

**FORBIDDING MONCHROMATIC AND RAINBOW CYCLES AND
FAMILIES OF CYCLES**

by

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ABSTRACT

In this dissertation, we avoid certain cycles and families of cycles in complete graphs. Introduced by Axenovich and Choi [2], the mixed Ramsey spectrum, $\text{MRS}(K_n; F, H)$, is the set of numbers k such that for some k -edge coloring of K_n there is neither a monochromatic copy of $F \subseteq K_n$ nor a rainbow copy of $H \subseteq K_n$.

The values for the following spectrums are shown. Let m and n be an integers. For $n > 1$, $\text{MRS}(K_n; K_3, K_3) = \{g(n), \dots, n - 1\}$, in which $g(n) \in \{\lceil 2 \log_5 n \rceil, \lceil 2 \log_5 n \rceil + 1\}$. For all m and n , where $3 \leq m \leq n$, $\{n + 2 - m, \dots, n + 1 - m + \binom{m-1}{2}\} \subseteq \text{MRS}(K_n; C_m, C_m)$. For $n \geq 4$, $\max \text{MRS}(K_n; C_4, C_4) = n$. Note: $\max \text{MRS}(K_n; C_4, C_4) = n$ was a result shown by Axenovich and Choi [1], we provide an alternate proof.

We extended the definition of the mixed Ramsey spectrum from graphs to families of graphs. For families of graphs \mathcal{F}, \mathcal{H} , $\text{MRS}(K_n; \mathcal{F}, \mathcal{H})$ is the set of numbers k such that for some k -edge coloring of K_n , there is no monochromatic copy of any $F \subseteq \mathcal{F}$ in K_n nor any rainbow copy of any $H \subseteq \mathcal{H}$ in K_n . It is shown that for all $n \geq 2$, $\text{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\}) = \{\lceil \log_2 n \rceil, \dots, n - 1\}$.

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Chapter 1. Introduction

All graphs considered in this work are finite, undirected, and simple.

A **k -edge coloring**, c , of the edge set, $E(G)$, of a graph G is a surjective mapping $c: E(G) \rightarrow [k]$ where $[k] = \{1, 2, \dots, k\}$; $c(uw)$ is the *color* of edge uw . For any subgraph $H \subseteq G$, $c[H]$ denotes the set of colors used to label the edges of H .

An edge colored graph is called **monochromatic** if all its edges have the same color. In contrast, an edge colored graph is called **rainbow** (elsewhere referred to as polychromatic or totally multicolored) if no color appears more than once.

A **forbidden subgraph** is a subgraph which cannot appear in a graph, G , and satisfies certain conditions. Forbidden subgraphs are of interest because they can be used to characterize certain graph properties. Some subgraphs are forbidden in the context of edge coloring. In other words, if a graph contains a forbidden subgraph, that subgraph cannot be edge colored with respect to a given coloring.

The **multicolor Ramsey number** (or simply Ramsey number), $R_k(F)$, defined as the smallest n such that for every k -edge coloring of K_n there is a monochromatic copy of $F \subseteq K_n$ [7]. We argue that the many other variants of Ramsey theory are equally thought provoking and valuable as the classical version.

Defined by Axenovich and Choi [2], the **mixed Ramsey spectrum**, $\text{MRS}(K_n; F, H)$, is the set of numbers k such that for some k -edge coloring of K_n , there is neither a monochromatic copy of $F \subseteq K_n$ nor a rainbow copy of $H \subseteq K_n$. Classical Ramsey problems are ostensibly intractable and elusive. Turning the problem inside out into a mixed Ramsey problem often gives us more tangible results and expedites research. Notice that unlike classical Ramsey problems, $\text{MRS}(K_n; F, H)$ *fixes* n and *finds* k .

The values in $\text{MRS}(K_n; F, H)$ are closely related to some more familiar types of Ramsey numbers [1]. Assuming $\text{MRS}(K_n; F, H)$ is nonempty, the value of $\max \text{MRS}(K_n; F, H)$ is less than or equal to the **anti-Ramsey number**, $\text{AR}(n; H)$, which is the maximum number of colors, k , such that there exists a k -edge coloring of K_n with no rainbow copy of $H \subseteq K_n$. Secondly, the value of $\min \text{MRS}(K_n; F, H)$ is related to the multicolor Ramsey number as

previously defined. $\min \text{MRS}(K_n; F, H)$ is also related to the **Gallai-Ramsey number**, $gr_k(G : H)$, which in [11], the authors define as the minimum integer N such that for all $n \geq N$, every k -edge coloring of K_n contains either a rainbow copy of G or a monochromatic copy of H . The following result shows the benefit in finding the maximum and minimum of a given mixed Ramsey spectrum where the forbidden monochromatic graph is not a star and the forbidden rainbow subgraph has minimum degree of at least 2.

Theorem 1.1 (Axenovich and Choi [2]): *Let F be a graph that is not a star, and let H be a graph with minimum degree at least two. Then, for any natural number n , $\text{MRS}(K_n; F, H)$ is a set of consecutive integers.*

The content of this dissertation is divided into four chapters. The last three chapters are separated into research findings from three different papers, all under the same umbrella of mixed Ramsey problems. The recent findings in chapter 2 were published in the "International Journal of Mathematics and Computer Science" by authors Derrick DeMars and Peter Johnson [5]. Several decades earlier (roughly 1976 - 1990), Vitaly Voloshin [14] was developing his ideas about mixed hypergraphs and their proper colorings. A mixed **hypergraph** is a triple $\mathcal{H} = (V; C, D)$ in which V , the set of vertices of \mathcal{H} , is a non-empty set and $C, D \subseteq 2^V$ are sets of subsets of V . These subsets are *hyperedges*, or *edges*. A proper coloring of \mathcal{H} is a coloring of V such that no $c \in C$ is rainbow (that is, 2 different elements of c bear the same color), and no $d \in D$ is monochromatic (that is 2 different elements of d bear different colors). After his name sake, we referred to the set $\{k: \text{there is a proper coloring of } \mathcal{H} \text{ with exactly } k \text{ colors appearing}\} = \text{VSPEC}(\mathcal{H})$ as the **Voloshin spectrum** of \mathcal{H} .

In an important family of mixed hypergraphs, the vertices are edges of an ordinary graph and the hyperedges are edge sets of particular subgraphs. For simple graphs G, X, Y , we set $V = E(G)$, the edge set of G , $C = \{E(X') : X' \text{ is a subgraph of } G \text{ isomorphic to } X\}$, and $D = \{E(Y') : Y' \text{ is a subgraph of } G \text{ isomorphic to } Y\}$. Then a proper coloring of $\mathcal{H} = (V; C, D)$ is a coloring of G 's edges such that no copy of X in G is rainbow and no copy of Y in G is monochromatic. We will denote the Voloshin spectrum of \mathcal{H} by

$\text{VSPEC}'(G; X, Y)$. Chapter 2 investigates for each integer $n > 1$, what is the Voloshin spectrum $\text{VSPEC}'(K_n; K_3, K_3)$?

Notice that $\text{VSPEC}'(K_n; B, A) = \text{MRS}(K_n; A, B)$ where A is the forbidden monochromatic subgraph of K_n and B is the forbidden rainbow subgraph of K_n . We found $\text{MRS}(K_n; K_3, K_3) = \text{VSPEC}'(K_n; K_3, K_3)$ and the proof of this result is found in chapter 2.

Theorem 1.2 (DeMars and Johnson [5]): *For all $n \geq 3$, $\text{MRS}(K_n; K_3, K_3) = \{g(n), \dots, n-1\}$ where $g(n) \in \{\lceil 2 \log_5 n \rceil, \lceil 2 \log_5 n \rceil + 1\}$.*

It should be noted that the value of $\min \text{MRS}(K_n; K_3, K_3)$ was obtained from results in Chung and Graham's classic work [4].

Theorem 1.3 (Chung and Graham [4]): *Let $f(k)$ be the largest value of n such that it is possible to edge color K_n , with k or fewer colors, so that every copy of $K_3 \subseteq K_n$ is neither monochromatic nor rainbow. Then*

$$f(k) = \begin{cases} 5^{\frac{k}{2}} & \text{if } k \text{ is even} \\ 2 \cdot 5^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

In chapter 3 we switched notation from that of chapter 2 to that of chapter 1. In chapter 3 the conclusions of chapter 2 led us to inquire about $\text{MRS}(K_n; C_4, C_4)$ and more generally $\text{MRS}(K_n; C_m, C_m)$. Two of our findings, that are proved in this chapter 3, are listed below.

Theorem 1.4: *For integers m and n , where $3 \leq m \leq n$,*

$$\left\{ n+2-m, \dots, n+1-m + \binom{m-1}{2} \right\} \subseteq \text{MRS}(K_n; C_m, C_m)$$

C_4 Max Theorem 1.5: *For every integer $n \geq 4$,*

$$\max \text{MRS}(K_n; C_4, C_4) = n.$$

In chapter 4, we extend the definition of the mixed Ramsey spectrum to not just subgraphs but families of graphs as well. In chapter 4, we extend the definition of the mixed Ramsey spectrum to not just subgraphs but families of graphs as well. If \mathcal{F}, \mathcal{H} are families of graphs then $\text{MRS}(K_n; \mathcal{F}, \mathcal{H})$ is the set of numbers k such that for some k -edge coloring of K_n , there is no monochromatic copy of any $F \subseteq \mathcal{F}$ in K_n nor any rainbow copy of any $H \subseteq \mathcal{H}$ in K_n .

In chapter 4 we let \mathcal{F} be the family of odd cycles and \mathcal{H} to be the family of cycles and found the following result.

Corollary 1.6: $\text{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\}) = \{\lceil \log_2 n \rceil, \dots, n-1\}$.

In [10], the authors define an edge coloring for a graph G as **rainbow-cycle-forbidding** if no cycle in G is rainbow with respect to that coloring. They also define a **JL-coloring** as a rainbow-cycle-forbidding edge coloring for a given graph G on n vertices with c components in which the maximum possible number of colors, $n - c$, appear. By the main result in [10], JL-colorings forbid monochromatic odd cycles.

Edge colorings of complete graphs which forbid rainbow K_3 's are known as **Gallai colorings**. All Gallai colorings are rainbow-cycle-forbidding. In this dissertation we will adapt the construction found in [10] to obtain rainbow-cycle-forbidding edge colorings which also forbid monochromatic odd cycles.

Chapter 2. Forbidding monochromatic and rainbow K_3 's in complete graphs

1. INTRODUCTION

In 1983 Chung and Graham [4] obtained a wonderful result that would now be regarded as a “mixed Ramsey” theorem.

Theorem 2.1: *For each positive integer k let $f(k)$ be the largest integer n such that the edges of K_n can be colored with no more than k colors appearing so that each K_3 subgraph has exactly 2 colors appearing on its edges. (That is, no $K_3 \subseteq K_n$ is either monochromatic or rainbow.) Then,*

$$f(k) = \begin{cases} 5^{\frac{k}{2}} & \text{if } k \text{ is even} \\ 2 \cdot 5^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

Recall a proper coloring of $\mathcal{H} = (V; C, D)$ is a coloring of G 's edges such that no copy of X in G is rainbow and no copy of Y in G is monochromatic. We will denote the Voloshin spectrum of \mathcal{H} by $\text{VSPEC}'(G; X, Y)$.

Suppose that X and Y are graphs with $|E(Y)| > 1$ and k is a positive integer. Ramsey's theorem implies that for all n sufficiently large, depending on Y and k , for every edge coloring of K_n with less than or equal to k colors appearing, there must be a monochromatic copy of Y somewhere in K_n . Therefore, if X and Y are given, it is natural for Ramsey theorists to ask: “for each positive integer k , what is the largest $n = f_{X,Y}(k)$ such that $\{1, \dots, k\} \cap \text{VSPEC}'(K_n; X, Y) \neq \emptyset$?”

Of course, the question would not be posed in this way! But this is the first question answered by Chung and Graham in Theorem 2.1, in the case $X = Y = K_3$. Our question is: “for each integer $n > 1$, what is the Voloshin spectrum $\text{VSPEC}'(K_n; K_3, K_3)$?” We will begin by showing the smallest and largest element of $\text{VSPEC}'(K_n; K_3, K_3)$.

2. SMALLEST ELEMENT OF $\text{VSPEC}'(K_n; K_3, K_3)$ OR $(\text{MRS}(K_n; K_3, K_3))$

Lemma 2.2. *The smallest element of $\text{VSPEC}'(K_n; K_3, K_3)$ is either $\lceil 2 \log_5 n \rceil$ or $\lceil 2 \log_5 n \rceil + 1$.*

Proof. Let f be as in Theorem 2.1, let the smallest element of $\text{VSPEC}'(K_n; K_3, K_3)$ be denoted $g(n)$, and let \mathcal{H}_n denote the mixed hypergraph of which $\text{VSPEC}'(K_n; K_3, K_3)$ is the Voloshin spectrum.

Claim. $g(n)$ is the value of k satisfying $f(k-1) < n \leq f(k)$.

Proof. By Theorem 2.1, if $f(k-1) < n \leq f(k)$, then there is a proper coloring of \mathcal{H}_n with no more than k colors appearing, and there is no proper coloring of \mathcal{H}_n with $k-1$ or fewer colors appearing, so there must be a proper coloring of \mathcal{H}_n with exactly k colors appearing. Therefore $k \in \text{VSPEC}'(K_n; K_3, K_3)$, so $g(n) \leq k$.

On the other hand, if there is a proper coloring of \mathcal{H} with exactly $r \leq k-1$ colors appearing, then n would be $\leq f(k-1)$. Since $f(k-1) < n \leq f(k)$, it follows that there is no such r ; therefore k is the smallest element of $\text{VSPEC}'(K_n; K_3, K_3)$. ■

Suppose that $k = g(n)$ is even. By Theorem 2.1 and our previous Claim,

$$2 \cdot 5^{\frac{k-2}{2}} = f(k-1) < n \leq f(k) = 5^{\frac{k}{2}}$$

“Solving” for k , we obtain

$$2 \log_5 n \leq k < 2 \log_5 n + 2(1 - \log_5 2)$$

Because k is an integer and $2(1 - \log_5 2) < 2$, it follows that $k \in \{\lceil 2 \log_5 n \rceil, \lceil 2 \log_5 n \rceil + 1\}$.

When $k = g(n)$ is odd we obtain

$$1 - 2 \log_5 2 + 2 \log_5 n \leq k < 2 \log_5 n + 1$$

whence $k \in \{\lceil 2 \log_5 n \rceil, \lceil 2 \log_5 n \rceil + 1\}$. ■

Given n , how does one decide whether $k = g(n)$ is $\lceil 2 \log_5 n \rceil$ or $\lceil 2 \log_5 n \rceil + 1$? It is the value of k such that $f(k-1) < n \leq f(k)$.

Example 2.3: For instance, if $n = 19$, $\lceil 2 \log_5 19 \rceil = 4$, and we see that $10 = 2 \cdot 5^{\frac{3-1}{2}} < 19 \leq 5^{\frac{4}{2}} = 25$, so $g(19) = 4$. Now consider $n = 51$. Then $\lceil 2 \log_5 n \rceil = 5$. Clearly $51 \not\leq f(5) = 50$, so $g(51) = 6$.

Now we will find the largest element of $\text{VSPEC}'(K_n; K_3, K_3)$. The following is well known (see [10]), but we supply a proof for the reader's convenience.

3. LARGEST ELEMENT OF $\text{VSPEC}'(K_n; K_3, K_3)$ OR $(\text{MRS}(K_n; K_3, K_3))$

Lemma 2.4. *Suppose G is a simple connected graph on n vertices and $E(G)$ is colored with n or more colors appearing. Then there is a rainbow cycle in G with respect to this coloring.*

Proof. Choose n edges of G with different colors. Let H be the subgraph of G induced by these edges. Then H is a subgraph with n edges on no more than n vertices. So H contains a cycle and that cycle is rainbow. ■

Corollary 2.5: *The greatest number of colors with which the edges of a simple connected graph on n vertices can be colored so that there is no rainbow cycle is less than or equal to $n - 1$.*

The following theorem is proved in [10]; we will supply a short proof here.

Theorem 2.6: *If G is a connected simple graph on $n \geq 1$ vertices, then there is a rainbow-cycle-forbidding edge coloring of G with exactly $n - 1$ colors appearing.*

Proof. The proof will be by induction on n . Clearly the conclusion holds when $n = 1$.

Suppose that $n > 1$. Let T be a spanning tree in G . Take any $e \in E(T)$; $T - e$ is the disjoint union of two trees, T_1 and T_2 . Let $R = V(T_1)$, $S = V(T_2)$. Then R and S partition $V(G)$ and the induced subgraphs $G[R]$, $G[S]$ are connected, since each has a spanning connected subgraph.

By the induction hypothesis, if $X \in \{G[R], G[S]\}$ then $E(X)$ can be colored with $|V(X)| - 1$ colors appearing so that there are no rainbow cycles in X . We arrange for the sets of colors on the edges of $G[R]$, $G[S]$ to be disjoint. We complete the coloring of $E(G)$ by coloring the edges of the edge cut $[R, S] = \{f \in E(G) : \text{one end of } f \text{ is in } R, \text{ the other in } S\}$ with a color not appearing in $G[R] \cup G[S]$.

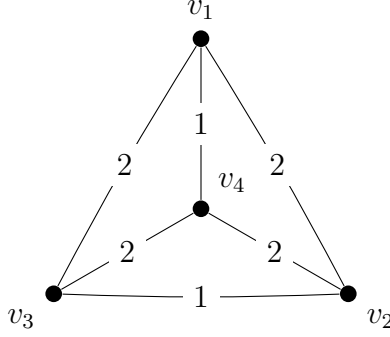


FIGURE 1. K_4 with a 2-edge-coloring such that every K_3 has exactly two colors.

Note that $e \in [R, S]$, so $[R, S]$ is non-empty. Therefore, the number of colors appearing on G is $|R| - 1 + |S| - 1 + 1 = |R| + |S| - 1 = n - 1$. There are no rainbow cycles in $G[R]$, nor in $G[S]$. If a cycle in G has a vertex in R and a vertex in S , then the cycle must have at least two edges in $[R, S]$, and so must have a color repeated on its edges. Thus the coloring of G is rainbow-cycle-forbidding. ■

This leads us to conclude the following.

Corollary 2.7: *For all $n \geq 1$, the largest element of $VSPEC'(K_n; K_3, K_3)$ is $n - 1$.*

4. $VSPEC'(K_n; K_3, K_3)$ OR $(MRS(K_n; K_3, K_3))$

Theorem 2.8: *For all $n \geq 1$, $VSPEC'(K_n; K_3, K_3) = \{k : g(n) \leq k \leq n - 1\}$ where $g(n) \in \{\lceil 2 \log_5 n \rceil, \lceil 2 \log_5 n \rceil + 1\}$.*

Proof. This proof is by induction on n . For $n = 1$, $VSPEC'(K_1; K_3, K_3) = \{0\}$. For $n = 2$, $VSPEC'(K_2; K_3, K_3) = \{1\}$. For $n = 3$, $VSPEC'(K_3; K_3, K_3) = \{2\}$. For $n = 4$, consider Figure 1 as it shows that the $\min(VSPEC'(K_4; K_3, K_3)) = 2$. This is consistent with Lemma 2.2, since $\lceil 2 \log_5 4 \rceil = 2$. Also, by Corollary 2.7, $n - 1 = 4 - 1 = 3$. So $VSPEC'(K_4; K_3, K_3) = \{2, 3\}$.

We will show that K_n is exactly k -edge-colorable (so that exactly 2 colors appear on each K_3 in K_n), when $g(n) < k < n - 1$ for $n > 4$. Note that $g(n - 1) \leq g(n) \leq k - 1$. Let $v \in V(K_n)$. By the induction hypothesis $K_n - v$ is exactly $(k - 1)$ -edge-colorable. Consider the join of $K_n - v$ and v , that is, K_n . Let all edges incident to v be colored

with a k th color. Clearly the resulting coloring of the edges of K_n with exactly k colors appearing admits neither monochromatic nor rainbow K_3 's. So $\text{VSPEC}'(K_n; K_3, K_3) = \{k: g(n) \leq k \leq n-1\}$ for $n \geq 1$. ■

Chapter 3. Forbidding monochromatic and rainbow C_4 's in complete graphs

5. PROOF OF THEOREM 1.4 AND RELATED COROLLARIES

Before showing the proof of Theorem 1.4, we will first define a join. The **join**, $G \vee H$, of two vertex-disjoint graphs G and H has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ [7].

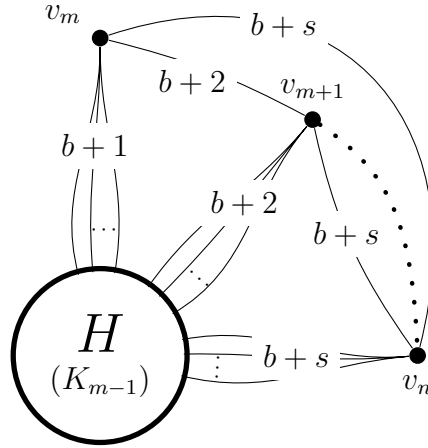


FIGURE 2. $K_{m-1} \vee v_m \vee \dots \vee v_n$; $s = n + 1 - m$

Proof of Theorem 1.4. Let $m, n \in \mathbb{Z}$ such that $3 \leq m \leq n$. Consider $K_{m-1} \subseteq K_n$; we will refer to this subgraph, K_{m-1} , as our base, H . In the following steps, we will construct an edge coloring on K_n forbidding monochromatic and rainbow C_m 's. First color the edges of H with color set B , where $B = \{1, \dots, b\}$ and $1 \leq b \leq \binom{m-1}{2}$ with all b colors appearing. Secondly, we will color all the edges joining H to v_m with color $b + 1$. Note that $V(K_n) \setminus V(H) = \{v_m, \dots, v_n\}$. In the coloring thus far, the complete graph $K_{m-1} \vee v_m$ contains neither monochromatic nor rainbow C_m 's.

Now concerning additional edges incident to vertices v_{m+1}, \dots, v_n . Color edges joining $K_{m-1} \vee v_m$ to v_{m+1} with color $b + 2$. Continue in this way: color the edges from $V(H) \cup \{v_m, \dots, v_{m+t}\}$ to v_{m+t+1} , $1 \leq t \leq n - m - 1$, with color $b + t + 2$. Notice, the value of $|c[K_n]|$ ranges from

- (1) $b + n + 1 - m = n + 2 - m$, when $b = 1$, to

$$(2) \ b + n + 1 - m = n + \binom{m-1}{2} + 1 - m, \text{ when } b = \binom{m-1}{2}.$$

To see that K_n , with any of these edge colorings, contains neither monochromatic nor rainbow m -cycles, consider an arbitrary $C \simeq C_m$ in K_n . Since $H \vee v_m$ contains neither monochromatic nor rainbow C_m 's, we may as well assume that $V(C)$ contains at least one vertex v_j , $m < j \leq n$.

Let $\alpha \in \{m+1, \dots, n\}$ be the largest integer such that $v_\alpha \in V(C)$. By the way the coloring c is defined, the two edges of C incident to v_α bear the same color, so C is not rainbow. Now, because $m = |V(C)| \geq 3$, C has an edge uw , $u, w \in V(C) \cup \{v_m, \dots, v_{\alpha-1}\}$; again, by appeal to the way the coloring is defined, the color on uw is different from the color shared by the two edges incident to v_α , so C is not monochromatic. ■

Theorem 3.1: For integers m and n , where $3 \leq m \leq n$,

$$k \in \text{MRS}(K_n; C_m, C_m) \implies$$

$$k + 1 \in \text{MRS}(K_{n+1}; C_m, C_m).$$

Proof. Let K_n be k -edge colored with all colors of $[k]$ appearing, with no monochromatic or rainbow $C_m \subseteq K_n$. Now join K_n to $v_{n+1} \in V(K_{n+1}) \setminus V(K_n)$ and color the new joining edges with $k+1$. With respect to the edge coloring, K_{n+1} clearly contains no monochromatic or rainbow $C_m \subseteq K_{n+1}$. Therefore $k+1 \in \text{MRS}(K_{n+1}; C_m, C_m)$. ■

Prior to beginning our work and unbeknownst to us, Axenovich and Choi had shown the following.

Proposition 3.2 (Axenovich and Choi [2]): *If a graph F is not a star, a graph H has minimum degree at least two, and $k \in \text{MRS}(K_n; F, H)$, then $k+1 \in \text{MRS}(K_{n+1}; F, H)$.*

Corollary 3.3: For integer n , where $n \geq 4$,

$$\{n-2, n-1, n\} \subseteq \text{MRS}(K_n; C_4, C_4).$$

6. CONCLUSIONS CONCERNING $\min \text{MRS}(K_n; C_4, C_4)$

We define the **extremal number** of a graph F as $\text{ex}(n; F) := \max\{|E(G)| : |V(G)| = n, F \not\subseteq G\}$ [3]. The multicolor Ramsey number of a graph F , $R_k(F)$, is defined as the smallest n such that for every k -edge coloring of K_n there is a monochromatic copy of $F \subseteq K_n$ [1].

Proposition 3.4: *For positive integers k^* and n , if*

$$k^* \text{ex}(n; F) < \binom{n}{2}$$

then

- (1) *in every k^* -edge coloring of K_n there is a color class with more than $\text{ex}(n; F)$ edges, so that color class contains a copy of F ,*
- (2) *$R_{k^*}(F) \leq n$, and*
- (3) *$k^* < \min \text{MRS}(K_n; F, F)$.*

Now consider when $F = C_4$.

Theorem 3.5 (Reiman [13]):

$$\text{ex}(n; C_4) < \frac{1}{4}n(1 + \sqrt{4n-3}) \tag{1}$$

Corollary 3.6: *Suppose that $n, k^* \geq 2$ are integers and $\frac{1}{k^*} \binom{n}{2} \geq \frac{n+n\sqrt{4n-3}}{4}$. Then*

- (1) *K_n cannot be k^* -edge colored forbidding monochromatic C_4 's,*
- (2) *$R_{k^*}(C_4) \leq n$, and*
- (3) *$k^* < \min \text{MRS}(K_n; C_4, C_4)$.*

Proof. This corollary follows easily by combining Proposition 3.4 and Theorem 3.5, with $F = C_4$ in the former. ■

Similar results were found by Axenovich and Choi.

Proposition 3.7 (Axenovich and Choi [2]): *We define $\eta(k; F, H) := \max\{n: \text{there is an edge coloring of } K_n \text{ forbidding monochromatic } F \text{ and rainbow } H \text{ using exactly } k \text{ colors}\}$.*

Then:

- (1) *If $\eta(k; F, H) = n$, then $\min \text{MRS}(K_n; F, H) \leq k$.*
- (2) *If $\eta(k; F, H) = n$ and $\eta(k'; F, H) < n$, for every $k' < k$, then $\min \text{MRS}(K_n; F, H) = k$.*
- (3) *In particular if η is strictly increasing in k , and $\eta(k; F, H) = n$, then $\min \text{MRS}(K_n; F, H) = k$.*
- (4) *$\eta(k; F, H) + 1 \leq R_k(F)$. Equality holds if there is a k -edge coloring of $K_{R_k(F)-1}$ with no monochromatic F and no rainbow H .*

7. PRELIMINARY LEMMAS FOR THE C_4 MAX THEOREM 1.5

Fact 1: *If G is connected, C is a cycle in G , and $e \in E(C)$, then $G - e$ is connected.*

Fact 2: *If G is connected and acyclic then $|V(T)| - 1 = |E(T)|$.*

Corollary 3.8: *If G is connected and $|V(G)| \leq |E(G)|$ then G contains a cycle subgraph.*

Corollary 3.9: *If G is a connected graph and if*

- (1) *$|V(G)| + 1 \leq |E(G)|$, then G contains at least two different cycle subgraphs, C_1 and C_2 . Furthermore,*
- (2) *$|V(G)| + 1 = |E(G)|$, then G satisfies the following: If e_1 is an edge on some cycle subgraph of $C_1 \subseteq G$, and e_2 is an edge (where $e_1 \neq e_2$) on some other cycle subgraph $C_2 \subseteq G - e_1$, then $G - (e_1 \cup e_2)$ is a tree.*

Erdős, Rubin, and Taylor [6] use the concept of the **generalized Θ graph**, denoted $\Theta_{r,s,t}$. The authors of [12] define $\Theta_{r,s,t}$ as “the graph consisting of two end vertices u and w meeting three internally vertex disjoint paths containing r , s , and t edges, respectively.”

Corollary 3.10: *If G is connected and $|V(G)| + 1 = |E(G)|$, then either*

- (1) *G contains two edge-disjoint cycles, or*
- (2) *G contains exactly three cycles, and together they form a Θ graph.*

Lemma 3.11. *Let K_n be edge colored with at least $n + 1$ colors appearing. Then K_n has a rainbow connected subgraph G such that $|V(G)| + 1 = |E(G)|$.*

Proof. Choose $n + 1$ edges of K_n bearing $n + 1$ distinct colors. Let H be the subgraph of K_n induced by these edges. Notice that H is rainbow. Then $n + 1 = |E(H)| = |V(K_n)| + 1 \geq |V(H)| + 1$.

Since $|E(H)| > |V(H)|$, some component H' of H , satisfies $|E(H')| > |V(H')|$. If $|E(H')| = |V(H')| + 1$, take $G = H'$. If $|E(H')| > |V(H')| + 1$, obtain G by removing edges from cycle subgraphs of H' (with each cycle after the first being a subgraph of the graph arrived at by the previous removal) to obtain a connected subgraph G such that $|E(G)| = |V(H')| + 1 = |V(G)| + 1$. Since H is rainbow, its subgraph G is rainbow. ■

Proposition 3.12: *Let K_n be edge colored with at least $n + 1$ colors appearing. Then K_n has a rainbow connected subgraph H bearing at least $|V(H)| + 1$ colors where either*

- (1) H contains two edge-disjoint cycles, or
- (2) H contains exactly three cycles and they form a Θ graph.

Rainbow C_3 Lemma 3.13: *3.13*

Suppose that K_n is edge colored, and for some integer m , there is a rainbow C_m subgraph of K_n . Then there is a rainbow $C_3 \subseteq K_n$ [9].

Proof. We can assume that $m > 3$. Let $e = uv$ be a chord of the rainbow C_m . There are two edge-disjoint paths, P and Q , on C_m with end vertices u and v . Because C_m is rainbow, the color of e in the edge coloring of K_n appears on at most one of P, Q . Therefore, either $P \cup e$ or $Q \cup e$ is a rainbow cycle of order $s < m$. If $s = 3$, we are done. Otherwise, we repeat the process in the argument preceding until we obtain a rainbow C_3 . ■

Lemma 3.14. *$k \notin \text{MRS}(K_4; C_4, C_4)$ for all $k \geq 5$.*

Proof. k -edge color K_4 where $k \geq 5$. $|E(K_4)| = 6$, so exactly one color is repeated exactly once if $n = 5$. Remove one edge, e , associated with said repeated color. Then $K_4 - e$ is rainbow. $K_4 - e$ contains contains a C_4 , which is necessarily a rainbow C_4 . ■

We define a **B_t graph** as the graph consisting of two edge disjoint triangles, joined by path of length $t - 1$.

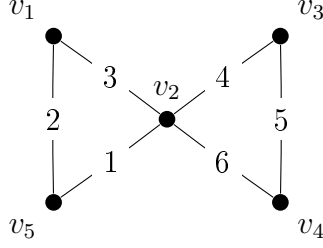


FIGURE 3. A rainbow subgraph, $B_1 \subseteq K_5$.

B_1 –Lemma 3.15: *If $n \geq 5$, K_n is edge colored, and $B_1 \subseteq K_n$ is rainbow, as depicted in Fig. 3, then $B_1 \cup v_1v_3$ contains a rainbow C_4 as a subgraph.*

Proof. Referring to Fig. 3, we can see that in order for there to be no rainbow $C_4 \subseteq B_1 \cup v_1v_3$, $c(v_1v_3) \in \{3, 5, 6\} \cap \{1, 2, 4\} = \emptyset$. Therefore, there is a rainbow $C_4 \subseteq B_1 \cup v_1v_3$. ■

Corollary 3.16: $k \notin \text{MRS}(K_5; C_4, C_4)$ for all $k \geq 6$.

Proof. k -edge color K_5 with $k \geq 6$ colors all appearing. By Proposition 3.12, we know there is a connected rainbow $H \subseteq K_5$ with $|V(H)| + 1$ colors appearing on H containing at least two cycles. Assume there are no C_4 's in H . Since $|V(H)| + 1 \leq 6$, either H contains a C_5 or H contains two C_3 's.

Case 1. If H contains a C_5 , then any edge in H bearing a sixth color (not on the C_5), must be a chord across C_5 , so H contains a rainbow C_4 .

Case 2. There are two C_3 's sharing either an edge or a vertex.

Case i. If the two C_3 's share an edge, then H contains a C_4 , which is rainbow because H is rainbow.

Case ii. If the two rainbow C_3 's share exactly one vertex, refer to B_1 –Lemma 3.15.

Corollary 3.17: *For $n \geq 5$, if we have K_n edge colored with any number of colors appearing, with no rainbow C_4 's, then we can conclude that there are no rainbow B_1 subgraphs in K_n .*

Corollary 3.18: $(n + i) \notin \text{MRS}(K_n; C_4, C_4)$ where $n \in \{4, 5\}$ and $i \geq 1$.

Proof. This claim combines those Lemma 4.7 and Corollary 4.9. ■

Given a nonempty set $S \subseteq V(G)$, the subgraph $G[S] \subseteq G$ is said to be **induced** by S if: $u, w \in S$ are adjacent in $G[S]$ if and only if u and w are adjacent in G [7].

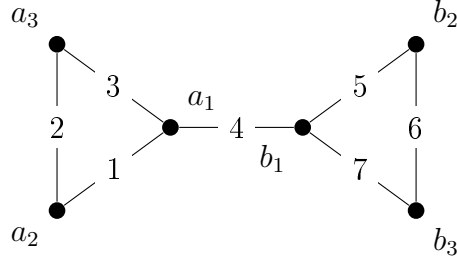


FIGURE 4. A rainbow subgraph, $B_2 \subseteq K_n$.

B_2 –Lemma 3.19: *Let $k \geq 7$, $n \geq 6$, and B_2 be the graph depicted in Fig. 4, and suppose that there is an edge coloring of K_n such that there are no rainbow copies of $C_4 \subseteq K_n$, but there is a rainbow copy of $B_2 \subseteq K_n$. Let B_2 be such a rainbow copy described as well as depicted in Fig. 4.*

Then every edge in $E(K_n[V(B_2)]) \setminus E(B_2)$ must be colored four (the color of edge a_1b_1). Consequently, there exists a monochromatic copy of C_4 in K_n , under this coloring.

Proof. Let c denote the coloring of $E(K_n)$. Since there are no rainbow C_4 's in K_n , $c(a_3b_2) \in \{3, 4, 5\} \cap \{2, 6, c(a_2b_3)\}$, and similarly $c(a_2b_3) \in \{1, 4, 7\} \cap \{2, 6, c(a_3b_2)\}$. It now follows that $c(a_3b_2) = c(a_2b_3) = 4$. Then $c(a_3b_1) \in \{1, 2, 4\} \cap \{6, 7, 4\} = \{4\}$. Given the symmetry of the situation, it follows that $c(a_3b_3) = c(a_1b_2) = c(a_1b_3) = c(a_2b_1) = c(a_2b_2) = 4$. So there exists a monochromatic $a_3b_2a_2b_3a_3 = C_4 \subseteq K_6$. ■

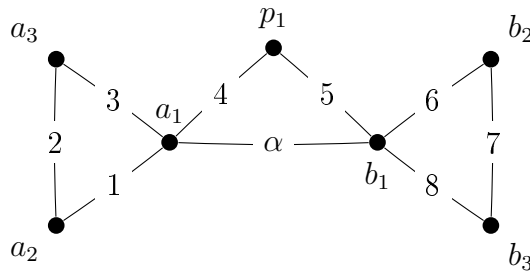


FIGURE 5. A rainbow subgraph $B_3 \subseteq K_n$ with an additional edge, a_1b_1 , where $c(a_1b_1) = \alpha$.

B_3 –Lemma 3.20: Assume $n > 6$. Edge color K_n . Let $B_3 \subseteq K_n$ be a connected rainbow subgraph containing two vertex disjoint C_3 's, where the C_3 's are connected by a path P and P contains exactly two edges. Then K_n contains a monochromatic or rainbow C_4 .

Proof. If $|V(P)| = 3$, as depicted in Fig. 5, we either have at least a rainbow $B_1 \subseteq K_n$ or a rainbow $B_2 \subseteq K_n$.

Case 1. If $\alpha \in \{1, 2, 3, 6, 7, 8\}$ then $a_1p_1b_1$ is rainbow, and either $a_1a_2a_3 \cup a_1p_1b_1$ is rainbow or $b_1b_2b_3 \cup a_1p_1b_1$ is rainbow. In both cases we have a rainbow $B_1 \subseteq K_n$ and should refer to B_1 –Lemma 3.15.

Case 2. If $\alpha \notin \{1, 2, 3, 6, 7, 8\}$ then we have a rainbow $B_2 \subseteq K_n$ and should refer to B_2 –Lemma 3.19. ■

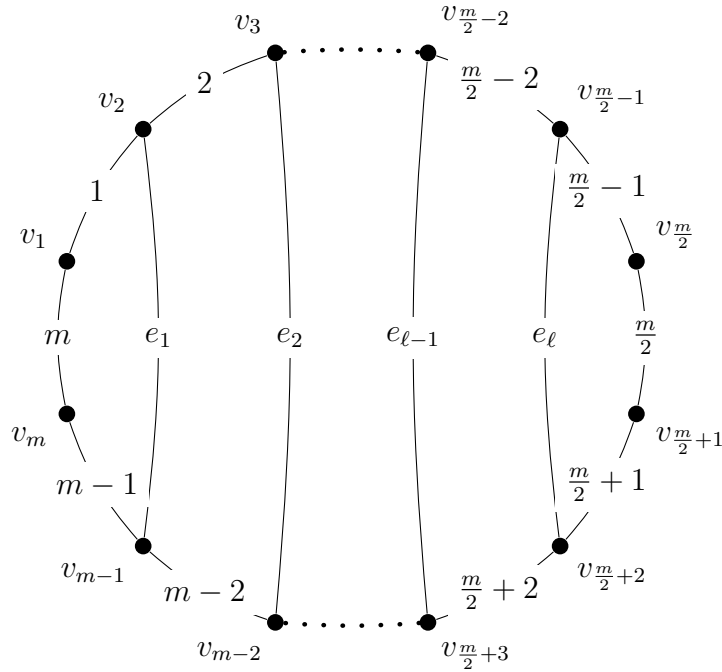


FIGURE 6. A rainbow $C_m \subseteq K_n$ (where m is even) with strategically placed chords, e_i , for integers $1 \leq i \leq \ell$.

Even Rainbow Cycle Lemma 3.21: 3.21

Assume $n \geq m \geq 6$ and m is even. Let K_n be edge colored and assume that $C_m \subseteq K_n$ is rainbow. Then there is a rainbow $C_4 \subseteq K_n$.

Proof. Let $C_m \subseteq K_n$ be m -edge colored with colors one through m , as depicted in Fig. 6. Suppose there is not a rainbow $C_4 \subseteq K_n$. Drop a chord, e_1 , across C_m from v_2 to v_{m-1} as depicted in Fig. 6. Since there is not a rainbow $C_4 \subseteq K_m$, $c(e_1) \in \{1, m-1, m\}$. If $m \geq 6$, draw another chord, e_2 , from v_3 to v_{m-2} : $c(e_2) \in \{c(e_1), 2, m-2\}$. If this process continues, there are $\frac{m-2-3}{2} + 1 = \ell$ chords drawn across C_m . So continue this process until a chord, e_ℓ , is drawn from $v_{\frac{m}{2}-1}$ to $v_{\frac{m}{2}+2}$. Because $c(e_\ell) \in \{c(e_{\ell-1}), \frac{m}{2}-2, \frac{m}{2}+2\} \cap \{\frac{m}{2}-1, \frac{m}{2}, \frac{m}{2}+1\}$ it follows that $c(e_\ell) = c(e_{\ell-1}) = \frac{m}{2}$. By a similar argument $c(e_{\ell-1}) = c(e_{\ell-2}) = \dots = \frac{m}{2}$. However, since $m \geq 6$, the C_4 on vertices $v_1v_2v_{m-1}v_m$ is rainbow. ■

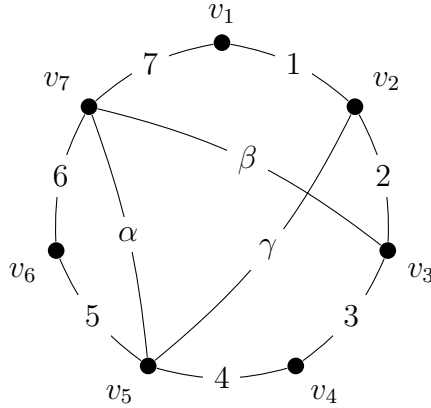


FIGURE 7. A rainbow $C_7 \subseteq K_n$ where $n \geq 7$ with chords colored α , β , and γ .

Rainbow C_7 Lemma 3.22: 3.22

Let $k, n \geq 7$ be integers and K_n is k -edge colored. If there is a rainbow $C_7 \subseteq K_n$, then there is a monochromatic $C_4 \subseteq K_n$.

Proof. Let $k, n \geq 7$ be integers. Suppose that K_n is k -edge colored and there is a rainbow C_7 as depicted in Fig. 7. Label the chords with colors α, β , and γ as depicted in Fig. 7.

We will examine the color of several C_4 's edges and assume that no C_4 is rainbow. Consider the cycles $v_1v_2v_3v_7$ and $v_3v_4v_5v_7$. We find $\beta \in \{1, 2, 7\}$ and $\beta \in \{3, 4, \alpha\}$. So $\beta = \alpha$ and $\alpha, \beta \in \{1, 2, 7\}$. Now examine cycles $v_2v_3v_4v_5$ and $v_1v_2v_5v_7$, this shows $\gamma \in \{2, 3, 4\}$ and $\gamma \in \{\alpha, 7, 1\}$, hence $\gamma = \alpha$ and $\alpha, \gamma \in \{2, 3, 4\}$. Since $\alpha = \beta$ and $\alpha = \gamma$, $\alpha, \beta, \gamma \in \{1, 2, 7\} \cap \{2, 3, 4\} = 2$. This leaves us with cycle $v_2v_5v_7v_3$ colored with only the color 2. That is, we have a monochromatic $C_4 \subseteq K_n$. ■

Odd Rainbow Cycle Lemma 3.23: 3.23

Assume $n \geq m \geq 9$ and m is odd. Let K_n be edge colored and suppose some $C_m \subseteq K_n$ is rainbow. Then there is a monochromatic or rainbow $C_4 \subseteq K_n$.

Proof. Let $C_m \subseteq K_n$ be m -edge colored numerically one through m . Suppose there is no rainbow $C_4 \subseteq K_n$.

Assume $m \geq 9$. Drop a chord on C_m so that one new cycle is a copy of C_4 . Since by our assumption C_4 is not rainbow. So C_{m-2} must be rainbow. Continue dropping chords on C_{m-2}, C_{m-4} , and so forth until we have a rainbow C_7 . By Rainbow C_7 Lemma 3.22, we have a monochromatic C_4 . ■

Rainbow Cycle Theorem 3.24: Assume $n \geq m \geq 6$. If K_n is edge colored and there is a rainbow $C_m \subseteq K_n$ then there is a either a monochromatic or rainbow $C_4 \subseteq K_n$.

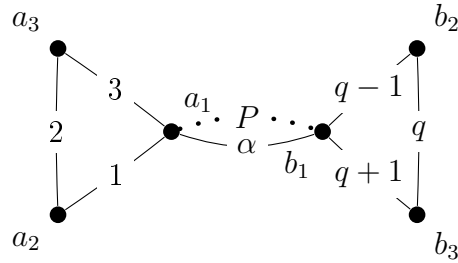


FIGURE 8. A rainbow subgraph $B_l \subseteq K_n$ with an additional edge, a_1b_1 , where $c(a_1b_1) = \alpha$.

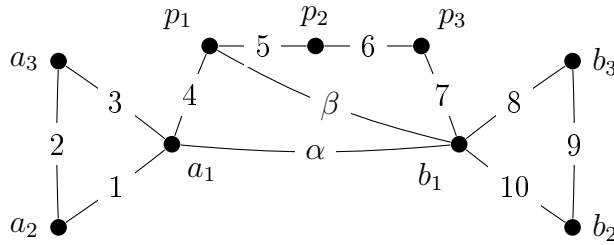


FIGURE 9. In Case ii in the proof of B_l -Lemma 3.25, if $\alpha \in \{8, 9, 10\}$, then either there is a rainbow C_4 in K_n , with edges colored 5, 6, 7, β , or there is a rainbow B_1 , with edges colored 1, 2, 3, 4, β, α .

B_l -Lemma 3.25: 3.25

Assume $n > 6$. Edge color K_n . Let $B_l \subseteq K_n$, where $l \geq 4$, be a connected rainbow subgraph containing two vertex disjoint C_3 's, where the C_3 's are connected by a path, P , with at least three edges, and $l = |E(P)|$. Then K_n contains a monochromatic or rainbow C_4 .

Proof. Suppose the ends of P are a_1 and b_1 . Then $a_1b_1 \in E(K_n) \setminus E(B_l)$ is an edge whose endpoints are vertices in each C_3 as depicted in Fig. 8. Let $c(a_1b_1) = \alpha$. Let $|V(B_l)| = q$. Either $\alpha \in \{1, 2, 3, q-1, q, q+1\}$ or not.

Case 1. Suppose $\alpha \notin \{1, 2, 3, q-1, q, q+1\}$, then we have a rainbow $B_2 \subseteq K_n$ and can use B_2 -Lemma 3.19 to conclude that K_n contains a monochromatic C_4

Case 2. If $\alpha \in \{1, 2, 3, q-1, q, q+1\}$, then the cycle $C = a_1p_1p_2 \dots b_1a_1$ is rainbow. Either $|E(P)| = 3$, $|E(P)| = 4$, or $|E(P)| \geq 5$.

Case i. Suppose $|E(P)| = 3$. This gives us a rainbow $B_1 \subseteq K_n$ and we can apply our B_1 -Lemma 3.15.

Case ii. Suppose $|E(P)| = 4$. Then $C \simeq C_5$. Without loss of generality, assume that $\alpha \in \{8, 9, 10\}$ (see Fig. 9) and consider the chord p_1b_1 of C in Fig. 9. Let its color be β . If $\beta \notin \{5, 6, 7\}$ then we have a rainbow C_4 in K_n . Otherwise, if $\beta \in \{5, 6, 7\}$ then we have a rainbow B_1 , whence, by B_1 -Lemma 3.15, there is a rainbow C_4 in K_n .

Case iii. Assume $|E(P)| \geq 5$. We can use our Rainbow Cycle Theorem 3.24 to show that K_n contains a rainbow or monochromatic C_4 .

■

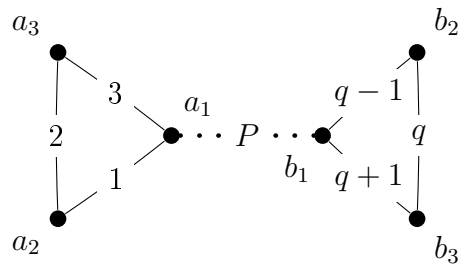


FIGURE 10. A rainbow subgraph $B_l \subseteq K_n$.

Proposition B_t 3.26: Assume $n > 6$. Edge color K_n . Let $B_t \subseteq K_n$ be a connected rainbow subgraph containing two edge disjoint C_3 's, where the C_3 's are connected by a path, P , of length $t - 1$. Then K_n contains either a monochromatic or rainbow C_4 .

Proof. This follows directly from the B_1 -Lemma 3.15, B_2 -Lemma 3.19, B_3 -Lemma 3.20, and B_t -Lemma 3.25. ■

8. PROOF OF C_4 MAX THEOREM 1.5

Theorem 3.27: For integers $n \geq 4$ and $i \geq 1$, $(n + i) \notin \text{MRS}(K_n; C_4, C_4)$.

Proof. Recall Corollary 3.17. This result states that $(n + i) \notin \text{MRS}(K_n; C_4, C_4)$ where $n \in \{4, 5\}$, and $i \geq 1$.

Now let K_n be $(n + i)$ -edge colored where $n \geq 6$ and $i \geq 1$. Assume that such a coloring forbids rainbow and monochromatic C_4 's. By Proposition 3.12, we know K_n contains at least two cycles in the same rainbow connected subgraph, say H . Let the collection of these rainbow cycles in H be denoted $rC(H)$. There are several cases we will consider where the largest rainbow cycle in $rC(H)$ is one of the following: C_m (for all $m \geq 6$), C_5 , or C_3 . There are also sub-cases we will soon address.

Case 1. Suppose the largest rainbow cycle in $rC(H)$ is C_m where $m \geq 6$. By the Rainbow Cycle Theorem 3.24, we have a monochromatic or rainbow $C_4 \subseteq K_n$.

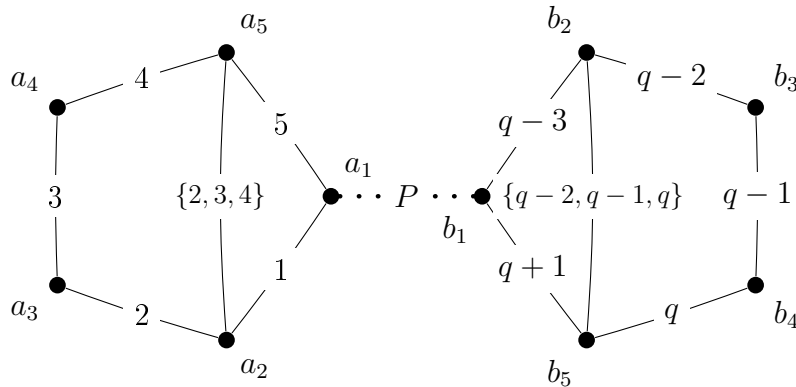


FIGURE 11. A rainbow $B_t \subseteq K_n$ among two mutually rainbow C_5 's each with a chord.

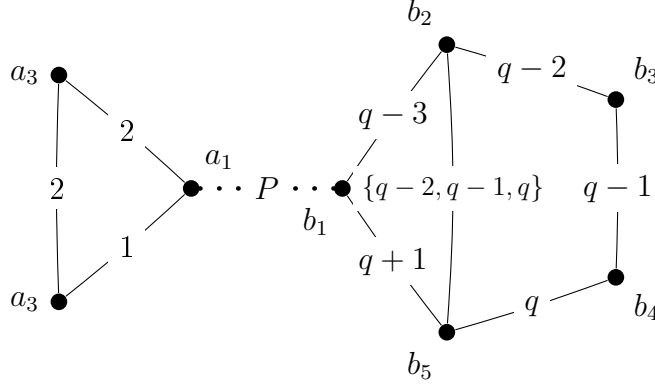


FIGURE 12. A rainbow $B_t \subseteq K_n$ among a rainbow C_3 and a rainbow C_5 with a chord.

Case 2. Suppose the largest rainbow cycle in $rC(H)$ is C_5 . By Proposition 3.12, let $C_\ell \in rC(H)$ be at least one other rainbow cycle in $rC(H)$. Either $\ell = 5$ or $\ell = 3$.

Case i. Let $\ell = 5$. The two C_5 's meet at the ends of a path, P , in H .

Draw two chords, as depicted in Fig. 11. By assumption there are no monochromatic or rainbow C_4 's, so we have a rainbow B_t graph. By Proposition B_t 3.26, we have monochromatic or rainbow copies of C_4 .

Case ii. Let $\ell = 3$. So a C_3 and a C_5 meet at the ends of path, P , in H .

Draw a chord across C_5 , as depicted in Fig. 12. By assumption there are no monochromatic or rainbow C_4 's, so we have a rainbow B_t graph. By Proposition B_t 3.26, we have monochromatic or rainbow copies of C_4 .

Case 3. Suppose the largest rainbow cycle in $rC(H)$ is C_3 . Then we have two C_3 's that are connected by a nonempty path in H , say P . It follows that we have a B_t graph which by Proposition B_t 3.26, we know that H has monochromatic or rainbow copies of C_4 .

Hence given a $(n + i)$ -edge colored K_n where $n > 6$ and $i \geq 1$, there is a monochromatic or rainbow $C_4 \subseteq K_n$. ■

It naturally follows from the previous theorems, in particular Theorem 1.4, that the C_4 Max Theorem 1.5 holds.

C_4 Max Theorem 1.5: *For every integer $n \geq 4$,*

$$\max \text{MRS}(K_n; C_4, C_4) = n.$$

Chapter 4. Forbidding monochromatic odd cycles and rainbow cycles in complete graphs

9. BALANCED BINARY TREES

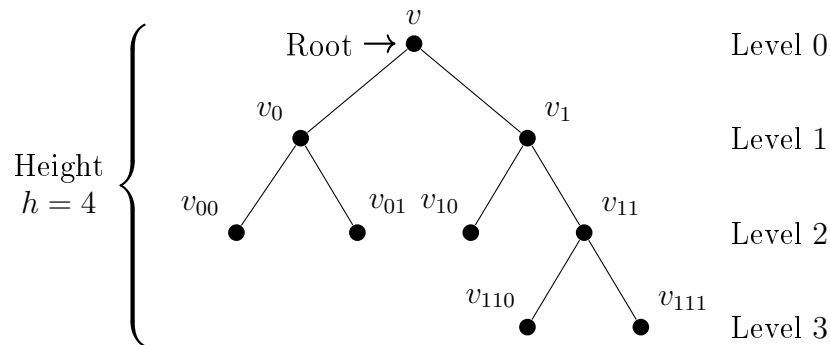


FIGURE 13. A balanced binary tree on nine vertices with five leaves, four levels, and a height of four.

For this dissertation a **balanced binary tree**, T , is an acyclic connected graph with a root vertex, v , and descendants presenting exclusively in pairs known as siblings or children. The vertex v is the only vertex to be located on level zero. The children of v , v_0 and v_1 , are on level one, v_0 's and v_1 's children are on level two, and so forth. The number of levels from level zero to the last level is known as the height of T . T 's final children are known as leaves and they are all located on the final level or the last two levels; see Fig 13.

Lemma 4.1. *The height of a balanced binary tree on q vertices is $\lceil \log_2 q \rceil + 1$.*

Proof. Let T be a balanced binary tree on q vertices and let level ℓ be the last level. Then $q = 1 + 2 + \dots + 2^{\ell-1} + 2t$ for some $t \in \{1, \dots, 2^{\ell-1}\}$. Therefore $2^\ell < q \leq 2^{\ell+1} - 1$. Since we start counting levels at zero, the height of T is $\ell + 1 = \lceil \log_2 q \rceil + 1$. ■

Our balanced binary trees are special cases of binary trees. In general, a binary tree on more than 3 vertices is a tree with one vertex of degree 2 (the root) and all other vertices of degrees either 3 or 1 (the leaves).

Lemma 4.2. *A binary tree with n leafs has $2n - 1$ vertices.*

Proof. Let q be the number of vertices of the tree. Then summing the degrees of those vertices, which will give us $2(q - 1)$, $q - 1$ being the number of edge. We have $2(q - 1) = n + 2 + 3(q - n - 1)$; solving we get $q = 2n - 1$. ■

10. PROOF OF MAIN RESULT

Our main result is Corollary 1.6, which follows from Theorems 4.7 and 4.10.

It is well known (see [10]) that for any coloring of the edges of K_n with n or more colors appearing there will be a rainbow cycle contained in K_n , whereas rainbow-cycle-forbidding colorings with $n - 1$ colors appearing are possible, and in every such coloring, monochromatic odd cycles are forbidden; so $n - 1$ is the largest integer in that spectrum. Our candidate for the smallest member is $k = \lceil \log_2 n \rceil$. We show that every integer between k and $n - 1$ is in the spectrum. If $X, Y \subseteq V(K_n)$ are disjoint then $[X, Y]$ denotes the set of edges in K_n with one end in X and one end in Y .

Theorem 4.3 (Gallai's Theorem [7]): *Suppose $n \geq 3$. Let k be a positive integer. In any k -edge coloring of K_n where there is no rainbow $K_3 \subseteq K_n$, there exists a partition of $V(K_n)$ into subsets V_1, V_2, \dots, V_t ($t \geq 2$) such that*

- (1) *for each pair i, j of integers with $1 \leq i < j \leq t$, all edges in $[V_i, V_j]$ are colored the same color,*
- (2) *the number of colors of the edges in the set $\bigcup_{1 \leq i < j \leq t} [V_i, V_j]$ is at most 2, and*
- (3) *no edge within the complete graph induced by V_i is colored with any of the $[V_i, V_j]$ colors.*

Lemma 4.4. *Suppose that G is a connected graph on $n > 1$ vertices. Then $V(G)$ can be partitioned into sets, A and B , satisfying the following*

- (1) $||A| - |B|| \leq 1$, and
- (2) $G[A]$ and $G[B]$ are connected,

if and only if G has a spanning tree, T , such that for some edge, $e \in E(T)$, such that, if T_1 and T_2 denote the two components of $T - e$, then $||V(T_1)| - |V(T_2)|| \leq 1$.

Proof. Suppose A, B partition $V(G)$, $||A| - |B|| \leq 1$, and $G[A], G[B]$ are connected. Note that $n > 1$ implies that $A \neq \emptyset \neq B$. Let T_1, T_2 be spanning trees in $G[A], G[B]$ respectively, so $A = V(T_1)$ and $B = V(T_2)$. Because G is connected, there must exist an edge e with one end in A and the other in B . Then $T = T_1 \cup T_2 \cup e$ is a tree satisfying the requirements given in the Lemma.

Conversely, if T and e satisfy those requirements, let $A = V(T_1)$ and $B = V(T_2)$. Then $||A| - |B|| \leq 1$, and $G[A], G[B]$ have spanning trees T_1, T_2 , respectively, and are therefore connected. ■

Theorem 4.5 (Hoffman, Horn, Johnson, and Owens [10]): *If G is a simple connected graph on n vertices, then there is a rainbow cycle forbidding edge coloring of G with $n - 1$ colors appearing.*

Lemma 4.6. *Suppose $n > 1$. An edge coloring of K_n is rainbow-cycle-forbidding if and only if it is a Gallai coloring.*

Proof. The forward implication is clear, since K_3 is a cycle. Now suppose that we have a coloring of the edges of K_n which is not rainbow-cycle-forbidding. We aim to show that there is a rainbow $K_3 = C_3$ in K_n , with respect to this coloring.

Let m be the smallest integer such that there is a rainbow C_m in K_n . If $m = 3$, we are done. Otherwise, consider any chord uv of this C_m . This chord makes, with the two edge-disjoint paths on the C_m with ends u, v two cycles, each of order $< m$. The color of uv can appear on at most one of those two uv paths, because the C_m is rainbow; but then there exists a smaller rainbow cycle in K_n , contradicting the choice of m . ■

Theorem 4.7: *For positive integers $n > 1$ and k , if $2^{k-1} < n \leq 2^k$ then $[k, \dots, n - 1] \subseteq \text{MRS}(K_n; \text{odd cycles, cycles})$.*

Proof. We will construct a balanced binary tree, T , representing a Gallai coloring, c . The vertices of T will be subsets of $V(K_n) = V$. The root will be the full vertex set V . For each vertex $X \subseteq V$, if $|X| = 1$ then X is a leaf of T . Otherwise, if $|X| > 1$, the two “children” of X at the next “level” of T will be sets Y, Z partitioning X , such that $||Y| - |Z|| \leq 1$. We will refer to Y and Z as “siblings.”

The edges of K_n will be colored as follows: for every pair Y, Z of siblings the edges $[Y, Z]$ will be colored with a single color that does not appear on any previously colored edge incident to a vertex in an ancestor of Y and Z .

We will enforce this restriction by the requirement that the sets of colors appearing on edges between siblings at different levels be disjoint. Thus, a color may appear on edges between different pairs of siblings, but it may not appear on edges between different pairs of siblings on different levels.

This requirement is not strictly necessary, but it does give us what we want. We shall see that every such coloring forbids rainbow cycles and monochromatic odd cycles, and the total number of colors appearing can be anything from $\lceil \log_2 n \rceil$ to $n - 1$.

To see this last claim, first note that the binary tree constructed will have n leafs, one for each vertex of K_n . Therefore, by Lemma 4.2, it will have $n - 1$ non-leafs, and each of these will have two sibling children, the edges between the sets of vertices corresponding to which will bear one of our colors. Thus we can arrange to have $n - 1$ colors appear in the coloring by making the colors assigned to the $n - 1$ sibling pairs distinct.

Now we can reduce the number of colors, one at a time, while honoring the requirement that the sets of colors assigned to the sets of sibling pairs at different levels be disjoint, by merging pairs of colors on the same level. For instance, if, at some stage, blue and burgundy both appear on (edges between sibling pairs on) same level (and therefore on no other level), we can recolor all burgundy edges blue, thus reducing the total number of colors appearing by one while preserving the disjointness of color sets on different levels.

We can continue counting down in this way until on each level after the zeroth only one color is assigned to sibling pairs on that level. At that point the number of different colors deployed is one less than the number of levels. By Lemma 4.1 that number is $\lceil \log_2 (2n - 1) \rceil - 1 = \lceil \log_2 n \rceil$ (Recall that $n > 1$) (Note that this number has not been proven to be $\min \text{MRS}(K_n; \text{odd cycles, cycles})$; that proof will come shortly.)

Observe that for any color appearing in any of colorings obtained as above, the subgraph of K_n induced by the set of edges bearing that color is union of vertex-disjoint complete

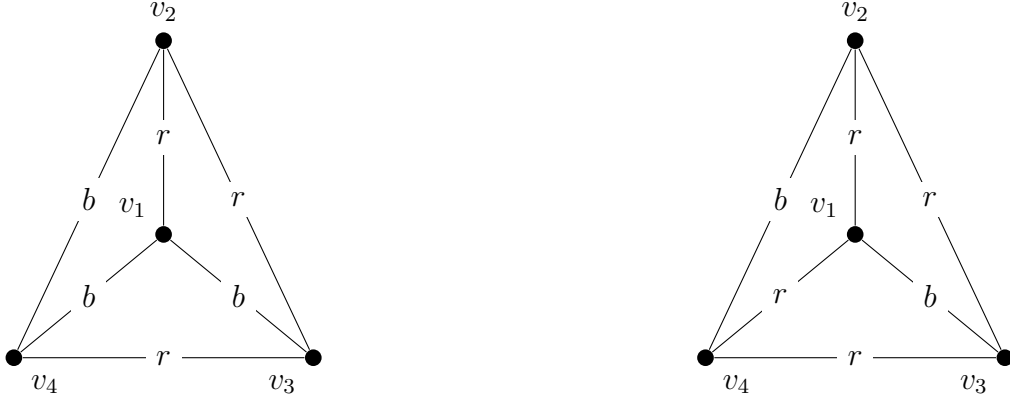


FIGURE 14. Essentially different two 2-edge colorings of K_4 that forbid monochromatic odd cycles.

bipartite graphs. Therefore, there are no monochromatic odd cycles in K_n with any of these colorings.

It remains to be seen that none of the colorings described allow a rainbow cycle in K_n . Let C be a cycle in K_n . Let X be a vertex of our bipartite graph therefore, a subset of $V(K_n)$ – such that $V(C) \subseteq X$ but, if Y, Z are the children of X , $V(C) \cap Y \neq \emptyset \neq V(C) \cap Z$.

Since C is a cycle, $E(C) \cap [Y, Z]$ must contain at least two edges. Therefore C is not rainbow.

■

Theorem 4.8: $R(K_3, K_3) = 6$ [8].

Lemma 4.9. $\min \text{MRS}(K_4; \{\text{odd cycles}\}, \{\text{cycles}\}) = 2$ and
 $\min \text{MRS}(K_5; \{\text{odd cycles}\}, \{\text{cycles}\}) = 3$

Proof. Clearly $\min \text{MRS}(K_4; \{\text{odd cycles}\}, \{\text{cycles}\}) > 1$, and Fig 14 gives two different edge colorings of K_4 with two colors that admit no monochromatic K_3 's and, obviously, no rainbow cycles. ("Obviously" because there are only two colors.)

Now suppose that the edges of K_5 are colored with red and blue so that no odd cycle in K_5 is monochromatic. Suppose that K_5 contains a monochromatic $K_{1,3}$ – suppose edges vx, vy, vz are colored red. If any of xy, xz, yz were red then there would be a red C_3 in the edge colored K_5 . Therefore all three of those edges are blue, so we have a monochromatic C_3 anyway.

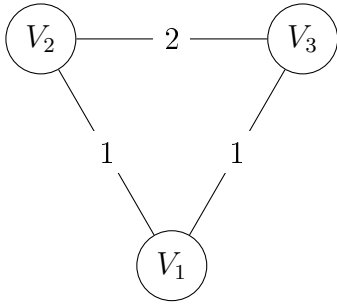


FIGURE 15. A Gallai partition with three parts and two colors.

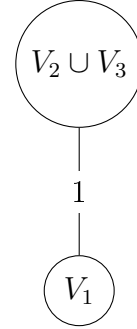


FIGURE 16. A Gallai partition with two parts and one color.

It follows that every vertex of K_5 is incident to two red and to two blue edges. The subgraph induced by the blue edges is therefore regular of degree two, so there must be a blue cycle in K_5 . It must be a C_4 , say on vertices v, w, x, y . Let z be the vertex of K_5 not in this C_4 . Of the four edges incident to z , two are blue, so there must be vertices of K_5 incident to three blue edges, a possibility that has already been ruled out. Thus no such coloring exists.

Since $3 = \lceil \log_2 5 \rceil$ there is an edge coloring of K_5 with 3 colors which forbids monochromatic odd cycles and rainbow cycles by Theorem 4.7. ■

Theorem 4.10: $\min \text{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\}) = k$ where $k = \lceil \log_2 n \rceil$.

Proof. The proof will be by induction on n . Assume that $n \geq 6$ and K_n is edge colored with k colors appearing so that rainbow cycles and monochromatic odd cycles are forbidden. Since there are no rainbow K_3 's in K_n , the coloring must be a Gallai coloring as described in Gallai's Theorem. The number of partitions K_n in that description, t , must be less than 6 as we know $R(K_3, K_3) = 6$ from Theorem 4.8. By Lemma 4.9, $t \neq 5$. So $t \in \{2, 3, 4\}$.

We lifted the following argument from Magnant and Nowbandegani [11]. Let $t = 3$. Since we are forbidding monochromatic odd cycles, we must use two colors among the three partitions, V_1, V_2 , and V_3 as depicted in the reduced graph in Fig 15. This put us in the $t = 2$ case where the edges from V_1 to $V_2 \cup V_3$ are colored with the color, 1, that was used twice, see Fig 16.

We are left with cases where $t \in \{2, 4\}$. Suppose $t = 2$ with partitioned vertex sets V_1 and V_2 . Let $n_1 = |V_1|$, $n_2 = |V_2|$, and $n_1 \leq n_2$. Let c_1 and c_2 be the number of colors in V_1 and V_2 respectively with $c_1 \geq \lceil \log_2 n_1 \rceil$ and $c_2 \geq \lceil \log_2 n_2 \rceil$. Now $n_2 \geq \frac{n}{2}$, so we have

$$k \geq c_2 + 1 \geq \lceil \log_2 n_2 \rceil + 1 \geq \left\lceil \log_2 \frac{n}{2} \right\rceil + 1 = \lceil \log_2 n \rceil - \lceil \log_2 2 \rceil + 1 = \lceil \log_2 n \rceil.$$

Now suppose $t = 4$ with partition vertex sets V_1, V_2, V_3 , and V_4 . Let $n_1 = |V_1|$, $n_2 = |V_2|$, $n_3 = |V_3|$, $n_4 = |V_4|$, and $n_1 \leq n_2 \leq n_3 \leq n_4$. Let c_1, c_2, c_3 , and c_4 be the number of colors in V_1, V_2, V_3 , and V_4 respectively with $c_1 \geq \lceil \log_2 n_1 \rceil$, $c_2 \geq \lceil \log_2 n_2 \rceil$, $c_3 \geq \lceil \log_2 n_3 \rceil$, and $c_4 \geq \lceil \log_2 n_4 \rceil$. Now $n_4 \geq \frac{n}{4}$,

$$k \geq c_4 + 2 \geq \lceil \log_2 n_4 \rceil + 2 \geq \left\lceil \log_2 \frac{n}{4} \right\rceil + 2 = \lceil \log_2 n \rceil - \lceil \log_2 4 \rceil + 2 = \lceil \log_2 n \rceil.$$

■

Corollary 1.6: $\text{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\}) = \{\lceil \log_2 n \rceil, \dots, n-1\}$.

Proof. This follows from Theorems 4.7 and 4.10.

■

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