^{(2-d-s)/2}\pi^{-d/2}\Gamma(s/2)^{-1}.$$

Note that the requirement of the smoothness parameter $\nu > 0$ for the Matérn class corresponds to the case of the Bessel kernel with $s > d$.

For $\alpha \in (0, 1/2)$, we introduce the following quantity which will now be needed throughout this article:

$$\mathcal{H}_\alpha(t) := \int_0^t \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi)\, \exp(-r|\xi|^2) \tag{4.2.6}$$

Proposition 4.2.9. *For the Bessel kernel $f_s(\cdot)$ with $s > 0$ defined in Example 4.2.7, it holds that*

$$\Upsilon_\alpha(0) = \frac{\Gamma\left(\frac{d}{2} - 1 + \alpha\right)\Gamma\left(\frac{s-d}{2} + 1 - \alpha\right)}{2^d \pi^{d/2}\, \Gamma(d/2)\, \Gamma(s/2)} \qquad \text{for all } s > d - 2(1-\alpha) > 0 \text{ and } d > 2, \tag{4.2.7}$$

and in particular when $\alpha = 0$, (4.2.7) simplifies to the following:

$$\Upsilon(0) = \frac{\Gamma\left(\frac{2+s-d}{2}\right)}{2^{d-1}\pi^{d/2}(d-2)\Gamma(s/2)}. \tag{4.2.8}$$

In addition,

$$\mathcal{H}_\alpha(t) < \infty \quad \forall t > 0 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} - \frac{(d-s)_+}{4} \quad \text{and} \quad s > d - 2 > 0, \tag{4.2.9}$$

where $a_+ := \max(a,0)$. *Moreover, for* $\alpha \in (0,1/2)$, *we have the following asymptotic behavior of* $\mathcal{H}_\alpha(t)$ *at* $t \to 0$*:*

$$
\mathcal{H}_\alpha(t) = \begin{cases}
\dfrac{\pi^{d/2}\Gamma\left((d-s)/2\right)}{\left((s-d)/2+1-2\alpha\right)\Gamma(d/2)} t^{(s-d)/2+1-2\alpha} \\
\qquad - \dfrac{\pi^{d/2}\Gamma\left((s-d)/2\right)}{(1-2\alpha)\Gamma(s/2)} t^{1-2\alpha} & d-2<s<d \text{ and } \alpha < \frac{1}{2}-\frac{1}{4}(d-s) \quad (4.2.10\text{-a}) \\
\qquad + O\left(t^{(s-d)/2+2(1-\alpha)}\right), \\
\dfrac{\pi^{d/2}}{(1-2\alpha)\Gamma\left(d/2\right)} t^{1-2\alpha}\log\left(\dfrac{1}{t}\right) \\
\quad + \dfrac{\pi^{d/2}\left(1-(1-2\alpha)\left[\psi(d/2)+2\gamma\right]\right)}{(1-2\alpha)^2\Gamma\left(d/2\right)} t^{1-2\alpha} & s = d \quad (4.2.10\text{-b}) \\
\qquad + O\left(t^2\log(t)\right), \\
\dfrac{\pi^{d/2}\Gamma\left((s-d)/2\right)}{(1-2\alpha)\Gamma(s/2)} t^{1-2\alpha} + O\left(t^{(s-d)/2+1-2\alpha}\right), & d<s<d+2 \quad (4.2.10\text{-c}) \\
\dfrac{\pi^{d/2}}{(1-2\alpha)\Gamma\left(d/2+1\right)} t^{1-2\alpha} + O\left(t^{2(1-\alpha)}\log(t)\right), & s = d+2 \quad (4.2.10\text{-d}) \\
\dfrac{\pi^{d/2}}{(1-2\alpha)\Gamma\left(d/2+1\right)} t^{1-2\alpha} + O\left(t^{2(1-\alpha)}\right), & s > d+2 \quad (4.2.10\text{-e})
\end{cases}
\tag{4.2.10}
$$

where $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ *refers to the* digamma function *and* $\gamma \approx 0.57721$ *to* Euler's constant*; see, e.g., 5.2.2 and 5.2.3 on p. 136 of [AR10].*

Proof. By the spherical coordinate integration formula and (4.2.4), for all $\alpha \in [0,1)$,

$$
\Upsilon_\alpha(0) = (2\pi)^{-d}\int_{\mathbb{R}^d}\frac{\mathrm{d}\xi}{|\xi|^{2(1-\alpha)}\left(1+|\xi|^2\right)^{s/2}} = (2\pi)^{-d}C_d\int_0^\infty \frac{r^{d-1}}{r^{2(1-\alpha)}\left(1+r^2\right)^{s/2}}\mathrm{d}r,
$$

where $C_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Now by the change of variables $z = r^2/(1+r^2)$, we can evaluate the above integral explicitly by transforming it to the Beta integral:

$$\begin{aligned}\int_0^\infty \frac{r^{d-1}}{r^{2(1-\alpha)}\left(1+r^2\right)^{s/2}}\mathrm{d}r &= \frac{1}{2}\int_0^1 z^{d/2+\alpha-2}(1-z)^{(s-d)/2-\alpha}\mathrm{d}z \\ &= \frac{\Gamma\left(d/2-1+\alpha\right)\Gamma\left((s-d)/2+1-\alpha\right)}{2\Gamma(s/2)},\end{aligned}$$

which is finite provided that $s > d-2(1-\alpha) > 0$. This proves (4.2.7) and from this, we easily deduce (4.2.8) by letting $\alpha = 0$ in (4.2.7) and by applying the formula $\Gamma(z+1) = z\Gamma(z)$, which holds for $z \in \mathbb{C}$ such that $\Re(z) > 0$.

It remains to prove (4.2.10), which then implies (4.2.9). From (4.2.6) and by the spherical coordinate integration formula, for all $t > 0$,

$$\begin{aligned}\mathcal{H}_\alpha(t) &= \int_0^t \mathrm{d}r\; r^{-2\alpha}\int_{\mathbb{R}^d}\mathrm{d}\xi\;\frac{\exp(-r|\xi|^2)}{(1+|\xi|^2)^{s/2}} \\ &= C_d\int_0^t \mathrm{d}r\; r^{-2\alpha}\int_0^\infty \mathrm{d}z\;\frac{\exp(-rz^2)}{(1+z^2)^{s/2}}z^{d-1} \\ &= \frac{C_d}{2}\int_0^t \mathrm{d}r\; r^{-2\alpha}\int_0^\infty \mathrm{d}u\;\exp(-ru)(1+u)^{-s/2}u^{d/2-1} \\ &=: \frac{C_d\Gamma\left(d/2\right)}{2}\int_0^t \mathrm{d}r\; r^{-2\alpha}I(r) = \pi^{d/2}\int_0^t \mathrm{d}r\; r^{-2\alpha}I(r).\end{aligned}$$

By [Old10, 13.4.4 on p.326], $I(r)$ is equal to the *confluent hypergeometric function*:

$$I(r) = U\left(\frac{d}{2}, \frac{2+d-s}{2}, r\right).$$

By 18.2.18 – 13.2.22 on p. 323 *ibid.*, we see that

$$I(r) = \begin{cases} \dfrac{\Gamma\left((d-s)/2\right)}{\Gamma(d/2)} r^{(s-d)/2} + \dfrac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r^{(s-d)/2+1}\right) & d-2 < s < d \quad 18.2.18, \\ -\dfrac{1}{\Gamma(d/2)} \left(\log(r) + \psi(d/2) + 2\gamma\right) + O\left(r\log(r)\right) & s = d \quad 18.2.19, \\ \dfrac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r^{(s-d)/2}\right) & d < s < d+2 \quad 18.2.20, \\ \dfrac{1}{\Gamma\left(d/2+1\right)} + O\left(r\log(r)\right) & s = d+2 \quad 18.2.21, \\ \dfrac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r\right) & s > d+2 \quad 18.2.22. \end{cases}$$

Then integrating the right-hand side of the above expressions against $\pi^{d/2} r^{-2\alpha} \mathrm{d}r$ over $[0,t]$ gives the five cases in (4.2.10). This completes the proof of Proposition 4.2.9. □

4.2.4 A discussion on Tessitore and Zabczyk's condition

Under the setting of the whole space, $\mathbb{R}^d$, Tessitore and Zabczyk [TZ98] prove the existence of an invariant measure for (4.1.1) in $L^2_\rho(\mathbb{R}^d)$ under the assumption that there exists a $\varphi \in L^2_\rho(\mathbb{R}^d) \cap L^2_{\widetilde{\rho}}(\mathbb{R}^d)$ where $\rho(\widetilde{\rho})^{-1} \in L^1(\mathbb{R}^d)$ and the solution starting from φ is bounded in probability in $L^2_{\widetilde{\rho}}(\mathbb{R}^d)$ and also that the spectral density $\hat{f}$ satisfies

$$\widehat{f} \in L^p(\mathbb{R}^d) \quad \text{where } \frac{d-2}{d} < \frac{1}{p}; \tag{4.2.11}$$

see Hypothesis 2.1 (*ibid.*). However as was illustrated in [TZ98, Theorem 3.3], in order to apply this theorem to a specific initial condition in L^2_ρ (or to have moments uniformly bounded in time), the following additional assumptions were imposed:

$$d \geqslant 3 \quad \text{and} \quad L_b^{-2} > \frac{\Gamma(d/2-1)2^{d/2-2}}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \left(\left|\mathcal{F}\left(\sqrt{\widehat{f}}\right)\right| * \left|\mathcal{F}\left(\sqrt{\widehat{f}}\right)\right| \right)(\zeta) |\zeta|^{2-d} \mathrm{d}\zeta, \tag{4.2.12}$$

where the convention of the Fourier transform is given in Remark 4.2.10. With these assumptions, they were able to prove that (4.1.1) starting from the constant 1 satisfies (4.1.9) and thus is bounded in probability, verifying the existence of an invariant measure. Moreover, the measure takes the form of (4.1.8) above with $\varphi = 1$. Lastly we mention that due to its difficult nature, (4.2.12) was not calculated for a specific $\widehat{f}$ (see Examples 4.2.11 and 4.2.12 below).

Remark 4.2.10. The Fourier transform may be defined differently depending how to handle the 2π constant. In this chapter (as in [CK19; CH19]), we use the convention that

$$\widehat{\phi}(\xi) = \mathcal{F}\phi(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\phi(x)\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}\psi(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(\xi)\mathrm{d}\xi. \tag{4.2.13}$$

Hence, Plancherel's theorem takes the form of

$$\int_{\mathbb{R}^d} \psi(x)\phi(x)\mathrm{d}x = (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}\mathrm{d}\xi. \tag{4.2.14}$$

Note that the authors in [TZ98] did not explicitly mention their convention of the Fourier transform. However, the proof of Theorem 3.3 (*ibid.*) suggests that the following convention has been used:

$$\widehat{\phi}(\xi) = \mathcal{F}\phi(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\phi(x)\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}\psi(x) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(\xi)\mathrm{d}\xi.$$

Hence, Plancherel's theorem takes the form, $\int_{\mathbb{R}^d} \psi(x)\phi(x)\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}\mathrm{d}\xi$, without the additional factor $(2\pi)^{-d}$. In particular, the spectral density γ (*ibid.*) corresponds to $(2\pi)^{-d/2}\widehat{f}$ in this chapter. Our equation (4.2.12), which is condition (3.4) [TZ98], takes into account this difference therefore explaining the slightly different factor in front of the integral in (4.2.12) from that in (3.4) (*ibid.*).

Recall the definition of $\mathcal{F}^{-1}$ in (4.2.13) above. We may rewrite the integral in the second condition of (4.2.12) as follows:

$$\Gamma(d/2-1)2^{d/2-2}\int_{\mathbb{R}^d}\left(\left|\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)\right| * \left|\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)\right|\right)(\zeta)|\zeta|^{2-d}\mathrm{d}\zeta.$$

We now consider a couple of situations and give two examples that illustrate some of the difficulties that may arise.

Case I: If both $\mathcal{F}^{-1}\left[\sqrt{\widehat{f}(\cdot)}\right](\xi)$ and $\widehat{f}$ are strictly positive for all $\zeta \in \mathbb{R}^d$, which is not a trivial assumption (see Example 4.2.11 below), then there is no ambiguity when taking the square root and one can remove the absolute value to see that

$$\begin{aligned}\Gamma(d/2-1)2^{d/2-2}&\int_{\mathbb{R}^d}\left(\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)*\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)\right)(\zeta)|\zeta|^{2-d}\mathrm{d}\zeta\\&=\frac{1}{(2\pi)^{3d/2}}\int_{\mathbb{R}^d}\widehat{f}(\zeta)|\zeta|^{-2}\mathrm{d}\zeta=(2\pi)^{d/2}\,\Upsilon(0),\end{aligned}$$

where we have applied Plancherel's theorem (see (4.2.14)) and the Fourier transform for Riesz kernel (in the generalized sense):

$$\mathcal{F}(|\cdot|^{-\alpha})(\xi)=\pi^{-d/2}2^{-\alpha}\frac{\Gamma\left((d-\alpha)/2\right)}{\Gamma(\alpha/2)}|\xi|^{-(d-\alpha)},\quad \text{for } \alpha\in(0,d) \text{ and } \xi\in\mathbb{R}^d.$$

Hence, the second condition in (4.2.12) can be equivalently written as $L_b^{-2} > (2\pi)^{d/2}\Upsilon(0)$. Comparing this with (4.1.10b), namely, $L_b^{-2} > 128\Upsilon(0)$, our condition is sharper when $d > \frac{14\log(2)}{\log(2\pi)} \approx 5.28$.

Example 4.2.11. One should note that the assumption that $\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x)$ is nonnegative, or equivalently that $\sqrt{\widehat{f}}$ is non-negative definite, is quite strong and may not be true even if f is non-negative and non-negative definite. Indeed, suppose that f was such that $\widehat{f}(\xi) = 2^{-2}\max\{2-|\xi|,0\}$. Then f is non-negative and non-negative definite, which is shown in Example 4.2.12 below. We will now show that in this case, $\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(\zeta)$ takes on both positive and negative values. Since $\sqrt{\widehat{f}(\xi)} = 2^{-1}\sqrt{\max\{2-|\xi|,0\}}$ is even, the inverse Fourier transform takes the following form:

$$\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x)=2\int_0^2 2^{-1}\sqrt{2-y}\cos(y\,x)\mathrm{d}y.$$

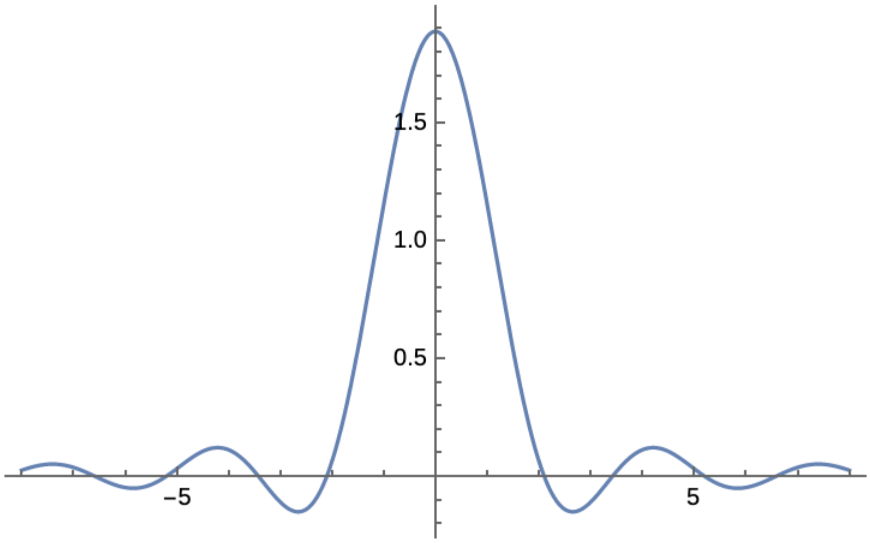


Figure 4.2: A plot of $\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x)$ for $\widehat{f}(\xi) = 2^{-2}\max\{2-|\xi|, 0\}$ with $-8 \leqslant x \leqslant 8$.

For simplicity, we will only consider the case $\zeta > 0$ and real. Therefore

$$\begin{aligned}\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x) &= 2\int_0^{\sqrt{2}} y^2\cos(2x - xy^2)\mathrm{d}y\\ &= 2\cos(2x)\int_0^{\sqrt{2}} y^2\cos(xy^2)\mathrm{d}y + 2\sin(2x)\int_0^{\sqrt{2}} y^2\sin(xy^2)\mathrm{d}y.\end{aligned}$$

Where in the first equality we used the change of variables, $y' = \sqrt{2-y}$. We now do an integration by parts to see that

$$\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x) = -\cos(2x)\int_0^{\sqrt{2}} \frac{\sin(y^2x)}{x}\mathrm{d}y + \sin(2x)\int_0^{\sqrt{2}} \frac{\cos(y^2x)}{x}\mathrm{d}y,$$

and lastly we apply the change of variables $y' = y\sqrt{2x/\pi}$ too see

$$\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\right)(x) = \frac{\sqrt{\frac{\pi}{2}}\left(-\cos(2x)\mathcal{S}\left(\frac{2\sqrt{x}}{\sqrt{\pi}}\right) + \sin(2x)\mathcal{C}\left(\frac{2\sqrt{x}}{\sqrt{\pi}}\right)\right)}{x^{3/2}},$$

where $\mathcal{S}$ and $\mathcal{C}$ are the Fresnel integrals (see [Olv+10, 7.2 (iii)]):

$$\mathcal{S}(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t \quad \text{and} \quad \mathcal{C}(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right)\mathrm{d}t.$$

Thus even if f is both non-negative and non-negative definite, we may still have problems with removing the absolute value sign.

Case II: Suppose $\widehat{f}(\cdot)$ is not strictly positive, but only nonnegative, namely, $\widehat{f}(\xi) = 0$ for some $\xi \in \mathbb{R}^d$. Then removing the absolute value in (4.2.12) becomes tricky, which will be illustrated in the following example.

Example 4.2.12. Let $d = 1$ and $g(x) = \frac{1}{2}1_{[-1,1]}(x)$. It is clear that $\widehat{g}(\xi) = \xi^{-1}\sin(\xi)$. Now set $f(x) = (g * g)(x) = 2^{-2}\max(2 - |x|, 0)$. It is clear that f is nonnegative. It is also nonnegative-definite because $\widehat{f}(\xi) = \widehat{g}(\xi)^2 = \xi^{-2}\sin^2(\xi) \geqslant 0$. But in this case, $\widehat{f}(\cdot)$ is only nonnegative (not strictly positive) with infinitely many zeros. Hence, when taking the square root of $\widehat{f}(\xi)$ as in (4.2.12), one needs to wisely choose the correct positive and negative branches:

1. Clearly, the signed version $\sqrt{\widehat{f}(\xi)} = \xi^{-1}\sin(\xi)$ is preferable since its inverse Fourier transform can be easily computed, which is equal to $g(x)$. Moreover, because this inverse Fourier transform $g(x)$ is clearly nonnegative, the absolute value signs in (4.2.12) do not pose any additional restrictions.

2. However, if one chooses the positive branches, namely, $\sqrt{\widehat{f}(\xi)} = |\xi^{-1}\sin(\xi)|$, then it is not clear how to compute its Fourier transform. In general, some bad choices of the positive/negative branches may make the conditions in (4.2.12) fail. For example, such choice may turn $\sqrt{\widehat{f}(\xi)}$ into a distribution, and then taking the absolute value of a distribution (unless it is a measure) may be problematic. Another issue that may arise is when $\sqrt{\widehat{f}(\xi)}$ is a well-defined function, taking on both positive and negative values and after taking the absolute value, the integral in (4.2.12) may blow up.

4.3 Moment Estimates – Proof of Theorem 4.1.2

We first state some known results and prove a moment bound in Corollary 4.3.3.

Theorem 4.3.1 (Theorem 1.2 [CH19])**.** *Suppose that* *(i) the initial deterministic measure μ satisfies the following integrability condition:*

$$\int_{\mathbb{R}^d} \exp\left(-a|x|^2\right) |\mu|(\mathrm{d}x) < \infty \quad \textit{for all } a > 0, \tag{4.3.1}$$

(ii) the spectral measure $\widehat{f}$ satisfies Dalang's condition (4.1.11),

Then (4.1.1) has a unique random field solution starting from μ. Moreover, the solution is $L^2(\Omega)$ continuous and is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Theorem 4.3.2 (Theorem 1.7 [CH19])**.** *Under the assumptions of Theorem 4.3.1, for any $t > 0$, $x \in \mathbb{R}^d$ and $p \geqslant 2$, the solution to (4.1.1), $u(t,x)$, given by (4.1.4) is in $L^p(\Omega)$ and*

$$\|u(t,x)\|_p \leqslant \left[\bar{\varsigma} + \sqrt{2}(G(t,\cdot) * |\mu|)(x)\right] H(t;\gamma_p)^{1/2}, \tag{4.3.2}$$

where $\bar{\varsigma} = L_0/L_b$, $\gamma_p = 32pL_b^2$ (see (4.1.3) for L_0 and L_b) and the function $H(t;\gamma_p)$ is nondecreasing in t (see [CH19] for the expression of the function H).

Corollary 4.3.3. *Under the same setting as Theorem 4.3.2, if the two conditions in (4.1.10) hold (see also (4.1.14)), then*

$$\|u(t,x)\|_p \leqslant C_p\Big(1 + (G(t,\cdot) * |\mu|)(x)\Big), \quad \textit{for all } p \textit{ such that } 1/p \in \left(64L_b^2\Upsilon(0), 1\right), \tag{4.3.3}$$

where $C_p = \left(\sqrt{2} \vee \bar{\varsigma}\right) \sup_{t\geqslant 0} H(t;\gamma_p)^{1/2} < \infty$.

Proof. Lemma 2.5 of [CK19] gives one sufficient condition, namely, $2\gamma_p\Upsilon(0) < 1$, under which the function $H((t;\gamma_p)$ is bounded in t. Therefore, by taking into account the expression of γ_p in Theorem 4.3.2, we see that as a direct consequence of (4.3.2), whenever

$$32pL_b^2 < \frac{1}{2\Upsilon(0)}, \tag{4.3.4}$$

we have the p-th moment bounded as given in (4.3.3). □

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Under condition (iii), we can apply Fubini's Theorem and the moment bound (4.3.3) below to see that for some constant $C > 0$ independent of t, which may vary from line to line, that

$$\begin{aligned}\mathbb{E}\left(\|u(t,\cdot;\mu)\|_\rho^2\right) &\leqslant C\,\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(1+(G(t,\cdot)*|\mu|)(x)\Big)^2\rho(x)\mathrm{d}x\right]\\ &= C\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\big(G(t,\cdot)*(1+|\mu|)\big)(x)\Big)^2\right]\rho(x)\mathrm{d}x\\ &= C\,\mathcal{G}_\rho(t;\mu)<\infty.\end{aligned}$$

This proves Theorem 4.1.2. □

Remark 4.3.4 (Restarted SHE)**.** Recall that the Markov property of the solution to (4.1.1) implies that for any $t \geqslant t_0 > 0$,

$$u(t+t_0,x;\mu) \overset{\mathcal{L}}{=} u\left(t,x;u\left(t_0,\cdot;\mu\right)\right) =: v(t,x), \tag{4.3.5}$$

where $\mathcal{L}$ refers to the equality in law and we have used $v(t,x)$ to simplify the notation. It is known that v satisfies the following restarted SPDE:

$$\begin{cases}\dfrac{\partial v}{\partial t}(t,x)-\dfrac{1}{2}\Delta v(t,x)=b(x,v(t,x))\dot{W}_{t_0}(t,x) & x\in\mathbb{R}^d,\,t>0,\\ v(0,x)=u(t_0,x;\mu), & x\in\mathbb{R}^d,\end{cases} \tag{4.3.6}$$

where $\dot{W}_{t_0}(t,x) := \dot{W}(t+t_0,x)$ denotes the time shifted noise, i.e.,

$$\int_0^t\int_{\mathbb{R}^d}W_{t_0}(\mathrm{d}s,\mathrm{d}y)=\int_{t_0}^{t+t_0}\int_{\mathbb{R}^d}W(\mathrm{d}s,\mathrm{d}y). \tag{4.3.7}$$

Under the conditions in (4.1.10), Theorem 4.3.2 and (4.3.5) imply immediately that

$$\|v(t,x)\|_q=\|u(t+t_0,x;\mu)\|_q\leqslant C_q\Big(1+(G(t+t_0,\cdot)*|\mu|)(x)\Big)=C_qJ_0(t+t_0,x;1+|\mu|), \tag{4.3.8}$$

for all $q \geqslant 2$ and $t > 0$, where the constant C_q does not depend on t. Moreover, under the assumptions of Theorem 4.1.2, we have $v(0,\cdot) \in L^2_\rho(\mathbb{R}^d)$ a.s. and

$$\mathbb{E}\left(\|v(t,x)\|_\rho^2\right) = \mathbb{E}\left(\|u(t+t_0,x;\mu)\|_\rho^2\right) \leqslant C\mathcal{G}_\rho(t+t_0;\mu) < \infty. \tag{4.3.9}$$

4.4 A Factorization Lemma

In this section, we establish a factorization lemma with corresponding moment estimates; see Lemmas 4.4.2 and 4.4.4 below. This factorization lemma was first discovered in [DKZ87]; check also Section 5.3.1 of [DZ14]. For $\alpha \in (0,1)$, $t > 0$ and $x \in \mathbb{R}^d$, define formally

$$\begin{aligned} \left(F_\alpha f\right)(t,x) &:= \int_0^t \int_{\mathbb{R}^d} (t-s)^{\alpha-1} G(t-s,x-y) f(s,y)\,\mathrm{d}s\mathrm{d}y \quad \text{and} \\ \left(Y_\alpha f\right)(t,x) &:= \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} G(t-s,x-y) f(s,y) W(\mathrm{d}s,\mathrm{d}y)\,. \end{aligned} \tag{4.4.1}$$

For F_α, the first step of the proof of [TZ98, Theorem 3.1] showed the following proposition:

Proposition 4.4.1. *Let ρ and $\tilde{\rho}$ be given as in condition (i) of Theorem 4.1.3 (see (4.1.17)). For any $q > 2$, $t_0 > 0$ and $\alpha \in (q^{-1}, 2^{-1})$, the operator F_α, as an operator from $L^q\big((0,t_0); L^2_{\tilde{\rho}}(\mathbb{R}^d)\big)$ to $L^2_\rho(\mathbb{R}^d)$, is compact.*

As for Y_α, we have the following two lemmas, which hold for both the non-restarted SHE ($t_0 = 0$) and the restarted SHE ($t_0 > 0$).

Lemma 4.4.2. *Suppose that μ — the initial condition for u — satisfies (4.3.1) and that $\widehat{f}$ satisfies Dalang's condition (4.1.11). Suppose that (4.2.6) is satisfied for some $\alpha \in (0, 1/2)$, i.e.,*

$$\mathcal{H}_\alpha(t) = \int_0^t \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right) < \infty \quad \textit{for all } t > 0.$$

Fix an arbitrary $t_0 \geqslant 0$. Let $v(t,x)$ be the solution to the restarted SHE (4.3.6) and $\dot{W}_{t_0}$ be the time-shifted noise (see (4.3.7)) when $t_0 > 0$ and let $v = u$ when $t_0 = 0$. Then

$$Y_v(s,y) := \left[Y_\alpha b\left(\circ, v(\cdot,\circ)\right)\right](s,y) = \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(s-r, y-z) b(z, v(r,z)) W_{t_0}(\mathrm{d}r, \mathrm{d}z) \tag{4.4.2}$$

has the following properties:

(1) for all $q \geqslant 2$, $s > 0$, $y \in \mathbb{R}^d$,

$$\|Y_v(s,y)\|_q^2 \leqslant H\left(s+t_0; 32qL_b^2\right) \, J_0^2(s+t_0, y; \mu^*) \, \mathcal{H}_\alpha(s) < \infty, \tag{4.4.3}$$

where $\mu^* = 1 + |\mu|$ and we refer to Theorem 4.3.2 for the function $H(t;\gamma)$;

(2) under the two conditions in (4.1.10), if the integral in (4.2.6) is finite for some $\alpha \in (64L_b^2\Upsilon(0),\ 1/2)$, then for any q with $1/q \in (64L_b^2\Upsilon(0),\ \alpha)$, the function $H\left(t; 32qL_b^2\right)$ in (4.4.3) is uniformly bounded in $t \geqslant 0$, i.e., $\sup_{t\geqslant 0} H\left(t; 32qL_b^2\right) < \infty$;

(3) under the two conditions in (4.1.10), if the integral in (4.2.6) is finite for some $\alpha \in (64L_b^2\Upsilon(0),\ 1/2)$, then for any q with $1/q \in (64L_b^2\Upsilon(0),\ \alpha)$ and for any nonnegative and $L^1(\mathbb{R}^d)$-function ρ, there exists a constant $\Theta = \Theta(q, L_b, L_0, \alpha)$, which does not depend on t, such that for $t > 0$,

$$\mathbb{E}\left(\int_0^t \|Y_v(s,\cdot)\|_\rho^q \, \mathrm{d}s\right) \leqslant \Theta \int_0^t \left[\mathcal{G}_\rho(s+t_0;\mu) \, \mathcal{H}_\alpha(s)\right]^{q/2} \mathrm{d}s, \tag{4.4.4}$$

which is finite provided that

$$\int_0^t \left[\mathcal{G}_\rho(s+t_0;\mu) \, \mathcal{H}_\alpha(s)\right]^{q/2} \mathrm{d}s < \infty. \tag{4.4.5}$$

Remark 4.4.3. Condition (4.4.5) is true for $t_0 > 0$ because $\mathcal{G}_\rho(t;\mu)$ is a continuous function for $t > 0$ and $\mathcal{H}_\alpha(s)$ is continuous and bounded for $s \in [0,t]$ thanks to (4.2.6). However, when $t_0 = 0$, the situation is much more trickier. For example, when the initial condition is the delta initial condition, we have that

$$\mathcal{G}_{\tilde{\rho}}(t;\delta_0)=\int_{\mathbb{R}^d}G(t,x)^2\rho(x)\mathrm{d}x=G(2t,0)\int_{\mathbb{R}^d}G(t/2,x)\rho(x)\mathrm{d}x<\infty,$$

where one can obtain the second equality via (4.1.20). Hence, when $s\to 0$, $\mathcal{G}_{\tilde{\rho}}(s;\delta_0)$ blows up with a rate $s^{-d/2}$. On the other hand, $\mathcal{H}_\alpha(s)$ goes to zero with a different rate. One needs to combine these two rates to check if condition (4.4.5) holds. By introducing t_0 and restarting the heat equation, one can avoid this issue, that being the potential singularity of $\mathcal{G}_{\tilde{\rho}}$ at $s=0$.

Proof. In the proof, we use C to denote a generic constant that may change its value at each appearance.

(1) We first prove (4.4.3). By the Burkholder-Davis-Gundy inequality and Minkowski's integral inequality, we see that

$$\begin{aligned}\|Y_v(s,y)\|_q^2\leqslant C\int_0^s\mathrm{d}r\,(s-r)^{-2\alpha}\iint_{\mathbb{R}^{2d}}&\mathrm{d}z_1\mathrm{d}z_2\,G(s-r,y-z_1)\left\|b\left(z_1,v(r,z_1)\right)\right\|_q\\&\times f(z_1-z_2)G(s-r,y-z_2)\left\|b\left(z_2,v(r,z_2)\right)\right\|_q.\end{aligned}$$

Note that for the Lipschitz condition in (4.1.3), we have that

$$|b\left(x,v\right)|\leqslant|b\left(x,v\right)-b\left(x,0\right)|+|b\left(x,0\right)|\leqslant L_b|v|+L_0\leqslant C(1+|v|),\quad C:=L_b\vee L_0.$$

We apply this and the moment bound (4.3.2) to $\left\|b(z_i,v(r,z_i))\right\|_q$ above to see that

$$\begin{aligned}\left\|b(z_i,v(r,z_i))\right\|_q&\leqslant C\left(1+\left\|v(r,z_i)\right\|_q\right)\\&=C\left(1+\left\|u(r+t_0,z_i)\right\|_q\right)\\&\leqslant CH\left(r+t_0;32qL_b^2\right)J_0\left(r+t_0,z_i;\mu^*\right)\\&\leqslant CH\left(s+t_0;32qL_b^2\right)J_0\left(r+t_0,z_i;\mu^*\right),\quad i=1,2,\ r\in(0,s),\end{aligned}\qquad(4.4.6)$$

where in the last step, we have used the fact that $H(t;\gamma)$ is a nondecreasing function; see Lemma 2.6 of [CK19]. Therefore, by denoting $C_s := H\left(s+t_0; 32qL_b^2\right)$,

$$
\begin{aligned}
\|Y_v(s,y)\|_q^2 &\leqslant CC_s \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\, f(z_1-z_2) \prod_{i=1}^2 \Big(G(s-r, y-z_i) J_0\left(r+t_0, z_i; \mu^*\right) \Big) \\
&= CC_s \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2) \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2 \\
&\quad \times f(z_1-z_2) \prod_{i=1}^2 \Big(G(s-r, y-z_i) G(r+t_0, z_i-\sigma_i) \Big) \\
&= CC_s \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2)\, G(s+t_0, y-\sigma_1) G(s+t_0, y-\sigma_2) \\
&\quad \times \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2 f(z_1-z_2) \prod_{i=1}^2 G\left(\frac{(r+t_0)(s-r)}{s+t_0}, z_i - \sigma_i \frac{r+t_0}{s+t_0} - \frac{s-r}{s+t_0} y \right) \\
&\leqslant CC_s (2\pi)^{-2d} \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2)\, G(s+t_0, y-\sigma_1) G(s+t_0, y-\sigma_2) \\
&\quad \times \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{(r+t_0)(s-r)}{s+t_0} |\xi|^2 \right),
\end{aligned}
$$

where we have applied (4.1.20) and Plancherel's theorem. Hence,

$$
\|Y_v(s,y)\|_q^2 \leqslant CC_s (2\pi)^{-2d} J_0^2\left(s+t_0, y; \mu^*\right) \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{(r+t_0)(s-r)}{s+t_0} |\xi|^2 \right).
$$

Because the function

$$
t_0 \mapsto \frac{r+t_0}{s+t_0} = 1 - \frac{s-r}{s+t_0} \quad \text{for } t_0 > 0,
$$

is nondecreasing in t_0 whenever $s > r > 0$, we can replace the two appearances of t_0 in the exponent of the above inequality by zero to see that

$$
\|Y_v(s,y)\|_q^2 \leqslant CC_s (2\pi)^{-2d} J_0^2(s+t_0, y; \mu^*) \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{r(s-r)}{s} |\xi|^2 \right). \tag{4.4.7}
$$

Furthermore, by symmetry of $r(s-r)/s$ and the fact that $r(s-r)/s \geqslant r/2$ for all $r \in [0, s/2]$, we see that the above double integral is bounded by

$$
\begin{aligned}
&\leqslant 2\int_0^{s/2} \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{r}{2}|\xi|^2\right)\\
&= 2^{2(1-\alpha)} \int_0^{s/4} \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right)\\
&\leqslant 2^{2(1-\alpha)} \int_0^{s} \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right)\\
&= 2^{2(1-\alpha)} \mathcal{H}_\alpha(s).
\end{aligned}
$$

Plugging the above bound back to (4.4.7) proves (4.4.3).

(2–3) Part (2) is a direct consequence of Theorem 4.3.2. It remains to prove (4.4.4). An application of Minkowski's inequality shows that

$$
\mathbb{E}\left(\|Y_v(s,\cdot)\|_\rho^q\right) = \left\| \int_{\mathbb{R}^d} Y_v(s,y)^2 \rho(y)\mathrm{d}y \right\|_{q/2}^{q/2} \leqslant \left(\int_{\mathbb{R}^d} \|Y_v(s,y)\|_q^2\, \rho(y)\mathrm{d}y \right)^{q/2}. \tag{4.4.8}
$$

By the definition of $\mathcal{G}_\rho(t;\mu)$ in (4.1.15) and by (4.4.3), we see that

$$
\int_{\mathbb{R}^d} \|Y_v(s,y)\|_q^2\; \rho(y)\mathrm{d}y \leqslant C\, \mathcal{G}_\rho(s+t_0;\mu)\mathcal{H}_\alpha(s).
$$

Plugging the above expression to the far right-hand side of (4.4.8) proves (4.4.4). Finally, the finiteness of the upper bound in (4.4.4) is guaranteed by condition (4.4.5). This completes the proof of Lemma 4.4.2. □

Lemma 4.4.4 (Factorization lemma)**.** *Suppose that μ — the initial condition for u — satisfies (4.3.1) and $\widehat{f}$ satisfies Dalang's condition (4.1.11). Assume that condition (4.2.6) is satisfied for some $\alpha \in (0, 1/2)$. Fix an arbitrary $t_0 \geqslant 0$. Let $v(t,x)$ be the solution to the restarted SHE (4.3.6) and $\dot{W}_{t_0}$ be the time-shifted noise (see (4.3.7)) when $t_0 > 0$ and let $v = u$ when $t_0 = 0$. Then the following factorization holds*

$$
\frac{\sin(\alpha\pi)}{\pi} \int_0^t (t-s)^{\alpha-1} \left[G(t-s,\cdot) * Y_v(s,\cdot)\right](x)\mathrm{d}s = \int_0^t \int_{\mathbb{R}^d} G(t-r,x-z) b(z,v(r,z)) W_{t_0}(\mathrm{d}r,\mathrm{d}z),
$$

for all $t > 0$ *and* $x \in \mathbb{R}^d$. *As a consequence,*

$$v(t,x) = [G(t,\cdot) * u(t_0,\cdot;\mu)](x) + \frac{\sin(\alpha\pi)}{\pi}[F_\alpha Y_v](t,x), \quad \textit{for all } t > 0 \textit{ and } x \in \mathbb{R}^d. \tag{4.4.9}$$

Proof. The lemma is straightforward provided that one can switch the orders of stochastic and ordinary integrals:

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1}[G(t-s,\cdot) * Y_v(s,\cdot)](x)\mathrm{d}s \\
&= \int_0^t \mathrm{d}s\,(t-s)^{\alpha-1}\int_{\mathbb{R}^d}\mathrm{d}y\, G(t-s,x-y) \\
&\quad \times \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(s-r,y-z) b(z,v(r,z)) W_{t_0}(\mathrm{d}r,\mathrm{d}z) \\
&= \int_0^t \mathrm{d}s\,(t-s)^{\alpha-1}\int_0^s\int_{\mathbb{R}^d}(s-r)^{-\alpha}G(t-r,x-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z) && (4.4.10)\\
&= \int_0^t\int_{\mathbb{R}^d} W(\mathrm{d}r,\mathrm{d}z) G(t-r,x-z) b(z,v(r,z)) \int_r^t \mathrm{d}s\,(s-r)^{-\alpha}(t-s)^{\alpha-1} && (4.4.11)\\
&= \frac{\pi}{\sin(\alpha\pi)}\int_0^t\int_{\mathbb{R}^d} G(t-r,x-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z),
\end{aligned}$$

where the last step is the *Beta integral* which requires that $\alpha \in (0,1)$. It remains to justify the two applications of the stochastic Fubini's theorem (see Theorem 5.30 of Chapter one in [Dal+09], or also [Wal86] or Theorem 4.33 of [DZ14]) in (4.4.10) and (4.4.11) in the following two steps.

Step 1. In this step, we justify the change of orders in (4.4.10). Note that t, x and s are fixed. It suffices to prove the following condition:

$$\begin{aligned}
I_1 := &\int_{\mathbb{R}^d}\mathrm{d}y\, G(t-s,x-y)\int_0^s \mathrm{d}r\,(s-r)^{-2\alpha}\iint_{\mathbb{R}^{2d}}\mathrm{d}z_1\mathrm{d}z_2 \\
&\times f(z_1-z_2)\left(\prod_{i=1}^2 G(s-r,y-z_i)\right)\mathbb{E}\left(\prod_{i=1}^2 b(z_i,v(r,z_i))\right) \\
= &\int_{\mathbb{R}^d}\mathrm{d}y\, G(t-s,x-y)\,\|Y_v(s,y)\|_2^2 \\
< &+\infty.
\end{aligned}$$

But this follows immediately from (4.4.3). Indeed

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{d}y\, G(t-s,x-y)\, \|Y_v(s,y)\|_2^2 \leqslant & C \int_{\mathbb{R}^d} \mathrm{d}y\, G(t-s,x-y) J_0^2(s+t_0,y;\mu^*)\mathcal{H}_\alpha(s) \\
= & C\mathcal{H}_\alpha(s) \int_{\mathbb{R}^d} \mathrm{d}y\, G(t-s,x-y) \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}z_1)\mu^*(\mathrm{d}z_2) \\
& \times G\left(s+t_0, y-z_1\right) G\left(s+t_0, y-z_2\right).
\end{aligned}
$$

Now we bound the three heat kernels using (4.1.20) as follows:

$$
\begin{aligned}
& G(t-s,x-y) \prod_{i=1}^2 G(s+t_0,y-z_i) \\
& = \frac{G\left(2(t-s),x-y\right)^2}{G\left(4(t-s,0\right)} \prod_{i=1}^2 G\left(s+t_0,y-z_i\right) \\
& \leqslant 2^d \frac{G\left(2(t-s),x-y\right)^2}{G\left(4(t-s),0\right)} \prod_{i=1}^2 G\left(2s+2t_0,y-z_i\right) \\
& = 2^d[4(t-s)]^{d/2} \prod_{i=1}^2 \Big[G\left(2s+2t_0,y-z_i\right) G\left(2(t-s),x-y\right) \Big] \\
& = 2^{2d}(t-s)^{d/2} \prod_{i=1}^2 \left[G\left(2(t+t_0),x-z_i\right) G\left(\frac{2(t-s)(s+t_0)}{t+t_0}, y-\frac{s+t_0}{t+t_0}(x-z_i)\right)\right] \\
& \leqslant 2^{2d}(t-s)^{d/2} \prod_{i=1}^2 \left[G\left(2(t+t_0),x-z_i\right) G\left(\frac{2(t-s)(s+t_0)}{t+t_0}, 0\right)\right] \\
& \leqslant C_{t,s,t_0} \prod_{i=1}^2 G\left(2(t+t_0),x-z_i\right).
\end{aligned}
$$

Therefore, $I_1 \leqslant C_{t,s,t_0}\, \mathcal{H}_\alpha(s)\, J_0^2\left(2(t+t_0),x;\mu^*\right) < \infty$.

Step 2. Similarly, as for (4.4.11), we need to show that

$$
\begin{aligned}
I_2 := & \int_0^t \mathrm{d}s\, (t-s)^{\alpha-1} \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2 \\
& \times f(z_1-z_2) \left(\prod_{i=1}^2 G(t-r,x-z_i)\right) \mathbb{E}\left(\prod_{i=1}^2 b(z_i,v(r,z_i))\right) < \infty.
\end{aligned}
$$

By the Cauchy Schwartz inequality, (4.4.6) and because $\alpha \in (0, 1/2)$,

$$
\begin{aligned}
I_2 \leqslant & C \int_0^t \mathrm{d}s\, (t-s)^{\alpha-1} \int_0^s \mathrm{d}r\, (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\, f(z_1 - z_2) \prod_{i=1}^2 \Big(G(t-r, x-z_i) J_0\left(r+t_0, z_i; \mu^*\right) \Big) \\
= & C' \int_0^t \mathrm{d}r\, (t-r)^{-\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\, f(z_1 - z_2) \prod_{i=1}^2 \Big(G(t-r, x-z_i) J_0\left(r+t_0, z_i; \mu^*\right) \Big).
\end{aligned}
$$

Now by the same arguments as those leading to (4.4.3) (with 2α there replaced by α), we see that

$$
I_2 \leqslant C J_0^2\left(t, x; \mu^*\right) \int_0^t \mathrm{d}r\, r^{-\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{r(t-r)}{t} |\xi|^2 \right),
$$

which is finite by (4.2.6) where we replace α with $\alpha/2$ and repeat the same steps right after (4.4.7). This completes the proof of Lemma 4.4.4. ☐

Finally, we characterize conditions (4.1.13) and (4.2.6) in the following lemma:

Lemma 4.4.5. *For all $\alpha \in (0, 1/2]$, we have the following properties:*

(1) $(2\pi)^{-d} \mathcal{H}_\alpha(t) \leqslant \Gamma(1-2\alpha) \Upsilon_{2\alpha}(0)$ *for all $t > 0$ and hence*

$$
\Upsilon_{2\alpha}(0) < \infty \quad \Longrightarrow \quad \mathcal{H}_\alpha(t) < \infty \quad \text{for all } t > 0; \tag{4.4.12}
$$

(2) $\lim_{t\to\infty} (2\pi)^{-d} \mathcal{H}_\alpha(t) = \Gamma(1-2\alpha) \Upsilon_{2\alpha}(0)$*;*

(3) if $\Upsilon(0) < \infty$, then the reverse implication of (4.4.12) holds.

Proof. We only need to consider the case when $\alpha > 0$. It is clear that the function $\mathcal{H}_\alpha(t)$ is nondecreasing. Hence, (2) implies (1). As for (2), by Fubini's theorem, we see that

$$
\lim_{t\to\infty} \mathcal{H}_\alpha(t) = \int_0^\infty \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) e^{-r|\xi|^2} = \Gamma(1-2\alpha) \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-2\alpha)}} = C \Upsilon_{2\alpha}(0), \tag{4.4.13}
$$

with $C := \Gamma(1-2\alpha)(2\pi)^d$. As for (3), for any $t>0$ we split the $\mathrm{d}r$ integral of (4.4.13) into two parts and see that

$$
\begin{aligned}
C\Upsilon_{2\alpha}(0) &= \int_0^\infty \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right) = \mathcal{H}_\alpha(t) + I_\alpha(t), \quad \text{with} \\
I_\alpha(t) &= \int_t^\infty \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right).
\end{aligned}
$$

Notice that

$$
I_\alpha(t) \leqslant t^{-2\alpha} \int_t^\infty \mathrm{d}r \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) e^{-r|\xi|^2} = t^{-2\alpha} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2} e^{-t|\xi|^2} \leqslant t^{-2\alpha} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2} = \frac{(2\pi)^d}{t^{2\alpha}} \Upsilon(0).
$$

Therefore,

$$
\Upsilon_{2\alpha}(0) \leqslant \frac{\mathcal{H}_\alpha(t)}{(2\pi)^d \Gamma(1-2\alpha)} + \frac{\Upsilon(0)}{\Gamma(1-2\alpha)\, t^{2\alpha}} < \infty, \quad \text{for all } t>0,
$$

which proves (3). □

4.5 Tightness and Construction – Proof of Theorem 4.1.3

4.5.1 Proof of part (a) of Theorem 4.1.3

We are now ready to prove part (a) of Theorem 4.1.3.

Proof of Theorem 4.1.3 (a). In this proof, $u(t,x)$ refers to $u(t,x;\mu)$. Fix $\tau>0$ and let $t_0 = \tau/2$. Throughout the proof, we have $t \geqslant \tau$. Let v be the solution to (4.3.6) that is restarted from $t-t_0$. Then (see Figure 4.3 for an illustration)

$$
v_t(s,x) \stackrel{\mathcal{L}}{=} u\left(s,x;u\left(t-t_0,\cdot;\mu\right)\right) \quad \text{for } s \geqslant 0 \text{ and } t \geqslant \tau. \tag{4.5.1}
$$

According to Assumption (i), we can choose and fix some admissible weight function $\tilde{\rho}$ such that (4.1.17) is satisfied. Hence, by Proposition 4.2.2, the following set

$$
\mathscr{K}_1(\Lambda) := \left\{ \left(G(t_0,\cdot) * y(\cdot)\right)(x) : \; \|y\|_{\tilde{\rho}} \leqslant \Lambda \right\} \quad \text{with } \Lambda > 0
$$

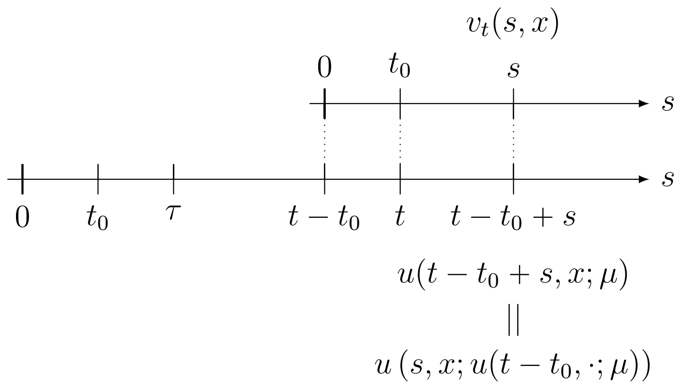


Figure 4.3: An illustration for the restarted SHE in (4.5.1).

is relatively compact in $L^2_\rho(\mathbb{R}^d)$.

Assumption (iii), i.e., (4.1.10), implies that the interval $(64L_b^2\Upsilon(0), 1/2)$ is not empty. Moreover, Assumption (iv), i.e., (4.1.13), guarantees that there exists a constant α in this interval, namely, $64L_b^2\Upsilon(0) < \alpha < 1/2$, such that (4.1.13) holds with α replaced by 2α, i.e., $\Upsilon_{2\alpha}(0) < \infty$. Now we can apply part (3) of Lemma 4.4.5, thanks to (4.1.10a), to see that $\Upsilon_{2\alpha}(0) < \infty$ if and only if (4.2.6) holds. Therefore, both Lemmas 4.4.2 and 4.4.4 (more precisely part (3) of Lemma 4.4.2) are applicable. In particular, Lemma 4.4.4 ensures that the following factorization is well defined:

$$v(t_0,x) = \Big(G(t_0,\cdot) * u(t-t_0,\cdot)\Big)(x) + \frac{\sin(\alpha\pi)}{\pi}\left[F_\alpha Y_v\right](t_0,x). \tag{4.5.2}$$

Part (3) of Lemma 4.4.2 shows that for any q in the following range,

$$64L_b^2\Upsilon(0) < \frac{1}{q} < \alpha < \frac{1}{2} \qquad \left(\text{or equivalently} \quad 2 < \frac{1}{\alpha} < q < \frac{1}{64L_b^2\Upsilon(0)}\right), \tag{4.5.3}$$

we can apply Proposition 4.4.1 to see that the following set

$$\mathscr{K}_2(\Lambda) := \left\{(F_\alpha h)(t_0,x) : \|h\|_{L^q\left((0,t_0);L^2_\rho(\mathbb{R}^d)\right)} \leqslant \Lambda\right\} \quad \text{with } \Lambda > 0$$

is relatively compact in $L^2_\rho(\mathbb{R}^d)$. Now for any $\Lambda > 0$, define the set $\mathscr{K}(\Lambda)$ as

$$
\begin{aligned}
\mathscr{K}(\Lambda) &:= \mathscr{K}_1(\Lambda) + \mathscr{K}_2(\Lambda) \\
&= \left\{ \big(G(t_0,\cdot) * y(\cdot)\big)(x) + (F_\alpha h)(t_0,x) : \; \|y\|_{\tilde{\rho}} \leqslant \Lambda \quad \text{and} \quad \|h\|_{L^q\left((0,t_0);L^2_{\tilde{\rho}}(\mathbb{R}^d)\right)} \leqslant \Lambda \right\}.
\end{aligned}
$$

Notice that from the factorization formula (4.5.2),

$$
\begin{aligned}
\mathbb{P}\left[v(t_0,\cdot) \notin \mathscr{K}(\Lambda)\right] &\leqslant \mathbb{P}\left[\left(\int_0^{t_0} \|Y_v(s,\cdot)\|_{\tilde{\rho}}^q \,\mathrm{d}s\right)^{1/q} > \frac{\pi\Lambda}{\sin(\alpha\pi)}\right] + \mathbb{P}\left[\|u(t-t_0,\cdot)\|_{\tilde{\rho}} > \Lambda\right] \\
&=: I_1 + I_2.
\end{aligned}
$$

By Chebyshev's inequality and (4.1.16), we see that

$$
I_2 \leqslant \frac{1}{\Lambda^2}\mathbb{E}\left(\|u(t-t_0,\cdot)\|_{\tilde{\rho}}^2\right) \leqslant \frac{1}{\Lambda^2}\mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu\right).
$$

Because $\mathcal{G}_{\tilde{\rho}}(t;\mu)$ is a continuous function for $t > 0$, and because it is also bounded at infinity, thanks to Assumption (ii) (see (4.1.18)), we have that

$$
\mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu\right) \leqslant \sup_{t\geqslant\tau} \mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu\right) = \sup_{t\geqslant t_0} \mathcal{G}_{\tilde{\rho}}\left(t;\mu\right) < \infty. \tag{4.5.4}
$$

Therefore, we can bound I_2 from above with a constant that does not depend on $t \geqslant \tau$, namely,

$$
I_2 \leqslant \frac{1}{\Lambda^2} \sup_{t\geqslant t_0} \mathcal{G}_{\tilde{\rho}}\left(t;\mu\right) < \infty.
$$

As for I_1, with the choice of α and q in (4.5.3), one can apply Chebyshev's inequality and part (3) of Lemma 4.4.2 to see that

$$
I_1 \leqslant \frac{\sin^q(\alpha\pi)}{\pi^q\Lambda^q}\mathbb{E}\int_0^{t_0} \|Y_v(s,\cdot)\|_{\tilde{\rho}}^q \,\mathrm{d}s \leqslant \frac{\sin^q(\alpha\pi)}{\pi^q\Lambda^q}\Theta \int_0^{t_0} \left(\mathcal{G}_{\tilde{\rho}}(s+t-t_0;\mu)\mathcal{H}_\alpha(s)\right)^{q/2} \mathrm{d}s,
$$

where the constant Θ does not depend on t. As we have seen from above, since $\Upsilon_{2\alpha}(0) < \infty$, we can apply Lemma 4.4.5 to bound $\mathcal{H}_\alpha(s)$ from above by the finite bound: $(2\pi)^d\,\Gamma(1-2\alpha)\Upsilon_{2\alpha}(0)$.

Hence, together with (4.5.4), we obtain the following upper bound for I_1 that is uniform in $t \geqslant \tau$:

$$I_1 \leqslant \frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma\left(1-2\alpha\right)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\geqslant t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0).$$

Combining these two upper bounds, we see that

$$\mathbb{P}\left[v(t_0,\cdot)\notin\mathscr{K}(\Lambda)\right] \leqslant \frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma\left(1-2\alpha\right)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\geqslant t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0)+\frac{1}{\Lambda^2}\sup_{t\geqslant t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu\right)<\infty,$$

with the upper bound holding uniformly for all $t \geqslant \tau$. Hence, for any $\epsilon > 0$, by choosing $\Lambda > 0$ big enough such that

$$\frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma\left(1-2\alpha\right)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\geqslant t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0)+\frac{1}{\Lambda^2}\sup_{t\geqslant t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu\right)<\epsilon,$$

we can ensure that

$$\mathbb{P}\left(u(t,\cdot)\in\mathscr{K}(\Lambda)\right)=\mathbb{P}\left(v(t_0,\cdot)\in\mathscr{K}(\Lambda)\right)\geqslant 1-\epsilon,\quad \text{for all } t\geqslant\tau,$$

which proves part (a) of Theorem 4.1.3. □

4.5.2 Proof of part (b) of Theorem 4.1.3

Proof. Fix an arbitrary $\tau > 0$ and denote

$$U(T) := \frac{1}{T}\int_\tau^{T+\tau}\mathcal{L}\left(u(t,\cdot;\mu)\right)\,\mathrm{d}t,\quad T>0.$$

We claim that the family of laws $U(T,\cdot)$ for $T > 0$ is tight in $L^2_\rho(\mathbb{R}^d)$. Indeed, for any $\epsilon \in (0,1)$, by part (a), there exists a compact set $\mathcal{K} \in L^2_\rho(\mathbb{R}^d)$ such that (4.1.19) holds. This implies that

$$U(T)\left(\mathcal{K}\right)=\frac{1}{T}\int_\tau^{T+\tau}\mathscr{L}(u(t,\cdot;\mu))\left(\mathcal{K}\right)\,\mathrm{d}t\geqslant\frac{1}{T}\int_\tau^{T+\tau}(1-\epsilon)\,\mathrm{d}t=1-\epsilon,\quad\text{for all } T>0.$$

Let $\{T_n\}_{n\in\mathbb{N}}$ be any deterministic sequence such that $T_n \uparrow \infty$. Since $\{U(T_n)\}_{n\geqslant 1}$ is a tight sequence of measures, then there exists a subsequence $\{U(T_{n_m})\}_{m\geqslant 1}$ that converges weakly to a measure, η, on $L^2_\rho(\mathbb{R}^d)$ (e.g. see [Bil99, Theorem 5.1]). Then one can apply the Krylov-Bogoliubov existence theorem (see, e.g., [DZ14, Theorem 11.7]) to conclude that the measure η is an invariant measure for $\mathscr{L}(u(t,\cdot;\mu))$, $t \geqslant \tau$. Finally, since τ can be arbitrarily close to zero, one can conclude part (b) of Theorem 4.1.3. $\square$

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